1. Triangular triangularization

$|\Delta A| \leq 3n\epsilon |L||U|$. [**Find proof**]. $\tilde{L}\tilde{U} = A + \Delta A$, $\frac{\|\Delta A\|}{\|L\|\|U\|} = O(\epsilon)$. [**Find proof**].

1.1. **With pivoting.** $L_{i,j} \leq 1; \|L\| = O(1)$; let Growth Factor $\rho = \frac{\max|U_{i,j}|}{\max|A_{i,j}|}$; $\|U\| = \rho \|A\|$; so $\tilde{L}\tilde{U} = A + \Delta A$, $\frac{\|\Delta A\|}{\|A\|} = O(\rho\epsilon)$.

Maximal instability: $\rho = 2^{m-1}$.

2. Computer Architecture

If M = num bits in mantissa, E = num bits in exp. Stores $\pm(1+f/2^M)2^{e-2^{E-1}+1}$. $\epsilon_M = 2^{-M-1} = 2^{-53}$.

Single prec: 1 $\pm$ bit, M= 23 bits, E = 8 bits. **Double prec**: M= 52 bits, E = 11 bits.

3. Eigenvalue algs

3.1. **Power iteration for $A^T = A$.** The series $v^{(i)} = \frac{A^i x}{\|A^i x\|}$ and $l^{(i)} = r(v^{(i)})$ converge to eigenpair corresponding to largest ew l_1, q_1: as $x = \sum a_i q_i$.

So, Applying A repeatedly takes x to dominant ev.

3.1.1. *Convergence.* Linear convergence of ev. $\left\|v^{(i)} - \pm q_1\right\| = O(|\frac{l_2}{l_1}|^i)$, $\left\|l^{(i)} - \pm l_1\right\| = \left\|v^{(i)} - \pm q_1\right\|^2$.

3.2. **Inverse iteration.** ev of A and $(A-pI)^{-1}$ same, ew l_i shifted and inverted to get ew $(l_i - p)^{-1}$. If p near l_j, using power iteration on $(A-pI)^{-1}$ gives fast convergence.

Good for finding ev if ew already known.

3.2.1. *Convergence.* Linear convergence of ev.
$\left\|v^{(i)} - \pm q_j\right\| = O(|\frac{p-l_j}{p-l_k}|^i)$, $\left\|l^{(i)} - \pm l_1\right\| = \left\|v^{(i)} - \pm q_j\right\|^2$.

3.2.2. *Alg.* Solve $(A-pI)w = v^{(k-1)}$; normalize to get $v^{(k)}$.

3.3. **Rayleigh quotient iteration.** Inverse iteration, where $l^{(i)} = R(v^{(i)})$ used as p (ew estimate).

3.3.1. *Convergence.* Cubic convergence of ev and ew. If $\left\|v^{(k)} - q_j\right\| \leq eps$ when $|l^{(k)} - l_j| \leq O(\epsilon^2)$. So $\left\|v^{(k+1)} - q_j\right\| =$
$O(|l^{(k)} - l_j| \left\|v^{(k)} - q_j\right\|) = O(\left\|v^{(k)} - (\pm q_j)\right\|^3)$. $|l^{(k+1)} - (l_j)| = O(\left\|v^{(k+1)} - q_j\right\|^2) = O(|l^{(k)} - (\pm q_j)|^3)$.

Gain 3 digits of accuracy in each iteration.

3.4. **Simultaneous iteration for $A = A^T$.** Aka Block power itern. $\langle v_i \rangle$ linearly independent; their matrix $V^{(0)}$. $\langle q_i \rangle$ orth ev of A; cols of $\tilde{Q}$.

3.5. **Convergence.** If $|l_1| > .. > |l_n| \geq |l_{n+1}|..$, Orth basis of $\langle A^k v_1^{(0)}, ..A^k v_n^{(0)} \rangle$ converges to $\langle q_1, ..q_n \rangle$: take $v_i = \sum_j a_j q_j$, do power iteration.

3.5.1. *Alg.* Take some $Q^0 = I$ or other orth cols, get $Z = AQ^{(k-1)}$; get $Q^{(k)}R^{(k)} = Z$. Defn: $A^{(k)} = (Q^{(k)})^T A Q^{(k)}$, $R'^{(k)} = \prod R^{(k)}$.

