Let $X \subseteq Mat_{3\times 3}(\mathbb{R})$ be the set

$$X = \{M \mid M^T = M,\ M^2 = M,\ TrM = 1\}$$

We will describe a bijection between $X$ and $\mathbb{RP}^2$. The above definition tells us that $X$ consists of matrices that are symmetric, idempotent and whose eigenvalues add up to one. Spectral theorem tells us that a real symmetric matrix is diagonizable. We can also show that the eigenvectors of symmetric matrices, with distinct eigenvalues, are orthogonal. Indeed, let $x$ and $y$ be eigenvectors of a symmetric matrix $M$, with eigenvalues $\lambda$ and $\mu$, $\lambda \neq \mu$:

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Mx, y\rangle = \langle xM^T, y\rangle = \langle x, My\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle$$

Therefore $(\lambda - \mu)\langle x, y\rangle = 0$, since $\lambda$ and $\mu$ are distinct $xy$=0, thus orthogonal. Next, we can show that the eigenvalues of $M$ can be only 0 and 1. Let $v$ be an eigenvector, of eigenvalue $\lambda$.

$$Mv = \lambda v$$

$$M^2 v = M(\lambda v) = \lambda M(v) = \lambda^2 v$$

As $\bar{v} \neq 0$ $(\lambda^2 - \lambda)v = 0 \iff \lambda^2 - \lambda = 0$. Then solving for $\lambda^2 - \lambda = 0$, we get that $\lambda$ can only be 0 or 1. Finally, the fact that $Tr(M) = 1$ tells us that eigenvalues of $M$ are 0,0 and 1. Therefore $\dim \ker M = 2$ and $\dim Im(M) = 1$. This is telling us that there is a whole plane, that is sent to zero vector by $M$, and all vectors in the image are sitting on a line.
Thus, applying a matrix operator $M$ to a vector, is equivalent to projecting a vector to a line in $R^3$. So $M : R^3 \to R^3$ is the operator of orthogonal projection on the line $Im(M)$.
Now, let's explicitly define a map $\phi$, which to given line in $R^3$ assigns a corresponding matrix operator, that will orthogonally project all the vectors in $R^3$ to that line.

$$\phi : \mathbb{RP}^2 \to Mat_{3\times 3} \quad \phi([x : y : z]) \to A$$

To explicitly find $A$, note that we first need to find a unit vector along a line $[x : y : z] \in \mathbb{RP}^2$, we can do that by normalizing coordinates. $n = \frac{1}{\sqrt{x^2+y^2+z^2}}[x, y, z]^T$. Finally, to orthogonally project any $v \in R^3$ along $n$, we apply $(vn)n = v\, n \otimes n$. We can then define $\phi$ as follows:

$$\phi([x : y : z]) = \frac{1}{\sqrt{x^2 + y^2 + z^2}^2}[x, y, z]^T \otimes [x, y, z] = \frac{1}{x^2 + y^2 + z^2}\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}$$

Now, we redefine $\phi : \mathbb{RP}^2 \to \mathbb{R}^6$, because $Mat_{3\times 3} \supseteq Sym_{3\times 3} \simeq \mathbb{R}^6$.

**Definition 0.0.1.** *Immersion Let $X$ and $Y$ be smooth manifolds,* $\dim X = n$, $\dim Y = k$. *Let $f : X \to Y$ be a smooth map. We say that $f$ is a*

- *submersion, if $df_p$ is surjective $\forall p \in X$*
- *immersion, if $df_p$ is injective $\forall pin X$ equivalently if $rank D_p f = \dim M, M = f(X)$*

**Definition 0.0.2.** *Let $f : X \to Y$ be a smooth map of smooth manifolds. We say that $f$ is an embedding if*

- *$f$ is an injective immersion*
- *$X$ is homeomorphic to $f(X) \subset Y$ (equipped with the subspace topology)*

Next, we argue that $\phi\mathbb{RP}^2 \to R^6$ is an embedding ($R^6$ because we are taking only non-symmetric lower triangular entries) Namely, we will show that the following function is an embedding.

$$\phi([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(x^2, xy, xz, y^2, yz, z^2)$$

First, we show that $\phi$ is well defined. Take two vectors $a$,$b \in [x : y : z]$ on the same line. If $a$ is given by $a = [a_1, a_2, a_3]$ then $b = [ka_1, ka_2, ka_3]$ for $k \in \mathbb{R}$. We need to show that $\phi([a]) = \phi([b])$.

$$\phi([a_1, a_2, a_3]) = \frac{(a_1^2, a_1a_2, a_2^2, a_2a_3, a_3^2)}{a_1^2 + a_2^2 + a_3^2}$$

$$\phi([ka, kb, kc]) = \frac{(k^2a_1^2, k^2a_1a_2, k^2a_2^2, k^2a_2a_3, k^2a_3^2)}{k^2a_1^2 + k^2a_2^2 + k^2a_3^2} = \frac{k^2(a_1^2, a_1a_2, a_2^2, a_2a_3, a_3^2)}{k^2(a_1^2 + a_2^2 + a_3^2)} = \frac{(a_1^2, a_1a_2, a_2^2, a_2a_3, a_3^2)}{a_1^2 + a_2^2 + a_3^2}$$

Now that we showed that $\phi$ is well-defined, we show that it is injective.

