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On the stabilization of fixed-point iterations arising in hierarchical control design

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Abstract: Fixed-point iterations are commonly used to break the algebraic loops involved in the distributed optimization among computational entities sharing only a partial knowledge. However, although this approach is appealingly simple and that it works astonishingly well in many practical situations, its use is rarely associated to an appropriate analysis of its convergence. In this paper, it is shown that this iteration can be rationally conducted using control theory in order to derive a provable stability under appropriate assumptions.

Keywords: Distributed NMPC, Fixed-point iterations, Hierarchical distributed control.

1. INTRODUCTION

Distributed control design is an attractive option for solving complex coupled decision using distributed simpler computations (Maestre and Negenborn, 2014; Scattolini, 2009; Negenborn and Maestre, 2014). It also enables a more robust design w.r.t potential local failures and or deliberate changes in the local design. It is obviously beyond the scope of this paper to give a detailed state-of-the-art of this wide domain of active research area. Indeed the survey in (Maestre and Negenborn, 2014), limited to the only MPC design, identified 35 different approaches that are quite difficult to compare on a scalar criterion. Instead, we characterize the area concerned by the present contribution by the following properties:

1) We focus on *hierarchical* control design in which the subsystems communicate only with a single coordinator. The latter can only send set-points to the local controllers of the subsystems.

2) We focus on situations where the coordinator has no knowledge regarding the equations at subsystems level and only exchanges information regarding the set-points and the potentially existant coupling signals between subsystems. For instance, this feature distinguishes the present scope from that of (Kozma et al., 2014; Doan et al., 2014) and the similar contributions where the issue is considered to be linked to the computational burden and not to the knowledge availability regarding the subsystems equations.

3) No linearity is presumed for the subsystem's dynamics. This is not to be viewed as a positive feature. Rather it simply underlines that the way the local decisions are made, for given set-points, is not the issue that this paper discusses. Rather the discussion focuses on the coordinator algorithm in a lack of knowledge context.

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The item 2) above is viewed as a step-ahead towards hierarchical Plug-and-play architecture (Riverso et al., 2013) (although in the latter, a decentralized, not hierarchical, architecture is considered). In such an architecture, the amount of modification that is to be undertaken following a change inside a subsystem is reduced.

In a hierarchical architecture, the coordinator updates the set-points to be adopted by the subsystems based on the issue of an iterative negotiation process with these subsystems. At each step of this negotiation process, a fixed-point-like procedure is used by using some current-iteration values of coupling signals. The coupling signals sent by the subsystems are used to update the ones on which the computation is to be based at the next negotiation step. This scheme obviously introduces a fixed-point iteration that should be stabilized by the coordinator algorithm. This is the main contribution of the present paper.

The paper is organized as follows: First an introductory example is proposed in Section 2 in order to justify the general problem stated in Section 3 which also introduces the working assumptions. The proposed solution is sketched in Section 4. Finally section 5 concludes the paper and gives hint for further investigation.

2. INTRODUCTORY EXAMPLE

In order to better understand the paradigm studied in the present contribution, let us consider the situation depicted in Figure 1. This figure shows two subsystems [2], denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$. Each subsystem i possesses its own decision variable $\boldsymbol{u}_i$. The outcomes of each system's decision is coupled with the decision of the other subsystem through the coupling variables v_1 and v_2. These coupling variables are given by:

[2] The framework is totally compatible with the presence of many subsystems. The use of only two subsystem is adopted here for the sake of simplicity.

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$$v_1 = \phi_1(\boldsymbol{u}_2, v_2) \text{ and } v_2 = \phi_2(\boldsymbol{u}_1, v_1) \tag{1}$$

which obviously means that there are two maps ν_1 and ν_2 such that:

$$v_1 := \nu_1(\boldsymbol{u}) \text{ and } v_2 = \nu_2(\boldsymbol{u}) \quad \Big[\Leftrightarrow v = \nu(\boldsymbol{u})\Big] \tag{2}$$

We assume that a centralized ideal optimization problem can be defined that depends on a set-point $r^d = (r_1^d, r_2^d)$ defining a centralized cost $J(\boldsymbol{u}|r^d)$ in the decision variable $\boldsymbol{u} := (\boldsymbol{u}_1, \boldsymbol{u}_2)$ so that the optimal centralized solution would be given by:

$$\boldsymbol{u}^*(r^d) := \arg\min_{\boldsymbol{u}} J(\boldsymbol{u}|r^d) \tag{3}$$

Moreover the corresponding coupling variables at the solution are denoted by $v^*(r^d) := \nu(\boldsymbol{u}^*(r^d))$. Finally, the corresponding optimal cost is denoted by $J^*(r^d)$. In the sequel, the following notation is adopted in order to simplify the expressions:

Notation 1. As the set-point r^d for the centralized problem is unchanged, the short notation $\boldsymbol{u}^*$, v^* and J^* are sometimes used to denote $\boldsymbol{u}^*(r^d)$, $v^*(r^d)$ and $J^*(r^d)$ respectively.

