A family of directional relation models for extended objects

Spiros Skiadopoulos, Nikos Sarkas, Timos Sellis, Manolis Koubarakis

***Abstract*—In this paper, we introduce a family of expressive models for qualitative spatial reasoning with directions. The proposed family is based on the cognitive plausible cone-based model. We formally define the directional relations that can be expressed in each model of the family. Then, we use our formal framework to study two interesting problems: computing the inverse of a directional relation and composing two directional relations. For the composition operator, in particular, we concentrate on two commonly used definitions, namely consistency-based and existential composition. Our formal framework allows us to prove that our solutions are correct. The presented solutions are handled in a uniform manner and apply to all the models of the family.**

***Index Terms*—Spatial databases and GIS, cone-based directional relations, inverse and composition operators.**

I. INTRODUCTION

The subject of this paper belongs to the broader research area of *Qualitative Spatial Reasoning* (QSR). The goal of QSR is to approach commonsense knowledge and reasoning about space using symbolic and not numerical methods. It is no surprise that QSR has found applications in many diverse scientific areas that concentrate on building successful intelligent systems: Geographic Information Systems [1], [2], Artificial Intelligence [3], [4], Databases [5], [6] and Multimedia [7] just to name a few. Most researchers have concentrated on the three main aspects of space, namely topology [8], [3], [1], [4], distance [2], [9] and orientation [10], [11], [12], [13], [14], [15], [16], [17]. The uttermost aim in these lines of research is to define new categories of spatial operators, as well as to build efficient algorithms for the automatic processing of queries using such operators.

In this paper, we consider extended objects and concentrate on *binary directional relations*. Such relations describe how a *primary object* a is placed relative to a *reference object* b utilizing a co-ordinate system (e.g., object a is *north of* object b). Early qualitative models for directional relations approximate an extended spatial object by a representative point (most commonly the centroid) [10], [2], [12], [16]. Typically, such models partition the space around the reference object b into a number of mutually exclusive areas. For instance, the *projection-based* model partitions the space using lines parallel to the axes (Fig. 1a) while the *cone-based* model partitions the space using lines with an origin angle ϕ (Fig. 1b). Depending on the adopted model the relation between two objects may change. For instance, consider Fig. 1: according to the projection-based model a is northeast of b (Fig. 1a) while according to the cone-based model a is north of b (Fig. 1b). Later models approximate an object by a representative area (most commonly the minimum bounding box) and express directions on these approximations [14], [15]. Unfortunately, models that approximate both the primary and the reference object may give misleading directional relations when objects are overlapping, intertwined, or horseshoe-shaped (for an extended discussion see [18]).

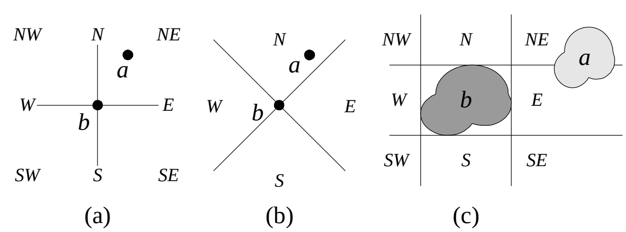

Fig. 1. Models of directional relations

More recently, Goyal [18] and Skiadopoulos and Koubarakis [17], [19] studied a model that expresses the directional relation by only approximating the reference object b (using its minimum bounding box) while using the exact shape of the primary object a. Intuitively, this model *(a)* partitions the plane around the reference object into 9 areas similarly to the projection-based model (these areas correspond to directional relations such as *north*, *northeast* etc.) and *(b)* records the areas occupied by the primary object. These areas provide the directional relation between the primary and the reference object. For instance, in Fig. 1c, object a is partly NE and partly E of object b. We denote this model by $\mathcal{P}DR$ (*P*rojection-based *D*irectional *R*elations). Clearly, the $\mathcal{P}DR$ model offers a more precise and expressive model than previous approaches that approximate objects using points or boxes [18].

However, the $\mathcal{P}DR$ model is not without weaknesses. The number of relations that can be expressed in the model is very large (511 relations). Furthermore, the $\mathcal{P}DR$ model partitions the reference space similarly to the projection-based model

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using lines parallel to the axes (Fig. 1c). Most people do not find this partition natural. Typically, people tend to organize surrounding space using lines with an origin angle similarly to the cone-based model (Fig. 1b). Hence, most people find the cone-based partition more intuitive and descriptive. The cognitive plausibility of the cone-based model has been verified by studies in the field of Cognitive Sciences (see for instance [20], [21]). Moreover, the cone-based partition is a typical approximation for the field of view of the human eye and camera lenses [22], [23]. For the above reasons, cone-based models have been used in Computer Vision [22], [24], Robot Navigation [25] and Geographic Information Systems [11], [26].

In this paper, we propose $\mathcal{CDR}$ (*C*one-based *D*irectional *R*elations), an alternative family of directional relation models that is based on the cognitive plausible cone-based model. In the $\mathcal{CDR}$ family of models, only the reference object is approximated by its minimum bounding box (as in $\mathcal{PDR}$), but the space around the reference object is partitioned into 5 areas using the cone-based model. The family contains an infinite number of models. Each model in the $\mathcal{CDR}$ family is identified by a unique value for ϕ ($0^\circ < \phi < 90^\circ$) that defines the origin angle of the space partitioning lines (see also Fig. 2a). In other words, for each particular application, by choosing a suitable value for ϕ, we can find an appropriate model in the $\mathcal{CDR}$ family. Moreover, $\mathcal{CDR}$ models result in a set of 31 relations, a significantly smaller set compared to $\mathcal{PDR}$ which has 511 relations.

We formally define the relations that can be expressed in the $\mathcal{CDR}$ family and focus on two interesting problems: computing the inverse of a directional relation and composing two directional relations. The inverse and composition operations, for various kinds of spatial relations, have received considerable attention in the literature [27], [1], [2], [15], [4], [12]. For the composition operator, in particular, research has mainly concentrated on two definitions, namely *existential* and *consistency-based* composition [8], [28]. Existential composition is the standard definition of composition from set theory [8], [28], [15], [4]. Consistency-based composition is a weaker interpretation useful in several domains [29], [30].

The inverse and composition operations are used as mechanisms for inferring new spatial relations from existing ones. Such mechanisms are important as they are in the heart of any system that retrieves collections of objects similarly related to each other using spatial relations. For instance, these inference mechanisms are very helpful when we need to detect inconsistencies in a given set of spatial relations [16], [4] or preprocess spatial queries and prune the search space [31]. Inverse and composition are also an essential part of Relation Algebras [32], [33], [34] so their formal study is a prerequisite to any algebraic approach to spatial reasoning. Moreover, composition is often used to identify classes of relations that have a tractable consistency problem [27], [4], [35].

The technical contributions of this paper can be summarized as follows:

- We propose the $\mathcal{CDR}$ family of directional relation models. The relations that can be expressed in each model of the family are formally defined. $\mathcal{CDR}$ models are based on the cognitive plausible cone-based model and can be customized to serve a wide variety of applications. Finally, $\mathcal{CDR}$ offers a small and easy to manage set of relations.
- We consider the inverse operation for directional relations in the $\mathcal{CDR}$ family. We present a method to compute the inverse of a relation and formally prove its correctness.
- We study the problem of composing two directional relations of the $\mathcal{CDR}$ family. We first present a method for consistency-based composition. To this end, we consider progressively more expressive classes of directional relations and present consistency-based composition algorithms for these classes. Our theoretical frameworks allows us to prove formally that our algorithms are correct. Finally, we consider the existential definition of composition. Contrary to consistency-based composition, we show that the binary relation resulting from the existential composition of two directional relations cannot always be expressed using the relations of the $\mathcal{CDR}$ family.

The rest of the paper is organized as follows. Section II defines the $\mathcal{CDR}$ family of directional relation models. In Section III, we study the inverse and composition problem for the directional relations in the $\mathcal{CDR}$ family. Finally, Section IV offers conclusions and discusses future research directions.

II. A FAMILY OF DIRECTIONAL RELATION MODELS

In this section, we present the $\mathcal{CDR}$ family of directional relation models. We consider the Euclidean space $\Re^2$. *Objects* are defined as non-empty and bounded sets of points in $\Re^2$. Let a be an object. The *minimum bounding box* of object a, denoted by $mbb(a)$, is the smallest rectangle, aligned with the axes, that encloses the object (Fig. 2a). Throughout this paper, we will consider objects that are formed by finite unions of objects that are homeomorphic to the *closed unit disk* [17]. This set of objects is denoted by REG^*. Objects in REG^* can be *disconnected* and have *holes*. However, class REG^* excludes points, lines and objects with emanating lines. A thorough discussion about REG^* and the way objects are modeled in REG^* appears in [36], [19].

To define a relation in $\mathcal{CDR}$ between a primary object a and a reference object b, we consider the minimum bounding box of object b and four rays originating from its four vertices. We denote by $r_1(b)$ (respectively $r_2(b)$, $r_3(b)$, $r_4(b)$) the ray that originates from the upper right (respectively the upper left, lower left, lower right) vertex of $mbb(b)$ (see also Fig. 2a). The *origin angle* of each ray is presented in Fig. 2a. Note that the origin angle has the same value for all rays, such that the plane is partitioned symmetrically. This angle, denoted by ϕ, is called the *characteristic angle* of the model and can have values in the interval $(0^\circ, 90^\circ)$[1].

Every possible value of ϕ specifies a new model in the $\mathcal{CDR}$ family. Such a model will be denoted by $\mathcal{CDR}(\phi)$. For instance, $\mathcal{CDR}(30^\circ)$ denotes the model of $\mathcal{CDR}$ where

[1]For values $\phi = 0^\circ$ and $\phi = 90^\circ$ the model degenerates to a subcase of $\mathcal{PDR}$.

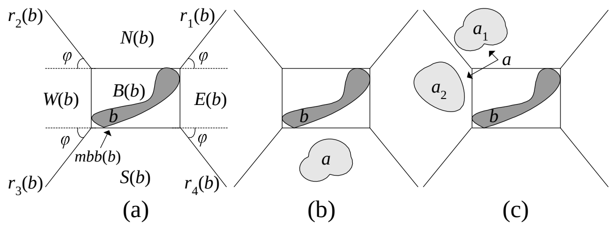


Fig. 2. Reference tiles and relations

$\phi = 30°$. Notice that the value of ϕ is fixed within a certain model $\mathcal{CDR}(\phi)$ and is not allowed to vary for different objects. The analysis that we present in this paper is valid for any model $\mathcal{CDR}(\phi)$ in the $\mathcal{CDR}$ family ($0° < \phi < 90°$). Wherever necessary in the material that follows, the value of ϕ appears as a parameter.

The minimum bounding box of the reference object b, along with the four rays divide the plane into 5 areas which we call *tiles* (Fig. 2a). The peripheral tiles correspond to the four directional relations *north*, *west*, *south* and *east*. These tiles will be denoted by $N(b)$, $W(b)$, $S(b)$ and $E(b)$ respectively. The central area corresponds to the object's minimum bounding box and is denoted by $B(b)$. Notice that *(i)* all tiles are closed, *(ii)* all tiles but $B(b)$ are unbounded, *(iii)* the union of all 5 tiles is $\Re^2$ and *(iv)* two distinct tiles have disjoint interiors but may share points in their boundaries, for instance, $W(b)$ and $B(b)$ share the left-side of the minimum bounding box of b. Even though tiles share some points along their borders, there is no ambiguity in defining relations in the $\mathcal{CDR}$ family because class REG^* does not contain objects that could lie entirely on the borderline (like lines, points and objects with emanating lines).

