Module - 5

Multiple Integrals

Introduction:-

When a function $f(x)$ is integrated w.r.to x b/w the limits a and b, we get the definite integral

$$\int_a^b f(x)\,dx.$$

If the integrand is a function $f(x,y)$ and if it is integrated w.r.to x and y repeatedly b/w the limits x_0 & x_1, and b/w the limits y_0 and y_1,

We get a double integral that is denoted by the symbol $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy$.

Extending the concept of double integral one step further, we get the triple integral, $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)\,dx\,dy\,dz$.

(or)

" The process of integration for one variable can be extended to the function of more than one variable. The generalization of definite integral is known as "multiple integral"."

Pblms

① Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx\,dy}{1+x^2+y^2}$

Soln: $I = \int_0^1 \left[\int_0^{\sqrt{1+y^2}} \frac{dx}{(\sqrt{1+y^2})^2+x^2} \cdot dx \right] dy$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$= \int_0^1 \left(\frac{1}{\sqrt{1+y^2}} \tan^{-1} \frac{x}{\sqrt{1+y^2}} \right)_0^{\sqrt{1+y^2}} dy$

$= \int_0^1 \frac{1}{\sqrt{1+y^2}} \frac{\pi}{4}\, dy$

$= \frac{\pi}{4} \int_0^1 \frac{dy}{\sqrt{1+y^2}} = \frac{\pi}{4} \log\left(y+\sqrt{1+y^2}\right)_0^1$

$= \frac{\pi}{4} \log(1+\sqrt{2})$

② $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\, dx\,dy = \int_0^a x^2 \left.\frac{y^2}{2}\right|_0^{\sqrt{a^2-x^2}} dx$

$= \frac{1}{2} \int_0^a x^2(a^2-x^2)\, dx$

$= \frac{1}{2} \left(\frac{x^3}{3}a^2 - \frac{x^5}{5} \right)_0^a$

$= \frac{a^5}{15}$

③ $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)\, dx\,dy = \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3} \right)_x^{\sqrt{x}} dx$

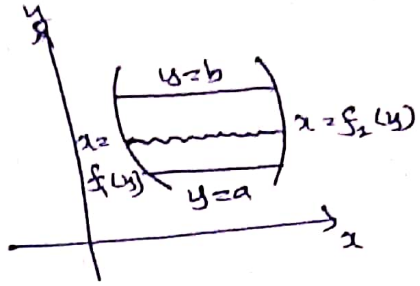
$= \int_0^1 \left(\frac{x^2 \cdot x}{2} + \frac{x\, x\sqrt{x}}{3} - \frac{x^4}{2} - \frac{x^4}{3} \right) dx$

$= \left(\frac{x^4}{4.2} + \frac{2x^{7/2}}{7\times 3} - \frac{x^5}{10} - \frac{x^5}{15} \right)_0^1$

$= \frac{1}{8} + \frac{2}{21} - \frac{1}{10} - \frac{1}{15} = \frac{3}{56}.$

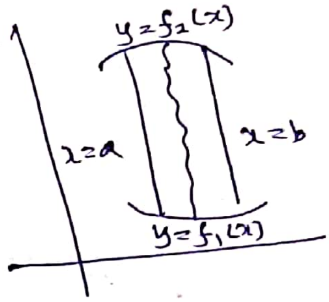
Region of Integration

Case (i) Strip parallel to x-axis



$$\int_a^b \int_{f_1(y)}^{f_2(y)} f(x,y)\, dx\, dy.$$

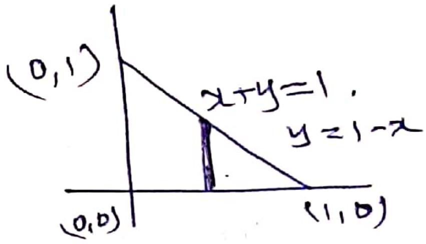
Case (ii) Strip parallel to y-axis



$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx.$$

1. Evaluate $\iint x^2 y\, dx\, dy$ over the region in the positive quadrant for which $x+y \le 1$.

Soln:-



x : 0 to 1

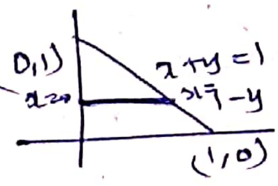
y : 0 to $1-x$.

$$\therefore \iint x^2 y\, dx\, dy = \int_0^1 \int_0^{1-x} x^2 y\, dy\, dx$$

$$= \int_0^1 \left[x^2 \frac{y^2}{2} \right]_0^{1-x} dx = \frac{1}{2}\int_0^1 x^2 (1-x)^2\, dx$$

$$= \frac{1}{2}\int_0^1 (x^2 + x^4 - 2x^3)\, dx = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right) = \frac{1}{60}$$

(or)



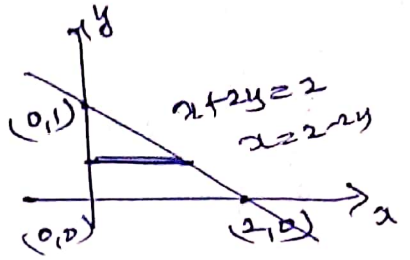
y: 0 to 1

x: 0 to $1-y$

$$\int_0^1 \int_0^{1-y} x^2 y\, dx\, dy = \frac{1}{60}.$$

② $\iint_R xy\,dx\,dy$ where R is the region bdd by the lines $x+2y=2$ lying in the first quadrant.

Soln:- $x+2y=2$



$x=0 \Rightarrow y=1 \quad (0,1)$

$y=0 \Rightarrow x=2 \quad (2,0)$

$y: 0$ to 1

$x: 0$ to $2-2y$

$$\iint_R xy\,dx\,dy = \int_0^1 \int_0^{2-2y} xy\,dx\,dy = \int_0^1 \left(\frac{x^2}{2}\right)_0^{2-2y} y\,dy = \frac{1}{2}\int_0^1 (2-2y)^2\, y\,dy$$

$$= \frac{1}{2}\int_0^1 (4+4y^2-8y)\,y\,dy$$

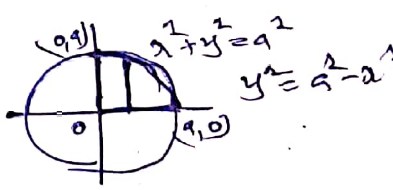
$$= 2\int_0^1 (y+y^3-2y^2)\,dy$$

$$= 2\left(\frac{y^2}{2}+\frac{y^4}{4}-\frac{2y^3}{3}\right)_0^1 = 2\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)$$

$$= 2\left(\frac{3}{4}-\frac{2}{3}\right) = 2\left(\frac{1}{12}\right) = \frac{1}{6}\ //.$$

③. Evaluate $\iint_R xy\,dx\,dy$; R is the first quadrant of the circle

H.W $x^2+y^2=a^2 \quad (x\geq 0, y\geq 0)$



Soln:- $x: 0$ to a

$y: 0$ to $\sqrt{a^2-x^2}$

$$\iint_R xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = \int_0^a \left(x\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{a^4}{8}.$$

4. Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ & $y=x$.

