

# SAMPLING AND ESTIMATION FOR (SPARSE) EXCHANGEABLE GRAPHS

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**ABSTRACT.** A recent sequence of papers has introduced sparse exchangeable graphs as a generalization of the graphon framework for dense graphs to a framework that works also for sparse graphs. We describe the sampling scheme naturally associated with sparse exchangeable graphs, and a general consistent estimator for the graphex parameter underlying these models, which plays a role analogous to that of the empirical cumulative distribution function in the sequential data setting. The sampling scheme is a modification of independent vertex sampling that throws away vertices that are isolated in the sampled subgraph. The estimator is a modified version of the empirical graphon estimator that is known to be a consistent estimator for dense exchangeable graphs. Our results may be viewed as a generalization of consistent estimation via the empirical graphon from the dense graph regime to also include sparse graphs.

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## 1. INTRODUCTION

This paper is concerned with the statistical analysis of real-world networks. For densely connected networks, the graphon framework has emerged as powerful tool for both theory and applications in network analysis; many of the models used in practice are within the remit of this framework (see [OR15] for a review). However, in most real-world situations, we are interested in modeling networks that are sparsely connected; i.e., networks that exhibit only a vanishing fraction of all possible links as the size of the observed network grows.

A recent sequence of papers [CF14; VR15; BCCH16] has introduced a family of random graph models that generalizes the graphon models for dense graphs to the sparse graph regime. In [VR15] it was argued that these models are strong candidates for statistical models of networks on the basis that they both allow for the sparsity that is expected of real-world processes and they admit the rich graph structure (such as small world world and scale free behaviour) observed in practice.

The present paper further develops this framework for statistical network modeling by answering, in the context of the framework, two closely related fundamental questions:

- (1) What is the notion of sampling naturally associated with the statistical network model? and
- (2) How can we use an observed dataset to estimate the statistical network model?

We will see that the answers to these questions significantly clarify both the meaning of the modeling framework, and its connection to classical statistics (i.e., conditionally i.i.d sequences) and the dense graph theory. These questions may be viewed as specific examples of a general approach to formalizing the problem of statistical analysis on structured data being carried out by Orbanz [Orb12].

To explain the results we must first recall the modeling framework of [VR15; BCCH16]. The basic setup introduces a family of finite, symmetric point processes  $\Gamma_s \subseteq [0, s] \times [0, s]$ , for  $s \in \mathbb{R}_+$ , where each  $\Gamma_s$  is interpreted as the edge set of a random graph whose vertices are points in the interval  $[0, s]$ . Hence, for  $\theta, \theta' \in [0, s]$ , there is an edge between  $\theta$  and  $\theta'$  if and only if  $(\theta, \theta') \in \Gamma_s$ . The edge set  $\Gamma_s$  determines a graph over its active vertex set: those elements  $\theta \in [0, s]$  such that  $\theta$  exhibits some edge in  $\Gamma_s$ . Accordingly,  $(\Gamma_s)_{s \in \mathbb{R}_+}$  are understood as  $(\mathbb{R}_+)$ -labeled graph-valued random variables that are nested in the sense that  $\Gamma_r \subseteq \Gamma_s$  whenever  $r \leq s$ . We will argue below that the indices  $s \in \mathbb{R}_+$  are properly understood as specifying the sample size of the corresponding observations  $\Gamma_s$ .

The natural parameter of the distributions of these graphs is a *graphex*  $\mathcal{W} = (I, S, W)$  defined on some locally finite measure space  $(\mathfrak{V}, \mathcal{B}_{\mathfrak{V}}, \nu)$  where  $I \in \mathbb{R}_+$ ,  $S : \mathfrak{V} \rightarrow \mathbb{R}_+$  is an integrable function, and  $W : \mathfrak{V}^2 \rightarrow [0, 1]$  is a symmetric function satisfying several weak integrability conditions we formalize later. (Without loss of generality, one can always take  $(\mathfrak{V}, \mathcal{B}_{\mathfrak{V}})$  to be the non-negative reals,  $\mathbb{R}_+$ , with its standard Borel structure, and take  $\nu$  to be Lebesgue measure  $\Lambda$ .) The component  $W$  is a natural generalization of the graphon of the dense graph models [VR15; BCCH16], and for this reason we refer to it as a *graphon*. Although the results of the present paper hold for general graphexes, for simplicity of exposition, we will temporarily restrict our attention to graphexes of the form  $\mathcal{W} = (0, 0, W)$ , giving a full treatment in subsequent sections.

