Let $X \subseteq Mat_{3\times3}(\mathbb{R})$

$$X = \{M \mid M^T = M,\ M^2 = M,\ TrM = 1\}$$

We will describe a bijection between X and $\mathbb{RP}^2$. The above statement tells us that matrices are symmetric, idempotent and sum of it's eigenvalues is one. Spectral theorem tells us that real symmetric matrix is diagonizable. We can also conclude that eigenvectors of symmetric matrices, with distinct eigenvalues, are orthogonal. Let x and y be eigenvectors of a symmetric matrix M, with distinct eigenvalues λ and μ:

$$\lambda(xy) = (\lambda x)y = (Mx)y = x(M^T y) = x(My) = x(\mu y) = \mu(xy)$$

Therefore $(\lambda - \mu)xy = 0$, since λ and μ are distinct xy=0, thus orthogonal. Next, we can show that the eigenvalues of M can be only 0 and 1. Let v be an eigenvector, and λ a corresponding eigenvalue.

$$Mv = \lambda v$$

$$M^2 v = M(\lambda v) = \lambda M(v) = \lambda^2 v$$

Then solving for $\lambda^2 - \lambda = 0$, we get that λ can only be 0 or 1. Finally, the fact that $Tr(M) = 1$ tells us that eigenvalues of M are 0,0 and 1. Therefore $dim(ker(M)) = 2$ and $dim(Im(M)) = 1$. This is telling us that there is a whole plane, that is sent to zero vector, and all vectors in the image are sitting on a line.
Thus, applying a matrix operator M to a vector, is equivalent to projecting a vector to a line in R^3. So $M : R^3 \to R^3$ is the operator of orthogonal projection on the line $Im(M)$.
Now, let's explicitly define a map ϕ which given a line in R^3 outputs a corresponding matrix operator, that will orthogonally project all the vectors in R^3 to that line.

$$\phi : \mathbb{RP}^2 \to Mat_{3\times3} \quad \phi([x : y : z]) \to A_{3\times3}$$

To explicitly find A, note that we first need to find a unit vector along a line $[x : y : z] \in \mathbb{RP}^2$, we can do that by normalizing coordinates. $n = \frac{1}{\sqrt{x^2+y^2+z^2}}[x, y, z]^T$. Finally, to orthogonally project any $v \in R^3$ along n, we apply $(vn)n = vn^2$. We can then define ϕ as follows:

$$\phi([x : y : z]) = n^2 = \frac{1}{\sqrt{x^2+y^2+z^2}^2}[x, y, z]^T[x, y, z] = \frac{1}{x^2+y^2+z^2}\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}$$

Next, we argue that $\phi\mathbb{RP}^2 \to R^6$ is an embedding (R^6 because we are taking only non-symmetric lower triangular entries) Namely, we will show that the following function is an embedding.

$$\phi([x : y : z]) \to \frac{1}{x^2+y^2+z^2}(x^2, xy, xz, y^2, yz, z^2)$$

We proved that ϕ is injective, and we know that $\mathbb{RP}^2$ is compact, so we proceed to proving that ϕ is an immersion.

We can use the definition with local charts to prove it. Consider the following charts and maps.

$$u_0\{[x:y:z], x \neq 0\} \simeq \mathbb{R}^2$$

$$u_1\{[x:y:z], y \neq 0\} \simeq \mathbb{R}^2$$

$$u_2\{[x:y:z], z \neq 0\} \simeq \mathbb{R}^2$$

$$\psi_0 : v_0 \to \mathbb{R}^2, \quad \psi_0([x:y:z]) \to (\frac{y}{x}, \frac{z}{x}), \quad \psi_0^{-1}(s,t) \to [1:s,t]$$

$$\psi_1 : v_1 \to \mathbb{R}^2, \quad \psi_1([x:y:z]) \to (\frac{x}{y}, \frac{z}{y}), \quad \psi_1^{-1}(s,t) \to [s:1:t]$$

$$\psi_2 : v_2 \to \mathbb{R}^2, \quad \psi_2([x:y:z]) \to (\frac{x}{z}, \frac{y}{z}), \quad \psi_2^{-1}(s,t) \to [s:t:1]$$

These local charts cover all the points in $\mathbb{RP}^2$, to prove that ϕ is an immersion we need to show the following, for all $p \in R^2$

$$rank(J(\phi \circ \psi_i^{-1}(s,t))) = 2$$

for $i \in 1, 2, 3$.
We check for ψ_0.

$$\phi \circ \psi_0^{-1}(s,t) = [1:s:t] \to (1, s, t, s^2, st, t^2)\frac{1}{1+s^2+t^2}$$

$$J(\phi \circ \psi_0^{-1}(s,t)) = \frac{1}{(1+s^2+t^2)^2}\begin{pmatrix} 0 & 0 \\ t^2-s^2+1 & 0 \\ 0 & s^2-t^2+1 \\ 2s(t^2+1) & 9 \\ t(t^2-s^2+1) & s(s^2+1-t^2) \\ 0 & 2t(s^2+1) \end{pmatrix}$$

To check for the line $[x:0:0]$, $x \neq 0$, we check when $(s,t) = (0,0)$. $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ More generally,

to check for $[x, 0, z]$, $x \neq 0$, we check for $(s,t) = (0,t)$, $\begin{pmatrix} 0 & 0 \\ t^2 & 0 \\ 0 & 1-t^2 \\ 0 & 0 \\ t(t^2+1) & 0 \\ 0 & 2t \end{pmatrix}$

Here we have that $\Delta_{2,3} \neq 0$, thus $rank = 2$

Also, we check when $t = 0$ $\begin{pmatrix} 0 & 0 \\ 1-s^2 & 0 \\ 0 & s^2+1 \\ 2s & 0 \\ 0 & s(s^2+1) \\ 0 & 0 \end{pmatrix} \Delta_{2,5} \neq 0$

Points in $\mathbb{RP}^2$ left to check are:

- $x \neq 0$, which we can check using ψ_1 and v_1
- After that, the only point left is $[0:0:1]$ which we check using ψ_2 and v_2.

We can conclude that: The map ϕ maps $\mathbb{RP}^2$ diffeomorphically onto $\mathbb{R}^9$ [HWG16] [JMC+19]

Bibliography

[HWG16] Zhiwu Huang, Jiqing Wu, and Luc Van Gool. Building deep networks on grassmann manifolds. *CoRR*, abs/1611.05742, 2016.

[JMC+19] Linxi Jiang, Xingjun Ma, Shaoxiang Chen, James Bailey, and Yu-Gang Jiang. Black-box adversarial attacks on video recognition models. *CoRR*, abs/1904.05181, 2019.