Numerical methods in scientific computing   2023

Exercise 6
Return by Monday 27.2.2023 23:59 to Moodle
Exercise session: Friday 3.3.2023

Problem 1. *(pen and paper)* (6 points)
Prove the recursion relation of the Richardson extrapolation

$$D(n,m+1)=\frac{4^m}{4^m-1}D(n,m)-\frac{1}{4^m-1}D(n-1,m)\ ,$$

using the expression

$$D(n,m)=L+\sum_{k=m}^{\infty}A(k,m)\left(\frac{h}{2^n}\right)^{2k}$$

and neither L nor $A(k,m)$ depend on h .

Problem 2. *(computer)* (6 points)
Calculate the derivative of function

$$f(x)=\sin(e^{-x^2})$$

at $x=1$ using Richardson extrapolation (RE). For RE parameters use[1]

$$N=5\ ,\ h=0.1\ .$$

Calculate the error of the value of the derivative and compare it with the error when the derivative is calculated using the central difference form

$$f\,'(x)\approx\frac{f(x+h)-f(x-h)}{2h}\ ,$$

using various step sizes $h=10^{-n}$, with $n=4,6,8,10,12,14$.

Problem 3. *(computer)* (6 points)
Plot the cubic spline interpolants of random data[2] with 5, 10, and 20 points. For each set of nodes use three different 'boundary conditions':

1 Remember that *N* is the size of the RE 'matrix' and *h* is the initial step size (see lecture notes, part 'Interpolation', slide 51).

2 From command line:
`gawk 'BEGIN {srand(); N=10; for (i=0;i<N;i++) print i,rand()}'` or by NumPy
`d=numpy.random.uniform(size=10)`

$$S''(x_0)=S''(x_N)=\begin{cases}0\\1\\10\end{cases}.$$

Problem 4. *(computer)* (6 points)
B-splines of order k for nodes $t_i, i=0,\dots,n$ are defined recursively as

$$N_i^0(t)=\begin{cases}1, & t\in[t_i,t_{i+1})\\0, & \text{otherwise}\end{cases}, \quad i=0,\dots,n-1$$

$$N_i^k(t)=\left(\frac{t-t_i}{t_{i+k}-t_i}\right)N_i^{k-1}(t)+\left(\frac{t_{i+k+1}-t}{t_{i+k+1}-t_{i+1}}\right)N_{i+1}^{k-1}(t), \quad i=0,\dots,n-k-1$$

$N_i^k(t)\neq 0$, only when $t\in[t_i,t_{i+k+1})$.

Write down the B-splines for $k=1,2,3$. You can use a symbolic math programs (Maxima or the like) to assist your work if you want to. Plot the splines. You may assume that the distance between the nodes is one and starting from zero; i.e. $t_i=i$.