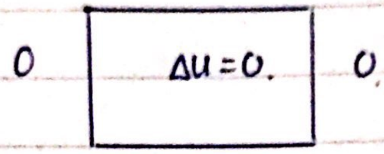
Bài Tập Về Nhà lần 16.

Sách Pinchover - Rubinstein, Trang 204 - 205.

7.4

$$\begin{cases} \Delta u = 0, & 0 < x, y < \pi \\ u(x,0) = u(x,\pi) = 1 & 0 \le x \le \pi \\ u(0,y) = u(\pi,y) = 0 & 0 \le y \le \pi \end{cases}$$

$\varphi_1(x) = u(x,\pi) = 1$



$\varphi_0(x) = u(x,0) = 1$

Từ $\begin{cases} u_{xx} + u_{tt} = \Delta u = 0 & 0 < x, y < \pi \\ \cancel{u(x,0) = u(x,\pi)} = u(0,y) = u(\pi,y) = 0 & 0 \le y \le \pi \end{cases}$

→ Ta có bài toán Sturm-Liouville.

$$\begin{cases} X'' - \text{const}\, X = 0 \\ X(0) = X(\pi) = 0 \end{cases}$$

→ $\text{Const} = -n^2 \qquad n = 1, 2, 3, \dots$

$X_n(x) = \sin(nx) \qquad n = 1, 2, 3, \dots$

$Y_n(y) = a_n\, sh(ny) + b_n\, sh(n(\pi - y))$

→ Chuỗi nghiệm của bài toán có dạng

$$u(x,y) = \sum_{n=1}^{\infty} \sin(nx)\left(a_n\, sh(ny) + b_n\, sh(n(\pi - y))\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} \sin(nx) . b_n\, sh(n\pi) = 1$$

$$\rightarrow \quad b_n = \frac{2}{\pi\, sh(n\pi)} \int_0^{\pi} \sin(nx) dx.$$

$$= \frac{-2}{n\pi\, sh(n\pi)} \cos(nx) \Big|_0^{\pi}$$

$$= \frac{2}{n\pi\, sh(n\pi)} \left(1 - \cos(n\pi)\right)$$

$$u(x,\pi) = \sum_{n=1}^{\infty} \sin(nx)\, a_n \sinh(n\pi) = 1$$

$$\to a_n = \frac{2}{\pi\, sh(n\pi)} \int_0^{\pi} \sin(nx)\,dx = \frac{2}{n\pi\, sh(n\pi)} \left(1 - \cos(n\pi)\right)$$

$$\to u(x,y) = \sum_{n=1}^{\infty} \frac{2\sin(nx)}{n\pi\, sh(n\pi)} \left(1 - \cos(n\pi)\right)\left(sh(ny) + sh(n(\pi - y))\right)$$

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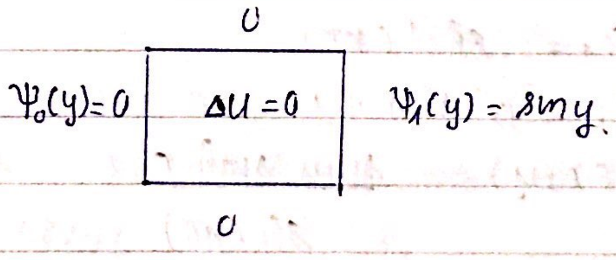
$$\begin{cases} \Delta u = 0 & 0 < x < \pi,\ 0 < y < \pi \\ u(x,0) = u(x,\pi) = 0 & 0 \le x \le \pi \\ u(0,y) = 0 & 0 \le y \le \pi \\ u(\pi,y) = \sin y & 0 \le y \le \pi \end{cases}$$



Từ $\begin{cases} \Delta u = 0 & 0 < x, y < \pi \\ u(x,0) = u(x,\pi) = 0 & 0 < x \le \pi \end{cases}$

→ Ta có bài toán Sturm - Liouville.

