St. Petersburg Paradox Resolutions

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Statement of the Game

A player will be offered a game where a fair coin (Pr$(H) = 0.5$) will be tossed until a tails is flipped. The initial stake is \$2 and the stake will be doubled at each stage if a heads is flipped. The player will receive the stake when the first tails is flipped and the game concludes. To give some examples, if tails is first flipped on the first toss, the player will receive \$2; if tails is first flipped on the second toss, the player will receive \$4.

The question of interest is how much would a 'rational' player pay to play this game?

A lower bound on the price is \$2 since in every possible outcome the player will receive at least \$2. However, \$2 is not fair from the perspective of the counter-party, as in the best case, they will stay even and the other cases, they will lose money.

Divergent Expected Value of the Game

A well known result is that the expected payout from this game is infinite. To see this, let P denote the payoff for the player.

$$\begin{aligned}
\mathbb{E}[P] &= \sum_{k=1}^{\infty} \text{Pr(First tails on toss } k\text{)Payoff(first tails on toss } k\text{)} \qquad (1)\\
&= \sum_{k=1}^{\infty} (1/2)^k 2^k \\
&= \sum_{k=1}^{\infty} 1 \\
&= \infty
\end{aligned}$$

Would any 'rational' player pay infinite dollars to pay this game? No, since in any conclusion of the game, the player will only receive a finite payoff. In other words, the expected payoff from the game is not a good framework to determine how 'rational' individuals would act when presented with this game.

Resolving the Paradox

In the subsequent sections, I will propose frameworks a 'rational' individual could take to come up with a reasonable finite price for the game.

Percentile Approach

Instead of finding the expected payoff from the game, one could conceive finding the median payoff from the game. That would be the payoff $\$p$ so that at least 50% of the time, the individual makes $\$p$. In this case, that's very easy to calculate–Percentile$(50, P) =$ Median$(P) = \$4$. For that matter, one could find the xth percentile payoff from the game too. For any fixed x, Percentile(x, P) is a plausible finite price for the game.

UTILITY OF WEALTH

A utility function $u(\cdot)$ on wealth is a measure of how happy an individual is with a given wealth level w. When one attempts to find a price for the St. Petersburg Paradox game using the raw expected value, one subconsciously assumes a linear utility of wealth, $u(w) = w$. While a linear utility function is a reasonable approximation at small changes in wealth levels, given the potentially very large payouts in this game that could really impact a person's savings, it's reasonable to consider a concave utility function that evokes diminishing returns to wealth.

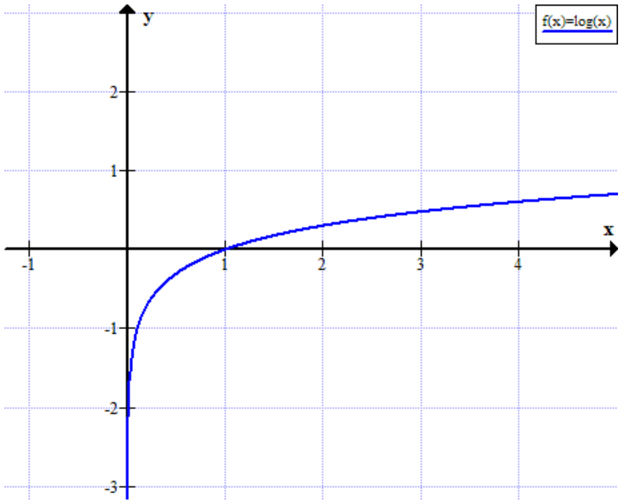

Figure 1. Log Utility

For intuition purposes, suppose you start with wealth level $w < \$1,000,000$ and receive two consecutive payments of $\$5$. You'd be about as happy when you receive the first payment as when you receive the second– the benefit to your quality of life would be the same in both cases. Next, suppose you receive two consecutive payments of $\$1$ billion. You would be very happy to receive the first payment and happy with the second payment too but less happy than after the first– the first payment would now have a much larger on your quality of life than the second.

A utility function like log utility plotted in Figure 1 mathematically captures the experience of the individual in both examples above. Small changes in wealth can be reasonably approximated as a first-order (ie., linear) approximation to the log function whereas for larger changes in wealth, one desires some concavity to the utility function to convey that each additional dollar causes less additional happiness as wealth levels rise.

