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Quantum Computation I - List of Exercises 2

The Evolution Operator

1. In the first list of exercises you showed that

$$R_{\hat{n}}(\phi) = e^{\frac{-i\phi\vec{\sigma}\cdot\hat{n}}{2}} = \cos(\phi/2)\mathbf{1} - i\vec{\sigma}\cdot\hat{n}\sin(\phi/2). \tag{1}$$

Now, let's connect it to Quantum Mechanics.

a: Given a generic two-level Hamiltonian

$$H = \begin{pmatrix} h_{00} & h_{01} \\ h_{01}^* & h_{11} \end{pmatrix},$$

where $h_{01} = h_R + ih_I$ with $h_{00}, h_{11}, h_R, h_I \in \mathbb{R}$, write it as function of the Pauli matrices and identity operator.

b: Show that the right hand side of Eq. (1) is a unitary operator.

c: Find the correspondence between $R_{\hat{n}}(\phi)$ (Eq.(1)) and $U(t,0) = e^{-iHt/\hbar}$.

d: Show that an arbitrary unitary operator U acting on $\mathcal{H}_2$ can be written as $U = e^{i\alpha}R_{\hat{n}}(\phi)$, where $\alpha \in \mathbb{R}$.

e: Find the values of α, ϕ, and $\hat{n}$ in which U becomes: i) the Hadamard, ii) S, iii) X, iv) Y, v) Z, and vi) T gates.

2. For a time independent Hamiltonian H show that the evolution operator can be written as

$$U(t,0) = \sum_n e^{-iE_n t/\hbar}|E_n\rangle\langle E_n|,$$

where $H|E_n\rangle = E_n|E_n\rangle$. What does this means?

Observables and measurements

1. Given the state

$$|\psi(\theta,\phi)\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle.$$

a: What are the probabilities of finding the system state in the eigenstates of the observables X, Y and Z with eigenvalues ± 1?

b: Whats are the average values of the operators X, Y, and Z?

c: The Heisenberg uncertainty relation is given by $\Delta A.\Delta B \geq \frac{|\langle\psi|[A,B]|\psi\rangle|}{2}$. Verify the validity of this relation for all combinations of operators X, Y, and Z.

FIG. 1. Bell's scenario. Alice can choose to measure Q or R, while Bob can choose between S or T. Bob and Alice are far apart so that the simultaneous measurements can not influence each other.

Bell's inequality

1. Classically, the quantities Q, R, S, and T are dihcotomic variables, i.e., they can have only ± 1 values. Show in this case that $B \equiv QS + RS + RT - QT = \pm 2$.

2. Before the measurement $p(q, r, s, t)$ describes the probability that $Q = q$, $R = r$, $S = s$, and $T = t$. Then, show that the average value of B can be written as $\mathbf{E}(B) = \sum_{qrst} p(q, r, s, t)(qs + rs + rt - qt) = \mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT) \leq 2$.

3. Now compare the result above with that one coming from Quantum Mechanics. For this purpose, the average values in classical physics are replaced by the ones in Quantum Mechanics. So, evaluate $\langle B \rangle$ considering the Bell state $|\psi_-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$ and the following operators $Q = Z_1$, $R = X_1$, $S = -\frac{Z_2 + X_2}{\sqrt{2}}$, and $T = \frac{Z_2 - X_2}{\sqrt{2}}$. Comment your result.

4. In order to observe the effect of entanglement on $\langle B \rangle$, consider the average on the state $|\psi\rangle = \sqrt{1-\alpha}|00\rangle + \sqrt{\alpha}|\psi_-\rangle$, with $0 \leq \alpha \leq 1$. Comment your result.

The inequality $\mathbf{E}(B) \leq 2$ *is called Clauser-Horne-Shimone-Holt (CHSH) inequality, a particular type of Bell's inequality.*