$$\begin{cases} Y'' + Const\, Y = 0 \\ Y(0) = Y(\pi) = 0 \end{cases}$$

$\rightarrow Const = n^2 \quad n = 1, 2, 3, \ldots$

$Y_n(y) = \sin(ny) \quad n = 1, 2, 3, \ldots$

$X_n(x) = a_n\, sh(nx) + b_n\, sh(n(\pi - x)) \quad n = 1, 2, 3, \ldots$

$\rightarrow$ Chuỗi nghiệm

$$u(x;y) = \sum_{n=1}^{\infty} \sin(ny)\left(a_n\, sh(nx) + b_n\, sh(n(\pi - x))\right)$$

$$u(0,y) = \sum_{n=1}^{\infty} \sin(ny)\, b_n\, sh(n\pi) = 0$$

$\rightarrow b_n = 0 \ \forall n = 1, 2, 3, \ldots$

$$u(\pi,y) = \sum_{n=1}^{\infty} \sin(ny).a_n\, sh(n\pi) = \sin y$$

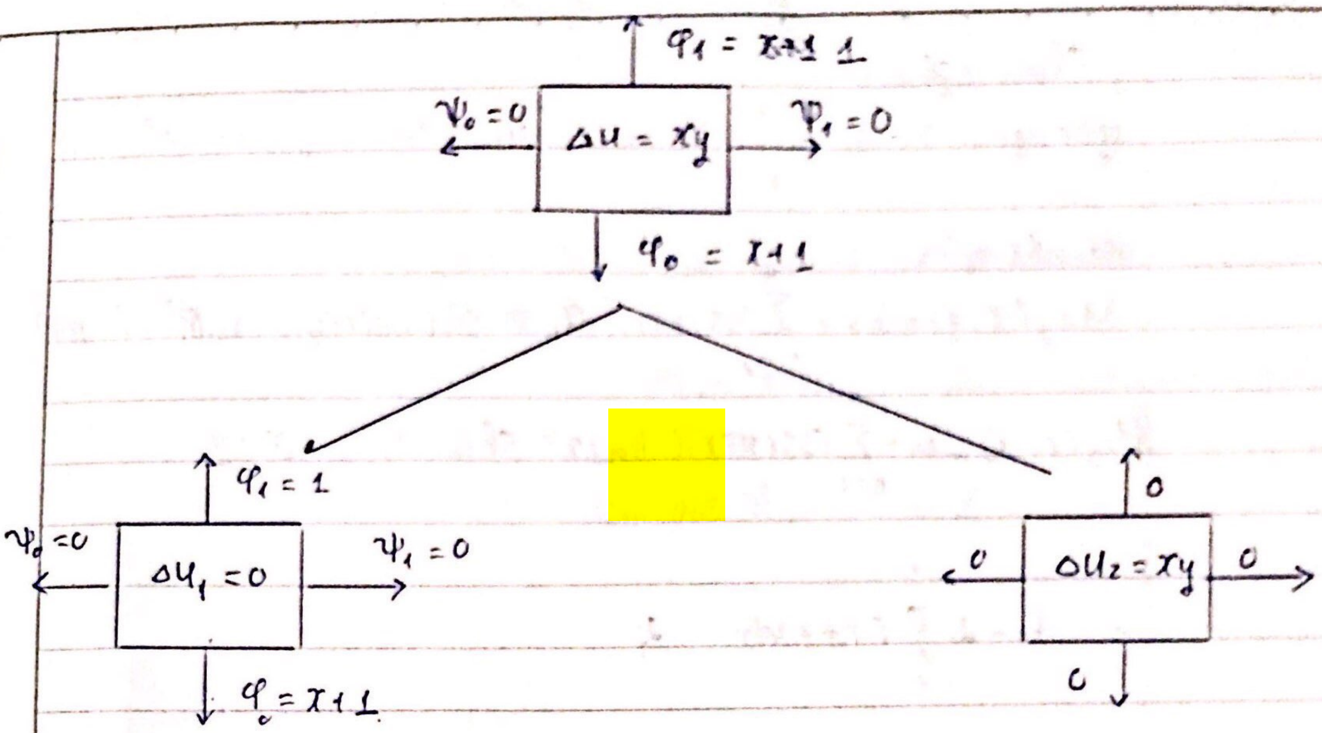
$\rightarrow$ Đồng nhất hệ số, ta được

$$\begin{cases} a_1 = sh^{-1}(n\pi) \\ a_n = 0 \ \forall n \neq 1 \end{cases}$$

$$\rightarrow u(x,y) = \frac{\sin y \sinh x}{sh(n\pi)}$$

Câu 2. ý (a) Đề 4 - GK - K63TH

$$\begin{cases} u_{xx} + u_{yy} = xy & -1 < x, y < 1 \\ u_x(-1,y) = u_x(1,y) = 0 & -1 \leq y \leq 1 \\ u_y(x,-1) = x+1, \ u_y(x,1) = 1 & -1 \leq x \leq 1 \end{cases}$$



