Bài Tập lần 21.

$$(*)\begin{cases} u_{tt} = 4\Delta u \\ u_{tt}(x,y,t) = 4\Delta u(x,y,t) \\ u(x,y,0) = \varphi(x,y) = 0 \\ u_t(x,y,0) = \psi(x,y) = \begin{cases} 0 & x<0,\ y\geq 0 \\ 1 & x\geq 0,\ y\geq 0 \\ -1 & x<0,\ y<0 \\ 0 & x\geq 0,\ y<0 \end{cases} \end{cases}$$

Ta xét 2 bài toán

(1) $\cancel{u_{1tt}(x,y,t)}$ $\begin{cases} u_{1tt}(x,y,t) = 4\Delta u_1(x,y,t) \\ u_1(x,y,0) = \varphi_1(x,y) = 0 \\ u_{1t}(x,y,0) = \psi_1(x,y) = \begin{cases} 1 & \text{khi } x\geq 0 \\ 0 & \text{khi } x<0 \end{cases} \end{cases}$

(2) $\begin{cases} u_{2tt}(x,y,t) = 4\Delta u_2(x,y,t) \\ u_2(x,y,0) = \varphi_2(x,y) = 0 \\ u_{2t}(x,y,0) = \psi_2(x,y) = \begin{cases} 0 & \text{khi } y\geq 0 \\ 1 & \text{khi } y<0 \end{cases} \end{cases}$

Khi đó nghiệm của bài toán (*) có dạng $u = u_1 - u_2$.

+) Giải bài toán (1) $\begin{cases} u_{1tt}(x,y,t) = 4\Delta u_1(x,y,t) \\ u_1(x,y,0) = \varphi_1(x,y) = 0 \\ u_{1t}(x,y,0) = \psi_1(x,y) = \begin{cases} 1 & \text{khi } x\geq 0 \\ 0 & \text{khi } x<0 \end{cases} \end{cases}$

Bằng phương pháp hạ thấp số chiều ta xét bài toán (1) trong trường hợp $\mathbb{R}^3$.

$$\begin{cases} u_{1tt}(x,y,z,t) = 4\Delta u_1(x,y,z,t) \\ u_1(x,y,z,0) = \varphi_1(x,y,z) = 0 \\ u_{1t}(x,y,z,0) = \psi_1(x,y,z) = \begin{cases} 1 & \text{khi } z\geq 0 \\ 0 & \text{khi } z<0 \end{cases} \end{cases}$$

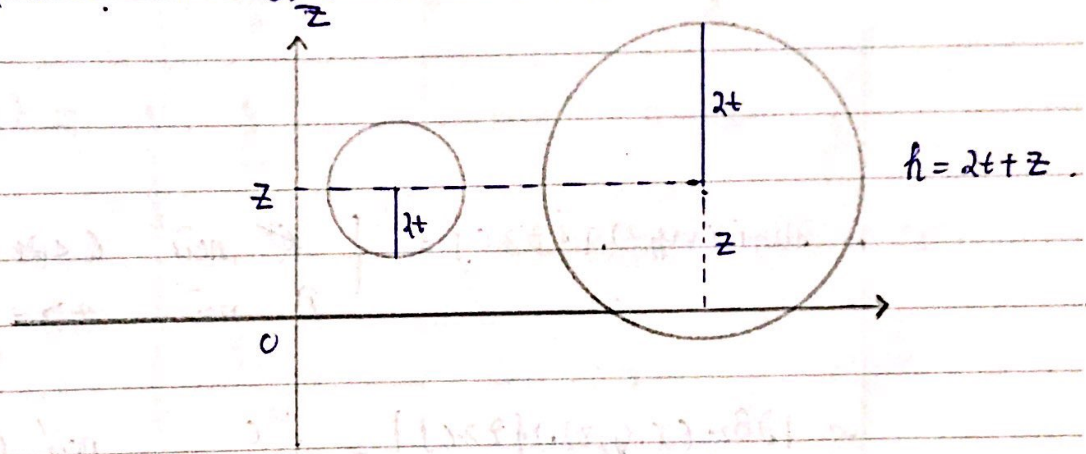
Áp dụng Công Thức Kirchhoff

$$u_1(x,y,z,t) = \partial_t\left[\frac{1}{16\pi t}\iint_{\partial B_{2t}(x,y,z)} \varphi(x,y,z)\,dS\right] + \frac{1}{16\pi t}\iint_{\partial B_{2t}(x,y,z)} \psi(x,y,z)\,dS$$

$$= \frac{1}{16\pi t}\iint_{\partial B_{2t}(x,y,z)\cap\{z\geq 0\}} dS$$ do $\varphi(x,y,z)=0$, $\psi(x,y,z) = \begin{cases} 1 & \text{khi } z\geq 0 \\ 0 & \text{khi } z<0\end{cases}$

