We will show that S^2 is a smooth manifold. Let's denote a sphere as:

$$M = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

Also, we define a disk with a center at (x_0, y_0) and radius ϵ as:

$$D_\epsilon(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < \epsilon^2\}$$

Thus we can cover the sphere with 6 charts and 6 functions, described as follows:

$$u_i^+ = S^2 \cap \{x_i > 0\}, \quad u_i^- = S^2 \cap \{x_i < 0\}$$

$$\varphi_i^+ : u_i^+ \to D_1(0,0), \quad \varphi(x_1, x_2, x_3) = (\dots \hat{x}_i \dots)$$

$$\varphi_i^- : u_i^- \to D_1(0,0), \quad \varphi(x_1, x_2, x_3) = (\dots \hat{x}_i \dots)$$

Then the atlas covering the sphere is:

$$A_1 = \{(u_i^\pm, \varphi_i^\pm) : i \in I\}, \quad I = \{1, 2, 3\}$$

For example $\varphi_3^+ : u_3^+ \to \mathbb{R}^2, \quad \varphi_3^+(x_1, x_2, x_3) = (x_1, x_2, \hat{x_3}) = (x_1, x_2)$

We first show that a function $\varphi_3^+ : u_3^+ \to \mathbb{R}$ is injective. Assume $\exists P_1$ and P_2 s.t. $P_1 = (p_1, p_2, p_3)$, and $P_2 = (p_1\prime, p_2\prime, p_3\prime)$, $P_1 \neq P2$, and $f(P_1) = f(P_2) \iff (p_1, p_2) = (p_1\prime, p_2\prime)$
Because P_1 and P_2 are the points on the sphere: $p_3\prime = \pm\sqrt{1 - p_1^2 - p_2^2} \quad \pm p_3\prime = \pm\sqrt{1 - p_1\prime^2 - p_2\prime^2}$
Since u_3^+ has only positive numbers for a third coordinate $p_3 = p_3\prime \implies (p_1, p_2, p_3) = (p_1\prime, p_2\prime, p_3\prime) \implies P_1 = P_2$, and we can conclude that φ_3^+ is injective.

To show that $\varphi_i^\pm$ is injective, we similarly assume that $\exists P_1$ and P_2, $P_1 \neq P_2$ and $\varphi_i^\pm(P_1) = \varphi_i^\pm(P_2)$. Let $K = \{1, 2, 3\}/i$. Then $\forall k \in K, a_k = a_k\prime \quad a_i = \pm\sqrt{1 - \sum(a_k^2)} = a_i\prime$. Since a_i and $a_i\prime$ are limited to only positive or only negative values $P_1 = P_2$, and $\varphi_i^\pm$ is injective.

To prove surjectivity, consider $(\varphi_i^\pm)^{-1}$ componentwise. It sends an arbitary point $(b_1, b_2) \in \mathbb{R}^2$ to a point $(a_1, a_2, a_3) \in S^2$, or componentwise

$$a_j = \begin{cases} b_j & j < i \\ \sqrt{1 - b_1^2 - b_2^2} & j = i \\ b_{j-1} & j > i \end{cases}$$

Since each component in both S^2 and the disk varies from 0 to 1, each point in the codomain is mapped to, and we have surjectivity for $\varphi_i^\pm$

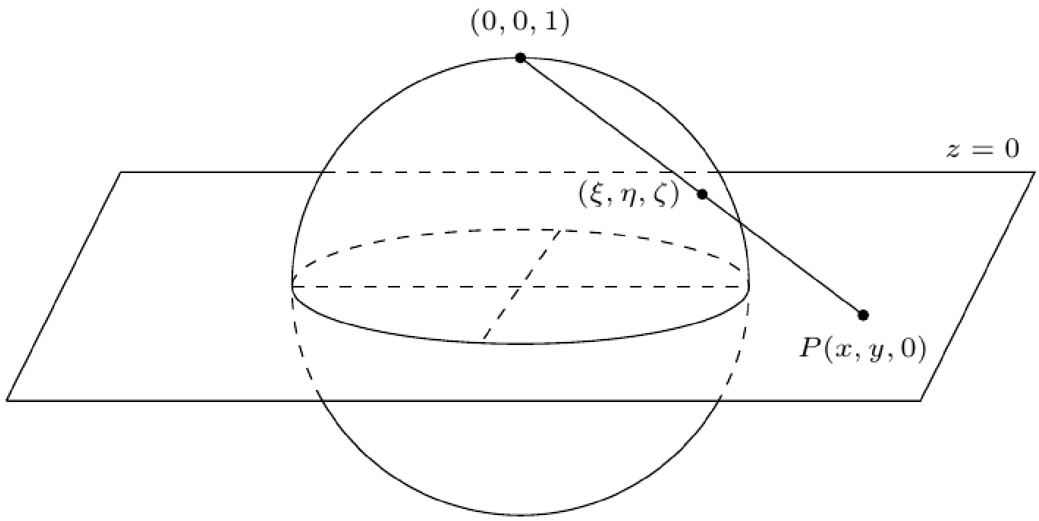
Lemma 0.0.1. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose the partial derivatives $\frac{\partial f_i}{\partial x_j}$ of f all exist and are continuous in a neighbourhood of a point $x \in U$. Then f is differentiable at x.*

