Exercise 12 Problem 1.

(a) If $f(t)$ is real, then $\tilde{f}(-\omega) = \tilde{f}(\omega)^*$

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{2\pi i \omega t} dt$$

$$\rightarrow \quad \tilde{f}(-\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt$$

Complex conjugate of $\tilde{f}$:

$$\tilde{f}(\omega)^* = \left(\int_{-\infty}^{\infty} f(t) e^{2\pi i \omega t} dt \right)^*$$

$$= \int_{-\infty}^{\infty} f(t)^* \left(e^{2\pi i \omega t}\right)^* dt$$

$$= \int_{-\infty}^{\infty} f(t)^* e^{-2\pi i \omega t} dt \qquad (1)$$

If $f(t)$ is real, then $f(t)^* = f(t)$

$$\rightarrow \tilde{f}(\omega)^* = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt = \tilde{f}(-\omega) \quad \therefore \; (2)$$

(b) If $f(t)$ is imaginary, then $\tilde{f}(-\omega) = -\tilde{f}(\omega)^*$

If $f(t)$ is purely imaginary, $f(t)^* = -f(t)$

$$\rightarrow \tilde{f}(\omega)^* = \int_{-\infty}^{\infty} (-f(t)) e^{-2\pi i \omega t} dt$$

$$= -\int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt$$

$$= -\tilde{f}(-\omega)$$

$$\Leftrightarrow \quad -\tilde{f}(\omega)^* = \tilde{f}(-\omega) \quad \therefore \qquad (1)$$