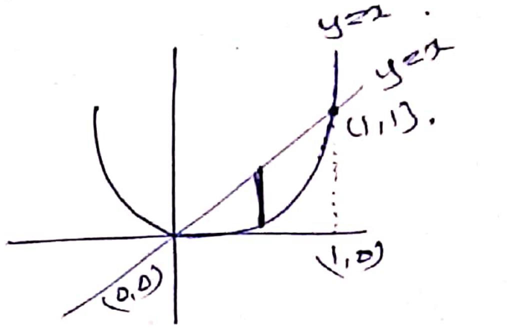
Soln:- $x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x=0$ & $x=1$

$y=0$ $y=1$.



x: 0 to 1

y: x^2 to x.

$$\int_0^1\int_{x^2}^{x}(x^2y+xy^2)\,dy\,dx = \int_0^1\left(x^2\frac{y^2}{2}+x\frac{y^3}{3}\right)_{x^2}^{x}dx$$

$$= \int_0^1\left(\frac{x^2x^2}{2}+\frac{x\,x^3}{3}-\frac{x^2x^4}{2}-\frac{x\,x^6}{3}\right)dx$$

$$= \int_0^1\left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$= \left.\frac{x^5}{10}+\frac{x^5}{15}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$= \frac{1}{10}+\frac{1}{15}-\frac{1}{14}-\frac{1}{24} = \frac{25}{150}-\frac{38}{336}$$

$$= \frac{1}{6}-\frac{19}{168} = \frac{168-114}{1008}$$

$$= \frac{3}{56}$$ //

5. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bdd by Parabola $y^2=x$ and the lines $y=0$ & $x+y=2$ lying in the 1st Quadrant.

$x+y=2$

$x=0$ $y=2$ (0,2)

$y=0$ $x=2$ (2,0)

$y^2+y=2$

$y^2+y-2=0$

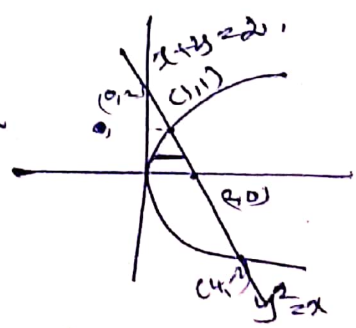
$y=1$ & $y=-2$

$x=1$ & $x=4$

$y=0$ to 1

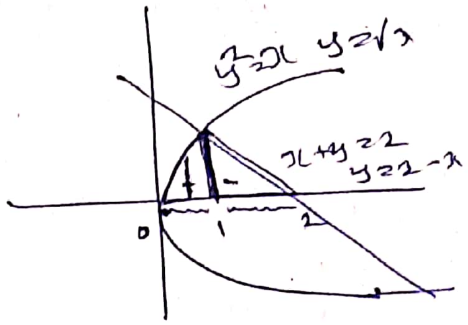
x: y^2 to $2-y$.

Ans $= 3/8$.



$$\int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx + \int_1^2 \int_0^{2-x} xy\,dy\,dx$$

$= 3/8$.

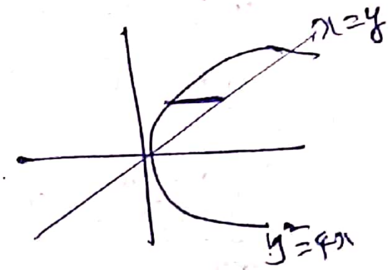


H.W

$$\iint_R (x^2+y^2)\,dx\,dy \text{ where } R \text{ is the region bdd by}$$

$y = x$ & $y^2 = 4x$.

Ans 21.94.



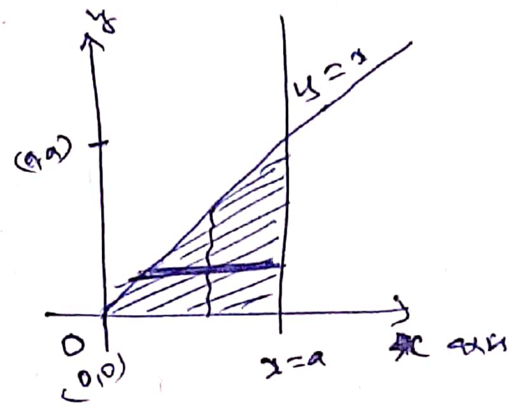
Change of order of Integration

1. Change the order of the integration $\int_0^a \int_0^x f(x,y)\,dy\,dx$

Soln:-

$$I = \int_0^a \int_0^x f(x,y)\,dy\,dx$$

The limits : $x=0,\ x=a,\ \&\ y=0\ ,\ y=x$



The changing the order of integration limits of x and y are

$y=0,\ y=a,\ x=y,\ x=a.$

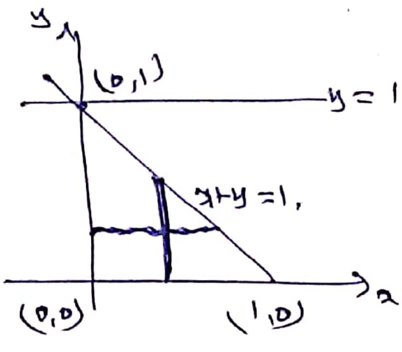
$$\therefore I = \int_0^a \int_y^a f(x,y)\,dx\,dy.$$

2. Change the order of Integration in $\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$

The limits are

x : 0 to $1-y$

y : 0 to 1



New Limits

x : 0 to 1

y : 0 to $1-x$

$$\therefore I = \int_0^1 \int_0^{1-x} f(x,y)\,dx\,dy.$$

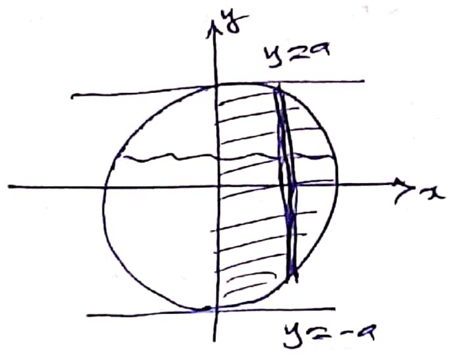
3. Change the order of integration $I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy.$

Limits : $x = 0$, $x = \sqrt{a^2-y^2}$

$x^2 = a^2 - y^2$

$x^2 + y^2 = a^2$.

