Introduction to Dynamics and Control

AERSP 304 - Dynamics and Control of Aerospace Systems

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The goals for today lecture are to:

- Review the definition of Frame of References
- Review the definition of DCMs
- Review time derivatives of vectors in different frames of reference

Reference Frames

A *frame* is a language used to convey dynamical information. A frame is a coordinate system defined by an origin and a set of basis vectors. It is used to describe the position, orientation, and motion of objects in space. Different frames can be used to describe the same physical situation from different perspectives.

A dynamical information example is the orientation of a rigid body in space, which can be described using a body-fixed frame and an inertial frame. Position and velocity are other examples of dynamical information that can be conveyed using frames of reference.
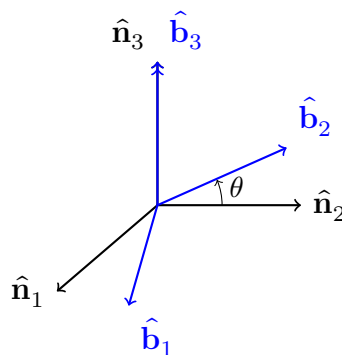
Frame N: $\{0, \hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3\}$ and Frame B: $\left\{0, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\right\}$

Frame N is said to be orthonormal and dextral (right-handed) if its basis vectors satisfy:

$$\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_3$$

Frame Orientation

How is the B oriented w.r.t. the frame N?

We are looking for a language translator to express the basis vectors of frame B in terms of the basis vectors of frame N. This language translator is called the Direction Cosine Matrix (DCM) and is denoted, in this case, by C_{BN}.

But before we jump into the DCMs, let's first understand how to express the coordinates of a vector in frame B in terms of the coordinates of the same vector in frame N. We want to find a relationship (which physical quantity is that?) between the coordinates of a vector expressed in frame N and frame B.

We have the information of frame N. We know how its basis vectors are oriented in space. We want to express the basis vectors of frame B in terms of the basis vectors of frame N.

In other words, if dealing only with unit vectors:

$$\hat{\mathbf{b}}_1 = (\qquad)\,\hat{\mathbf{n}}_1 + (\qquad)\,\hat{\mathbf{n}}_2 + (\qquad)\,\hat{\mathbf{n}}_3$$

Let's look at the 2D sketch of the frames again to get some intuition.
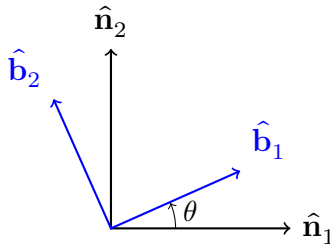
From the sketch, we can see that:

$$\hat{\mathbf{b}}_1 = \cos\theta\,\hat{\mathbf{n}}_1 + \sin\theta\,\hat{\mathbf{n}}_2$$

$$\hat{\mathbf{b}}_2 = -\sin\theta\,\hat{\mathbf{n}}_1 + \cos\theta\,\hat{\mathbf{n}}_2$$

From there, we conclude that the physical quantity that relates the coordinates of a vector expressed in frame N to the coordinates of the same vector expressed in frame B is build on the projection of $\hat{\mathbf{b}}_i$ onto $\hat{\mathbf{n}}_j$.

The mathematical representation of a projection is the dot product:

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{n}}_1 = |\hat{\mathbf{b}}_1|\,|\hat{\mathbf{n}}_1|\,\cos\theta = \cos\theta$$

Direction Cosine Matrix

The Direction Cosine Matrix (DCM) is a matrix that describes the orientation of one frame relative to another. It is constructed using the dot products of the basis vectors of the two frames. The DCM maps coordinates from frame N to frame B and is denoted by C_{BN}.

Let $\hat{\mathbf{r}}$ be a vector expressed in frame N as:

$$\hat{\mathbf{r}} = x_N\,\hat{\mathbf{n}}_1 + y_N\,\hat{\mathbf{n}}_2 + z_N\,\hat{\mathbf{n}}_3$$

and in frame B as:

$$\hat{\mathbf{r}} = x_B\,\hat{\mathbf{b}}_1 + y_B\,\hat{\mathbf{b}}_2 + z_B\,\hat{\mathbf{b}}_3$$

To find the relationship between the coordinates in frame B and frame N, we can take the dot product of both sides of the equation with each of the basis vectors of frame B. Taking the dot product with $\hat{\mathbf{b}}_1$:

