Non Linear Programming: Homework 2

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1 2.12 Identification of convex sets

1.1 Hint

For problem 2.12, give a brief justification for each of your answers. With the possible exception of part (g), you shouldn't need more than a sentence or two.

1.2 a

A slab, $\left\{x : \alpha \leq a^T x \leq \beta\right\}$, is convex. This is because it is the intersection of two half-spaces, which are convex sets.

1.3 b

A rectangle, $\{x : \alpha_i \leq x_i \leq \beta_i\}$, is convex. It is the intersection of convex sets. This is because it is the intersection of halfspaces formed by vertical and horizontal hyperplanes, which are convex sets.

1.4 c

A wedge, which is in the interesection of two halfspaces, is convex as it is the intersection of two convex sets.

1.5 d

Consider the set S of points which are closer to a given point p than to a given set Y. It is the intersection of halfspaces, or in degenerate cases, even the entire space. So, it is convex.

1.6 e

Take sets S and T. Consider the set X of points closer to S than to T. Note that $S \subseteq X$. This is not convex. Consider S and T to be 2 concentric rings in R^2.

Acknowledgement. Got this example from discussion with nAgarAjan naTarAjan.

1.7 f

Take the set $X = \{x : x + S_2 \subseteq S_1\}$, with S_1 convex. This set is convex. If S_2 is larger than S_1, $X = \emptyset$. Else, we can use the definition of convexity to show that X is convex.

1.8 g

Take the set $X = \{x : \|x - a\| \leq \theta \|x - b\|\}$, with $a \neq b$ and $\theta \in [0, 1]$. Consider the boundary cases: at $\theta = 1$, X is a halfspace, and at $\theta = 0$, it is a line. For intermediate values, it seems to be a hyperbola-like. By geometric intuition, I conjecture that this is a convex set.

2 2.17 (parts b and c only). Image of polyhedral sets under perspectve function

2.1 Hint

Be careful on problem 2.17! Each has three different possibilities depending upon specific values of f, g, or h.

$P(C) = \{v/t : (v, t) \in C, t > 0\}$.

2.2 b Hyperplane

$C = \left\{(v, t) : f^T v + gt = h\right\}; f, g \neq 0$.

$P(C) = \left\{v : tf^T v = h - gt, \forall t > 0\right\}$. If $g > 0$: $P(C) = \left\{v : tf^T v \leq h\right\}$. This is a halfspace below the hyperplane $f^T v = h$. If $g < 0$: $P(C) = \left\{v : tf^T v \geq h\right\}$. This is a halfspace above the hyperplane $f^T v = h$.

2.3 c Halfspace

$C = \left\{(v, t) : f^T v + gt \leq h\right\}; f, g \neq 0$.

$P(C) = \left\{v : tf^T v \leq h - gt, \forall t > 0\right\}$. Depending on whether g is +ve or -ve, P(C) becomes the same halfspaces described in part (b).

3 2.19: Preimages of convex sets under linear fractional functions.

3.1 The problem setup

$f(x) = \frac{(Ax+b)}{c^T x+d}$, $dom(f) = \left\{x : c^T x + d > 0\right\}$.

3.2 a Halfspace

$C = \left\{ y : g^T y \leq h \right\}, g \neq 0.$

$$\begin{aligned} g^T f(x) &\leq h \\ g^T \frac{(Ax+b)}{c^T x + d} &\leq h \\ (g^T Ax + g^T b) &\leq h(c^T x + d) \\ g^T Ax &\leq h(c^T x + d) - g^T b \end{aligned}$$

So, $f^{-1}(C) = \left\{ x : g^T Ax \leq h(c^T x + d) - g^T b \right\}$, a halfspace.

3.3 b Polyhedron

$C = \{y : Gy \leq h\}$. Look at C as an intersection of halfspaces $C_i = \left\{ y : g_{i,:}^T y \leq h_i \right\}, g_{i,:} \neq 0$. Using the previous result, we have:

$$f^{-1}(C_i) = \left\{ x : g_{i,:}^T Ax \leq h_i(c^T x + d) - g^T b \right\} f^{-1}(C) = \cap_i f^{-1}(C_i). \qquad (1)$$

The inverse image is also a polyhedron.