$y = -a$ to $y = a$



After changing the new Limits are

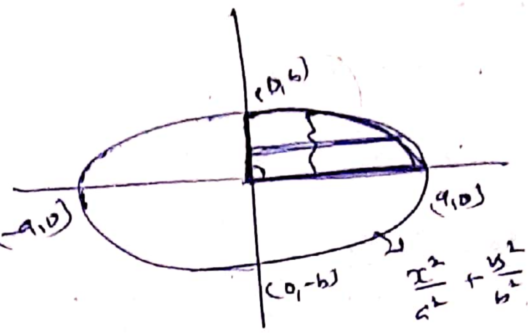
$x = 0$ to a

$y : -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$\therefore I = \int_{0}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\, dx\, dy.$

5

4. $$\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy.$$



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$y=\frac{b}{a}\sqrt{a^2-x^2}$

Given Limits

$y=0$ & $y=b$

$x=0$ & $x=\frac{a}{b}\sqrt{b^2-y^2}$

$$x^2=\frac{a^2}{b^2}(b^2-y^2)$$

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}.$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The region is the quadrant in the ellipse

New Limits

$x: 0$ to a

$y: 0$ to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a x\left(\frac{y^2}{2}\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\frac{b^2}{a^2}\int_0^a x(a^2-x^2)\,dx = \frac{b^2}{2a^2}\int_0^a (xa^2-x^3)\,dx$$

$$= \frac{b^2}{2a^2}\left(\frac{x^2a^2}{2}-\frac{x^4}{4}\right)_0^a$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right) = \frac{a^4b^2}{4\cdot 2a^2}$$

$$= \frac{a^2b^2}{8}.$$

H.W

5. $$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy.$$

$$\int_0^a \int_0^x \frac{x}{x^2+y^2}\,dy\,dx = \int_0^a \left(\tan^{-1}\frac{y}{x}\right)_0^x dx$$

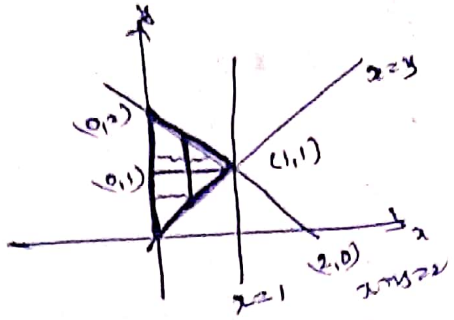
$$= \int_0^a (\tan^{-1}1-\tan^{-1}0)\,dx$$

$$= \int_0^a \left(\frac{\pi}{4}-0\right)dx = \frac{\pi}{4}\cdot(x)_0^a$$

$$= \frac{\pi a}{4}.$$

$$\boxed{\begin{aligned}\int\frac{x\,dx}{x^2+a^2} &= \frac{1}{2}\log(x^2+a^2)+c\\ \int\frac{dx}{x^2+a^2} &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c\end{aligned}}$$

(6) $\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx$



$y = x$ & $y = 2-x$

$x = 0$ & $x = 1$.

New Limits

$y: 0$ to 1 | $y = 1$ to 2

$x: 0$ to y | $x: 0$ to $2-y$

$$I = \int_0^1 \int_0^y \frac{x}{y}\, dy\, dx + \int_1^2 \int_0^{2-y} \frac{x}{y}\, dx\, dy$$

$$= \int_0^1 \frac{1}{y}\left(\frac{x^2}{2}\right)_0^y dy + \int_1^2 \frac{1}{y}\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{y \times 2} dy + \int_1^2 \frac{1}{2y}(2-y)^2 dy$$

$$= \left.\frac{y^2}{4}\right)_0^1 + \frac{1}{2}\int_1^2 \frac{1}{y}(4 + y^2 - 4y)\, dy$$

$$= \frac{1}{4} + \frac{1}{2}\int_1^2 4\frac{dy}{y} + y\,dy - 4\,dy$$

$$= \frac{1}{4} + \frac{1}{2}\left(4\log y + \frac{y^2}{2} - 4y\right)_1^2$$

$$= \frac{1}{4} + \frac{1}{2}\left(4\log 2 + 2 - 8 - 0 - \frac{1}{2} + 4\right)$$

$$= \frac{1}{4} + \frac{1}{2}\left[4\log 2 - \frac{5}{2}\right].$$

$$= \frac{1}{4} + 2\log 2 - \frac{5}{4} \;// = \log 4 - 1$$

H.W

(1) $\int_0^a \int_y^a \frac{x+y}{x^2+y^2}\, dx\, dy$.

(2) $\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\, dx\, dy$

(Ans) $\frac{\pi a}{4} + \frac{a}{2}\log 2$.

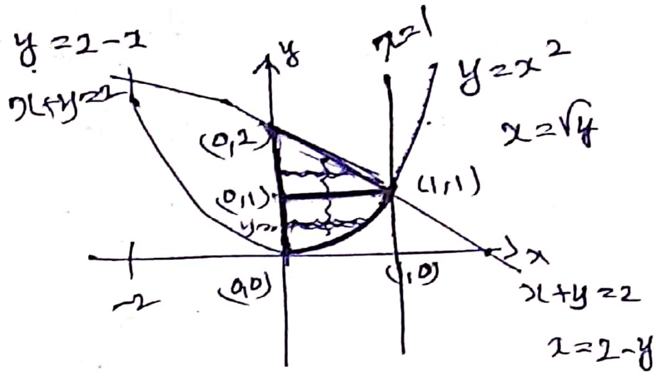
(3) $\int_0^1 \int_y^{2-y} xy\, dx\, dy$

(7) change the order of integration and evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$$

Soln:- Given limits $x=0$ & $x=1$

$y=x^2$ & $y=2-x$



pts of intersection

$x^2 = 2-x$

$\Rightarrow x^2+x-2=0$

$\Rightarrow x=1$ & $x=-2$

$x=1, \quad y=1 \quad (1,1)$

$x=-2, \quad y=4 \quad (-2,4)$

change the order

$y: 0$ to 1 $\qquad$ $y: 1$ to 2

$x: 0$ to $\sqrt{y}$ $\qquad$ $x: 0$ to $2-y$

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy \, dx\, dy = \int_0^1 \int_0^{\sqrt{y}} xy \, dx\, dy + \int_1^2 \int_0^{2-y} xy \, dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} y\, dy + \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} y \, dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \int_1^2 \frac{(2-y)^2 y}{2} dy$$

$$= \left(\frac{y^3}{6}\right)_0^1 + \frac{1}{2}\int_1^2 (4+y^2-4y) y\, dy$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{4y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\right)_1^2$$