The function φ_i is continous by 0.0.1 (polynomials are infinitely differentiable).
The space is Hausdorff since it is a subset of $\mathbb{R}^3$, therefore $\varphi_i^{\pm}$ are homeomorphisms.
We now show that A is a smooth atlas. First we consider the transition function $\varphi_3 \circ \varphi_1^{-1}$: $u_1^+ \to u_3^+$

$$(x, y) \to (\sqrt{x^2 + y^2}, x, y) \to (\sqrt{x^2 + y^2}, x)$$

But this is infinitely differentiable since it is a polynomial. Similarly, we can show that all transition functions are C^∞

Stereographic projection is a map that will allow us to embed the sphere with smooth structure by using 2 charts.



$$v^+ = S^2 \setminus \{(0, 0, -1)\} \quad v^- = S^2 \setminus \{(0, 0, 1)\}$$

We think of map ϕ^- as drawing a line through the point the north pole $(0, 0, 1)$ and (x, y, z), then the output is the point where the line intersects z plane. Similarly for ϕ^+, we project from the south pole $(0, 0, -1)$. We get the maps explicitly by parametrizing the line:

$$l : (0, 0, 1) + t((x_1, x_2, x_3) - (0, 0, 1)) = (x_1 t, x_2 t, x_3 t - t + 1)$$

l intersects $x_3 = 0$ when $x_3 t - t + 1 =$, thus $t = \frac{1}{(1-x_3)}$ It follows that the point of intersection is $(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0)$ Similarly we can explicitly find ϕ^+

$$\phi_+ : v^+ \to \mathbb{R}^2, \quad (x_1, x_2, x_3) \to (\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3})$$

$$\phi_- : v^- \to \mathbb{R}^2, \quad (x_1, x_2, x_3) \to (\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3})$$

We can find the inverses in the similar way. Consinder points on the $z = 0$ plane in $\mathbb{R}^3$ $(\alpha, \beta, 0)$.

We parametrize the line through this point and north pole:

$$l : (0,0,1) + t((\alpha,\beta,0) - (0,0,1)) = (\alpha t, \beta t, -t+1)$$

We need to see when are these points going to be on the sphere , i.e.

$$(\alpha t)^2 + (\beta t)^2 + (-t+1)^2 = 1$$

$$\alpha^2 t^2 + \beta^2 t^2 + t^2 - 2t + 1 = 1$$

$$t^2(\alpha^2 + \beta^2 + 1 - \frac{2}{t}) = 0$$

$$\therefore t = \frac{2}{\alpha^2 + \beta^2 + 1}$$

$$\phi_+^{-1} : \mathbb{R}^2 \to v^+ \quad (\alpha,\beta) \to (\frac{2\alpha}{1+\alpha^2+\beta^2}, \frac{2\beta}{1+\alpha^2+\beta^2}, -\frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1})$$

$$\phi_-^{-1} : \mathbb{R}^2 \to v^- \quad (\alpha,\beta) \to (\frac{2\alpha}{1+\alpha^2+\beta^2}, \frac{2\beta}{1+\alpha^2+\beta^2}, \frac{\alpha^2+\beta^2-1}{\alpha^2+\beta^2+1})$$

