Real numbers x and y have their machine presentations $\hat{x}$ and $\hat{y}$.

Relative errors

$$r_x = \frac{x - \hat{x}}{x}, \quad r_y = \frac{y - \hat{y}}{y}$$

$$\leftrightarrow \begin{cases} x - \hat{x} = r_x x \\ y - \hat{y} = r_y y \end{cases}$$

$$\leftrightarrow \begin{cases} x - r_x x = \hat{x} \\ y - r_y y = \hat{y} \end{cases}$$

$$\leftrightarrow \begin{cases} x = \dfrac{\hat{x}}{1 - r_x} \\ y = \dfrac{\hat{y}}{1 - r_y} \end{cases} \quad (1)$$

For product xy:

$$xy = \frac{\hat{x}\hat{y}}{(1 - r_x)(1 - r_y)} \leftrightarrow \hat{x}\hat{y} = (1 - r_x)(1 - r_y)\, xy \quad (2)$$

Definition of relative error r_{xy} is

$$r_{xy} = \frac{xy - \hat{x}\hat{y}}{xy} \quad \text{insert (2)}$$

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$$\rightarrow = \frac{xy - (1 - r_x)(1 - r_y)\, xy}{xy}$$

$$r_{x \cdot y} = 1 - (1 - r_x)(1 - r_y)$$

$$= 1 - (1 - r_x - r_y + r_x r_y)$$

$$= r_x + r_y - r_x r_y$$

For quotient x/y using (1):

$$\frac{x}{y} = \frac{\hat{x}(1 - r_x)}{(1 - r_y)\hat{y}} = \frac{\hat{x}}{\hat{y}} \frac{(1 - r_x)}{(1 - r_y)}$$

$$\Rightarrow \frac{\hat{x}}{\hat{y}} = \frac{x}{y} \frac{(1 - r_y)}{(1 - r_x)} \quad (3)$$

Definition of relative error

$$r_{\frac{x}{y}} = \frac{\frac{x}{y} - \frac{\hat{x}}{\hat{y}}}{\frac{x}{y}} \quad \text{insert (3)}$$

$$= \frac{\frac{x}{y} - \frac{x}{y} \frac{(1 - r_x)}{(1 - r_y)}}{\frac{x}{y}}$$

$$= 1 - \frac{(1 - r_x)}{(1 - r_y)}$$

$$= \frac{\cancel{1 - r_y} - (1 - r_x)}{\cancel{1 - r_y}}$$

$$= \frac{r_x - r_y}{1 - r_y}$$