Informally, if a primary object a is included (in the set-theoretic sense) in tile $S(b)$ of some reference object b (Fig. 2b), then we say that a *is south of* b and we write $a\ S\ b$. Similarly, we can define *north* (N), *west* (W), *east* (E) and *bounding box* (B) relations. If a primary object a lies partly in tile $N(b)$ and partly in tile $W(b)$ of some reference object b (Fig. 2c) then we say that *a is partly north and partly west of b* and we write $a\ N{:}W\ b$.

The general definition of a basic directional relation in our framework is as follows.

Definition 1: A *basic directional relation* is an expression $R_1{:}\cdots{:}R_k$ where *(i)* $1 \leq k \leq 5$, *(ii)* $R_1, \ldots, R_k \in \{N, W, S, E, B\}$ and *(iii)* $R_i \neq R_j$ for every i, j such that $1 \leq i, j \leq k$ and $i \neq j$. A basic directional relation $R_1{:}\cdots{:}R_k$ is called *single-tile* if $k = 1$; otherwise it is called *multi-tile*.

Example 1: Expressions S and $N{:}W$ are basic directional relations. The first is single-tile relation, while the second is a multi-tile. Objects involved in these relations are shown in Fig. 2b and Fig. 2c respectively.

In order to avoid confusion, we will write the single-tile elements of a basic directional relation according to the following order: N, W, S, E and B. Thus, we always write $N{:}W{:}B$ instead of $W{:}B{:}N$ or $N{:}B{:}W$. Moreover, for a relation such as $N{:}W{:}B$ we will often refer to N, W and B as its *tiles*.

The set of basic directional relations (single or multi-tile) in every $\mathcal{CDR}$ model contains $\sum_{i=1}^{5} \binom{5}{i} = 31$ elements. We will use $\mathcal{B}^*$ to denote this set. Relations in $\mathcal{B}^*$ are jointly exhaustive and pairwise disjoint, and can be used to represent *definite information* about directions. Thus, relations in $\mathcal{B}^*$ express precise knowledge like object a is north of b, denoted by $a\ N\ b$. Using the relations of $\mathcal{B}^*$ as our basis, we can define the *powerset* $2^{\mathcal{B}^*}$ of $\mathcal{B}^*$ which contains 2^{31} relations. Elements of $2^{\mathcal{B}^*}$ are called *directional relations* and can be used to represent not only definite but also *indefinite information* about directions. Thus, relations in $2^{\mathcal{B}^*}$ also express imprecise knowledge like object a is *either* partly north and partly west *or* entirely west of object b, denoted by $a\ \{N{:}W, W\}\ b$. In general, expression $a \cup_{i=1}^{n} R_i\ b$ denotes that object a is related to b with some relation among $R_1, \ldots, R_n$. We will use Q, Q_1, Q_2, ... to denote *directional relations* and R, R_1, R_2, ... to denote *basic directional relations*, either single-tile or multi-tile.

Let us now highlight the advantages of the proposed model.

Cognitive plausibility. $\mathcal{CDR}$ models are based on the cone-based partition of space that is close to the human perception of direction as shown by Cognitive Science studies [20], [21]. Informally, the cone-base partition is a typical approximation for the field of view of camera lenses and the human eye [22], [23]. Cone-based models have been used in Computer Vision [22], [24], Robot Navigation [25] and Geographic Information Systems [11], [26].

Applicability. The $\mathcal{CDR}$ models can be used in a wide set of applications that use directions like Geographic Information Systems and Robot Navigation. In this paper, we have focused on the Geographic Information Systems paradigm and use cardinal direction relations (like *West*). The model can also be used in other applications by simply renaming the appropriate relations (for instance using *Left* instead of *West*). For completeness, let us also give a robot navigation example [25]. Consider a set of robots, equipped with perceptual capabilities, exploring an unknown area. This process can be optimized if robots move towards unexplored areas. Thus, every robot should know the position and the explored area of every other robot. To this end, the robots could compose and broadcast a complete metric map of their vicinity. This solution is costly mainly because metric maps are hard to compute and broadcast. Alternatively, every robot could *(a)* locate the landmarks and the robots in its vicinity, *(b)* identify their relations (using the $\mathcal{CDR}$ model) and *(c)* broadcast this information to the other robots. This qualitative solution does not require full metric mapping capabilities, requires significantly smaller bandwidth and typically is sufficient for the exploring task. Similarly, robots can use $\mathcal{CDR}$ relations to retain their formation.

Customization. The $\mathcal{CDR}$ models are distinguished according to the parameter ϕ ($0° < \phi < 90°$) that defines the origin angle of the space separating lines (Fig. 2a). For each particular application, by choosing a suitable ϕ, we can find an appropriate model in the $\mathcal{CDR}$ family. For instance, in a Robot Navigation application where the field of view angle of the lenses used in robot's vision system is $30°$, we may choose to use the $\mathcal{CDR}(30°)$ model.

Small set of relations. The $\mathcal{CDR}$ models can express a small

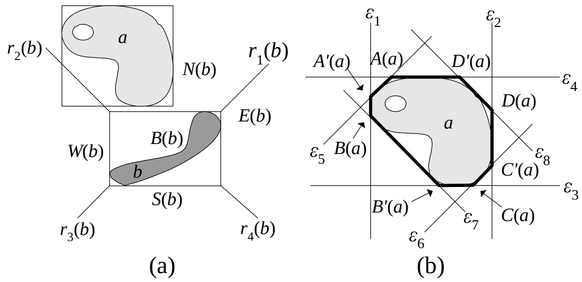


Fig. 3. The minimum bounding octagon

and easy to use set of 31 jointly exhaustive and pairwise disjoint relations. This number is significantly smaller than the respective set of $\mathcal{P}DR$ relations that contains 511 relations.

The next section defines formally the relations that can be expressed in the $\mathcal{C}DR$ family.

A. Defining directional relations formally

Intuitively, in order to derive the basic directional relation between a primary object a and a reference object b, one needs to identify the tiles of the plane induced by b where object a lies. However, such an intuitive (but informal) definition is generally inadequate for the study of a spatial model.

For the $\mathcal{P}DR$ model, two objects are related through a single-tile directional relation, iff their minimum bounding boxes are related with the same relation. This observation allows the definition of single-tile relations using sets of conditions involving the vertex coordinates of the objects' minimum bounding boxes [17]. This more elaborate definition was subsequently used to study the inverse, composition and consistency checking problems for that particular model.

We will attempt to derive such a definition for the proposed family of models. We can easily demonstrate that the above observation does not hold for the proposed $\mathcal{C}DR$ family. For instance, in Fig. 3a notice that while $a\ N\ b$, we have $mbb(a)\ N{:}W\ b$. Thus, the minimum bounding box provides a crude approximation of an object in the $\mathcal{C}DR$ family. However, a more precise approximation exists and can be constructed as follows.

Let us consider an arbitrary model $\mathcal{C}DR(\phi)$. We further refine the minimum bounding box of an object by using lines parallel to rays $r_1, \ldots, r_4$. After we form the minimum bounding box around the object, we also form four lines, tangent to the object and parallel to the rays. We use those lines to clip the corners of the minimum bounding box. By doing so, we come up with a new approximation which we call *minimum bounding octagon* (Fig. 3b). The minimum bounding octagon is formally defined in Definition 2 and we will later see that is appropriate for defining relations in $\mathcal{C}DR(\phi)$. Notice that the minimum bounding octagon is a special eight-corner approximation belonging to the general class of minimum bounding n-corner approximations, for $n = 8$ [37].

Notation 1: We denote by:

- O_x (respectively O_y) the x (respectively y) coordinate of a point O.
- $\varepsilon_1|\varepsilon_2$ the intersection point of lines ε_1 and ε_2.

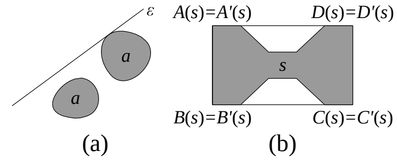


Fig. 4. Tangency and a degenerated minimum bounding octagon

- $(\varepsilon)_x$ (respectively $(\varepsilon)_y$) the x coordinate (respectively y coordinate) of the intersection point of line ε with the x-axis (respectively y-axis), i.e., $(\varepsilon)_x = (\varepsilon|x)_x$ and $(\varepsilon)_y = (\varepsilon|y)_y$.

Since set REG^* includes disconnected objects, we also need an appropriate definition of tangency. So, a line is *tangent* to an object in REG^* if it is tangent to one of its components and the whole object lies on a single side of the line. For example, in Fig. 4 line ε is tangent to the composite object a.

Definition 2: Let $\mathcal{C}DR(\phi)$ be an arbitrary model in the $\mathcal{C}DR$ family (where ϕ is the characteristic angle of the model). The *minimum bounding octagon* of an object $a \in REG^*$, denoted by $mbo_\phi(a)$, is the polygon created by lines $\varepsilon_1, \ldots, \varepsilon_8$, where:

(i) $\varepsilon_1, \ldots, \varepsilon_8$ are tangential to a.
(ii) ε_1, ε_2 are parallel to the y axis and $(\varepsilon_1)_x < (\varepsilon_2)_x$.
(iii) ε_3, ε_4 are parallel to the x axis and $(\varepsilon_3)_y < (\varepsilon_4)_y$.
(iv) ε_5, ε_6 form angle ϕ with the x axis and $(\varepsilon_5)_x < (\varepsilon_6)_x$.
(v) ε_7, ε_8 form angle $180° - \phi$ with the x axis and $(\varepsilon_7)_x < (\varepsilon_8)_x$.

The points forming the polygon presented in a counterclockwise order are $A(a) = \varepsilon_4|\varepsilon_5$, $A'(a) = \varepsilon_1|\varepsilon_5$, $B(a) = \varepsilon_1|\varepsilon_7$, $B'(a) = \varepsilon_3|\varepsilon_7$, $C(a) = \varepsilon_3|\varepsilon_6$, $C'(a) = \varepsilon_2|\varepsilon_6$, $D(a) = \varepsilon_2|\varepsilon_8$ and $D'(a) = \varepsilon_4|\varepsilon_8$ (Fig. 3b).

Example 2: In certain cases, depending on the shape of object a, $mbo_\phi(a)$ can degenerate to a polygon having 3, 4, 5, 6 or 7 vertices. For instance, in Fig. 4b the minimum bounding octagon of object s has only 4 vertices. This fact does not affect our analysis.

To define the minimum bounding octagon, we only need to specify 4 points (instead of 8)[2]. More specifically, vertices A', B', C' and D' can be computed using vertices A, B, C and D as follows:

$$\begin{array}{cc} A'_x = B_x & A'_y = A_y - \tan\phi(A_x - B_x) \\ B'_x = B_x + \frac{1}{\tan\phi}(B_y - C_y) & B'_y = C_y \\ C'_x = D_x & C'_y = C_y + \tan\phi(D_x - C_x) \\ D'_x = D_x - \frac{1}{\tan\phi}(A_y - D_y) & D'_y = A_y \end{array}$$

Note also that from the minimum bounding octagon we can easily compute the minimum bounding box. For instance, in Fig. 3, the $mbb(a)$ is the box (with its sides aligned with the axes) specified by points $(B_x(a), C_y(a))$ and $(D_x(a), A_y(a))$.

