Let us write $G(p,n) = \{$p-dimensional (vector) subspaces of $\mathbb{R}^{n\times p}\}$. A hyperplane $V \subseteq \mathbb{R}^{n\times p}$ is specified by $n \times p$ matrix $A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a_p}] \in Mat_{n\times p}$ where $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_p\}$ is a basis for V. I.e., given $A \in Mat_{n\times p}$, rk$A = p$, we get a p-hyperplane $V \subseteq \mathbb{R}^{n\times p}$ by $V =$span $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_p\}$. Conversely, given any p-dim subspace $V \subseteq \mathbb{R}^{n\times p}$, there is a $n \times p$ matrix A with rk$(A) = p$, from which V is obtained in the above way. Two matrices, A and B, determine the same subspace $V \iff \exists g \in GL(p)$, such that $B = Ag$. $GL(p)$ stands for general linear group of degree p over real field.

We thus have the following setup. Let the set of all $\ddot{2}$-frames $\ddot{\text{b}}$e

$$F(p,n) = \{A \mid rkA = p\} \subseteq Mat_{n\times p}(\mathbb{R}) \simeq \mathbb{R}^{n\times p}$$

and consider on it the equivalence relation

$$B \sim A \text{ if } \exists\, g \in GL(p), \text{ s.t. } B = Ag$$

We have described a bijection of sets

$$F(p,n)/\!\sim\, \simeq G(p,n).$$

We will show that $G(p,n)$ is a manifold equipped with a natural smooth structure. To achieve that we need to prove that:

- $G(p,n)$ has a countable base
- $G(p,n)$ is Hausdorff
- $G(p,n)$ is locally euclidean

Proposition 0.0.1. *$F(p,n)$ is an open subset of* $\mathbb{R}^{p\times n}$

Lemma 0.0.1. *The rank of an $m \times n$ matrix is $r \iff$ some $r \times r$ minor does not vanish, and every $(r+1) \times (r+1)$ minor vanishes.*

Since we know that for $M \in F(p,n)^{\complement}$, rk$M < p$ lemma **??** tells us all $p \times p$ minors of an arbitary element $A \in F(p,n)^{\complement}$ vanish. Let's denote the determinant of each minor of A with $S_i,\ i \in (1, 2, \dots, \binom{n}{p})$ Then consider a continuous map $\psi : Mat_{n\times p} \to \mathbb{R}^{\binom{n}{p}},\quad \psi(M) \to (S_1, S_2, \dots, S_{\binom{n}{p}})$. We can express $F(p,n)^{\complement} = \psi^{-1}(\vec{0})$ because all minors vanish (det=0). A point $\vec{0} \in \mathbb{R}^{\binom{n}{p}}$ is a closed set, and because continuity preserves the closedness, $F(p,n)^{\complement}$ is closed in $Mat_{n\times p}$, an since its complement is closed, $F(p,n)$ is an open subset of $Mat_{n\times p}(\mathbb{R})$

Proposition 0.0.2. *$\sim$ is an open equivalence relation on $F(p,n)$*

In other words we need to show that the map $\pi : F(p,n) \to F(p,n)/\!\sim$ is an open map. Then π is a quotient map and is equipped with $F(p,m)/\!\sim$ is equipped with quotient topology.

Lemma 0.0.2. *A subset of a quotient space is open if and only if its preimage under the canonical projection map is open in the original topological space.*

Let U be an open in $F(p,n)$. Then for every $g \in GL(p)$ the set $Ug = \{xg|x \in U\}$ is an open subset of $F(p,n)$. Therefore $\pi^{-1}\pi(U) = \bigcup_{g \in G} Ug$ is an open in $F(p,n)$ because the union of open sets is open. And by **??** $\pi(U) = [U]$ is open in $G(p,n)$. π is a canonical quotient map, and $F(p,n)/\sim$ is open in $\mathbb{R}^{n \times p}$.

