4 B)

Taylor series:

$$f(x) = f(0) + \sum_{i=1}^{n-1} \frac{x^i}{i!} f^{(i)}(0) + R(n)$$

$$R(n) = \frac{x^n}{n!} f^{(n)}(\xi) \quad 0 \le \xi \le x$$

Function 1:

$$f(x) = \frac{\cos(x) - 1}{x^2}$$

f(0):

$$f(x) = \frac{a(x)}{b(x)}$$

L'Hopital's rule needed to solve indeterminate value $\frac{0}{0}$ at $x = 0$:

$$\lim_{x \to 0} \frac{a(x)}{b(x)} = \lim_{x \to 0} \frac{a'(x)}{b'(x)} = \lim_{x \to 0} \frac{a''(x)}{b''(x)}$$

$$= \lim_{x \to 0} \frac{-\sin(x)}{2x} = \lim_{x \to 0} \frac{-\cos(x)}{2}$$

$$\rightarrow f(0) = -\frac{1}{2}$$

Taylor series for $k = 12$ (Wolfram Alpha)

$$f(x) = -\frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \frac{x^6}{40320} + \frac{x^{10}}{479.001.600} - \frac{x^{12}}{87178291.200}$$

Generalized series:

$$f(x) = -\frac{1}{2} - \sum_{n=1}^{\infty} \frac{x^{2n}}{2(1+n)!} (-1)^n$$

Function 2:

$$f(x) = \frac{e^{x} - e^{-x}}{2x} = \frac{a(x)}{b(x)}$$

L'Hôpital:

$$\lim_{x \to 0} \frac{a(x)}{b(x)} = \lim_{x \to 0} \frac{a'(x)}{b'(x)}$$

$$= \lim_{x \to 0} \frac{e^{x} + e^{x}}{2}$$

$$= \lim_{x \to 0} e^{x} \to f(0) = 1$$

$$f'(x) = \frac{(e^x+e^{-x})\,2x - (e^x - e^{-x})\,2}{4x^2}$$

$$= \frac{xe^x + xe^{-x} - e^x + e^{-x}}{2x^2} = \frac{a(x)}{b(x)}$$

L'Hôpital:

$$\lim_{x\to 0} \frac{a(x)}{b(x)} = \lim_{x\to 0} \frac{a'(x)}{b'(x)}$$

$$= \lim_{x\to 0} \frac{xe^x + \cancel{e^x} - xe^{-x} + \cancel{e^{-x}} - \cancel{e^x} - \cancel{e^{-x}}}{4x}$$

$$= \lim_{x\to 0} \frac{x(e^x - e^{-x})}{4x} \to f'(0) = 0$$

Taylor series for k = 10 (Wolfram Alpha)

$$f(x) = 1 + \frac{x^2}{6} + \frac{x^4}{120} + \frac{x^6}{5040} + \frac{x^8}{362880} + \frac{x^{10}}{6227020800}$$

Generalized Taylor series:

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n+1)!}$$

Function 1 errors $\quad 0 \le \xi \le x$

$R(n)$:

$$R(1) = \frac{-\xi\sin(\xi) - 2\cos(\xi) + 2}{\xi^2}$$

$$R(2) = \frac{-(\xi^2-6)\cos(\xi) + 4x\sin(\xi) - 6}{2\xi^2}$$

$$R(3) = \frac{\xi(\xi^2-18)\sin(\xi) + 6(\xi^2-4)\cos(\xi) + 24}{6\xi^2}$$

Function 2 errors $\quad 0 \le \xi \le x$

$R(n)$:

$$R(1) = \frac{e^{-\xi}(e^{2\xi}(\xi-1) + \xi + 1)}{2\xi}$$

$$R(2) = \frac{e^{-\xi}(-\xi^2 + e^{2\xi}(\xi^2 - 2\xi + 2) - 2\xi - 2)}{4\xi}$$

$$R(3) = \frac{e^{-\xi}(\xi^3 + 3\xi^2 + e^{2\xi}(\xi^3 - 3\xi^2 + 6\xi - 6) + 6\xi + 6}{12\xi}$$