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Quantum Computation I - List 1

1. Show the Cauchy-Schwarz inequality

$$|\langle\psi|\phi\rangle| \leq \||\psi\rangle\|.\||\phi\rangle\|. \tag{1}$$

2. Show the triangular inequality

$$\||\psi\rangle + |\phi\rangle\| \leq \||\psi\rangle\| + \||\phi\rangle\|. \tag{2}$$

3. Find the orthogonal state to $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$, where $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$.

4. Prove the following properties:

   (a) $\left(A^\dagger\right)^\dagger = A$

   (b) $(\lambda A)^\dagger = \lambda^* A^\dagger$

   (c) $(A + B)^\dagger = A^\dagger + B^\dagger$

   (d) $(AB)^\dagger = B^\dagger A^\dagger$

5. Show that:

   (a) $|\phi\rangle\langle\psi|$ is linear;

   (b) $(|\phi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\phi|$

6. The Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$ and the identity operator $\mathbf{1}$ below are represented in the standard vector basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In the quantum computing field the Pauli matrices are usually denoted by $\{X = \sigma_x, Y = \sigma_y, Z = \sigma_z\}$, since they are one qubit quantum gates. Write such matrices in the outer product notation.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

7. By using the outer product notation show the properties of the Pauli spin matrices described below:

   (a)

$$[\sigma_l, \sigma_j] = 2i\varepsilon_{ljk}\sigma_k, \qquad l, j, k = x, y, z, \tag{4}$$

   where ε_{ijk} is the Levi-Civita symbol and the Einstein summation convention is being used.

(b)

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij},$$

where $\{A, B\} = AB + BA$ is the anti commutation relation.

(c)

$$\sigma_i{}^\dagger = \sigma_i,$$

(d)

$$\mathrm{tr}\sigma_i = 0.$$

8. Let the vector operator $\overrightarrow{\sigma}$ be defined by

$$\overrightarrow{\sigma} = \sigma_x\widehat{i} + \sigma_y\widehat{j} + \sigma_z\widehat{k}.$$

Obtain the operator $\overrightarrow{\sigma}^2$ in the outer product notation and compare it to the identity operator. The Casimir operator of an algebra is defined as the one which commutes with all elements of the algebra. In the case of *su(2)* algebra, defined by equation (4), $\overrightarrow{\sigma}^2$ is its Casimir operator.

9. Given the projection operator $\mathbf{P}$, show that:

(a) $\mathbf{Q} = \mathbf{1} - \mathbf{P}$ is also a projection operator;

(b) $[\mathbf{Q}, \mathbf{P}] = 0$;

(c) the vector $\mathbf{P}|\phi\rangle$ is orthogonal to the vector $\mathbf{Q}|\phi\rangle$.

10. Find the eigenvalues and eigenvectors of

$$H = \begin{pmatrix} h_{00} & h_{01} \\ h_{01} & h_{11} \end{pmatrix},$$

where $h_{i,j}$ $(i, j = 0, 1)$ are real constants.

11. Obtain the eigenvalues and the normalized eigenvectors of

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, J_y = \frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The matrices J_x, J_y, and J_z are similar to Pauli matrices X, Y, and Z for 3 dimensions, i.e., the generators of qutrits rotations.

12. Consider two orthonormal basis defined by $\{|u_i\rangle\}$ and $\{|v_j\rangle\}$, with $i, j = 1, .., n$. Show that

$$U = \sum_{k=1}^{n} |u_i\rangle\langle v_i|$$

is unitary.

13. Given the Pauli spin matrices represented in the eigenstates basis of the σ_z operator

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

do a change of basis to the eigenstates of the following operators i)σ_x, ii)σ_y, and iii)$\vec{\sigma} \cdot \hat{n}$, where $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$.

14. Do the Pauli spin matrices in the representation introduced in the items i), ii), and iii) of the exercise above obey the usual commutation relation? Justify your answer.

15. Prove the identity

$$(\vec{a} \cdot \vec{\sigma}) \cdot \left(\vec{b} \cdot \vec{\sigma}\right) = \vec{b} \cdot \vec{a}\mathbf{1} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}, \tag{6}$$

where $\vec{a}$ and $\vec{b}$ are vectors.

16. Prove that

$$R(\phi) = e^{\frac{-i\phi\vec{\sigma}\cdot\hat{n}}{2}} = \cos(\phi/2)\mathbf{1} - i\vec{\sigma} \cdot \hat{n}\sin(\phi/2), \tag{7}$$

where ϕ is the rotation angle, $\hat{n}$ is a unity vector in some direction.

17. Show that the operator $U = e^{-iA}$ is unitary if A is Hermitean. Given the eigenvalue equation for A

$$A|\lambda\rangle = \lambda|\lambda\rangle,$$

show that $|\lambda\rangle$ are eigenstates of U with eigenvalues $e^{-i\lambda}$.
Hint: A function of an operator $f(A)$ is defined by the expansion of f in Taylor series of A.

18. Show that

$$e^{(A+B)} = e^A e^B e^{-[A,B]/2},$$

if $[A,[A,B]] = [B,[A,B]] = 0$. Hint: see page 137 of W. H. Louisell, *Quantum Statistical Properties of Radiation* (1973).

19. Show that the Bell states are a vector basis for a 4-dimensional vector space.

$$|\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \qquad |\phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}. \tag{8}$$

20. Evaluate the tensor product of matrices

(a) All possible combinations of $\{\mathbf{1}, X, Y, Z\} \otimes \{\mathbf{1}, X, Y, Z\}$.

(b) All possible combinations of $\{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|\} \otimes \{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|\}$.

(c) $CNOT = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes X$.