Using the minimum bounding octagon, we can formally define single-tile relations.

Definition 3: Let *(i)* $\mathcal{C}DR(\phi)$ be an arbitrary model in the $\mathcal{C}DR$ family ($0° < \phi < 90°$), *(ii)* a and b be two objects in

[2]Similarly to the minimum bounding box, where we need 2 points (instead of 4).

REG^* and *(iii)* $A(a)$, $B(a)$, $C(a)$, $D(a)$ and $A(b)$, $B(b)$, $C(b)$, $D(b)$ the vertices of $mbo_\phi(a)$ and $mbo_\phi(b)$ respectively. Relations N, W, S, E and B are defined as follows.

$a\ N\ b$ iff $\tan(\phi)B_x(a) + B_y(a) \geq \tan(\phi)B_x(b) + A_y(b)$, $\tan(\phi)C_x(a) - C_y(a) \leq \tan(\phi)D_x(b) - A_y(b)$ and $C_y(a) \geq A_y(b)$

$a\ W\ b$ iff $\tan(\phi)D_x(a) + D_y(a) \leq \tan(\phi)B_x(b) + A_y(b)$, $\tan(\phi)C_x(a) - C_y(a) \leq \tan(\phi)B_x(b) - C_y(b)$ and $D_x(a) \leq B_x(b)$

$a\ S\ b$ iff $\tan(\phi)A_x(a) - A_y(a) \geq \tan(\phi)B_x(b) - C_y(b)$, $\tan(\phi)D_x(a) + D_y(a) \leq \tan(\phi)D_x(b) + C_y(b)$ and $A_y(a) \leq C_y(b)$

$a\ E\ b$ iff $\tan(\phi)A_x(a) - A_y(a) \geq \tan(\phi)D_x(b) - A_y(b)$, $\tan(\phi)B_x(a) + B_y(a) \geq \tan(\phi)D_x(b) + C_y(b)$ and $B_x(a) \geq D_x(b)$

$a\ B\ b$ iff $C_y(a) \geq C_y(b)$, $A_y(a) \leq A_y(b)$, $B_x(a) \geq B_x(b)$ and $D_x(a) \leq D_x(b)$

A single-tile relation between a primary object a and a reference object b can also be defined using the minimum bounding octagon of a ($mbo_\phi(a)$) and the minimum bounding box of b ($mbb(b)$). Notice that the above definition is essentially equivalent to Definition 3 since, as we have previously seen, $mbb(b)$ can be easily computed from $mbo_\phi(b)$. We have expressed Definition 3 using only minimum bounding octagons for two reasons. First, using the same type of approximation for both objects a and b results in a more simple and uniform definition. More importantly, the minimum bounding octagon of the reference object b ($mbo_\phi(b)$) will be more useful in subsequent computations. For instance, it is easy to verify that $mbb(b)$ can only be used to compute relations where b acts as a reference object while $mbo_\phi(b)$ can be used to compute any relation involving b (regardless if b acts as a primary or a reference object).

Multi-tile directional relations are defined as follows:

Definition 4: Let a and b be two objects in REG^* and $R = R_1{:}\cdots{:}R_k$ a multi-tile directional relation. Then, $a\ R_1{:}\cdots{:}R_k\ b$ holds iff there exist objects $a_1, \ldots, a_k \in REG^*$ such that $a_1\ R_1\ b, \ldots, a_k\ R_k\ b$ and $a = a_1 \cup \cdots \cup a_k$.

Example 3: In Fig. 2c, we have $a\ N{:}W\ b$ since there exist objects a_1 and a_2 in REG^* such that $a_1\ N\ b$, $a_2\ W\ b$ and $a = a_1 \cup a_2$.

To avoid overloading Fig. 2, we have not illustrated $mbo_\phi(a)$, $mbo_\phi(a_1)$ and $mbo_\phi(a_2)$. In the rest of the paper, we will also omit the minimum bounding octagon of the primary object whenever the relation can be easily seen. Finally, in Definition 4, notice that for every i, j such that $1 \leq i, j \leq k$ and $i \neq j$, a_i and a_j have disjoint interiors but may share points in their boundaries.

In the following section, we will study the problems of computing the *inverse* of a directional relation and the *composition* of two directional relations. Our results are valid for every model $\mathcal{CDR}(\phi)$ in the $\mathcal{CDR}$ family.

III. Inverse and composition

In this section, we will study the problem of computing the inverse and the composition of directional relations in the $\mathcal{CDR}$ family. We first present a method for computing the inverse of a $\mathcal{CDR}$ relation and then a method for composing two $\mathcal{CDR}$ relations. The presented solutions are handled in a uniform manner and apply to all the $\mathcal{CDR}(\phi)$ models of the $\mathcal{CDR}$ family. Let us first define the inverse of a relation.

Definition 5: Let Q be a directional relation in $2^{\mathcal{B}^*}$. The *inverse* of relation Q, denoted by $inv(Q)$, is another directional relation which satisfies the following. For arbitrary objects $a, b \in REG^*$, $a\ inv(Q)\ b$ holds, iff $b\ Q\ a$ holds.

Two definitions of composition appear in the literature. The first one is the standard existential definition from set theory [8], [4].

Definition 6: Let Q_1 and Q_2 be directional relations in $2^{\mathcal{B}^*}$. The *existential composition* of relations Q_1 and Q_2, denoted by $Q_1; Q_2$, is another directional relation from $2^{\mathcal{B}^*}$ which satisfies the following. For arbitrary objects a and c, $a\ Q_1; Q_2\ c$ holds if and only if there exists an object b such that $a\ Q_1\ b$ and $b\ Q_2\ c$ hold.

The second definition is as follows [8], [28].

Definition 7: Let Q_1 and Q_2 be directional relations in $2^{\mathcal{B}^*}$. The *consistency-based composition* of relations Q_1 and Q_2, denoted by $Q_1 \circ Q_2$, is another directional relation from $2^{\mathcal{B}^*}$ which satisfies the following. $Q_1 \circ Q_2$ contains all relations $R \in \mathcal{B}^*$ such that there exist objects $a, b, c \in REG^*$ such that $a\ Q_1\ b$, $b\ Q_2\ c$ and $a\ R\ c$ hold.

The consistency-based definition of composition is weaker that the existential definition. Observe that $R_1; R_2 \subseteq R_1 \circ R_2$ holds. The above definitions are important and have attracted the interest of many researchers since they can be used as a mechanism for inferring new information from existing one [8], [28], [4].

In this section, we first present a method to compute the inverse of a $\mathcal{CDR}$ relation (Lemmata 1, 2 and Theorem 1). Then, we study consistency-based composition. We consider progressively more expressive classes of directional relations and give consistency-based composition algorithms for these classes (Lemmata 3, 4, 5 and Theorem 2). Finally, we consider the existential definition of composition and show that the binary relation resulting from the existential composition of some directional relations *cannot be expressed* using the $\mathcal{CDR}$ relations. Our theoretical framework allows us to *prove formally* that our solutions are correct.

As we discussed in Section II-A, relations in the $\mathcal{CDR}$ family are defined using the minimum bounding octagon while $\mathcal{PDR}$ relations are defined using the minimum bounding box. When handling the inverse and composition problems, this difference is crucial and renders unapplicable the *mbb*-based technique developed in [17] for the $\mathcal{PDR}$ model. This led us to develop a new *mbo*-based strategy for handling the inverse and composition problems for the $\mathcal{CDR}$ family.

During the study of the inverse and composition problems, we will use the informal, inclusion-based definition of basic directional relations. The formal definition involving the minimum bounding octagons of objects is used implicitly. We note that the *mbo*-based definition is not without use, as (a) it is implicitly used and (b) it is an integral part of the framework that will be required for the further study of the proposed models (e.g., for the study of the consistency checking and variable elimination problems).

Before we present our results, we introduce the necessary notation.

Notation 2: Let $R_1, \ldots, R_k$ be single-tile directional relations. We denote by $\delta(R_1, \ldots, R_k)$ the disjunction of all basic directional relations that can be constructed by combining the single tile relations $R_1, \ldots, R_k$. For instance, $\delta(N, W, B)$ stands for the following directional relation:

$$\{N,\ W,\ B,\ N{:}W,\ N{:}B,\ W{:}B,\ N{:}W{:}B\}$$

Moreover, we define:

$$\delta(R_1{:}\cdots{:}R_k) = \delta(R_1, \ldots, R_k) \text{ and}$$
$$\delta(\delta(R_{11}, \ldots, R_{1k_1}), \ldots, \delta(R_{m1}, \ldots, R_{mk_m})) = \delta(R_{11}, \ldots, R_{1k_1}, \ldots, R_{m1}, \ldots, R_{mk_m}).$$

We denote by U_{dir} the universal directional relation, i.e., $U_{dir} = \delta(N, W, S, E, B)$.

Notation 3: Let $R \in \{N, W, S, E\}$ be a single tile. We denote by $R^{\leftarrow}$ (respectively $R^{\rightarrow}$, $R^{\downarrow}$) the tile that we meet by moving counter-clockwise (respectively clockwise, diametrically) from tile R. Given a relation R expressions $R^{\leftarrow}$, $R^{\rightarrow}$ and $R^{\downarrow}$ are defined in the following table.

R	N	W	S	E
$R^{\leftarrow}$	W	S	E	N
$R^{\rightarrow}$	E	N	W	S
$R^{\downarrow}$	S	E	N	W

Notation 4: Let $R_1, \ldots, R_k$ be basic directional relations. The ***tile-union*** of $R_1, \ldots, R_k$, denoted by $\textit{tile-union}(R_1, \ldots, R_k)$, is the basic directional relation that consists of all the tiles in relations $R_1, \ldots, R_k$. Furthermore, we denote by $Combine(Q_1, \ldots, Q_k)$ (where $Q_1, \ldots, Q_k \in 2^{\mathcal{B}^*}$) the directional relation $\{R \in \mathcal{B}^* : R = \textit{tile-union}(s_1, \ldots, s_k) \wedge s_1 \in Q_1 \wedge \cdots \wedge s_k \in Q_k\}$.

Example 4: Consider two basic directional relations, $N{:}W$ and $N{:}E{:}B$. Then,

$$\textit{tile-union}(N{:}W, N{:}E{:}B) = N{:}W{:}E{:}B.$$

Furthermore, consider two directional relations, $\{N, N{:}W\}$ and $\{S, S{:}E\}$. Then, $Combine(\{N, N{:}W\}, \{S, S{:}E\}) =$

$$\left\{\begin{array}{c} \textit{tile-union}(N, S), \\ \textit{tile-union}(N, S{:}E), \\ \textit{tile-union}(N{:}W, S), \\ \textit{tile-union}(N{:}W, S{:}E) \end{array}\right\} = \left\{\begin{array}{c} N{:}S, \\ N{:}S{:}E, \\ N{:}W{:}S, \\ N{:}W{:}S{:}E \end{array}\right\}.$$

A. Computing the inverse of a directional relation

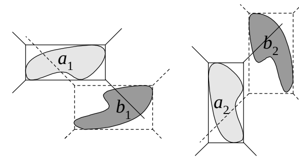
Before we proceed, we present a useful proposition. Proposition 1 reveals the inherent symmetry in the $\mathcal{CDR}$ family and simplifies the proofs of Lemmata 1 and 2 that follow.

