Let us write $G(2,4) = \{$2-dimensional (vector) subspaces of $\mathbb{R}^4\}$ A plane $V \subseteq \mathbb{R}^4$ is specified by 4×2 matrix $A = [\vec{a}_1, \vec{a}_2] \in Mat_{4\times 2}$ where $\{\vec{a}_1, \vec{a}_2\}$ is a basis for V. I.e., given $A \in Mat_{4\times 2}$, $rkA = 2$, we get a 2-plane $V \subseteq \mathbb{R}^4$ by $V =$span $\{\vec{a}_1, \vec{a}_2\}$. Conversely, given any 2-dim subspace $V \subseteq \mathbb{R}^4$, there is a 4×2 matrix A with $rk(A) = 2$, from which V is obtained in the above way. Two matrices, A and B, determine the same subspace $V \iff \exists g \in GL_2(\mathbb{R})$, such that $B = Ag$.
We thus have the following setup. Let

$$F(2,4) = \{A \mid rkA = 2\} \subseteq Mat_{2\times 4}(\mathbb{R}) \simeq \mathbb{R}^8$$

and consider on it the equivalence relation.

$$B \sim A \text{ if } \exists\, g \in GL_2(\mathbb{R}),\ \text{s.t. } B = Ag$$

We have described a bijection of sets

$$F(2,4)/\!\sim\, \simeq G(2,4)$$

In this problem we argue that $G(2,4)$ can be equipped with a natural smooth structure.

First we show that $F(2,4) \subset Mat_{2\times 4}(\mathbb{R}) \simeq \mathbb{R}^8$ is an open subset

Lemma 0.0.1. *The rank of an $m \times n$ matrix is $r \iff$ some $r \times r$ minor does not vanish, and every $(r+1) \times (r+1)$ minor vanishes.*

Since we know that $rk(F(2,4)^\complement) < 2$, lemma 0.0.1 tells us all minors of an arbitary element $A \in F(2,4)^\complement$ vanish. Let's denote the determinant of each minor of A with $S_i,\ i \in (1,6)$ Then consider a continious map $\psi : Mat_{2\times 4} \to \mathbb{R}^6, \quad \psi(M) \to (S_1, S_2, S_3, S_4, S_5, S_6)$. We can express $F(2,4)^\complement = \psi^{-1}(0,0,0,0,0,0)$ because all minors vanish (det=0). A point $(0,0,0,0,0,0) \in \mathbb{R}^6$ is a closed set, and because continuity preserves the closeness, $F(2,4)^\complement$ is closed in $Mat_{2\times 4}$, and since its complement is closed $F(2,4)$ is an open subset of $Mat_{2\times 4}(\mathbb{R})$

Next we show that $\simeq$ is an open equivalence relation on $F(2,4)$. In other words we need to show that the map $\pi : F(2,4) \to F(2,4)/\!\sim$ is an open map.
Consider an open $u \subset F(2,4)$. Then the set $gu = \{gx | x \in u\}$ is an open subet of $F(2,4)/\!\sim$. Therefore $\pi^{-1}\pi(u) = \cup\, gu$ is an open in $F(2,4)$ because the union of open sets is open.
We have that π is a quotient map.

Lemma 0.0.2. *if $\beta = \{\beta_\alpha\}_\alpha$ is a base for a topology T on a topological space S, and if $f : S \to X$ is an open map, then the collection $\{f(\beta_\alpha)\}_\alpha$ is a base for the topology on X.*

Proof: Let v be an open in X and $y \in V$. Choose $x \in f^{-1}(y)$. Since $f^{-1}(V)$ is open there is a basis element $u \in \beta$ s.t. $x \in u \subset f^{-1}(v)$ which implies that $y \in f(u) \subset v$. Since y is arbitary, and $f(U) \subset f(\beta)$ the collection $\{f(\beta_\alpha)\}_\alpha$ is a base for the topology on X.

We have that $F(2,4)$ has a second countable base since the subsets keep the properties of the space ($\mathbb{R}^8$). Thus by lemma 0.0.2, we have that $F/{\sim}(2,4)$ is second countable.

