FINITE ELEMENT METHODS

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CONTENTS :

MOTIVATION

- Finite Element Methods (FEM) is used to solve problems in Physics and Continuum Mechanics governed by boundary value differential equations.
- In order to understand the mathematics behind FEM, we need to first understand Functional Analysis.
- One can do this by taking a model problem and show different possible mathematical formulations.
- We shall see *Sturm-Liouville* problem to understand how FEM is used to solve such a problem.

THE STURM-LIOUVILLE PROBLEM

- Let us consider elongational deformations of a string Ω of length L. One of its end is fixed and a force f(x) is acting along its length where 0 ≤ x ≤ L. The resulting length variation is denoted by u(x) at a point x. u(x) is described below (using the standard Sturm-Liouville problem)

$$\begin{cases} -(pu')' + qu = f & \text{in } (0, L) \\ u(0) = 0 & \text{(fixed left end)} \\ u'(L) = 0 & \text{(free right end)} \end{cases} \qquad (\mathrm{P}_1)$$

ASSUMPTIONS IN STURM-LIOUVILLE PROBLEM

- p and q are functions that represent physical characteristics of the material which the string is made of. We shall assume that p and q satisfy respectively for certain constants α, β, A and B :

$$A \geq p(x) \geq \alpha > 0 \text{ and } B \geq q(x) \geq \beta \geq 0, \ \forall x \in [0, L]$$

- We shall also assume that p is continuous in (0, L) and also differentiable there, except eventually, at a certain number of points, and that p' is also bounded, that is, there exists C > 0 such that $|p'(x)| \leq C$ at every point $x \in [0, L]$ where $p'(x)$ is well defined.
- We further assume that all the operations performed for this purpose are feasible and well defined.

SIMPLIFYING STURM-LIOUVILLE PROBLEM

- Let us then multiply both sides of the differential equation with a ”test function” v, and then integrate over the interval (0, L). Using integration by parts we obtain:

$$-pu'v|_0^L + \int_0^L pu'v'\,dx + \int_0^L quv\,dx = \int_0^L fv\,dx$$

- The expression ”test function” means that v is supposed to behave somehow like the solution u, in the sense that u itself could be one of such functions. We shall simply say that v sweeps a function space V

VARIATIONAL FORM OF STURM-LIOUVILLE PROBLEM

- Since u'(L) = 0,

$$\int_0^L pu'v'dx + \int_0^L quvdx = \int_0^L fvdx, \ \forall v \in V \qquad (\mathrm{P}_2)$$

- f together with every function in V and its first order derivative are square integrable in the interval (0, L). Thus we define V:

$$V = \{v \,/ v, v' \in \mathcal{L}^2(0, L), v(0) = 0\}$$

where $L^2(0, L) = \{f \,/ f^2$ has a finite (Lebesgue) integral in $(0, L)\}$

- We have written the Sturm-Liouville problem (P_1) in the variational form (P_2).

ADVANTAGES OF VARIATIONAL FORM (P_2)

- Highest derivative order reduces from 2 to 1.
- $u'(L) = 0$ is used in (P_2) but not in (P_1).
- As p is material dependent function, it can be discontinuous, thus it can cause problems in (P_1) but not in (P_2).
- (P_2) is more generalized than (P_1) .
- We can simplify (P_2) using some assumptions and concepts that will later be explained to understand that the choice of space V relies on rigorous mathematics concepts from functional analysis. Thus first we need to understand functional analysis.

INTRODUCTION TO FUNCTIONAL ANALYSIS

- Functional Analysis is a branch of mathematical analysis which studies transformations of functions and their algebraic properties.
- Functional Analysis has strong parallels with Linear Algebra
- We will see mathematical concepts that will help us find the solution for (P_2) using FEM.

