Exercise 10 Problem 3

a) Uniform distribution

$$p(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Moment generating function MGF:

$$M(t) = \int_1 e^{tx} p(x)\, dx$$

$$= \int_0^1 e^{tx}\, dx$$

$$= \Big/_0^1 \frac{1}{t} e^{tx}$$

$$= \frac{1}{t}(e^t - e^0)$$

$$= \frac{1}{t}(e^t - 1)$$

Use Taylor series $e^t \approx 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \dots$

$$\rightarrow M(t) = 1 + \frac{1}{2!}t + \frac{1}{3!}t^2 + \dots$$

Cumulant generating function CGF:

$$K(t) = \ln[M(t)]$$

Use the series $\ln(1+u) = u - u^2/2 + u^3/3 - u^4/4 + \dots$

where $u = \frac{1}{2}t + \frac{1}{6}t^2 + \frac{1}{24}t^3 + \dots$

$$-u^2/2 = -\frac{1}{8}t^2 - \frac{1}{72}t^4 - \dots$$

$$+u^3/3 = \frac{1}{24}t^3 + \frac{1}{648}t^6 + \dots$$

$$-u^4/4 = -\frac{1}{64}t^4 - \frac{1}{5184}t^8 - \dots$$

$$\rightarrow K(t) = \frac{1}{2}t + \left(\frac{1}{6} - \frac{1}{8}\right)t^2 + \frac{1}{12}t^3 + \dots$$

$$= \frac{1}{2}t + \frac{1}{24}t^2 + \frac{1}{12}t^3 + \dots$$

Derivatives:

$$K'(t) = \frac{1}{2} + 2\cdot\frac{1}{24}t + 3\cdot\frac{1}{12}t^2 + \dots$$

$$= \frac{1}{2} + \frac{1}{12}t + \frac{1}{4}t^2 + \dots$$

$$K'(0) = \frac{1}{2} = c_1$$

$$K''(t) = \frac{1}{12} + 2\cdot\frac{1}{4}t + \dots$$

$$= \frac{1}{12} + \frac{1}{2}t + \dots$$

$$K''(0) = \frac{1}{12} = c_2$$

$$K'''(t) = \frac{1}{2} + \dots$$

$$K'''(0) = \frac{1}{2} = c_3$$