Proposition 1: Consider a basic directional relation $R_1{:}\cdots{:}R_k$. Let us assume that its inverse is a directional relation that can be represented as a function of the five tiles N, W, S, E, B, i.e., $inv(R_1{:}\cdots{:}R_k) = f(N, W, S, E, B)$. Then:

(i) $inv(R_1^{\leftarrow}{:}\cdots{:}R_k^{\leftarrow}) = f(N^{\leftarrow}, W^{\leftarrow}, S^{\leftarrow}, E^{\leftarrow}, B) = f(W, S, E, N, B)$

(ii) $inv(R_1^{\rightarrow}{:}\cdots{:}R_k^{\rightarrow}) = f(N^{\rightarrow}, W^{\rightarrow}, S^{\rightarrow}, E^{\rightarrow}, B) = f(E, N, W, S, B)$

(iii) $inv(R_1^{\downarrow}{:}\cdots{:}R_k^{\downarrow}) = f(N^{\downarrow}, W^{\downarrow}, S^{\downarrow}, E^{\downarrow}, B) = f(S, E, N, W, B)$



(a) (b)

Fig. 5. Proving Proposition 1

Proof: ***Case (i).*** This is due to the symmetry of the directional relations of $\mathcal{CDR}$. To give an example, let us observe Fig. 5a. For this spatial configuration, we have that $a_1\ N{:}W\ b_1$, $b_1\ S{:}E\ a_1$ and, thus $S{:}E \in inv(N{:}W)$. Now, consider rotating the configuration of Fig. 5a by 90° counter-clockwise (Fig. 5b). The effect of this rotation is that the tiles in our relations have also been rotated. Specifically, N became W ($N^{\leftarrow}$), W became S ($W^{\leftarrow}$), S became E ($S^{\leftarrow}$) and E became N ($E^{\leftarrow}$). Notice that, in Fig. 5b, we have that $a_2\ W{:}S\ b_2$, $b_2\ N{:}E\ a_2$ and, thus, $N{:}E \in inv(W{:}S)$. Note that these expressions can be derived directly from the expressions $a_1\ N{:}W\ b_1$, $b_1\ S{:}E\ a_1$ and $S{:}E \in inv(N{:}W)$ concerning Fig. 5a by directly applying the aforementioned substitutions. ***Cases (ii) and (iii).*** These cases also hold due to the symmetry of directional relations. To verify this, we have to rotate the plane clockwise by 90° and 180° respectively. ■

We now present and formally prove Lemma 1, for computing the inverse of single-tile relations.

Lemma 1: Let $R \in \{N, W, S, E\}$ be a single-tile directional relation. Then:

(i) $inv(R) = \delta(R^{\leftarrow}, R^{\rightarrow}, R^{\downarrow}) - \{R^{\leftarrow}, R^{\rightarrow}\}$

(ii) $inv(B) = U_{dir} - \{N, W, S, E\}$

Proof: ***Case (i).*** We will first prove that the expression of Lemma 1(i) holds for $R = N$, i.e., $inv(N) = \delta(W, S, E) - \{W, E\}$. Let a, b be two objects in REG^* such that $a\ N\ b$. Since a is *north* of b, we can intuitively understand and easily verify that no part of b can in turn lie *north* or inside the *minimum bounding box* of object a. Therefore, no part of b can lie inside tiles $N(a)$ and $B(a)$ and consequently, tiles N and B cannot appear in directional relation $inv(N)$, which implies that

$$inv(N) \subseteq \delta(W, S, E) = \{W, S, E, W{:}S, W{:}E, S{:}E, W{:}S{:}E\}$$

Let us now consider every basic relation in $\delta(W, S, E)$ and check whether it belongs to $inv(N)$ or not.

1) Relation S: Fig. 6a demonstrates that $b\ S\ a$ is possible, thus, $S \in inv(N)$.
2) Relation $W{:}S$: Fig. 6b shows that $b\ W{:}S\ a$ is possible, thus, $W{:}S \in inv(N)$.
3) Relation $S{:}E$: Similarly with relation $W{:}S$, we can show that $S{:}E \in inv(N)$.
4) Relation $W{:}E$: Fig. 6c depicts the not so obvious possibility that $b\ W{:}E\ a$, thus, $W{:}E \in inv(N)$.
5) Relation $W{:}S{:}E$: Fig. 6d shows that $b\ W{:}S{:}E\ a$ is possible, thus, $W{:}S{:}E \in inv(N)$.

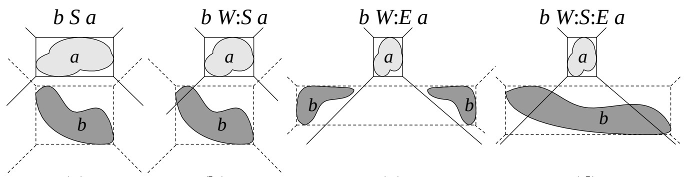


Fig. 6. Proving Lemma 1(i) for $R = N$

6) Relation W: It is not possible to create a spatial configuration such that $b\ W\ a$, thus, $W \notin inv(N)$.
7) Relation E: Similarly with relation W, it is not possible to create a spatial configuration such that $b\ E\ a$ and, thus, $E \notin inv(N)$.

Therefore, we have that $inv(N) = \delta(W, S, E) - \{W, E\}$ (M). By applying Proposition 1 to Expression (M), we also have:

$$inv(W) = \delta(S, N, E) - \{S, N\}, \quad inv(S) = \delta(E, W, N) - \{E, W\} \quad \text{and} \quad inv(E) = \delta(N, S, W) - \{N, S\}.$$

The above expressions and Expression (M) are captured by Lemma 1(i).

Case (ii). In order to compute $inv(B)$ we apply the same procedure as with Case (i). Let a, b be two objects in REG^* such that $a\ B\ b$. In this case, we cannot eliminate any tiles from $inv(B)$, so as a starting point we will consider that

$$inv(B) \subseteq U_{dir}$$

By examining every basic relation in U_{dir}, in the same way we did while proving Case (i), we conclude that $inv(B) = U_{dir} - \{N, W, S, E\}$. ■

The following Lemma can be used to compute the inverse of multi-tile relations.

Lemma 2: Let $R = R_1{:}\cdots{:}R_k$ $(2 \leq k \leq 5)$ be a multi-tile directional relation. Let also $\overline{R} = \{N, W, S, E, B\} - \{R_1, \ldots, R_k\}$. Then:

(i) $inv(R) = \delta(\overline{R})$, if $B \notin \{R_1, \ldots, R_k\}$

(ii) $inv(R) = \delta(\overline{R}, B) - \overline{R}$, if $B \in \{R_1, \ldots, R_k\}$

Proof: ***Case (i).*** We will first prove that the expression of Lemma 2(i) holds for $R = N{:}W$, i.e., $inv(N{:}W) = \delta(S, E, B)$. Let a, b be two objects in REG^* such that $a\ N{:}W\ b$. Since a is *north* and *west* of b, we can intuitively understand and easily verify that no part of b can in turn lie *north* or *west* of object a. Therefore, no part of b can lie within tiles $N(a)$ and $W(a)$. Consequently, we can exclude tiles N and W from directional relation $inv(N{:}W)$. Thus, as a starting point we can use the following expression:

$$inv(N{:}W) \subseteq \delta(S, E, B) = \{S, E, B, S{:}E, S{:}B, E{:}B, S{:}E{:}B\}.$$

Let us now consider every basic relation in $\delta(S, E, B)$ and check whether it belongs to $inv(N{:}W)$ or not.

1) Relation E: Fig. 7a shows that $b\ E\ a$ is possible, thus, $E \in inv(N{:}W)$.
2) Relation $E{:}B$: Fig. 7b illustrates that $bE{:}B\ a$ is possible, thus, $E{:}B \in inv(N{:}W)$.

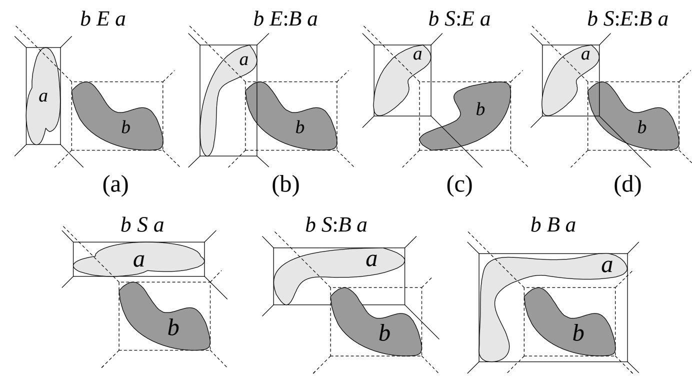


Fig. 7. Proving Lemma 2(i) for $R = N{:}W$

3) Relation $S{:}E$: Fig. 7c shows that $b\ S{:}E\ a$ is also feasible, thus, $S{:}E \in inv(N{:}W)$.
4) Relation $S{:}E{:}B$: Fig. 7d shows that $b\ S{:}E{:}B\ a$ is feasible, thus, $S{:}E{:}B \in inv(N{:}W)$.
5) Relation S: Fig. 7e illustrates that $b\ S\ a$ is possible, thus, $S \in inv(N{:}W)$.
6) Relation $S{:}B$: Fig. 7f shows that $b\ S{:}B\ a$ is feasible, thus, $S{:}B \in inv(N{:}W)$.
7) Relation B: Fig. 7g illustrates that $b\ B\ a$ is possible, thus, $B \in inv(N{:}W)$.

Therefore, $inv(N{:}W) = \delta(S, E, B)$ (T1). By applying Proposition 1 to Expression (T1), we have:

$$inv(W{:}S) = \delta(N, E, B), \ inv(N{:}E) = \delta(W, S, B) \ \text{and} \ inv(S{:}E) = \delta(N, W, B).$$

The above expressions verify that Lemma 2(i) also holds for relations $W{:}S$, $N{:}E$ and $S{:}E$.

We will now prove that the expression of Lemma 2(i) holds for $R = N{:}S$, i.e., $inv(N{:}S) = \delta(W, E, B)$. To this end, we will follow the same procedure as with relation $N{:}W$.

Let a and b be two objects in REG^* such that $a\ N{:}S\ b$. Since no part of object b can lie inside tiles $N(a)$ and $S(a)$, we can exclude tiles N and S from directional relation $inv(N{:}S)$. Therefore, as a starting point we can use expression:

$$inv(N{:}S) \subseteq \delta(W, E, B) = \{W, E, B, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}.$$

By examining every relation in $\delta(W, E, B)$, we conclude that $inv(N{:}S) = \delta(W, E, B)$ (T2). Furthermore, by applying Proposition 1 to Expression (T2), we can prove that Lemma 2(i) also holds for relation $W{:}E$.

Let us now prove that the expression of Lemma 2(i) holds for $R = N{:}W{:}E$, i.e., $inv(N{:}W{:}E) = \delta(S, B)$. We will follow the same procedure as with relation $N{:}W$.

Let a and b be two objects in REG^* such that $a\ N{:}W{:}E\ b$. Since no part of object b can lie inside tiles $N(a)$, $W(a)$ and $E(a)$, we can exclude tiles N, W and E from directional relation $inv(N{:}W{:}E)$. Therefore, as our starting point we can use expression:

$$inv(N{:}W{:}E) \subseteq \delta(S, B) = \{S, B, S{:}B\}.$$

By examining every relation in $\delta(S,B)$, we conclude that $inv(N{:}W{:}E) = \delta(S,B)$ (T3). Furthermore, by applying Proposition 1 to Expression (T3), we can prove that Lemma 2(i) also holds for relations $N{:}W{:}S$, $N{:}S{:}E$ and $W{:}S{:}E$. Lastly, it is easy to verify that the expression of Lemma 2(i) also holds for $R = N{:}W{:}S{:}E$, i.e., $inv(N{:}W{:}S{:}E) = \delta(B) = B$.

