Bài tập lần 17

Sách thầy Hợp

Bài 30

$$\begin{cases} \Delta u = 0 & x^2+y^2 \le 4 \\ u|_\Gamma = 3-4y^2-4xy^2 \end{cases}$$

Đổi biến $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\Rightarrow v(r,\theta) = u(r\cos\theta, r\sin\theta)$ T/m bài toán

$$\frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} = 0 \quad 0 \le r < 2,\ 0 < \theta < 2\pi$$

ĐK biên Dirichlet $v(2,\theta) = 3 - 4.2^2\sin^2\theta - 4.2.\cos\theta\, 2^2\sin^2\theta$

$$= 3 - 16\sin^2\theta - 32\cos\theta\sin^2\theta.$$

Chuỗi nghiệm của $v(r,\theta)$ có dạng

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(n\theta) + b_n\sin(n\theta)] r^n$$

$$v(2,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(n\theta) + b_n\sin(n\theta)] 2^n$$

$$= 3 - 16\sin^2\theta - 32\cos\theta\sin^2\theta$$

$$= 3 - 8(1-\cos2\theta) - 16\cos\theta(1-\cos2\theta)$$

$$= -5 + 8\cos(2\theta) - 16\cos\theta + 16\cos\theta\cos(2\theta)$$

$$= -5 + 8\cos2\theta - 16\cos\theta + 8\cos3\theta + 8\cos\theta$$

$$= -5 - 8\cos\theta + 8\cos2\theta + 8\cos3\theta$$

Đồng nhất hệ số, Ta được

$a_0 = -5$ $\quad a_2 = 2$ $\quad a_n = 0\ \forall n \ne \{0,1,2,3\}$

$a_1 = -4$ $\quad a_3 = 1$ $\quad b_n = 0\ \forall n$

$\Rightarrow v(r,\theta) = -5 - 4r\cos\theta + 2r^2\cos2\theta + r^3\cos3\theta.$

Thử lại

$v_r = -4\cos\theta + 4r\cos2\theta + 3r^2\cos3\theta$

$v_{rr} = 4\cos2\theta + 6r\cos3\theta$

$v_{\theta\theta} = 4r\cos\theta - 8r^2\cos2\theta - 9r^3\cos3\theta$

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$$\rightarrow \frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} = -\frac{4\cos\theta}{r} + 4\cos 2\theta + 3r\cos 3\theta + 4\cos 2\theta + 6r\cos 3\theta$$

$$+ \frac{4\cos\theta}{r} - 8\cos 2\theta - 9r\cos 3\theta = 0 \ (T/m)$$

$$v(2,\theta) = -5 - 8\cos\theta + 8\cos 2\theta + 8\cos 3\theta$$

$$= -3 - 16\sin^2\theta - 32\cos\theta\sin^2\theta \quad (T/m)$$

Bài 31

$$\begin{cases} \Delta u = 2x & x^2+y^2<1 \\ u|_{\Gamma} = x - x^3 - 2xy^2 \end{cases}$$

Ta xác định hàm v T/m $v''(x) = 2x$

$$\rightarrow v'(x) = x^2 + C_1$$

$$\rightarrow v(x) = \frac{x^3}{3} + C_1x + C_2$$

Chọn $C_1 = 1, C_2 = 0 \rightarrow v(x) = \frac{x^3}{3} + x$

Khi đó đặt $w = u - v$ thì w T/m bài toán

$$w_{xx} + w_{yy} = u_{xx} - v'' + u_{yy} = 2x - 2x = 0$$

$$w|_{\Gamma} = x - x^3 - 2xy^2 - \frac{x^3}{3} - x = -\frac{4}{3}x^3 - 2xy^2$$

Đổi biến $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\rightarrow v(r,\theta) = w(r\cos\theta, r\sin\theta)$ T/m bài toán

$$\frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} = 0 \quad 0 \le r < 1, 0 < r < 2\pi$$

$$v(1,\theta) = -\frac{4}{3}\cos^3\theta - 2\cos\theta\sin^2\theta$$

Chuỗi nghiệm của $v(r,\theta)$ có dạng

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(n\theta) + b_n\sin(n\theta)] r^n$$

$$v(1,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(n\theta) + b_n\sin(n\theta)]$$