$$= \frac{1}{16\pi t}\cdot\left|\partial B_{2t}(x,y,z)\cap\{z\geq 0\}\right|$$

TH1: Nếu $\cancel{z>0}$. (x,y,z) T/m $z>0$



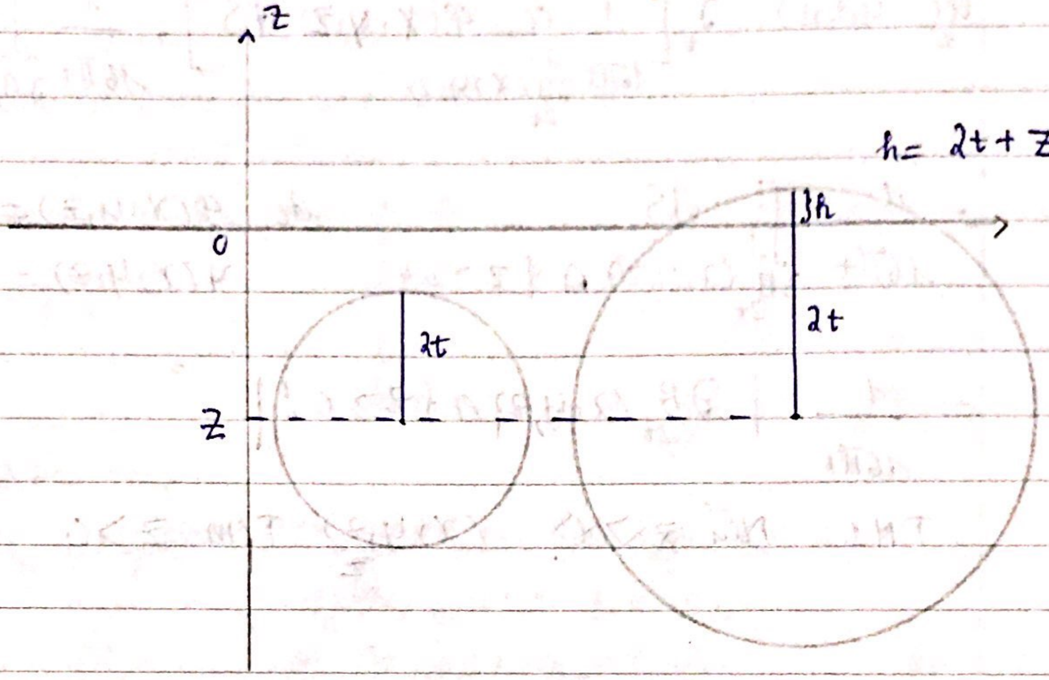
$\to \partial B_{2t}(x,y,z)\cap\{z\geq 0\} = \begin{cases} \partial B_{2t}(x,y,z) & \text{nếu } 0<2t<z \Leftrightarrow 0<t<\frac{z}{2} \\ D_t & \text{nếu } 2t>z \Leftrightarrow t>\frac{z}{2}\end{cases}$

$\to |\partial B_{2t}(x,y,z)\cap\{z\geq 0\}| = \begin{cases} 4\pi\cdot(2t)^2 = 16\pi t^2 & \text{nếu } 0<t<\frac{z}{2} \\ 2\pi Rh = 2\pi\cdot 2t\cdot(2t+z) = 4\pi t(2t+z) & \text{nếu } t>\frac{z}{2}\end{cases}$

$\to u_1(x,y,z,t) = \begin{cases} \frac{16\pi t^2}{16\pi t} = t & \text{nếu } 0<t<\frac{z}{2} \\ \frac{4\pi t(2t+z)}{16\pi t} = \frac{2t+z}{4} & \text{nếu } t>\frac{z}{2}\end{cases}$

$\cancel{TH2}$ $\to u_1(x,y,t) = \begin{cases} t & \text{nếu } 0<t<\frac{x}{2}, x>0 \\ \frac{2t+x}{4} & \text{nếu } t>\frac{x}{2}, x>0\end{cases}$

TH2 Nếu (x,y,z) T/M $z<0$

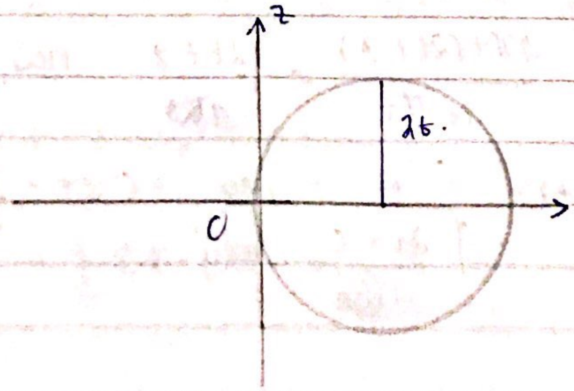


$\Rightarrow \partial B_{2t}(x,y,z) \cap \{z \geq 0\} = \begin{cases} \emptyset & \text{nếu } 0 < 2t < |z| \Leftrightarrow 0 < t < -\frac{z}{2} \\ D_t & \text{nếu } t > -\frac{z}{2} \end{cases}$

