Laplace Transform

Unit - II

Definition

Laplace transform of a function $f(t)$ for $t>0$ is defined as

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt = F(s) \quad \text{---} (1)$$

Improper integral

Here L is called the Laplace transformation operator. The parameter s is a real positive number.

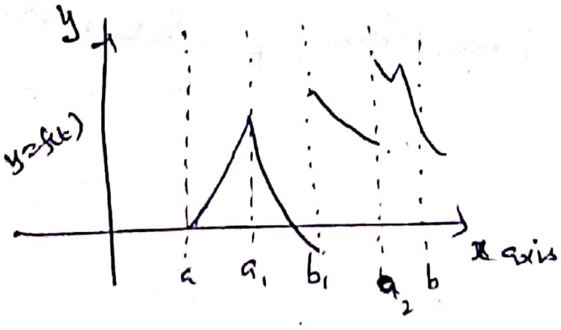
* Laplace transform exist if the integral is convergent for some values of s.

* e^{-st} is called the kernel of the transform.

Piecewise (or) Sectional Continuity

If f is a real valued function defined in some interval, we say that f is piecewise continuous if $f(t)$ is continuous everywhere except for a finite number (more precisely countable) of points in the given interval.

Example:-



Exponential Order

A function $f(t)$ is said to be of exponential order if

$$\lim_{t \to \infty} e^{-t} f(t) = 0.$$

Ex:- Consider the function $f(t) = t^2$ is of exponential order.

$$\lim_{t \to \infty} e^{-t} t^2 = \lim_{t \to \infty} \frac{t^2}{e^t} \quad \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{t \to \infty} \frac{2t}{e^t} \quad \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{t \to \infty} \frac{0}{e^t} = 0.$$

|||ly t^n, $\sin t$, $\cos t$ are of exponential order.

* Consider the function $f(t) = (e^t)^2$ is not of exponential order since $\lim_{t \to \infty} e^{-t} (e^t)^2 = \infty$. $\frac{(e^t)^{2}}{e^t}$ $e^{\infty} = \infty$.

Sufficient Condition for the existence of Laplace Transforms

* If $f(t)$ is piecewise continuous in a closed ~~every finite~~ interval $[a, b]$ and is of exponential order then the Laplace transform of $f(t)$ exists.

Some Results

1. $L(1) = \frac{1}{s}$ where $s>0$.

Proof:-

Defn of L.T

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[1] = \int_0^\infty e^{-st} \cdot dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{s}[0-1] = \frac{1}{s}.$$

$e^{\infty} = \infty$
$e^{-\infty} = 0$
$e^{0} = 1$
$e^{1} = e$

Note:- $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$ $\quad \Gamma n+1 = ?$

2. $L[t^n] = \frac{n!}{s^{n+1}}$ if n is +ve integer

$= \frac{\Gamma n+1}{s^{n+1}}$ if n is not an integer.

Proof:- $L[t^n] = \int_0^\infty t^n e^{-st}\, dt$

$st = x$

$s\,dt = dx \Rightarrow dt = \frac{dx}{s}$

$t \to 0 \Rightarrow x \to 0$

$t \to \infty \Rightarrow x \to \infty$

$$\therefore L[t^n] = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x} \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

Note:- Definition of Gamma fn

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx.$$

Definition of Gamma function

$$\Gamma n+1 = \int_0^\infty e^{-x} x^n\, dx$$

$$\therefore L[t^n] = \frac{\Gamma n+1}{s^{n+1}}.$$

If n is a +ve integer then

$$\Gamma(n+1) = n!$$

$\therefore$ $L[t^n] = \dfrac{n!}{s^{n+1}}$ if n is a +ve integer.

$= \dfrac{\Gamma(n+1)}{s^{n+1}}$ if n is not an integer.

Particular cases $\qquad 0! = 1$

(i) If $n=0$, $\quad L[1] = \dfrac{1}{s}$

If $n=1$, $\quad L[t] = \dfrac{1}{s^2}$

If $n=2$, $\quad L[t^2] = \dfrac{2}{s^3}$.

3. $L[e^{at}] = \dfrac{1}{s-a}$ if $s-a>0$

Proof:- $\quad L[e^{at}] = \displaystyle\int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left(\frac{e^{-(s-a)t}}{-(s-a)}\right)_0^\infty$$

$$= -\frac{1}{s-a}\left(e^{-\infty} - e^{0}\right) = -\frac{1}{s-a}(0-1)$$

$$= \frac{1}{s-a}.$$

H.W

4. $L[e^{-at}] = \frac{1}{s+a}$, $s+a>0$.

5. $L[\sin at] = \frac{a}{s^2+a^2}$, $s>0$

Proof:- $L[\sin at] = \int_0^\infty \sin at \, e^{-st} \, dt$

$= \left.\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right)_0^\infty$

$= 0 - \frac{1}{s^2+a^2}(0-a)$

$= \frac{a}{s^2+a^2}$

(or)

$= I.P \int_0^\infty e^{-st} e^{iat} \, dt$

$= I.P \int_0^\infty e^{-(s-ia)t} \, dt$

$= I.P \left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty$

$= I.P \left[\frac{0-1}{-(s-ia)}\right]$

$= I.P \frac{1}{s-ia} \times \frac{s+ia}{s+ia}$

$= I.P \frac{s+ia}{s^2+a^2} = \frac{a}{s^2+a^2}$.

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

6. H.W $L[\cos at] = \frac{s}{s^2+a^2}$ $s>0$.

$L[\cos at] = \int_0^\infty e^{-st} \cos at \, dt$

$= \left.\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right)_0^\infty$

$= 0 - \frac{1}{s^2+a^2}(-s) = \frac{s}{s^2+a^2}$ if $s>0$.

7. $L[\sinh at] = \dfrac{a}{s^2-a^2}$, $s > |a|$.

$\sinh t = \dfrac{e^{t}-e^{-t}}{2}$

$$L[\sinh at] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right]$$ by linear property.

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{\cancel{s}+a \cancel{-s}+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

$\cosh at = \dfrac{e^{at}+e^{-at}}{2}$

8. $L[\cosh at] = \dfrac{s}{s^2-a^2}$

Linear Property :-

If $L[f(t)] = F(s)$ & $L[g(t)] = G(s)$, then

$$L[c_1 f(t) \pm c_2 g(t)] = c_1 L[f(t)] \pm c_2 L[f(t)].$$

where c_1 & c_2 are constants.

Proof :- $L[c_1 f(t) \pm c_2 g(t)] = \int_0^\infty e^{-st}[c_1 f(t) \pm c_2 g(t)]\,dt$

$$= c_1 \int_0^\infty e^{-st} f(t)\,dt \pm c_2 \int_0^\infty e^{-st} g(t)\,dt$$

$$= c_1 L[f(t)] \pm c_2 L[g(t)].$$

Change of Scale Property

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof:-

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$$

Put $at = u$ $\quad t \to 0 \Rightarrow u \to 0$

$a\,dt = du$ $\quad t \to \infty \Rightarrow u \to \infty$

$dt = \frac{du}{a}$

$$L[f(at)] = \int_0^\infty e^{-\frac{su}{a}} f(u) \frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\, du$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

First shifting Property

If $L[f(t)] = F(s)$ then

1. $L[e^{at} f(t)] = F(s-a)$
2. $L[e^{-at} f(t)] = F(s+a)$.