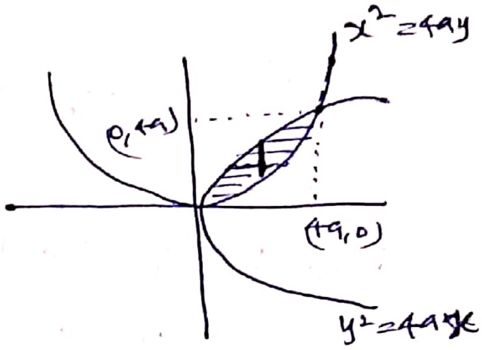
$$= \frac{1}{6} + \frac{1}{2}\left(8+4-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right)$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

(8) $\displaystyle\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Given limits $x=0$, $x=4a$

$y=\dfrac{x^2}{4a}$, $y=2\sqrt{ax}$



pts of intersection

$\dfrac{x^2}{4a}=2\sqrt{ax}$

$\dfrac{x^4}{16a^2}=4ax \Rightarrow x^3=64a^3$

$\Rightarrow x=4a$

$y=\dfrac{(4a)^2}{4a}=4a$

$\therefore$ $(4a,4a)$ pt of intersection

New limits $y: 0$ to $4a$

$x: \dfrac{y^2}{4a}$ to $2\sqrt{ay}$

$\therefore\ I=\displaystyle\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy=\int_0^{4a}(x)_{y^2/4a}^{2\sqrt{ay}}\,dy$

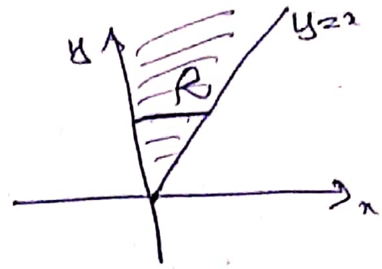
$=\displaystyle\int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$

$=\left(2\sqrt{a}\,\dfrac{y^{3/2}}{3/2}-\dfrac{y^3}{3\times 4a}\right)_0^{4a}$

$=2\sqrt{a}\,\dfrac{4a\sqrt{4a}}{3/2}-\dfrac{(4a)^{\cancel{3}\,2}}{3\times\cancel{4a}}=\dfrac{32a^2}{3}-\dfrac{16a^2}{3}$

$=\dfrac{16a^2}{3}$ //.

9. $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$



Limits $x = 0$ to ∞

$y = x$ to ∞

New Limits

$y : 0$ to ∞

$x : 0$ to y

$\therefore\ I = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dx\, dy = \int_0^{\infty} \frac{e^{-y}}{y} (x)_0^y\, dy$

$= \int_0^{\infty} e^{-y}\, dy = (-e^{-y})_0^{\infty} = -(0-1) = 1.$

10. HW $\int_0^{a}\int_{x^2/a}^{2a-x} xy\, dy\, dx = \frac{3}{8}a^4$

11. $\int_0^{1}\int_{\sqrt{y}}^{2-y} xy\, dx\, dy = \frac{7}{24}$

12. $\int_0^{a}\int_{a-y}^{\sqrt{a^2-y^2}} y\, dx\, dy = \frac{a^3}{6}$

14. $\int_0^{1}\int_x^{1} \frac{x}{x^2+y^2}\, dx\, dy$

Ans: $\log 4$.

13. $\int_0^{a}\int_0^{b/a\sqrt{a^2-x^2}} x^2\, dy\, dx$

15. $\int_0^{1}\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

Ans: $\frac{2-\sqrt{2}}{2}$

Change of Variables between Cartesian and polar Co-ordinates

(1) To change cartesian co-ordinates (x,y) to polar co-ordinates (r,θ).

Put $x = r\cos\theta$, $y = r\sin\theta$, so that $x^2+y^2 = r^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Then. $$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta.$$

(2) To change cartesian co-ordinates (x,y,z) to spherical co-ordinates (r,θ,ϕ).

Put $x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$ So that $x^2+y^2+z^2 = r^2$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = r^2\sin^2\phi$$

$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)\,|J|\,dr\,d\theta\,d\phi.$$

3. To change cartesian co-ordinates (x, y, z) to cylindrical coordinates (r, θ, z)

Put $x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$J = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\,|J|\,dr\,d\theta\,dz$$

Pblms

(i) $$\int_0^{\pi/2} \cos^n\theta\,d\theta = \int_0^{\pi/2} \sin^n\theta\,d\theta = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots 1 & \text{if } n \text{ is odd} \end{cases}$$

(ii) $$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \begin{cases} \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots} \times \frac{\pi}{2}, & \text{if } m+n \text{ eve} \\ \qquad '' & \text{others} \end{cases}$$

① By changing into polar coordinates evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{x^2+y^2}$$

Soln:-

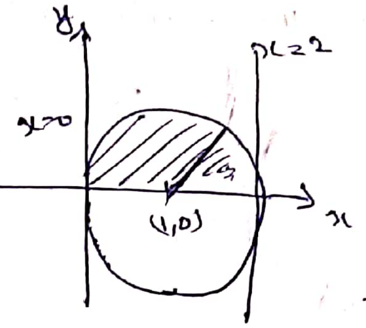
$y = 0$ to $y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2$

$x^2 + y^2 = 2x$

$x : 0$ to $x = 2$.

To change $(x,y) \to (r,\theta)$

$x = r\cos\theta, \quad y = r\sin\theta$



$x^2+y^2 = 2x \Rightarrow x^2-2x+y^2=0$

$x^2-2x+1)+y^2 = 1$

$(x-1)^2+y^2=1$

$(1,0)$ radius 1

$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

θ varies from 0 to $\pi/2$.