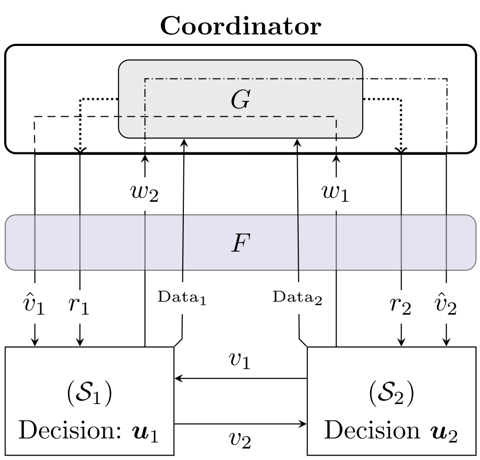

Fig. 1. Distributed control scheme including two subsystems and one coordinator. Data$_1$ and Data$_2$ contains information regarding the solutions of the local problems.

Unfortunately, centralized decision cannot be obtained because it requires a total knowledge of the whole system at one centralized decision node. Instead a distributed Plug & Play architecture is commonly preferred such as the one depicted in Figure 1 where a coordinator sends set-points r_i to each subsystems together with current estimations $\hat{v}_i$ of the coupling variables; receives the corresponding decisions and appropriately update the signals to send. Such architectures enable changes in the design/sizing at some subsystems level while keeping unchanged the coordination algorithm as well as the local controllers in all the remaining unchanged subsystems (Maestre and Negenborn, 2014).

Remark 2.1. In the formulation above, the states of the subsystems are not explicitly mentioned because all the iterations that are studied in this contribution are to be performed for a given state vector (frozen time). The results of the iterations is then applied and the whole framework is repeated at the next decision instant where the new value of the state vector is considered as a constant parameter during the next iterations (round of negotiation) and so one. In this context, the decision variables $\boldsymbol{u}_i$ and the coupling variable v_i are profiles over some prediction horizon rather than instantaneous values.

In this contribution, it is assumed that the cost function is partially separable in the following sense:

Assumption 1. (Partial cost separability). There are local cost functions $J_i^{(r_i,v_i)}(\boldsymbol{u}_i)$ in the local decision variable $\boldsymbol{u}_i$, parametrized by the pairs (r_i, v_i), such that:

$$J(\boldsymbol{u}|r^d) = J_1^{(r_1^d,\nu_1(\boldsymbol{u}))}(\boldsymbol{u}_1) + J_2^{(r_2^d,\nu_2(\boldsymbol{u}))}(\boldsymbol{u}_2) \tag{4}$$

In the sequel, a minimizer of $J_i^{(r_i,v_i)}(\cdot)$, if any, is denoted by $\boldsymbol{u}_i^0(r_i, v_i)$. ♡

This assumption simply means that the centralized cost is the sum of local costs that are almost exclusively defined by local variables except the exogenous variables v_1 and v_2.

Remark 2.2. Note that in general, one has (because of the coupling):

$$\boldsymbol{u}_i^0(r_i^d, \nu_i(\boldsymbol{u}^*)) \neq \boldsymbol{u}_i^* \tag{5}$$

Said differently, even if one *feeds* the local optimizer with the set-point r_i^d and the globally optimal coupling signal $v_i^* = \nu_i(\boldsymbol{u}^*)$, then the local optimal solution $\boldsymbol{u}_i^0(r_i^d, v_i^*)$ does not necessarily correspond to the local part of the centralized optimal solution $\boldsymbol{u}_i^*$. This is because the local problem does not take into account the impact of the local decision, say $\boldsymbol{u}_1$ on the other subsystems cost through the coupling-induced variable $v_2 := \nu_2(\boldsymbol{u}_1, \boldsymbol{u}_2)$. ♡

A direct consequence of Assumption 1 is stated in the following proposition:

Proposition 2.1.