Mathematically speaking, one typically imposes that the utility function $u(\cdot)$ is monotonically increasing (ie., $u'(w) > 0$) and exhibits diminishing returns to wealth ($u''(w) < 0$).

Utility of Wealth in the St. Petersburg Paradox

Suppose that the individual facing the St. Petersburg Paradox game starts with wealth w_0. Also, let W denote the individuals wealth after the game and let c denote the price to play the game.

$$\begin{aligned}
\mathbb{E}[u(W)] &= \sum_{k=1}^{\infty} \text{Pr(First tails on toss } k\text{)}\; u(\text{Wealth(first tails on toss } k\text{))} \\
&= \sum_{k=1}^{\infty} (1/2)^k u(w_0 + 2^k - c)
\end{aligned}$$

To find an individuals fair price for the game, one searches for the c such that:

$$
\begin{aligned}
u(w_0) &= \mathbb{E}[u(W)] \\
&= \sum_{k=1}^{\infty} (1/2)^k u(w_0 + 2^k - c)
\end{aligned}
$$

In other words, one searches for the price of the game c so that the individual is indifferent between playing the game and not playing the game. If $u(w_0) > \mathbb{E}[u(W)]$, the individual will prefer to not play the game and if $u(w_0) < \mathbb{E}[u(W)]$, the individual will prefer to play the game. One can rearrange the above indifference condition:

$$
\begin{aligned}
0 &= \mathbb{E}[\Delta u(\text{Playing the game})] \\
&= \mathbb{E}[u(W) - u(w_0)] \\
&= \mathbb{E}[u(W)] - u(w_0) \\
&= \sum_{k=1}^{\infty} (1/2)^k (u(w_0 + 2^k - c) - u(w_0))
\end{aligned} \tag{2}
$$

Interpreting the condition in this format, c is fairly chosen for the individual if the expected change in utility from playing the game is 0. If $0 < \mathbb{E}[\Delta u(\text{Playing the game})]$, then the expected change in utility from the playing the game is positive and so the individual will want to play. If $0 > \mathbb{E}[\Delta u(\text{Playing the game})]$, then the expected change in utility from the playing the game is negative and so the individual will not want to play.

Comparative Statics on the Cost of the Game

Of note, the cost that establishes the indifference condition depends on the initial wealth w_0. In other words, the amount of wealth an individual would be willing to pay to play the game is a function $c(w_0)$. Interestingly, how c varies with w_0 depends on the shape of the utility function and the initial wealth w_0, even when one imposes that $u(\cdot)$ is increasing and concave. One can consider the function,

$$
0 = F(c(w_0), w_0) = \sum_{k=1}^{\infty} (1/2)^k (u(w_0 + 2^k - c) - u(w_0))
$$

For convenience,

$$
\begin{aligned}
F_c &= -\sum_{k=1}^{\infty} (1/2)^k u'(w_0 + 2^k - c) \\
F_{w_0} &= \sum_{k=1}^{\infty} (1/2)^k (u'(w_0 + 2^k - c) - u'(w_0))
\end{aligned}
$$

Notice that $F_c < 0 \; \forall c$ since we assumed that $u'(\cdot) > 0$. Thus, we can apply the implicit function theorem to say that,

$$
\frac{\mathrm{d}c(w_0)}{\mathrm{d}w_0} = -\frac{F_{w_0}}{F_c}
$$

The sign of the derivative $\frac{\mathrm{d}c(w_0)}{\mathrm{d}w_0}$ will be determined by the sign of F_{w_0}, and is not globally conclusive. Again, the sign depends on the shape of $u(\cdot)$ (ie., where it's 'concave') and w_0. To see this, let $\bar{k}$ denote the largest k such that $2^k < c$ so that $u'(w_0 + 2^k - c) > u'(w_0)$ for $k \leq \bar{k}$. Note that $u(\cdot)$ is concave which means that $x > y \implies u'(x) < u'(y)$. Thus,

$$
\begin{aligned}
\text{sign}(\frac{\mathrm{d}c(w_0)}{\mathrm{d}w_0}) &= \text{sign}(F_{w_0}) \\
&= \text{sign}(\sum_{k=1}^{\infty}(1/2)^k(u'(w_0+2^k-c)-u'(w_0))) \\
&= \text{sign}(\underbrace{\sum_{k=1}^{\bar{k}}(1/2)^k(u'(w_0+2^k-c)-u'(w_0))}_{\text{positive terms}} + \underbrace{\sum_{k=\bar{k}+1}^{\infty}(1/2)^k(u'(w_0+2^k-c)-u'(w_0))}_{\text{negative terms}})
\end{aligned}
$$

The sign of $\frac{\mathrm{d}c(w_0)}{\mathrm{d}w_0}$ depends on the size of the positive terms and negative terms which depend on the shape of $u(\cdot)$ and w_0.