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{b}}_1 = x_B\,(\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_1) + y_B\,(\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_1) + z_B\,(\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{b}}_1)$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{b}}_1 = x_B$$

since the basis vectors are orthonormal.
Similarly, taking the dot product with $\hat{\mathbf{b}}_2$ and $\hat{\mathbf{b}}_3$ gives:

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{b}}_2 = y_B$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{b}}_3 = z_B$$

Now, substituting the expression for $\hat{\mathbf{r}}$ in terms of frame N into these equations, we get:

$$x_B = x_N\,(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{b}}_1) + y_N\,(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{b}}_1) + z_N\,(\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{b}}_1)$$

$$y_B = x_N\,(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{b}}_2) + y_N\,(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{b}}_2) + z_N\,(\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{b}}_2)$$

$$z_B = x_N\,(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{b}}_3) + y_N\,(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{b}}_3) + z_N\,(\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{b}}_3)$$

The result is the following matrix equation:

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{n}}_1 & \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{n}}_2 & \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{n}}_3 \\ \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{n}}_1 & \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{n}}_2 & \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{n}}_3 \\ \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{n}}_1 & \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{n}}_2 & \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{n}}_3 \end{bmatrix}}_{C_{BN}} \begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix}$$

The matrix C_{BN} is the Direction Cosine Matrix (DCM) that maps coordinates from frame N to frame B.

What is the orientation of frame N w.r.t. frame B? The answer is given by the inverse of the DCM:

$$\begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix} = C_{NB} \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}, \quad \text{where} \quad C_{NB} = (C_{BN})^{-1}.$$

Since DCMs are orthogonal matrices, their inverse is equal to their transpose:

$$C_{NB} = (C_{BN})^T.$$

Let's now consider another frame A rotating w.r.t. frame B. What is the orientation of the frame A w.r.t. frame N?

The DCM that maps coordinates from frame N to frame A is given by the product of the DCMs that map coordinates from frame N to frame B and from frame B to frame A:

$$\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = C_{AN} \begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix}, \quad \text{where} \quad C_{AN} = C_{AB}\, C_{BN}.$$

The notation starts at the rightmost subindex and goes to the leftmost subindex. We want to go to the left from N to A, so we first go from N to B and then from B to A.

Independent Parameters

Let a DCM be defined by the following elements:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Since DCMs are orthogonal matrices, they satisfy the following properties:

$$C^T C = C C^T = I$$

where I is the identity matrix. This leads to the following constraints on the elements of the DCM:

$$C_{11}^2 + C_{21}^2 + C_{31}^2 = 1$$
$$C_{12}^2 + C_{22}^2 + C_{32}^2 = 1$$
$$C_{13}^2 + C_{23}^2 + C_{33}^2 = 1$$
$$C_{11}C_{12} + C_{21}C_{22} + C_{31}C_{32} = 0$$
$$C_{11}C_{13} + C_{21}C_{23} + C_{31}C_{33} = 0$$
$$C_{12}C_{13} + C_{22}C_{23} + C_{32}C_{33} = 0$$

These relations can also be achieved by trying to project a frame in itself and knowing that the dot product of orthonormal basis vectors is either 1 (if the vectors are the same) or 0 (if the vectors are different):

$$\hat{\mathbf{b}}_1 = (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_1)\,\hat{\mathbf{b}}_1 + (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_1)\,\hat{\mathbf{b}}_2 + (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{b}}_1)\,\hat{\mathbf{b}}_3$$
$$\hat{\mathbf{b}}_2 = (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2)\,\hat{\mathbf{b}}_1 + (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_2)\,\hat{\mathbf{b}}_2 + (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{b}}_2)\,\hat{\mathbf{b}}_3$$
$$\hat{\mathbf{b}}_3 = (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_3)\,\hat{\mathbf{b}}_1 + (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_3)\,\hat{\mathbf{b}}_2 + (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{b}}_3)\,\hat{\mathbf{b}}_3$$

That means:

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_1 = 1, \quad \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_2 = 1, \quad \hat{\mathbf{b}}_3 \cdot \hat{\mathbf{b}}_3 = 1$$

$$\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2 = 0, \quad \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_3 = 0, \quad \hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_3 = 0$$

So, in conclusion, the total number of independent parameters to define a DCM is 3. A common representation of a DCM is given by the use of Euler angles, which are three angles that describe the orientation of a rigid body with respect to a fixed coordinate system.

