Numerical Methods in Scientific Computing, spring 2023

Problem sheet 12: Fourier transforms

Return code and solutions by Monday 24.4.2023 23:59 to Moodle.

N.B. Fixed the form of the semicircular hammer in 3b. Clarified the problem statement in 3 and 4.

1. (pen and paper, 6 points) Refresh your memory about Fourier transforms by showing the following properties for the Fourier transform $\mathcal{F}$ of function $f(t)$, $\mathcal{F}(f) = \tilde{f}(\omega)$

$$\tilde{f}(\phi) = \int_{-\infty}^{\infty} f(t) e^{2\pi i \phi t} \mathrm{d}t \tag{1}$$

and its inverse transform

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\phi) e^{-2\pi i \phi t} \mathrm{d}\phi : \tag{2}$$

   (a) if $f(t)$ is real, then $\tilde{f}(-\omega) = \tilde{f}(\omega)^*$ (asterisk meaning complex conjugation)
   (b) if $f(t)$ is imaginary, then $\tilde{f}(-\omega) = -\tilde{f}(\omega)^*$
   (c) if $f(t)$ is even, $f(-t) = f(t)$, then $\tilde{f}(\omega)$ is even
   (d) if $f(t)$ is odd, $f(-t) = -f(t)$, then $\tilde{f}(\omega)$ is odd
   (e) if $f(t)$ is real and even, then $\tilde{f}(\omega)$ is real and even
   (f) if $f(t)$ is real and odd, then $\tilde{f}(\omega)$ is imaginary and odd
   (g) if $f(t)$ is imaginary and even, then $\tilde{f}(\omega)$ is imaginary and even
   (h) if $f(t)$ is imaginary and odd, then $\tilde{f}(\omega)$ is real and odd
   (i) scaling of coordinate: $t \to at$, $\tilde{f}(\omega) \to \frac{1}{|a|}\tilde{f}\left(\frac{\omega}{a}\right)$
   (j) shifting of coodinate: $t \to t - t_0$, $\tilde{f}(\omega) \to \tilde{f}(\omega) e^{2\pi i \omega t_0}$
   (k) the inverse scaling relation: $\omega \to b\omega$, $f(t) \to \frac{1}{|b|} f\left(\frac{t}{b}\right)$
   (l) the inverse shifting relation: $\omega \to \omega - \omega_0$, $f(t) \to f(t) e^{-2\pi i \omega_0 t}$

   You'll need equations 1 or 2 to show the properties 1a-1l. Each proof is very short.

   These kinds of mathematical properties are important whenever you are developing scientific software, especially if you don't know what the right answer should be. If you are, for example, calculating the Fourier transform of a signal that should be real and even and you see that it has imaginary components, then you know that either your signal is wrong, or your Fourier transform isn't working properly.

2. (computer, 6 points) The teaching lab of the physics department has two tuning forks that are used for demonstration purposes. The forks are tuned to Stuttgart pitch – a frequency of 440 Hz – that is, the A note above middle C. (Note that earlier conventions for the note exist, e.g. an earlier French convention used 435 Hz, whereas the tuning forks used by Georg Friedrich Händel were tuned even lower, whereas Ludwig van Beethoven used higher tones.)

   The file `samples.tar.xz` attached (decompress with `tar Jxf samples.tar.xz`) contain three samples.

   (a) Using a Fourier transform, what are the important frequencies you find in each case?
   (b) Can you find higher harmonics, what are their frequencies?
   (c) Running a windowed transform, can you see differences in the ratios of the harmonics between the initial sound – when the fork was rung – and the later stages when fork is still ringing?

You can use library implementations of the Fourier transform, e.g. fft in Octave/Matlab or NumPy, and fftpack in SciPy. Note that both NumPy and SciPy use the FFTPACK library which is a lot slower than FFTW which is available in Python via PyFFTW.

3. (computer, 6 points) Higher harmonics (and the way they decay) turn out to be very important for the sound an instrument makes. The sound of a piano is created by strings, which are hit by hammers operated through a mechanism controlled by the keyboard. In addition to the fundamental frequency, the hammers also induce oscillations at higher harmonics, which are damped faster than the fundamental harmonic.

   Study a model of a piano, in which the sound that the hammer makes is determined by the shape of the hammer that presses down on the string. Assuming that the width of the hammer w is about $1/50^{\text{th}}$ of the length of the string l, $w = l/50$, and that the hammer is centered mid-string at $x = 0$, study the wave form induced by three kinds of hammers

   (a) a flat hammer: $y(x) = A$ for $|x| \leq w$; $y(x) = 0$ otherwise
   (b) a round hammer: $y(x) = A\sqrt{w^2 - x^2}$ for $|x| \leq w$; $y(x) = 0$ otherwise
   (c) a Gaussian hammer: $y(x) = A \exp\left(-\frac{(4\ln 2)x^2}{w^2}\right)$ for $x \in [-\frac{l}{2}, \frac{l}{2}]$; that is, the width parameter w defines the full width at half maximum (FWHM) of the Gaussian, while the hammer in principle spans the whole length of the string – and more.

   Note, however, that the exponential term has the value 1 at $x = 0$, and that it attains the value ϵ at $x = \sqrt{\frac{-\ln\epsilon}{4\ln 2}}w \approx 3.6056w$ for double precision ($\epsilon \approx 2.2204 \times 10^{-16}$), so in reality the effective width of the hammer is limited.

   The displacement $y(x)$ induced by the hammer at $t = 0$ acts as an initial condition to the wave equation for the strings, wherein all frequencies undergo their own evolution in time. To project the displacements onto the fundamental components of the string, one needs to do a Fourier transform. Remember that the string is tied at both ends, so the amplitude has to be zero at the end points.

   Although it's quite possible that the transforms can be done analytically in closed form at least in some of the cases above, the task is likely easiest to accomplish numerically. Moreover, since the signal is nonzero only on a small interval of the string, it may be smartest to do the transform yourself as you will avoid unnecessary work computing zeros. You may also want to use a Fourier series in terms of sines and cosines instead of the complex, exponential formulation in the transformation, as it has a clearer physical interpretation. Remember to check that your results are converged with respect to the spatial discretization.

   Compute the Fourier transform of the wave $y(x)$ with the forms (a)-(c). **Visualize the transforms**. **Comment on the differences**.

   Hint: For simplicity, you can set $A = 1$ since it's just an overall multiplier. You can also switch to dimensionless units, $z = x/l$, as this should make the implementation simpler.

4. (computer, 6 points) Continuing problem 3, study the dependence of the spectrum on the position of the hammer along the string. That is, instead of being centered at $x = 0$, the center of the hammer can be anywhere along the string $x \in [-\frac{l}{2}, \frac{l}{2}]$. Next, plot the intensity $\left|\tilde{f}(\kappa)\right|^2$ of the fundamental harmonic, and the next 9 harmonics.

   According to Parseval's theorem, the integral of the squared amplitude (essentially, the energy stored in the vibration of the string) is the same in real space and in Fourier space, that is,

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} \left|\tilde{f}(\kappa)\right|^2 \mathrm{d}\kappa;$$

   so the total intensity in the signal should be the same in each case. For each type of hammer, which position of the hammer yields the largest amplitude for the fundamental harmonic? Vice versa, where are the anharmonicities the strongest, e.g. which position yields the lowest squared amplitude for the fundamental frequency?