(i) Scaling of coordinate

$t \to at \, , \quad \tilde{f}(\omega) \to \frac{1}{|a|} \tilde{f}\left(\frac{\omega}{a}\right)$

Let $\tilde{g}(\omega) = \int_{-\infty}^{\infty} f(at) \, e^{2\pi i \omega t} dt$

Change variable $u = at$

$\Leftrightarrow t = \frac{u}{a} \, , \quad dt = \frac{du}{a}$

$$\to \quad \tilde{g}(\omega) = \int_{-\infty}^{\infty} f(u) \, e^{2\pi i \omega \frac{u}{a}} (du/a)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(u) \, e^{2\pi i \frac{\omega}{a} u} du$$

$$= \frac{1}{a} \tilde{f}\left(\frac{\omega}{a}\right) \qquad (1)$$

If $a < 0$

$$\tilde{g}(\omega) = \frac{1}{a} \int_{\infty}^{-\infty} f(u) e^{2\pi i \frac{\omega}{a} u} du$$

$$= -\frac{1}{a} \int_{-\infty}^{\infty} f(u) e^{2\pi i \frac{\omega}{a} u} du$$

$$= -\frac{1}{a} \tilde{f}\left(\frac{\omega}{a}\right) \qquad (2)$$

Combining (1) and (2)

$$\tilde{g}(\omega) = \frac{1}{|a|} \tilde{f}\left(\frac{\omega}{a}\right) \quad \therefore$$

(j) Shifting of coordinate

$t \to t - t_0, \quad \tilde{f}(\omega) \to \tilde{f}(\omega) e^{2\pi i \omega t_0} = \tilde{g}(\omega)$

Change variable $u = t - t_0$, $\infty - t_0 = \infty$, $-\infty - t_0 = -\infty$

$\Leftrightarrow t = u + t_0$, $dt = d(u + t_0)$
$= du$

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} f(t - t_0)\, e^{2\pi i \omega (u + t_0)} du$$

$$= \int_{-\infty}^{\infty} f(u) e^{2\pi i \omega u} \cdot e^{2\pi i \omega t_0} du$$

factor $e^{2\pi i \omega t_0}$ is constant w.r.t. u

$$\to \tilde{g}(\omega) = \tilde{f}(\omega) e^{2\pi i \omega t_0} \quad \therefore$$

(k) Inverse scaling: $\omega \to b\omega$, $f(t) \to \frac{1}{|b|} f(\frac{t}{b})$

Let $g(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-2\pi i \omega t} d\omega$

Change variable $\phi = b\omega$

$\Leftrightarrow \omega = \frac{\phi}{b}$, $d\omega = \frac{d\phi}{b}$

$$\to g(t) = \int_{-\infty}^{\infty} \tilde{f}(\phi) e^{-2\pi i \frac{\phi}{b} t} (d\phi / b)$$

$$= \frac{1}{b} \int_{-\infty}^{\infty} \tilde{f}(\phi) e^{-2\pi i \phi \frac{t}{b}} d\phi$$

$$= \frac{1}{b} f(\frac{t}{b})$$

As in (i) if $b < 0$ there is a minus sign in front

$$\to g(t) = \frac{1}{|b|} f(\frac{t}{b}) \quad \therefore$$

(l) Inverse shifting: $\omega \to \omega - \omega_0$, $f(t) \to f(t)e^{-2\pi i \omega_0 t}$
$= g(t)$

Change variable $\phi = \omega - \omega_0$
$\Leftrightarrow \quad \omega = \phi + \omega_0$, $d\omega = d(\phi + \omega_0)$
$= d\phi$

$$g(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega - \omega_0) e^{-2\pi i(\phi + \omega_0)t} \, d\phi$$

$$= \int_{-\infty}^{\infty} \tilde{f}(\phi) e^{-2\pi i \phi t} e^{-2\pi i \omega_0 t} \, d\phi$$

$$= f(t) e^{-2\pi i \omega_0 t} \quad \therefore$$