Exercise 7 Problem 4

Romberg element: $R(n+1, m+1) = R(n+1, m) + \frac{1}{4^{m+1}-1}\left[R(n+1, m) - R(n, m)\right]$

Simpson's rule $\int_{x_1}^{x_3} f(x)\,dx = h\left[\frac{1}{3}f_1 + \frac{4}{3}f_2 + \frac{1}{3}f_3\right] + \varepsilon$

$$R(1,1)\Big|_{m,n=0} = R(1,0) + \frac{1}{4-1}\left[R(1,0) - R(0,0)\right]$$

$$R(0,0) = \frac{1}{2}(x_3 - x_1)\left[f_1 + f_3\right] \qquad x_3 - x_1 = 2h$$

$$R(1,0) = \frac{1}{2}R(0,0) + h\,f(x_1 + h) \qquad f_2 = f(x_1 + h)$$

$$= \frac{1}{2}\left\{h\left[f_1 + f_3\right]\right\} + h f_2$$

$$= h\left[\frac{1}{2}f_1 + f_2 + \frac{1}{2}f_3\right]$$

$$R(1,1) = h\left[\frac{1}{2}f_1 + f_2 + \frac{1}{2}f_3\right] + \frac{1}{3}\left\{h\left[\frac{1}{2}f_1 + f_2 + \frac{1}{2}f_3\right] - h\left[f_1 + f_3\right]\right\}$$

$$= h\left[\left(\frac{1}{2} + \frac{1}{6} - \frac{1}{3}\right)f_1 + \left(1 + \frac{1}{3}\right)f_2 + \left(\frac{1}{2} + \frac{1}{6} - \frac{1}{3}\right)f_3\right]$$

$$= h\left[\frac{1}{3}f_1 + \frac{4}{3}f_2 + \frac{1}{3}f_3\right]$$