Problème de Ménages

Vasco Villas-Boas

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INTRODUCTION

The *Problème de Ménages*, which translates to "Problem of Households", is an exercise in combinatorics. Formulated by Edoard Lucas in 1891, it asks for the number of ways that n couples can sit around a circular table of $2n$ seats such that the seatings alternate "boy, girl, boy, girl,..." and no couples are seated side-by-side. I provide intuition to understand the "sexist" solution to the problem first stated by Touchard (1934) and proved by Kapalansky (1943)- this solution first seats the women, by convention, and then counts all satisfactory ways that the men can sit. I also look at the Bogart and Doyle's (1986) "non-sexist" solution- this solution counts all of the possible seatings all at-once instead of giving preference to the women. The solutions rely heavily on the solution to the *Stars and Bars Problem* and on the *Inclusion-Exclusion Principle*. I'll introduce these two topics and build to the solution of the *Problème de Ménages*.

Finally, I extend the *Problème de Ménages* to a counting problem of etiquette that I've experience in my life in the inspiration for this writing. My grandmother likes to sit all married couples separately and all dating couples together and I'd like to count many ways this is possible. Formally, suppose there are n couples that will occupy $2n$ seats around a circular table. I'd like to count the number of seatings such that men and women alternate and such that some fixed subset of the couples D (where $k := |D|$) sit together.

STARS AND BARS PROBLEM

The *Stars and Bars Problem* asks how many ways you can throw n indistinguishable balls at k distinguishable bins. To match terminology, the *stars* are the balls and the *bars* are the bins.

Figure 1. Throwing 5 Indistinguishable Balls at 5 Distinguishable Bins

To gain intuition, let's look at a a specific example. Consider counting the number of ways that we can throw 5 indistinguishable balls at 5 distinguishable bins. We can reduce the problem to counting the number of strings of five "stars" and four "bars" like in Figure 1. The number of "stars" to the left of the first "bar" notes how many balls go into the first bin, the number of stars between the first and second bars notes the number of balls that go to the second bin, the number of stars between the second and third bars denote the number of balls that go the second bin, etc. In Figure 1, we've distributed one ball to the first bin, one ball to the second bin, three balls to the third bin, and zero balls to the fourth and fifth bins.

The solution to counting the number of strings with 5 stars and 4 bars is fairly straightforward: $\frac{9!}{5!4!} = \binom{9}{5}$.

Back to the general problem: how many ways can we throw n indistinguishable balls at k distinguishable bins. This will reduce to a *stars and bars problem* with n stars and $k-1$ bars. As a result, the solution is $\binom{n+k-1}{n}$.

Number of Monomials that Have Degree Less Than n

Here I give a slightly more complicated application of the *stars and bars* combinatorics approach. Suppose we have $x := (x_1, ..., x_k)$ for $k \in \mathbb{Z}^+$. How many monomials can you form by multiplying elements of x such that the sum of the exponents of the monomial is less than or equal to n for $n \in \mathbb{Z}^+$?

To answer this question, first introduce $x' := (1, x_1, ..., x_k)$. The problem reduces to finding the number of monomials we can form by multiplying elements of x' such that the sum of the exponents of the monomial is equal to n. To see this, in the reduction, the monomials of x and x' always turn out the same after mathematical simplification. The only difference is that for any monomial formed by the terms of x that has sum of degrees $s < n$, we augment the monomial of x' by raising $1^{n-s} = 1$.

The problem of x' reduces to a stars and bars problem of n stars and k bars where each element of x' becomes a bin and we have n balls ("degrees") to distribute amongst these elements. We in turn state that the solution to the original problem is $\binom{n+k}{n}$.

Selecting Non-consecutive Elements from an Array

Suppose we have an array $a := [1, 2, ..., n]$ for $n \in \mathbb{Z}^+$. How many ways can you select k elements from a such that no two consecutive elements are selected? So that the problem is interesting, we impose $2k < n$.

This problem reduces to a *stars and bars problem* of $n - k - (k - 1) = n - 2k + 1$ "stars" and k "bars". To see this, for each of the k elements of a picked, the following one cannot be picked. This means for each of the first $k - 1$ elements picked, we're by construction picking paris of the chosen element and the one after. We don't have this problem in the case of the kth element chosen since we have no more elements left to pick. For the remaining $n - 2k - (k - 1)$ elements, we choose if they go before the first element chosen, between the first and the second, between the second and the third, etc.

The answer to this problem is $\binom{n-k+1}{n-2k+1} = \binom{n-k+1}{k}$.