We begin by giving a construction of the graphex process for a graphex of the form  $\mathcal{W} = (0, 0, W)$ : Let  $\Pi$  be a Poisson (point) process on  $\mathbb{R}_+ \times \mathfrak{V}$  with intensity  $\Lambda \otimes \nu$ , i.e., for two intervals  $J_1, J_2 \subseteq \mathbb{R}_+$  and two measurable subsets  $B_1, B_2 \subseteq \mathfrak{V}$ , the number of points of  $\Pi$  in  $J_1 \times B_1$  and in  $J_2 \times B_2$  are Poisson random variables with mean  $|J_1|\nu(B_1)$  and  $|J_2|\nu(B_2)$  respectively, where  $|J_i| = \Lambda(J_i)$  is the length of the interval, and these variables are even independent when  $J_1 \times B_1 \cap J_2 \times B_2 = \emptyset$ . (If  $\nu$  is also Lebesgue measure on  $\mathbb{R}_+$ , then  $\Pi$  is then simply a unit-rate Poisson process on  $\mathbb{R}_+^2$ .) Write  $\{(\theta_i, \vartheta_i)\}_{i \in \mathbb{N}}$  for the points of  $\Pi$ , let  $\Pi_s$  be the restriction of  $\Pi$  to  $[0, s]^2$ , and let  $(\zeta_{\{i,j\}})_{i \leq j \in \mathbb{N}}$  be an i.i.d. collection of uniform random variables in  $[0, 1]$ .

For every  $s \in \mathbb{R}_+$ , define the size- $s$  random edge set  $\Gamma_s$  on  $[0, s]$  to be exactly the set of distinct pairs  $(\theta_i, \theta_j)$  where  $\theta_i, \theta_j \leq s$  and  $\zeta_{\{i,j\}} \leq W(\vartheta_i, \vartheta_j)$ . In other words, for every distinct pair of points  $(\theta, \vartheta), (\theta', \vartheta') \in \Pi_s$ , the edge set  $\Gamma_s$  includes the edge  $(\theta, \theta')$  independently with probability  $W(\vartheta, \vartheta')$ . The vertex set of the graph corresponding to the edge set  $\Gamma_s$  is defined to be those points that appear in some edge; hence, this model does not allow for isolated vertices. The entire family of graphs  $\Gamma_s$ , for  $s \in \mathbb{R}_+$ , is a projective family with respect to subset restriction, i.e.,  $\Gamma_r = \Gamma_s \cap [0, r]^2$  for every  $r, s \in \mathbb{R}_+$  with  $r \leq s$ . See Fig. 1 for an illustration of the

generative model for a general graphex  $\mathcal{W}$  defined on  $\mathbb{R}_+$  with Lebesgue measure. A rigorous definition is provided in Section 2.

We refer to  $\Gamma$  as the *graphex process* generated by  $\mathcal{W}$ : we also use this nomenclature for the family  $(\Gamma_s)_{s \in \mathbb{R}_+}$ . Formally, we will represent locally finite random subsets, such as  $\Gamma$  and each  $\Gamma_s$ , by simple point processes. Note that the  $\Gamma$  has the property that its distribution is invariant to the action of the maps  $(x, y) \mapsto (\phi(x), \phi(y))$ , where  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is measure preserving. A random graph with this property is called a sparse exchangeable graph [CF14]. In Section 2, we quote the result due to Veitch and Roy [VR15] and Borgs, Chayes, Cohn, and Holden [BCC16], building off work by Kallenberg [Kal90], that proves that every (sparse) exchangeable graph is a graphex process generated by some (potentially random) graphex. For a finite labeled graph  $G$ , such as each  $\Gamma_s$ , for  $s \in \mathbb{R}_+$ , we will write  $\mathcal{G}(G)$  to denote the *unlabeled*<sup>1</sup> graph corresponding to  $G$ .

