Exercise 8 Problem 3

Using Newton's method for minimization

Boundary condition $\lambda_0 = 0$

Known variables
$$\begin{cases} f(0) \\ f'(0) \\ f(\lambda_{max}) \\ \lambda_{max} \\ \Delta\lambda_0 = -\dfrac{f'(\lambda_0)}{f''(\lambda_0)} \end{cases}$$

First iteration creates maximum step, the following steps form a geometric series

First iteration:
$$\lambda_1 = \lambda_0 - \Delta\lambda_0$$
$$= \lambda_{max}$$

Second iteration:
$$\lambda_2 = \lambda_1 - \frac{f'(\lambda_1)}{f''(\lambda_1)}$$
$$= \lambda_{max} - \frac{f'(\lambda_{max})}{f''(\lambda_{max})}$$

Factor in geometric series
$$q = \frac{\Delta\lambda_1}{\Delta\lambda_0} = -\frac{f'(\lambda_{max})}{f''(\lambda_{max})\lambda_{max}}$$

Sum of geometric series results into minimum point λ_*

$$\lambda_* = \frac{a}{1-q} = \frac{\lambda_{max}}{1 + \dfrac{f'(\lambda_{max})}{f''(\lambda_{max})\lambda_{max}}}$$

Use Taylor series approximations in two directions:

$$1: \begin{cases} f(\lambda_{max}) \approx f(0) + f'(0)\lambda_{max} \\ f'(\lambda_{max}) \approx f'(0) + f''(0)\lambda_{max} \end{cases}$$

$$2: \begin{cases} f(0) \approx f(\lambda_{max}) - f'(\lambda_{max})\lambda_{max} \\ f'(0) \approx f'(\lambda_{max}) - f''(\lambda_{max})\lambda_{max} \end{cases}$$

$$\Leftrightarrow \begin{cases} f'(\lambda_{max}) \approx \dfrac{f(\lambda_{max}) - f(0)}{\lambda_{max}} \\ f''(\lambda_{max}) \approx \dfrac{f'(\lambda_{max}) - f'(0)}{\lambda_{max}} \end{cases}$$

Use above for solving λ_*

$$\lambda_* \approx \frac{\lambda_{max}}{1 + \dfrac{\dfrac{f(\lambda_{max}) - f(0)}{\lambda_{max}}}{f'(\lambda_{max}) - f'(0)}}$$

$$= \frac{\lambda_{max}}{1 + \dfrac{f(\lambda_{max}) - f(0)}{\lambda_{max}(f'(\lambda_{max}) - f'(0))}}$$

insert $f'(\lambda_{max})$

$$= \frac{\lambda_{max}}{1 + \dfrac{f(\lambda_{max}) - f(0)}{\lambda_{max}\left(\dfrac{f(\lambda_{max}) - f(0)}{\lambda_{max}} - f'(0)\right)}}$$

$$= \frac{\lambda_{max}}{1 + \dfrac{f(\lambda_{max}) - f(0)}{f(\lambda_{max}) - f(0) - \dfrac{f'(0)}{\lambda_{max}}}}$$

$$= \frac{\lambda_{max}}{\dfrac{f(\lambda_{max}) - f(0) - \dfrac{f'(0)}{\lambda_{max}} + f(\lambda_{max}) - f(0)}{f(\lambda_{max}) - f(0) - \dfrac{f'(0)}{\lambda_{max}}}}$$

$$= \frac{\lambda_{max}\left(f(\lambda_{max}) - f(0)\right) - f'(0)}{2\left(f(\lambda_{max}) - f(0)\right) - \dfrac{f'(0)}{\lambda_{max}}}$$