NORMED FUNCTION SPACES

- Let V be a real vector space with null vector $\vec{0}_V$. V is said to be a normed space if it is equipped with a mapping $\|\cdot\|: V \to R$ having the following properties:

  (i). $\|u\| \geq 0, \forall\ u \in V$ and $\|u\| = 0$ if and only if $u = \vec{0}_V$

  (ii). $\|\alpha u\| = |\alpha| \|u\|, \forall \alpha \in R, \forall\ u \in V$

  (iii). $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|, \forall\ u_1, u_2 \in V$

EXAMPLES OF NORMED FUNCTION SPACES:

- Following norms for the space $C^0[0, L]$ of continuous functions in the closed interval [0, L]:

$$\|v\|_{0,\infty} \stackrel{def}{=} \max_{x\in[0,L]} |v(x)| \qquad \text{(Maximum norm or } \mathcal{L}^{\infty}\text{-norm)}$$

$$\|v\|_{0,1} \stackrel{def}{=} \int_0^L |v(x)|\, dx \qquad (\mathcal{L}^1\text{-norm})$$

$$\|v\|_{0,2} \stackrel{def}{=} \left[\int_0^L |v(x)|^2\, dx\right]^{\frac{1}{2}} \qquad (\mathcal{L}^2\text{-norm})$$

INNER PRODUCT SPACES

- We shall be particularly concerned about a class of normed spaces whose norm is defined through a given inner product. We recall that a real vector space V is said to be equipped with an inner product $(\cdot|\cdot)$, if this is a mapping from $V \times V$ onto R that satisfies the following properties:

(i). $(u|u) \geq 0$, $\forall\ u \in V$ and $(u|u) = 0$ if and only if $u = \vec{0}_V$

(ii). $(v|u) = (u|v)$, $\forall u, v \in V$

(iii). $(\alpha_1 u_1 + \alpha_2 u_2 | v) = \alpha_1 (u_1 | v) + \alpha_2 (u_2 | v)$, $\forall \alpha_1, \alpha_2 \in R$ and $\forall\ u_1, u_2, v \in V$

- Example -

$$\|v\| \stackrel{def}{=} (v|v)^{\frac{1}{2}}, \forall v \in V$$

IMPORTANT LEMMAS AND DEFINITIONS:

Lemma 2.2 (The Friedrichs-Poincaré inequality) *There exists a constant C_p such that*

$$\|v\|_{0,2} \leq C_p \, |v|_{1,2} \, , \forall v \in V$$

Remark 2.3 *Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there exist two strictly positive constants C_1 and C_2 such that for every v in the vector space V we have*

$$C_1 \|v\|_a \leq \|v\|_b \leq C_2 \|v\|_a$$

Definition 2.4 *Let V be a real vector space equipped with inner product $(\cdot|\cdot)$. Two vectors u and v belonging to V are said to be orthogonal if $(u|v) = 0$. In this case we write $u \perp v$.*

Definition 2.5 *Let S be a subset of a real vector space V equipped with an inner product $(\cdot|\cdot)$. The orthogonal of S denoted by $S^{\perp}$ is the set of elements in V which are orthogonal to every element of S.*

Proposition 2.7 *Let v_0 be a given element of V, $v_0 \neq \vec{0}_V$. Then $\forall v \in V$ there exists a unique splitting of the form $v = \alpha v_0 + u$ where $u \in v_0^{\perp}$, $\alpha \in \mathbb{R}$.*

CLOSED SETS IN NORMED SPACES

- Let S: N→V be a given sequence (v_1, v_2, ..., v_n, ...) of elements of V also denoted by $\{v_n\}_n$ or simply $\{v_n\}$.
- A **set** A in a **normed** linear **space** is **closed** if it contains all its boundary points. To understand this completely we need to through some definitions.

Definition 2.8 *A sequence* $\{v_n\}$ *contained in a normed real space* V *is said to converge to an element* $v \in V$ *if* $\forall \varepsilon > 0$ *there exists an integer* $N(\varepsilon) \in \mathbb{N}$ *such that* $\|v_n - v\| < \varepsilon \ \forall n > N(\varepsilon)$. *In this case we write* $v_n \to v$ *in* V.