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$$= -\frac{4}{3}\cos^3\theta - 2\cos\theta\sin^2\theta$$

$$= -\frac{4}{3}\cdot\frac{1}{4}(\cos 3\theta + 3\cos\theta) - \cos\theta(1-\cos 2\theta)$$

$$= -\frac{1}{3}(\cos 3\theta + 3\cos\theta) - \cos\theta + \frac{1}{2}(\cos 3\theta + \cos\theta)$$

$$= -\frac{3}{2}\cos\theta + \frac{1}{6}\cos 3\theta.$$

→ Đồng nhất hệ số, Ta được

$a_1 = -\frac{3}{2}$, $a_3 = \frac{1}{6}$, $a_n = 0\ \forall n \neq \{1,3\}$

$b_n = 0\ \forall n$

$\Rightarrow v(r,\theta) = w(r\cos\theta, r\sin\theta) = -\frac{3}{2}r\cos\theta + \frac{1}{6}r^3\cos 3\theta$

$\Rightarrow u(r\cos\theta, r\sin\theta) = w(r\cos\theta, r\sin\theta) + v(r\cos\theta, r\sin\theta)$

$$= -\frac{3}{2}r\cos\theta + \frac{1}{6}r^3\cos 3\theta + \frac{r^3\cos^3\theta}{3} + r\cos\theta$$

$$= -\frac{1}{2}r\cos\theta + \frac{r^3}{6}(\cos 3\theta + 2\cos^3\theta)$$

Thử lại $v(r,\theta) = u(r\cos\theta, r\sin\theta)$

$v_r = -\frac{1}{2}\cos\theta + \frac{r^2}{2}(\cos 3\theta + 2\cos^3\theta)$

$v_{rr} = r(\cos 3\theta + 2\cos^3\theta)$

$v_{\theta\theta} = \frac{1}{2}r\cos\theta + \frac{r^3}{6}(-9\cos 3\theta + 12\cos\theta\sin^2\theta - 6\cos^3\theta)$

$\rightarrow \frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} = r\cos^3\theta + 2r\cos^3\theta + 2r\cos\theta\sin^2\theta - r\cos^3\theta$

$= 2r\cos\theta$ (T/M)

$v(1,\theta) = -\frac{1}{2}(\cos\theta) + \frac{1}{6}(\cos 3\theta + 2\cos^3\theta)$

$= \cos\theta - \cos^3\theta - 2\cos\theta\sin^2\theta$ (T/M)

Sách Pinchover - Rubinstein, Trang 205, 207

7.7

a) $\Delta u = u_{xx} + u_{yy} = 0$

Trong hệ tọa độ cực

Đổi biến $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Khi đó

$v(r,\theta) = u(r\cos\theta, r\sin\theta)$ Thỏa mãn

$$\frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} = 0$$

(b) $\begin{cases} \Delta u = 0 & x^2 + y^2 < 6 \\ u|_{\Gamma} = y + y^2 \end{cases}$

Đổi biến $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\to v(r,\theta) = u(r\cos\theta, r\sin\theta)$ Tìm bài toán.

$$\begin{cases} \frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} = 0 \\ v(\sqrt{6},\theta) = \sqrt{6}\sin\theta + 6\sin^2\theta. \end{cases}$$

Chuỗi Nghiệm $v(r,\theta)$ có dạng

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(n\theta) + b_n\sin(n\theta)]\, r^n$$

$$v(\sqrt{6},\theta) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(n\theta) + b_n\sin(n\theta)](\sqrt{6})^n = \sqrt{6}\sin\theta + 6\sin^2\theta$$

$$= \sqrt{6}\sin\theta + 3(1 - \cos 2\theta) = 3 + \sqrt{6}\sin\theta - 3\cos 2\theta$$

Đồng nhất hệ số Ta được $a_0 = 3$, $a_2 = -\frac{1}{2}$, $b_1 = 1$

$a_n = 0 \ \forall n \neq \{0, 2\}$, $b_n = 0 \ \forall n \neq 1$

$$\to v(r,\theta) = 3 - \frac{r^2}{2}\cos 2\theta + r\sin\theta.$$

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$= 3 - r^2\cos^2\theta + \frac{r^2}{2} + r\sin\theta$

$= 3 - x^2 + \frac{x^2+y^2}{2} + y$

$\Rightarrow u(x,y) = 3 + y + \frac{1}{2}(x\ \ y^2 - x^2)$

Thử lại

$\left.\begin{array}{l} u_{xx} = 0 \ \ -1 \\ u_{yy} = \ \ 1 \end{array}\right\} \Rightarrow u_{xx} + u_{yy} = 0$ (T/m)

