Title

Chris Monico
Department of Mathematics and Statistics
Texas Tech University
e-mail: c.monico@ttu.edu

Draft of April 26, 2017

We consider Conway's Game Of Life as a function

$$\lambda : \mathbb{F}_2^{\mathbb{Z}\times\mathbb{Z}} \longrightarrow \mathbb{F}_2^{\mathbb{Z}\times\mathbb{Z}}.$$

Some of the questions we are interested in are the following:

1. Let $\kappa(n)$ denote the number of elements of $\mathbb{F}_2^{\mathbb{Z}\times\mathbb{Z}}$ vanishing outside of $[0, n-1]\times[0, n-1]$ which lie in the image of λ. Determine good bounds for κ.

2. Given an element Y of $\mathbb{F}_2^{\mathbb{Z}\times\mathbb{Z}}$ with compact support (i.e., vanishing outside of a bounded set), determine if Y is in the image of λ. This is the *Image Decision Problem*. Show that this problem is NP-complete.

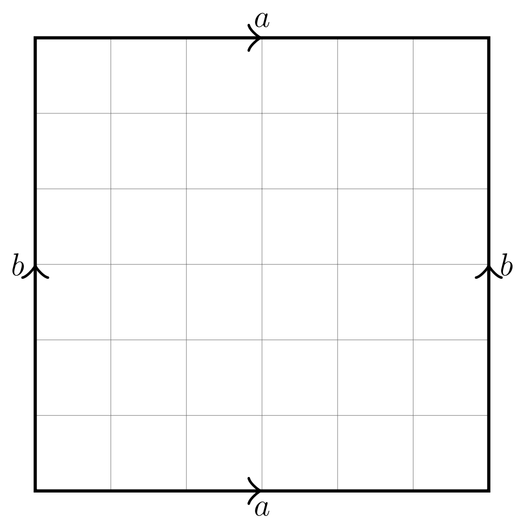
3. Given an element Y in the image of λ with compact support, find $X \in \mathbb{F}_2^{\mathbb{Z}\times\mathbb{Z}}$ for which $\lambda(X) = Y$. This is the *Inverse Problem*.

4. Determine a good lower bound on the computational complexity of solving both of the previous problems.

Notice immediately: it should be fairly straightforward to prove that $\kappa(n)/2^{n^2} \to 0$ as $n \to \infty$. Such an argument could be based on the observation that λ restricted to an $(n+2)\times(n+2)$ grid is not one-to-one. Therefore, it is not onto either, and so there are patterns which are not in the image of λ. Let α denote the proportion of, say 5×5 grids which are not in the image of λ. Then the proportion of $5N \times 5N$ grids which are in the image of λ is not more than $(1-\alpha)^{N^2}$, which tends to 0 as $N \to \infty$ since $\alpha > 0$.

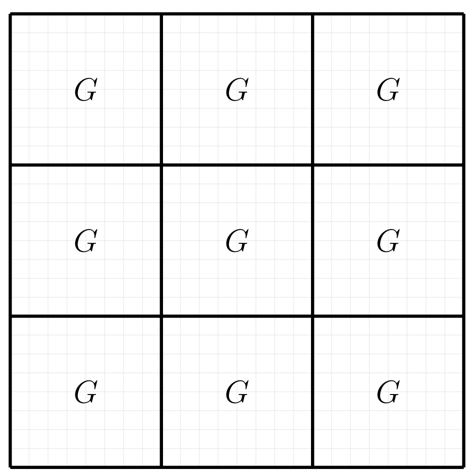
1 Size of the image

First consider the Game of Life on a torus; that is, on a finite grid with edges identified as in the following figure:



The Game of Life on a grid with edges identified in this way is generally not one-to-one, and hence not onto (since the domain and codomain are both finite with equal sizes). Suppose that with an $n \times n$ grid with edges identified in this way, the particular grid G is not in the image.

Consider the following $(3n) \times (3n)$ grid in the plane as usual (without edges identified):



Can we show, under suitable assumptions about the size of G (probably must be at least 2×2 or 3×3) that this grid is not in the image of λ? Perhaps we need 5×5 copies of G instead of 3×3 to ensure compatible bounday conditions. Specifically, could we arrive at a contradiction by showing that if this grid is in the image of λ that G itself was in the image of GOL on the torus?