$\Rightarrow |\partial B_{2t}(x,y,z) \cap \{z \geq 0\}| = \begin{cases} 0 & \text{nếu } 0 < t < -\frac{z}{2} \\ 2\pi r h = 2\pi . 2t.(2t+z) = 4\pi t(2t+z), & t > -\frac{z}{2} \end{cases}$

$\Rightarrow u_1(x,y,z,t) = \begin{cases} 0 & \text{nếu } 0 < t < -\frac{z}{2} \\ \frac{2t+z}{4} & \text{nếu } t > -\frac{z}{2} \end{cases} \Rightarrow u_1(x,y,t) = \begin{cases} 0, & 0 < t < -\frac{x}{2}, x < 0 \\ \frac{2t+x}{4} & t > -\frac{x}{2}, x < 0 \end{cases}$

TH3: Nếu (x,y,z) TM $z = 0$.



$\rightarrow |\partial B_{2t}(x,y,z) \cap \{z \geq 0\}| = \frac{1}{2} \cdot 4\pi \cdot (2t)^2 = 8\pi t^2$

$\rightarrow u_1(x,y,z,t) = \frac{t}{2}$ khi $z = 0$.

$\rightarrow u_1(0,y,t) = \frac{t}{2}$.

Tóm lại

$$u_1(x,y,t) = \begin{cases} t & \text{nếu } 0<t<\frac{x}{2},\ x>0 \\ \frac{2t+x}{4} & \text{nếu } t>\frac{x}{2},\ x>0 \\ \frac{t}{2} & \text{nếu } x=0 \\ 0 & \text{nếu } 0<t<-\frac{x}{2},\ x<0 \\ \frac{2t+x}{4} & \text{nếu } t>-\frac{x}{2},\ x<0 \end{cases}$$

+) Giải bài toán (2) $\begin{cases} u_{2tt}(x,y,t) = 4\Delta u_2(x,y,t) \\ u_2(x,y,0) = 0 \\ u_{2t}(x,y,0) = \psi_2(x,y) = \begin{cases} 0 & \text{khi } y \geq 0 \\ 1 & \text{khi } y < 0. \end{cases} \end{cases}$

Bằng phương pháp hạ thấp số chiều, ta xét bài toán (2) trong $\mathbb{R}^3$

$u_{2tt}(x,y,z,t) = 4\Delta u_2(x,y,z,t)$

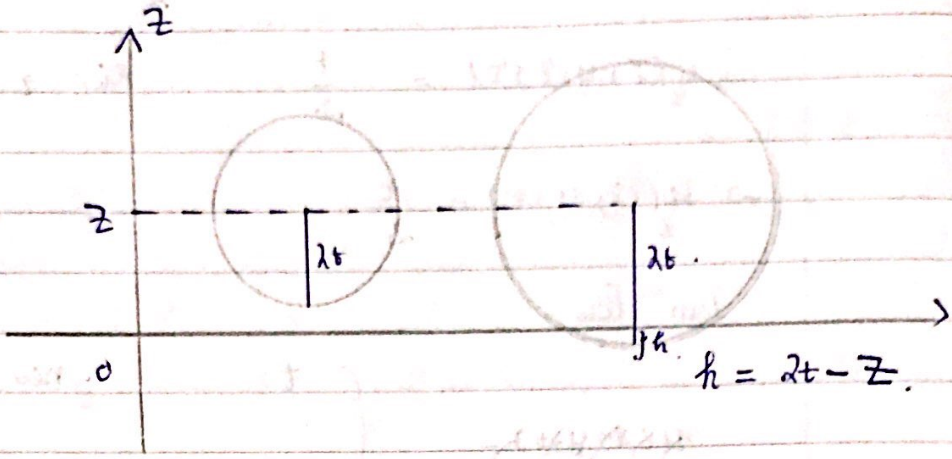
$u_2(x,y,z,0) = 0$

$u_{2t}(x,y,z,0) = \psi_2(x,y,z) = \begin{cases} 0 & \text{khi } z \geq 0 \\ 1 & \text{khi } z < 0 \end{cases}$

Áp dụng CT Kirchhoff $\rightarrow$

$u_2(x,y,z,t) = \frac{1}{16\pi t} |\partial B_{2t}(x,y,z,t) \cap \{z<0\}|$

+) TH1. Nếu (x,y,z) TM $z>0$.