~~$u|_\Gamma = y + y^2$ (T/m)~~

$v(\sqrt{6},\theta) = 3 - 6\cos^2\theta + 3 + \sqrt{6}\sin\theta$

$= 6\sin^2\theta + \sqrt{6}\sin\theta \Rightarrow u|_\Gamma = y + y^2$ (T/m)

7.14

~~$\{\Delta u = 0 \quad x^2+y^2 < 4$~~

7.22 d)

$\begin{cases} \Delta u = 0 & x^2+y^2 < 36 \\ u(x,y) = a \end{cases} \begin{cases} x & x < 0 \\ 0 & \text{nếu ngược lại} \end{cases}$

Đổi biến $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\Rightarrow v(r,\theta) = u(r\cos\theta, r\sin\theta)$ T/m².

$\frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} = 0$

$v(6,\theta) = \begin{cases} 6\cos\theta & \text{khi } +\frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \\ 0 & \text{khi } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{cases}$

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Chuỗi nghiệm $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)] r^n$

$u(6,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)] 6^n$

$\rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(6,\theta)\,d\theta = \frac{1}{2\pi}\left(\int_{-\pi}^{-\frac{\pi}{2}} 6\cos\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi} 6\cos\theta\,d\theta\right) = \frac{1}{2\pi}(-6-6) = \frac{-6}{\pi}$

$a_n = \frac{1}{6^n\pi}\int_{-\pi}^{\pi} u(6,\theta)\cos(n\theta)\,d\theta = \frac{1}{\pi 6^n}\left(\int_{-\pi}^{-\frac{\pi}{2}} 6\cos\theta\cos n\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi} 6\cos\theta\cos n\theta\,d\theta\right)$

$= \frac{1}{\pi 6^n} \cdot 12 \int_{\frac{\pi}{2}}^{\pi} \cos\theta\cos(n\theta)\,d\theta$

$= \frac{6}{\pi . 6^n}\int_{\frac{\pi}{2}}^{\pi}(\cos(n+1)\theta + \cos(n-1)\theta)\,d\theta = \frac{6}{\pi . 6^n}\left(\frac{\sin(n+1)\theta}{n+1} + \frac{\sin(n-1)\theta}{n-1}\right)\Bigg|_{\frac{\pi}{2}}^{\pi}$

$= -\frac{1}{\pi 6^{n-1}}\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right)$ $(n \neq 1)$

$a_1 = \frac{1}{6\pi}\left(\int_{-\pi}^{-\frac{\pi}{2}} 6\cos^2\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi} 6\cos^2\theta\,d\theta\right) = \frac{1}{6\pi}\left(\frac{3\pi}{2} + \frac{3\pi}{2}\right)$

$= \frac{1}{2}$

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$$b_n = \frac{1}{6^n\pi}\int_{-\pi}^{\pi} v(6,\theta)\sin(n\theta)\,d\theta$$

$$= \frac{1}{6^n\pi}\left(\int_{-\pi}^{-\frac{\pi}{2}} 6\cos\theta\sin(n\theta)\,d\theta + \int_{\frac{\pi}{2}}^{\pi} 6\cos\theta\sin(n\theta)\,d\theta\right)$$

$$= \frac{3}{6^n\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}}[\sin(n+1)\theta+\sin(n-1)\theta]\,d\theta + \int_{\frac{\pi}{2}}^{\pi}[\sin(n+1)\theta+\sin(n-1)\theta]\,d\theta\right]$$

$$= \frac{3}{6^n\pi}\left(\left(\frac{-\cos((n+1)\theta)}{n+1} - \frac{\cos((n-1)\theta)}{n-1}\right)\Bigg|_{-\pi}^{-\frac{\pi}{2}} + \left(\frac{-\cos((n+1)\theta)}{n+1} - \frac{\cos((n-1)\theta)}{n-1}\right)\Bigg|_{\frac{\pi}{2}}^{\pi}\right)$$

$$= \frac{3}{6^n\pi}\left(\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}\right) = 0$$

$$b_1 = \frac{1}{6\pi}\left(\int_{-\pi}^{-\frac{\pi}{2}} 3\sin 2\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi} 3\sin 2\theta\,d\theta\right) = \frac{1}{6\pi}(3-3) = 0$$

$$\Rightarrow u(r,\theta) = \frac{-6}{\pi} + \frac{1}{2}3\cos\theta + \sum_{n=2}^{\infty} \frac{-1}{\pi 6^{n-1}}\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right) r^n$$