$A^k = Q^{(k)} R'^{(k)}$: By induction: $A^k = AQ^{(k-1)} R'^{(k-1)} = Q^{(k)} R'^{(k)}$.

3.6. **QR algorithm or iteration.** Not QR factorization. Get $Q^{(k)}R^{(k)} = A^{(k-1)}$; $A^{(k)} = R^{(k)}Q^{(k)} = (Q^{(k)})^T A^{(k-1)} Q^{(k)}$: Similarity transformation. Works for all A with distinct $|l_i|$; easy analysis for $A = A^T$.

Defn: $R'^{(k)} = \prod R^{(k)}$, $Q'^{(k)} = \prod_k Q^{(k)}$: same as $Q^{(k)}$ in Simult itern alg.

3.7. **Convergence for $A = A^T$.** Same as Simultaneous iteration starting with I. $A^k = Q^{(k)} R'^{(k)}$: So, finds orth bases for A^k.

$A^{(k)} = (Q'^{(k)})^T A Q'^{(k)}$; $A_{i,i}^{(k)}$ are $R(Q_i'^{(k)})$; as $Q_i'^{(k)}$ converges, $A_{i,i}^{(k)} \to l_i$, off diagonal entries tend to 0; so approaches Schur factorization.

Linear convergence rate: $max_j \frac{l_{j+1}}{l_j}$.

4. Hermitian matrix

Aka Self Adjoint Operator. Symmetric matrix: $A = A^T$ generalizes to Hermitian matrix $A = A^*$; analogous to $R \subseteq C$. Skew symmetric matrix: $A = -A^T$, generalizes to skew Hermitian.

Any B $= \frac{B+B^*}{2} + \frac{B-B^*}{2}$: Hermitian + Skew Hermitian.

4.0.1. *Positive definite matrix (pd) properties.* $x^*Ax \in R; x^*Ax > 0$. So, $A^* = A$.

Eigenvalues $l > 0$: $lx^*x = x^*Ax > 0$. If $A = A^*$, all eigenvalues $l > 0$, A is +ve definite.

4.1. **Cholesky factorization.** $A = R^*R$. As $A = LDU^* = UDL^*$, $L = U^*$. So, $A = LDL^* = LD^{1/2}D^{1/2}L^* = R^*R$; $d_{j,j} > 0$ as $a_{j,j} > 0$; $r_{j,j} = \sqrt{d_{j,j}} > 0$ chosen.

By SVD, $\|R\|^2 = \|A\|$.

4.1.1. *Symmetric Elimination Algorithm.* Do Gaussian elimination + extra column ops to maintain symmetry at each step.

$A = \begin{pmatrix} a_{1,1} & A_{2,1}^* \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{A_{2,1}}{a_{1,1}} & I \end{pmatrix} \begin{pmatrix} a_{1,1} & 0 \\ 0 & A_{2,2} - \frac{A_{2,1}A_{2,1}^*}{a_{1,1}} \end{pmatrix} \begin{pmatrix} 1 & \frac{A_{2,1}}{a_{1,1}} \\ 0 & I \end{pmatrix} = LDL^*$. Get R^*R by doing $LD^{1/2}$ at each step.

So, every Hermitian PDM has R^*R factorization. It is also unique: $r_{j,j} = \sqrt{d_{j,j}} > 0$ fixed by defn; it inturn fixes rest of R.

4.1.2. *Code and Opcount.* R=A; Repeat: do symmetric elimination on submatrix $R_{i+1,i+1}$; do $R_i^*/\sqrt{r_{i,i}}$. Only Upper part of R stored.

Opcount: $\sum_{k=1}^{m} \sum_{j=k+1}^{m} 2(m-j) \approx \frac{m^3}{3}$ flops.

4.1.3. *Stability.* By SVD: $\|R\|_2 = \|R^*\|_2 = \|A\|_2$; *so* $\|R\| \leq \sqrt{m}\,\|A\|$ [**Check**]. . So, R never grows large. So, backward stable : get $\hat{R} * \hat{R}$ for perturbed A. Forward error in R large; but R and R^* diabolically correlated.