Lemma 0.0.3. *if $\beta = \{\beta_\alpha\}_\alpha$ is a base for a topology $\mathcal{T}$ on a topological space S, and if $f : S \to X$ is an open map, then the collection $\{f(\beta_\alpha)\}_\alpha$ is a base for the topology on X.*

Proof: Let V be an open in X and $y \in V$. Choose $x \in f^{-1}(y)$. Since $f^{-1}(V)$ is open there is a basis element $U \in \beta$ s.t. $x \in U \subset f^{-1}(V)$ which implies that $y \in f(U) \subset V$. Since y is arbitary, and $f(U) \subset f(\beta)$ the collection $\{f(\beta_\alpha)\}_\alpha$ is a base for the topology on X.

We have that $F(p,n)$ has a second countable base since it is a subspace of $\mathbb{R}^{n \times p}$. Thus by lemma **??**, we have that the base of $G(p,n)$ is second countable.

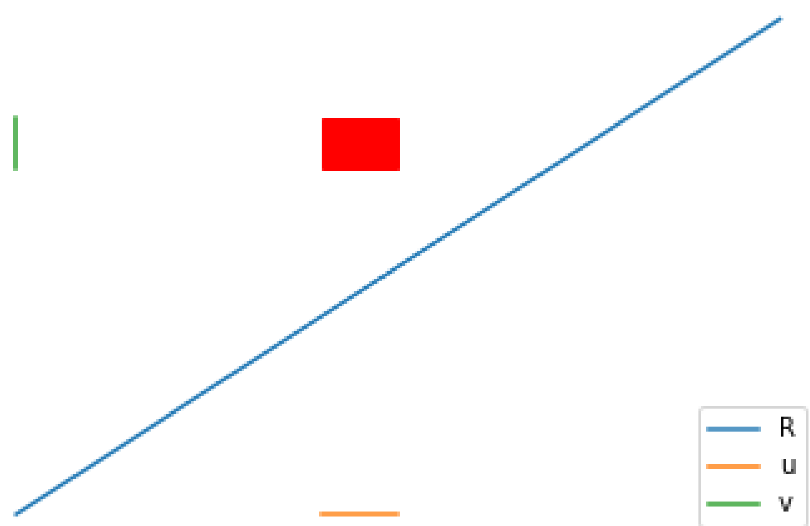
Proposition 0.0.3. *The graph of the equivalence relation on $F(p,n)$ is a closed subset of $F(p,n) \times F(p,n)$. i.e. $R = \{(A,B) \in F(p,n) \times F(p,n) \mid A = Bg\}$ is closed.*

We can consider R as a set of matrices $[AB] = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_p, \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p]$ of rank p. Lemma **??** tells us that every $(p+1) \times (p+1)$ minor of an element in R must vanish. Consider the map that assigns to (A,B) the values of all $(p+1) \times (p+1)$ minors

$$\psi : F(p,n) \times F(p,n) \to \mathbb{R}^{\binom{n}{3}(p+2)}$$

Since ϕ is continuous (as all of its components are polynomials) and $R = \psi^{-1}(0)$, then R is closed.

Example 0.0.1. For example take $G(2,4)$, then $\phi : Mat_{4\times 2} \to \mathbb{R}^{16}$



Proposition 0.0.4. *$G(p,n)$ is Hausdorff.*

That is because R is closed in $F(p,n) \times F(p,n)$, $(F(p,n) \times F(p,n)) \setminus R = R^{\complement}$ is open. $\implies \forall (x,y) \in R^{\complement}$ there is a basic open set $u \times v$ containing (x,y) s.t. $(u \times v) \cap R = \emptyset \implies$ $\forall x, y$ s.t. $(x,y) \notin R, \exists$ u around x and v around y s.t. $u \cap v = \emptyset$ Thus for any two points $[x] \neq [y] \in F(p,n)/\sim$ there exist disjoint neighborhood of x and y and $F(2,4)/\sim$ which is exactly the definition of Hausdorff property.