Summarizing, we have proven that Lemma 2(i) holds for all multi-tile relations that do not contain tile B.

Case (ii). This case can be proven by applying the same procedure as with Case (i). We start by verifying that the expression of Lemma 2(ii) holds for relations $N{:}B$, $N{:}W{:}B$, $N{:}S{:}B$, $N{:}W{:}E{:}B$ and $N{:}W{:}S{:}E{:}B$. Then, by applying Proposition 1, we can verify that Lemma 2(ii) holds for all multi-tile relations that include tile B. ■

To compute the inverse of an arbitrary directional relation we can use the following theorem in combination with Lemmata 1 and 2.

Theorem 1: Let $Q = \cup_{i=1}^{k} R_i$ be a directional relation in $2^{\mathcal{B}^*}$, where R_i are basic directional relations. Then, $inv(Q) = \cup_{i=1}^{k}(inv(R_i))$. Note that $inv(R_i)$ can be computed using Lemmata 1 and 2.

Proof: From the definition of inverse (Definition 5), we have:

$$inv(Q) = \{R \in \mathcal{B}^* : (\exists a, b \in REG^*)(a\ Q\ b \wedge b\ R\ a)\}.$$

Since $Q = \cup_{i=1}^{k} R_i$, we have $a\ Q\ b = a\ R_1\ b \vee \cdots \vee a\ R_k\ b$. Therefore,

$$\begin{aligned} inv(Q) = \{R \in \mathcal{B}^* : (\exists a, b \in REG^*) \\ (a\ R_1\ b \vee \cdots \vee a\ R_k\ b) \wedge b\ R\ a\}. \end{aligned}$$

By distributing $\wedge$ over $\vee$, we have:

$$\begin{aligned} inv(Q) = \{R \in \mathcal{B}^* : (\exists a, b \in REG^*) \\ (a\ R_1\ b \wedge b\ R\ a) \vee \cdots \vee (a\ R_k\ b \wedge b\ R\ a)\}. \end{aligned}$$

Thus, $inv(Q) = \cup_{i=1}^{k}(inv(R_i))$ holds. ■

B. Computing the composition of two directional relations

Before we address the composition problem, we present Proposition 2, which serves the same purpose as Proposition 1 did while studying the inverse problem, i.e., it reveals the inherent symmetry in the $\mathcal{CDR}$ family and simplifies the proofs of the relevant Lemmata.

Proposition 2: Consider two basic directional relations $R_1 = R_{11}{:}\cdots{:}R_{1k}$ and $R_2 = R_{21}{:}\cdots{:}R_{2m}$. Let us assume that their composition is a directional relation that can be represented as a function of the five tiles N, W, S, E, B, i.e., $R_1 \circ R_2 = f(N, W, S, E, B)$. Then:

(i) $R_{11}^{\leftarrow}{:}\cdots{:}R_{1k}^{\leftarrow} \circ R_{21}^{\leftarrow}{:}\cdots{:}R_{2m}^{\leftarrow} = f(N^{\leftarrow}, W^{\leftarrow}, S^{\leftarrow}, E^{\leftarrow}, B) = f(W, S, E, N, B)$

(ii) $R_{11}^{\rightarrow}{:}\cdots{:}R_{1k}^{\rightarrow} \circ R_{21}^{\rightarrow}{:}\cdots{:}R_{2m}^{\rightarrow} = f(N^{\rightarrow}, W^{\rightarrow}, S^{\rightarrow}, E^{\rightarrow}, B) = f(E, N, W, S, B)$

(iii) $R_{11}^{\downarrow}{:}\cdots{:}R_{1k}^{\downarrow} \circ R_{21}^{\downarrow}{:}\cdots{:}R_{2m}^{\downarrow} = f(N^{\downarrow}, W^{\downarrow}, S^{\downarrow}, E^{\downarrow}, B) = f(S, E, N, W, B)$

Proof: This is due to the symmetry of the directional relations of $\mathcal{CDR}$. ■

We will address the composition problem one step at a time. First, we consider the case of composing two single-tile relations.

Lemma 3: Let $R \in \{N, W, S, E\}$ be a single-tile directional relation. Then:

(i) $R \circ R = R$ (ii) $R \circ R^{\downarrow} = U_{dir}$
(iii) $R \circ R^{\leftarrow} = R \circ R^{\rightarrow} = R \circ B = \delta(R, R^{\leftarrow}, R^{\rightarrow}, B)$
(iv) $B \circ R = \delta(R, R^{\leftarrow}, R^{\rightarrow})$ (v) $B \circ B = B$

Proof: ***Case (i).*** We will first prove that the expression of Lemma 3(i) holds for $R = N$, i.e., $N \circ N = N$. Let a, b and c be three objects in REG^* such that $a\ N\ b$ and $b\ N\ c$ holds. Fig. 8a presents objects b and c such that $b\ N\ c$. Since $a\ N\ b$, object a lies inside tile $N(b)$ (the dotted area of Fig. 8a). Notice that tile $N(b)$ can only lie inside tile $N(c)$ and, consequently, object a can only lie inside tile $N(c)$. More formally, we have that $a \subseteq N(b) \subseteq N(c)$. In other words, if $a\ N\ b$ and $b\ N\ c$ then $a\ N\ c$, thus, $N \circ N = N$ (S1) holds. By applying Proposition 2 to Expression (S1), we also have:

$$W \circ W = W,\ E \circ E = E \text{ and } S \circ S = S.$$

All the above expressions and Expression (S1) are captured by Lemma 3(i).

Case (ii). We will first prove that the expression of Lemma 3(ii) holds for $R = N$, i.e., $N \circ S = U_{dir}$. Let a, b and c be three objects in REG^* such that $a\ N\ b$ and $b\ S\ c$ holds. Fig. 8b presents objects b and c such that $b\ S\ c$. Object a lies inside tile $N(b)$ (the dotted area of Fig. 8b). Notice that area $N(b)$ can intersect with all five tiles of object c, namely $N(c)$, $W(c)$, $S(c)$, $E(c)$ and $B(c)$. Consequently, object a can lie within any of these five tiles or any combination of them. In other words, if $a\ N\ b$ and $b\ S\ c$ then $a\ \delta(N, W, S, E, B)\ c$, or $a\ U_{dir}\ c$ and, thus, $N \circ S = U_{dir}$ (S2) holds. By applying Proposition 2 to Expression (S2), we can easily verify that Lemma 3(ii) holds.

Case (iii). We will first prove that expression $R \circ R^{\leftarrow} = \delta(R, R^{\leftarrow}, R^{\rightarrow}, B)$ holds for $R \in \{N, W, S, E\}$. Fig. 8c helps us verify that the expression holds for $R = N$, i.e., $N \circ W = \delta(N, W, E, B)$ (S3). Then, by applying Proposition 2 to Expression (S3), we can prove that the expression holds for every $R \in \{N, W, S, E\}$.

In a similar manner, we can also prove that $R \circ R^{\rightarrow} = \delta(R, R^{\leftarrow}, R^{\rightarrow}, B)$ holds for $R \in \{N, W, S, E\}$. We will now prove that expression $R \circ B = \delta(R, R^{\leftarrow}, R^{\rightarrow}, B)$ holds for $R \in \{N, W, S, E\}$. Fig. 8d helps us verify that the expression holds for $R = N$, i.e., $N \circ B = \delta(N, W, E, B)$ (S4). Then, by applying Proposition 2 to Expression (S4), we can prove that the expression holds for every $R \in \{N, W, S, E\}$.

Therefore, we have proven that Lemma 3(iii) holds.

Case (iv). Fig. 8e helps us verify that expression $B \circ R = \delta(R, R^{\leftarrow}, R^{\rightarrow})$ holds for $R = N$, i.e., $B \circ N = \delta(N, W, E)$ (S5). By applying Proposition 2 to Expression (S5), we can verify that Lemma 3(iv) holds.

Case (v). This case is trivial and Fig. 8f helps us verify that $B \circ B = B$. ■

We will now turn our attention to the composition of a single-tile with a multi-tile directional relation. To this end, we use Algorithm Compose^{SM} (Fig. 9). The algorithm

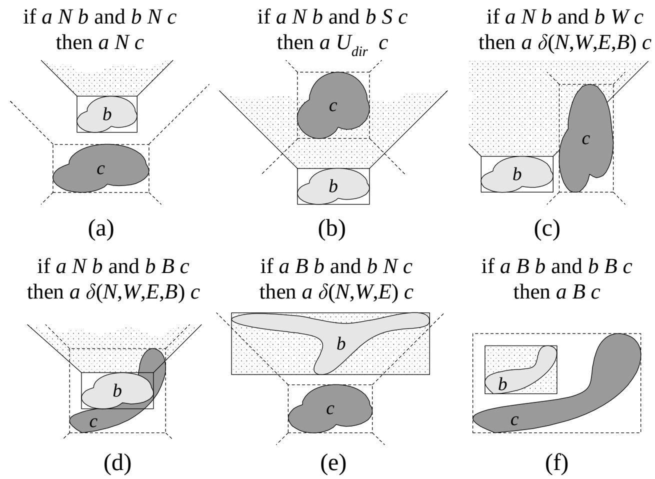


Fig. 8. Proving Lemma 3

takes as inputs a single-tile directional relation R_1 and a multi-tile relation $R_2 = R_{21}{:}\cdots{:}R_{2k}$ ($k \geq 2$) and returns the composition $R_1 \circ R_2$. The following is an example of Algorithm COMPOSE^{SM} in operation.

Example 5: Let $R_1 = N$ and $R_2 = N{:}B = R_1{:}B$. Using Algorithm COMPOSE^{SM} (Line 3), we have $N \circ N{:}B = N$. This can be verified using Fig. 10c.

Algorithm COMPOSE^{SM}
Input: A single-tile relation R_1 and a multi-tile relation $R_2 = R_{21}{:}\cdots{:}R_{2k}$, $2 \leq k$.
Output: The composition $R_1 \circ R_2$.
Method:

1. <u>If</u> ($R_1 = B$ and $R_2 = R_{21}{:}B$) <u>Return</u> $\delta(R_{21}, R_{21}^{\leftarrow}, R_{21}^{\rightarrow}, B)$
2. <u>If</u> ($R_1 = B$) <u>Return</u> U_{dir}
3. <u>If</u> ($R_2 = R_1{:}B$) <u>Return</u> R_1
4. <u>If</u> ($R_2 = R_1{:}R_1^{\leftarrow}$ or $R_2 = R_1{:}R_1^{\leftarrow}{:}B$) <u>Return</u> $\delta(R_1, R_1^{\leftarrow})$
5. <u>If</u> ($R_2 = R_1{:}R_1^{\rightarrow}$ or $R_2 = R_1{:}R_1^{\rightarrow}{:}B$) <u>Return</u> $\delta(R_1, R_1^{\rightarrow})$
6. <u>If</u> ($\{R_1, R_1^{\downarrow}\} \in R_2$ or $\{R_1, R_1^{\leftarrow}, R_1^{\rightarrow}\} \in R_2$) <u>Return</u> $\delta(R_1, R_1^{\leftarrow}, R_1^{\rightarrow})$
7. <u>If</u> ($B \in R_2$) <u>Return</u> $\delta(R_1, R_1^{\leftarrow}, R_1^{\rightarrow}, B)$
8. <u>Return</u> U_{dir}

Fig. 9. Algorithm COMPOSE^{SM}

The following lemma establishes the correctness Algorithm COMPOSE^{SM}.