Proof:-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= F(s+a).$$

Problems

Find the L.T of the following

(1) $\sin^2 kt$ (3) $4e^{5t}+6t^3-3\sin 4t+2\cos 2t$

(2) $\cos^3 at$ (4) $\sin t \sin 2t \sin 3t$ (5) $t^{3/2}$.

Proof:-

(1) $L(\sin^2 kt) = L\left(\frac{1-\cos 2kt}{2}\right)$

$L(\cos at) = \frac{s}{a^2+s^2}$

$= \frac{1}{2} L(1) - L(\cos(2k)t)$

$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4k^2}\right]$

$\cos^3 A = \frac{1}{4}(3\cos A + \cos 3A)$

(ii) $L(\cos^3 at) = L\left(\frac{\cos 3at + 3\cos at}{4}\right)$

$= \frac{1}{4}\left[L(\cos 3at) + 3L(\cos at)\right]$

$= \frac{1}{4}\left[\frac{s}{s^2+9a^2} + \frac{3s}{s^2+a^2}\right]$

(3) $L[4e^{5t}+6t^3-3\sin 4t+2\cos 2t]$

$= 4L[e^{5t}] + 6L[t^3] - 3L[\sin 4t] + 2L[\cos 2t]$

$= 4\frac{1}{s-5} + \frac{6\times 3!}{s^4} - 3\cdot\frac{4}{s^2+4^2} + \frac{2\times s}{s^2+4}$

(4) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\sin 2t \sin 3t = \frac{1}{2}[\cos(-t) - \cos(5t)] = \frac{1}{2}[\cos t - \cos 5t]$

$\sin t \sin 2t \sin 3t = \frac{1}{2}[\sin t\cos t - \sin t \cos 5t]$.

$$= \frac{1}{4}\left[\sin 2t - \sin 6t - \sin(-4t)\right]$$

$$= \frac{1}{4}\left[\sin 2t - \sin 6t + \sin 4t\right]$$

$$= \frac{1}{4}\left[\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right]$$

(5) $$L[t^{3/2}] = \frac{\overline{|3/2+1}}{s^{3/2+1}} = \frac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{5/2}}$$

$\overline{|n+1} = n.(n-1)\ldots \frac{1}{2}\sqrt{\pi}$.

$$= \frac{3\sqrt{\pi}}{4 \cdot s^{5/2}}.$$

First shifting Property

(2) Find the L.T of the following

(1) $e^{2t}\sin 3t$ (2) $e^{-t}\cos^2 t$. (3) $\cosh 2t \cos 2t$

$\cancel{(4)}$ $\cos(\omega t + 2)$ (5) $e^{-t}(3\sinh 2t - 2\cosh 3t)$. $L[f(t)] = F(s)$

(6) $L[e^{4t}\sin 2t \cos t]$. $L[e^{at} f(t)] = F(s-a)$

Soln:-

(1) $$L[e^{2t}\sin 3t] = L(\sin 3t)\Big|_{s\to s-2}$$

$$= \frac{3}{s^2+9}\Big|_{s\to s-2}$$

$$= \frac{3}{(s-2)^2+9}$$

(2) $$L[e^{-t}\cos^2 t] = L\left[\frac{1+\cos 2t}{2}\right]\Big|_{s\to s+1}$$

$$= \frac{1}{2}\left(L[1] + L[\cos 2t]\right)\Big|_{s\to s+1}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]_{s\to s+1}$$

$$= \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+4}\right].$$

(3) $L[\cosh 2t \cos 2t] = L\left[\frac{e^{2t}+e^{-2t}}{2} \cdot \cos 2t\right]$

$$= \frac{1}{2}\left[L[e^{2t}\cos 2t] + L[e^{-2t}\cos 2t]\right]$$

$$= \frac{1}{2}\left[L[\cos 2t]_{s\to s-2} + L[\cos 2t]_{s\to s+2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4}\Big|_{s\to s-2} + \frac{s}{s^2+4}\Big|_{s\to s+2}\right]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{s+2}{(s+2)^2+4}\right]$$

$L[\cos(wt+2)] = L[\cos wt \cos 2 - \sin wt \sin 2]$

$$= \cos 2 \cdot \frac{s}{s^2+w^2} - \sin 2 \, \frac{w}{s^2+w^2}$$

(5) $L[e^{-t}(3\sinh 2t - 2\cosh 3t)]$

$$= 3\left[L(\sinh 2t)\right]_{s\to s+1} - 2L(\cosh 3t)_{s\to s+1}]$$

$$= 3 \cdot \frac{2}{s^2-4} - \frac{2s}{s^2-9}\Big|_{s\to s+1}$$

$$= \frac{6}{(s+1)^2-4} - \frac{2(s+1)}{(s+1)^2-9}$$

(6) $L[e^{4t}\sin 2t \cos t] = L[\sin 2t \cos t]_{s\to s-4}$

$\sin A \cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$

$$= \frac{1}{2}L[\sin 3t + \sin t]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right].$$

Laplace transform of Unit Step function.

Defn

Unit Step function (or) Heaviside's unit step function.

The unit step function is defined as

$$U(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \quad \text{where } a \geq 0 \end{cases}$$

Laplace Transform of $U(t-a)$ is

$$L[U(t-a)] = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= 0 + \left(\frac{e^{-st}}{-s}\right)_a^\infty = 0 - \left(\frac{e^{-sa}}{-s}\right) = \frac{e^{-sa}}{s}$$

Laplace transform of Unit Impulse function

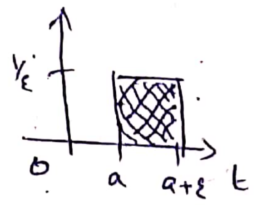
Unit Impulse function (or) Dirac-delta function

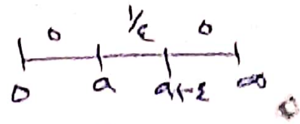
The Unit Impulse function is defined as

$$\delta(t-a) = \lim_{\epsilon \to 0} \delta_\epsilon(t-a)$$

where

$$\delta_\epsilon(t-a) = \begin{cases} \frac{1}{\epsilon}, & a < t < a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$





Laplace transform

$$L[\delta_\varepsilon(t-a)] = \int_0^\infty e^{-st}\,\delta_\varepsilon(t-a)\,dt = \int_0^a e^{-st}(0)\,dt + \int_a^{a+\varepsilon} e^{-st}\,\frac{1}{\varepsilon}\,dt + \int_{a+\varepsilon}^{\infty} e^{-st}\,0\,dt$$

$$= \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st}\,dt = \frac{1}{\varepsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\varepsilon}$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-s(a+\varepsilon)} - e^{-sa}}{-s}\right]$$

$$= \frac{1}{\varepsilon}\,\frac{e^{-sa}}{s}\left(1 - e^{-s\varepsilon}\right)$$

$$= \frac{e^{-sa}}{s}\left(\frac{1 - e^{-s\varepsilon}}{\varepsilon}\right)$$

Now $L[\delta(t-a)] = L\left[\lim_{\varepsilon\to 0} \delta_\varepsilon(t-a)\right]$

$$= \lim_{\varepsilon\to 0} L[\delta_\varepsilon(t-a)]$$

$$= \lim_{\varepsilon\to 0} \frac{e^{-sa}}{s}\left(\frac{1-e^{-s\varepsilon}}{\varepsilon}\right)$$

$$= \frac{e^{-sa}}{s}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(1 - \left(1 - \frac{s\varepsilon}{1!} + \frac{s^2\varepsilon^2}{2!} - \dots\right)\right)$$

$$= \frac{e^{-sa}}{s}\lim_{\varepsilon\to 0}\left(+\frac{s}{1!} - \frac{s^2\varepsilon}{2!} \dots\right)$$

$$= \frac{e^{-sa}}{\cancel{s}}\left(+\cancel{s}\right)$$

$$\boxed{L[\delta(t-a)] = e^{-sa}}$$

Change of Scale

1. $\underset{F(s)}{L[f(t)]} = \frac{e^{-1/s}}{s}$ find $L[e^{-t} f(3t)]$.