Dominos on a Circle

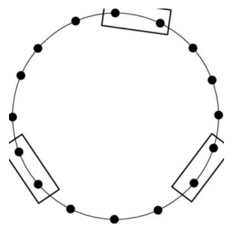

Figure 2. Non-overlapping Dominos on a Circle

Suppose we have a circle of size n dots. How many ways can we can put k directionless dominos (that cover exactly two dots) over the dots on the circle such that no dominos overlap? Again, to make the problem interesting, we impose $2k < n$. Figure 2 shows a sample legal placement of dominos with $n = 16$ dots and $k = 3$ dominos.

This problem is very similar to the prior one in that we're picking k non-consecutive dots on a circle of n dots- the k dots we pick can be interpreted as the clock-wise first element of each of the k non-overlapping dominos placed. The main difference to the prior problem is that we're now picking from elements on a circle which adds some subtleties.

To solve this problem, first see that in a legal placement of dominos, $2k$ dots will lie under a domino and $n - 2k$ dots will not lie under a domino. Let's "linearize" the dominos: arbitrarily place down a domino on any two dots and call the second domino the one that appears clockwise after the first, the third the one that appears clockwise after the second, ..., and the kth the one that appears clockwise before the first.

We have $n - 2k$ dots that must be placed in one of k bins: between the first and second domino, between the second and third domino, ..., between the kth and first domino. This observation suggests that after fixing the first domino, we have a reduction to the *stars and bars problem* of $n - 2k$ "stars" and $k - 1$ "bars" where the "stars" are the locations of the uncovered dots and the "bars" are the locations of the remaining $k - 1$ dominos. We know that the answer to that problem is $\binom{n-k-1}{n-2k} = \binom{n-k}{k-1}$.

Note that we arbitrarily placed a domino and declared it the first and there are n places to put that first domino. Also, note that for any placement of dominos, any of the k dominos can be labeled as the first domino. Thus, we say that our answer to the original question is $\frac{n}{k}\binom{n-k-1}{k-1} = \frac{n}{n-k}\binom{n-k}{k}$.

INCLUSION-EXCLUSION PRINCIPLE

As Wikipedia explains, "the *Inclusion-Exclusion Principle* is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets". Mathematically, for finite sets A, B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Visually, in Figure 3, we see that when counting the number of elements in the union of A and B, when we add the number of elements of A to the number in B, we double-count the number of elements in the intersection of A and B and so must subtract that out to find the cardinality of the union.

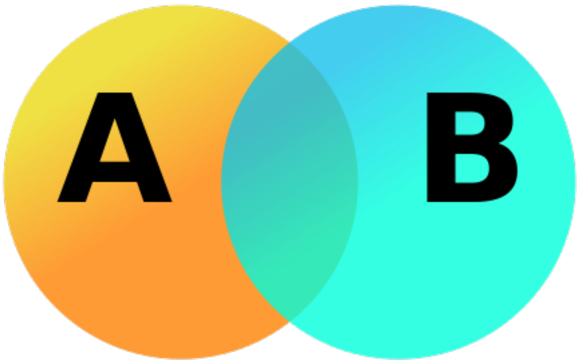

Figure 3. Inclusion-Exclusion Principle Visually

Suppose we're trying to find the cardinality of the union of sets $A_1, ..., A_n$ for $n \in \mathbb{Z}^+$. We can find this union using the inclusion-exclusion principle as follows:

$$|\cup_{i=1}^{n} A_i| = \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|+1} |\cap_{j \in J} A_j| \tag{1}$$

It will also be useful for us to see the inclusion-exclusion principle in its complementary form (where $\bar{A}_i$ denotes the complement of set A_i with respect to set of all elements S):

$$|\cap_{i=1}^{n} \bar{A}_i| = |S - \cup_{i=1}^{n} A_i| = |S| - \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|} |\cap_{j \in J} A_j| \tag{2}$$

Four People Lining Up for Doughnuts

Suppose six individuals of different heights line up for doughnuts. How many ways can the six individuals line up so that no three consecutive individuals are in ascending height order[1]?

To solve the problem using the *inclusion-exclusion principle*, define the following sets. Let A_1 denote the orderings where slots 1,2,3 are in ascending order, let A_2 denote the orderings where slots 2,3,4 are in ascending order, let A_3 denote the orderings where slots 3,4,5 are in ascending order, and let A_4 denote the orderings where 4,5,6 are in ascending order.