The optimal centralized quadruplet $(\boldsymbol{u}_1^*, v_1^*, \boldsymbol{u}_2^*, v_2^*)$ is an optimal solution to the following constrained extended optimization problem:

$$\mathcal{P}_1(r^d): \quad \min_{(\boldsymbol{u}_1,v_1,\boldsymbol{u}_2,v_2)} \Big[J_1^{(r_1^d,v_1)}(\boldsymbol{u}_1) + J_2^{(r_2^d,v_2)}(\boldsymbol{u}_2)\Big] \tag{6}$$

$$\text{under} \quad \begin{array}{l} v_1 = \phi_1(\boldsymbol{u}_2, v_2) \\ v_2 = \phi_2(\boldsymbol{u}_1, v_1) \end{array} \tag{7}$$

Proof. Note first of all that $(\boldsymbol{u}_1^*, v_1^*, \boldsymbol{u}_2^*, v_2^*)$ satisfies by definition the constraints (7). Therefore, it is an admissible solution to $\mathcal{P}_1(r^d)$. Assume that there exists $(\boldsymbol{u}_1, v_1, \boldsymbol{u}_2, v_2)$ such that:

$$J_1^{(r_1^d,v_1)}(\boldsymbol{u}_1) + J_2^{(r_2^d,v_2)}(\boldsymbol{u}_2) < J_1^{(r_1^d,v_1^*)}(\boldsymbol{u}_1^*) + J_2^{(r_2^d,v_2^*)}(\boldsymbol{u}_2^*) \tag{8}$$

using the fact that $v_i = \nu_i(\boldsymbol{u})$ and $v_i^* = \nu_i(\boldsymbol{u}^*)$, one obviously has because of (4):

$$J(\boldsymbol{u}|r^d) = J_1^{(r_1^d,v_1)}(\boldsymbol{u}_1) + J_2^{(r_2^d,v_2)}(\boldsymbol{u}_2) \tag{9}$$

$$J(\boldsymbol{u}^*|r^d) = J_1^{(r_1^d,v_1^*)}(\boldsymbol{u}_1^*) + J_2^{(r_2^d,v_2^*)}(\boldsymbol{u}_2^*) \tag{10}$$

which together with (8) leads to

$$J(\boldsymbol{u}|r^d) < J(\boldsymbol{u}^*|r^d) \tag{11}$$

which contradicts the optimality of $\boldsymbol{u}^*$. □

Proposition 2.1 shows that solving the centralized problem is equivalent to solving $\mathcal{P}_1(r^d)$ which corresponds to a totally separable cost under coupled constraints on the extended decision variables $(\boldsymbol{u}, v)$. This new problem can be viewed as a *resource sharing problem* that can be efficiently solved by several approaches such as the Bundle Method (BM) (Lamoudi et al., 2014; Pflaum et al., 2014) in the case of general convex costs, the Dantzig-Wolfe decomposition (Bourdais et al. (2014)) in the case of linear subsystems coupled through constraints. The latter can also be handled using primal-dual iterations as shown in (Doan et al., 2014). Many other candidate approaches can be found in the excellent recent survey (Maestre and Negenborn, 2014).

Unfortunately, the majority of the approaches that might be used to address the problem do not accommodate the fact that the maps ϕ_1 and ϕ_2 involved in the coupling constraints (7) are not known by the coordinator which is generally responsible for the coupling constraint fulfillment. Moreover the formulation (6)-(7) is still a centralized formulation as the decision variables $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are local variables which are non accessible to the coordinator.

A first step in the re-formulation of (6)-(7) into a coordinator-related optimization problem is to parametrize $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ according to:

$$\boldsymbol{u}_i = \boldsymbol{u}_i^0(r_i, v_i) \tag{12}$$

namely, $\boldsymbol{u}_i$'s are restricted to be the minimizers of the local costs $J_i^{(r_i,v_i)}(\cdot)$ for some *modified* set-points r_i and some values v_i of the coupling variables. By doing so, the new optimization problem becomes:

$$\mathcal{P}_2(r^d):$$
$$\min_{(\boldsymbol{r}_1,v_1,\boldsymbol{r}_2,v_2)} J_c^{(r^d,v)}(r) := \Big[\bar{J}_1^{(r_1^d,v_1)}(r_1) + \bar{J}_2^{(r_2^d,v_2)}(r_2)\Big] \tag{13}$$

$$\text{under} \quad \begin{aligned} v_1 &= F_1(r_2, v_2) := \phi_1(\boldsymbol{u}_2^0(r_2, v_2), v_2) \\ v_2 &= F_2(r_1, v_1) := \phi_2(\boldsymbol{u}_1^0(r_1, v_1), v_1) \end{aligned} \tag{14}$$

where

$$\bar{J}_1^{(r_1^d,v_1)}(r_1) := J_1^{(r_1^d,v_1)}(\boldsymbol{u}_1^0(r_1, v_1)) \tag{15}$$

$$\bar{J}_2^{(r_2^d,v_2)}(r_2) := J_1^{(r_1^d,v_1)}(\boldsymbol{u}_2^0(r_2, v_2)) \tag{16}$$

Regarding this new formulation, some comments are worth mentioning:

1) The costs $\bar{J}_i^{(r_i^d,v_i)}(r_i)$ that are involved in the evaluation of the global cost from the coordinator are still evaluated for the original set-points r_i^d but the set-points r_i sent to the subsystems may be different in accordance with Remark 2.2. Said differently, the costs $\bar{J}_i^{(r_i^d,v_i)}(r_i)$ returned by the subsystems to the coordinator are their contribution to the global cost evaluated at the optimal solution $u_i^0(r_i, v_i)$ given their modified set-points r_i.

