Hot-Hand Principle or Fallacy?

Vasco Villas-Boas

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The "hot-hand principle", the occurrences of streaks in "random" sequences, has been debated academic literature. In tennis, for instance, if a player has won 10 points in a row a believer in the "hot-hand principle" would think the likelihood they win the next point is larger than the probability they win the typical point. A non-believer would think the likelihood they win the next point is equal to the probability they win the typical point. Papers such as Gilovich, Vallone, and Tversky (1985), argue that the "hot-hand principle" is a fallacy and Sanjurju and Miller (2018), in a beautiful result, argue that GVT's selection of streaks to examine is biased, invalidating their work. I want to present ST (2018)'s key insight.

Suppose you flip a fair coin ("H" or "T") 50 times. Every time you flip an "H", you write down the result of the next flip. What is the expected proportion of "H" you will write down? It's actually less than 1/2. To see that, consider the experiment run with 3 total flips as done in the ST (2018) paper.

3-flip sequence	# of recorded flips	proportion of "H" on recorded flips
TTT	0	-
TTH	0	-
THT	1	0
HTT	1	0
THH	1	1
HTH	1	0
HHT	2	1/2
HHH	2	1

Table 1. Streaks with 3 Coin Flips

Since each 3-flip sequence is an equally likely outcome of 3 total coin flips, the expected proportion of "H" on recorded flips is $5/12 < 1/2$. What does this mean? We know that flips of a fair coin are independent and the expected proportion of "H" is less than 1/2. Suppose we're trying to see if tennis service points of a player are subject to the "hot-hand principle" and our null hypothesis is that all service points are won iid with probability p. After a streak, the player doesn't need to win a fraction of serves that's greater than p to be considered "hot", in fact, they can win a fraction of serves that's less than p and be considered "hot".

Let's see what's going on mathematically. Suppose that we have a sequence of n iid random variables $D := (X_1, ..., X_n)$ where each $X_i \sim \text{Bernoulli}(p)$. Interpret each X_i as a coin flip that can land "H" or "T" and $\text{P}(X_i = \text{"H"}) = p$. We want to find the expected probability of "H" for trials that follow k consecutive "H"s. Let $S_k(D) := \{i \text{ s.t. the } i\text{th flip follows } k \text{ consecutive "H"s}\}$. Then, we want to show that for any randomly selected subsequence of D with k consecutive "H", the conditional likelihood of getting "H" on the next flip is smaller than p, the unconditional likelihood of getting an "H" on the next flip.

Proof.

Let t be some arbitrary element of $S_k(D)$ and let i be a uniformly random chosen element from $S_k(D)$. We want to show that $\text{Pr}(X_t = \text{"H"} | i = t) < \text{Pr}(X_t = \text{"H"}) = p$.

We know that:

$$(t \in S_k(D) \ \ \& \ \ X_t = \text{"H"}) \Rightarrow t+1 \in S_k(D)$$
$$(t \in S_k(D) \ \ \& \ \ X_t = \text{"T"}) \Rightarrow t+1 \notin S_k(D)$$

So, if $X_t =$ "H", the cardinality of $S_k(D)$ is larger than if $X_t =$ "T". That implies, the likelihood that our uniformly randomly selected index i from $S_k(D)$ equals t is smaller in the case that $X_t =$ "H". So, $\Pr(i = t | X_t = \text{"H"}) < \Pr(i = t | X_t = \text{"T"})$. Now,

$$\begin{aligned}
&\Pr(i = t | X_t = \text{"H"}) < \Pr(i = t | X_t = \text{"T"}) \\
&\Rightarrow \frac{\Pr(X_t = \text{"H"} | i = t)\Pr(i = t)}{\Pr(X_t = \text{"H"})} < \frac{\Pr(X_t = \text{"T"} | i = t)\Pr(i = t)}{\Pr(X_t = \text{"T"})} \\
&\Rightarrow \frac{\Pr(X_t = \text{"H"} | i = t)}{\Pr(X_t = \text{"H"})} < \frac{\Pr(X_t = \text{"T"} | i = t)}{\Pr(X_t = \text{"T"})} \\
&\Rightarrow \frac{\Pr(X_t = \text{"H"} | i = t)}{p} < \frac{1 - \Pr(X_t = \text{"H"} | i = t)}{1 - p} \\
&\Rightarrow (1 - p)\Pr(X_t = \text{"H"} | i = t) < p(1 - \Pr(X_t = \text{"H"} | i = t)) \\
&\Rightarrow p > \Pr(X_t = \text{"H"} | i = t)
\end{aligned}$$

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REFERENCES

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