Soln:- w.k.t by change of scale property

$L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Here $a = 3$.

So $L[f(3t)] = \frac{1}{3} F\left(\frac{s}{3}\right)$

$= \frac{1}{\cancel{3}} \frac{e^{-3/s}}{s/\cancel{3}} = \frac{1}{s} e^{-3/s}$

$L[e^{-t} f(3t)] = L[f(3t)]\Big|_{s \to s+1}$

$= \left[\frac{1}{s} e^{-3/s}\right]_{s \to s+1}$

$= \frac{1}{s+1} e^{-3/(s+1)}$.

Laplace transform of Unit ~~Impulse~~ Step function

The Unit Step function is defined as

$$U(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

The Laplace Transform is

$L[U(t-a)] = \int_0^\infty e^{-st} U(t-a)\, dt$

$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} (1)\, dt$

$= \int_a^\infty e^{-st}\, dt = \left(\frac{e^{-st}}{-s}\right)_a^\infty = \left(\frac{0 - e^{-sa}}{-s}\right)$

$= \frac{e^{-sa}}{s}$.

Differentiation of L. Transforms : Multiplication by t

Theorem:-

1. If $L[f(t)] = F(s)$ then

$$L[t\,f(t)] = -\frac{d}{ds}F(s) = -\frac{d}{ds}L[f(t)]$$

2. If $L[f(t)] = F(s)$ then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$$

① Find the Laplace transform of the following :

1) $t\cos at$ 2) $t\,e^{at}\sin at$ 3) $t(3\sin 2t - 2\cos 2t)$

4) $t^3 e^{-3t}$ 5) $(1+te^{-t})^3$ 6) $t\sin 3t\cos 2t$.

Soln:-

1) $$L[t\cos at] = -\frac{d}{ds}L[\cos at]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

2. $$L[t\,e^{at}\sin at] = -\frac{d}{ds}L[e^{at}\sin at]$$

$$= \left[-\frac{d}{ds}L[\sin at]\right]_{s\to s-a}$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\Bigg]_{s\to s-a}$$

$$= -\left[\frac{-a\cdot 2s}{(s^2+a^2)^2}\right]_{s\to s-a}$$

$$= \frac{2a(s-a)}{((s-a)^2+a^2)^2} \;//.$$

3. $L[t(3\sin 2t - 2\cos 2t)]$

$= -\frac{d}{ds}\left[3L(\sin 2t) - 2L[\cos 2t)\right]$

$= -\frac{d}{ds}\left[\frac{3\times 2}{s^2+4} - \frac{2s}{s^2+4}\right]$

$= -\left(\frac{-6\times 2s}{(s^2+4)^2} - 2\left(\frac{s^2+4(1) - s\,2s}{(s^2+4)^2}\right)\right)$

$= \frac{12s}{(s^2+4)^2} + \frac{2(4-s^2)}{(s^2+4)^2}$

4. $L[t^3 e^{-3t}] = (-1)^3 \frac{d^3}{ds^3} L[e^{-3t}]$

$= -\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$

$= -\frac{d^2}{ds^2}\left(\frac{-1}{(s+3)^2}\right) = \frac{d}{ds}\frac{-2}{(s+3)^3}$

$= \frac{6}{(s+3)^4}.$

$L[e^{-3t} t^3] = L[t^3]\Big|_{s\to s+3}$

$= \frac{6}{s^4}\Big|_{s\to s+3}$

$= \frac{6}{(s+3)^4}.$

5. $L[1+te^{-t}]^3$

$= L(1+t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t})$

$= L[1] + L[e^{-3t}t^3] + 3L[e^{-t}t] + 3L[e^{-2t}t^2]$

$= \frac{1}{s} + \frac{6}{s^4}\Big|_{s\to s+3} + 3\frac{1}{s^2}\Big|_{s\to s+1} + 3\times\frac{2}{s^3}\Big|_{s\to s+2}$

$= \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3}.$

6. $L[t \sin 3t \cos 2t] = -\frac{d}{ds} L[\sin 3t \cos 2t]$

$= \frac{-1}{2}\frac{d}{ds} L[\sin 5t + \sin t]$

$= -\frac{1}{2} \frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$

$= -\frac{1}{2}\left[\frac{-10s}{(s^2+25)^2} + \frac{-2s}{(s^2+1)^2}\right]$

$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$.

2. P.T $\int_0^\infty t^2 e^{-4t} \sin 2t \, dt = \frac{11}{500}$.

W.K.T $\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$

$\int_0^\infty e^{-4t} \underbrace{t^2 \sin 2t}_{f(t)} \, dt = L[t^2 \sin 2t]_{s=4}$

$= (-1)^2 \frac{d^2}{ds^2} L[\sin 2t] \Big|_{s=4}$

$= \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)\Big|_{s=4}$

$= 2\frac{d}{ds}\left(-\frac{2s}{(s^2+4)^2}\right)\Big|_{s=4}$

$= -4\left(\frac{(s^2+4)^2 - s \cdot 2(s^2+4) \cdot 2s}{(s^2+4)^4}\right)\Big|_{s=4}$

$= -4\left(\frac{s^2+4 - 4s^2}{(s^2+4)^3}\right) = -4\left(\frac{4-3s^2}{(s^2+4)^3}\right)_{s=4}$

$= -4\left(\frac{4-48}{(20)^3}\right) = \frac{44 \times 4}{8000} = \frac{11}{500}$ //.

H.W P.T $\int_0^\infty e^{-3t} t \cos t \, dt = \frac{2}{25}$

Integration of Transform: Division by t

Theorem:-

* If $L[f(t)] = F(s)$ ~~and if~~ $\frac{f(t)}{t}$ ~~has a limit as~~ ~~$t \to 0$~~ then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\,ds = \int_s^\infty L[f(t)]\,ds$

* $L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \int_s^\infty F(s)\,ds\,ds$

Pblm:- Find the L.T of the following

1) $L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty L(e^{-at} - e^{-bt})\,ds$

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right)\right]_s^\infty = \left[\log\left(\frac{1+a/s}{1+b/s}\right)\right]_s^\infty$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right) = -\log\left(\frac{s+a}{s+b}\right) = \log\left(\frac{s+b}{s+a}\right)$$

2. $L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at] - L[\cos bt]\,ds$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left.\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2(1+a^2/s^2)}{s^2(1+b^2/s^2)}\right)\right]_s^\infty$$

$$= 0 - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

3. $L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty L[1-\cos at]\,ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+a^2)\right)_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+a^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2}{s^2+a^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{1}{1+a^2/s^2}\right]_s^\infty$$

$$= \frac{1}{2}\left(0 - \log \frac{s^2}{s^2+a^2}\right) = \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right).$$

