Time Series
Ch2:Probability Models

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2017-3-24

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2.2 STOCHASTIC PROCESSES

- **Definition 2.1** A collection of random variables $\{\mathbf{X}(t) : t \in \mathcal{R}\}$ is called a **stochastic process**.
- in general, $\{\mathbf{X}(t) : 0 \leqq t < \infty\}$ and $\{\mathbf{X_t} : t = 1, 2, ..., n\}$ are used to define a continuous-time and a discrete-time stochastic process, respectively.
- To describe the underlying probability model, we can consider the joint distribution of the process; that is, for any given set of times $(t_1, ..., t_n)$, consider the joint distribution of $(X_{t_1}, ..., X_{t_n})$, called the finite-dimensional distribution .

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- **Definition 2.2** Let T be the set of all vectors $\{t = (\mathrm{t}_1, ..., t_n)' \in T^n : h_1 < ... < t_n, n = 1, 2, ...\}$. Then the **(finite-dimensional) distribution functions** of the stochastic process $\{X_t, t \in T\}$ are the functions $\{F_t(\cdot), t \in T\}$ defined for $t = (t_1, ..., t_n)'$ by

$$F_t(x) = P(X_{t_1} \leqq x_1, ..., X_{t_n} \leqq x_n), x = (x_1, ..., x_n)' \in R^n$$

where $t(i)$ and $x(i)$ are the $(n-1)$-component vectors obtained by deleting the ith components of t and x, respectively.

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- **Theorem 2.1 (Kolmogorov's Consistency Theorem)** The probability distribution functions $\{F_t(\cdot), t \in T\}$ are the distribution functions of some stochastic process if and only if for any $n \in \{1, 2, ...\}$, $t = (ti, ..., tn)' \in \mathcal{T}$ and $1 \leqq i \leqq n$,

$$\lim_{x_i \to \infty} F_t(x) = F_{t(i)}(x(i)),$$

where $t(i)$ and $x(i)$ are the $(n-1)$-component vectors obtained by deleting the ith components of t and x, respectively.

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- **Definition 2.3** $\{X_t\}$ is said to be **strictly stationary** if for all n, for all $(t_1, ..., t_n)$, and for all τ,

$$(X_{t_1}, ..., X_{t_n}) \stackrel{\mathsf{d}}{=} (X_{t_1+\tau}, ..., X_{t_n+\tau}),$$

where $\stackrel{\mathsf{d}}{=}$ denotes equality in distribution.

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- **Definition 2.4** Let $\{X_t : t \in T\}$ be a stochastic process such that vr(Xt) ¡ oo for all t e T. Then the **auto covariance function** $\gamma_X(\cdot,\cdot)$ of $\{X_t\}$ is defined by

$$\gamma_X(r,s) = cov(X_r, X_s) = E[(X_r - EX_r)(X_s - EX_s)], \quad r,s \in T$$

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- **Definition 2.5** $\{X_t\}$ is said to be **weakly stationary (second-orderstationary, wide-sense stationary)** if

$(i) E(X_t) = \mu$ for all t.

$(ii) cov(X_t, X_{t+\tau}) = \gamma(\tau)$ for all t and for all τ.

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- **Definition 2.6** Let $\{X_t\}$ be a stationary process. Then

$(i)\gamma(\tau) = cov(X_t, X_{t+\tau})$ is called the **autocovariance function**.

$(ii)\rho(\tau) = \gamma(\tau)/\gamma(0)$ is called the **autocorrelation function**.

2.3 EXAMPLES

- Let $X_t = A\cos\theta t + B\sin\theta t, \quad A, B \sim (0, \sigma^2)$ i.i.d. Since EX_t=0, it follow that

$$cov(X_{t+h}, X_t) = E(X_{t+h}X_t)$$

$$= E(A\cos\theta(t+h) + B\sin\theta(t+h))(A\cos\theta t + B\sin\theta t)$$

$$= \sigma^2\cos\theta h.$$

Hence the process is stationary.

2.4 SAMPLE CORRELATION FUNCTION

- **Definition 2.7** If the sample average formed from a sample path of a process converges to the underlying parameter of the process, the process is called **ergodic**.
- For ergodic processes, we do not need to observe separate independent replications of the entire process in order to estimate its mean value or other moments. One sufficiently long sample path would enable us to estimate the underlying moments. In this book, all the time series studied are assumed to be ergodic.

2.4 SAMPLE CORRELATION FUNCTION

- In practice,$\gamma(\tau)$ and $\rho(\tau)$ are unknown and they have to be estimated from the dat . This leads to the following definition.
- **Definition 2.8** Let $\{X_t\}$ be a given time series and be its sample mean. Then

(i)C$_k = \sum_{t=1}^{n-k}(X_t - \bar{X})(X_{t+k} - \bar{X})$ is called the **sample autocovariance function**.

$(ii) r_k = C_k/C_0$ is called the **sample autocorrelation function(ACF)**.

- By definition, $r_0 = 1$.Intuitively, C_k approximates $ma(k)$ and r_k approximates $\rho(k)$.

Thank You for your Attention