c) Normal distribution

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Moment generating function MGF:

$$M(t) = \int e^{tx} p(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{(tx - \frac{1}{2}x^2)} dx$$

Completing the square exponential

$$tx - \frac{1}{2}x^2 = -\frac{1}{2}(x^2 - 2xt + t^2) + \frac{1}{2}t^2$$

$$= -\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2$$

$$M(t) = e^{\frac{1}{2}t^2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx$$

The integral is a normal distribution with mean t so it equals $\sqrt{2\pi}$

$$\rightarrow M(t) = e^{\frac{1}{2}t^2}$$

$$M'(t) = e^{\frac{1}{2}t^2} \cdot t$$

$$c_1 = M'(0) = 1.0 = 0$$

$$M''(t) = e^{\frac{1}{2}t^2} + t \cdot e^{\frac{1}{2}t^2} t$$

$$= e^{\frac{1}{2}t^2}(1 + t^2)$$

$$c_2 = M''(0) = 1.1 = 1$$

$$M'''(t) = e^{\frac{1}{2}t^2} \cdot 2t + te^{\frac{1}{2}t^2}(1+t^2)$$

$$= 2te^{\frac{1}{2}t^2} + te^{1/2 t^2} + t^2 te^{1/2 t^2}$$

$$= te^{1/2 t^2}(3 + t^2)$$

$$c_3 = M'''(0) = 0$$

Comparing cumulants and moments

Moments are defined as

$\mu_n = E[X^n]$

Conversions between moments and cumulants

$c_1 = \mu_1$, first cumulant = mean

$c_2 = \mu_2 - \mu_1^2$, second cumulant = variance

$c_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$ third cum. = skewness