We show that the graph of the equivalence relation on $F(2,4)$ is a closed subset of $F(2,4) \times F(2,4)$. I.e $R = \{(x,y) \in F(2,4) \times F(2,4) \mid y = gx\}$ is closed. We can consider R as a set of matrices. $[AB] = [\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2]$ of rank 2. 0.0.1 tells us that every 3×3 minor of an element in R must vanish. So if we take a map:

$$\psi : F(2,4) \times F(2,4) \to \mathbb{R}^{16}$$

ψ takes the determinant of every 3×3 minor. ψ is continuous and thus $R = \psi^{-1}(0,0,\ldots,0)$ is closed.

From the previous assertion we can conclude that $G(2,4)$ is Hausdorff.
That is because R is closed in $F(2,4) \times F(2,4)$, $(F(2,4) \times F(2,4)) \setminus R$ is open. $\implies \forall (x,y) \in (F(2,4) \times F(2,4)) \setminus R$ there is a basic open set $u \times v$ containing (x,y) s.t. $(u \times v) \cap R = \emptyset \implies \forall x \nsim y \in (F(2,4) \times F(2,4)) \setminus R, \exists$ u around x and v around y s.t. $u \cap v = \emptyset$ Thus for any two points $[x] \neq [y] \in F(2,4)/{\sim}$ there exist disjoint neighborhood of x and y and $F(2,4)/{\sim}$ (Nice pics insert)

For $A \in Mat_{4\times 2}$ denote by $A_{i,j}$ the 2×2 matrix, formed by the $i-th$ and $j-th$ rows of A ($1 \leq i \leq j \leq 4$). The set

$$v_{i,j} = \{A \mid det(A_{i,j}) \neq 0\} \subset F(2,4)$$

is open, because its complement is closed. We also have that $\forall g \in GL_2(\mathbb{R})$ if $A \in v_{i,j}$ then $Ag \in vi,j$, because the product of invertible matrices is invertible. Next, define $u_{i,j} = v_{i,j}/{\sim} = \pi(v_{i,j}) \subset G(2,4)$. $u_{i,j}$ is open since the equivalence relation is open.π is an open map. Thus $[A]$ has a canonical representative: $A \sim AA_{i,j}^{-1}$ For example if a minor $A_{2,4}$ is invertible we have that $A \sim A_{1,3}A_{2,4}^{-1} = \begin{pmatrix} * & * \\ 1 & 0 \\ * & * \\ 0 & 1 \end{pmatrix}$ A map $\phi_{3,4} : V_{3,4} \to Mat_{2\times 2}(\mathbb{R}) \simeq \mathbb{R}^4, \quad \phi_{3,4}(A) = A_{1,2}A_{3,4}^{-1}$ We have continuity because we are just multiplying by an invertible matrix. To show bijection we will explicitly write the inverse. Since $v_{3,4} \subset G(2,4)$, $\phi_{3,4}([A])^{-1} = A_{1,2}A_{3,4}^{-1}g = A_{1,2}$ Since $\cup v_{i,j}$ covers $F(2,4)$, $u_{i,j}$ covers $G(2,4)$ Finally, we check transition maps.

$$\phi_{1,2}([A])^{-1} \to A_{3,4}A_{1,2}^{-1}, \quad \phi_{1,2}^{-1}(u) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}$$

$$\phi_{2,4}([A]) \to A_{1,3}A_{2,4}^{-1}, \quad \phi_{2,4}^{-1}(v) = \begin{pmatrix} v_{1,1} & v_{1,2} \\ 1 & 0 \\ v_{2,1} & v_{2,2} \\ 0 & 1 \end{pmatrix}$$

$$\phi_{2,4} \circ \phi_{1,2}^{-1}(v) = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ v_{1,1} & v_{1,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ v_{2,1} & v_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ v_{1,2}v_{2,1} & v_{1,1} + v_{1,2}v_{2,2} \end{pmatrix}$$

Hence $G(2,4)$ can be equipped with the structure of a 4 dimensional smooth manifold

Next, we need to generalize the statements above, so that we get that the dimension of grassmannian is$G(k,n) = n * k - k^2$. We have as many charts as $k \times k$ minors.

Bibliography