Euler Angles

Euler angles are a set of three angles that describe the orientation of a rigid body in three-dimensional space. They are commonly used in aerospace engineering to represent the orientation of aircraft and spacecraft. The three Euler angles are typically denoted as roll (ϕ), pitch (θ), and yaw (ψ).

The Euler Theorem states that any orientation of a rigid body can be achieved by a sequence of three rotations about the axes of a coordinate system. The most common sequence of rotations is the Z-Y-X sequence.

Let a position vector $\bar{\mathbf{r}}_p$ be defined as:

$$\bar{\mathbf{r}}_p = x\,\hat{\mathbf{n}}_1 + y\,\hat{\mathbf{n}}_2 + z\,\hat{\mathbf{n}}_3$$

Or simply $\left[\bar{\mathbf{r}}_p\right]_N$. The representation in the body frame is given by $\left[\bar{\mathbf{r}}_p\right]_B$:

$$\left[\bar{\mathbf{r}}_p\right]_B = C_{BN}\left[\bar{\mathbf{r}}_p\right]_N$$

Here, $\bar{\mathbf{r}}_p$ is the information we are trying to describe and the frames N and B are the languages we are using to describe it. Therefore, the position vector can have different expressions that represent the same information:

$$\left[\bar{\mathbf{r}}_p\right]_N = (\quad)\,\hat{\mathbf{n}}_1 + (\quad)\,\hat{\mathbf{n}}_2 + (\quad)\,\hat{\mathbf{n}}_3$$

$$\left[\bar{\mathbf{r}}_p\right]_B = (\quad)\,\hat{\mathbf{b}}_1 + (\quad)\,\hat{\mathbf{b}}_2 + (\quad)\,\hat{\mathbf{b}}_3$$

Time Derivative of a Vector in Different Frames

Let's first look at the time derivative of a vector expressed in a single frame of reference. Considering a 2D case:
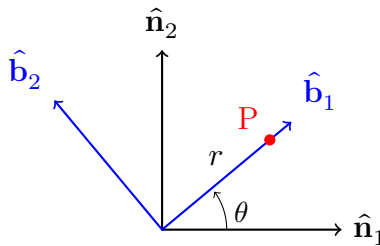
The position vector of point P can be defined in two different ways, depending on the frame of reference used:

$$\bar{\mathbf{r}} = r\,\hat{\mathbf{b}}_1$$

or

$$\bar{\mathbf{r}} = r\cos\theta\,\hat{\mathbf{n}}_1 + r\sin\theta\,\hat{\mathbf{n}}_2$$

If we consider the observer to be in the body frame B, the time derivative of the position vector, in the B frame is given by:

$$^B\dot{\bar{\mathbf{r}}} = \dot{r}\,\hat{\mathbf{b}}_1$$

However, if the observer, then we will be defining the time derivative of the position vector in the B frame, or inertial velocity in the B frame, as given by:

$$^N\dot{\bar{\mathbf{r}}} = \dot{r}\,\hat{\mathbf{b}}_1 + r\,\dot{\hat{\mathbf{b}}}_1 = {}^N\dot{\bar{\mathbf{r}}}_B$$

At this point, we need to find $\dot{\hat{\mathbf{b}}}_1$. For that we will use the Transport Theorem.

Transport Theorem

The Transport Theorem relates the time derivative of a vector as observed in two different frames of reference. It states that the time derivative of a vector $\hat{\mathbf{b}}_1$ in frame N is equal to the time derivative of the vector in frame B plus the cross product of the angular velocity of frame B relative to frame N and the vector $\hat{\mathbf{b}}_1$ itself:

$$\boxed{{}^N\dot{\hat{\mathbf{b}}}_1 = {}^B\dot{\hat{\mathbf{b}}}_1 + \bar{\omega}_{B/N} \times \hat{\mathbf{b}}_1}$$

However, the frame B is fixed, so the time derivative of its basis vectors in its own frame is zero. Therefore:

$$^N\dot{\hat{\mathbf{b}}}_1 = \bar{\omega}_{B/N} \times \hat{\mathbf{b}}_1$$

The term $\bar{\omega}_{B/N}$ is directly the angular velocity of frame B w.r.t. frame N.