We're looking for the quantity $|\cap_{i=1}^{4} \bar{A}_i|$. By equation 2, we know that

[1] This problem is courtesy of the *Art of Problem Solving*.

$$
\begin{aligned}
|\cap_{i=1}^4 \bar{A}_i| &= |S| - \sum_{\emptyset \neq J \subseteq \{1,..,4\}} (-1)^{|J|} |\cap_{j \in J} A_j| \\
&= |S| - \sum_{i=1}^4 |A_i| + \sum_{i,j \in \{1,...,4\}, i<j} (|A_i \cap A_j|) - \sum_{i,j,k \in \{1,..,4\}, i<j<k} (|A_i \cap A_j \cap A_k|) + |A_1 \cap A_2 \cap A_3 \cap A_4|
\end{aligned}
$$

Now we need to calculate the cardinalities requisited. $|S|$, the total number of orderings for 6 people, equals $6! = 720$. $|A_i|$, the number of orderings where $i, i+1, i+2$ are in order, equals $\binom{6}{3}3! = 120$. $|A_1 \cap A_2|$, the number of orderings where 1,2,3,4 are in order, equals $\binom{6}{4}2! = 30$. $|A_1 \cap A_3|$, the number of orderings where 1,2,3,4,5 are in order, equals $\binom{6}{5}1! = 6$. $|A_1 \cap A_4|$, the number of orderings where 1,2,3 and 4,5,6 are in order, equals $\binom{6}{3} = 20$. $|A_2 \cap A_3|$, the number of orderings where 2,3,4,5 are in order, equals, $\binom{6}{4}2! = 30$. $|A_2 \cap A_4|$, the number of orderings where 2,3,4,5,6 are in order, equals $\binom{6}{5}1! = 6$. $|A_3 \cap A_4| = \binom{6}{4}2! = 30$. $|A_1 \cap A_2 \cap A_3|$, the number of orderings that have 1,2,3,4,5 in order, equals $\binom{6}{5}1! = 6$. $|A_1 \cap A_2 \cap A_4| = 1$. $|A_1 \cap A_3 \cap A_4| = 1$. $|A_2 \cap A_3 \cap A_4| = \binom{6}{5}1! = 6$. Finally, $|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$.

Thus, we say the answer is,

$$
\begin{aligned}
|\cap_{i=1}^4 \bar{A}_i| &= 720 - 4 * 120 + 30 + 6 + 20 + 30 + 6 + 30 - 6 - 1 - 1 - 6 + 1 \\
&= 349
\end{aligned}
$$

Palindromes in License Plates

Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, what is the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left)[2]?

Let A_1 denote the set of license plates that have a three-letter palindrome and A_2 denote the set of license plates that have a three-number palindrome. Let X denote a randomly chosen license plate.

$$
\begin{aligned}
\Pr(X \in A_1 \cup A_2) &= \Pr(X \in A_1) + \Pr(X \in A_2) - \Pr(x \in A_1 \cup A_2) \\
&= \frac{26^2}{26^3} + \frac{10^2}{10^3} - (\frac{26^2}{26^3})(\frac{10^2}{10^3}) \\
&= \frac{7}{52}
\end{aligned}
$$

Counting Derangements

Consider a plane that has n seats assigned to n passengers. What is the probability that if the n passengers sit randomly, no one ends up in their assigned seat? An ordering where no passenger ends up in their seat is called a *derangement*.

Let A_i for $i \in \{1, .., n\}$ denote the event that person i ends up in their assigned seat. Also, let X denote a randomly selected seating of the n individuals.

We're looking for the quantity $\Pr(X \in \cap_{i=1}^n \bar{A}_i)$. We'll answer the question using set cardinality. Let S denote the set of all possible seatings so that $|S| = n!$. We need to find the count $|\cap_{i=1}^n \bar{A}_i|$, because $\Pr(X \in \cap_{i=1}^n \bar{A}_i) = \frac{|\cap_{i=1}^n \bar{A}_i|}{|S|}$. By equation 2,

$$
|\cap_{i=1}^n \bar{A}_i| = |S - \cup_{i=1}^n A_i| = |S| - \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|} |\cap_{j \in J} A_j|
$$

Looking at that summation, each contributing term depends simply on the cardinality of J by symmetry- there's nothing special in this setting about a particular set of people sitting in their seats. Observe that for a particular cardinality of J,

[2] This problem is also courtesy of the *Art of Problem Solving*.