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{x^2+y^2} = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta \; \cancel{r}\, dr\, d\theta}{r^{\cancel{2}}}$$

$$= \int_0^{\pi/2}\int_0^{2\cos\theta} \cos\theta\, (r)_0^{2\cos\theta}\, d\theta$$

$$= \int_0^{\pi/2} 2\cos^2\theta\, d\theta = \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

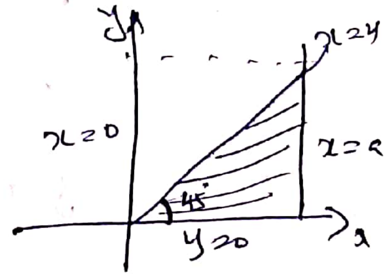
$$= \left(\theta + \frac{\sin 2\theta}{2}\right)_0^{\pi/2} = \frac{\pi}{2} \; //.$$

(2) $$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx$$

Ans: $\frac{3\pi a^4}{4}$

(3) $$\int_0^{a}\int_y^{a} \frac{x^2\, dx\, dy}{\sqrt{x^2+y^2}}$$ polar

Given limits $y=0, y=a, x=y$ & $x=a$



$x = a = r\cos\theta$, ~~y~~

$\Rightarrow r = \frac{a}{\cos\theta}$

$= a\sec\theta$

$\theta : 0$ to $\pi/4$

$\therefore r : 0$ to $a\sec\theta$

$$\int_0^a\int_y^a = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta \; r\,dr\,d\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} r^2\cos^2\theta\,dr\,d\theta = \int_0^{\pi/4}\left(\frac{r^3}{3}\right)_0^{a\sec\theta}\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/4}\frac{a^3\sec^3\theta}{3}\cos^2\theta\,d\theta = \frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta$$

$$= \frac{a^3}{3}\left(\log(\sec\theta+\tan\theta)\right)_0^{\pi/4}$$

$$= \frac{a^3}{3}\left(\log(\sqrt{2}+1) - 0\right) = \frac{a^3}{3}\log(\sqrt{2}+1) \;//$$

(4) Transform the double integral $\displaystyle\int_0^a\int_{-\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$ into polar coordinates and then evaluate it.

Soln:- $x=0$ to a

$y=-\sqrt{ax-x^2}$ to $\sqrt{a^2-x^2}$

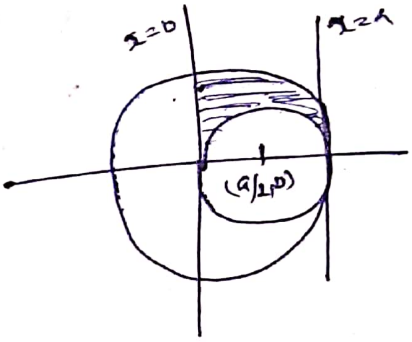
$y^2=ax-x^2$ $\quad$ $y^2=a^2-x^2$

$x^2+y^2=ax,$ $\quad$ $y^2+x^2=a^2$

$x^2-ax+y^2=0$

$x^2-ax+\frac{a^2}{4}+y^2=\left(\frac{a}{2}\right)^2$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$



To change $x=r\cos\theta$, $y=r\sin\theta$

$x^2+y^2=r^2$, $dx\,dy = r\,dr\,d\theta$

$x^2+y^2=ax$ $\Rightarrow r=a\cos\theta$ $\qquad$ $x^2+y^2=a^2$ $\Rightarrow r=a$

$$\therefore \int_0^{\pi/2}\int_{a\cos\theta}^{a}\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = \int_0^{\pi/2}\int_{a\cos\theta}^{a} -\frac{1}{2}\,\frac{-2r}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}\left(2\sqrt{a^2-r^2}\right)_{a\cos\theta}^{a} d\theta$$

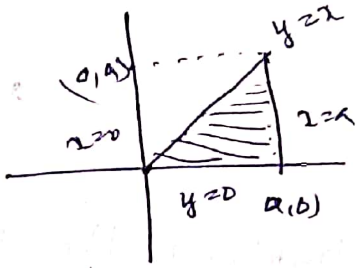
$$= -\int_0^{\pi/2} 0-\sqrt{a^2-a^2\cos^2\theta}\, d\theta = \int_0^{\pi/2} a\sin\theta\, d\theta$$

$$= a.$$

5. $$\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\, dx\, dy.$$

$y=0,\ y=a$

$x=y,\ x=a$



$x=r\cos\theta,\ y=r\sin\theta$

$x^2+y^2=r^2,\ dx\,dy = r\,dr\,d\theta$

$x=a=r\cos\theta$

$r=a\sec\theta$

$x=y$

$r\cos\theta = r\sin\theta$

$\tan\theta=1$

$\theta=\pi/4$

$\theta: 0\to\pi/4$

$r: 0\to a\sec\theta$

$$I=\int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{r^3}\, r\,dr\,d\theta = \frac{a}{\sqrt{2}}.$$

Evaluation of triple integrals

1. $$\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dx\,dy\,dz$$

$$I = \int_0^1\int_0^{1-x} \left(e^z\right)_0^{x+y} dx\,dy$$

$$= \int_0^1\int_0^{1-x} \left(e^{x+y}-1\right) dx\,dy = \int_0^1 \left(e^{x+y}-y\right)_0^{1-x} dx$$

$$= \int_0^1 \left(e-(1-x)-(e^x-0)\right) dx = \int_0^1 \left(e-1+x-e^x\right) dx$$

$$= \left(ex - x + \frac{x^2}{2} - e^x\right)_0^1 = \frac{1}{2}.$$

2. $$\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$I = \int_0^a\int_0^{\sqrt{a^2-x^2}} \sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1}1 - \sin^{-1}0\right) dy\,dx = \int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2}\,dy\,dx$$

$$= \frac{\pi}{2}\int_0^a \left(y\right)_0^{\sqrt{a^2-x^2}} dx = \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}\right]_0^a$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right] = \frac{\pi^2 a^2}{8}.$$

3. $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz = \frac{abc}{3}[a^2+b^2+c^2]$.

H.W

4. $\int_0^1\int_0^2\int_0^3 (x+y+z)\,dx\,dy\,dz = 18$

5. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx = \frac{\pi^2}{8}$.

6. Find the Volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Soln: $V = 8 \times$ Volume in an octant

z varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

x varies from $x=0$ to $x=a$.

$$\therefore V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx = 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\,dy\,dx$$

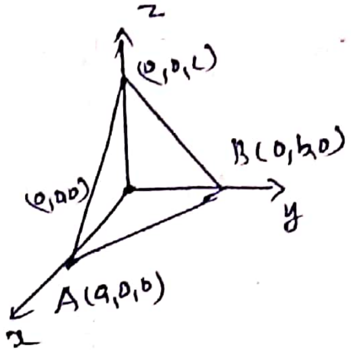
$$= 8\int_0^a \left(\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} + \frac{y}{2}\sqrt{a^2-x^2-y^2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left(\frac{a^2-x^2}{2}\,\frac{\pi}{2}\right) dx = 2\pi\left(a^2x - \frac{x^3}{3}\right)_0^a$$

$$= 2\pi\left(a^3 - \frac{a^3}{3}\right) = \frac{4\pi a^3}{3}.$$

$$\boxed{\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}}$$

7. Find the Volume of the tetrahedron bdd by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.



Soln:-

$$V = \iiint_V dz\,dy\,dx \quad \text{---} \ (1)$$

x : 0 to a

y : 0 to $b\left(1-\frac{x}{a}\right)$

z : 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$$\therefore (1) \Rightarrow \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx = \int_0^a \int_0^{b(1-\frac{x}{a})} (z)_0^{c(1-\frac{x}{a}-\frac{y}{b})}\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$$= c\int_0^a \left(\left(1-\frac{x}{a}\right)y - \frac{y^2}{2b}\right)_0^{b(1-\frac{x}{a})} dy\,dx$$