Lemma 4: Let R_1 be a single-tile and $R_2 = R_{21}{:}\cdots{:}R_{2k}$ be a multi-tile directional relation. Then, $R_1 \circ R_2$ can be computed by Algorithm COMPOSE^{SM}.

Proof: Every line of the Algorithm computes the composition of a set of pairs of basic directional relations. Particularly, Lines 1 and 2 compute the composition for $R_1 = B$. The rest of the Algorithm computes the composition for $R_1 \in \{N, W, S, E\}$. Therefore, we will examine each line of the algorithm individually and verify that it correctly computes the relevant composition.

Line 1. This <u>If</u> statement states that $B \circ R_{21}{:}B = \delta(R_{21}, R_{21}^{\leftarrow}, R_{21}^{\rightarrow}, B)$ where $R_{21} \in \{N, W, S, E\}$. We will first prove that this expression holds for $R_{21} = N$, i.e., $B \circ N{:}B = \delta(N, W, E, B)$. Let a, b and c be three objects in REG^* such that $a\ B\ b$ and $b\ N{:}B\ c$ holds. Fig. 10a presents objects b and c such that $b\ N{:}B\ c$. Since $a\ B\ b$, object a lies inside tile $B(b)$ (the dotted area of Fig. 10a). Notice that area $N(b)$ can intersect with tiles $N(c)$, $W(c)$, $E(c)$ and $B(c)$. Consequently, object a can lie within any of these four tiles or any combination of them. In other words, if $a\ B\ b$ and $b\ N{:}B\ c$ then $a\ \delta(N, W, E, B)\ c$. Thus, $N \circ N{:}B = \delta(N, W, E, B)$ (E1) holds. By applying Proposition 2 to Expression (E1), we also have:

$$B \circ W{:}B = \delta(N, W, S, B),\ \ B \circ E{:}B = \delta(N, S, E, B) \ \text{ and } \ B \circ S{:}B = \delta(W, S, E, B)$$

All the above expressions and Expression (E1) are captured by Line 1 of Algorithm COMPOSE^{SM}.

Line 2. The condition of this <u>If</u> statement is satisfied when $R_1 = B$ and $R_2 \neq \{N{:}B\ , W{:}B,\ S{:}B,\ E{:}B\}$, otherwise the condition of Line 1 would have been satisfied. Line 2 holds for $R_2 = N{:}W$, i.e., $B \circ N{:}W = U_{dir}$. This can be verified using Fig. 10b. By also using Fig. 10b, we can verify that the composition of B with any relation that is made up of at least two adjacent tiles (i.e., $R_2 \neq \{N{:}B\ , W{:}B,\ S{:}B,\ E{:}B\}$) is equal to U_{dir}

Line 3. This <u>If</u> statement states that $R_1 \circ R_1{:}B = R_1$, $R_1 \in \{N, W, S, E\}$. To prove that this expression holds for $R_1 = N$, i.e., $N \circ N{:}B = N$ (E2), we use Fig. 10c. Then, by applying Proposition 2 to Expression (E2), we can verify that Line 3 is correct for all $R_1 \in \{N, W, S, E\}$.

Line 4. The condition of this <u>If</u> statement is satisfied when $R_2 \in \{R_1{:}R_1^{\leftarrow}, R_1{:}R_1^{\leftarrow}{:}B\}$. We will only prove that Line 4 is correct for $R_2 = R_1{:}R_1^{\leftarrow}$, i.e., $R_1 \circ R_1{:}R_1^{\leftarrow} = \delta(R_1{:}R_1^{\leftarrow})$. The proof for $R_2 = R_1{:}R_1^{\leftarrow}{:}B$ is similar. Line 4 holds for $R_1 = N$, i.e., $N \circ N{:}W = \delta(N, W)$ (E3). This can be verified using Fig. 10d. By applying Proposition 2 to Expression (E3), we can also verify that Line 4 is correct for all $R_1 \in \{N, W, S, E\}$.

Line 5. The proof is similar to the proof of Line 4.

Line 6. The condition of this <u>If</u> statement is satisfied when relation R_2 contains tiles $\{R_1, R_1^{\downarrow}\}$ or $\{R_1, R_1^{\leftarrow}, R_1^{\rightarrow}\}$. We will first prove that Line 6 holds for $R_1 = N$. Then, R_2 contains tiles $\{N, S\}$ or $\{N, W, E\}$. We will concentrate on the two most representative relations of this group, namely $R_2 = N{:}S$ and $R_2 = N{:}W{:}E$, since the proofs for the other relations of the group are almost identical to one of these two. Fig. 10e helps us verify that $N \circ N{:}W{:}E = \delta(N, W, E)$ (E4) and Fig. 10f that $N \circ N{:}S = \delta(N, W, E)$ (E5). Then, by applying Proposition 2 to Expressions (E4) and (E5), we can verify that Line 6 holds for all $R_1 \in \{N, W, S, E\}$.

Line 7. The <u>If</u> condition of this statement requires that $B \in \{R_{21}, \ldots, R_{2k}\}$. However, for the Algorithm to reach Line 7, the conditions of the <u>If</u> statements in Lines 3-6 must have not been satisfied. These four statements provide the composition for all relations R_2 that contain tile R_1. Therefore, relation R_2 includes tile B but not tile R_1. In other words, $R_2 \in \{R_1^{\leftarrow}{:}B,\ R_1^{\rightarrow}{:}B,\ R_1^{\downarrow}{:}B,\ R_1^{\leftarrow}{:}R_1^{\rightarrow}{:}B,\ R_1^{\leftarrow}{:}R_1^{\downarrow}{:}B,\ R_1^{\rightarrow}{:}R_1^{\downarrow}{:}B,\ R_1^{\leftarrow}{:}R_1^{\rightarrow}{:}R_1^{\downarrow}{:}B\}$. We will concentrate relations of the form $R_2 = R_1^{\downarrow}{:}B$ and prove that $R_1 \circ R^{\downarrow}{:}B = \delta(R_1, R_1^{\leftarrow}, R_1^{\rightarrow}, B)$ (E6) holds (the proofs for the rest of the relations are similar). Expression (E6) holds for $R_1 = N$, i.e., $N \circ S{:}B = \delta(N, W, E, B)$. This can be verified using Fig. 10g. Then, by applying Proposition 2 we can prove that Expression (E6) holds for every $R_1 \in \{N, W, S, E\}$.

Line 8. If the execution of the Algorithm reaches Line 8, then relation R_2 cannot contain tiles R_1 and B, i.e.,

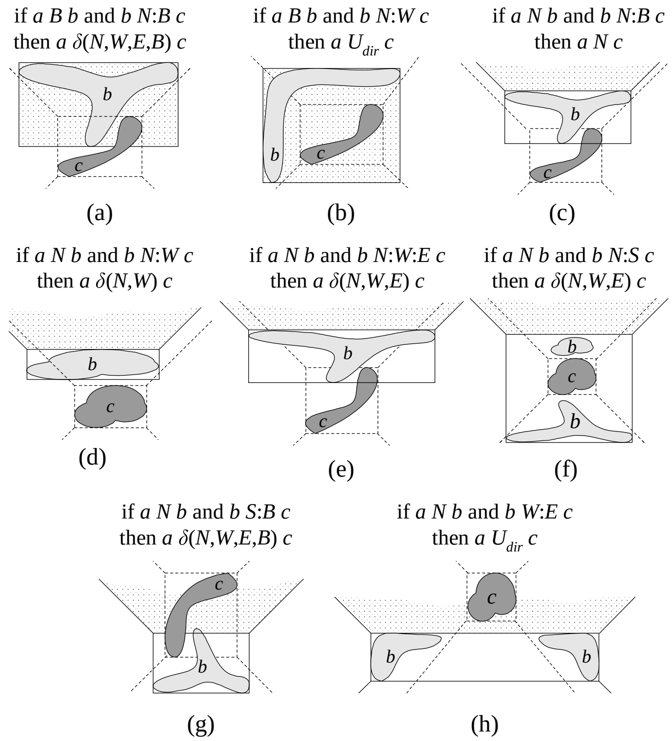


Fig. 10. Proving Lemma 4

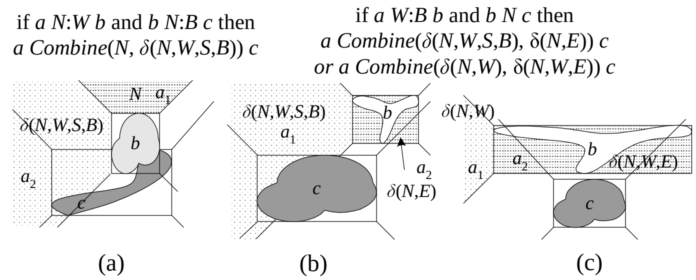


Fig. 11. Illustrations of Examples 6 and 7

$R_2 \in \{R_1^{\leftarrow}{:}R_1^{\rightarrow}, R_1^{\leftarrow}{:}R_1^{\downarrow}, R_1^{\rightarrow}{:}R_1^{\downarrow}, R_1^{\leftarrow}{:}R_1^{\rightarrow}{:}R_1^{\downarrow}\}$. When tile $R_1^{\downarrow}$ is present in relation R_2, the proof is similar to the proof of Lemma 3(ii), which states that $R_1 \circ R_1^{\downarrow} = U_{dir}$, so we will not consider these relations. Instead, we will prove expression $R_1 \circ R_1^{\leftarrow}{:}R_1^{\rightarrow} = U_{dir}$, which is not apparent. Fig. 10h shows that the expression holds for $R_1 = N$, i.e., $N \circ W{:}E = U_{dir}$. Then, by applying Proposition 2, we can prove that Line 8 is correct for all $R_1 \in \{N, W, S, E\}$. ■

Summarizing our progress so far, we have presented Lemma 3 that can be used to compute the composition of two single-tile relations, and then Algorithm COMPOSE^{SM} that provides the composition of a single-tile and a multi-tile relation. In other words, we are able to compute the composition of a single-tile and a basic (single-tile or multi-tile) directional relation. The next logical step is to address the problem of composing a multi-tile and a basic relation. Let us study two specific examples that will help us understand the method used to compute the composition of such relations.

Example 6: Let us compute $N{:}W \circ N{:}B$. Let a, b and c be three objects in REG^* such that $a\ N{:}W\ b$ and $b\ N{:}B\ c$. To compute $N{:}W \circ N{:}B$, we have to find all possible relations between a and c. According to Definition 4, $a\ N{:}W\ b$ implies that there exist objects a_1 and a_2, such that $a_1\ N\ b$, $a_2\ W\ b$ and $a = a_1 \cup a_2$. We can handle the composition problem for each of the two components of object a separately and then use the corresponding results to create the directional relation $N{:}W \circ N{:}B$.