$k := |J|$, there are $\binom{n}{k}$ ways to pick a subset of $\{1, ..., n\}$ of size k people that are seated in their seat. For each set of k people sitting in their seats, there are $(n-k)!$ ways to sit the other guys. Thus, we can say that:

$$\begin{aligned}
|\cap_{i=1}^n \bar{A}_i| &= |S| - \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|} |\cap_{j \in J} A_j| \\
&= |S| - \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= |S| - \sum_{k=1}^{n} (-1)^k \frac{n!}{k!}
\end{aligned}$$

Then,

$$\begin{aligned}
\Pr(X \in \cap_{i=1}^n \bar{A}_i) &= \frac{|\cap_{i=1}^n \bar{A}_i|}{|S|} \\
&= \frac{|S| - \sum_{k=1}^n (-1)^k \frac{n!}{k!}}{|S|} \\
&= \frac{n! - \sum_{k=1}^n (-1)^k \frac{n!}{k!}}{n!} \\
&= 1 - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \\
&\approx \frac{1}{e}
\end{aligned}$$

"SEXIST" SOLUTION TO THE PROBLÈME DE MÉNAGES

Recall that the *Problème de Ménages* asks for the number of ways that n couples can sit around a circular table of $2n$ seats such that the seatings alternate "boy, girl, boy, girl,..." and no couples are seated side-by-side.

In the "sexist" solution, by convention, we first seat the n women. There are $2n!$ ways to seat the women, since any seating of women can be permuted and the women can sit on the "odd" or "even" seats as they must alternate.

For each seating of women, we need to count the number of ways that we can seat the men so that no couple is seated next to each other. In that spirit, let's fix a seating of women. Let A_i for $i \in \{1, ..., n\}$ denote the set of orderings where couple i is seated together and let S denote all possible seatings with the women fixed in this way. We're looking for the quantity $|\cap_{i=1}^n \bar{A}_i|$. By the principle of inclusion-exclusion,

$$|\cap_{i=1}^n \bar{A}_i| = |S| - \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|} |\cap_{j \in J} A_j|$$

The domino problem from earlier is very useful here: as a reminder the problem stated how many ways can we can put k directionless dominos (that cover exactly two dots) over the $2n$ dots on the circle such that no dominos overlap? Recall that the answer to that problem is $\frac{2n}{2n-k}\binom{2n-k}{k}$.

Like in the derangement problem, each term in the summation above depends only on the cardinality of the set J. For any cardinality of J, $|J| = k$, we can think that we're placing the k dominos over women that will be sitting with their partner, the men will take the other seat. For each placement of dominos, the remaining $n-k$ men may sit wherever they like (ie., there are $(n-k)!$ ways they can sit). For each k, we're counting the number of ways that at least k couples are seated together given the fixed seating of women. That's because we have our k couples that sit together and then some other couples could end up sitting together by chance in the random assignment of the remaining $n-k$ men. As a result,

$$
\begin{aligned}
|\cap_{i=1}^n \bar{A}_i| &= |S| - \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|} |\cap_{j\in J} A_j| \\
&= n! - \sum_{k=1}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!
\end{aligned}
$$

That final expression is equal to the number of legal ways to seat the men given the fixed seating of women. We must multiply that value by $2n!$, the number of ways to sit the women first, to get the final answer:

$$2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!$$

”Non-sexist” Solution to the Problème de Ménages

In the ”non-sexist” solution, we sit couples at once. That’s different from the ”sexist” solution where we fixed a seating of women before proceeding.

Like before, let S be the set of all possible seatings where men and women alternate, and define $A_i \subseteq S$ for $i \in \{1, ..., n\}$ to be the set of orderings where couple i is seated together. Note that $|S| = 2n!^2$ since there are two spots the women could take $n!$ ways to arrange the women and $n!$ ways to arrange the men amongst their spots. We’re looking for the quantity $|\cap_{i=1}^n \bar{A}_i|$. By the principle of inclusion-exclusion,

$$|\cap_{i=1}^n \bar{A}_i| = |S| - \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|} |\cap_{j\in J} A_j|$$