Assume that $\phi$ is not injective, then there are unit vectors $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$ lying on different lines such that $\phi([a]) = \phi([b])$ In other words $a \in [x : y : z]$ and $b \in [x\prime : y\prime : z\prime]$

$$\phi([a_1, a_2, a_3]) = (a_1^2, a_1a_2, a_2^2, a_2a_3, a_3^2)$$

$$\phi([b_1, b_2, b_3]) = (b_1^2, b_1b_2, b_2^2, b_2b_3, b_3^2)$$

From $\phi([a_1, a_2, a_3]) = \phi([b_1, b_2, b_3])$ we have that $b_1 = \pm a_1$, $b_2 = \pm a_2$, $b_3 = \pm a_3$, and we know that all $b_i$ have the same sign. Therefore we either have $b = [a_1, a_2, a_3]$ or $b = [-a_1, -a_2, -a_3]$ which both lie on the line $[x : y : z]$. So we have that $b \in [x : y : z]$ which is a contradiction.
We proved that $\phi$ is injective, and we know that $\mathbb{RP}^2$ is compact, so we proceed to proving that $\phi$ is an immersion, once we have that we can claim that $\phi$ is an embedding.

We can use the definition with local charts to prove it. Consider the following charts and maps.

$$u_0 = \{[x : y : z], x \neq 0\} \simeq \mathbb{R}^2$$

$$u_1 = \{[x : y : z], y \neq 0\} \simeq \mathbb{R}^2$$

$$u_2 = \{[x : y : z], z \neq 0\} \simeq \mathbb{R}^2$$

$$\psi_0 : RP^2 \to \mathbb{R}^2, \quad \psi_0([x : y : z]) = (\frac{y}{x}, \frac{z}{x}), \quad \psi_0^{-1}(s, t) = [1 : s, t]$$

$$\psi_1 : RP^2 \to \mathbb{R}^2, \quad \psi_1([x : y : z]) = (\frac{x}{y}, \frac{z}{y}), \quad \psi_1^{-1}(s, t) = [s : 1 : t]$$

$$\psi_2 : RP^2 \to \mathbb{R}^2, \quad \psi_2([x : y : z]) = (\frac{x}{z}, \frac{y}{z}), \quad \psi_2^{-1}(s, t) = [s : t : 1]$$

These local charts cover all the points in $\mathbb{RP}^2$, to prove that $\phi$ is an immersion we need to show the following, for all $p \in R^2$

$$rank(J(\phi \circ \psi_i^{-1}(s,t))) = 2$$

for $i \in 1,2,3$.
We check for $\psi_0$.

$$\phi \circ \psi_0^{-1}(s,t) = [1:s:t] \to (1,s,t,s^2,st,t^2)\frac{1}{1+s^2+t^2}$$

$$J(\phi \circ \psi_0^{-1}(s,t)) = \frac{1}{(1+s^2+t^2)^2}\begin{pmatrix} -2s & -2t \\ -s^2+t^2+1 & -2st \\ -2st & s^2-t^2+1 \\ 2s(t^2+1) & -2s^2t \\ t(-s^2+t^2+1) & s(s^2-t^2+1) \\ -2st^2 & 2t(s^2+1) \end{pmatrix}$$

To see that the rank is always 2 we can chek the the determinant of minor $\Delta_{4,6} = 4st(s^2+t^2+1)$ which is only zero when $st = 0$. But when both $s = 0$ and $t = 0$ equal to zero, the determinant of the minor $\Delta_{2,3} = 1$, and if $\Delta_{2,3}$ is zero only if $s = 1$ and $t = 0$ or $s = 0$ and $t = 1$. But when that is the case $\Delta_{1,5} \neq 0$. In conclusion there will always be a $2 \times 2$ minor with non-zero determinant, which means that our matrix has rank 2. Similarly we can check that $rankJ(\phi_i) = 2$
**We can conclude that: The map $\phi$ maps $\mathbb{RP}^2$ is an immersion, and thus embedding to $\mathbb{R}^6$**
**Think what happens if we consider:**

$$X = \{M \mid M^2 = M = M^T, trM = k\} \subset Mat_{n\times n}$$

There is an immersion from $G(k,n)$ to $\mathbb{R}^{n^2}$

$$\phi : G(k,n) \to X, \quad \phi([U_{n\times k}]) \to A_{n\times n}$$

$$\phi([U_{n\times k}]) = UU^T$$

If bases of $U$ are unit vectors we are good with this formula. If no we need to first normalize each vector in a matrix $U$. Where one matrix represent a projection of 2 dimensional plane to a set of two dimensional subspaces, namely:

$$A\ B = C$$

such that $A_{n\times n} \in X$, $B \in Mat_{n\times k}$ and $C \in [U_{n\times k}]$

$$\phi G(k,n) \to X,$$

$$P = A(A^TA)^{-1}A^T$$

**We can quotient to get grassmann Normalize and Quotient over G**

[HWG16] [JMC+19]

# Bibliography

[HWG16] Zhiwu Huang, Jiqing Wu, and Luc Van Gool. Building deep networks on grassmann manifolds. *CoRR*, abs/1611.05742, 2016.

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