3.4 c Ellipsoid

$C = \left\{ y : y^T P^{-1} y \leq 1 \right\}, P \in S_{++}^n.$

$$\begin{aligned} f(x)^T P^{-1} f(x) &\leq h \\ (Ax+b)^T P^{-1}(Ax+b) &\leq h(c^T x + d)^2 \end{aligned}$$

So, $f^{-1}(C) = \left\{ x : (Ax+b)^T P^{-1}(Ax+b) \leq h(c^T x + d)^2 \right\}$, some ellipsoid in R^m. Note that this ellipsoid is displaced from 0.

3.5 d Solution set of linear matrix inequality

$C = \{y : \sum y_i A_i \leq B\}; A_i, B \in S^p.$

Let 1_i be an $n \times p^2$ matrix with ith row being 1's and all other elements being 0. Let $h(A_i)$ be the vector derived by stacking up the elements of A_i in any some way. Let $A = [h(A_1)..h(A_n)]$. So, $C = \{y : Ay \leq h(B)\}$.

This is a polyhedron. As described in answer to a previous question, $f^{-1}(C)$ is also a polyhedron.

4 2.20 or 2.22: +ve solutions of linear equations

Lemma 4.0.1. $\forall x : Ax = b \implies c^T x = d$ *iff* $\exists l : c = A^T l, d = b^T l$.

Proof. If $\exists l : c = A^T l, d = b^T l$, we have that, $\forall x : Ax = b$, $c^T x = l^T Ax = l^T b = d$.

The proving the implication in the reverse direction is more interesting. Suppose that $\forall x : Ax = b \implies c^T x = d$. Take any x_n in the null space of A; ie: $Ax_n = 0$. If Ax = b, we have that $A(x + x_n) = b$. So, we also have that $c^T(x + x_n) = d$. So, c is orthogonal to the null space of A. But, the component of R^m orthogonal to this is just the row space of A, which is $R(A^T)$. So, c can be expressed as a linear combination of A's rows: $\exists l : c = A^T l$. Thence, we also derive $l^T b = l^T Ax = c^T x = d$. □

Notation. When we use inequalities with vectors, we mean to use elementwise inequalities.

Theorem 4.0.2. *pa: There exists x satisfying* $x > 0, Ax = b$.
pb: $\nexists l : A^T l \geq 0, A^T l \neq 0, b^T l \leq 0$.
Then $pa \equiv pb$.

Proof. Consider $\neg pa$. Suppose that $\exists l : A^T l = c \geq 0, b^T l = d \leq 0$. Then, using the lemma proved earlier, we see that this is equivalent to saying $\forall x : Ax = b \implies c^T x = d \leq 0 \implies \neg(x > 0)$. So, $\neg pa \equiv \neg pb$. □

5 2.23 Strict separability

The following are disjoint closed non-strictly separable convex sets.
$A = \{x : x \in R^2, x_2 \leq 1\}$.
$B = \{x : x \in R^2, x_2 \geq 1/x_1\}$.

Acknowledgement. Heard this example from nAgarAjan naTarAjan.

6 2.24 Supporting hyperplanes

6.1 a

$S = \{x \in R^2_+ : x_1 x_2 \geq 1\}$. Boundary of S, $bnd(S) = \{x \in R^2_+ : x_1 x_2 = 1\}$. Take halfspaces defined by supporting hyperplanes at $x \in bnd(S)$: $H(x) = \left\{y \in R^2_+ : \frac{(y_2 - x_2)}{y_1 - x_1} \geq -x_1^{-2}\right\}$. *Then*, $S = \cap_{x \in bnd(S)} H(x)$.

6.2 b

$C = \{x \in R^n : \|x\|_\infty \leq 1\}$. Boundary $bnd(C) = \{x : \max(x) = 1\}$. Take $x \in bnd(C)$. Then, $\forall i : x_i = 1$, define $H_i = \{y \in R^n : y_i = 1\}$. These are the supporting hyperplanes of C at x.