Substituting this result back into the expression for the time derivative of the position vector in frame N, we get:

$$^N\dot{\bar{\mathbf{r}}} = \dot{r}\,\hat{\mathbf{b}}_1 + r\,\dot{\hat{\mathbf{b}}}_1$$
$$^N\dot{\bar{\mathbf{r}}} = \dot{r}\,\hat{\mathbf{b}}_1 + r\left(\bar{\omega}_{B/N} \times \hat{\mathbf{b}}_1\right)$$

The term $^N\dot{\bar{\mathbf{r}}}$ is the inertial velocity and can be expressed in any frame of reference.

The angular velocity vector $\bar{\omega}_{B/N}$ is perpendicular to the plane formed by $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$. Therefore, its direction is along the axis $\hat{\mathbf{b}}_3$ (out of the page) and its magnitude is given by $\dot{\theta}$:

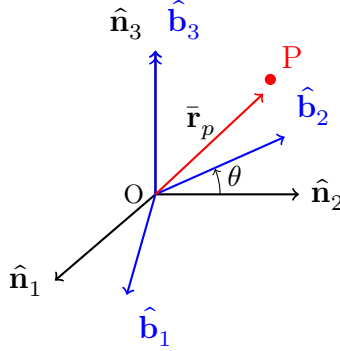$$\bar{\omega}_{B/N} = \dot{\theta}\,\hat{\mathbf{b}}_3$$

Now, we can compute the cross product:

$$\dot{\hat{\mathbf{b}}}_1 = \dot{\theta}\,\hat{\mathbf{b}}_3 \times \hat{\mathbf{b}}_1 = \dot{\theta}\,\hat{\mathbf{b}}_2$$

Substituting this result back into the expression for the time derivative of the position vector, for an observer in frame N and expressed in the B frame, we get:

$$^N\dot{\bar{\mathbf{r}}} = \dot{r}\,\hat{\mathbf{b}}_1 + r\,\dot{\theta}\,\hat{\mathbf{b}}_2$$

Let's now consider the 3D general case:

Let the position vector of point P be defined as:

$$\bar{\mathbf{r}}_p = x\,\hat{\mathbf{b}}_1 + y\,\hat{\mathbf{b}}_2 + z\,\hat{\mathbf{b}}_3$$

In order to find the time derivative of $\bar{\mathbf{r}}_p$, we need to apply the product rule of differentiation:

$$\dot{\bar{\mathbf{r}}}_p = \dot{x}\,\hat{\mathbf{b}}_1 + x\,\dot{\hat{\mathbf{b}}}_1 + \dot{y}\,\hat{\mathbf{b}}_2 + y\,\dot{\hat{\mathbf{b}}}_2 + \dot{z}\,\hat{\mathbf{b}}_3 + z\,\dot{\hat{\mathbf{b}}}_3$$

Now, let's consider that we have an observer in the frame B and we want to express the velocity in the frame B. In this case, the time derivatives of the basis vectors are zero because they are fixed in frame B:

$$^B\dot{\bar{\mathbf{r}}}_p = \dot{x}\,\hat{\mathbf{b}}_1 + \dot{y}\,\hat{\mathbf{b}}_2 + \dot{z}\,\hat{\mathbf{b}}_3$$

However, if the observer is in the frame N, the terms with the time derivatives of the basis vectors are not zero. Therefore, we need to use the Transport Theorem to find those terms:

$$^N\dot{\bar{\mathbf{r}}}_p = \dot{x}\,\hat{\mathbf{b}}_1 + x\,\dot{\hat{\mathbf{b}}}_1 + \dot{y}\,\hat{\mathbf{b}}_2 + y\,\dot{\hat{\mathbf{b}}}_2 + \dot{z}\,\hat{\mathbf{b}}_3 + z\,\dot{\hat{\mathbf{b}}}_3$$

$$^N\dot{\bar{\mathbf{r}}}_p = \dot{x}\,\hat{\mathbf{b}}_1 + \dot{y}\,\hat{\mathbf{b}}_2 + \dot{z}\,\hat{\mathbf{b}}_3 + x\,(\bar{\omega}_{B/N} \times \hat{\mathbf{b}}_1) + y\,(\bar{\omega}_{B/N} \times \hat{\mathbf{b}}_2) + z\,(\bar{\omega}_{B/N} \times \hat{\mathbf{b}}_3)$$

This expression can be rewritten as:

$$^N\dot{\bar{\mathbf{r}}}_p = {}^B\dot{\bar{\mathbf{r}}}_p + \bar{\omega}_{B/N} \times \bar{\mathbf{r}}_p$$

and represents the inertial velocity of point P expressed in frame B as observed from frame N.