Non Linear Programming: Homework 4

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1 3.32 Products and quotients of convex functions on R

Remark. Consider functions on R. Assume differentiability too. Let D be the differential operator with respect to x.

1.1 a

Theorem 1.1.1. *If f, g are convex, both nondecreasing (or nonincreasing) +ve functions, then fg is convex.*

Proof. $D^2f(x)g(x) = D^2f(x)g(x) + 2Df(x)Dg(x) + f(x)D^2g(x) \geq 0$. All terms in this sum are non-negative from the assumptions. □

1.2 b

Theorem 1.2.1. *If f, g are concave, +ve, with f nondecreasing and g nonincreasing, fg is concave.*

Proof. $D^2f(x)g(x) = D^2f(x)g(x) + 2Df(x)Dg(x) + f(x)D^2g(x) \geq 0$. All terms in this sum are non-positive from the assumptions. □

1.3 c

Theorem 1.3.1. *If f is convex, nondecreasing, +ve. g is concave, nonincreasing, +ve. Then, f/g is convex.*

Proof. $D^2f(x)/g(x) = D^2f(x)/g(x) - 2Df(x)Dg(x)/g(x)^2 -$
$f(x)D^2g(x)/g(x)^2 - f(x)D^2g(x)/g(x)^2 + 2f(x)(Dg(x))^2/g(x)^3 \geq 0$. All terms in this sum are non-negative from the assumptions. □

2 3.36 (a)-(d) Conjugate functions

2.1 Hint

Problem 3.36 concerns conjugate functions, and is likely to be quite difficult for some of you. Conjugates often defy intuition. I recommend a two-step approach:

* First, determine the *domain* of the conjugate, by trying to find general conditions under which the supremum is unbounded. Many conjugate functions have bounded domains even when the original function does not.

* Then, choose a point inside the domain and determine the supremum.

Do not be surprised by strange results! (Though verify them!)

Notation. Given f(x), conjugate function is $f'(y) = \sup_{x \in dom(x)}(y^T x - f(x))$. Assume that y is not 0.

2.2 a max

$f'(y) = \sup_{x \in dom(x)}(y^T x - f(x)) = \infty$. This is a special case of part b below.

2.3 b Sum of top k values

Let $y_{(i)}$ denote entry of y corresponding to ith largest entry of x, $x_{[i]}$. $f'(y) = \sup_{x \in dom(x)} \sum_{i=1}^{k}(y_{(i)} - 1)x_{[i]} + \sum_{i=k+1}^{n} y_{(i)}x_{[i]} = \infty$, when $n > k$.

2.4 c Piecewise linear fn

$f'(y) = \sup_{x \in dom(x)} \min_i (y - a_i)x - b_i$. This will correspond to x which is either be unbounded if $y > a_n$, or a point where two linear pieces whose slopes are such that $a_i \leq y \leq a_{i+1}$ intersect.

2.5 d Powers

$f'(y) = \sup_{x \in dom(x)} yx - x^p$. To find the supremum, set the gradient to 0 to get: $y - px^{p-1} = 0$. This is indeed the maximum because the hessian happens to be $-px^{p-2} \leq 0$ for $p > 1$. So, the maximizing x is $g(y) = (\frac{y}{p})^{\frac{1}{p-1}}$. So, $f'(y) = yg(y) - g(y)^p$.

For $p < 0$, $f'(y) = \infty$.

3 3.49 Log concavity

3.1 a Logistic fn

$\log f(x) = x - \log(1 + e^x)$. We see that $D^2(\log f(x)) = -e^{3x}/(1 + e^x) \leq 0$, so f(x) is log concave.

3.2 b Harmonic mean

From the slides, we know that $g(x) = x^{-1}$ is log concave. It is also decreasing. So, $\log g(x)$ is concave and decreasing. Consider $h : R^n_{++} \to R$: $h(x) = \sum x_i^{-1}$. This, being a sum of concave functions, is concave. Applying the rules of function composition, we have that $\log g(h(x))$ is concave. So, f(x) is log concave.

3.3 c Product over sum

From the slides, we know that $g(x) = x^{-1}$ is log concave. It is also decreasing. So, $\log g(x)$ is concave and decreasing. Consider $h : R^n_{++} \to R$: $h(x) = \sum x_i$ is also concave. So, $\log g(h(x)) = (\sum x_i)^{-1}$ is log concave. Also, functions $f_i(x) = x_i$ are all log concave. Product of log concave functions is log concave. So, $f(x) = \frac{\prod_i x_i}{\sum_i x_i}$ is also log concave.

3.4 d Determinant over trace

We use part c and claim that $g(\lambda) = \frac{\prod_i \lambda_i}{\sum_i \lambda_i}$ is log concave in λ. So, $\log g(\lambda)$ is concave and increasing (seen by taking the derivative and seeing its non-negativity). The eigenvalue function, $\lambda(X)$ is convex.

So, by composition rules, $f(X) = \frac{det(X)}{tr(X)} = g(\lambda(X))$ is log concave.

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Let p be a degree k polynomial on R, with all roots $\{r_i\}$ being real. So, $p(x) = \prod_{i=1}^k (x - r_i)$ and $\log p(x) = \sum_{i=1}^k \log(x - r_i)$. $\log(x - r_i)$ is a concave function (composition of concave fn with affine transformation). So, $\log p(x)$ is concave where it is +ve and x exceeds all r_i.