Ta chia bài toán ban đầu thành 2 bài toán như sau khi đó $u = u_1 + u_2$

+) Xét bài toán 1

$$\begin{cases} \Delta u_1 = 0 \qquad -1 < x, y < 1 & (1) \\ u_{1x}(-1,y) = u_{1x}(1,y) = 0 \qquad -1 \le y \le 1 & (2) \\ u_{1y}(x,-1) = x+1,\ u_{1y}(x,1) = 1 \qquad -1 \le x \le 1 & (3) \end{cases}$$

Ta xác định cơ sở của không gian nghiệm của bài toán {(1), (2)} gồm các phần dạng $u_1(x,y) = X(x)Y(y)$

Do $\Delta u_1 = 0 \Rightarrow \dfrac{X''}{X} = \dfrac{-Y''}{Y} = Const$

$u_{1x}(-1,y) = X'(-1)Y(y) = 0 \to X'(-1) = 0$

$u_{1x}(1,y) = X'(1)Y(y) = 0 \to X(1) = 0$

→ BT Sturm - Liouville. $\begin{cases} X'' - Const\, X = 0 \\ \cancel{X'(0)}\ X'(-1) = X'(1) = 0 \end{cases}$

+) $Const = 0 \Rightarrow X_0(x) = 1 \to Y_0(y) = a_0 + b_0 y$.

+) $Const = -(n\pi)^2 \to X_n(x) = \cos(n\pi x)$ $\quad n = 1, 2, 3, \ldots$

→ $\dfrac{-Y''}{Y} = -(n\pi)^2 \to Y_n = a_n \operatorname{ch}(n\pi(1+y)) + b_n \operatorname{ch}(n\pi(1-y))$, $n = 1, 2, 3$.

$\rightarrow$ ~~$u_1(x,y) =$~~

$\rightarrow$ Chuỗi nghiệm

$$u_1(x,y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \cos(n\pi x)[a_n ch(n\pi(1+y)) + b_n ch(n\pi(1-y)))$$

$$\rightarrow u_{1y}(x,y) = b_0 + \sum_{n=1}^{\infty} \cos(n\pi x)[a_n n\pi sh(n\pi(1+y)) - b_n n\pi sh(n\pi(1-y)))$$

$$u_{1y}(x,1) = b_0 + \sum_{n=1}^{\infty} \cos(n\pi x) a_n n\pi sh(2n\pi) = 1$$

$$\rightarrow \begin{cases} b_0 = 1 \\ a_n = 0 \quad \forall n = 1,2,3,\dots \end{cases}$$

$$\rightarrow u_{1y}(x,-1) = 1 + \sum_{n=1}^{\infty} -\cos(n\pi x) b_n n\pi sh(2n\pi) = x+1$$

$$\rightarrow \sum_{n=1}^{\infty} \cos(n\pi x) - b_n n\pi sh(2n\pi) = x$$

$$\rightarrow b_n = \frac{-1}{n\pi sh(2n\pi)} \int_{-1}^{1} x\cos(n\pi x)\,dx = 0.$$

(Do $f(x) = x\cos(n\pi x)$ là hàm lẻ.)

$\rightarrow u_1(x,y) = a_0 + y$

+) Xét bài toán 2.

$$\begin{cases} \Delta u_2 = a\,xy \\ u_{2x}(-1,y) = u_{2x}(1,y) = 0 \\ u_{2y}(x,-1) = u_{2y}(x,1) = 0 \end{cases}$$

$\rightarrow$ Ta có bài toán Sturm-Liouville
$$\begin{cases} \Delta u_2 = const\, u_2 \\ X''Y + XY'' = const\, XY \\ X'(-1) = X'(1) = 0 \\ Y'(-1) = Y'(1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X'' - \lambda_1 X = 0, & X'(-1) = X'(1) = 0 \\ Y'' - \lambda_2 Y = 0, & Y'(-1) = Y'(1) = 0 \end{cases}$$

$\lambda_{10} = 0, \quad X_0(x) = 1$

$\lambda_{1n} = -(n\pi)^2, \quad X_n(x) = \cos(n\pi x) \quad n = 1, 2, 3, \ldots$

$\lambda_{20} = 0, \quad Y_0(y) = 1$

$\lambda_{2m} = -(m\pi)^2, \quad Y_n(y) = \cos(m\pi y) \quad n = 1, 2, 3, \ldots$

$$\Rightarrow Const = \lambda_1 + \lambda_2 = \begin{cases} 0, & n = 0 \\ -(n^2 + m^2)\pi^2 & n = 1, 2, 3 \ldots \end{cases}$$

$\Rightarrow$ Chuỗi nghiệm

$$u_2(x,y) = (a_0 + b_0 x) + (c_0 + d_0 y) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \cos(n\pi x) . \cos(m\pi y)$$