Fig. 11a shows two objects b and c such that $b\ N{:}B\ c$. The heavily dotted area corresponds to tile $N(b)$, where object a_1 lies ($a_1\ N\ b$) and the lightly dotted area corresponds to tile $W(b)$, where object a_2 lies ($a_2\ W\ b$). Since $a_1\ N\ b$ and $b\ N{:}B\ c$, we have $a_1\ N \circ N{:}B\ c$ and using Algorithm COMPOSE^{SM}, we can compute that $a_1\ N\ c$ (see also Fig. 11a). Similarly, since $a_2\ W\ b$ and $b\ N{:}B\ c$, we have $a_2\ \delta(N, W, S, B)\ c$.

Let us see how we can use these results to calculate the possible relations between a and c. Since $a = a_1 \cup a_2$, then if for example $a_1\ N\ c$ and $a_2\ W{:}S\ c$, we have that $a\ \textit{tile-union}(N, W{:}S)\ c$ or $a\ N{:}W{:}S\ c$. Likewise, if $a_1\ N\ c$ and $a_2\ N{:}W{:}B\ c$, then $a\ \textit{tile-union}(N, N{:}W{:}B)\ c$ or $a\ N{:}W{:}B\ c$. Therefore, directional relation $N{:}W \circ N{:}B$, includes all basic relations created by taking the $\textit{tile-union}$ of each relation in $\{N\}$ with every relation in $\delta(N, W, S, B)$. In other words, $N{:}W \circ N{:}B = \textit{Combine}(N, \delta(N, W, S, B))$.

The result, as well as the procedure we have used in order to compute $N{:}W \circ N{:}B$ can be captured by expression $N{:}W \circ N{:}B = \textit{Combine}(N \circ N{:}B, W \circ N{:}B)$. One could be tempted to generalize this expression and use it to compute the composition of any two relations:

$$R_{11}{:}\cdots{:}R_{1k} \circ R_2 = \textit{Combine}(R_{11} \circ R_2, \ldots, R_{1k} \circ R_2), \qquad (C)$$

but unfortunately it does not always produce the correct result. Let us see another example that will help clarify why the aforementioned expression fails.

Example 7: Let us compute $W{:}B \circ N$. If we use Expression (C) we have $W{:}B \circ N = \textit{Combine}(W \circ N, B \circ N)$. Since $W{:}S \in \textit{Combine}(W \circ B, W \circ N)$, it follows from the above equation that there exists a spatial configuration such that $a\ W{:}B\ b$, $b\ N\ c$ and $a\ W{:}S\ c$. It is easy to verify that such a configuration does not exist, thus, Expression (C) cannot be applied to $W{:}B \circ N$.

To compute the correct composition, let a, b and c be three objects in REG^* such that $a\ W{:}B\ b$ and $b\ N\ c$. According to Definition 4, $a\ W{:}B\ b$ implies that there exist objects a_1 and a_2, such that $a_1\ W\ b$, $a_2\ B\ b$ and $a = a_1 \cup a_2$. Fig. 11b and Fig. 11c depict two spatial configurations involving objects b and c, such that $b\ N\ c$. In both cases, the lightly dotted area to tile $W(b)$ (i.e., the area where object a_1 lies), while the heavily dotted area corresponds to tile $B(b)$ (i.e., the area where object a_2 lies).

- For the configuration of Fig. 11b, we have that $a_1\ \delta(N, W, S, B)\ c$ and $a_2\ \delta(N, E)\ c$. Thus, we have $a\ \textit{Combine}(\delta(N, W, S, B), \delta(N, E))\ c$ and $\textit{Combine}(\delta(N, W, S, B), \delta(N, E)) \subseteq W{:}B \circ N$.
- For the configuration of Fig. 11c, we have that $a_1\ \delta(N, W)\ c$ and $a_2\ \delta(N, W, E)\ c$. Thus, we

Algorithm COMPOSE^M

Input: A multi-tile relation $R_1 = R_{11}:\cdots:R_{1k}$ ($2 \leq k$) and a basic relation R_2

Output: The composition $R_1 \circ R_2$.

Method:

$R \in \{N, W, S, E\}$

$C = Combine(R_{11} \circ R_2, \ldots, R_{1k} \circ R_2)$

If ($R_2 = R$)

  If ($R_1 = R^{\leftarrow}{:}B$) Return $C - \{R^{\leftarrow}{:}R^{\downarrow}, R^{\leftarrow}{:}B, R^{\leftarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 = R^{\rightarrow}{:}B$) Return $C - \{R^{\rightarrow}{:}R^{\downarrow}, R^{\rightarrow}{:}B, R^{\rightarrow}{:}R^{\downarrow}{:}B\}$

If ($R_2 = R{:}B$)

  If ($R_1 = R^{\leftarrow}{:}B$) Return $C - \{R^{\leftarrow}{:}R^{\downarrow}\}$

  If ($R_1 = R^{\rightarrow}{:}B$) Return $C - \{R^{\rightarrow}{:}R^{\downarrow}\}$

If ($R_2 \in \{R{:}R^{\downarrow}, R{:}R^{\downarrow}{:}B\}$)

  If ($R_1 = R{:}R^{\leftarrow}$) Return $C - \{R^{\leftarrow}{:}B, R^{\leftarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 = R{:}R^{\rightarrow}$) Return $C - \{R^{\rightarrow}{:}B, R^{\rightarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 = R^{\leftarrow}{:}R^{\downarrow}$) Return $C - \{R^{\leftarrow}{:}B, R{:}R^{\leftarrow}{:}B\}$

  If ($R_1 = R^{\rightarrow}{:}R^{\downarrow}$) Return $C - \{R^{\rightarrow}{:}B, R{:}R^{\rightarrow}{:}B\}$

  If ($R_1 = R{:}R^{\downarrow}$) Return $C - \delta(R^{\leftarrow}, R^{\rightarrow})$

  If ($R_1 = R{:}R^{\leftarrow}{:}R^{\downarrow}$) Return $C - \delta(R^{\leftarrow}, R^{\rightarrow}, B) - \{R{:}R^{\leftarrow}{:}B, R^{\leftarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 = R{:}R^{\rightarrow}{:}R^{\downarrow}$) Return $C - \delta(R^{\leftarrow}, R^{\rightarrow}, B) - \{R{:}R^{\rightarrow}{:}B, R^{\rightarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 \in \{R{:}R^{\downarrow}{:}B, R{:}R^{\leftarrow}{:}R^{\downarrow}{:}B, R{:}R^{\rightarrow}{:}R^{\downarrow}{:}B\}$) Return $C - \delta(R^{\leftarrow}, R^{\rightarrow}, B)$

  If ($R_1 \in \{N{:}W{:}S{:}E, N{:}W{:}S{:}E{:}B\}$) Return $C - \delta(R^{\leftarrow}, R^{\rightarrow}, B)$

If ($R_2 \in \{R{:}R^{\leftarrow}{:}R^{\rightarrow}, R{:}R^{\leftarrow}{:}R^{\rightarrow}{:}B\}$)

  If ($R_1 = R^{\leftarrow}{:}R^{\rightarrow}$) Return $C - \{R^{\downarrow}\}$

  If ($R_1 \in \{R^{\leftarrow}{:}R^{\downarrow}, R{:}R^{\leftarrow}{:}R^{\downarrow}\}$) Return $C - \{R^{\downarrow}{:}B, R^{\rightarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 \in \{R^{\rightarrow}{:}R^{\downarrow}, R{:}R^{\rightarrow}{:}R^{\downarrow}\}$) Return $C - \{R^{\downarrow}{:}B, R^{\leftarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 = R^{\leftarrow}{:}R^{\rightarrow}{:}R^{\downarrow}$) Return $C - \{R^{\downarrow}, R^{\downarrow}{:}B, R^{\leftarrow}{:}R^{\downarrow}{:}B, R^{\rightarrow}{:}R^{\downarrow}{:}B\}$

  If ($R_1 \in \{R^{\leftarrow}{:}R^{\rightarrow}{:}B, R^{\leftarrow}{:}R^{\rightarrow}{:}R^{\downarrow}{:}B\}$) Return $C - \{R^{\downarrow}, R^{\downarrow}{:}B\}$

  If ($R_1 = N{:}W{:}S{:}E$) Return $C - \{R^{\leftarrow}{:}R^{\downarrow}{:}B, R^{\rightarrow}{:}R^{\downarrow}{:}B\}$

Return C

Fig. 12. Algorithm COMPOSE^M

have $a \quad Combine(\delta(N, W), \delta(N, W, E)) \quad c$ and $Combine(\delta(N, W), \delta(N, W, E)) \subseteq W{:}B \circ N$.

Summarizing, we have $Combine(\delta(N, W, S, B), \delta(N, E)) \cup Combine(\delta(N, W), \delta(N, W, E)) \subseteq W{:}B \circ N$. It is not hard to verify, that any other spatial configuration such that $a\ W{:}B\ b$ and $b\ N\ c$ would produce composition results that are a subset of those produced by the configurations of Fig. 11b and Fig. 11c. Thus, $W{:}B \circ N = Combine(\delta(N, W, S, B), \delta(N, E)) \cup Combine(\delta(N, W), \delta(N, W, E))$ or equivalently $W{:}B \circ N = Combine(W \circ N, B \circ N) - \{W{:}S, W{:}B, W{:}S{:}B\}$.

Summarizing Examples 6 and 7, we can distinguish two cases. For some pairs of relations $R_1 = R_{11}:\cdots:R_{1k}$ and R_2, like $N{:}W$ and $N{:}B$ of Example 6, Expression (C) can be applied directly. For the other cases, there are pairs, like $W{:}B$ and N of Example 7, that Expression (C) cannot be applied directly. Fortunately, as we will see later, we always have:

$$R_1 \circ R_2 \subseteq Combine(R_{11} \circ R_2, \ldots, R_{1k} \circ R_2)$$

Based on this observation, we present Algorithm COMPOSE^M (Fig. 12), that can be used to compute the composition of a multi-tile and a basic directional relation. Algorithm COMPOSE^M takes as inputs a multi-tile directional relation $R_1 = R_{11}:\cdots:R_{1k}$ ($k \geq 2$) and a basic relation R_2. Initially, the algorithm computes set $C = Combine(R_{11} \circ R_2, \ldots, R_{1k} \circ R_2)$. Then, it removes from set C all relations that cannot belong to the composition $R_1 \circ R_2$.

The following lemma demonstrates the correctness of Algorithm COMPOSE^M .

Lemma 5: Let R_1 and R_2 be two basic directional relations. Then $R_1 \circ R_2$ can be computed by Algorithm COMPOSE^M.

Proof: To demonstrate the correctness of Algorithm COMPOSE^M, we will present the steps we followed in order to create it. As we discussed earlier, for some relation pairs $R_1 = R_{11}:\cdots:R_{1k}$ and R_2 we can directly compute their composition using Expression (C) (Example 6), while for other pairs we must compute their composition from first principals (like in Example 7). For these pairs, the composition is equal to a subset of $Combine(R_{11} \circ R_2, \ldots, R_{1k} \circ R_2)$ and therefore can be described using an expression of the form $R_1 \circ R_2 = Combine(R_{11} \circ R_2, \ldots, R_{1k} \circ R_2) - S$, where S is a set of basic directional relations.

Based on this observation, we present Table I. This table presents the composition of all 26 multi-tile relations and relations B, N, $N{:}B$, $N{:}W$, $N{:}W{:}B$, $N{:}W{:}E$, $N{:}W{:}E{:}B$, $N{:}W{:}S{:}E$ and $N{:}W{:}S{:}E{:}B$. In Table I, we use a star ($\star$) to denote that the composition can be computed using expression $R_1 \circ R_2 = Combine(R_{11} \circ R_2, \ldots, R_{1k} \circ R_2)$. In case where the composition is computed using expression $R_1 \circ R_2 = Combine(R_{11} \circ R_2, \ldots, R_{1k}) - S$, we simply write the set S. The complete transitivity table can be derived from Table I using Proposition 2.