Log Utility

My last stop on the topic expected utility is to write the indifference condition of Equation 2 with the log utility function (ie., $u(\cdot) = \log(\cdot)$).

$$
0 = \sum_{k=1}^{\infty}(1/2)^k(\log(w_0+2^k-c)-\log(w_0)) \tag{3}
$$

This equation specifies the indifference condition that c must satisfy for someone with log utility, which is a commonly chosen utility function. I would like to show that the sum in Equation 3 converges. One can show sum converges by the ratio test. For notational purposes, let a_k denote the kth term of the sum.

$$
\begin{aligned}
\lim_{k\to\infty}|\frac{a_{k+1}}{a_k}| &= \lim_{k\to\infty}\frac{a_{k+1}}{a_k} \\
&= \lim_{k\to\infty}\frac{(1/2)^{k+1}(\log(w_0+2^{k+1}-c)-\log(w_0))}{(1/2)^k(\log(w_0+2^k-c)-\log(w_0))} \\
&= \lim_{k\to\infty}\frac{(1/2)(\log(w_0+2^{k+1}-c)-\log(w_0))}{(\log(w_0+2^k-c)-\log(w_0))} \\
&= \lim_{k\to\infty}\frac{(1/2)(k+1)}{k} \\
&= 1/2
\end{aligned}
$$

Since $\lim_{k\to\infty}|\frac{a_{k+1}}{a_k}| < 1$, that implies the sum in Equation 3 converges absolutely.

COUNTER-PARTY CREDIT RISK

If someone plays the St. Petersburg Paradox game, there is a non-zero chance that the player receives an arbitrarily large payoff. By the game's definition, the person offering the game must be able to pay an arbitrarily large sum of money to the player. In reality, the person offering the game does not have infinite wealth; rather they have wealth $B < \infty$. That means the person playing the game can receive at most $\$B$ as payout. Thus, realistically, one can write that

$$
\text{Payoff(first tails on toss } k) := 2^k\mathbb{1}_{2^k\le B} + B2^k\mathbb{1}_{2^k>B}
$$

Again, letting P denote the payoff for the player playing the St. Petersburg Paradox game, one can solve for $\mathbb{E}[P]$

$$
\begin{aligned}
\mathbb{E}[P] &= \sum_{k=1}^{\infty} \text{Pr(First tails on toss } k\text{)Payoff(first tails on toss } k\text{)} \\
&= \sum_{k=1}^{\lfloor \log(B) \rfloor} (1/2)^k 2^k + \sum_{k=\lfloor \log(B) \rfloor + 1}^{\infty} (1/2)^k B \\
&= \lfloor \log(B) \rfloor + (1/2)^{(\lfloor \log(B) \rfloor + 1)} (2B) \\
&< \infty
\end{aligned}
$$

When imposing a finite maximum credit on the player offering the game, the expected payoff from the game is suddenly finite. If one calculates the cost of the game from the expected payoff and imposes a credit maximum, one gets a finite cost. If one desires to operate in the realm of expected utility, one can also solve the indifference condition in Equation 2 to find the fair cost of the game but with this modified payout structure.

Lack of Ergodicity

The St. Petersburg Paradox game is not *ergodic*. An *ergodic* activity is one where the the outcome of many people compeleting it once is the same as one person completing it many times. Said differently, *ergodicity* is the property of a stochastic random variable detailing that its expected value is the same as its long term average.

Russian Roulette

To gain intuition, the Russian roulette is not an ergodic game. Consider the following extreme situation, you put a single bullet in a gun with six slots and spin the cylinder containing the bullets. You point the gun at your head and fire, if you get hit with the bullet, you die; if you don't get hit, you make $\$600$. If many people play this game, they will on average collect $\$500$. If you play this game six times, you'll die. In other words, the collective experience of many people playing the Russian roulette is not the same as the experience of one person playing it many times. When you sign up for a game and want to determine a fair price, what you care about is the time-average, not the collective-average. When playing a game many times, one expects to experience something closer to the time-average, not the collective-average. It's often easier compute the collective-average but that's only equivalent when the game is ergodic.