The first contribution of the present paper is the identification of a sampling scheme that is naturally associated with the graphex processes:

**Definition 1.1.** A *p-sampling* of an unlabeled graph  $G$  is obtained by selecting each vertex of  $G$  independently with probability  $p \in [0, 1]$ , and then returning the edge set of the random vertex-induced subgraph of  $G$ .

It is important to note that only the edge set of the vertex-induced subgraph is returned; in other words, vertices that are isolated from the other sampled vertices are thrown away. The key fact about this sampling scheme is that: For  $s > 0$  and  $r \in [0, s]$ , if  $G_r$  is an  $r/s$ -sampling of  $\mathcal{G}(\Gamma_s)$  then  $G_r \stackrel{d}{=} \mathcal{G}(\Gamma_r)$ . This result justifies the interpretation of the parameter  $s$  as a sample size.

In the estimation problem for the graphex process, the observed dataset is a realization of the random sequence of graphs  $G_1, G_2, \dots$  such that  $G_k = \mathcal{G}(\Gamma_{s_k})$ , and  $s_1, s_2, \dots$  is some sequence of sizes such that  $s_k \uparrow \infty$  as  $k \rightarrow \infty$ . The task is to take such an observation and return an estimate for  $\mathcal{W}$ , where  $\mathcal{W}$  is the graphex that generated  $(\Gamma_s)_{s \in \mathbb{R}_+}$ . Both the formulation and solution of this problem depend on whether the sizes  $s_k$  are included as part of the observations.

We first treat the simpler case where the sizes are known. To formalize the estimation problem we must introduce a notion of when one graphex is a good approximation for another. Intuitively, our notion is that, for any fixed  $s$ , a size- $s$  random graph generated by an estimator should be close in distribution to a size- $s$  random graph generated by the true graphex. Let  $\text{uKEG}(\mathcal{W}, s)$  be the distribution of an unlabeled size- $s$  graphex process, i.e., the distribution of  $\mathcal{G}(\Gamma_s)$  where  $\Gamma$  is generated by  $\mathcal{W}$ . Approximation is then formalized by the following notion of convergence:

**Definition 1.2.** Write  $\mathcal{W}_k \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$ , when  $\text{uKEG}(\mathcal{W}_k, s) \rightarrow \text{uKEG}(\mathcal{W}, s)$  weakly as  $k \rightarrow \infty$ , for all  $s \in \mathbb{R}_+$ .

Our goal in the estimation problem is then to take a sequence of observations and use these to produce a sequence of graphexes  $\mathcal{W}_1, \mathcal{W}_2, \dots$  that are consistent in the sense that  $\mathcal{W}_k \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$ . This is a natural analogue of the definition of consistent

<sup>1</sup>The unlabelled graph corresponding to a labelled graph  $G$  is the equivalence class of graphs isomorphic to  $G$ . Restricting ourselves to finite unlabelled graphs, we can represent the unlabelled graphs formally in terms of their homomorphism counts,  $(N_F)$ , where  $F$  ranges over the countable set of all finite simple graphs whose vertex set is  $[n]$  for some  $n \in \mathbb{N}$ , and  $N_F$  is the number of homomorphisms from  $F$  to  $G$ .

