Shrinkage methods: Least angles regression and Lasso

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October 7, 2009

Outline

1. Outline
2. Least Angle regression
3. LAR and Lasso: the connection
   - Lasso
   - The connection
4. Conclusion

The problem and the notation

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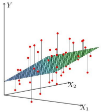

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- Given N data points.

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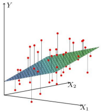

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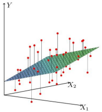

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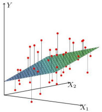

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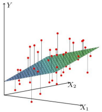

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- Assume feature vectors have norm 1. $XDD^{-1}\beta \approx y$

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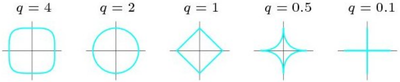

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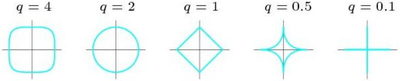

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- Least angle regression.

From: [2], [1].

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Forward selection

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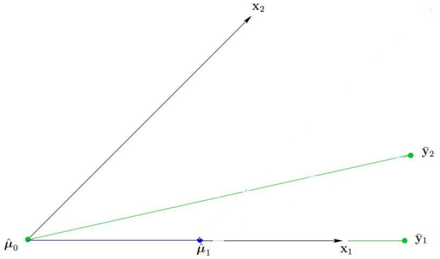
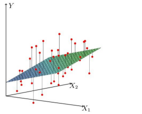
- Grow A one feature at a time: Select $x_j = argmax_i\{\langle x_i, r\rangle\}$.





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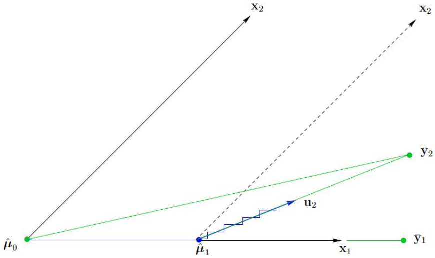
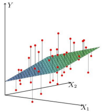
LAR Algorithm





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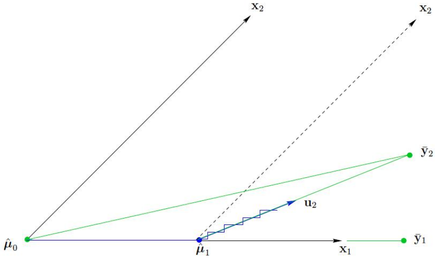

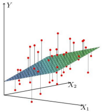

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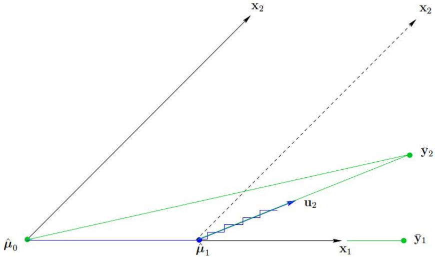

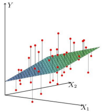

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- Project residue r on subspace(A), get u_i.
- Set $\beta_A = \beta_A + \gamma_i$.
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  until you find $x_k : \langle x_k, r\rangle = \langle x_j, r\rangle$.

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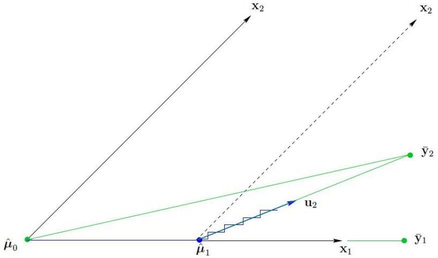

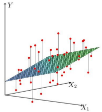

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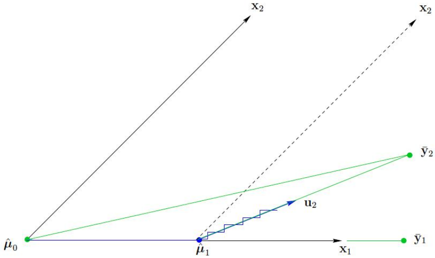

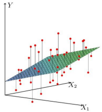

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- $A = A \cup \{x_k\}$.
- Note! At any point, the residue makes same angle with all $x_i \in A$.