Back to the St. Petersburg Paradox

For any nontrivial cost to play, the St. Petersburg Paradox game is not ergodic because if at any point you lose all of your money, you don't have the opportunity to play again.

When thinking about repeatedly playing a game many times over time, it's often helpful to think about the growth factor and exponential growth rate of wealth rather than the absolute change in wealth from one period to the next. Define the growth factor as,

$$
\begin{aligned}
r_t &:= \frac{w_t}{w_{t-1}} = \frac{w_{t-1} + p_t - c}{w_{t-1}} \\
g_t &:= \log(r_t)
\end{aligned}
$$

The growth factor from period $t-1$ to t is equal to the ratio of wealth in period t, w_t to the wealth in period $t-1$, w_{t-1}. The wealth in period t is equal to the wealth in period $t-1$ plus the payout of the game in period t, p_t, minus the cost of playing the game, c. The exponential growth rate is simply the log of the growth factor.

Collective-Average Exponential Growth Rate

For N people playing the game, the collective-average growth factor at time 0 is equal to

$$
\mathbb{E}_N[r_1] = \frac{1}{N} \sum_{i=1}^{N} \frac{w_0 + P_i - c}{w_0}
$$

where P_i is the payout for person i. To find the collective-average growth rate, we can take the limit of $\mathbb{E}_N[r_1]$ as $N \to \infty$. Let k_{max} be the largest number of heads flips seen out of the N players and n_k be the number of the N players that flip tails for the first time on the kth flip. One can re-index the sum on k, the number of flips until one flips tails for the first time:

$$\begin{aligned}
\mathbb{E}[r_1] &= \lim_{N\to\infty} \mathbb{E}_N[r_1] \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \frac{w_0 + P_i - c}{w_0} \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{k_{\text{max}}} n_k \frac{w_0 + 2^k - c}{w_0} \\
&= \lim_{N\to\infty} \sum_{k=1}^{k_{\text{max}}} \frac{n_k}{N} \frac{w_0 + 2^k - c}{w_0} \\
&= \sum_{k=1}^{\infty} (1/2)^k \frac{w_0 + 2^k - c}{w_0} \\
&= (1/w_0) \sum_{k=1}^{\infty} (1/2)^k (w_0 + 2^k - c) \\
&= \infty
\end{aligned}$$

From step 3 to step 4, I use the fact that as $N \to \infty$, $k_{\text{max}} \to \infty$ (ie., as you have more people, the maximum number of the number of flips it takes someone until they flip tails, goes to infinity) and $n_k/N \to (1/2)^k$ (ie., as you have more people, the fraction of people flipping tails for the first time on the kth flip approaches the probability of flipping tails for the first time on the kth flip, $(1/2)^k$). Thus, the exponential collective-average growth rate, $g_{\text{coll}} := \log(\mathbb{E}[r_1]) = \infty$.

Time-Average Exponential Growth Rate

To calculate the time-average exponential growth rate, we will consider observing T outcomes of the gamble with wealth w_0 but will aggregate them in a different manner than above. Suppose we observe T outcomes of the gamble with initial wealth w_0 and let each outcome account for $1/T$ units of time. Now, a good estimate[1] of the time-average exponential growth rate of the game is $\hat{g}_{T,\text{time}} := \sum_{t=1}^{T}(r_t^{1/T} - 1)$. Analogous to the collective-average, define n_k to be the number of rounds for which the first tails was flipped on the kth toss, and define k_{max} to be the biggest number of flips, out of the T rounds, it took to flip tails for the first time. We will try to compute the time-average exponential growth rate g_{time}.

