Morse Lemma. Suppose that the point $a \in \mathbb{R}^k$ is a nondegenerate critical point of the function f, and

$$(h_{ij}) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right)$$

is the Hessian of f at a. Then there exists a local coordinate system $(x_1, \ldots, x_k)$ around a such that

$$f = f(a) + \sum h_{ij} x_i x_j$$

near a.
or

$$f = -x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_\lambda^2 + \ldots + x_n^2 + f(a)$$

where λ is the index of f at a.

Theorem 1.1 *Let f be a smooth function in a neighborhood N_x of $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Suppose $f(0, \ldots, 0) = 0$. Then, there exist n smooth functions $g_i, \ldots, g_n$ defined on N_x such that $g_i(0, \ldots, 0) = \frac{\partial f}{\partial x_i}(0, \ldots, 0)$ for every i, and*

$$f(x_i, \ldots, x_n) = \sum_{i=1}^{n} (x_i, \ldots, x_n)$$

Theorem 1.2 *(Inverse Function Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function on an open set U containing $a \in \mathbb{R}^n$. Suppose that $\det J_f(a) \neq 0$. Then there is an open set $V \subset \mathbb{R}^n$ containing a and an open set $W \subset \mathbb{R}^n$ containing $f(a)$ such that $f : V \to W$ is a diffeomorphism.*

Proof. Let p_0 be a nondegenerate critical point of the function $f : M \to \mathbb{R}$, where M is an n-manifold. The degeneracy of the point p_0 on f is determined independent of our choice of a local coordinate system. Therefore, we may assume that when we pick a local coordinate system $(x_1, \ldots, x_n)$ defined in a neighborhood N_{p_0},

$$\frac{\partial^2 f}{\partial x_1^2}(p_0) \neq 0 \tag{1.1}$$

or that we may pick a suitable linear transformation of the local coordinate system such that equation 1.1 is true. We may further assume that p_0

corresponds to the origin $(0,\ldots,0) \in \mathbb{R}^n$ on the local coordinate system and that $f(p_0) = 0$, replacing f with $f - f(p_0)$ if necessary.

By Theorem 1.1, there exit n smooth functions $g_i, \ldots, g_n$ defined on N_{p_0} such that

$$g_i(0,\ldots,0) = \frac{\partial f}{\partial x_i}(0,\ldots,0) \tag{1.2}$$

and

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} (x_1,\ldots,x_n) \tag{1.3}$$

But since p_0 is a critical point, equation 1.2 turns out to be zero on both sides at p_0. So we can apply Theorem 1.1 again to get n smooth functions $h_{i1},\ldots,h_{in}$ for every i that is defined on N_{p_0} such that

$$\sum_{j=1}^{n} x_j h_{ij}(x_1,\ldots,x_n) = g_i(x_1,\ldots,x_n) \tag{1.4}$$

By plugging equation 1.4 into equation 1.3, we get

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j h_{ij}(x_1,\ldots,x_n) \tag{1.5}$$

We may assume that $h_{ij} = h_{ji}$, rewriting h_{ij} as $H_{ij} = \frac{h_{ij}+h_{ji}}{2}$ if necessary. Furthermore,

$$(h_{ij}(0,\ldots,0))_{n\times n} = \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0,\ldots,0)_{n\times n}\right) \tag{1.6}$$

And since we assumed equation 1.1 to be true, then $h_{11}(0,\ldots,0) \neq 0$. h_{11} is a smooth, hence continuous function, and so h_{11} is not zero in a neighborhood of the origin. Let us call this neighborhood $\bar{N}_0$

Our ultimate goal is to express f in the standard quadratic form of the equation from the lemma. We do this by eliminating all terms which are not of the form $\pm x_i^2$ via induction over $k \leq n$ steps. While we are currently dealing with $k = 1$, in the general case of k, we wish to express f as a sum of terms such that k terms are of the form $\pm x_i^2$ and the rest of the terms depend on coordinate in the set $x_i | i \neq k$. To this end, let

$$G(x_1,\ldots,x_n) = \sqrt{|h_{11}(x_1,\ldots,x_n)|} \tag{1.6}$$

G is a smooth, non-zero function of $x_1, \ldots, x_n$ on $\bar{N}_0$.

Now suppose by induction that there exists a local coordinate system $(y_1, \ldots, y_n)$ defined on $\bar{N}_0$ such that

$$y_i = x_i (\neq 1) \tag{1.7}$$

$$y_1 = G * (x_1 + \sum_{i>1}^{n} \frac{x_i h_{1i}}{h_{11}}) \tag{1.8}$$

It follows from the Inverse Function Theorem that $y_1, \ldots, y_n$ is a local coordinate system defined on a smaller neighborhood $\tilde{N}_0 \subset \bar{N}_0$, since the determinant of the Jacobian of the transformation from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_m)$ may be verified to be nonzero.

When we square y_1, we get

$$y_1^2 = \pm h_{11} x_1^2 \pm 2 \sum_{i=2}^{n} x_1 x_i h_{1i} \pm \frac{\left(\sum_{i=2}^{n} x_i h_{1i}\right)^2}{h_{11}} \tag{1.9}$$

where the signs are either positive or negative, depending on the sign of h_{11}. Using equation 1.5, we can verify that f can be expressed in the following way with respect to this coordinate system on the restricted domain $\tilde{N}_0$.

$$f = \pm y_1^2 + \sum_{i=2}^{n} \sum_{j=2}^{n} x_i x_j h_{ij} - \frac{\left(\sum_{i=2}^{n} x_i h_{1i}\right)^2}{h_{11}} \tag{1.10}$$

where the sign of the y_1^2 term is positive or negative, depending on the sign of h_{11}. Staying consistent with our goals, we notice that the first term is in the standard quadratic form seen in the Morse Lemma formulation, whereas the rest of the terms depend on local coordinates x_i whereby $i \neq k$ $(k = 1)$. By induction from $k = 1$ to $k = n$, we prove the Morse Lemma. ■