4. $L\left[\frac{\sin 3t \cos t}{t}\right] = \int_s^\infty \frac{1}{2} L[\sin 4t] + L[\sin 2t]\Big)\,ds$

$$= \frac{1}{2}\int_s^\infty \frac{4}{s^2+16} + \frac{2}{s^2+4}\Big]\,ds$$

$$= \frac{1}{2}\left(\cancel{4}\,\frac{1}{\cancel{4}}\tan^{-1}\left(\frac{s}{4}\right) + \cancel{2}\cdot\frac{1}{\cancel{2}}\tan^{-1}\left(\frac{s}{2}\right)\right)_s^\infty$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{4} + \frac{\pi}{2} - \tan^{-1} s/2\right).$$

$$= \frac{1}{2}\left(\pi - \tan^{-1}\frac{s}{4} - \tan^{-1}\frac{s}{2}\right).$$

5. Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$ $\qquad \int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$

$$\int_0^\infty e^{-0t}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = L\left[\frac{\cos 6t - \cos 4t}{t}\right]_{s=0}$$

$$= \int_s^\infty L[\cos 6t] - L[\cos 4t]\Big]_{s=0} = \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds \Big|_{s=0}.$$

$$= \frac{1}{2}\left(\log \frac{s^2+36}{s^2+16} \right)_s^\infty \Bigg|_{s=0}$$

$$= -\frac{1}{2} \log \frac{s^2+36}{s^2+16} \Bigg)_{s=0}$$

$$= \frac{1}{2} \log \frac{s^2+16}{s^2+36} \Bigg|_{s=0}$$

$$= \frac{1}{2} \log \frac{16}{36} = \log \frac{4}{6} = \log 2/3.$$

6. Evaluate $\int_0^\infty e^{-t} \frac{\sin^2 t}{t}\, dt$

$\int_0^\infty e^{-st} f(t)\, dt = F(s)$
$= L[f(t)].$

$$= L\left[\frac{\sin^2 t}{t}\right]_{s=1}$$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right) ds \Bigg|_{s=1}$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+4}\, ds \Bigg|_{s=1}$$

$$= \frac{1}{2}\left(\log s - \frac{1}{2}\log s^2+4 \right)_s^\infty \Bigg|_{s=1}$$

$$= \frac{1}{4} \log \frac{s^2}{s^2+4} \Bigg)_s^\infty \Bigg|_{s=1}$$

$$= -\frac{1}{4} \log \frac{s^2}{s^2+4} \Bigg|_{s=1}$$

$$= -\frac{1}{4} \log \frac{1}{5}$$

$$= \frac{1}{4} \log 5.$$

H.W

1. $L\left[\frac{\sin at}{t}\right] \to \cot^{-1}\left(\frac{s}{a}\right)$

$\frac{4s}{(s^2-4)^2}$ 2. $t \sinh 2t$, $t e^{-t} \cos 2t \to \frac{s^2+2s-3}{(s^2+2s+5)^2}$, $\frac{e^{-t} \sin t}{t} \to \cot^{-1}(s+1)$, $\frac{e^{-at} - \cos at}{t} \to \log\left(\sqrt{\frac{s^2+b^2}{s^2+a^2}}\right)$

3. $\int_0^\infty t e^{-3t} \sin t\, dt = 3/50$

~~4.~~ $\cancel{\int_0^\infty e^{-t} \frac{\sin^2 t}{t}\, dt}$.

Thm If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$.

1. Find the L.T of $e^{-t}\left[\int_0^t t\cos t\,dt\right]$.

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = L\left[\int_0^t t\cos t\,dt\right]_{s\to s+1}$$

$$= \frac{1}{s} L[t\cos t]_{s\to s+1}$$

$$= \frac{1}{s}\left[-\frac{d}{ds} L[\cos t]\right]_{s\to s+1}$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= -\frac{1}{s}\left[\frac{s^2+1-s.2s}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= -\frac{1}{s}\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s\to s+1} = \frac{1}{s+1}\left[\frac{(s+1)^2-1}{((s+1)^2+1)^2}\right].$$

2. $$L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right] = L\left[\int_0^t \frac{\sin t}{t}\,dt\right]_{s\to s+1}$$

$$= \frac{1}{s} L\left[\frac{\sin t}{t}\right]_{s\to s+1} = \frac{1}{s}\int_s^\infty L[\sin t]\,ds\Big|_{s\to s+1}$$

$$= \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\,ds\Big|_{s\to s+1}$$

$$= \frac{1}{s}\left(\tan^{-1}s\right)_s^\infty\Big|_{s\to s+1}$$

$$= \frac{1}{s}\left[\frac{\pi}{2}-\tan^{-1}s\right]_{s\to s+1} = \frac{1}{s}\cancel{\cot^{-1}s}\Big]_{s\to s+1}$$

$$= \frac{1}{s+1}\left[\frac{\pi}{2}-\tan^{-1}(s+1)\right]$$

HW

3. $L\left[\int_0^t \frac{e^t\sin t}{t}\,dt\right] = \frac{\pi}{2} - \tan^{-1}(s-1).$

4. $L\left[\int_0^t t e^{-t}\sin t\,dt\right] = \frac{1}{s}\frac{2(s+1)}{(s^2+2s+2)}.$

Initial Value Theorem

H.W

1. IVT & FVT $3e^{-2t}$
2. $t^2 e^{-3t}$

If $L[f(t)] = F(s)$, then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

Final value Theorem

If $L[f(t)] = F(s)$ then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

1. Verify the initial Value theorem for $f(t) = ae^{-bt}$.

I.V.T is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$.

Proof:-

L.H.S $\quad \lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt} = a.$

R.H.S $\quad \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\,L[f(t)] = \lim_{s\to\infty} s\,L[ae^{-bt}]$

$$= \lim_{s\to\infty} s\, a \frac{1}{s+b} = \lim_{s\to\infty} \cancel{s}.a \frac{1}{\cancel{s}(1+b/s)}$$

$$= a.$$

L.H.S. = R.H.S

2. Verify the initial & final value theorem for the fn

$$f(t) = 1 + e^{-t}(\sin t + \cos t).$$

Solu:-

$$F(s) = L[f(t)] = L[1 + e^{-t}(\sin t + \cos t)]$$

$$= \frac{1}{s} + L[e^{-t}\sin t] + L[e^{-t}\cos t]$$

$$= \frac{1}{s} + \frac{1}{s^2+1}\Big|_{s\to s+1} + \frac{s}{s^2+1}\Big|_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

IVT.

$$\lim_{t\to 0} f(t) = 1 + e^{0}(\sin 0 + \cos 0) = 1 + 1(0+1) = 2.$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right)$$

$$= \lim_{s\to\infty} \left(1 + \frac{s}{s^2((1+1/s)^2+1/s^2)} + \frac{s(1+1/s)}{s((1+1/s)^2+1/s^2)}\right)$$

$$= 1 + 1 = 2.$$

FVT

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

L.HS
$$\lim_{t\to\infty} [1 + e^{-t}(\sin t + \cos t)] = 1.$$

R.HS
$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} 1 + \frac{s}{(s+1)^2+1} + \frac{s(s+1)}{(s+1)^2+1}$$

$$= 1. \quad LHS = RHS$$

3. Verify IVT for $e^{-2t}\sin t$.

LHS
$$\lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-2t}\sin t = 0.$$

R.HS
$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\, L[e^{-2t}\sin t]$$

$$= \lim_{s\to\infty} s\, L[\sin t]_{s\to s+2}$$

$$= \lim_{s\to\infty} s \left.\frac{1}{s^2+1}\right|_{s\to s+2}$$

$$= \lim_{s\to\infty} \left(\frac{s}{(s+2)^2+1}\right) = \lim_{s\to\infty} \frac{s}{s^2+4s+5} \quad \frac{\infty}{\infty}$$

for L Hospital's rule

$$= \lim_{s\to\infty} \frac{1}{2s+4} = \frac{1}{\infty} = 0.$$

L.HS = R.HS

Laplace Transform of Periodic Functions

Thm:- If $f(t)$ is periodic function with Period T, then the Laplace transform of Periodic function is

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

1. Find the Laplace transform of

$$f(t) = \begin{cases} 1 & 0<t<a \\ -1 & a<t<2a \end{cases} \quad \text{and}$$

$f(t+T) = f(t)$
T - non zero constant

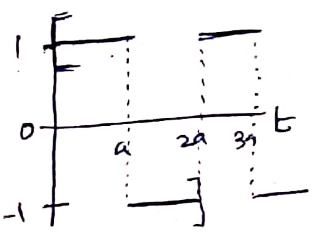
$f(t+2a) = f(t)$ for all t.

