We will show that S^2 is a smooth manifold. Let's denote a sphere as:

$$M = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

Also, we define a disk with a center at (x_0, y_0) and radius ϵ as:

$$D_\epsilon(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < \epsilon^2\}$$

Thus we can cover the sphere with 6 charts and 6 functions, described as follows:

$$u_i^+ = S^2 \cap \{x_i > 0\}, \quad u_i^- = S^2 \cap \{x_i < 0\}$$

$$\varphi_i^+ : u_i^+ \to D_1(0,0), \quad \varphi(x_1, x_2, x_3) = (\dots \hat{x}_i \dots)$$

$$\varphi_i^- : u_i^- \to D_1(0,0), \quad \varphi(x_1, x_2, x_3) = (\dots \hat{x}_i \dots)$$

Then the atlas covering the sphere is:

$$A_1 = \{(u_i^\pm, \varphi_i^\pm) : i \in I\}, \quad I = \{1, 2, 3\}$$

For example $\varphi_3^+ : u_3^+ \to \mathbb{R}^2, \quad \varphi_3^+(x_1, x_2, x_3) = (x_1, x_2, \hat{x_3}) = (x_1, x_2)$

We first show that a function $\varphi_3^+ : u_3^+ \to \mathbb{R}$ is injective. Assume $\exists P_1$ and P_2 s.t. $P_1 = (p_1, p_2, p_3)$, and $P_2 = (p_1\prime, p_2\prime, p_3\prime)$, $P_1 \neq P2$, and $f(P_1) = f(P_2) \iff (p_1, p_2) = (p_1\prime, p_2\prime)$
Because P_1 and P_2 are the points on the sphere: $p_3\prime = \pm\sqrt{1 - p_1^2 - p_2^2} \quad \pm p_3\prime = \pm\sqrt{1 - p_1\prime^2 - p_2\prime^2}$
Since u_3^+ has only positive numbers for a third coordinate $p_3 = p_3\prime \implies (p_1, p_2, p_3) = (p_1\prime, p_2\prime, p_3\prime) \implies P_1 = P_2$, and we can conclude that φ_3^+ is injective.

To show that $\varphi_i^\pm$ is injective, we similarly assume that $\exists P_1$ and P_2, $P_1 \neq P_2$ and $\varphi_i^\pm(P_1) = \varphi_i^\pm(P_2)$. Let $K = \{1, 2, 3\}/i$. Then $\forall k \in K, a_k = a_k\prime \quad a_i = \pm\sqrt{1 - \sum(a_k^2)} = a_i\prime$. Since a_i and $a_i\prime$ are limited to only positive or only negative values $P_1 = P_2$, and $\varphi_i^\pm$ is injective.

To prove surjectivity, consider $(\varphi_i^\pm)^{-1}$ componentwise. It sends an arbitary point $(b_1, b_2) \in \mathbb{R}^2$ to a point $(a_1, a_2, a_3) \in S^2$, or componentwise

$$a_j = \begin{cases} b_j & j < i \\ \sqrt{1 - b_1^2 - b_2^2} & j = i \\ b_{j-1} & j > i \end{cases}$$

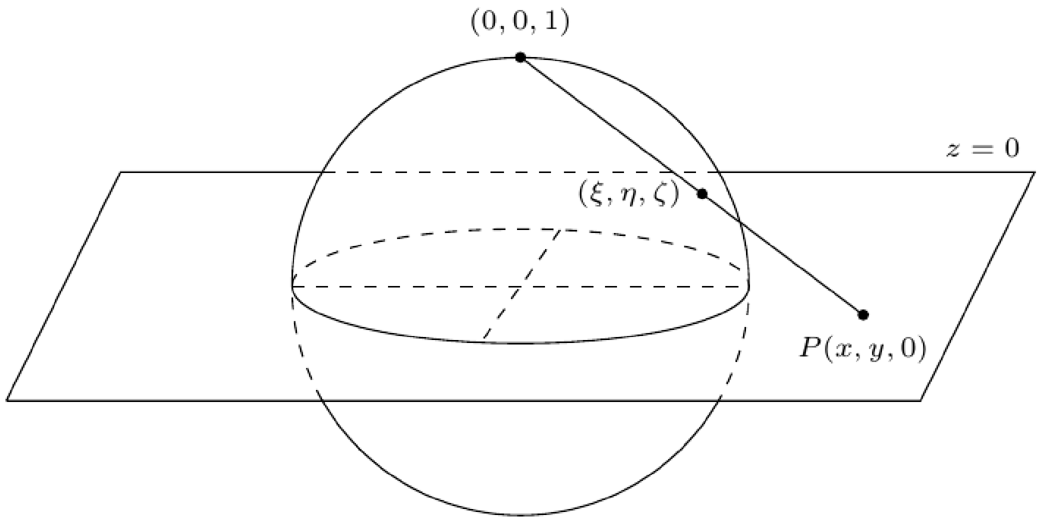
Since each component in both S^2 and the disk varies from 0 to 1, each point in the codomain is mapped to, and we have surjectivity for $\varphi_i^\pm$
We can easily check for continuity of φ_i since for each open $\beta_\epsilon(x_1, x_2) \in D_1(0,0) \ \exists \ \beta_\delta(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) \in S^2$ such that $\epsilon = (\epsilon_1, \epsilon_2)$ and $\delta = (\epsilon_1, \epsilon_2, \sqrt{1 - \epsilon_1^2 - \epsilon_2^2})$
Similarly φ_i^{-1} is continuous because $\forall \beta_\epsilon(x, y, z) \in S^2 \ \exists \ \beta_\delta(x, y) \in D_1(0,0)$ where $\delta = (\epsilon_1, \epsilon_2)$
The space is hausdorff since it is a subset of $\mathbb{R}^3$, therefore $\varphi_i^\pm$ are homeomorphisms.

