MPRI Algorithms Lab: The Steiner Tree Problem

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October 5, 2025

1 Problem Modeling

1.1 Problem Definition: Minimum Steiner Tree

The following is the standard formal definition of the Minimum Steiner Tree problem, as found in the compendium.

Instance: A complete graph $G = (V, E)$, a metric given by edge weights $w : E \to \mathbb{Z}^+$, and a subset of required vertices $S \subset V$.

Solution: A Steiner tree, i.e., a subtree of G that includes all the vertices in S.

Measure: The sum of the weights of the edges in the subtree, which is to be minimized.

Good News: The problem is approximable within a ratio of $1 + \ln(3)/2 \approx 1.55$. In your project file, this approximation ratio is simplified to 2.

Bad News: The problem is APX-complete.

Garey and Johnson: ND12.

1.2 NP-completeness Proof

We show that the decision version of the Minimum Steiner Tree problem is NP-complete.

1. Membership in NP. Given a subgraph T', we can verify in polynomial time that it is connected, acyclic, contains all required vertices R, and that its total weight $\sum_{e \in T'} w(e) \leq K$. Hence, the problem is in NP.

2. NP-hardness via reduction from X3C. We reduce from the *Exact Cover by 3-Sets* (X3C) problem, which is NP-complete.

X3C instance: a set X with $|X| = 3q$ and a collection $C = \{C_1, \ldots, C_n\}$ of 3-element subsets of X. **Question:** does there exist $C' \subseteq C$ such that every element of X appears in exactly one member of C'?

Construction of Steiner Tree instance:

$$V = \{v_0\} \cup \{v_{C_i} \mid C_i \in C\} \cup \{v_{x_j} \mid x_j \in X\},$$

with edges and weights:

- (v_0, v_{C_i}) with weight 1 for each $C_i \in C$;
- (v_{C_i}, v_{x_j}) with weight 1 for each $x_j \in C_i$.

Terminals: $R = \{v_0\} \cup \{v_{x_1}, \ldots, v_{x_{3q}}\}$, and budget $K = 4q$.

Equivalence:

($\Rightarrow$) If X3C has an exact cover C', then taking all edges (v_0, v_{C_i}) and (v_{C_i}, v_{x_j}) for $C_i \in C'$ yields a tree of total weight $q + 3q = 4q$, satisfying the Steiner constraint.

($\Leftarrow$) Conversely, if there exists a Steiner tree of weight $\leq 4q$, it must contain exactly $3q$ edges connecting element vertices and q edges connecting the corresponding v_{C_i} to v_0. These q sets are disjoint and cover X, forming an exact cover.

Thus, the reduction holds in polynomial time, and the Minimum Steiner Tree problem is NP-complete.

References

[1] Richard M. Karp. Reducibility Among Combinatorial Problems. In M. Jünger et al. (eds.), *50 Years of Integer Programming 1958–2008*, pp. 219–241. Springer, 2010. (Reprint of the original 1972 paper).