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2.1

1. $A := \{1,2,3,4\}$ $B = \{2,5\}$

a. $A \cup B := \{1,2,3,4,5\}$ b. $A \cap B = \{2\}$ c. $A - B := \{1,3,4\}$ $|A-B| = 3$

d. $A \Delta B := \{1,3,4,5\}$ $|A \Delta B| = 4$

e. $A \times B := \{(1,2),(1,5),(2,2),(2,5),(3,2),(3,5),(4,2),(4,5)\}$ $|A \times B| = 8$

f. $2^B := \{\emptyset, \{2\}, \{5\}, \{2,5\}\}$. $|2^B| = 4$

2. False

$2^\emptyset = \{\emptyset\} \neq \emptyset$. $2^\emptyset$ contains $\emptyset$ while $\emptyset$ is empty

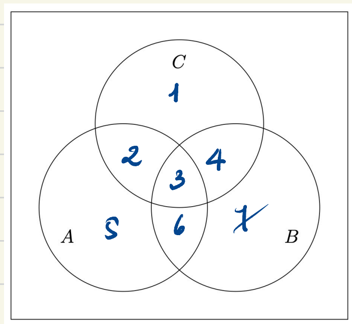
3. $2^{2^{\{\emptyset\}}}$

$2^{\{\emptyset\}} = \{\emptyset, \{\emptyset\}\} \Rightarrow 2^{2^{\{\emptyset\}}} = 2^{\{\emptyset, \{\emptyset\}\}}$

$= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} = \{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\}, \{\{\}\}\}\}$

4.

exp.	shaded region
$A \cap B \cap C$	3
$A \cup B \cup C$	1,2,3,4,5,6,7
$A \cap (B \cup C)$	2,3,6
$A - (B \cap C)$	2,5,6
$(A \cup B) - C$	5,6,7
$A - (B - C)$	2,3,5
$(A \Delta B) \Delta C$	1,3,5,7
$(A \cap B) \cup (A \cap C)$	2,3,6
$A \Delta (B \Delta C)$	1,3,5,7
$A \Delta (B \cap C)$	2,4,5,6



5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$(A \Delta B) \Delta C = A \Delta (B \Delta C)$

6. $A \Delta C = 1,5$ $\Rightarrow$ $A \Delta C = C \Delta A \rightarrow$ commutativity

$C \Delta A = 1,5$

$(A \Delta B) \Delta C = 1,3,5,7 \Rightarrow (A \Delta B) \Delta C = A \Delta (B \Delta C)$

$A \Delta (B \Delta C) = 1,3,5,7$ $\Rightarrow$ associadivity

$A \Delta A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$

2.2 $A := 1,2,3,4$

$R := \{(1,2),(2,3),(3,4)\}$

a. Add $(1,1),(2,2),(3,3),(4,4)$

b. Add $(2,1),(3,2),(4,3)$

c. Add $(1,3),(1,4),(2,4)$

d. Add $(1,1),(2,2),(3,3),(4,4),(2,1),(3,2),(4,3),(1,3),(1,4),(2,4)$

class $= \{1,2,3,4\}$

8. $R := \{(1,2)(2,3), (3,1), (4,4)\}$

a. Add $(1,1), (2,2), (3,3)$

b. Add $(2,1), (3,2), (1,3)$

c. Add $(1,1), (1,3), (2,1), (2,2), (3,2), (3,3)$

d. Add $(1,1), (1,3), (2,1), (2,2), (3,2), (3,3)$

2 classes: $\{1,2,3\}$ & $\{4\}$

9. $\{(2,2), (2,3), (3,2), (3,3)\}$

10. Let $a, b, c \in A$

① Reflexivity: $a \leq a \Rightarrow a \equiv a$

② Symmetry: suppose $a \equiv b \Leftrightarrow a \leq b$ & $b \leq a \Leftrightarrow b \equiv a$

③ Transivity: suppose $a \equiv b$ & $b \equiv c \Leftrightarrow a \leq b$ & $a \leq c$

$c \leq b$ & $b \leq a$

$\Leftrightarrow a \equiv c$

①②③ $\Rightarrow \equiv$ is equivalent relation

2.3 $A := \{1,2,3,4\}$ $B := \{2,5\}$

11. $f: B \to A$ one-to-one

$f(2) = 1$ $\Rightarrow f := \{(2,1), (5,3)\}$

$f(5) = 3$

for the first element of B (2), there are 4 choices to map to A

for the second one, since one element in A is already mapped, there are 3 choices left

$\Rightarrow$ total number of one-to-one function from B to A is $4 \times 3 = 12$