[1]To see why this estimate makes sense, suppose that we have an average exponential growth rate of g and we observe T growth factors r_1, ..., r_T for T evenly spaced periods of length $1/T$. We expect $e^g = \Pi_{t=1}^{T} r_t^{(1/T)} \implies g = \sum_{t=1}^{T} \log(r_t^{1/T}) \approx \sum_{t=1}^{T}(r_t^{1/T} - 1)$ taking a first order Taylor series approximation for $\log(x)$ around $x = 1$.

$$
\begin{aligned}
g_{\text{time}} &= \lim_{T\to\infty} \hat{g}_{T,\text{time}} \\
&= \lim_{T\to\infty} \sum_{t=1}^{T} (r_t^{1/T} - 1) \\
&= \lim_{T\to\infty} \sum_{t=1}^{T} ((\frac{w_0 + p_t - c}{w_0})^{1/T} - 1) \\
&= \lim_{T\to\infty} \sum_{k=1}^{k_{\text{max}}} n_k ((\frac{w_0 + 2^k - c}{w_0})^{1/T} - 1) \\
&= \sum_{k=1}^{\infty} \lim_{T\to\infty} n_k ((\frac{w_0 + 2^k - c}{w_0})^{1/T} - 1) \\
&= \sum_{k=1}^{\infty} (1/2)^k \lim_{T\to\infty} T((\frac{w_0 + 2^k - c}{w_0})^{1/T} - 1) \\
&= \sum_{k=1}^{\infty} (1/2)^k \log(\frac{w_0 + 2^k - c}{w_0}) \\
&= \sum_{k=1}^{\infty} (1/2)^k (\log(w_0 + 2^k - c) - \log(w_0))
\end{aligned}
$$

In the 4th step, I make a switch analogous to the one in collective-average growth factor computation to make the sum over k. In the 6th step, I use the fact that $\log(x) = \lim_{n\to\infty} n(x^{1/n} - 1)$.

A fair price[2] c for the game would be one where the exponential growth rate is 0. In other words, someone playing this game over time would not make or lose money in the long-run by playing many times. That means, a fair price c for the game satisfies:

$$
\begin{aligned}
0 &= g_{\text{time}} \\
&= \sum_{k=1}^{\infty} (1/2)^k (\log(w_0 + 2^k - c) - \log(w_0)) \qquad (4)
\end{aligned}
$$

Interestingly, Equation 4 that sets the fair price c in the time-average setting is the exact same as Equation 3 that sets the fair price c in the log utility case. Imposing log utility on an individual is arbitrary but trying to maximize wealth accumulation over time is widely desirable. Again, the sum in Equation 4 converges[3] for any finite w_0 per my earlier discussion in the log utility section and therefore we can report a fair price $c < \infty$ for the game. It's nice that we can reach a finite fair price for the game using wealth accumulation dynamics without imposing any sort of utility function.

An Alternate Derivation

My second comment is that my derivation is potentially non-intuitive and I find it helpful to think about it in a different manner too to help shed light on what's going on. If an individual starts with wealth w_0, after one round of the St. Petersburg Paradox game, they experience growth factor r_1 and reach wealth w_1. After a second iteration of the game, they experience growth factor r_2 and reach wealth w_2. We can write that our wealth at time T equals $w_T = w_0 \Pi_{t=1}^{T} r_t$. The average growth factor per round over the T rounds is $\bar{r}_{T,\text{time}} := (\Pi_{t=1}^{T} r_t)^{1/T}$. We can take the log of $\bar{r}_{T,\text{time}}$ to reach the time-average exponential growth over the T rounds

[2] If the exponential growth rate is greater than zero, the individual will accumulate wealth by playing the game many times and so will want to play. If the exponential growth rate is less than zero, the individual will lose their wealth over time by playing the game, and so will not want to play.

[3] The collective-average exponential growth rate diverges whereas the time-average exponential growth rate converges. Thus, clearly, they're not equal in the setting of the St. Petersburg Paradox.

$$
\begin{aligned}
\bar{g}_{T,\text{time}} &:= \log(\bar{r}_{T,\text{time}}) \\
&= \frac{1}{T}\sum_{t=1}^{T}\log(r_t) \\
&= \sum_{t=1}^{T}(\log(w_{t-1}+p_t-c)-\log(w_{t-1}))
\end{aligned}
$$

where p_t is the payoff of the game in round t. To keep advancing, we'd like to take the change sum to be over k (ie., the number of flips it takes to flip tails for the first time) as we take the limit of $T \to \infty$. The trouble is that the wealth level at the start of the round is different for each round $w_0, w_1, w_2, \ldots$. Thus, it's not necessarily true that the growth factor's come from identical distributions where each growth factor corresponds to a certain number of coin flips needed to realize the first tail. I need to make the initial argument that so long as c is properly chosen[4] so that the average exponential growth rate is 0, I can expect to realize an infinite number of lotteries with wealth arbitrarily close to w_0. For these rounds with wealth close to w_0, the growth factors are (nearly) identically distributed where each growth factor corresponds to a certain number of coin flips needed to realize the first tail.