Solution:- The given function is a periodic function with Period $2a$.

$$L[f(t)] = \frac{1}{1-e^{-s2a}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}(1)\,dt + \int_a^{2a} e^{-st}(-1)\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a} \right]$$



$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-sa}-1}{-s} - \left(\frac{e^{-2as}-e^{-sa}}{-s}\right) \right]$$

$$= \frac{1}{s(1-e^{-2as})} \left[1-e^{-sa} + e^{-2as} - e^{-sa} \right]$$

$$= \frac{1}{s(1-e^{-2as})} \left[1-2e^{-sa} + e^{-2as} \right]$$

$$= \frac{1}{s(1-e^{-2as})} \left[1-e^{-sa} \right]^2$$

$$= \frac{\cancel{(1-e^{-sa})^2}}{s(1+e^{-as})\cancel{(1-e^{-as})}}$$

$$= \frac{1-e^{-sa}}{s(1+e^{-as})} \quad // \quad \frac{\cancel{e^{sa/2}}\left(e^{\frac{sa}{2}}-e^{-\frac{sa}{2}}\right)}{s\,\cancel{e^{sa/2}}\left(e^{\frac{sa}{2}}+e^{-\frac{sa}{2}}\right)} = \frac{1}{s}\tanh\left(\frac{sa}{2}\right)$$

2. Find the Laplace Transform of a Square wave function given by

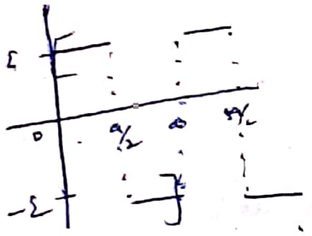
$$f(t) = \begin{cases} E, & 0 \le t \le a/2 \\ -E, & a/2 \le t \le a \end{cases}$$

and $f(t+a) = f(t)$ for all t.

Soln:- The given function is a periodic function with Period a. Hence,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^a e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} E\, dt - \int_{a/2}^{a} e^{-st} E\, dt \right]$$



$$= \frac{E}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{a/2} - \left(\frac{e^{-st}}{-s} \right)_{a/2}^{a} \right]$$

$$= \frac{E}{1-e^{-as}} \left[\left(\frac{e^{-sa/2}}{-s} + \frac{1}{s} \right) - \left(\frac{e^{-sa}}{-s} + \frac{e^{-sa/2}}{+s} \right) \right]$$

$$= \frac{E}{s(1-e^{-as})} \left[-e^{-sa/2} + 1 + e^{-sa} - e^{-sa/2} \right]$$

$$= \frac{E}{s(1-e^{-as})} \left(1 + e^{-sa} - 2e^{-sa/2} \right)$$

$$= \frac{E}{s(1-e^{-as})} \left(1 - e^{-sa/2} \right)^2$$

2. $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ 0, & \pi<t<2\pi \end{cases}$

$$= \frac{E\, (1-e^{-sa/2})^{\cancel{2}}}{s\, \cancel{(1-e^{-as/2})}\, (1+e^{-sa/2})}$$

1. $f(t) = \begin{cases} t & 0<t<1 \\ 2-t & 1<t<2 \end{cases}$

$f(t+2) = f(t)$

Ans: $\frac{1}{s^2} \tanh\left(\frac{s}{2}\right)$

$$= \frac{E}{s} \frac{(1-e^{-sa/2})}{(1+e^{-sa/2})} \quad // \quad = \cancel{\frac{E\, e^{-sa/4}(1-e^{sa/2})}{s\, e^{-sa/4}(1+e^{sa/2})}}$$

$$= \frac{E}{s} \frac{e^{-sa/4}(e^{sa/4} - e^{-sa/4})}{e^{-sa/4}(e^{sa/4} + e^{-sa/4})} = \frac{E}{s} \tanh\left(\frac{sa}{4}\right) //$$

$\frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} = \tanh\theta$.

(3) Find the Laplace transform of the Half-wave rectifier function

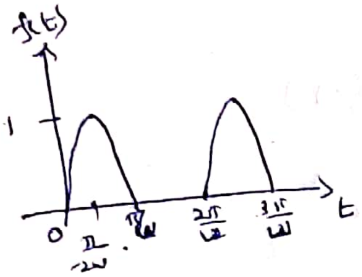
$$f(t) = \begin{cases} \sin\omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

and $f(t+\frac{2\pi}{\omega}) = f(t)$ for all t.

Soln:-

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-s\frac{2\pi}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt$$



$$= \frac{1}{1-e^{-s\frac{2\pi}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt + 0 \right]$$

$$= \frac{1}{1-e^{-s\frac{2\pi}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s\sin\omega t - \omega\cos\omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-\pi s/\omega}}{s^2+\omega^2} (0+\omega) - \frac{1}{s^2+\omega^2}(-\omega) \right]$$

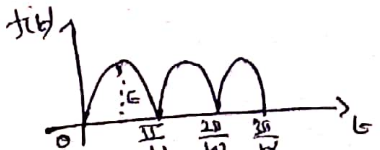
$$= \frac{\omega}{1-e^{-\frac{2\pi s}{\omega}}(s^2+\omega^2)} (e^{-\pi s/\omega}+1)$$

$$= \frac{\omega\,\cancel{(e^{-\frac{\pi s}{\omega}}+1)}}{(s^2+\omega^2)\,\cancel{(1+e^{-\frac{\pi s}{\omega}})}\,(1-e^{-\frac{\pi s}{\omega}})}$$

$$= \frac{\omega}{(s^2+\omega^2)(1-e^{-\frac{\pi s}{\omega}})}.$$

H.W Full Sine wave rectifier

$f(t) = E\sin\omega t \quad 0 < t < \frac{\pi}{\omega}$ and the period $T = \frac{\pi}{\omega}$



Ans:- $\dfrac{E\omega\,(e^{-\pi s/\omega}+1)}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}$.

Inverse Laplace Transforms

Definition If the Laplace transform of $f(t)$ is $F(s)$, ie.,

$L[f(t)] = F(s)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is given by

$$L^{-1}[F(s)] = f(t)$$

where L^{-1} is called the inverse Laplace transform operator.