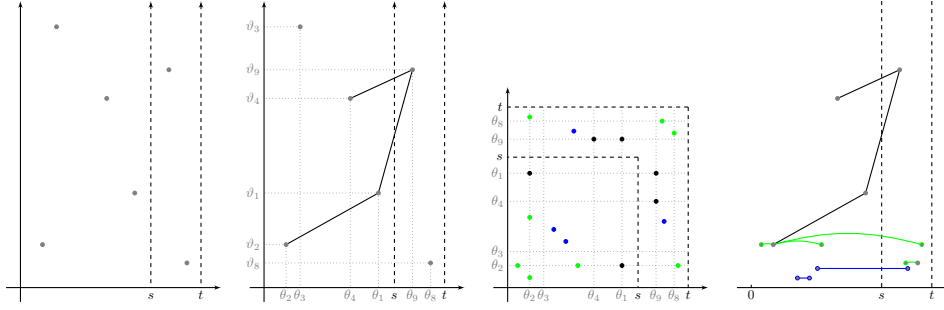


FIGURE 1. Generative process of a graphex process generated by a graphex  $\mathcal{W} = (I, S, W)$  defined on  $\mathbb{R}_+$  with Lebesgue measure, observed at sizes  $s$  and  $t$ . First panel: a (necessarily truncated) realization of the latent Poisson process  $\Pi_t$  on  $[0, t] \times \mathbb{R}_+$ . A countably infinite number of points lie above the six points visualized. Second panel: Edges due to the graphon component  $W$  are sampled by connecting each distinct pair of points  $(\theta_i, \vartheta_i), (\theta_j, \vartheta_j) \in \Pi_t$  independently with probability  $W(\vartheta_i, \vartheta_j)$ . Integrability conditions on  $W$  imply that only a finite number of edges will appear, despite there being an infinite number of points in  $\Pi_t$ . Assume the three edges are the only ones. Third panel: The edge set  $\Gamma_t$  represented as an adjacency measure on  $[0, t]^2$ . The edges in the graphon component appear as (symmetric pairs of) black dots; the edges corresponding to the star component  $S$  appear in green; the isolated edges (from the  $I$  component) appear in blue. At size  $s$ , only the edges in  $[0, s]^2$  (inner dashed black line) appear in the graph. The edges  $\{\theta_j, \sigma_{jk}\}$  of the star ( $S$ ) component of the process (green) centered at  $\theta_j$  are realizations of a rate- $S(\vartheta_j)$  Poisson process  $\{\sigma_{jk}\}$  along the line through  $\theta_j$  (show as green dots along grey dotted lines). Hence, at size  $t$ , each point  $\theta_i$  is the center of  $\text{Poi}(t S(\vartheta_i))$  star process rays. The edges  $\{\rho_i, \rho'_j\}$  generated by the isolated edge ( $I$ ) component of the process (blue) are a realization of a rate- $I$  Poisson process on the upper (or lower) triangle of  $[0, t]^2$ , reflected. At size  $t$ , there are  $\text{Poi}(t^2 I)$  isolated edges due to this part of the graphex. The final panel shows the graphs corresponding to the sampled adjacency measure at sizes  $s$  and  $t$ .

estimation used for the convergence of the empirical cumulative distribution function in the i.i.d. sequence setting, and of the definition of consistent estimation used for the convergence of the empirical graphon in the dense graph setting.

Let  $v(G)$  denote the number of vertices of graph  $G$ . Our estimator is the *dilated empirical graphon*

$$\hat{W}_{(G_k, s_k)} : [0, v(G_k)/s_k]^2 \rightarrow \{0, 1\}, \quad (1.1)$$

defined by transforming the adjacency matrix of  $G_k$  into a step function on  $[0, v(G_k)/s_k]^2$  where each pixel has size  $1/s_k \times 1/s_k$ ; see Fig. 2. Intuitively, when the generating graphex is  $\mathcal{W} = (0, 0, W)$ , we have  $s_k \uparrow \infty$  as  $k \rightarrow \infty$ , and the estimator is an increasingly higher and higher resolution pixel picture of the generating graphon. Formally, given a non-empty finite graph  $G$  with  $n$  vertices labeled  $1, \dots, n$ , we define the empirical graphon  $\tilde{W}_G : [0, 1]^2 \rightarrow \{0, 1\}$  by partitioning  $[0, 1]$  into adjacent intervals  $I_1, \dots, I_n$  each of length  $1/n$  and taking  $\tilde{W}_G = 1$  on  $I_i \times I_j$  if  $i$  and  $j$  are connected in  $G$ , and taking  $\tilde{W}_G = 0$  otherwise. The dilated empirical graphon with dilation  $s$  is then defined by  $\hat{W}_{(G, s)}(x, y) = \tilde{W}_G(x/s, y/s)$ . To map an unlabeled graph to a (dilated) empirical graphon we must introduce a labeling of the vertices.