Definition 2.9 *A subset S of a normed real vector space V is said to be a* ***closed set*** *if every sequence $\{v_n\} \subset S$ that converges to an element $v \in V$ is such that $v \in S$.*

Proposition 2.10 $\forall S \subset V$, $S^{\perp}$ *is a closed subspace of V.*

Definition 2.11 *Let S be a subset of a normed real vector space V. The* ***closure*** *of S denoted by $\bar{S}$ is the set of all the elements in V that are limits of a sequence of elements contained in S.*

Definition 2.12 *A subset D of a normed vector space V is said to be dense in another subset C of V, if $D \subset C$ and $\forall c \in C$ and $\forall \varepsilon > 0$ there exists an element $d \in D$ such that $\|c - d\| < \varepsilon$.*

COMPLETE NORMED SPACES

- The concept of convergence in normed vector spaces implicitly involves the limit of sequences. However, a converging sequence also fits the following one:

Definition 2.13 *A sequence* $\{v_n\}$ *of a normed space* V *is a* ***Cauchy sequence*** *if* $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ *such that* $\|v_m - v_n\| < \varepsilon$, $\forall m, m > N(\varepsilon)$.

- If every Cauchy sequence converges in a normed space, then it is a complete normed space.

Definition 2.15 *A complete normed space, whose norm is associated with an inner product is called a* ***Hilbert space****.*

Definition 2.16 *A complete normed space whose norm is not associated with an inner product is called a* ***Banach space****.*

Proposition 2.20 *If a normed vector space V is complete, then a subspace W of V is complete if and only if it is closed.*

Proposition 2.21 *If a subspace W of a complete normed space V is dense in V, then the latter is the completion of W with respect to the given norm.* ■

ORTHOGONAL PROJECTION

- When the vector space has an inner product and is complete (is a Hilbert space) the concept of orthogonality can be used. An **orthogonal projection** is a **projection** for which the range and the null space are **orthogonal** subspaces.
- To solve the problem (P_2) we need to have an approximation for u such that:

$$\|u - u_W\| \leq \|u - w\|, \forall w \in W, \text{ where } u_W \in W. \tag{2.2}$$

Theorem 2.22 (The Classical Theorem of the Orthogonal Projection) *If V is a Hilbert space and W is a closed subspace of V for every $u \in V$ there exists a unique $u_W \in W$ that minimizes $\|u - v\|$ over W.*

BOUNDED LINEAR FUNCTIONALS

Definition 2.23 *A functional $F : V \to \mathbb{R}$, where V is a normed vector space is bounded if*

$$\exists \mathcal{C} > 0 \textit{ such that } |F(v)| \leq \mathcal{C} \|v\|, \forall v \in V$$

Proposition 2.24 *A linear functional F on a normed vector space V is continuous at every $v \in V$ if and only if it is bounded.*

Theorem 2.26 (The Riesz Representation Theorem) *Let V be a Hilbert space for inner product $(\cdot|\,\cdot)$ and $F : V \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique $v_F \in V$ such that $F(v) = (v_F|\, v)\,, \forall v \in V$ and moreover*

$$\sup_{\substack{v \in V \\ v \neq \vec{0}_V}} \frac{F(v)}{\|v\|} = \|v_F\|\,.$$

INTRODUCTION TO FINITE ELEMENTS METHODS

- FEM is an integral part of design and development of many engineering systems.
- Most of the systems that occur in nature, such as solid mechanics, fluid mechanics, are represented using gradients. Along with such differential equations, we have additional constraints on it called the boundary conditions (BC).
- These Boundary conditions with the differential equations are called boundary value problems.
- So FEM is used for approximation for such boundary value problems. FEM converts boundary value problems to linear system of equations and simplifies the problem.

ABSTRACT LINEAR VARIATIONAL PROBLEM

- Let us start with an abstract linear variational problem. Let V be a vector space equipped with inner product $(\cdot|\ \cdot)$ and $a : V \times V \to R$ and $F : V \to R$ be two mappings satisfying the following assumptions:

Assumption 1

(a). *a is bilinear, i.e., linear with respect to each argument;*

(b). *a is bounded:* $\exists M > 0$ *such that* $a(u,v) \leq M \|u\| \|v\|, \forall u, v \in V$;

(c). *a is symmetric:* $a(u,v) = a(v,u)$, $\forall u, v \in V$;

(d). *a is coercive:* $\exists \mu > 0$ *such that* $a(v,v) \geq \mu \|v\|^2, \forall v \in V$.