Table of Inverse Laplace Transform

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
$L[t^n] = \frac{n!}{s^{n+1}}$	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$
$L[\sin at] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$
$L[\cos at] = \frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
$L[\sinh at] = \frac{a}{s^2-a^2}$	$L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$
$L[\cosh at] = \frac{s}{s^2-a^2}$	$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$,

Linear Property

If $F(s)$ and $G(s)$ are the Laplace transforms of the functions of $f(t)$ and $g(t)$ respectively, then

$$L^{-1}[c_1 F(s) \pm c_2 G(s)] = c_1 L^{-1}[F(s)] \pm c_2 L^{-1}[G(s)],$$

where c_1, c_2 are constants.

Proof:- We know that

$$L[c_1 f(t) \pm c_2 g(t)] = c_1 F(s) \pm c_2 G(s).$$

Now, Taking inverse Laplace on both sides

$$c_1 f(t) \pm c_2 g(t) = L^{-1}[c_1 F(s) \pm c_2 G(s)]$$

W.K.T $\quad f(t) = L^{-1}[F(s)] \;\&\; g(t) = L^{-1}[G(s)].$

$$\therefore L^{-1}[c_1 F(s) \pm c_2 G(s)] = c_1 L^{-1}[F(s)] \pm c_2 L^{-1}[G(s)].$$

First Shifting Theorem

If $f(t) = L^{-1}[F(s)]$ then $L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]$.

$L^{-1}[F(s-a)] = e^{at} L^{-1}[F(s)]$.

Proof:-

W.K.T $\quad L[e^{-at} f(t)] = F(s+a).$

$$e^{-at} f(t) = L^{-1}[F(s+a)]$$

$$e^{-at} L^{-1}[F(s)] = L^{-1}[F(s+a)].$$

Example:- Find the inverse Laplace transform $f(t)$ of each of the following function $F(s)$.

(1) $\dfrac{3}{s+4}$ (2) $\dfrac{8s}{s^2+16}$ (3) $\dfrac{1}{2s-5}$

(4) $\dfrac{3s-8}{4s^2+25}$ (5) $\dfrac{6}{s^2+4}$ (6) $\dfrac{s}{2s^2-8}$

Soln:-

(1) $$L^{-1}\left[\frac{3}{s+4}\right] = 3\,L^{-1}\left[\frac{1}{s+4}\right] = 3e^{-4t}.$$

(2) $$L^{-1}\left[\frac{8s}{s^2+16}\right] = 8\,L^{-1}\left[\frac{s}{s^2+4^2}\right] = 8\cos 4t.$$

(3) $$L^{-1}\left[\frac{1}{2s-5}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s-5/2}\right] = \frac{1}{2}e^{5/2\,t}$$

(4) $$L^{-1}\left[\frac{3s-8}{4s^2+25}\right] = 3\,L^{-1}\left[\frac{s}{4s^2+25}\right] - 8\,L^{-1}\left[\frac{1}{4s^2+25}\right]$$

$$= \frac{3}{4}L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - \frac{8\times 2}{4\times 5}L^{-1}\left[\frac{5/2}{s^2+(5/2)^2}\right]$$

$$= \frac{3}{4}\cos 5/2\,t - \frac{2\times 2}{5}\sin 5/2\,t.$$

(5) $$L^{-1}\left[\frac{6}{s^2+4}\right] = 6\,L^{-1}\left[\frac{1}{s^2+2^2}\right] = 3\,L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= 3\sin 2t.$$

(6) $$L^{-1}\left[\frac{s}{2s^2-8}\right] = \frac{1}{2}L^{-1}\left[\frac{s}{s^2-4}\right] = \frac{1}{2}\cosh 2t.$$

2. ILT of

(1) $\dfrac{s}{(s+a)^2+b^2}$ (3) $\dfrac{4s+12}{s^2+8s+16}$ (2) $\dfrac{5}{(s+2)^5}$

(4) $\dfrac{s+2}{s^2-4s+13}$ (5) $\dfrac{3s+2}{4s^2+12s+9}$

Soln:-

(1)

$$L^{-1}\left[\frac{s}{(s+a)^2+b^2}\right] = L^{-1}\left[\frac{s+a-a}{(s+a)^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] - L^{-1}\left[\frac{a}{(s+a)^2+b^2}\right]$$

$$= e^{-at} L^{-1}\left[\frac{s}{s^2+b^2}\right] - \frac{a}{b} e^{-at} L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$- \frac{a}{b} e^{-at} \sin bt$$

(3)

$$L^{-1}\left[\frac{4s+12}{s^2+8s+16}\right] = 4 L^{-1}\left[\frac{s+3}{(s+4)^2}\right] = 4 L^{-1}\left[\frac{s}{(s+4)^2}\right] + 12 L^{-1}\left[\frac{1}{(s+4)^2}\right]$$

$$= 4 L^{-1}\left[\frac{s+4-4}{(s+4)^2}\right] + 12 L^{-1}\left[\frac{1}{(s+4)^2}\right]$$

$$= 4 L^{-1}\left[\frac{s+4}{(s+4)^2}\right] - 16 L^{-1}\left[\frac{1}{(s+4)^2}\right] + 12 L^{-1}\left[\frac{1}{(s+4)^2}\right]$$

$$= 4 L^{-1}\left[\frac{1}{s+4}\right] - 4 L^{-1}\left[\frac{1}{(s+4)^2}\right]$$

$$= 4 e^{-4t} - 4 e^{-4t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= 4e^{-4t} - 4e^{-4t} t$$

(2)

$$L^{-1}\left[\frac{5}{(s+2)^5}\right] = 5 L^{-1}\left[\frac{1}{(s+2)^5}\right]$$

$$= 5e^{-2t} L^{-1}\left[\frac{1}{s^5}\right]$$

$$= 5e^{-2t} \frac{t^4}{4!}$$

$$= \frac{5}{24} t^4 e^{-2t}.$$

(4) $$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = L^{-1}\left[\frac{s+2}{(s-2)^2+9}\right] = L^{-1}\left[\frac{s+2-2+2}{(s-2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]$$

$$= e^{2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] + \frac{4}{3}e^{2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t.$$

(5) $$L^{-1}\left[\frac{3s+2}{4s^2+12s+9}\right] = L^{-1}\left[\frac{3s}{4s^2+12s+9}\right] + 2L^{-1}\left[\frac{1}{4s^2+12s+9}\right]$$

$$= \frac{3}{4}L^{-1}\left[\frac{s}{s^2+3s+\frac{9}{4}}\right] + \frac{2}{4}L^{-1}\left[\frac{1}{s^2+3s+\frac{9}{4}}\right]$$

$$= \frac{3}{4}L^{-1}\left[\frac{s}{(s+\frac{3}{2})^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+\frac{3}{2})^2}\right]$$

$$= \frac{3}{4}L^{-1}\left[\frac{s+3/2-3/2}{(s+3/2)^2}\right] + \frac{1}{2}e^{-3/2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= \frac{3}{4}L^{-1}\left[\frac{s+3/2}{(s+3/2)^2}\right] - \frac{9}{8}L^{-1}\left[\frac{1}{(s+3/2)^2}\right] + \frac{1}{2}e^{-3/2t}\,t.$$

$$= \frac{3}{4}L^{-1}\left[\frac{1}{s+3/2}\right] - \frac{9}{8}t\,e^{-3/2t} + \frac{1}{2}e^{-3/2t}\cdot t$$

$$= \frac{3}{4}e^{-3/2t} - \frac{5}{8}t\,e^{-3/2t}.$$

Result :- $L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]$.

Note :- The above result can be used when $F(s)$ is a log fn, inverse trigonometry functions.