12. onto $g: A \to B$

$g(1) = 2$

$g(2) = 5$ $\Rightarrow g := \{(1,2), (2,5), (3,2), (4,5)\}$

$g(3) = 2$

$g(4) = 5$

Using inclusion-exclusion

$$2^4 - \binom{2}{1} 1^4 = 16 - 2 = 14$$

2.4 6.1.6

1. $|x| \geq 0$

- Base case: Consider empty string ε. $|\varepsilon| = 0$. Since 0 is non-negative, $|\varepsilon| \geq 0$.
- Inductive case: Assume the statement is true for a string x & $|x| = n$, i.e, $|x| \geq 0$
  When we append a character c to $x \to$ new string xc. $|xc| = |x| + 1 \geq 1$
  Hence, if $|x| \geq 0$ then $|xc| \geq 0$. By induction $|x| \geq 0$ holds for all strings x

2. $|xy| = |x| + |y|$

- Base case: Let x be any string & y be empty string ε. Then $x\varepsilon = x$ & $|x\varepsilon| = |x|$
  Since $|\varepsilon| = 0$, $|x| + |\varepsilon| = |x| + 0 = |x|$.

- Inductive case: Assume the statement is true for a string y of length n.
  ie.: $|xy| = |x| + |y|$. Consider appending a character c to $y \rightarrow$ new string yc
  By inductive hypothesis, $|xy| = |x| + |y|$. Then $|x(yc)| = |xy| + 1 = (|x| + |y|) + 1$
  $= |x| + (|y| + 1) = |x| + |yc|$
  By induction, $|xy| = |x| + |y|$ holds for all x & y

3. Right Cancellation. If $xz = yz$ then $x = y$
   - Base case: Let $z = \varepsilon$. If $x\varepsilon = y\varepsilon$, then $x = y$. Because appending empty string to a string does not change the string so the statement holds for the base case.
   - Inductive case: Assume that right cancellation holds for a string z of length n.
     ie: if $xz = yz$ then $x = y$. Consider a string zc where c is a single character.
     If $x(zc) = y(zc)$, by the definition of string equality, we must have $xz = yz$ & also $xc = yc$. By the inductive hypothesis, $xz = yz$ implies $x = y$. Since $x = y$, the added c at the end of both strings does not change the equality, thus $xc = yc$ is trivially true
   - By induction, the right cancellation law holds for all string x, y & z

4. Left Cancellation: If $xy = xz$ then $|y| = |z|$
   - Base case: Let $x = \varepsilon$, y & z are arbitrary strings. If $\varepsilon y = \varepsilon z$ then $y = z$ because any string prefixed by an empty string remained unchanged.

   - Inductive case: Assume that left cancellation holds for a string x length n,
     ie: if $xy = xz$ then $y = z$. Consider a string cx where c is a single character. If $(cx)y = (cx)z$, by the definition of string concatenation & equality, the first character of both strings must be c and the remainder must be equal, i.e: $xy = xz$. By the inductive hypothesis $xy = xz$ implies $y = z$.
     By induction, the left cancellation holds for all string x, y & z.

6.1.7.

1. - Base case: If $x = \varepsilon$, then x^R is also $= \varepsilon$. $\varepsilon^R = \varepsilon$
   - Inductive case: If a string $x \neq \varepsilon$ & $x = ay$ where a is a first char of x, and y is the rest of the string, then the reversal of x, denoted by x^R is defined as $x^R = y^R a$. This means we take the first char of x, reverse the remaining y then append a to the end of y^R

2a $|x^R| = |x|$
   - Base case: $x = \varepsilon \Leftrightarrow |\varepsilon^R| = |\varepsilon| = 0$
   - Inductive case: Suppose the statement is true for any string y such that $|y^R| = |y|$. Let $x = ay$ where a is a single char. By inductive hyp., $|y^R| = y$. Now, $x^R = y^R a$ so $|x^R| = |y^R| + |a|$ $\Rightarrow |x^R| = |y| + 1$. Since $|x| = |y| + 1$, $|x^R| = |x|$
     By induction, $|x^R| = |x|$ holds for any string x

b. $(x^R)^R = x$:
   - Base case: $x = \varepsilon \Leftrightarrow (\varepsilon^R)^R = \varepsilon^R = \varepsilon$
   - Inductive case: Suppose the statement is true for any string y such that $(y^R)^R = y$. Let $x = ay$ where a is a single char. By def. of reversal, $x^R = y^R a$. Consider $(x^R)^R$ which by the inductive case gives $(y^R a)^R$. By the inductive def. of reversal $\Rightarrow a^R (y^R)^R$. Since $a^R = a$, we have $a(y^R)^R$. By the inductive hyp., $(y^R)^R = y \Rightarrow a(y^R)^R = ay = x$. By induction $(x^R)^R = x$ holds for any string x