2) Obviously $\mathcal{P}_2(r^d)$ is no more equivalent to $\mathcal{P}_1(r^d)$ as the choice of the decision variables $\boldsymbol{u}_i$ is restricted to the minimizers $\boldsymbol{u}_i^0(r_i, v_i)$ of the local costs in which the new decision variables r_i (which generally encompass less degrees of freedom that $\boldsymbol{u}_i$) replaces $\boldsymbol{u}_i$.

3) The maps $F_i(\cdot)$ involved in the expression of the coupling constraints (14) are still unknown to the coordinator which only gets their values at the pairs (r_i, v_i) it sends to the subsystems.

Based on the formulation (13)-(14), the management of the set-points r_1 and r_2 by the coordinator in order to solve the optimization problem $\mathcal{P}_2(r^d)$ can be described by the following steps [see Figure 1]:

Step 0: Initialization. Choose $r_i^{(0)}$, $v_i^{(0)}$. Set $\sigma \leftarrow 0$.
Step 1: Coordinator sends $(r_i^{(\sigma)}, v_i^{(\sigma)})$ to subsystems.
Step 2: Each subsystem i performs the following:

(1) Update the values of (r_i, v_i) according to
$$(r_i, v_i) \leftarrow (r_i^{(\sigma)}, v_i^{(\sigma)}) \tag{17}$$
(2) Solve local problem defined by (r_i, v_i):
$$\boldsymbol{u}_i^0(r_i, v_i) \leftarrow \arg\min_{\boldsymbol{u}_i}\big[J_i^{(r_i,v_i)}(\boldsymbol{u}_i)\big] \tag{18}$$
(3) Compute the corresponding cost:
$$\bar{J}_i^{(\sigma)} \leftarrow \bar{J}_i^{(r_i^d,v_i)}(r_i) \tag{19}$$
(4) Evaluate the coupling variable v_j, $j \neq i$ using (14):
$$w_j^{(\sigma)} \leftarrow F_j(r_i, v_i) \tag{20}$$
This is a candidate value for the next update for v_j to be amended by the coordinator in a later step [**Step 3.1** hereafter]. Note that this evaluation can be done locally as the map F_j is known to subsystem i
(5) Repeat the above three steps (2)-(4) for $r_i + \delta_i$ in order to get an estimation of the sensitivities w.r.t r_i:
$$\Big[\partial\bar{J}_i^{(\sigma)}/\partial r\Big](r_i, v_i) \quad ; \quad \Big[\partial w_j^{(\sigma)}/\partial r\Big](r_i, v_i) \tag{21}$$
Note that this step justifies the parametrization (12) through which the dimensions of the variables w.r.t which the sensitivities are computed is drastically reduced. Note also that if the underlying problem is linear, exact sensitivities can be obtained as a by-product of the local problem's solution without finite-difference approximation.
(6) Send the data $\bar{J}_i^{(\sigma)}$, $w_j^{(\sigma)}$, $\left[\dfrac{\partial\bar{J}_i^{(\sigma)}}{\partial r}\right]$ and $\left[\dfrac{\partial w_j^{(\sigma)}}{\partial r}\right]$ to the coordinator.

Step 3: Based on the so collected data, the coordinator performs the following tasks:

(1) For all i, compute a filtered update $v_i^{(\sigma+1)}$ using the previous value $v_i^{(\sigma)}$ and the received value $w_i^{(\sigma)}$
$$v_i^{(\sigma+1)} \leftarrow A_i v_i^{(\sigma)} + B_i w_i^{(\sigma)} \tag{22}$$
A possible choice is $A_i = \theta$ and $B_i = 1 - \theta$ for some $\theta \in (0, 1)$.
(2) For all i, update the value $r_i^{(\sigma+1)}$ according to:
$$r_i^{(\sigma+1)} \leftarrow r_i^{(\sigma)} + \gamma_i^{(\sigma)} \tag{23}$$
where $\gamma_i^{(\sigma)}$ is a *control* term whose computation is the object of the next section.

