LINEAR ALGEBRA: ANSWER TO HOMEWORK 2

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1. Question

Prove that for any $A \in \mathbb{C}^{m\times n}$,

$$\|A\|_\infty = \max_i \|a_i^*\|_1,$$

where a_i^* is the i-th row of A.

1.1. Answer.

Notation. $a_{i,j}$ indicates the element of A at row i and column j. a_i^* is the i-th row of A. x_i and $(Ax)_i$ indicates the ith element of vector Ax.

Theorem 1.1.1. $A \in \mathbb{C}^{m\times n}$,

$$\|A\|_\infty = \max_i \|a_i^*\|_1$$

Proof.

$$\begin{aligned}
\|A\|_\infty &= \sup \frac{\|Ax'\|_\infty}{\|x'\|_\infty} \\
&= \sup \frac{\|A\alpha x\|_\infty}{\|\alpha x\|_\infty},\ \textit{with}\ \|x\|_\infty = 1 \texttt{ or } \max(|x_i|) = 1 \texttt{ constant } \alpha \\
&= \sup \|Ax\|_\infty
\end{aligned}$$

We now see when $\|Ax\|_\infty$ is maximum, when $\|x\|_\infty = 1$.

$$\begin{aligned}
\|Ax\|_\infty &= \max_{i,x} |(Ax)_i| \texttt{ By definiton} \\
&= \max_{i,x} |a_i^* x| \\
&= \max_{i,x} |\sum_{j=1}^n a_{i,j}x_i| \leq \max_{i,x} \sum_{j=1}^n |a_{i,j}x_i| \\
&\quad \texttt{We now maximize with respect to } x_i \\
&= \max_{i,x} \sum_{j=1}^n |a_{i,j}||x_i| \texttt{ We ensure: } \quad sign(a_{i,j}) = sign(x_i) \\
&= \max_i \sum_{j=1}^n |a_{i,j}| \texttt{ As: } \quad \max(|x_i|) = 1 \texttt{ We choose: } \quad |x_i| = 1 \\
&= \max_i |a_i^*|_1
\end{aligned}$$

So:

$$\|A\|_\infty = \max_i \|a_i^*\|_1$$

□

2. Question

Assume $A \in \mathbb{C}^{n\times n}$ and $\exists\ p \geq 1$, s.t. $\|A\|_p < 1$, where $\|.\|_p$ is a vector-induced matrix norm.

(1) Prove that $I - A$ is invertible.
(2) Assuming that the series $\sum_{k=0}^{\infty} A^k$ converges, prove that:

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

(3) Prove that:

$$\|A\|_q \|A^{-1}\|_q \geq 1, \quad \forall 1 \leq q < \infty.$$

(4) Prove that:

$$\frac{1}{1+\|A\|_p} \leq \|(I-A)^{-1}\|_p \leq \frac{1}{1-\|A\|_p}.$$

2.1. **Answer.**

Notation. 0 is also used to indicate the 0 vector, whose dimension may be inferred from the context.

2.1.1. *1.*

Theorem 2.1.1. *$I - A$ is invertible.*

Proof.

$$\begin{array}{rcl}
\|x-Ax\|_p + \|Ax\|_p & \geq & \|x - Ax + Ax\|_p\texttt{:Triangle inequality} \\
\|x-Ax\|_p & \geq & \|x\|_p - \|Ax\|_p \\
\|(I-A)x\|_p & \geq & \|x\|_p - \|Ax\|_p \\
\texttt{But, as } \|A\|_p < 1, \forall x\texttt{:}\quad \dfrac{\|Ax\|_p}{\|x\|_p} & < & 1 \\
\therefore \|(I-A)x\|_p & > & \|x\|_p - \|x\|_p = 0
\end{array}$$

So, I-A does not map any $x \neq 0$ to the 0 vector. So, the null space of I-A consists only of 0. □

2.1.2. *2.*

Theorem 2.1.2. *Assuming that the series $\sum_{k=0}^{\infty} A^k$ converges,*

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Proof.

$$\begin{aligned}(I-A)(I-A)^{-1} &= (I-A)(\sum_{k=0}^{\infty} A^k) \\ &= I(\sum_{k=0}^{\infty} A^k) - A(\sum_{k=0}^{\infty} A^k) \\ &= \sum_{k=0}^{\infty} A^k - \sum_{k=1}^{\infty} A^k \\ &= A^0 = I\end{aligned}$$