Example :- Find ILT for the following functions

(1) $\log\left(\frac{s+1}{s-1}\right)$ (2) $\log\left(\frac{s+1}{s}\right)$ (3) $\log\left(1+\frac{a}{s}\right)$

(4) $\log\left(\frac{s^2+1}{s^2+4}\right)$ (5) $\tan^{-1}\left(\frac{2}{s^2}\right)$ (6) $\cot^{-1}\left(\frac{s+3}{2}\right)$

Soln :-

1. $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t} L^{-1}\left(\log(s+1) - \log(s-1)\right)'$.

$= -\frac{1}{t}\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s-1}\right)\right]$

$= -\frac{1}{t}\left[e^{-t} - e^{t}\right]$.

$F(s) = \log(s+1) - \log(s-1)$

$F'(s) = \frac{1}{s+1} - \frac{1}{s-1}$

2. $L^{-1}\left[\log\left(\frac{s+1}{s}\right)\right] = -\frac{1}{t} L^{-1}\left[\log(s+1) - \log s\right]'$.

$= -\frac{1}{t}\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s}\right)\right]$

$= -\frac{1}{t}\left[e^{-t} - 1\right]$

3. $L^{-1}\left[\log\left(1+\frac{a}{s}\right)\right] = -\frac{1}{t}\left(L^{-1}\left[\frac{1}{s+a}\right] - L^{-1}\left[\frac{1}{s}\right]\right)$

$= -\frac{1}{t}\left(e^{-at} - 1\right)$.

4. $L^{-1}\left[\log\left(\frac{s^2+1}{s^2+4}\right)\right] = -\frac{1}{t} L^{-1}\left(\frac{2s}{s^2+1} - \frac{2s}{s^2+4}\right)$

$= -\frac{2}{t}\left(\cos t - \cos 2t\right)$

5. $L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{-4s}{s^4+4}\right]$

$$\boxed{L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{2a^2}\sinh at \sin at}$$

$F(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

$F'(s) = \frac{1}{1+\left(\frac{2}{s^2}\right)^2}\cdot\left(-\frac{4}{s^3}\right)$

$= \frac{-4}{s^3\left(\frac{s^4+4}{s^4}\right)} = \frac{-4s}{s^4+4}$

$= \frac{4}{t} L^{-1}\left[\frac{s}{s^4+4}\right]$

$= \frac{4}{t}\cdot\frac{1}{2}\sinh t \sin t = \frac{2}{t}\sinh t \sin t$.

(6) $L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right] =$

$$F(s) = \cot^{-1}\left(\frac{s+3}{2}\right)$$

$$F'(s) = \frac{-1}{1+\left(\frac{s+3}{2}\right)^2}\cdot\left(\frac{1}{2}\right)$$

$$= \frac{-1}{1+\frac{(s+3)^2}{4}}\cdot\frac{1}{2} = \frac{-2}{4+(s+3)^2}.$$

$$L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right] = -\frac{1}{t}\,L^{-1}\left[\frac{-2}{4+(s+3)^2}\right]$$

$$= \frac{1}{t}\,e^{-3t}\,L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= \frac{1}{t}\,e^{-3t}\sin 2t.$$

Inverse Laplace Transforms using Partial Fractions

Example:- Find ILT

(1) $\dfrac{s-2}{s^2+5s+6}$

Soln:- $L^{-1}\left[\dfrac{s-2}{s^2+5s+6}\right]$

Consider $\dfrac{s-2}{s^2+5s+6} = \dfrac{A}{s+3} + \dfrac{B}{s+2}$

$$\frac{s-2}{\cancel{s^2+5s+6}} = \frac{A(s+2)+B(s+3)}{\cancel{(s+3)(s+2)}}$$

$$s-2 = A(s+2)+B(s+3)$$

$s=-2$: $-4 = B$

$s=-3$: $-5 = -A \Rightarrow A = 5$

$$L^{-1}\left[\frac{s-2}{s^2+5s+6}\right] = L^{-1}\left[\frac{5}{s+3}\right] - L^{-1}\left[\frac{4}{s+2}\right]$$

$$= 5e^{-3t} - 4e^{-2b}.$$

2. $L^{-1}\left[\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right]$

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$$

$s=2$, $\quad 4 = -3B \Rightarrow B = -\frac{4}{3}$

$s=3$, $\quad C = 7/2$

$s=-1$, $\quad A = -\frac{1}{6}$.

$$L^{-1}\left[\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right] = -\frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right] - \frac{4}{3}L^{-1}\left[\frac{1}{s-2}\right] + \frac{7}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}.$$

3. $L^{-1}\left[\dfrac{s+1}{(s^2+1)(s^2+4)}\right]$

$$\frac{s+1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\frac{s+1}{\cancel{(s^2+1)(s^2+4)}} = \frac{(As+B)(s^2+4) + (Cs+D)(s^2+1)}{\cancel{(s^2+1)(s^2+4)}}$$

$$s+1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

coeff s^3 $\quad 0 = A + C \Rightarrow C = -A$

coeff s^2 $\quad 0 = B + D \Rightarrow D = -B$

coeff s $\quad 1 = 4A + C \Rightarrow 4A - A = 1 \Rightarrow A = 1/3$

$C = -1/3$

constant $\quad 1 = 4B + D \Rightarrow 4B - B = 1 \Rightarrow B = 1/3$

$D = -1/3$.

$$\therefore L^{-1}\left[\frac{s+1}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{\frac{1}{3}s + \frac{1}{3}}{s^2+1}\right] + L^{-1}\left[\frac{-\frac{1}{3}s - \frac{1}{3}}{s^2+4}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{3}L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{3\times 2}L^{-1}\left[\frac{1\times 2}{s^2+4}\right]$$

$$= \frac{1}{3}\cos t + \frac{1}{3}\sin t - \frac{1}{3}\cos 2t - \frac{1}{6}\sin 2t.$$

4. $L^{-1}\left[\dfrac{5s^2-7s+17}{(s-1)(s^2+4)}\right]$

$$\frac{5s^2-7s+17}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

$$5s^2-7s+17 = A(s^2+4) + (Bs+C)(s-1)$$

coeff s^2, $5 = A+B$

coeff s, $-7 = -B+C$

coeff constant $17 = 4A - C$

$A + C = -2$
$4A - C = 17$
$5A = 15$
$A = 3$.

solving this we get $A=3$, $B=2$ & $C=-5$.

$$\therefore L^{-1}\left[\frac{5s^2-7s+17}{(s-1)(s^2+4)}\right] = 3L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{2s-5}{s^2+4}\right]$$

$$= 3e^{t} + 2L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{5}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= 3e^{t} + 2\cos 2t - \frac{5}{2}\sin 2t.$$

5. $L^{-1}\left[\dfrac{1}{s(s+1)^2}\right]$

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$1 = A(s+1)^2 + Bs(s+1) + C(s)$$

$s=-1$ $\quad 1 = -C$

$s=0$ $\quad 1 = A$.

$s=1$ $\quad B = -1$

$$L^{-1}\left[\frac{1}{s(s+1)^2}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$

$$= 1 - e^{-t} - te^{-t}.$$

H.W

1. $L^{-1}\left[\dfrac{7s-11}{(s+1)(s-2)^2}\right] = -2e^{-t} + 2e^{2t} + e^{2t}t$

2. $L^{-1}\left[\dfrac{s^2+1}{s^3+3s^2+2s}\right] = \dfrac{1}{2} - 2e^{-t} + \dfrac{5}{2}e^{-2t}$

3.

Convolution of two functions

Definition:-

Let $f(t)$ and $g(t)$ be two functions defined for $t \geq 0$.

