Morse Theory

Veronika Starodub

Miloš Vukadinović

Nikolay Ninov

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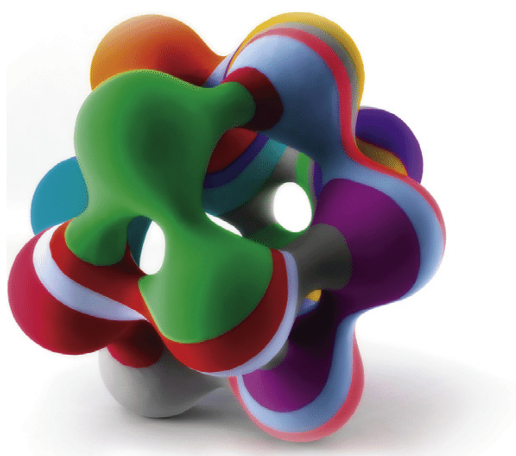

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1 Introduction

Ex. Show that the function $f : S^2 \to \mathbb{R}, f(x, y, z) = z$ is a Morse function.

f is smooth on S^2 since it extends to a smooth map on all of $\mathbb{R}^3$. We can map R^3 to S^2 with the stereographic projection two functions:

$\phi_1(x_1, x_2, x_3) = (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3})$ and $\phi_2(y_1, y_2, y_3) = (\frac{y_1}{1+y_3}, \frac{y_2}{1+y_3})$

Therefore, to get $S^2 \to R^3$ we can take ϕ_1^{-1} and ϕ_2^{-1}

$\phi_1^{-1}(x_1, x_2, x_3) = (\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1})$

$\phi_1^{-1}(y_1, y_2, y_3) = (\frac{2y_1}{y_1^2+y_2^2+1}, \frac{2y_2}{y_1^2+y_2^2+1}, \frac{1-y_1^2+y_2^2}{y_1^2+y_2^2+1})$

Then, we take $g_1 = f \circ \phi_1^{-1}$ and $g_2 = f \circ \phi_2^{-1}$

$g_1(x, y) = \frac{x^2+y^2-1}{x^2+y^2+1}$ and $g_2(x, y) = \frac{1-x^2+y^2}{x^2+y^2+1}$

Now, we can compute jacobian of g_1 and g_2, find critical points, and check that determinant of hessian matrix is non-zero.

$\nabla g_1(x, y, z) = 0$ iff $(x, y, z) = (0, 0, -1)$

$\nabla g_2(x, y, z) = 0$ iff $(x, y, z) = (0, 0, 1)$ We have two critical points, and now we compute the hessian at them.

$$H(g_1) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \implies \det(H(g_1)) = 16$$

$$H(g_2) = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \implies \det(H(g_2)) = 16$$

Both critical points are non-degenerate, therefore f is a Morse function.■