□

2.1.3. *3.*

Theorem 2.1.3.

$$\|A\|_q \|A^{-1}\|_q \geq 1, \ \forall 1 \leq q < \infty, \ \forall.$$

Proof. It is shown in [1] that: $\|A\|_q \|B\|_q \geq \|AB\|_q$. So:

$$\|A\|_q \|A^{-1}\|_q \geq \|AA^{-1}\|_q = 1, \ \forall 1 \leq q < \infty.$$

□

2.1.4. *4.*

Notation. $\|.\|_p$ will be abbreviated to $\|.\|$.

Lemma 2.1.4.

$$\frac{1}{1+\|A\|_p} \leq \|(I-A)^{-1}\|_p.$$

Proof.

$$\begin{aligned}\|(I-A)^{-1}(I-A)\| &\leq \|(I-A)^{-1}\|\|(I-A)\| : \ \texttt{By Theorem shown earlier.} \\ 1 &\leq \|(I-A)^{-1}\|\|(I-A)\| \\ \frac{1}{\|(I-A)\|} &\leq \|(I-A)^{-1}\|\end{aligned}$$

But:

$$\begin{aligned}\|(I-A)\| &\leq \|I\| + \|-A\| : \ \texttt{Triangle inequality} \\ &= \|I\| + \|A\| \\ \frac{1}{\|(I-A)\|} &\geq \frac{1}{1+\|A\|}\end{aligned}$$

So, we have the result. □

Lemma 2.1.5.

$$\|(I-A)^{-1}\|_p \leq \frac{1}{1-\|A\|_p}.$$

Proof.

$$
\begin{aligned}
\frac{\|Ax\|_p}{\|x\|_p} &\leq \|A\|_p \\
\|Ax\|_p &\leq \|A\|_p \|x\|_p \\
\frac{\|A^2x\|_p}{\|Ax\|_p} &\leq \|A\|_p \\
\|A^2x\|_p &\leq \|A\|_p\|Ax\|_p \leq \|A\|_p^2\|x\|_p \\
\therefore \|A^kx\|_p &\leq \|A\|_p^k\|x\|_p \texttt{Provable by induction} \\
\therefore \|A^k\|_p &\leq \|A\|_p^k \\
\|(I-A)^{-1}\| &= \|\sum_{k=0}^{\infty} A^k\| \\
&\leq \sum_{k=0}^{\infty} \|A^k\| \texttt{ Triangle inequality} \\
&= \sum_{k=0}^{\infty} \|A\|_p^k \\
&= \frac{1}{1-\|A\|_p}
\end{aligned}
$$

□

From the above lemmata, we have the proof.

3. Question

Consider the following procedure to approximate the SVD of a given square matrix $A = U\Sigma V^T$, where $A, U, \Sigma, V \in \Re^{n\times n}$:

(1) Initialize U, Σ, V to I.
(2) Assuming U, Σ fixed, compute V and orthogonalize it.
(3) Assume U, V fixed, compute Σ. Ensure that Σ is diagonal and positive.
(4) Assuming Σ, V fixed, compute U and orthogonalize it.
(5) If $\|A - U\Sigma V^T\|_F \geq tol$, repeat steps (ii)-(iv).

For simplicity, assume that A is an invertible matrix.