Assumption 2

(e). *F is linear;*

(f). *F is bounded.*

SOLUTION TO THE LINEAR VARIATIONAL PROBLEM

- Consider the given problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = F(v), \forall \in V \end{cases} \tag{3.1}$$

- We need to find an approximation for u. Let W ⊂ V, we wish to solve the following:

$$\begin{cases} \text{Find } u_W \in W \text{ such that} \\ a(u_W,w) = F(w), \forall w \in W \end{cases} \tag{3.2}$$

THEOREMS USED TO SOLVE 3.1 AND 3.2

- Theorem 3.1 : Both problems (3.1) and (3.2) have a unique solution provided V is a Hilbert space equipped with inner product $(\cdot|\ \cdot)$ and W is closed. Moreover in this case the following "error" bound applies:

$$\|u - u_W\| \leq \sqrt{\frac{M}{\mu}} \inf_{w \in W} \|u - w\| \,. \qquad (3.3)$$

where $M = \max(A,B)$, where $A = ||p||_{0,\infty}$, $B = ||q||_{0,\infty}$ and $\mu = \alpha C_1^2$,

$C_1 = (1 + C_p^{\ 2})^{-0.5}$ where C_p is the constant from Friedrichs-Poincaré inequality

- Theorem 3.2 : Let $E(v) = 0.5 * a(v, v) - F(v)$. If V is a Hilbert space for inner product $(\cdot|\ \cdot)$, there exists a unique $u \in V$ that minimizes E over V, that is, the solution of

$$\begin{cases} Find\ u \in V\ \ such\ that \\ E(u) \leq E(v), \forall v \in V \end{cases} \qquad (3.4)$$

Moreover u is precisely the solution of (3.1).

REALIZING (P_2) AS 3.1

- Writing (P_2) as 3.1,

$$a(u,v) = \int_0^L pu'v'dx + \int_0^L quvdx \qquad (3.5)$$

- V is a closed subspace of $H^1(0, L)$ equipped with the $||\cdot||_{1,2}$ norm. $H^1(0, L)$ is Sobolev space consisting of functions v such that $v(0) = 0$ where $H^1(0, L) = \{v / v, v' \in L^2(0, L)\}$. Hence V is Hilbert space.

- There exists a constant C such that,

$$F(v) \leq \mathcal{C} \left\| v \right\|_{1,2}, \forall v \in V \qquad (3.6)$$

FINDING THE SOLUTION FOR (P_2)

- We see all assumptions are fulfilled by the function *a* in 3.5. Also according to 3.1, now (P_2) has a unique solution. Also the solution u ∈ V is the one that minimizes the total energy of the string under the action of the forces corresponding to functional F that is

$$E(u) \leq E(v), \forall v \in V$$

with

$$E(v) = \frac{1}{2}\left[\int_0^L p(v')^2 dx + \int_0^L qv^2 dx\right] - F(v).$$

RITZ APPROXIMATION METHOD

- Let W = V_N, where N is the dimension of V_N , then we have the following approximation problem:

$$\begin{cases} \text{Find } u_N \in V_N \text{ such that} \\ a(u_N, v) = F(v), \forall v \in V_N \end{cases} \tag{3.7}$$

- According to Theorem 3.1, 3.7 has unique solution.

Moreover, if $\{\varphi_i\}_{i=1}^N$ is a basis of V_N then there exists a unique set of N constants $c_1, c_2, \ldots, c_N$ such that $u_N = \sum_{j=1}^{n} c_j \varphi_j$.

RITZ APPROXIMATION METHOD

- Since $v \in V_N$, 3.7 becomes

$$a\left(\sum_{j=1}^{N} c_j \varphi_j, \varphi_i\right) = F(\varphi_i) \text{ for } i = 1, 2, \ldots, N,$$

or yet by linearity:

$$\sum_{j=1}^{N} c_j a(\varphi_j, \varphi_i) = F(\varphi_i) \text{ for } i = 1, 2, \ldots, N, \tag{3.8}$$

This is nothing but the linear system of equations

$$A\vec{c} = \vec{f} \tag{3.9}$$

- Solving 3.7 is equivalent to inverting matrix in 3.9. As matrix A is symmetric and PSD, we can easily solve it. However, the choice of basis is the key here. Main concerns regarding choice of basis are computational complexity and error control.

PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS

- First, we subdivide the interval [0, L] into N equal intervals (or "elements") of length h = L/N. Then we associate with every interval end (or "node") $x_i = ih$, a function φ_i having the following properties for i = 0, 1, . . . , N :

1. φ_i restricted to any interval (x_{j-1}, x_j) is a polynomial of degree less than or equal to one, $j = 1, 2, \ldots, N$.

2. $\varphi_i(x_j) = \delta_{ij}$.

PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS

- The set $\{\varphi_i\}$ forms a basis for following space.

$$V_h = \left\{ v \Big/ v \in C^0[0,L], v(0) = 0,\ v|_{[x_{i-1},x_i]} \in P_1,\ \text{ for } i = 1,2,\ldots,N \right\}$$

where P_k denotes the space of polynomials (in one variable) of degree less than or equal to k.

- Now we consider the Ritz method here and consider V_N as V_h. This gives the following approximation of (P_2):

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v) = F(v), \forall v \in V_h \end{cases} \tag{3.10}$$

PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS

Clearly problem (3.10) is of the form (3.2) with $W = V_h$. Hence according to Theorem 3.1 it has a unique solution, and moreover we have:

$$\|u - u_h\|_{1,2} \leq \sqrt{\frac{M}{\mu}} \inf_{v \in V_h} \|u - v\|_{1,2} \,. \tag{3.11}$$

- So to find the infimum in the inequality, we can use the theorem of orthogonal projections. So the infimum is the orthogonal projection onto V_h of exact solution u. After solving we get the following bound:

$$\|u - u_h\|_{0,2} \leq \tilde{C} M \sqrt{\frac{M}{\mu}} \frac{h^2}{\pi^2} \left(1 + \frac{h^2}{\pi^2}\right) |u|_{2,2}$$

$\tilde{C}$ is the constant of inequality.

FINAL SOLUTION FROM THE METHOD

u tends to u_h as h tends to 0 and we have our final inequality as,

$$\left\| u - u_h \right\|_{0,2} \leq C_0 h \left| u \right|_{1,2} . \tag{3.15}$$

Where C_0 is a constant.

HIGHER ORDER METHODS

- There are some other FEM that are applied to variants of (P_1):

For $0 < \alpha \leq p \leq A$ and $0 \leq q \leq B$:

$$\begin{cases} -(pu')' + qu = f \\ u(0) = u(L) = 0 \end{cases} \quad \textbf{(The Dirichlet problem)} \tag{3.17}$$

For $0 < \alpha \leq p \leq A$ and $0 \leq \beta \leq q \leq B$:

$$\begin{cases} -(pu')' + qu = f \\ (pu')(0) = (pu')(L) = 0 \end{cases} \quad \textbf{(The Neumann problem)} \tag{3.18}$$

EXTENSION TO MULTI-DIMENSIONAL PROBLEMS

- We can apply FEM to multi-dimensional problems also using several mathematical concepts like Green's formulae, Lagrange Finite Element Methods, etc.
- These all methods involve dealing with partial derivatives and integrating small units/elements with curved planes.
- These methods are comparable with linear dimensional methods.

CONCLUSION

- Overall we started with the *Sturm-Liouville* problem and then simplified it. Then to further approximate the solution, we first looked at basic concepts from Functional Analysis. Then we saw Finite Element Methods and ways to use FEM to approximate solutions. Finally we came up with a solution by converting approximation problem to linear system of equations.
- It was clearly visible that FEM used many concepts from Functional Analysis like Normed Spaces, Inner Product Spaces, Functionals, etc. Thus to thoroughly understand FEM, we first need to understand the basics of Functional Analysis, which we understood in the course and implemented in this project. Even though this course was online, we still understood the concepts and its application in different fields, one of which is shown in this project.

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THANK YOU !