Step 4: $\sigma \leftarrow \sigma + 1$.
Step 5: Goto Step 1 unless the maximum number of iterations is reached or the following terminating condition is satisfied:

$$\max\Big\{\max_i \|v_i^{(\sigma)} - v_i^{(\sigma-1)}\|, \max_i \|r_i^{(\sigma)} - r_i^{(\sigma-1)}\|\Big\} \leq \epsilon \tag{24}$$

in which case the iterations terminate.

To summarize, the updating rules used by the coordinator are given by:

$$v^+ = \theta v + (1-\theta)w \tag{25}$$
$$r^+ = r + \gamma \tag{26}$$
$$w = F(r,v) \tag{27}$$
$$\varepsilon = \frac{1}{2}\|w-v\|^2 \tag{28}$$
$$J = g(r,v) := \bar{J}_c^{(r^d,v)}(r) \tag{29}$$

Note that equations (25)-(26) are represented by the block G in Figure 1 while the computation of w through (27) is represented by the block F in the same figure. Note that in order to complete the definition of the coordinator behavior, one needs to define the increment γ involved in the updating rule (26). In the next section, it is shown that this can be done by viewing (25)-(28) as an uncertain dynamical system with state (r,v), manipulated input γ and regulated output vector $y = (J,\varepsilon)$.

Note however that for the sake of simplicity, the problem stated in the next section is the continuous-time version of the problem discussed in the present section. The discrete-time version that should be applied in the real-context can be derived from the continuous version either via standard Ordinary Differential Equation integration algorithm or by considering the discrete-time version of the development proposed in the next section. For a first presentation, our choice is to prevent the additional discrete-time related complication in order to get straightforward understanding of the proposed solution.

3. PROBLEM STATEMENT

We consider the controlled system given by:

$$\dot{v} = \zeta(r,v)(w-v) \quad ; \quad w = F(r,v) \tag{30}$$
$$\dot{r} = \gamma \tag{31}$$
$$\varepsilon = \frac{1}{2}\|w-v\|^2 \tag{32}$$
$$J = g(r,v) \tag{33}$$

where $r \in \mathbb{R}^{n_r}$, $v \in \mathbb{R}^{n_v}$ are state variables while ε and J represent two scalar outputs to be controlled in a way that is stated in the sequel.

Assume that the system (30)-(33) meets the following assumption:

Assumption 2.
There is a compact sets $\mathbb{S} \subset \mathbb{R}^{n_r}$, $\mathbb{V} \subset \mathbb{R}^{n_v}$ such that:

$$\forall (r,v) \in \mathbb{S}\times\mathbb{V}, \qquad F(r,v) \in \mathbb{V} \tag{34}$$

Moreover, the following inequality holds $\forall (r,v) \in \mathbb{S}\times\mathbb{V}$:

$$\left\|[F(r,v)-v]^T \frac{\partial F}{\partial r}(r,v)\right\| \geq \beta_1 \|F(r,v)-v\|^2 \tag{35}$$
$$\left\|\frac{\partial F}{\partial r}(r,v)\right\| \leq \beta_2 \quad ; \quad \left\|\frac{\partial F}{\partial v}(r,v)\right\| \leq \beta_3 \tag{36}$$
$$\left\|\frac{\partial^2 F}{\partial r \partial v}(r,v)\right\| \leq \beta_4 \quad ; \quad \left\|\frac{\partial^2 F}{\partial v^2}(r,v)\right\| \leq \beta_5 \tag{37}$$

♡

The design problem can be stated as follows:

Problem statement

Find a map $\zeta(v)$ together with a feedback law $\gamma = K(r,v)$ s.t the resulting closed-loop defined by (30)-(33) converges to the solution of the following optimization problem:

$$\min_{(r,v)\in\mathbb{S}\times\mathbb{V}} [J(r,v)] \text{ under } \varepsilon(r,v) = 0 \tag{38}$$

Moreover the feedback law has to use the only available information which are (r,v) and the values at (r,v) of the following maps: F, g, $\dfrac{\partial F}{\partial r}$ and $\dfrac{\partial g}{\partial r}$.

In terms of the introductory example of Section 2, the constraint $\varepsilon = 0$ simply means that $w = v$ meaning that the fixed point iteration is stabilized and that the search for the minimum of the global centralized cost in the reduced dimensional degrees of freedom is done on the manifold of the fixed-point's solutions. Finally, the restrictions included in the second part of the statement refer to the information available at the coordinator level via communication with the subsystem's problems solvers.