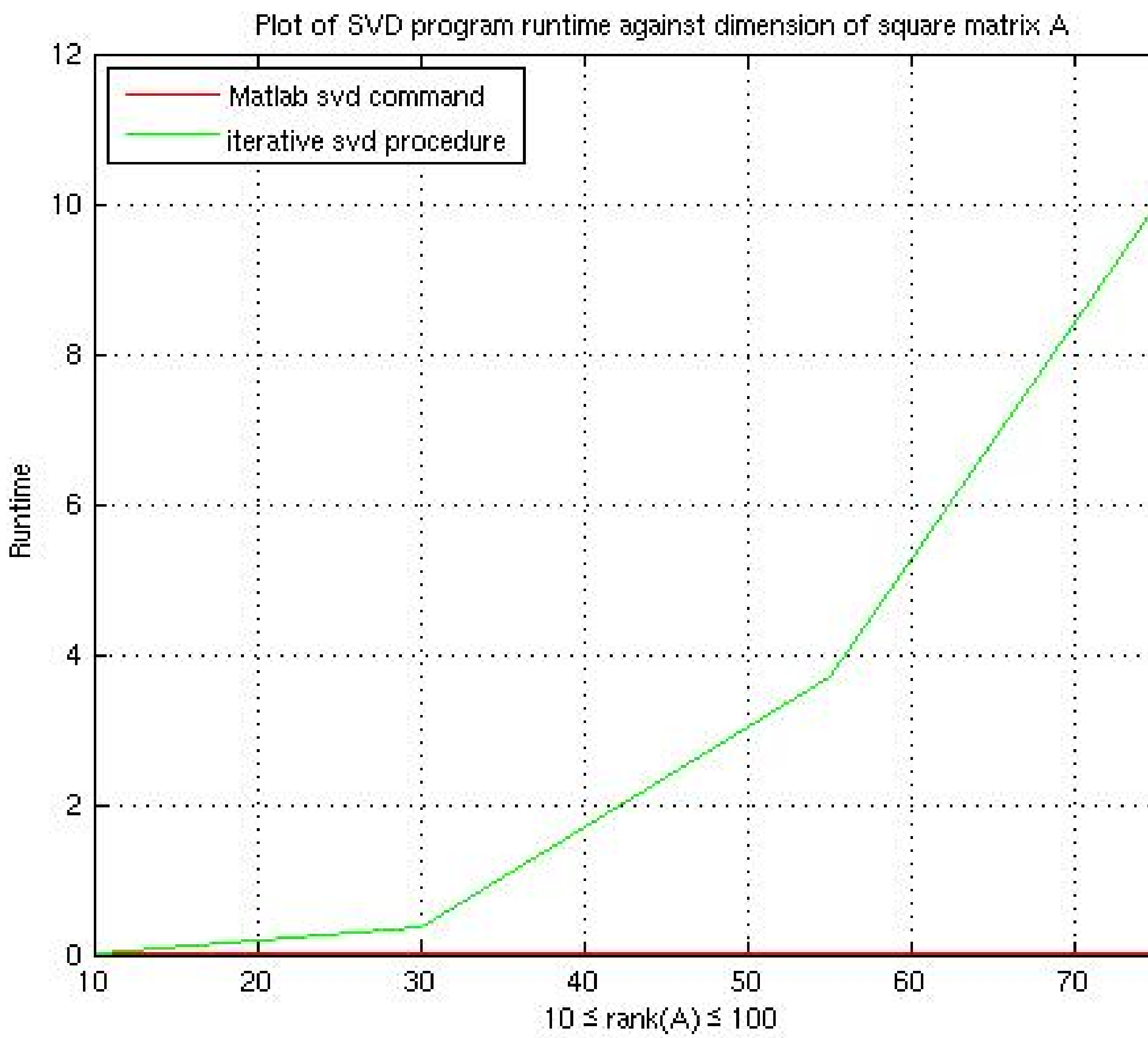
(1) Implement the above given procedure using Matlab and verify that it converges to the SVD given by Matlab's `svd` command. For verification, run the above given procedure with $tol = 1e-5$ for 10 different 50x50 random matrices. Compress all your matlab code in one file and email it to the TA with the subject "NLA:HW2". You can use Matlab's `qr` function for orthogonalization. YOU CAN USE MATLAB's STANDARD FUNCTIONS, BUT DO NOT USE ANY CODE FROM THE WEB.
(2) Compare the time required by this procedure to that of Matlab's `svd` command. Generate a plot of the time required by your implementation to that of Matlab's `svd` command while varying size of input matrix from 10 to 100. Average your results over 10 different runs. Use matlab's `tic` and `toc` command to measure the elapsed time.

3.1. **Answer.** The code has been emailed to the TA.

Remark 3.1.1. It turns out that for the procedure to work, a certain way of calculating V should be used:

V = inv(A)*U*S; Won't lead to convergence. V = (inv(S)*U'*A)'; Gives different V, leads to convergence!

The graph is shown below (The faster procedure corresponds to the Matlab svd command):



References

[1] Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra*. Siam, 1997.