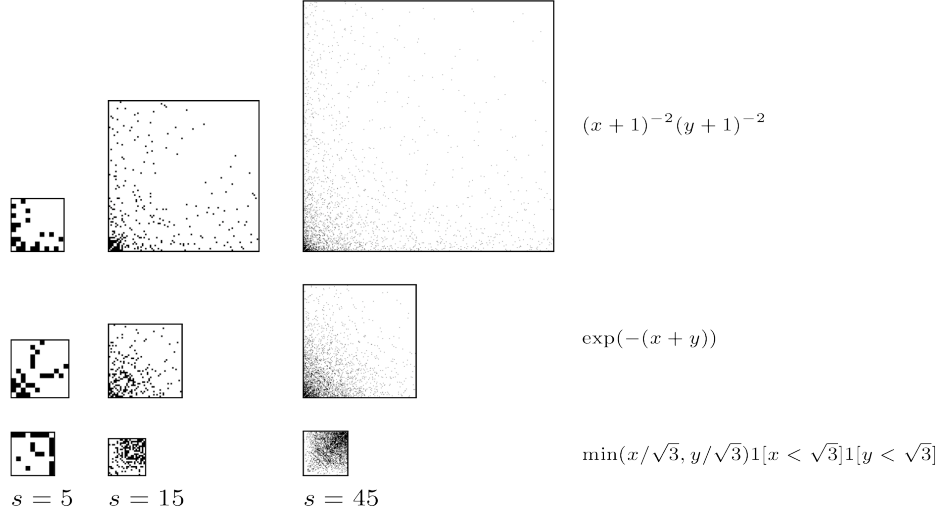


FIGURE 2. Realizations of dilated empirical graphons of graphex processes generated by  $(0, 0, W)$  for  $W$  given in the rightmost column, at observation sizes given in the bottom row. Note that the ordering of the vertices used to define the estimator is arbitrary. Here we have suggestively ordered the vertices according to the latent values from the process simulations; with this ordering the dilated empirical graphons are approximate pixel pictures of the generating graphon where the resolution becomes finer as the observation size grows. All three graphons satisfy  $\|W\|_1 = 1$ , and thus the expected number of edges (black pixels) at each size  $s$  is  $\frac{1}{2}s^2$  in each column. Note that the rate of dilation is faster for sparser graphs; as established in [VR15], the topmost graphex process used for this example is sparser than the middle graphex process, and the graphon generating the bottom graphex process is compactly supported and thus corresponds to a dense graph.

Notice that if  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a measure-preserving transformation,  $\phi \otimes \phi$  is the map  $(\phi \otimes \phi)(x, y) = (\phi(x), \phi(y))$ ,  $\mathcal{W} = (I, S, W)$ , and  $\mathcal{W}' = (I, S \circ \phi, W \circ (\phi \otimes \phi))$ , then  $\text{uKEG}(\mathcal{W}, s) = \text{uKEG}(\mathcal{W}', s)$  for all  $s \in \mathbb{R}_+$ . In particular, the dilated empirical graphon functions corresponding to different labelings of the vertices of  $G$  are related by obvious measure-preserving transformations in this way. For the purposes of this paper, graphexes that give rise to the same distributions over graphs are equivalent. We then define the empirical graphon of an unlabeled graph to be the empirical graphon of that graph with some arbitrary labeling, and we define the dilated empirical graphon similarly. These functions may be thought as arbitrary representatives of the equivalence class on graphons given by equating two graphons whenever they correspond to isomorphic graphs.

The first main estimation result is that  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} W$  in probability as  $k \rightarrow \infty$ . That is, for every infinite sequence  $N \subseteq \mathbb{N}$  there is a further infinite subsequence  $N' \subseteq N$  such that  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} W$  almost surely along  $N'$ . Subject to an additional technical constraint (implied by integrability of  $W$ ) the convergence in probability may be replaced by convergence almost surely. Note that consistency holds for observations generated by an arbitrary graphex  $\mathcal{W} = (I, S, W)$ , not just those with the form  $\mathcal{W} = (0, 0, W)$ ; see Fig. 3.