$h = 2t - z$.

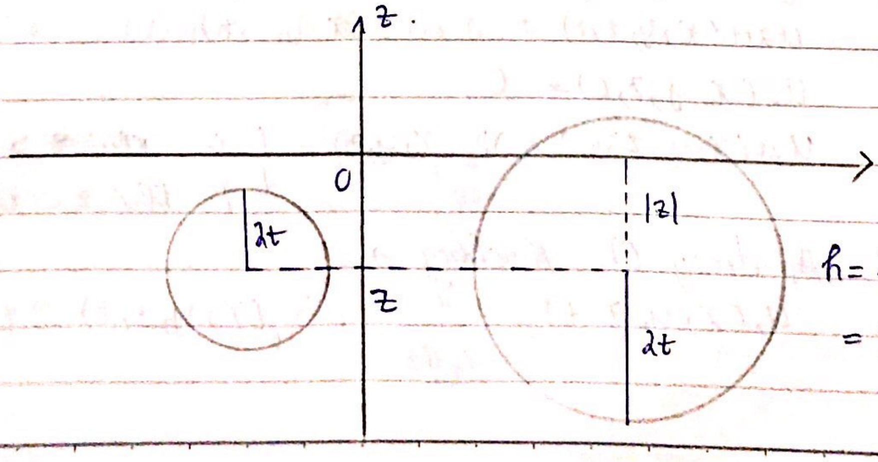
$$\rightarrow \partial B_{2t}(x,y,z,t) \cap \{z<0\} = \begin{cases} \emptyset & \text{nếu } 0<2t<z \Leftrightarrow 0<t<\frac{z}{2} \\ D_t & \text{nếu } 2t>z \Leftrightarrow t>\frac{z}{2} \end{cases}$$

$$\rightarrow |\partial B_{2t}(x,y,z,t) \cap \{z<0\}| = \begin{cases} 0 & \text{nếu } 0<t<\frac{z}{2} \\ 2\pi r h = 2\pi.2t.(2t-z) = 4\pi t(2t-z) & \text{nếu } t>\frac{z}{2} \end{cases}$$

$$\rightarrow u_2(x,y,z,t) = \begin{cases} 0 & \text{nếu } 0<t<\frac{z}{2} \\ \frac{2t-z}{4} & \text{nếu } t>\frac{z}{2} \end{cases}$$

$$\rightarrow u_2(x,y,t) = \begin{cases} 0 & \text{nếu } 0<t<\frac{y}{2},\ y>0 \\ \frac{2t-y}{4} & \text{nếu } t>\frac{y}{2},\ y>0. \end{cases}$$

+) TH2 Nếu (x,y,z) T/M $z<0$.



$h = 2t + |z|$
$= 2t - z$.

$$\to \partial B_{2t}(x,y,z,t) \cap \{z<0\} = \begin{cases} \partial B_{2t}(x,y,z,t) & \text{nếu } 0<2t<|z| \Leftrightarrow 0<t<-\frac{z}{2} \\ D_t & \text{nếu } 2t>|z| \Leftrightarrow t>-\frac{z}{2} \end{cases}$$

$$\to |\partial B_{2t}(x,y,z,t) \cap \{z<0\}| = \begin{cases} 4\pi.(2t)^2 = 16\pi t^2 & \text{nếu } 0<t<-\frac{z}{2} \\ 2\pi rh = 4\pi t(2t-z) & \text{nếu } t>-\frac{z}{2} \end{cases}$$

$$\to u_2(x,y,z,t) = \begin{cases} t & \text{nếu } 0<t<-\frac{z}{2} \\ \frac{2t-z}{4} & \text{nếu } t>-\frac{z}{2} \end{cases}$$

$$\to u_2(x,y,t) = \begin{cases} t & \text{nếu } 0<t<-\frac{y}{2},\ y<0 \\ \frac{2t-y}{4} & \text{nếu } t>-\frac{y}{2},\ y<0 \end{cases}$$

+) TH3: Nếu (x,y,z) TM $z=0$

$\to u_2(x,y,z,t) = \frac{t}{2} \Rightarrow u_2(x,y,t) = \frac{t}{2}$

Tóm lại

$$u_2(x,y,t) = \begin{cases} 0 & \text{nếu } 0<t<\frac{y}{2},\ y>0 \\ \frac{2t-y}{4} & \text{nếu } t>\frac{y}{2},\ y>0 \\ \frac{t}{2} & \text{nếu } y=0 \\ t & \text{nếu } 0<t<-\frac{y}{2},\ y<0 \\ \frac{2t-y}{4} & \text{nếu } t>-\frac{y}{2},\ y<0 \end{cases}$$

$\to u(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$ Với $u_1(x,y,t)$ và $u_2(x,y,t)$ xác định như trên.