4. PROPOSED SOLUTION

4.1 Recalls and preliminary results

The solution proposed in this contribution is based on the following result:

Proposition 4.1. (Alamir (2015)).
Consider a scalar uncertain system of the form

$$\dot{x} = \alpha[u_c - h] \tag{39}$$

where
1) $\alpha > \alpha_{min} > 0$
2) $u_c \in [u_{min}, u_{max}]$ and
3) $h \in [h_{min}, h_{max}]$ such that:

$$u_{max} - h_{max} \geq \varrho_+ > 0 \tag{40}$$
$$h_{min} - u_{min} \geq \varrho_- > 0 \tag{41}$$

Take some $\lambda > 0$. Take any $\lambda_f > 0$ satisfying:

$$\lambda_f < \left[\min\left\{\frac{\min\{\varrho_+,\varrho_-\}}{u_{max}-u_{min}}, \frac{\alpha_{min}}{4}\right\}\right] \times \lambda \tag{42}$$

If the dynamics of the unknown term h satisfies:

$$\left|\frac{dh}{dt}\right| \leq \delta_h \tag{43}$$

then the dynamic state feedback law defined by:

$$\dot{z} = \lambda_f\left[S(\lambda(x_d - x) + z) - z\right] \tag{44a}$$
$$u_c = S(\lambda(x_d - x) + z) \tag{44b}$$

(where S is the saturation function on $[u_{min}, u_{max}]$) leads to a tracking error $e_x = x - x_d$ that satisfies:

$$\lim_{t\to\infty} |x(t) - x_d| \leq \frac{\delta_h}{\lambda\lambda_f} \tag{45}$$

and hence, the tracking error can be made as small as desired by taking high values of λ. ♡

The following result shows that if we take $x = \varepsilon$ then the problem of stabilizing the fixed-point iteration can be put in the framework of Proposition 4.1:

Lemma 4.1. If a constant gain $\zeta(r,v)=\zeta$ is used in (30) then the dynamic of ε can be put in the form (39) with the following properties:

$$\alpha := 1 \tag{46}$$
$$u_c := \left[(w-v)^T\frac{\partial F}{\partial r}(r,v)\right]\gamma \tag{47}$$
$$h := \zeta(w-v)^T\left[I-\frac{\partial F}{\partial v}(r,v)\right](w-v) \tag{48}$$
$$h_{max} := 4\zeta\rho_{\mathbb{V}}^2(1+\beta_3) \tag{49}$$
$$h_{min} := -h_{max} \tag{50}$$
$$\delta_h := \zeta^2\left[8\beta_5\rho_{\mathbb{V}}^3+8(1+\beta_3)^2\rho_{\mathbb{V}}^2\right]+ \zeta\|\gamma\|\left[4\beta_4\rho_{\mathbb{V}}^2+4\beta_2(1+\beta_3)\rho_{\mathbb{V}}\right] \tag{51}$$

where $\rho_{\mathbb{V}} := \sup\{\|v\| \text{ s.t } v\in\mathbb{V}\}$ is the radius of $\mathbb{V}$. ♡

Proof. Since
$$\dot\varepsilon = (w-v)^T(\dot w-\dot v) \tag{52}$$
using $w=F(r,v)$ obviously leads to $\dot\varepsilon = u_c-h$ where u_c and h are given by (47) and (48) respectively. This also means that (39) in this case with $\alpha=1$. This proves (46)-(48). To prove (49), (50) and (51), let us write h in the following compact form
$$h=\zeta T_1^TT_2T_1 \tag{53}$$
where
$$T_1 = w-v \tag{54}$$
$$T_2 = I-\frac{\partial F}{\partial v}(r,v) \tag{55}$$
then we has
$$\frac{dh}{dt} = \zeta\left[T_1^T\left(T_2^T+T_2\right)\dot T_1+T_1^T\dot T_2T_1\right] \tag{56}$$
straightforward computations involving the inequalities (36)-(37) show that:
$$\|T_1\| \le 2\rho_{\mathbb{V}} \tag{57}$$
$$\|T_2\| \le 1+\beta_3 \tag{58}$$
$$\|\dot T_1\| \le \beta_2\|\gamma\|+2\left(1+\beta_3\right)\rho_{\mathbb{V}}\zeta \tag{59}$$
$$\|\dot T_2\| \le \beta_4\|\gamma\|+2\beta_5\rho_{\mathbb{V}}\zeta \tag{60}$$
Combining (57) and (58) gives
$$\left\|T_1^TT_2T_1\right\| \le \rho_{\mathbb{V}}^2(1+\beta_3) \tag{61}$$
which obviously leads to the upper bound $h_{\max}$ defined by (49). Hence, it remains to prove (51).