$$= c\int_0^a \left(1-\frac{x}{a}\right)^2 b - \frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx = \frac{bc}{2}\cancel{\left(1-\frac{x^2}{2a}\right)_0^a}$$

$$= \frac{bc}{2}\,\frac{\left(1-\frac{x}{a}\right)^3}{-\frac{3}{a}} = -\frac{abc}{6}\left(1-\frac{x}{a}\right)^3$$

$$= \frac{abc}{6} \text{ cubic units.}$$

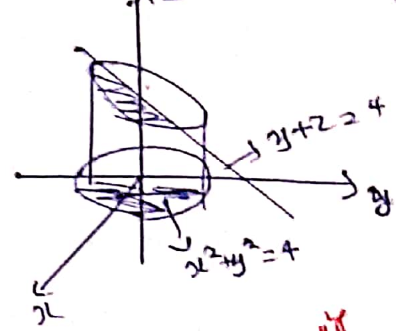
8. H.W Evaluate $\iiint_V dx\,dy\,dz$ where V is the Volume of the tetrahedran whose vertices are $(0,0,0)$, $(0,1,0)$, $(1,0,0)$ & $(0,0,1)$

(or)

Find the Volume of the tetrahedran bdd by the co-ordinate planes and the plane $x+y+z=1$.

9. Find the volume bdd by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ & $z=0$.

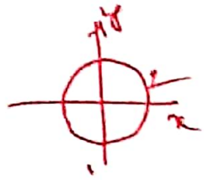
Soln:- Let $V = \iiint\limits_V dz\,dy\,dx$



Here z varies from 0 to $4-y$

x : -2 to 2

y : $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$



$$\therefore V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (z)_0^{4-y}\,dy\,dx = \iint (4-y)\,dy\,dx$$

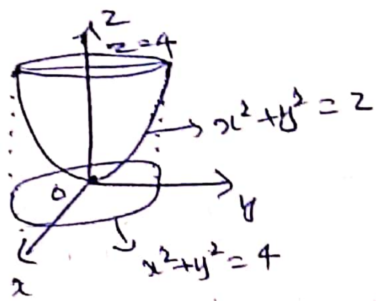
$$= \int_{-2}^{2}\left(4y-\frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \int_{-2}^{2}\left[\left(4\sqrt{4-x^2}-\cancel{\frac{4-x^2}{2}}\right)-\left(-4\sqrt{4-x^2}-\cancel{\frac{4-x^2}{2}}\right)\right]$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\,dx = 16\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 16\left[\frac{4}{2}\sin^{-1}\frac{x}{2}+\frac{x}{2}\sqrt{4-x^2}\right]_0^2$$

$$= 16\left(2\cdot\frac{\pi}{2}\right) = 16\pi \text{ cubic units.}$$

10. Find the volume of the region bdd by the paraboloid $z=x^2+y^2$ and the plane $z=4$.



Soln:- z varies from x^2+y^2 to $z=4$.

The projection of the region on the xy plane will give the area of the circle

$x^2+y^2=4$

$$V=\iiint_V dz\,dy\,dx=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (z)_{x^2+y^2}^{4}\,dy\,dx=\iint(4-x^2-y^2)\,dy\,dx$$

$$=\int_{-2}^{2}\left(4y-x^2y-\frac{y^3}{3}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$=4\int_{0}^{2}\frac{2}{3}(4-x^2)^{3/2}\,dx \qquad x=2\sin\theta, \quad dx=2\cos\theta\,d\theta$$

$$=\frac{8}{3}\int_0^{\pi/2}(4\cos^2\theta)^{3/2}\,2\cos\theta\,d\theta$$

$$=\frac{128}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta=\frac{128}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=8\pi \text{ cubic units.}$$

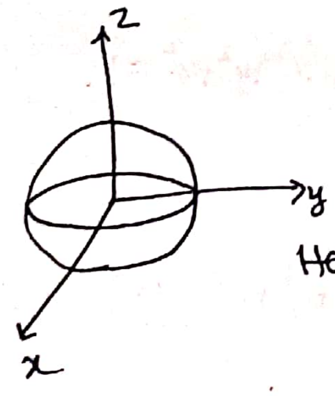
$\longrightarrow$ $\boxed{\text{Note}}$ next page (x)

Change cartesian $\rightarrow$ spherical

1. Evaluate $\iiint \frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dx\,dy\,dz$ over the region bdd by the sphere $x^2+y^2+z^2=1$.

Soln:- Let us transform this integral in spherical polar co-ordinates by taking

$$x=r\sin\theta\cos\phi$$
$$y=r\sin\theta\sin\phi$$
$$z=r\cos\theta \qquad , \quad x^2+y^2+z^2=r^2$$



$$dx\,dy\,dz=(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

Here ϕ varies from 0 to 2π

θ varies from 0 to π

r varies from 0 to 1

Note:- If the region of integration is a sphere $x^2+y^2+z^2=a^2$ with centre $(0,0,0)$ and radius a, then the limits of r, θ & ϕ are

	Complete sphere	hemi sphere	Positive octant of the sphere
r	$0 \to a$	$0 \to a$	$0 \to a$
θ	$0 \to \pi$	$0 \to \pi/2$	$0 \to \pi/2$
ϕ	$0 \to 2\pi$	$0 \to 2\pi$	$0 \to \pi/2$

r. eq. (4)

$$\therefore \quad I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \frac{1}{\sqrt{1-r^2}}\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$r = \sin t$

$dr = \cos t\, dt$

$r \to 0 \Rightarrow t \to 0$

$r \to 1 \Rightarrow t = \pi/2$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}} \cos t\, dt \, \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi/2} \sin^2 t\, dt \, \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] \sin\theta\, d\theta\, d\phi = \frac{\pi}{4}\int_0^{2\pi} (-\cos\theta)_0^{\pi}\, d\phi$$

$$= \frac{\pi}{4}\int_0^{2\pi} -(-1-1)\, d\phi = \frac{\pi}{2}\int_0^{2\pi} d\phi = \pi^2 .$$

(2) H.W. $\iiint (x^2+y^2+z^2)\, dx\, dy\, dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$.

Ans: $\frac{4\pi}{5}$.

(3) Ex. $\iiint (x^2+y^2+z^2)\, dx\, dy\, dz$, sphere $x^2+y^2+z^2=a^2$

Ans: $\frac{4\pi a^5}{5}$.

