Let $X \subseteq Mat_{3 \times 3}(\mathbb{R})$

$$X = \{M \mid M^T = M, \ M^2 = M, \ TrM = 1\}$$

We will describe a bijection between X and $\mathbb{RP}^2$. The above statement tells us that matrices are symmetric, idempotent and sum of it's eigenvalues is one. Spectral theorem tells us that real symmetric matrix is diagonizable. We can also conclude that eigenvectors of symmetric matrices, with distinct eigenvalues, are orthogonal. Let x and y be eigenvectors of a symmetric matrix M, with distinct eigenvalues λ and μ:

$$\lambda(xy) = (\lambda x)y = (Mx)y = x(M^T y) = x(My) = x(\mu y) = \mu(xy)$$

Therefore $(\lambda - \mu)xy = 0$, since λ and μ are distinct $xy=0$, thus orthogonal. Next, we can show that the eigenvalues of M can be only 0 and 1. Let v be an eigenvector, and λ a corresponding eigenvalue.

$$Mv = \lambda v$$

$$M^2 v = M(\lambda v) = \lambda M(v) = \lambda^2 v$$

Then solving for $\lambda^2 - \lambda = 0$, we get that λ can only be 0 or 1. Finally, the fact that $Tr(M) = 1$ tells us that eigenvalues of M are 0,0 and 1. Therefore $dim(ker(M)) = 2$ and $dim(Im(M)) = 1$. This is telling us that there is a whole plane, that is sent to zero vector, and all vectors in the image are sitting on a line.

Thus, applying a matrix operator M to a vector, is equivalent to projecting a vector to a line in R^3. So $M : R^3 \to R^3$ is the operator of orthogonal projection on the line $Im(M)$.

Now, let's explicitly define a map ϕ which given a line in R^3 outputs a corresponding matrix operator, that will orthogonally project all the vectors in R^3 to that line.

$$\phi : \mathbb{RP}^2 \to Mat_{3 \times 3} \quad \phi([x : y : z]) \to A_{3 \times 3}$$

To explicitly find A, note that we first need to find a unit vector along a line $[x : y : z] \in \mathbb{RP}^2$, we can do that by normalizing coordinates. $n = \frac{1}{\sqrt{x^2+y^2+z^2}}[x, y, z]^T$. Finally, to orthogonally project any $v \in R^3$ along n, we apply $(vn)n = vn^2$. We can then define ϕ as follows:

$$\phi([x : y : z]) = n^2 = \frac{1}{\sqrt{x^2 + y^2 + z^2}^2}[x, y, z]^T [x, y, z] = \frac{1}{x^2 + y^2 + z^2}\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}$$

Bibliography