The structure of Algorithm COMPOSE^M reflects the results of Table I. The composition of most relation pairs is equal to C, while a handful of pairs produce results equal to $C - S$. So, the Algorithm is mainly a list of rules describing these exceptions. Let us now see how these rules can be derived from Table I. Consider for instance the composition of relations $W{:}B$ and N. According to Table I, we have that $W{:}B \circ N = Combine(W \circ N, B \circ N) - \{W{:}S, W{:}B, W{:}S{:}B\}$. By applying Proposition 2 to this expression, we conclude that:

$$S{:}B \circ W = Combine(S \circ W, B \circ W) - \{S{:}E, S{:}B, S{:}E{:}B\}$$
$$N{:}B \circ E = Combine(N \circ E, B \circ E) - \{N{:}W, N{:}B, N{:}W{:}B\}$$
$$E{:}B \circ S = Combine(E \circ S, B \circ S) - \{N{:}E, E{:}B, N{:}E{:}B\}$$

We can easily verify that the above expressions are equivalent to this single expression:

$$R^{\leftarrow}{:}B \circ R = Combine(R^{\leftarrow} \circ R, B \circ R) - \{R^{\leftarrow}{:}R^{\downarrow}, R^{\leftarrow}{:}B, R^{\leftarrow}{:}R^{\downarrow}{:}B\}, \quad R \in \{N, W, S, E\}.$$

The final version of the Algorithm, as presented in Fig. 12, contains all the rules that can be derived from Table I as If statements. ■

The following example demonstrates how Algorithm COMPOSE^M is used.

Example 8: The four outer If statements of Algorithm COMPOSE^M regard the pattern of relation R_2. $R_2 = R$ implies that R_2 consists of a single peripheral tile. $R_2 = R{:}B$ implies that R_2 consists of a single peripheral tile and tile B. Similarly, $R_2 \in \{R{:}R^{\downarrow}, R{:}R^{\downarrow}{:}B\}$ implies that R_2 consists of two non-adjacent tiles and possibly tile B, while $R_2 \in \{R{:}R^{\leftarrow}{:}R^{\rightarrow}, R{:}R^{\leftarrow}{:}R^{\rightarrow}{:}B\}$ implies that R_2 consists of three adjacent peripheral tiles and possibly tile B. For instance, the pattern of relation $R_2 = N{:}S$ is $R{:}R^{\downarrow}$, where $R = N$. Having determined the pattern of relation R_2 and assigned a value to R, we proceed, if necessary, to the inner If statements and substitute the value of R that we determined. For example, let us consider the composition $N{:}W \circ N{:}S$. As we mentioned earlier, the pattern of relation R_2 is $R{:}R^{\downarrow}$, where $R = N$. As a consequence, the condition of the third outer If statement is

R_1/R_2	B	N	$N{:}B$	$N{:}W$, $N{:}W{:}B$	$N{:}S$	$N{:}S{:}B$	$N{:}W{:}E$, $N{:}W{:}E{:}B$	$N{:}W{:}S{:}E$, $N{:}W{:}S{:}E{:}B$
$N{:}W$	$\star$	$\star$	$\star$	$\star$	$\{W{:}B, W{:}S{:}B\}$	$\{W{:}B, W{:}S{:}B\}$	$\star$	$\star$
$N{:}S$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E\}$	$\{W, E, W{:}E\}$	$\star$	$\star$
$N{:}E$	$\star$	$\star$	$\star$	$\star$	$\{E{:}B, S{:}E{:}B\}$	$\{E{:}B, S{:}E{:}B\}$	$\star$	$\star$
$N{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$W{:}S$	$\star$	$\star$	$\star$	$\star$	$\{W{:}B, N{:}W{:}B\}$	$\{W{:}B, N{:}W{:}B\}$	$\{S{:}B, S{:}E{:}B\}$	$\star$
$W{:}E$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\{S\}$	$\star$
$W{:}B$	$\star$	$\{W{:}S, W{:}B$ $W{:}S{:}B\}$	$\{W{:}S\}$	$\star$	$\star$	$\star$	$\star$	$\star$
$S{:}E$	$\star$	$\star$	$\star$	$\star$	$\{E{:}B, N{:}E{:}B\}$	$\{E{:}B, N{:}E{:}B\}$	$\{S{:}B, W{:}S{:}B\}$	$\star$
$S{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$E{:}B$	$\star$	$\{S{:}E, E{:}B, S{:}E{:}B\}$	$\{S{:}E\}$	$\star$	$\star$	$\star$	$\star$	$\star$
$N{:}W{:}S$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E, W{:}B, E{:}B, N{:}W{:}B, W{:}S{:}B, W{:}E{:}B\}$	$\{W, W{:}E, W{:}B, E{:}B, N{:}W{:}B, W{:}S{:}B, W{:}E{:}B\}$	$\{S{:}E{:}B\}$	$\star$
$N{:}W{:}E$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$N{:}W{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$N{:}S{:}E$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E, W{:}B, E{:}B, N{:}E{:}B, S{:}E{:}B, W{:}E{:}B\}$	$\{E, W{:}E, W{:}B, E{:}B, N{:}E{:}B, S{:}E{:}B, W{:}E{:}B\}$	$\{W{:}S{:}B\}$	$\star$
$N{:}S{:}B$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\{W, E, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\star$	$\star$
$N{:}E{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$W{:}S{:}E$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\{S, S{:}B, W{:}S{:}B, S{:}E{:}B\}$	$\star$
$W{:}S{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$W{:}E{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\{S, S{:}B\}$	$\star$
$S{:}E{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$N{:}W{:}S{:}E$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\{W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\{W{:}S{:}B, S{:}E{:}B\}$	$\star$
$N{:}W{:}S{:}B$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\{W, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\star$	$\star$
$N{:}W{:}E{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$N{:}S{:}E{:}B$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\{E, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\star$	$\star$
$W{:}S{:}E{:}B$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\{S, S{:}B\}$	$\star$
$N{:}W{:}S{:}E{:}B$	$\star$	$\star$	$\star$	$\star$	$\{W, E, W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\{W{:}E, W{:}B, E{:}B, W{:}E{:}B\}$	$\star$	$\star$

TABLE I
PROVING LEMMA 5

satisfied, so we proceed to the relevant inner If statements. By substituting $R \rightarrow N$, we notice that the first inner If statement is satisfied, since $R_1 = R{:}R^{\leftharpoondown} = N{:}W$. Therefore, we have that $N{:}W \circ N{:}S = Combine(N \circ N{:}S, W \circ N{:}S) - \{W{:}B, W{:}S{:}B\}$.

To compute the composition of two arbitrary directional relations we use the following theorem.

Theorem 2: Let $Q_1 = \cup_{i=1}^{k} R_{1i}$ and $Q_2 = \cup_{j=1}^{m} R_{2j}$ be two directional relations in $2^{\mathcal{B}^*}$ where all R_{1i}, R_{2j} are basic directional relations. Then, $R_1 \circ R_2 = \{R \in \mathcal{B}^* : R \in R_{1i} \circ R_{2j}\}$. Note that $R_{1i} \circ R_{2j}$ can be computed using Lemmata 3, 4 and 5.

Proof: Based on Definition 7, we have that:

$$Q_1 \circ Q_2 = \{R \in \mathcal{B}^* : (\exists a, b, c \in REG^*) \\ (a\ Q_1\ b \wedge b\ Q_2\ c \wedge a\ R\ c)\}$$

Since $Q_1 = \cup_{i=1}^{k} R_{1i}$ and $Q_2 = \cup_{j=1}^{m} R_{2j}$, we have $a\ Q_1\ b = a\ R_{11}\ b \vee \cdots \vee a\ R_{1k}\ b$ and $b\ Q_2\ c = b\ R_{21}\ c \vee \cdots \vee b\ R_{2m}\ c$. Therefore,

$$Q_1 \circ Q_2 = \{R \in \mathcal{B}^* : (\exists a, b, c \in REG^*)(a\ R_{11}\ b \vee \cdots \\ \vee a\ R_{1k}\ b) \wedge (b\ R_{21}\ c \vee \cdots \vee b\ R_{2m}\ c) \wedge a\ R\ c\}.$$

Finally, by distributing $\wedge$ and $\vee$, we have that:

$$\begin{aligned} Q_1 \circ Q_2 &= \{R \in \mathcal{B}^* : (\exists a, b, c \in REG^*) \\ &\qquad (\textstyle\bigvee_{i,j}(a\ R_{1i}\ b \wedge b\ R_{2j}\ c \wedge a\ R\ c)\} \\ &= \{R \in \mathcal{B}^* : R \in R_{1i} \circ R_{2j}\} \end{aligned}$$

■

Let us now leave the consistency based definition of composition and consider the standard notion of existential composition from set theory (Definition 6). Similarly to many models of spatial relations [8], [33], [17], the language of $\mathcal{CDR}$ *is not expressive enough* to capture the binary relation which is the result of the existential composition of directional relations. This is illustrated by the following example.

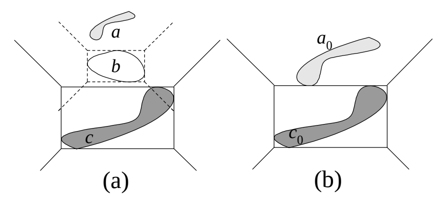


Fig. 13. Illustrations of Example 9

Example 9: Consider object variables a, b, c and directional relations $a\ N\ b$ and $b\ N\ c$. The only directional relation implied by these two constraints is $a\ N\ c$ (see Fig. 13a). This is captured by the fact that $N \circ N = N$ (see Lemma 3). Let us now assume that $(N; N) = N$ also holds. Then, for each pair of objects a_0 and c_0 such that $a_0\ N\ c_0$, there exists an object $b_0 \in REG^*$ such that $a_0\ N\ b_0$ and $b_0\ N\ c_0$. However, Fig. 13b shows two such objects a_0 and c_0 such that $a_0\ N\ c_0$ and it is impossible to find an object $b_0 \in REG^*$ such that $a_0\ N\ b_0$ and $b_0\ N\ c_0$.

If we consider Fig. 13b, we will notice that the semantics of existential composition imply that object a lies completely on the north tile of c (i.e., $a\ N\ c$ holds), and the minimum bounding boxes of objects a and c do not touch. Intuitively, the second constraint is not expressible in the language of directional relations presented in Section II. It is an open question to define an appropriate set of relations that could be used to augment the language of $\mathcal{CDR}$ such that the constraints needed to define the result of existential composition are expressible.

IV. Conclusions

In this paper, we have introduced a family of directional relation models. We have formally defined the relations that can be expressed in the family and studied the inverse and the composition (consistency-based and existential) of directional relations. We have presented methods to compute the inverse and consistency-based composition while we have demonstrated that the result of existential composition cannot be expressed. The aforementioned methods apply to all the models of the family. Further research could concentrate on the extension of $\mathcal{CDR}$ language so that existential composition is definable and the study of algorithms for *(i)* computing the minimal network of a set of directional constraints, *(ii)* for enforcing consistency and *(iii)* performing variable elimination (a task which relates to existential composition).

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