Bài tập lần 13

Câu 3 - Đề 4 GK - K64 TT

$$\begin{cases} u_t(x,t) = 2u_{xx}(x,t) & x>0,\ t>0 \\ u_x(0,t) = 0 & t \geq 0 \\ u(x,0) = \cos x = \varphi(x) & 0 \leq x < \infty \end{cases}$$

- Do với điều kiện biên $u_x(0,t) = 0$, ta thác triển chẵn hàm $\varphi(x)$ thành $\varphi^*(x)$ và thu được công thức nghiệm của bài toán có dạng

$$u(x,t) = \frac{1}{\sqrt{4\pi 2t}} \int_0^{+\infty} \cos y \left[e^{-\frac{(x-y)^2}{4.2t}} + e^{-\frac{(x+y)^2}{4.2t}} \right] dy$$

$$= \frac{1}{2\sqrt{2\pi t}} \int_0^{+\infty} \cos y \left[e^{-\frac{(x-y)^2}{8t}} + e^{-\frac{(x+y)^2}{8t}} \right] dy$$

$$= \frac{1}{2\sqrt{2\pi t}} \left(\int_0^{+\infty} \cos y\, e^{-\frac{(x-y)^2}{8t}} dy + \int_0^{+\infty} \cos y\, e^{-\frac{(x+y)^2}{8t}} dy \right)$$

+) Xét $I_1 = \int_0^{+\infty} \cos y\, e^{-\frac{(x-y)^2}{8t}} dy$

Đặt $x - y = z \Rightarrow \begin{cases} dy = -dz \\ y = 0 \Rightarrow z = x,\ y = +\infty \Rightarrow z = -\infty \end{cases}$

$$\Rightarrow I_1 = \int_{-\infty}^{x} \cos(x-z)\, e^{-\frac{z^2}{8t}} dz$$

$$= \int_{-\infty}^{x} \cos x \cos z\, e^{-\frac{z^2}{8t}} dz + \int_{-\infty}^{x} \sin x \sin z\, e^{-\frac{z^2}{8t}} dz$$

+) Xét $I_2 = \int_0^{+\infty} \cos y\, e^{-\frac{(x+y)^2}{8t}} dy$

Đặt $x + y = z \Rightarrow \begin{cases} dy = dz \\ y = 0 \Rightarrow z = x,\ y = +\infty \Rightarrow z = +\infty \end{cases}$

$$\Rightarrow I_2 = \int_x^{+\infty} \cos(z-x)\, e^{-\frac{z^2}{8t}} dz = \int_x^{+\infty} \cos x \cos z\, e^{-\frac{z^2}{8t}} dz + \int_x^{+\infty} \sin x \sin z\, e^{-\frac{z^2}{8t}} dz$$

$\Rightarrow u(x,t) = \frac{1}{2\sqrt{2\pi t}}(I_1 + I_2)$

$= \frac{1}{2\sqrt{2\pi t}}\left(\int_{-\infty}^{+\infty} \cos x \cos z\, e^{-\frac{z^2}{8t}} dz + \int_{-\infty}^{+\infty} \sin x \sin z\, e^{-\frac{z^2}{8t}} dz\right)$

Do ~~cos~~ $f_1(z) = \cos x \cos z . e^{-\frac{z^2}{8t}}$ ~~dz~~ là hàm chẵn theo z

$f_2(z) = \sin x \sin z\, e^{-\frac{z^2}{8t}}$ là hàm lẻ theo z

$\Rightarrow u(x,t) = \frac{1}{2\sqrt{2\pi t}}\left(2\int_0^{+\infty} \cos x . \cos z\, e^{-\frac{z^2}{8t}} dz + 0\right)$

$= \frac{\cos x}{\sqrt{2\pi t}} \int_0^{+\infty} \cos z\, e^{-\frac{z^2}{8t}} dz$

$= \frac{\cos x}{\sqrt{2\pi t}} \cdot \frac{\sqrt{\pi}}{2\frac{1}{2\sqrt{2t}}} . e^{-\frac{(2\sqrt{2t})^2}{4}}$

$= \cos x . e^{-2t}$

Vậy $u(x,t) = \cos x . e^{-2t}$

Câu 3 - Đề 2 GK - KG2TT

$$\begin{cases} u_t = 9u_{xx} & x > 0,\ t > 0 \\ u(0,t) = 0 & x \geq 0 \\ u(x,0) = e^x \chi_{[1,2]}(x) = \begin{cases} e^x & \text{khi } 1 \leq x \leq 2 \\ 0 & \text{khi } 0 \leq x < 1,\ 2 < x \end{cases} \end{cases}$$

Do Với đk biên $u(0,t) = 0$, Ta thác triển lẻ hàm số $u(x,0) = \varphi(x)$ thành $\varphi^+(x)$ và thu được nghiệm của bài toán có dạng

Thứ

Ngày . No.

$$u(x,t)=\frac{1}{\sqrt{4.9\pi t}}\left(\int_0^{+\infty}\varphi(y)\left[e^{-\frac{(x-y)^2}{4.9t}}-e^{-\frac{(x+y)^2}{4.9t}}\right]dy\right)$$

$$=\frac{1}{6\sqrt{\pi t}}\left(\int_0^{+\infty}\varphi(y)\left[e^{-\frac{(x-y)^2}{36t}}-e^{-\frac{(x+y)^2}{36t}}\right]dy\right)$$