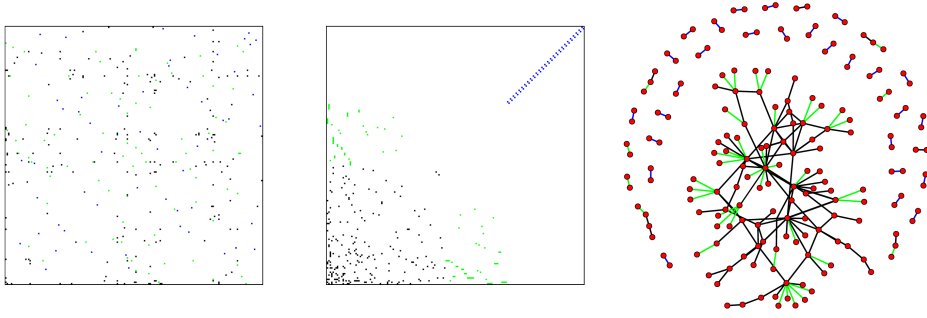


FIGURE 3. Realization of unlabeled graphex process generated by  $\mathcal{W} = (I, S, W)$  at size  $s = 15$  (right panel), and associated dilated empirical graphon (left and center panels). The generating graphex is  $W = (x+1)^{-2}(y+1)^{-2}$ ,  $S = 1/2 \exp(-(x+1))$ , and  $I = 0.1$ . The observation size is  $s = 15$ . The dilated empirical graphex is pictured as two equivalent representations  $\hat{W}_{(G,15)}$  and  $\hat{W}'_{(G,15)}$ , each with support  $[0, 12]^2$  (180 vertices at size 15). Edges from the  $W$  component are shown in black, edges from the  $S$  component are shown in green, and edges from the  $I$  component are shown in blue. Recall that the ordering of the dilated empirical graphon is arbitrary, so the left and center panels depict different representations of the same estimator. The leftmost panel shows the dilated empirical graphon with a random ordering. The middle panel shows the dilated empirical graphon sorted to group the  $I$ ,  $S$ , and  $W$  edges, with the  $W$  edges sorted as in Fig. 2. The middle panel gives some intuition for why the dilated empirical graphon is able to estimate the entire graphex triple: When a graphex process is generated according to  $\hat{W}_{(G,15)}$  with latent Poisson process  $\Pi$ , the disjoint structure of the dilated graphon regions due to the  $I$ ,  $S$ , and  $W$  components induces a natural partitioning of  $\Pi$  into independent Poisson processes that reproduce the independence structure used in the full generative model Eq. (2.1).

We now turn to the setting where the observation sizes  $s_1, s_2, \dots$  not included as part of the observations. In this case, there are two natural models for the dataset. The first is to treat the observed graphs  $G_k$  as realizations of  $\mathcal{G}(\Gamma_{s_k})$  for some fixed (but unknown) sequence  $s_k \uparrow \infty$  as  $k \rightarrow \infty$ . The other natural model is to take  $G_1, G_2, \dots$  to be the sequence of all distinct graph structures taken on by  $(\Gamma_s)_{s \in \mathbb{R}_+}$ ; in this case, for all  $k$ , we take  $G_k = \mathcal{G}(\Gamma_{\tau_k})$ , where  $\tau_k$  is the latent size at the  $k$ th occasion that the graph structure changes. In this latter case, we call  $\mathcal{G}(\Gamma) = (\mathcal{G}(\Gamma_{\tau_1}), \mathcal{G}(\Gamma_{\tau_2}), \dots)$  the graph sequence of  $\Gamma$ .

Intuitively,  $\mathcal{G}(\Gamma)$  is the graphex process  $\Gamma$  with the size information stripped away. In this sense, the graph sequence of  $\Gamma$  is the random object naturally associated to  $\mathcal{W}$  when the sizes are unobserved. Thus, in this setting, convergence in distribution of the graph sequences generated by the estimators gives a natural notion of consistent estimation.