Again, each term of the summation depends only on the cardinality of set J chosen by symmetry. Also, the domino problem is again useful. For any cardinality of set J, $|J| = k$ that defines a set of k couples we want seated together, there are $\frac{2n}{2n-k}\binom{2n-k}{k}$ ways to pick k spots for k couples to sit together, there are $\binom{n}{k}$ ways to pick k couples to take those spots, there are $k!$ ways to arrange the k couples amongst those spots, there are 2 ways to pick the spot that the women will take, and there are $(n-k)!$ ways to place the remaining $n-k$ women in their open spots and $(n-k)!$ ways to place the remaining $n-k$ men in their open spots. Thus,

$$
\begin{aligned}
|\cap_{i=1}^n \bar{A}_i| &= |S| - \sum_{\emptyset \neq J \subseteq \{1,..,n\}} (-1)^{|J|} |\cap_{j\in J} A_j| \\
&= |S| - \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{2n}{2n-k} \binom{2n-k}{k} k! 2 (n-k)!^2 \\
&= 2n!^2 - \sum_{k=1}^{n} (-1)^k 2n! \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)! \\
&= 2n!(n! - \sum_{k=1}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!) \\
&= 2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!
\end{aligned}
$$

That aligns with the count we found using the ”sexist” solution.

EXTENSION TO THE PROBLÈME DE MÉNAGES

If you recall, in the introduction, I proposed an extension to the *Problème de Ménages*: suppose there are n couples that will occupy $2n$ seats around a circular table; how many seatings are there such that men and women alternate and such that some fixed subset of the couples D (where $k := |D|$) sit together? Let's call the answer x.

We can't exactly proceed as above. Without loss of generality, suppose that we require that couples $\{1, ..., k\}$ sit together and the rest do not. Then, define S as all sets of seatings that alternate men and women. If we try to define $A_i \subseteq S$ for $i \in \{1, ..., n\}$ as the set of seatings where couple i sits together, we cannot proceed in the same way. That's because this time, we're looking for the quantity $|(\cap_{i=1}^{k} A_i) \cap (\cap_{i=k+1}^{n} \bar{A}_i)|$.

Consider an alternative way of thinking about the problem. Now let's redefine S to be the set of all seatings that alternate men and women and have the k desired couples seated together. Now define, $B_i \subseteq S$ for $i \in \{k+1, ..., n\}$ as the set of seatings in S that also have couple i seated together. We're looking for the quantity $| \cap_{i=k+1}^{n} \bar{B}_i|$. By the principle of inclusion-exclusion,

$$| \cap_{i=k+1}^{n} \bar{B}_i| = |S| - \sum_{\emptyset \neq J \subseteq \{k+1,..,n\}} (-1)^{|J|} | \cap_{j \in J} B_j|$$

First note that $|S| = \frac{2n}{2n-k}\binom{2n-k}{k} k! 2(n-k)!^2$. We use the fact that there are $\frac{2n}{2n-k}\binom{2n-k}{k}$ ways to pick k spots for the k couples to sit together (see the domino problem), $k!$ ways to arrange the couples in those k spots, 2 ways to pick where the women sit, and $(n-k)!$ ways to sit the remaining $n-k$ women (vis-a-vis for the remaining men). Like in the *Problème de Ménages*, each term of the summation depends only on the magnitude of the set J. For $|J| = m$, we're looking for the number of ways that $k+m$ couples can sit together (those initial k couples sit together and m more). For each m, there are $\binom{n-k}{m}$ ways to pick the additional m couples that will be seated together, $\frac{2n}{2n-(k+m)}\binom{2n-(k+m)}{k+m}$ ways to pick the $k+m$ spots for these couples to sit together, $(k+m)!$ ways to arrange the $k+m$ couples in these spots, 2 ways to pick where the women sit, and $(n-(k+m))!$ ways to sit the remaining women (vis-a-vis for the remaining men). Thus,

$$\begin{aligned}
| \cap_{i=k+1}^{n} \bar{B}_i| &= |S| - \sum_{\emptyset \neq J \subseteq \{k+1,..,n\}} (-1)^{|J|} | \cap_{j \in J} B_j| \\
&= \frac{2n}{2n-k}\binom{2n-k}{k} k! 2(n-k)!^2 - \sum_{m=1}^{n-k} (-1)^m \binom{n-k}{m} \frac{2n}{2n-(k+m)} \binom{2n-(k+m)}{k+m} (k+m)! 2(n-(k+m))!^2 \\
&= \sum_{m=0}^{n-k} (-1)^m 2 \binom{n-k}{m} \frac{2n}{2n-(k+m)} \binom{2n-(k+m)}{k+m} (k+m)!(n-(k+m))!^2 \\
&= 2(n-k)! \sum_{m=0}^{n-k} (-1)^m \frac{(k+m)!}{m!} (\frac{2n}{2n-(k+m)}) \binom{2n-(k+m)}{k+m} (n-(k+m))!
\end{aligned}$$

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