See how the correlation evolves

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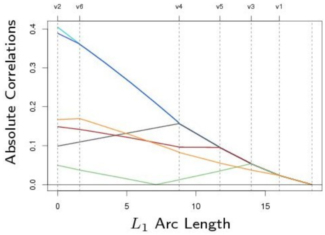
- $\hat{c}(\beta)$: the vector of correlations of the residue with $\{x_i\}$



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Improvement over forward stepwise (greedy)

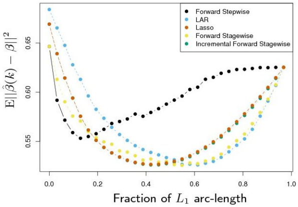

FIGURE 3.16. *Comparison of LAR and lasso with forward stepwise, forward stagewise (FS) and incremental forward stagewise (FS_0) regression. The setup*

How to balance love of sparsity with desire for a good fit?

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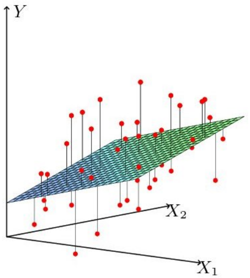

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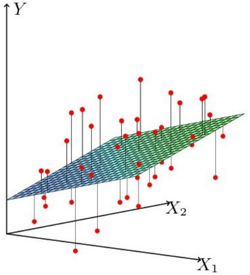

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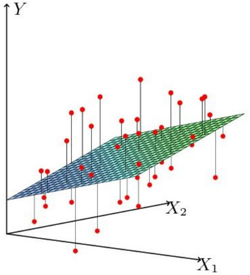

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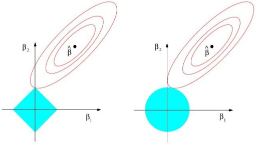
- Same as $f(\hat{\beta}) = \min_{\beta} \|y - X\beta\|_2^2 + \lambda \sum_i |\beta_i|$.

Lasso for sparsity



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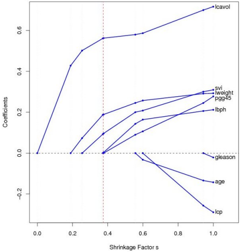

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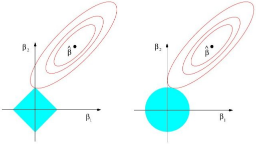

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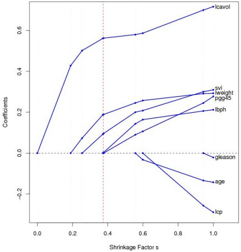

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- Quiz: What t will reduce the problem to least squares?

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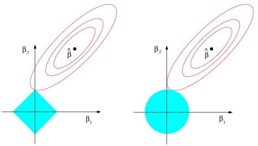

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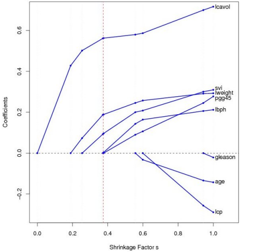

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- $t = \sum |\beta_i^*|$.

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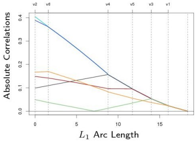

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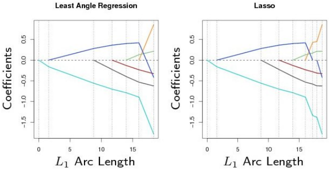

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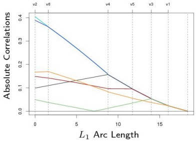

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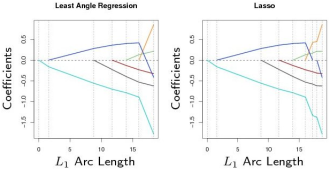

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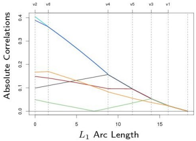

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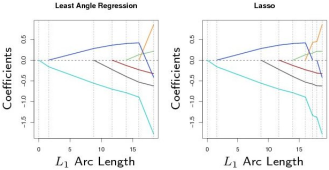

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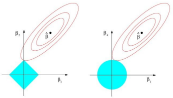
- So $|\beta_j|$ can begin to decrease, and it can change sign.
- The LARS fix: Drop coefficients which hit 0 out of 'active set'.