Since we explicitly found an inverse, the function is bijection.
And it's easy to see that ϕ is continuous by lemma 0.0.1 (denom never zero) Thus we equipped the sphere with the following atlas:

$$A_2 = \{(v^\pm, \phi_\pm)\}$$

We can check that the transition maps from A_1 to A_2 are smooth.
First we check for $\varphi_3^+ \circ \phi_+^{-1}$ where

$$\varphi_3^+ : S^2 \cap x_3 > 0 = u_3^+ \to D_1(0,0) \subset \mathbb{R}^2$$

$$\phi_+^{-1} : \mathbb{R}^2 \to v^+ = S^2 \setminus \{(0,0,-1)\}$$

Now, since we require that the domain of φ_3^+ is positive, the following is how our transition function will look like

$$\varphi_3^+ \circ \phi_+^{-1} : D_1(0,0) \to D_1(0,0), \quad \varphi_3^+(\phi_+^{-1}(x_1,x_2)) = (\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2})$$

$$\phi_+^{-1} \circ \varphi_3^+ : D_1(0,0) \to D_1(0,0), \quad \phi_+(\varphi_3^+((x_1,x_2))^{-1}) = (\frac{x_1}{1+\sqrt{1-x_1^2-x_2^2}}, \frac{x_1}{1+\sqrt{1-x_1^2-x_2^2}})$$

It is continuous componentwise and thus smooth. Next, we list the domain and the image of the rest of transition maps.

$$\varphi_3^- \circ \phi_+^{-1} : \mathbb{R}^2 \setminus \bar{D}_1(0,0) \to D_1(0,0) \setminus (0,0)$$

$$\varphi_2^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 > 0\} \to D_1(0,0)$$

$$\varphi_1^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 > 0\} \to D_1(0,0)$$

$$\varphi_2^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 < 0\} \to D_1(0,0)$$

$$\varphi_1^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 < 0\} \to D_1(0,0)$$

$$\varphi_3^+ \circ \phi_-^{-1} : \mathbb{R}^2 \setminus \bar{D_1}(0,0) \to D_1(0,0) \setminus (0,0)$$

$$\varphi_3^- \circ \phi_-^{-1} : D_1(0,0) \to D_1(0,0)$$

$$\varphi_2^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 > 0\} \to D_1(0,0)$$

$$\varphi_1^+ \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 > 0\} \to D_1(0,0)$$

$$\varphi_2^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_2 < 0\} \to D_1(0,0)$$

$$\varphi_1^- \circ \phi_+^{-1} : \mathbb{R}^2 \cap \{x_1 < 0\} \to D_1(0,0)$$

Definition 0.0.1. *We say that $g : V \times V \to \mathbb{R}$ is an* ***inner product*** *on V if for $u, v, w \in V$*

- $g(u, v) = g(v, u)$ *symmetry*
- $g(u + v, w) = g(u, w) + g(v, w)$ *billinearity*
- $g(u, v) \geq 0$ *positive definite*

Definition 0.0.2. *Every vector space has a* ***dual vector space*** *which consists of all linear forms on V, together with the vector space structure of pointwise addition and scalar multiplication.*

Definition 0.0.3. *Given a vector space V with a basis B, the dual set of B is a set B^* of vectors in V^* space. Vectors from B^* form biorthogonal system with B. If the dual set span V^* then B^* is called the* ***dual basis*** *for the basis B. Namely, given $B = \{v_i\}$ and $B^* = \{v^i$, being biorthogonal means that the elements pair to have an inner product equal to 1 if the indexes are equal, and equal to 0 otherwise.*

$$v^i v_j = \delta^i_j = \begin{cases} 1 \ \textit{if } i = j \\ 0 \ \textit{if } i \neq j \end{cases}$$

where δ^i_j is the Kronecker delta symbol.

Example 0.0.1. Consider a vector space on $\mathbb{R}^3$ with the basis $B = \{e_1, e_2, e_3\}$, and the usual inner product $< u, v >= u^T v$. Then the dual basis is $B^* = B^T$

Example 0.0.2. Given the basis $B = \{f_1, f_2, f_3\}$, where $f_1 = e_1$, $f_2 = 2e_1 + e_2$ and $f_3 = e_1 - e_3$ we will find the inner product on a space spanned by B, and the dual basis.
Let's denote the space spanned by the given basis with V. Each element $u \in V$ can be written as $u = [f_1\ f_2\ f_3][x_1\ x_2\ x_3]^T = [e_1\ e_2\ e_3] \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} [x_1\ x_2\ x_3]$ Then take $u, v \in V$ and their inner

product: $\langle u,v\rangle = ([e_1\ e_2\ e_3]\begin{pmatrix}1&2&1\\0&1&0\\0&0&-1\end{pmatrix}[x_1\ x_2\ x_3])^T([e_1\ e_2\ e_3]\begin{pmatrix}1&2&1\\0&1&0\\0&0&-1\end{pmatrix}[y_1\ y_2\ y_3]) = [x_1\ x_2\ x_3]^T\begin{pmatrix}1&2&1\\2&5&2\\1&2&1\end{pmatrix}[y_1\ y_2\ y_3]$ Next, we find the dual basis;