(4) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical polar co-ordinates.

Soln:-

x : 0 to 1

y : 0 to $\sqrt{1-x^2}$ → positive octant sphere radius -1

z : 0 to $\sqrt{1-x^2-y^2}$

∴ Limits are r : 0 to 1

θ : 0 to $\pi/2$

ϕ : 0 to $\pi/2$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{1}{\sqrt{1-r^2}} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \frac{\pi}{4} \sin\theta \, d\theta\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} (-\cos\theta)_0^{\pi/2} \, d\phi = \frac{\pi}{4} \int_0^{\pi/2} (1)\, d\phi$$

$$= \frac{\pi^2}{8}.$$

(5). Evaluate the integration $\iiint xyz\, dx\, dy\, dz$ taken throughout the volume for which $x, y, z > 0$ and $x^2+y^2+z^2 \le 9$.

r : $0 \to 3$

θ : $0 \to \pi/2$

ϕ : $0 \to \pi/2$

$x, y, z > 0$ positive octant.

radius $= 3$.

Ans: $\frac{243}{16}$.

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta. \end{cases}$$

(b) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$.

Soln:-

$x : 0$ to 1

$y : 0$ to $\sqrt{1-x^2}$

$z : \sqrt{x^2+y^2}$ to 1

$z = \sqrt{x^2+y^2}$

$z^2 = x^2+y^2$

$\Rightarrow r^2\cos^2\theta = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi$

$r^2\cos^2\theta = r^2\sin^2\theta$

$\Rightarrow \cos^2\theta = \sin^2\theta$

$\Rightarrow \theta = \frac{\pi}{4}$

$z = 1$

$r\cos\theta = 1$

$\Rightarrow r = \frac{1}{\cos\theta}$

$= \sec\theta$

The region of integration is common to the cone $z^2 = x^2+y^2$ and the cylinder $x^2+y^2=1$ bdd by the plane $z=1$ in the +ve octant

$\therefore$ The Limits $r : 0$ to $\sec\theta$

$\theta : 0$ to $\pi/4$

$\phi : 0$ to $\pi/2$

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{r^2}} = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} r\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4} \left(\frac{r^2}{2}\right)_0^{\sec\theta} \sin\theta\,d\theta\,d\phi = \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/4} \sin\theta\sec^2\theta\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4} \frac{1}{2}\sec\theta\tan\theta\,d\theta\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sec\theta)_0^{\pi/4}\,d\phi = \frac{1}{2}\int_0^{\pi/2} (\sqrt{2}-1)\,d\phi$$

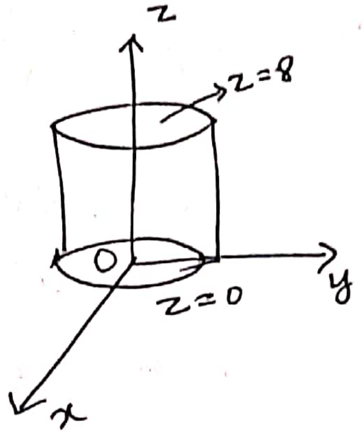
$$= (\sqrt{2}-1)\frac{\pi}{4}$$ //

Change into cartesian → cylindrical.

① Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^8 2yz \, dz\,dy\,dx$ by changing into cylindrical polar co-ordinates.

Soln:-

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^8 2yz\,dz\,dy\,dx$$



Here $z=0$, $z=8$.

$y=0$, $y=\sqrt{4-x^2}$

$\Rightarrow x^2+y^2=4$

$x=0$, $x=2$

To change into cylindrical co-ordinates

Put $x=r\cos\theta$, $y=r\sin\theta$, $z=z$

Here r : 0 to 2

θ : 0 to $\pi/2$

z : 0 to 8

$$I = \int_0^{\pi/2} \int_0^2 \int_0^8 2r\sin\theta \, z\, dz\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \left(\frac{z^2}{\cancel{2}}\right)_0^8 \cancel{2}r^2\sin\theta\,dr\,d\theta$$

$$= 64 \int_0^{\pi/2} \int_0^2 r^2\sin\theta\,dr\,d\theta$$

$$= 64 \int_0^2 (-\cos\theta)_0^{\pi/2} r^2\,dr = 64\int_0^2 r^2\,dr$$

$$= 64\left(\frac{r^3}{3}\right)_0^2 = \frac{512}{3} \text{ cubic units.}$$

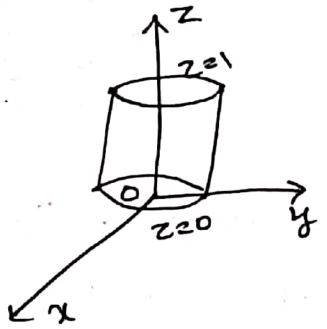
② By changing into cylindrical co-ordinates, evaluate the integral $\iiint (x^2+y^2+z^2)\, dx\,dy\,dz$ taken over the region of space defined by $x^2+y^2 \le 1$ and $0 \le z \le 1$.

Soln:-

$$I = \iiint_V (x^2+y^2+z^2)\, dx\, dy\, dz$$

where V is the Volume of cylinder

$x^2+y^2=1$ and $z=1$.



Here, r : 0 to 1

θ : 0 to 2π

z : 0 to 1

$$\therefore \iiint_V (x^2+y^2+z^2)\, dx\,dy\,dz = \int_0^1 \int_0^{2\pi} \int_0^1 (r^2+z^2)\, r\, dr\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r^3+rz^2)\, dr\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^4}{4} + \frac{r^2z^2}{2}\right)_0^1 d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \left[\frac{1}{4} + \frac{z^2}{2}\right] d\theta\, dz$$

$$= \int_0^1 \left[\frac{\theta}{4} + \frac{\theta z^2}{2}\right]_0^{2\pi} dz = \int_0^1 \left(\frac{\pi}{2} + \pi z^2\right) dz$$

$$= \frac{5\pi}{6} \text{ cubic units.}$$

Gamma and Beta Functions

Definition (1) The definite integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ exists only when $n>0$ and when it exits, it is a function of n and called Gamma function and denoted by $\Gamma(n)$.

Thus $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

(2) The definite integral $\int_0^1 x^{m-1} (1-x)^{n-1}\, dx$ exists only when $m>0$ and $n>0$ and when it exits, it is a function of m and n and called Beta function and defined by $\beta_{m,n} = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$.