Of the T rounds, let's divide them into sets T_c, T_f where T_c contains the rounds that had initial wealth close to w_0 and T_f the complement of T_c. We can focus on computing the average exponential growth rate by just focusing on the rounds with wealth near w_0. Again, we define k_{max} be the largest number of heads flips seen out of the $|T_c|$ rounds and n_k be the number of the $|T_c|$ rounds that have tails for the first time on the kth flip.

$$
\begin{aligned}
g_{\text{time}} &= \lim_{T\to\infty}\bar{g}_{T,\text{time}} \\
&= \lim_{T\to\infty}\frac{1}{|T_c|}\sum_{t\in T_c}\log(r_t) \\
&= \lim_{T\to\infty}\frac{1}{|T_c|}\sum_{t\in T_c}(\log(w_{t-1}+p_t-c)-\log(w_{t-1})) \\
&= \lim_{T\to\infty}\sum_{k=1}^{k_{\text{max}}}\frac{n_k}{|T_c|}(\log(w_0+2^k-c)-\log(w_0)) \\
&= \sum_{k=1}^{\infty}(1/2)^k(\log(w_0+2^k-c)-\log(w_0))
\end{aligned}
$$

In the fourth step, I change the sum to be over k as I've done twice already earlier in this section. In the last step, I use the fact that as $T \to \infty$, $k_{\text{max}} \to \infty$ and $\frac{n_k}{|T_c|}$ approaches the probability of having tails flipped for the first time on the kth flip (ie., $(1/2)^k$). This derivation yields the same time-average growth rate. To compare the two derivations, in the first, I consider an infinite number of executions of the St. Petersburg Paradox game (each outcome occurring with the right relative frequency) happening in a 1 unit period of time while in the latter I consider an infinite number of executions of the St. Petersburg Paradox game (each outcome occurring with the right relative frequency) as time goes to infinity. In the first derivation, after each round, I reset the wealth to w_0 and go again whereas in the second derivation I consider only the lotteries that happened with wealth close to w_0[5]. While both derivations are quite similar, as I allude to in a footnote, the second rests on the assumption that we will have an infinite number of lotteries with wealth close to w_0. This is not a good assumption if there doesn't exist a price c that forces the exponential growth rate to be 0.

Resolution with Arbitrary Payoff Function

One can consider replacing the traditionally exponential payoffs of the St. Petersburg paradox with some arbitrary increasing payoff function $p(k)$. In other words, during the playing of the St. Petersburg Paradox game, if the coin comes

[4] I actually dislike this alternative derivation because advancing rests on this assumption. There will be cases in the next section where there's no c that lets the growth rate be 0 and so this derivation can't hold in those cases.

[5] This characteristic of both derivations is funny given that wealth will only be w_0 for one period of time but we aim to calculate the long term growth rate of the game with initial wealth w_0.

up tails for the first time on the kth flip, the player will receive $p(k)$. Log utility and the time-average approaches imply that a fair price c for the game satisfies:

$$0 = \sum_{k=1}^{\infty} (1/2)^k (\log(w_0 + p(k) - c) - \log(w_0))$$

One can consider letting $p(k) = 2^{2^k}$ or any other function that grows more quickly than the traditional payoff function 2^k. In this case, for any $w_0 < \infty$, and for any admissible $c < \infty$, the sum on the right hand side diverges. In practice, from a log utility perspective, for any *admissible* $c < \infty$, that means that the expected change in utility from playing the game with this payoff function is positive and therefore an individual with this utility function will choose to play the game. From a time-average perspective, for any *admissible* $c < \infty$, the expected exponential growth rate is greater than 0 and so an individual concerned with wealth accumulation will choose to play the game.

I want to quickly articulate what I mean by having an *admissible* cost c to the game. The log function is useful in this setting because it specifies a firm upper bound on c– namely, $c < w_0 + \inf_k(p(k))$ since $\log(x)$ is undefined for non-positive x. I define an *admissible* cost c to be one that satisfies the inequality above. An individual with log-utility or one concerned with wealth accumulation would never conclude to pay an *inadmissible* cost for the St. Petersburg Paradox game since their criterion would become undefined.

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