$$\alpha_1(e_1)=1 \quad \alpha_2(e_1)=0 \quad \alpha_3(e_1)=0$$

$$\alpha_1(2e_1+e_2)=0 \quad \alpha_2(e_1+e_2)=1 \quad \alpha_3(2e_1+e_2)=0$$

$$\alpha_1(e_1-e_3)=0 \quad \alpha_2(e_1-e_3)=0 \quad \alpha_3(e_1-e_3)=1$$

Solving this system of equations we get:

$$\alpha_1(e_1)=1 \quad \alpha_2(e_1)=0 \quad \alpha_3(e_1)=0$$

$$\alpha_1(e_2)=-2 \quad \alpha_2(e_2)=1 \quad \alpha_3(e_2)=0$$

$$\alpha_1(e_3)=1 \quad \alpha_2(e_3)=0 \quad \alpha_3(e_3)=-1$$

Thus our dual basis is $B^* = \begin{pmatrix}1&0&0\\-2&1&0\\1&0&-1\end{pmatrix}$

Definition 0.0.4. *For any smooth function f on a Riemannian manifold (M,g), the* ***gradient*** *of f is the vector field ∇f such that for any vector field X,*

$$g(\nabla f, X) = \partial_x f \quad \textit{that is} \quad g_x((\nabla f)_x, X_x) = (\partial_X f)(x)$$

where $g_x(,)$ denotes the inner product of tangent vectors at x defined by the metric g and $\partial_X f$ is the function that takes any point $x \in M$ to the directional derivative of f in the direction X, evaluated at x.

The gradient is dual to the differential df. The value of the gradient at a point is a tangent vector, a vector at each point while the value of the derivative at a point is a cotangent vector - a linear function on vectors. They are related as follows:

$$g(gradf, -) = (gradf)^T G - = df \tag{1}$$

Take a chart $v \subseteq \mathbb{R}^n$, and $f : v \to \mathbb{R}$. Then $df = \frac{\partial f}{\partial x_1}dx_1 \dots \frac{\partial f}{\partial x_n}dx_n = [\frac{\partial f}{\partial x_1}\ \dots\ \frac{\partial f}{\partial x_n}][dx_1\ \dots\ dx_n]^T = Jf[dx_1\ \dots\ dx_n]^T$ but from 1 we see that this is equal to $(gradf)^T G[dx_1\ \dots\ dx_n]^T$. Thus, we can conclude that p

$$gradf = (G^{-1})^T\ [\frac{\partial f}{\partial x_1}\ \dots\ \frac{\partial f}{\partial x_n}]^T$$

Example 0.0.3. Consinder a sphere $S^2 \subseteq \mathbb{R}^3$, then tangent space of the sphere is given by $T_pS^2 = \{v \mid pv = 0\}$ and tangent bundle by $TS^2 = \{(p,v) \mid p^2 = 1, pv = 0\}$. We will define a metric on a sphere in stereographic projection coordinates, and calculate the gradient for a function $f: S^2 \to \mathbb{R} \quad f(x,y,z) = z$. We consider a stereographic projection map defined previously a ϕ_-, and we will address it in this example as just ϕ. For some $\eta, \omega \in$ coordinate chart ($\mathbb{R}^2$) the dot product will be defined as $\langle \eta, \omega \rangle = (d\phi^{-1}(\eta))^T d\phi^{-1}(\omega) = (J\phi^{-1}\eta)^T(J\phi^{-1}\omega) = \eta^T(J\phi^{-1})^T J\phi^{-1}\omega$

We can calculate $J\phi^{-1} = \frac{2}{(\alpha^2+\beta^2+1)^2}\begin{bmatrix} -\alpha^2+\beta^2+1 & -2\alpha\beta \\ -2\alpha\beta & \alpha^2-\beta^2+1 \\ 2\alpha & 2\beta \end{bmatrix}$ Then, $(J\phi^{-1})^T J\phi^{-1} = \frac{4}{(\alpha^2+\beta^2+1)^4}\begin{bmatrix} \alpha^4+\beta^4+2\beta^2+2\alpha^2\beta^2+1 & 0 \\ 0 & \alpha^4+\beta^4+2\alpha^2+2\alpha^2\beta^2+1 \end{bmatrix}$

$$(G^{-1})^T = \begin{bmatrix} \frac{2a^4+a^2b^2+a^2+2b^4+b^2+2}{4} & 0 \\ 0 & \frac{2a^4+a^2b^2+a^2+2b^4+b^2+2}{4} \end{bmatrix}$$

$$grad = (G^{-1})^T J(\phi^{-1} \circ f)$$

Bibliography