Proposition 0.0.5. *$G(p,n)$ is locally euclidean.*

Now that we have Hausdorff property and secound countable basis, we need to prove that every point lying on a manifold has a neighbourhood that is homeomorphic to an open in $\mathbb{R}^n$. Then we can claim that $G(p,n)$ is a manifold.
First we define charts. Take $A \in Mat_{n \times p}$ denote by A_k, ($k \in$ all possible picks of p from the set $[1, \ldots, n]$) the $p \times p$ minor, formed by the k_1th $\ldots k_p$th rows of A. The set

$$U_k = \{A \mid \det(A_k) \neq 0\} \subset F(p,n)$$

is open, because its complement is closed. We also have that $\forall g \in GL(p)$ if $A \in U_k$ then $Ag \in U_k$. Indeed, because $\det(Ag) = \det(A)\det(g)$, $\det((Ag)_{i,j}) \neq 0$ which means Ag will belong to a set U_k Next, define

$$V_k = U_k/\sim = \pi(U_k) \subset G(p,k)$$

The set V_k is open since the equivalence relation is open. i.e. π is an open map.
U_k has a canonical representative $A \sim \widehat{AA_k^{-1}}$. $\widehat{\cdot}$ discards all the rows whose index is in k. Similarly V_k has a canonical representative: $[A] \sim [\widehat{AA_k^{-1}}]$

Example 0.0.2. Following the previous example consider $[A] \in G(2,4)$. If a minor $A_{2,4}$ is invertible we have that $[A] \sim [\widehat{AA_{2,4}^{-1}}] = \widehat{\begin{bmatrix} * & * \\ 1 & 0 \\ * & * \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$. Since charts $\bigcup U_k$ cover $F(p,n)$, charts V_k cover $G(p,n)$ (because π is open).

Now we define homeomorphisms between charts V_k and opens in $R^{p \times (n-p)}$ as follows:

$$\phi_k : V_k \subset G(p,n) \to Mat_{(n-p)\times p}(\mathbb{R}) \simeq \mathbb{R}^{(n-p)\times p}, \quad \phi_k([A]) = \widehat{AA_k^{-1}}$$

We can show that ϕ is well defined.
Let $A, A' \in [A]$ we will show that ϕ is well defined. Equivalently $\phi_k(A) = \phi_k(A'])$ Since A and A' are in the same class, we have that $A' = Ag, \quad g \in GL(p)$, $\phi_k(A) = AA_k^{-1}$.

$$\phi_k(A') = \phi_k(Ag) = Ag((Ag)_k)^{-1} = Ag(A_k g)^{-1} = Agg^{-1}A_k^{-1} = AIA_k^{-1} = AA_k^{-1} = \phi_k(A)$$

ϕ is continuous because matrix multiplication is continous. Next, we can see that ϕ is surjective and ϕ^{-1} is continuous by explicitly defining inverse.

$$\phi_k^{-1}(\begin{pmatrix} — \alpha_1 — \\ \vdots \\ — \alpha_{n-p} — \end{pmatrix}) = \begin{pmatrix} 1_1 \\ \vdots \\ 1_p \\ \alpha_1 \\ \vdots \\ \alpha_{n-p} \end{pmatrix}$$

Finally, to show that ϕ is a homeomorphism, we have left to shot that ϕ is injective.
Assume that there ϕ_k is not injective then there are $A \in [A]$ and $B \in [B]$ such that there is **no** $g \in GL(p)$ for which $Ag = B$. i.e. $AA_k^{-1} = BB_k^{-1} \iff AA_k^{-1}B_k = B$ but $A_k^{-1}B_k \in GL(p)$ thus we reach contradiction. Therefore ϕ_k is homeomorphism