We now show that A is a smooth atlas. First we consider the transition function $\varphi_3 \circ \varphi_1^{-1}$: $u_1^+ \to u_3^+$

$$(x, y) \to (\sqrt{x^2 + y^2}, x, y) \to (\sqrt{x^2 + y^2}, x)$$

But this is infinitely differentiable since it is a polynomial. Similarly, we can show that all transition functions are C^∞

Stereographic projection is a map that will allow us to embed the sphere with smooth structure by using 2 charts.



$$v^+ = S^2 \setminus \{(0, 0, -1)\} \quad v^- = S^2 \setminus \{(0, 0, 1)\}$$

We think of map ϕ^- as drawing a line through the point the north pole $(0, 0, 1)$ and (x, y, z), then the output is the point where the line intersects z plane. Similarly for ϕ^+, we project from the south pole $(0, 0, -1)$. We get the maps explicitly by parametrizing the line:

$$l : (0, 0, 1) + t((x_1, x_2, x_3) - (0, 0, 1)) = (x_1 t, x_2 t, x_3 t - t + 1)$$

l intersects $x_3 = 0$ when $x_3 t - t + 1 =$, thus $t = \frac{1}{(1-x_3)}$ It follows that the point of intersection is $(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0)$ Similarly we can explicitly find ϕ^+

$$\phi_+ : v^+ \to \mathbb{R}^2, \quad (x_1, x_2, x_3) \to (\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3})$$

$$\phi_- : v^- \to \mathbb{R}^2, \quad (x_1, x_2, x_3) \to (\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3})$$

We can find the inverses in the similar way. Consinder points on the $z = 0$ plane in $\mathbb{R}^3$ $(\alpha, \beta, 0)$. We parametrize the line through this point and north pole:

$$l : (0, 0, 1) + t((\alpha, \beta, 0) - (0, 0, 1)) = (\alpha t, \beta t, -t + 1)$$

We need to see when are these points going to be on the sphere , i.e.

$$(\alpha t)^2 + (\beta t)^2 + (-t+1)^2 = 1$$

$$\alpha^2 t^2 + \beta^2 t^2 + t^2 - 2t + 1 = 1$$

$$t^2(\alpha^2 + \beta^2 + 1 - \frac{2}{t}) = 0$$

$$\therefore t = \frac{2}{\alpha^2 + \beta^2 + 1}$$

$$\phi_+^{-1} : \mathbb{R}^2 \to v^+ \quad (\alpha, \beta) \to (\frac{2\alpha}{1+\alpha^2+\beta^2}, \frac{2\beta}{1+\alpha^2+\beta^2}, -\frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1})$$

$$\phi_-^{-1} : \mathbb{R}^2 \to v^- \quad (\alpha, \beta) \to (\frac{2\alpha}{1+\alpha^2+\beta^2}, \frac{2\beta}{1+\alpha^2+\beta^2}, \frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1})$$

Since we explicitly found an inverse, the function is bijection.

Lemma 0.0.1. *For any space X, a map $f : X \to \mathbb{R}^n$ is continuous if and only if each of the coordinate maps $f_i : X \to \mathbb{R}$ are continuous. (prove or find source)*

And it's easy to see that ϕ is continuous componentwise. (denom never zero) Thus we equipped the sphere with the following atlas:

$$A_2 = \{(v^{\pm}, \phi_{\pm})\}$$

We can check that the transition maps from A_1 to A_2 are smooth.
First we check for $\varphi_3^+ \circ \phi_+^{-1}$ where

$$\varphi_3^+ : S^2 \cap x_3 > 0 = u_3^+ \to D_1(0,0) \subset \mathbb{R}^2$$

$$\phi_+^{-1} : \mathbb{R}^2 \to v^+ = S^2 \setminus \{(0,0,-1)\}$$

Now, since we require that the domain of φ_3^+ is positive, the following is how our transition function will look like

$$\varphi_3^+ \circ \phi_+^{-1} : D_1(0,0) \to D_1(0,0), \quad \varphi_3^+(\phi_+^{-1}(x_1, x_2)) = (\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2})$$

$$\phi_+^{-1} \circ \varphi_3^+ : D_1(0,0) \to D_1(0,0), \quad \phi_+(\varphi_3^+((x_1,x_2))^{-1}) = (\frac{x_1}{1+\sqrt{1-x_1^2-x_2^2}}, \frac{x_1}{1+\sqrt{1-x_1^2-x_2^2}})$$

It is continous componentwise and thus smooth. Next, we list the domain and the image of the rest of transition maps.

$$\varphi_3^- \circ \phi_+^{-1} : \mathbb{R}^2 \setminus \bar{D}_1(0,0) \to D_1(0,0) \setminus (0,0)$$

$$\varphi_2^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 > 0\} \to D_1(0,0)$$

$$\varphi_1^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 > 0\} \to D_1(0,0)$$

$$\varphi_2^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 < 0\} \to D_1(0,0)$$

$$\varphi_1^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 < 0\} \to D_1(0,0)$$

$$\varphi_3^+ \circ \phi_-^{-1} : \mathbb{R}^2 \setminus \bar{D}_1(0,0) \to D_1(0,0) \setminus (0,0)$$

$$\varphi_3^- \circ \phi_-^{-1} : D_1(0,0) \to D_1(0,0)$$

$$\varphi_2^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 > 0\} \to D_1(0,0)$$

$$\varphi_1^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 > 0\} \to D_1(0,0)$$

$$\varphi_2^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 < 0\} \to D_1(0,0)$$

$$\varphi_1^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 < 0\} \to D_1(0,0)$$

Bibliography