Note :-

1. $\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx = (-e^{-x})_0^\infty = 1$

2. $\beta(1,1) = \int_0^1 x^0 (1-x)^0\, dx = 1.$

3. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$= -(x^{n-1} e^{-x})_0^\infty + \int_0^\infty (n-1)\, e^{-x} x^{n-2}\, dx$$

I. by Parts

$$= (n-1)\int_0^\infty e^{-x} x^{n-2}\, dx$$

$$= (n-1)\Gamma(n-1)$$

4. $\beta(m,n) = \beta(n,m)$

Corollary :- 1. $\Gamma(n+1) = n!$ where n is a +ve integer

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= \dots$$

$$= n(n-1)\dots 3.2.1\,\Gamma 1$$

$$= n! \quad (\because \Gamma 1 = 1.)$$

Note:

1. $\Gamma(n)$ does not exits (i.e $= \infty$) when n is 0 or a negative integer

2. when n is a negative fraction, $\Gamma(n)$ is defined by using the recurrence formula. i.e., when $n < 0$, but not an integer, $\Gamma(n) = \frac{1}{n}\Gamma(n+1)$.

eg: $\Gamma(-3.5) = \frac{1}{(-3.5)}\Gamma(-2.5)$

$$= \frac{1}{-3.5}\,\frac{1}{-2.5}\,\Gamma -1.5 = \frac{1}{-3.5}\,\frac{1}{-2.5}\,\frac{1}{-1.5}\,\Gamma -0.5$$

$$= \frac{\Gamma 0.5}{(3.5)(2.5)(1.5)(0.5)}$$

$\Gamma(0.5)$ Value taken from the gamma table.

Problems

① P.T $\Gamma\frac{1}{2} = \sqrt{\pi}$

Soln:- By defn, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2}\, dt$

$t = x^2$

$dt = 2x\,dx$

$$= \int_0^\infty e^{-x^2} \frac{1}{x} 2x\,dx$$

$$= 2\int_0^\infty e^{-x^2}\,dx$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 2\int_0^\infty e^{-x^2}\,dx \cdot 2\int_0^\infty e^{-y^2}\,dy$$

(x & y are dummy variables)

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$0 \le x < \infty,\ 0 \le y < \infty$

∴ the entire first quadrant in the xy plane.

$x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r : 0$ to ∞

$\theta : 0$ to $\pi/2$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$r^2 = \alpha$

$2r\,dr = d\alpha$

$$= 4\int_0^{\pi/2} d\theta \left(-\frac{1}{2}e^{-r^2}\right)_0^\infty$$

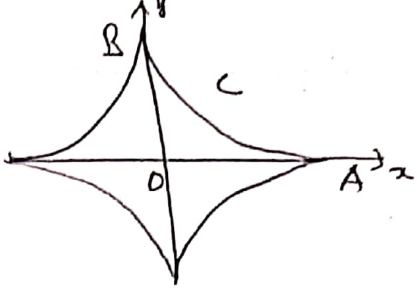
$$= 2\int_0^{\pi/2} d\theta = \pi$$

$$\therefore \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

2. Relation b/w Gamma and Beta functions

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}.$$

① Find the area of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ using Gamma function. (or) $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{a}\right)^{2/3}=1.$



Soln:-

Required Area $= 4 \times$ Area of $OABC$

$$= 4\iint_{OABC} dx\,dy$$

Put $\left(\frac{x}{a}\right)^{2/3}=X$ & $\left(\frac{y}{a}\right)^{2/3}=Y$

i.e., $x = aX^{3/2}$ & $y = aY^{3/2}$

$dx = \frac{3}{2}aX^{1/2}\,dX$ $dy = \frac{3}{2}aY^{1/2}\,dY$ ∴ The region of integration in the XY plane is given by $X\geq 0, Y\geq 0$ & $X+Y\leq 1$.

$$\therefore\ A = 4\times\frac{9}{4}a^2\int_0^1\int_0^{1-Y} X^{1/2}Y^{1/2}\,dX\,dY$$

$$= 9a^2\int_0^1 Y^{1/2}\left(\frac{2}{3}X^{3/2}\right)_0^{1-Y} dY$$

$$= \frac{9a^2\times 2}{3}\int_0^1 (1-Y)^{3/2}\,Y^{1/2}\,dY$$

$$= 6a^2\,\beta\left(\frac{3}{2},\frac{5}{2}\right) = 6a^2\times\frac{\Gamma(3/2)\,\Gamma(5/2)}{\Gamma(4)} = \frac{6a^2\,\frac{1}{2}\sqrt{\pi}\,\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{3!}$$

$$= \frac{3}{8}\pi a^2 \text{ sq. units}$$

② ST the volume of the region of space bdd by the coordinate planes and the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$ is $\frac{abc}{90}$.

Soln:-

Vol $= \iiint_V dz\,dy\,dx$, V is the region.

Put $\sqrt{\frac{x}{a}}=X$, $\sqrt{\frac{y}{b}}=Y$, $\sqrt{\frac{z}{c}}=Z$

$x = aX^2$, $y = bY^2$, $z = cZ^2$

$dx = 2aX\,dX$

$dy = 2bY\,dY$

$dz = 2cZ\,dZ$

$\therefore$ Vol $= 8abc \iiint XYZ\,dX\,dY\,dZ.$

Required Region is $X \geq 0, Y \geq 0, Z \geq 0$ & $X+Y+Z=1.$

$$V = 8abc \int_0^1 \int_0^{1-X} \int_0^{1-X-Y} XYZ\,dZ\,dY\,dX$$

$$= 8abc \int_0^1 \int_0^{1-X} XY \left(\frac{Z^2}{2}\right)_0^{1-X-Y} dY\,dX$$

$$= 4abc \int_0^1 \int_0^{1-X} XY(1-X-Y)^2\,dY\,dX \qquad 1-X=a$$

$$= 4abc \int_0^1 X(1-X)^4\,\beta(2,3)\,dX$$

$$= 4abc \frac{\Gamma 2\,\Gamma 3}{\Gamma 5} \int_0^1 X(1-X)^4\,dX$$

$$\left[\int_0^a y^{m-1}(a-y)^{n-1}\,dy = a^{m+n+1}\,\beta(m,n)\right]$$

$$= 4abc\,\frac{1\times 2}{24} \times \beta(2,5)$$

$$= \frac{abc}{90}.$$

(3) $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$, taken over the region of space in the the octant bdd by the sphere $x^2+y^2+z^2=1$.

Soln: Let $x^2 = X,\ y^2 = Y,\ z^2 = Z$ $\qquad X+Y+Z=1$

$dx = \frac{dX}{2\sqrt{X}},\ dy = \frac{dY}{2\sqrt{Y}},\ dz = \frac{dZ}{2\sqrt{Z}}$

$V = \frac{\pi^2}{8}$ cubic units