The convolution of $f(t)$ and $g(t)$ is defined as the integral $\int_0^t f(u)\, g(t-u)\, du$ and is denoted by $f(t) * g(t)$.

i.e, $$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$$

Convolution Theorem

statement:-

If $F(s)$ and $G(s)$ are the Laplace Transforms of $f(t)$ and $g(t)$ respectively, then $L[f(t) * g(t)] = F(s)\, G(s)$.

Note:-

$$\therefore L^{-1}[F(s)\, G(s)] = f(t) * g(t)$$
$$= L^{-1}[F(s)] * L^{-1}[G(s)].$$

1. **Example:-** Using convolution theorem find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Soln:- W.K.T $L^{-1}[F(s)\, G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+a^2}\right]$$
$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$
$$= 1 * \frac{1}{a}\sin at$$
$$= \frac{1}{a}\int_0^t 1 \cdot \sin a(t-u)\, du$$
$$= \frac{1}{a}\left(\frac{\cancel{-}\cos a(t-u)}{\cancel{-}a}\right)_0^t$$
$$= \frac{1}{a^2}[1 - \cos at].$$

2) $L^{-1}\left[\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] * L^{-1}\left[\dfrac{1}{s^2+b^2}\right]$

$$= \cos at * \frac{1}{b}\sin bt$$

$\sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$

$$= \frac{1}{b}\int_0^t \underbrace{\cos au}_{\cos B}\ \underbrace{\sin b(t-u)}_{\sin A}\, du$$

$$= \frac{1}{b}\int_0^t \frac{1}{2}\ \sin(au+bt-bu) - \sin(au-bt+bu)\, du$$

$$= \frac{1}{2b}\left[\frac{-\cos(au+bt-bu)}{a-b} + \frac{\cos(au-bt+bu)}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[\frac{-\cos at}{a-b} + \frac{\cos at}{a+b} + \frac{\cos bt}{a-b} - \frac{\cos bt}{a+b}\right]$$

$$= \frac{1}{2b}\left[\frac{-\cos at\,(a+b) + \cos at\,(a-b) + \cos bt\,(a+b) - \cos bt\,(a-b)}{a^2-b^2}\right]$$

$$= \frac{1}{2b(a^2-b^2)}\left[-\cancel{a\cos at} - b\cos at + \cancel{a\cos at} - b\cos at + \cancel{a\cos bt} + b\cos bt - \cancel{a\cos bt} + b\cos bt\right]$$

$$= \frac{b\cos bt - b\cos at}{b(a^2-b^2)} = \frac{\cos bt - \cos at}{a^2-b^2}.$$

3) Using convolution theorem, find $L^{-1}\left[\dfrac{1}{s^2(s^2+25)}\right]$.

Soln:

$$L^{-1}\left[\frac{1}{s^2(s^2+25)}\right] = L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s^2+25}\right]$$

$$= t * \frac{1}{5}\sin 5t$$

$$= \frac{1}{5}\int_0^t u \sin 5(t-u)\, du$$

$$= \frac{1}{5}\left[u\left(\frac{-\cos 5(t-u)}{-5}\right) - 1\left(\frac{-\sin 5(t-u)}{25}\right)\right]_0^t = \frac{1}{5}\left[\frac{t}{5} - \frac{\sin 5t}{25}\right]$$

$$= \frac{t}{25} - \frac{\sin 5t}{125}$$

$$L^{-1}\left[\frac{16}{(s-2)(s+2)^2}\right] = 16\, L^{-1}\left[\frac{1}{s-2}\right] * L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= 16 e^{2t} * t e^{-2t} \qquad f*g = g*f$$

$$= 16\left[t e^{-2t} * e^{2t}\right]$$

$$= 16 \int_0^t u e^{-2u} e^{2(t-u)}\,du = 16 e^{2t} \int_0^t u e^{-4u}\,du$$

$$= 16 e^{2t}\left[u \frac{e^{-4u}}{-4} - \frac{e^{-4u}}{16}\right]_0^t = 16 e^{2t}\left[\frac{-t e^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right]$$

$$= e^{2t} - e^{-2t} - 4te^{-2t}.$$

5. $$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt} = \int_0^t e^{-au} e^{-b(t-u)}\,du$$

$$= e^{-bt}\int_0^t e^{-(a-b)u}\,du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t = e^{-bt}\left[\frac{e^{-(a-b)t}}{-(a-b)} + \frac{1}{a-b}\right]$$

$$= \frac{1}{a-b}\left[e^{-bt} - e^{-at}\right].$$

6. $$L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = L^{-1}\left[\frac{s}{s^2+4}\cdot\frac{s}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+4}\right] * L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$= \cos 2t * \cos 2t$$

$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$

$$= \int_0^t \cos 2u \cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos(2u + 2t - 2u) + \cos(2u - 2t + 2u)\right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos(4u - 2t) + \cos 2t\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(4u-2t)}{4} + u\cos 2t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin 2t}{4} + t\cos 2t + \frac{\sin 2t}{4}\right]$$

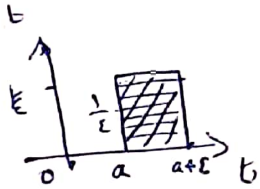
$$= \frac{1}{2}\left[\frac{1}{2}\sin 2t + t\cos 2t\right]$$

$$U(t-a) = \begin{cases} 0 & t<a \\ 1 & t\ge a \end{cases}$$

Laplace transform of Unit Impulse function (or) Dirac-delta function

The unit impulse function or Dirac-delta function is defined as.

$$\delta(t-a) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-a)$$



where

$$\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & a<t<a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$L[\delta_\varepsilon(t-a)] = \int_0^\infty e^{-st}\,\delta_\varepsilon(t-a)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}\,dt + \int_{a+\varepsilon}^{\infty} e^{-st}(0)\,dt$$

$$= \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st}\,dt$$

$$= \frac{1}{\varepsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\varepsilon}$$

$$= \frac{1}{\varepsilon}\left(\frac{e^{-s(a+\varepsilon)} - e^{-sa}}{-s}\right)$$

$$= \frac{e^{-sa}}{\varepsilon}\left(\frac{1-e^{-s\varepsilon}}{s}\right)$$

$$= \frac{e^{-sa}}{s}\left(\frac{1-e^{-s\varepsilon}}{\varepsilon}\right)$$

$$L[\delta(t-a)] = L\left[\lim_{\varepsilon\to 0}\delta_\varepsilon(t-a)\right]$$

$$= \lim_{\varepsilon\to 0} L[\delta_\varepsilon(t-a)]$$

$$= \lim_{\varepsilon\to 0} \frac{e^{-sa}}{s}\,\frac{1-e^{-s\varepsilon}}{\varepsilon}$$

$$= \frac{e^{-sa}}{s}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[1-\left(1-\frac{\varepsilon s}{1!}+\frac{\varepsilon^2 s^2}{2!}-\cdots\right)\right]$$

$$= \frac{e^{-sa}}{s}\lim\frac{1}{\varepsilon}\left[\frac{\varepsilon s}{1!}-\frac{\varepsilon^2 s^2}{2!}\cdots\right]$$

$$= \frac{e^{-sa}}{s}\lim_{\varepsilon\to 0}\left(\frac{s}{1!}-\frac{\varepsilon s^2}{2!}+\cdots\right)$$

$$= \frac{e^{-sa}}{s}[s]$$

$$= e^{-sa}.$$