Lasso solutions: Observations



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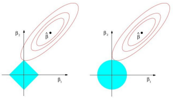

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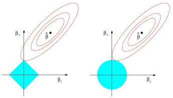

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- Note: $f(\hat{\beta}) = \min_\beta \|y - X\beta\|_2^2 + \lambda \sum_i |\beta_i|$.
- Set $\nabla f(\beta) = 0$.

Lasso solutions: Observations

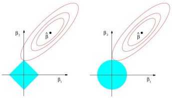

FIGURE 3.11. *Estimation picture for the lasso (left) and ridge regression (right). Shown are contours of the error and constraint functions. The solid blue areas are the constraint regions $|\beta_1| + |\beta_2| \le t$ and $\beta_1^2 + \beta_2^2 \le t^2$, respectively, while the red ellipses are the contours of the least squares error function.*

- Note: $f(\hat{\beta}) = \min_\beta \|y - X\beta\|_2^2 + \lambda \sum_i |\beta_i|$.
- Set $\nabla f(\beta) = 0$.
- B := features in sparse solution.

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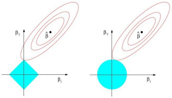

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  $\forall j \in B : x_j^T(y - X\beta) = \lambda * sgn(\beta_j)$.
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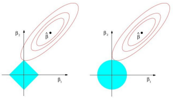

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- Remarkable! $\lambda \rightarrow$ upper bound on correlation of the residue with x_j.

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- The LARS fix: If β_j hits 0 for $j \in A$, it is about to change sign.

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- Experimental observation

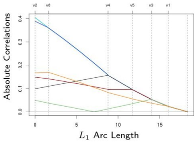

FIGURE 3.14. *Progression of the absolute correlations during each step of the LAR procedure, using a simulated data set with six predictors. The labels at the top of the plot indicate which variables enter the active set at each step. The step length are measured in units of L_1 arc length.*

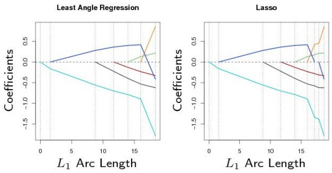

FIGURE 3.15. *Left panel shows the LAR coefficient profiles on the simulated data, as a function of the L_1 arc length. The right panel shows the Lasso profile. They are identical until the dark-blue coefficient crosses zero at an arc length of about 18.*

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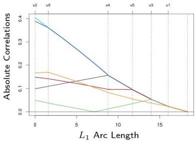

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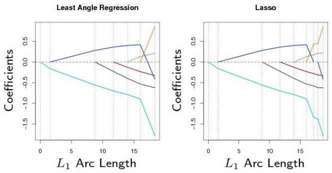

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- Geometric intuition: When does a coefficient start decreasing, even when correlation is positive? When you add feature v1 which is not independent of A. As $\beta_1 \uparrow, \beta_6 \downarrow$.

Outline

1. Outline
2. Least Angle regression
3. LAR and Lasso: the connection
   - Lasso
   - The connection
4. Conclusion

What did we learn?

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- Least Angle regression (LAR): get sparse solutions, like forward stepwise regression; but modified to be non greedy.

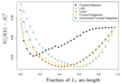

FIGURE 3.16. *Comparison of LAR and lasso with forward stepwise, forward stagewise (FS) and incremental forward stagewise (FS_0) regression. The setup*

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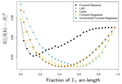

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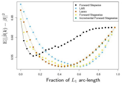

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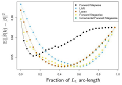

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- Lasso for sparse solutions.
- The sparse solutions are sometimes different.
- LARS: LAR modified to solve Lasso. Very efficient! Also looked at how Lasso works with new eyes ★★.

References

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Bye!

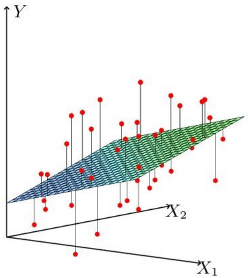
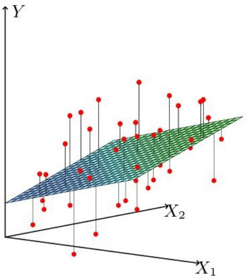


FIGURE 3.1. *Linear least squares fitting with $X \in \mathbb{R}^2$. We seek the linear function of X that minimizes the sum of squared residuals from Y.*

Bye!



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Ask us some questions!