LINEAR ALGEBRA: ANSWER TO HOMEWORK 7

VISHVAS VASUKI

1. 24.1

$A \in C^{m\times m}$.

1.1. a.

Theorem 1.1.1. *$k \in C$. l is an ew of A. Then, l-k is an ew of A-kI.*

Proof. As l is an ew of A, $det(A - lI) = 0$. So, $det(A - kI - lI + kI) = 0$. So $det(A - kI - (l - k)I) = 0$. So, l-k is an ew of A-kI. □

1.2. b.

Theorem 1.2.1. *The following claim is false: "A is real. l is an ew of A. Then, so is -l."*

Proof. A = [a]. Now, Ax = ax for any 1×1 x. So, a is an eigenvalue of A. Also, there cannot be anyother ew as the eigenspace of a spans the entire space. □

1.3. c.

Theorem 1.3.1. *A is real. l is an ew of A. Then, so is $\bar{l}$.*

Proof. Let P be the characteristic polynomial. As A is real, the coefficients in P are real. As l is an ew of A, $P(l) = 0$.

$P(l) = \sum a_i l^i = 0 = \overline{\sum a_i l^i} = \sum a_i \bar{l^i} = \sum a_i \bar{l}^i = P(\bar{l})$.

So, $\bar{l}$ is also an ew of A. □

1.4. d.

Theorem 1.4.1. *l is an ew of A. A is nonsingular. Then, l^{-1} is ew of A^{-1}.*

Proof. $\exists x \neq 0 : Ax = lx$. So, $xl^{-1} = A^{-1}x$. Thus, l^{-1} is ew of A^{-1}. □

1.5. e.

Theorem 1.5.1. *The following claim is false: If all ews of A are 0, A = 0.*

Proof. Take $A = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$. $A - lI = \left(\begin{smallmatrix} -l & a \\ 0 & -l \end{smallmatrix}\right)$; and the characteristic polynomial is $l^2 = 0$. So, all eigenvalues of A are 0; but $A \neq 0$. □

1.6. **f.**

Theorem 1.6.1. *$A = A^*$. l is an ew of A. Then $|l|$ is a singular value of A.*

Proof. As a consequence of a theorem is stated in [1], which follows directly from the theorem about the existance of the Schur factorization, A is unitarily diagonalizable.

So, $A = QLQ^*$. Rearranging, and fixing the signs of the columns of Q and L to ensure that $L_{i,i} \geq 0$, and that they occur in descending order, we arrive at $A = Q'\Sigma Q'^*$, where the elements of the diagonal matrix Σ are the same as the elements of the diagonal matrix L. But, this is the unique SVD of A.

So, $|l|$ is a singular value of A. □

1.7. **g.**

Theorem 1.7.1. *A is diagonalizable. All its ew's are equal. Then A is a diagonal matrix.*

Proof. Let $A = SLS^{-1}$ be the eigenvalue decomposition. But, we know that $L = lI$. So, $A = lSIS^{-1} = lI$. So, A is a diagonal matrix. □

2

Theorem 2.0.2. *Let $\hat{x}$ be the solution of hermitian positive definite system $Ax = b$ via Cholesky Factorization (Algorithm 23.1, Trefethen and Bau). Let $\hat{x}$ be the exact solution to the following perturbed system: $(A + \delta A)\hat{x} = b$. Show that $\frac{\|\delta A\|_\infty}{\|A\|_\infty} \leq 3n^2\epsilon_m$.*

Remark 2.0.3. You can use the error analysis for LU factorization discussed in the class.

Proof. Using the result from the error analysis for LU factorization, we know that $|\delta A| \leq 3n\epsilon|L||U|$. But, as A is positive definite and Hermitian, $|L||U| = |L||DL^*| = |L||D^{0.5}D^{0.5}L^*| = |L||D^{0.5}||D^{0.5}L^*| = |LD^{0.5}||D^{0.5}L^*| = |R||R^*|$, where $D^{0.5}$ involves taking the +ve square roots of $\{D_{i,i}\}$, which means that $|D^{0.5}L^*| = |D^{0.5}||L^*|$ and $|LD^{0.5}| = |L||D^{0.5}|$.

So, $\||\delta A|\|_\infty = \|\delta A\|_\infty \leq 3n\epsilon \||R||R^*|\|_\infty \leq 3n\epsilon \||R|\|_\infty \||R^*|\|_\infty = 3n\epsilon \|R\|_\infty \|R^*\|_\infty \leq 3n^2\epsilon \|R\|_2 \|R^*\|_2$ (Using facts proved in exercise 3.2.). We know that $\|R\|_2 = \|R^*\|_2$ (using SVD). So, $\|\delta A\|_\infty \leq 3n^2\epsilon \|R\|_2^2 = 3n^2\epsilon \|A\|_2 \leq 3n^{5/2}\epsilon \|A\|_\infty$. (Using a fact from the last section of lecture 23 of [1].)

Remark 2.0.4. We proved a slightly weaker bound above.

□

References

[1] Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra.* Siam, 1997.