Vì $\varphi(y)=\begin{cases} e^y & \text{khi } 1\le y\le 2 \\ 0 & \text{khi } 0\le y<1,\ y>2\end{cases}$

$$\Rightarrow u(x,t)=\frac{1}{6\sqrt{\pi t}}\left(\int_1^{2}\varphi(y)\left[e^{-\frac{(x-y)^2}{36t}}-e^{-\frac{(x+y)^2}{36t}}\right]dy\right)$$

$$=\frac{1}{6\sqrt{\pi t}}\left(\int_1^{2}\varphi(y)e^{-\frac{(x-y)^2}{36t}}dy-\int_1^{2}\varphi(y)e^{-\frac{(x+y)^2}{36t}}dy\right)$$

+) Xét $I_1=\int_1^2\varphi(y)e^{-\frac{(x-y)^2}{36t}}dy=\int_1^2 e^y e^{-\frac{(x-y)^2}{36t}}dy$

$$=\int_1^2 e^{\frac{-x^2+2xy-y^2+36yt}{36t}}dy$$

$$=\int_1^2 e^{\frac{-x^2+(x+18t)^2-(x+18t)^2+2y(x+18t)-y^2}{36t}}dy$$

$$=\int_1^2 e^{9t+x}\,e^{-\frac{(x+18t-y)^2}{36t}}dy$$

Đặt $\frac{x+18t-y}{6\sqrt{t}}=z \Rightarrow dy=-6\sqrt{t}\,dz.$

$$\rightarrow I_1=e^{9t+x}\,6\sqrt{t}\int_{\frac{x+18t-2}{6\sqrt{t}}}^{\frac{x+18t-1}{6\sqrt{t}}}e^{-z^2}dz.$$

Thứ

Ngày

No.

$$= e^{(9t+x)} 6\sqrt{t}\left(\int_0^{\frac{x+18t-1}{6\sqrt{t}}} e^{-z^2}dz - \int_0^{\frac{x+18t-2}{6\sqrt{t}}} e^{-z^2}dz\right)$$

$$= e^{(9t+x)} 6\sqrt{t}\,\frac{\sqrt{\pi}}{2}\left(\mathrm{erf}\left(\frac{x+18t-1}{6\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x+18t-2}{6\sqrt{t}}\right)\right)$$

+) Xét $I_2 = -\int_1^2 \varphi(y).e^{-\frac{(x+y)^2}{36t}}dy = -\int_1^2 e^{y-\frac{(x+y)^2}{36t}}dy$

$$= -\int_1^2 e^{\frac{-x^2-2xy-y^2+36ty}{36t}}dy$$

$$= -\int_1^2 e^{\frac{-x^2+(x-18t)^2-2y(x-18t)-y^2-(x-18t)^2}{36t}}dy$$

$$= -\int_1^2 e^{9t-x}.e^{-\frac{(x-18t+y)^2}{36t}}dy$$

Đặt $\frac{x-18t+y}{6\sqrt{t}} = z \Rightarrow dy = 6\sqrt{t}dz$

$$\to I_2 = -\int_{\frac{x-18t+1}{6\sqrt{t}}}^{\frac{x-18t+2}{6\sqrt{t}}} 6\sqrt{t}e^{9t-x}e^{-z^2}dz$$

$$= -6\sqrt{t}e^{(9t-x)}\left(\int_0^{\frac{x-18t+2}{6\sqrt{t}}} e^{-z^2}dz - \int_0^{\frac{x-18t+1}{6\sqrt{t}}} e^{-z^2}dz\right)$$

$$= -6\sqrt{t}e^{(9t-x)}\frac{\sqrt{\pi}}{2}\left(\mathrm{erf}\left(\frac{x-18t+2}{6\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x-18t+1}{6\sqrt{t}}\right)\right)$$

$$u(x,t) = \frac{1}{6\sqrt{\pi t}} \left(I_1 + I_2 \right)$$

$$= \frac{1}{6\sqrt{\pi t}} \cdot 3\sqrt{\pi t} \left\{ e^{(9t+x)} \left[\operatorname{erf}\left(\frac{x+18t-1}{6\sqrt{t}} \right) - \operatorname{erf}\left(\frac{x+18t-2}{6\sqrt{t}} \right) \right] - e^{(9t-x)} \left[\operatorname{erf}\left(\frac{x-18t+2}{6\sqrt{t}} \right) - \operatorname{erf}\left(\frac{x-18t+1}{6\sqrt{t}} \right) \right] \right\}$$

$$= \frac{e^{9t+x}}{2} \left[\operatorname{erf}\left(\frac{x+18x-1}{6\sqrt{t}} \right) - \operatorname{erf}\left(\frac{x+18t-2}{(6\sqrt{t})} \right) \right] - \frac{e^{9t-x}}{2} \left[\operatorname{erf}\left(\frac{x-18t+2}{6\sqrt{t}} \right) - \operatorname{erf}\left(\frac{x-18t+1}{6\sqrt{t}} \right) \right]$$