Example 0.0.3. Let $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad [A] \in V_{3,4}$

$$AA_{3,4}^{-1} = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -9 & 5 \\ -\frac{9}{2} & \frac{5}{2} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the above multiplication is continuous by **??** and we can exlcude rows 3 and 4 so that we get result in R^4. Then the restriction to R^4 is also continuous.

$$\phi_{3,4}([A]) = \begin{pmatrix} -9 & 5 \\ -\frac{9}{2} & \frac{5}{2} \end{pmatrix}$$

Next the inverse map $\phi_{3,4}(\beta)^{-1}$ $\beta \in Mat_{2\times 2} = \phi_{3,4}(A_{1,2}A_{3,4}^{-1}g) = [A]$ for some matrix A, such that $A_{1,2} = \beta$. But if we pick $g = A_{3,4}$ then $\phi_{3,4}(\beta) = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ More generally

$$\phi_{i,j}^{-1} : \mathbb{R}^4 \to v_{i,j} \subset G(2,4) \quad \phi_{i,j}^{-1}(\beta) \to \begin{bmatrix} \beta \\ I_{2\times 2} \end{bmatrix} = [\alpha]$$

Such that $\alpha[i :] = \beta[1 :]$, $\alpha[j :] = \beta[2 :]$, $\alpha[(I \setminus \{i,j\})[1]] = I[1 :]$ and $\alpha[(I \setminus \{i,j\})[2]] = I[2 :]$

Example 0.0.4. $\phi_{3,4}^{-1}(\alpha) = [A]$ as defined in **??**

$$\phi_{3,4}^{-1} \begin{pmatrix} -9 & 5 \\ -\frac{9}{2} & \frac{5}{2} \end{pmatrix} = \begin{bmatrix} -9 & 5 \\ -\frac{9}{2} & \frac{5}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can confirm that $\alpha = \begin{bmatrix} -9 & 5 \\ -\frac{9}{2} & \frac{5}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$ span the same subspace. Because if we take $g = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ then $\alpha g = A$

Since $\bigcup U_{i,j}$ covers $F(2,4)$, $\cup v_{i,j}$ covers $G(2,4)$ Finally, we check transition maps.

$$\phi_{1,2}([A])^{-1} = A_{3,4}A_{1,2}^{-1}, \quad \phi_{1,2}^{-1}(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}$$

$$\phi_{2,4}([A]) = A_{1,3}A_{2,4}^{-1}, \quad \phi_{2,4}^{-1}(v) = \begin{pmatrix} v_{1,1} & v_{1,2} \\ 1 & 0 \\ v_{2,1} & v_{2,2} \\ 0 & 1 \end{pmatrix}$$

$$\phi_{2,4}\circ\phi_{1,2}^{-1}(v) = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v_{1,1} & v_{1,2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ v_{2,1} & v_{2,2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ v_{1,1} & v_{1,2} \end{pmatrix} \begin{pmatrix} v_{2,2} & -1 \\ -v_{2,1} & 0 \end{pmatrix} \frac{1}{-v_{2,1}} =$$

$$-\frac{1}{v_{2,1}} \begin{pmatrix} v_{2,2} & -1 \\ v_{1,1}v_{2,2} - v_{1,2}v_{2,1} & -v_4 \end{pmatrix}$$

Now let's check the transition map $\phi_{3,4} \circ \phi_{2,3}^{-1}$

$$\phi_{3,4} \circ \phi_{2,3}^{-1}(v) = \begin{pmatrix} v_{1,1} & v_{1,2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ v_{2,1} & v_{2,2} \end{pmatrix} = -\frac{1}{v_{2,1}} \begin{pmatrix} v_{1,1}v_{2,2} + v_{1,2}v_{2,1} & -v_{1,1} \\ v_{2,2} & -1 \end{pmatrix}$$

Hence $G(2,4)$ can be equipped with the structure of a 4 dimensional smooth manifold. However, we can also consider $G(2,4)$ as a quotient of Stiefel Manifold. We will define a Stiefel manifold as a set of orthonormal k frames.

