LINEAR ALGEBRA: ANSWER TO HOMEWORK 1

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1. Question

2.1 Show that if a matrix A is both triangular and unitary, it is diagonal.

1.1. **Answer.**

Notation. $a_{i,j}$ indicates the element of A at row i and column j. a_i indicates the ith column of A.

Remark 1.1.1. Note that A, being triangular, is square. Let A be a $m \times m$ matrix.

Lemma 1.1.2. *If A is both upper triangular and unitary, then* $a_{1,j} = 0$ *for all* $j \neq 1$*. So, for all* $j \neq 1$, $a_{j,1} = a_{1,j} = 0$*. That is, the non diagonal elements in the first column and row are 0.*

Proof. Consider any $j \neq 1$. As A is unitary, $A^*A = I$.
So, $a_j^* a_1 = I_{j,1} = 0$.
So, $\sum_{i=1}^{m} a_{i,j} a_{i,1} = 0$.
But, as A is upper triangular, for all $i > 1, a_{i,1} = 0$.
So, $a_{1,j} a_{1,1} = 0$. This holds even if $a_{1,1} \neq 0$. Hence, $a_{1,j} = 0$. □

Lemma 1.1.3. *Suppose that A is both upper triangular and unitary. Suppose that if* $i \neq j$ *and either* $i \leq l$*, or if* $j \leq l$, $a_{i,j} = 0$*. That is, the non diagonal elements in the first l columns and rows are 0. Then, all non-diagonal elements in the first l+1 columns and rows are 0.*

Proof. Consider any $j > l + 1$. As A is unitary, $A^*A = I$.
So, $a_j^* a_{l+1} = I_{j,l+1} = 0$.
So, $\sum_{i=1}^{m} a_{i,j} a_{i,l+1} = 0$.
But, as A is upper triangular, for all $i > l + 1, a_{i,l+1} = 0$. Also, from our assumption, for all $i \leq l, a_{i,l+1} = 0$.
So, $a_{l+1,j} a_{l+1,l+1} = 0$. This holds even if $a_{l+1,l+1} \neq 0$. Hence, $a_{l+1,j} = 0$. Thence, we have the result. □

Theorem 1.1.4. *If a matrix A is both upper triangular and unitary, it is diagonal.*

Proof. **Base Case**: Suppose that A is both upper triangular and unitary. In our lemmas, we have already shown that all non diagonal elements of the first column and row will be 0.

Induction: Suppose that the non diagonal elements in the first l columns and rows are 0. Then, we have shown in our lemmas that all non-diagonal elements in the first l+1 columns and rows are 0.

Conclusion: Then, by the principle of mathematical induction, all non-diagonal elements in all of the first m columns and rows are 0. □

Theorem 1.1.5. *If a matrix A is both lower triangular and unitary, it is diagonal.*

Proof. As A is unitary, $AA^* = I$. As A is lower triangular, A^* is both upper triangular and unitary, because of which we may apply the theorem we proved for such matreces to conclude that A^* is diagonal. Note that A is diagonal if and only if A^* is diagonal. Hence, A is diagonal. □

2. Question

2.2 The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$, $||\sum_{i=1}^n x_i||^2 = \sum_{i=1}^n ||x_i||^2$.

2.0.1. *1.* Prove this for case n=2 by an explicit computation of $||x_1 + x_2||^2$.

2.0.2. *2.* Show that this computation also establishes the general case by induction.

2.1. **Answer.**

2.1.1. *1.*

Theorem 2.1.1. *For a set of n=2 orthogonal vectors $\{x_i\}$, $||\sum_{i=1}^n x_i||^2 = \sum_{i=1}^n ||x_i||^2$.*

Proof.

$$||x_1 + x_2||^2 = (x_1 + x_2)^*(x_1 + x_2) \tag{1}$$
$$= (x_1^* + x_2^*)(x_1 + x_2) \tag{2}$$
$$= x_1^*x_1 + x_2^*x_1 + x_1^*x_2 + x_2^*x_2 \tag{3}$$
$$= x_1^*x_1 + x_2^*x_2 : \texttt{ Due to orthogonality.} \tag{4}$$
$$= \sum_{i=1}^{2} ||x_i||^2 \tag{5}$$

□

2.1.2. *2.*

Theorem 2.1.2. *For a set of n orthogonal vectors $\{x_i\}$, $||\sum_{i=1}^n x_i||^2 = \sum_{i=1}^n ||x_i||^2$.*

Proof. **Base case**: For n=2, the statement has been proved to be true.

Induction: Suppose that it is true for n. Then, consider a set of n+1 orthogonal vectors $\{x_i\}$. Then:

$$||\sum_{i=1}^{n+1} x_i||^2 = ||\sum_{i=1}^{n} x_i + x_{n+1}||^2 \tag{6}$$
$$= ||\sum_{i=1}^{n} x_i||^2 + ||x_{n+1}||^2 : \texttt{ Using theorem for n=2} \tag{7}$$
$$= \sum_{i=1}^{n+1} ||x_i||^2 : \texttt{ Applying inductive hypothesis.} \tag{8}$$
(9)

Thus, by the principle of mathematical induction, we have the result. □

3. Question

2.6 If u and v are m-vectors, the matrix $A = I + uv^*$ is known as a rank-one perturbation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + auv$ for some scalar a, and give an expression for a. For what u and v is A singular? If it is singular, what is null(A)?

3.1. **Answer.**

Theorem 3.1.1. *If $A = I + uv^*$ is nonsingular, then its inverse has the form $A^{-1} = I + auv$ for some scalar a.*

Proof.

$$A(I + auv^*) = (I + uv^*)(I + auv^*) \tag{10}$$
$$= I + uv^* + auv^* + uv^*auv^* \tag{11}$$
$$= I + uv^* + auv^* + auv^*uv^* \tag{12}$$
$$= I + uv^* + uv^*a(1 + v^*u) \tag{13}$$
$$= I \texttt{ If } a = \frac{-1}{(1 + v^*u)} \tag{14}$$
$$\tag{15}$$

□

Remark 3.1.2. We have found above, the expression $a = \frac{-1}{(1+v^*u)}$.

Corollary 3.1.3. *a fails to exist if $v^*u = -1$. For all other cases, A^{-1} may be found using the value of a found above.*

Theorem 3.1.4. *For values of v and u where $v^*u = -1$, A is singular. Null(A) is 1-dimensional, and has u as its basis.*

Proof.

$$Ax = (I + uv^*)x \tag{16}$$
$$= x + uv^*x \tag{17}$$
$$Au = u + uv^*u \tag{18}$$
$$\tag{19}$$

Hence, Au = 0 if $v^*u = -1$. u is assumed to be non zero. So, A is singular in such a case. We have noted in the corollary above that in all other cases, A is non-singular. Hence, Null(A) is 1-dimensional, and has u as its basis. □