

Bài tập cuối 12

Bài 68

$$\int U_t = 9U_{xx}$$

$$U(x,0) = \sin 2x \cos 4x = \varphi(x) \quad x \in \mathbb{R}$$

Ngheem của bài toán có dạng

$$u(x,t) = \frac{1}{6\sqrt{\pi t}} \int_{-\infty}^{+\infty} \sin 2y \cos 4y e^{-\frac{(x-y)^2}{36t}} dy$$

$$= \frac{1}{12\sqrt{\pi t}} \int_{-\infty}^{+\infty} (\sin 6y - \sin 2y) e^{-\frac{(x-y)^2}{36t}} dy$$

$$\Rightarrow \text{Đặt } x-y=k \Rightarrow dy = -dk$$

$$y = -\infty \Rightarrow k = +\infty, \quad y = +\infty \Rightarrow k = -\infty$$

$$\Rightarrow u(x,t) = \frac{1}{12\sqrt{\pi t}} \int_{+\infty}^{-\infty} [\sin 6(x-k) - \sin 2(x-k)] e^{-\frac{k^2}{36t}} dk$$

$$= \frac{1}{12\sqrt{\pi t}} \int_{-\infty}^{+\infty} (\sin 6x \cos 6k - \cos 6x \sin 6k - \sin 2x \cos 2k + \sin 2k \cos 2x) e^{-\frac{k^2}{36t}} dk$$

Ta có các hàm $f_1(k) = \sin 6x \cos 6k e^{-\frac{k^2}{36t}}$ } là các hàm chẵn theo k
 $f_2(k) = -\sin 2x \cos 2k e^{-\frac{k^2}{36t}}$
 $g_1(k) = \cos 6x \sin 6k e^{-\frac{k^2}{36t}}$ } là các hàm lẻ theo k
 $g_2(k) = \sin 2k \cos 2x e^{-\frac{k^2}{36t}}$

$$\Rightarrow u(x,t) = \frac{1}{12\sqrt{\pi t}} \Rightarrow \int_{-\infty}^{+\infty} (f_1(k) + f_2(k)) dk = 2 \int_0^{+\infty} (f_1(k) + f_2(k)) dk$$

$$\int_{-\infty}^{+\infty} (g_1(k) + g_2(k)) dk = 0$$

$$\Rightarrow u(x,t) = \frac{1}{6\sqrt{\pi t}} \int_0^{+\infty} (\sin 6x \cos 6k - \sin 2x \cos 2k) e^{-\frac{k^2}{36t}} dk$$

$$= \frac{1}{6\sqrt{\pi t}} \left(\sin 6x \int_0^{+\infty} \cos 6k e^{-\frac{k^2}{36t}} dk - \sin 2x \int_0^{+\infty} \cos 2k e^{-\frac{k^2}{36t}} dk \right)$$

$$= \frac{1}{6\sqrt{\pi t}} \left(\frac{\sin 6x \sqrt{\pi}}{2} e^{-\frac{1}{4 \cdot \frac{1}{36t}}} - \frac{\sin 2x \sqrt{\pi}}{2} e^{-\frac{1}{4 \cdot \frac{1}{36t}}} \right)$$

$$= \frac{\sin 6x \sqrt{\pi}}{6\sqrt{\pi t}} - \frac{\sin 2x \sqrt{\pi}}{6\sqrt{\pi t}}$$

$$\rightarrow u(x,t) = \frac{1}{2} (\sin 6x e^{-320t} - \sin 2x e^{-36t})$$

Bài 70

$$\begin{cases} u_t = 4u_{xx}, & x \geq 0 \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R} \end{cases}$$

Ngheem của bài toán có dạng

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-y^2} e^{-\frac{(x-y)^2}{4t}} dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2 - 2xy + y^2}{4t}} dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{4t} + \frac{2xy}{4t} - \frac{y^2}{4t}} dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{4t} + \frac{x^2}{4t(16t+1)} - \frac{x^2}{4t(16t+1)} + \frac{2x \cdot y\sqrt{16t+1}}{\sqrt{16t}\sqrt{16t+1}} - \frac{y^2(16t+1)}{4t}} dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{4t(16t+1)} - \left(\frac{x}{\sqrt{16t}\sqrt{16t+1}} - \frac{y\sqrt{16t+1}}{\sqrt{16t}} \right)^2} dy$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t(16t+1)}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{16t}\sqrt{16t+1}} - \frac{y\sqrt{16t+1}}{\sqrt{16t}} \right)^2} dy$$

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t(16t+1)}} \text{ đặt } \frac{x}{\sqrt{16t}\sqrt{16t+1}} - \frac{y\sqrt{16t+1}}{\sqrt{16t}} = k$$

$$\rightarrow dy = -\frac{\sqrt{16t}}{\sqrt{16t+1}} dk$$

$$\rightarrow u(x,t) = \frac{1}{4\sqrt{\pi t}} e^{-\frac{x^2}{16t+1}} \frac{4\sqrt{t}}{\sqrt{16t+1}} \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi$$

$$= \frac{2}{\sqrt{\pi}\sqrt{16t+1}} e^{-\frac{x^2}{16t+1}} \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{16t+1}} e^{-\frac{x^2}{16t+1}} \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{16t+1}} e^{-\frac{x^2}{16t+1}}$$

$$\rightarrow u(x,t) = \frac{1}{\sqrt{16t+1}} e^{-\frac{x^2}{16t+1}}$$

Bài 73

$$\begin{cases} u_t = 4u_{xx} & t \geq 0 \\ u(x,0) = x^2 - 2x + 3 & x \in \mathbb{R} \end{cases}$$

$$u(x,0) = x^2 - 2x + 3 \quad x \in \mathbb{R}$$

Ngheem của bài toán có dạng

$$u(x,t) = \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{+\infty} (y^2 - 2y + 3) e^{-\frac{(x-y)^2}{16t}} dy$$