$$V(n,k) \subseteq F(n,k); \quad V(n,k) = \{A \mid rkA = 2,\ A^T A = I\}$$

$$\begin{tikzcd} V(2,4) \arrow[r, hook] \arrow[dr, dashed, "p"'] & F(2,4) \arrow[d, "\pi"] \\ & G(2,4) \end{tikzcd}$$

Map p is surjective because every subspace has an orthonormal basis. I.e. starting with any basis we can construct an orthonormal one via Gram-Schmidt algorithm. Now, if we redefine the map p such that $p : F(2,4)/_{O(2)} \to G(2,4)$, where $O(2)$ is the orthogonal group of $2-frames$, we will have a bijection. Therefore there are two ways we can consider $G(2,4)$

$$G(2,4) = F(2,4)/_{Gl_2(\mathbb{R})} \quad G(2,4) = V(2,4)/_{O(2)}$$

Chapter 1

Tangent Spaces

Let's define tangent spaces on $G(2,4)$. First consider the equivalence relation defined as:

$$(p,(U,\phi),v) \sim (p,(V,\psi),w) \text{ if } w = J(\psi \circ \phi^{-1})_{\phi(p)} v$$

Consider tanget space on $G(2,4)$ at X, it's given by

$$T_p X = \{(p,(U,\phi),v) \mid (U,\phi) \text{ is a chart around } p, v \in \mathbb{R}^n\}/\sim$$

Hence a tangent vector to $G(2,4)$ at $p \in G(2,4)$ is an equivalece class $[(p;(U;\phi);v)]$ Whenever we have fixed the chart $(U;\phi)$, the class is represented just by a vector $v \in \mathbb{R}^n$. To represent the same equivalence class in another chart, we multiply $v \in \mathbb{R}^n$ by the respective Jacobi matrix. Jacobi matrix of transition map $\phi_{3,4} \circ \phi_{2,3}^{-1}$ is given by:

$$J(\phi_{3,4} \circ \phi_{2,3}^{-1}) = \begin{pmatrix} -v_{2,2} & 1 & 0 & -v_{1,1} \\ \frac{1}{v_{2,1}} & 0 & -\frac{v_{1,1}}{v_{2,1}^2} & 0 \\ 0 & 0 & \frac{v_{2,2}}{v_{2,1}^2} & -\frac{1}{v_{2,1}} \\ 0 & 0 & -\frac{1}{v_{2,1}^2} & 0 \end{pmatrix}$$

Example 1.0.1. Consider the matrix a from the example **??**. We have that $\phi_{3,4}([A]) = \begin{pmatrix} -9 & 5 \\ -\frac{9}{2} & \frac{5}{2} \end{pmatrix}$ and $\phi_{2,3} = \begin{pmatrix} 2 & 0 \\ 0.4 & 1.8 \end{pmatrix}$

$$J(\phi_{3,4} \circ \phi_{2,3}^{-1}) = \begin{pmatrix} -1.8 & 1 & 0 & 0 \\ 2.5 & 0 & -12.5 & 0 \\ 0 & 0 & 11.25 & -2.5 \\ 0 & 0 & -6.25 & 0 \end{pmatrix}$$

Then

$$(p,(U,\phi_{2,3}),v) \sim (p,(V,\phi_{3,4}),w) \text{ if } w = J(\phi_{3,4} \circ \phi_{2,3}^{-1})_{\phi_{2,3}(p)} v$$

How to project on such tangent spaces ?