Combining (57), (58) and (59) gives
$$\left\|T_1^T\left(T_2^T+T_2\right)\dot T_1\right\| \le 4\beta_2(1+\beta_3)\rho_{\mathbb{V}}\|\gamma\|+8(1+\beta_3)^2\rho_{\mathbb{V}}^2\zeta \tag{62}$$
Similarly, combining (57) and (60) gives
$$\left\|T_1^T\dot T_2T_1\right\| \le 4\beta_4\rho_{\mathbb{V}}^2\|\gamma\|+8\beta_5\rho_{\mathbb{V}}^3\zeta \tag{63}$$
Combining (62) and (63) obviously leads to the upper bound δ_h defined by (51).

□

Note that Lemma 4.1 does not yet enable to use proposition 4.1 since we need to show that (40)-(41) holds. Moreover, the definition (51) involves the variable quantity $\|\gamma\|$ which might induce a non bounded behavior of the tracking error in closed-loop. The following lemma shows that by adopting a variable gain $\zeta(r,v)=\sigma\|F(r,v)-v\|^2$, these issues can be settled.

Lemma 4.1. If $\zeta(r,v)=\sigma\|F(r,v)-v\|^2$ is used in (39) with some fixed $\sigma>0$ and if $F(r,v)\neq v$ then the conditions of Proposition 4.1 hold for all $\varrho>0$ with the following properties:

$$\alpha := 1 \tag{64}$$
$$u_c := \left[(w-v)^T\frac{\partial F}{\partial r}\right]\gamma \tag{65}$$
$$h := \zeta(r,v)(w-v)^T\left[I-\frac{\partial F}{\partial v}\right](w-v) \tag{66}$$
$$h_{max} := 8\sigma\rho_{\mathbb{V}}^4(1+\beta_3) \tag{67}$$
$$h_{min} := -h_{max} \tag{68}$$
$$u_{max} := h_{max}+\varrho \tag{69}$$
$$u_{min} := -h_{max}-\varrho \tag{70}$$
$$\varrho_+ := \varrho_- = \varrho \tag{71}$$
$$\delta_h := p_1\sigma+p_2\sigma^2 \tag{72}$$
where
$$p_1 := 2\left(1+\beta_3+2\frac{\beta_4}{\beta_1}\right)\varrho\rho_{\mathbb{V}}^2+4(1+\beta_3)\frac{\beta_2}{\beta_1}\varrho\rho_{\mathbb{V}} \tag{73}$$
$$p_2 := 32\beta_5\rho_{\mathbb{V}}^7+32(1+\beta_3)\left(2+2\beta_3+\frac{\beta_4}{\beta_1}\right)\rho_{\mathbb{V}}^6+ 32(1+\beta_3)^2\frac{\beta_2}{\beta_1}\rho_{\mathbb{V}}^5 \tag{74}$$

Proof. Note that (64)-(66) holds as in Lemma 4.1 with $\zeta(r,v)$ replacing ζ in (66). Moreover, (67) comes from (49) by noticing that $\zeta(r,v)\le 2\sigma\rho_{\mathbb{V}}^2$. To prove (69)-(71), it is sufficient to notice that (35) of Assumption 2 together with the fact that $F(r,v)\neq v$ and the fact that γ is not bounded show that u_c is unbounded and therefore, the control u_c can dominate any value of $h\in[-h_{max}-\varrho,+h_{max}+\varrho]$ and this for any $\varrho>0$. Therefore, it remains to prove (72).

To do so, one can use the bound already computed in Lemma 4.1 (for $\zeta(r,v)=\zeta$) by adding the bound on the additional term that comes from the non vanishing derivative of $\zeta(r,v)$, namely:
$$|\dot h| \le 4\sigma^2\rho_{\mathbb{V}}^4\left[8\beta_5\rho_{\mathbb{V}}^3+8(1+\beta_3)^2\rho_{\mathbb{V}}^2\right]+ \|\zeta(r,v)\gamma\|\left[4\beta_4\rho_{\mathbb{V}}^2+4\beta_2(1+\beta_3)\rho_{\mathbb{V}}\right]+ 2|\dot\zeta(r,v)|\|T_1^TT_2T_1\| \tag{75}$$
In order to go further, we need upper bounds on $\|\zeta(r,v)\gamma\|$ and $|\dot\zeta(r,v)|$. Regarding $\|\zeta(r,v)\gamma\|$ note that thanks to (35), one can assign to u_c any values inside $[u_{min},u_{max}]$ with γ such that:
$$\|\gamma\| \le \frac{h_{max}+\varrho}{\beta_1\|w-v\|^2}$$
This together with the definition of $\zeta(r,v)$ leads to
$$\|\zeta(r,v)\gamma\| \le \frac{\sigma}{\beta_1}\left(h_{max}+\varrho\right) \tag{76}$$
As for $|\dot\zeta(r,v)|$, one obviously has:
$$|\dot\zeta(r,v)| = 2\sigma|\dot\epsilon| \le 2\sigma\left(|u_c|+|h|\right) \le 2\sigma\left(2h_{max}+\varrho\right) \tag{77}$$
Using (61), (76) and (77) in (75) gives:

$$\begin{aligned}|\dot{h}| \le 4\sigma^2\rho_{\mathbb{V}}^4\left[8\beta_5\rho_{\mathbb{V}}^3+8(1+\beta_3)^2\rho_{\mathbb{V}}^2\right]+\\ \frac{\sigma}{\beta_1}\left(h_{max}+\varrho\right)\left[4\beta_4\rho_{\mathbb{V}}^2+4\beta_2(1+\beta_3)\rho_{\mathbb{V}}\right]+\\ 2\sigma\left(2h_{max}+\varrho\right)\rho_{\mathbb{V}}^2(1+\beta_3)\end{aligned} \quad (78)$$

which obviously gives the upper bound δ_h defined by (72) with (73)-(74) by using (67) and regrouping the terms according to the orders of σ. □

Now we have all we need to state the main result of this contribution:

Proposition 4.2. Assume that

(1) Assumptions 1 and 2 are satisfied
(2) (λ, λ_f) satisfies the inequality (42) with the parameters given by (67)-(71)
(3) The cost function $J = g(r, v)$ is lower bounded.

Consider the dynamic feedback $\gamma = \Gamma(r, v, z)$ with internal state z given by

$$\dot{r} = \Gamma(r, v, z) \quad (79)$$

$$\dot{v} = \sigma\|F(r, v) - v\|^2(F(r, v) - v) \quad (80)$$

$$\dot{z} = \lambda_f\left[S(-\lambda\varepsilon + z) - z\right] \quad (81)$$

where $\Gamma(r, v, z)$ is defined by

$$\Gamma(r, v, z) := \arg\min_{\gamma}\left[\frac{\partial g}{\partial r}(r, v)\right]\gamma \quad (82)$$

$$\text{under } \left[\left[F(r, v) - v\right]^T\frac{\partial F}{\partial r}(r, v)\right]\gamma = S(-\lambda\varepsilon + z) \quad (83)$$

then the closed-loop system converges to an approximate local solution to the optimization problem (38). Moreover, the precision of the approximation can be made as high a desired by decreasing the ratio σ/λ. ♡

Proof. Since the result is obvious if $F(r, v) = v$, we assume that $F(r, v) \neq v$.

Note that Lemma 4.1 shows that under the assumptions of the proposition, the constraint (83) can be permanently enforced leading to the following asymptotic properties:

$$\lim_{t\to\infty}\|v - F(v, r)\|^2 \le \delta_h(\sigma)/(\lambda_f\lambda) \qquad \delta_h(\sigma) = O(\sigma) \quad (84)$$

which means that the fixed point equation $v = F(r, v)$ is approximately satisfied. Moreover, the approximation error can be made as small as possible by reducing the ration σ/λ.

Regarding the constrained minimization of the cost function $J = g(r, v)$, it depends on the evolution of the relative disposition of the two vectors:

$$G_J(r, v) = \frac{\partial g}{\partial r}(r, v);\ G_S(r, v) := \left[\frac{\partial F}{\partial r}(r, v)\right]^T\left[F(r, v) - v\right] \quad (85)$$

two situations have to be distinguished:

1) **Either** G_J has continuously a non vanishing component on $G_S^{\perp}$ in which case, the cost function g continuously decreases and since it is lower bounded, it converges towards some local minimum at which the gradient G_J vanishes.

2) **Or** G_J becomes asymptotically parallel to G_S in which case, the system reaches a point where no decrease in the cost function is possible without violating the constraints. This case corresponds to a solution that activates the constraint, in which the gradient is not necessarily vanishing but there is no descent direction that is compatible with the constraints. This is a typical solution to the KKT conditions. □

5. CONCLUSION AND FUTURE WORK

In this paper, the stability of fixed-point iterations arising in hierarchical control schemes is investigated and a solution is proposed for the coordinator updating rules.

Future investigations concern the derivation of a discrete-time realistic instantiation of the proposed algorithm. The impact of the communication delay and the time needed in the negotiation process on the performance and stability of the proposed scheme should also be deeply investigated. Finally, Real-life implementation on industrial cryogenic refrigerator is scheduled in order to address the practitioners desire to get a fully plug and play architecture.

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