Chapter 2

$Gr(2,4)$ is compact

We'll endow $R^{n\times k}$ with the norm induced by the Frobenius inner product:

$$||A|| = \sqrt{tr(A^TA)} = \sqrt{\sum_{i,j} a_{i,j}^2}$$

$$< A, B >= \sqrt{tr(A^TB)}$$

We define the compact Stiefel manifold as

$$St(2,4) = \{A \in F(2,4), \quad A^TA = I_k\}$$

It's closed because under the continuous map $A \to A^TA$ it maps to I_k, and it's bounded by $\sqrt{k}$ because each column of A has norm 1, so $||A|| = \sqrt{k}$.

Chapter 3

Optimization on Grassmanian as quotient mfld

Def If the manifold $X \in \mathbb{R}^N$ happens to be the level set $X = f^{-1}(c)$ of a smooth function $f : U \to \mathbb{R}^k$ over a regular value $c \in \mathbb{R}^k$, then

$$T_pX = \ker df_p$$

We show that defining function is $h : \mathbb{R}^{n\times p} \to Sym(p) : X \to h(X) = X^TX - I_p$ and regular value 0. Thus tangent spaces are defined as:

$$T_xSt(n,p) = kerDh(X) = \{V \in \mathbb{R}^n \times p : X^TV + V^TX = 0\}$$

Also, more explicitly we can arrive to:

$$T_xSt(n,p) = \{X\Omega + XB_\perp : \Omega \in Skew(p),\ B \in \mathbb{R}^{(n-p)\times p}\}$$

$$R_X(V) = (X+V)(I_p + V^TV)^{-\frac{1}{2}}$$

$$Proj_x(U) = (I - XX^T)U + X\frac{X^TU - U^TX}{2}$$

$$gradf(X) = Proj_X(grad\bar{f}(X)) = grad\bar{f}(X) - Xsym(X^Tgrad\bar{f}(X))$$

Wher $sym(M) = \frac{M+M^T}{2}$

Gradient Desccent

$$x_{k+1} = R_{x_k}(-\alpha_k gradf(x_k))$$

where $alpha_k$ is the step size, and x_1 is some random point.

Chapter 4

Next Steps

Next, we need to generalize the statements above, so that we get that the dimension of grassmannian is$G(k, n) = nk - k^2$. We have as many charts as $k \times k$ minors. Every $n \times k$ matrix has $\binom{n}{p}$ $k \times k$ minors.

Chapter 5

Rayleigh Quotient Newton's Method

Consider a Rayleigh quotient function $\bar{f} = \frac{x^T M x}{x^T x}$, if we restrict the points to lie on the sphere the function comes down to $\bar{f} = x^T M x$. Also we can easily calculate that the $grad\bar{f} = 2Mx$ Since the gradient on the manifold is given as $gradf = Pgrad\bar{f}$ we can caculate that $gradf = (I - xx^T)2Mx = 2Mx - (x^T Mx)x$, Next we find the differential of that gradient: $D(gradf) = 2M - 2x^T Mx + 4Mxx^T$. We take the canonical dot product defined as $g_c(\Delta, \Delta) = tr\Delta^T(I - \frac{1}{2}YY^T)\Delta$ Therefore $g_c(D(gradf), \eta) = 2M\eta - 2x^T Mx\eta + 2Mxx^T\eta$. Next to get the hessian:

$$Hess(f) = Pg_c(D(gradf, \eta)) = 2(PM\eta - Px^T Mx\eta + PMxx^T\eta)$$

. $P = (I - xx^T)$ is a projection operator, and notice that we can write

$$PMP\eta = PM(I - xx^T)\eta = PM\eta - PMxx^T\eta$$

Also in the term $-Px^T Mx\eta$ we can remove P and put η in front becayse $x^T Mx$ is a scalar. Therefore we get

$$Hessf = PMP\eta - \eta x^T Mx$$

Finally we go to the newton iteration

$$Hess(f)\eta = -grad(f)$$

and replace and get:

$$PMP\eta - \eta x^T Mx = -PMx \tag{5.1}$$

Therefore our Newton iteration is:

1. Solve **??**

2. Set x$_{new} = R_(x_k)(\eta_k)$