**Definition 1.3.** Write  $\mathcal{W}_k \rightarrow_{\text{GS}} \mathcal{W}$  as  $k \rightarrow \infty$  when  $\mathcal{G}(\Gamma^k) \xrightarrow{d} \mathcal{G}(\Gamma)$  as  $k \rightarrow \infty$ , for  $\Gamma^k$  generated by  $\mathcal{W}_k$  and  $\Gamma$  generated by  $\mathcal{W}$ .

The notion of consistent estimation corresponding to this convergence is that, for any fixed  $\ell \in \mathbb{N}$ , the distribution of the length  $\ell$  prefix of the graph sequence generated by the estimator should be close to the distribution of the length  $\ell$  prefix of the

graph sequence generated by  $\mathcal{W}$ . Convergence in distribution of every finite-size prefix is equivalent to convergence in distribution of the entire sequence.

To explain our estimator for this setting, we will need the following concept:

**Definition 1.4.** Let  $c \in \mathbb{R}_+$  and let  $\mathcal{W} = (I, S, W)$  be a graphex. A  $c$ -dilation of  $\mathcal{W}$  is the graphex  $\mathcal{W}^c = (c^2 I, cS(\cdot/c), W(\cdot/c, \cdot/c))$ .

The key fact about  $c$ -dilations is that  $\text{uKEG}(\mathcal{W}, s) = \text{uKEG}(\mathcal{W}^c, s/c)$  for all  $s \in \mathbb{R}_+$ , and thus also  $\mathcal{G}(\Gamma) \stackrel{d}{=} \mathcal{G}(\Gamma^c)$  whenever  $\Gamma$  is generated by  $\mathcal{W}$  and  $\Gamma^c$  is generated by  $\mathcal{W}^c$ . That is, the law of the graph sequence is invariant to dilations of the generating graphex. This means, in particular, that the dilation of a graphex is not an identifiable parameter when the observation sizes are not included as part of the observation. The obvious guess for the estimator in this setting is then the estimator for the known-sizes setting with the dilation information stripped away. That is, our estimator is the dilated empirical graphon modulo dilation; i.e., it is simply the empirical graphon  $\tilde{W}_{G_k} : [0, 1]^2 \rightarrow [0, 1]$  defined above. In this setting, the empirical graphon is acting as a representative of its equivalence class under the relation that equates graphons that generate graph sequences with the same laws.

The main estimation result is that if either

- (1) There is some (possibly random) sequence  $(s_k)$ , independent from  $\Gamma$ , such that  $s_k \uparrow \infty$  a.s. and  $G_k = \mathcal{G}(\Gamma_{s_k})$  for all  $k \in \mathbb{N}$ , or
- (2)  $(G_1, G_2, \dots) = \mathcal{G}(\Gamma)$ ,

then  $\tilde{W}_{G_k} \rightarrow_{\text{GS}} W$  in probability as  $k \rightarrow \infty$ . Subject to an additional technical constraint (implied by integrability), the convergence in probability may be strengthened to convergence almost surely.

Our estimation results are inspired by Kallenberg's development of the theory of estimation for exchangeable arrays [Kal99]. Restricted to the graph setting (that is, 2-dimensional arrays interpreted as adjacency matrices), and translated into modern language, that paper introduced the empirical graphon (although not named as such) and introduced the idea of encoding valid estimation as  $W_k \rightarrow_{\text{est}} W$  as  $k \rightarrow \infty$  when the graphs generated by  $W_k$  converge in distribution to the graphs generated by  $W$ . The estimation results of the present paper may be seen as a generalization of [Kal99] to the sparse graph regime.

The present paper is also closely related to the recent paper [BCCH16]. Specialized to the case  $\mathfrak{V} = \mathbb{R}_+$  equipped with Lebesgue measure, that paper extends the cut distance between compactly supported graphons—a core tool in the limit theory of dense graphs—to arbitrary integrable graphons. Convergence in the cut distance then gives a notion of limit for sequences of graphons. This is extended to a notion of convergence for sequences of (sparse) graphs by saying that a sequence  $G_1, G_2, \dots$  converges in the stretched cut distance sense if and only if  $\tilde{W}_{(G_1, \sqrt{e(G_1)})}, \tilde{W}_{(G_2, \sqrt{e(G_2)})}, \dots$  converges with respect to the cut distance. That is, each graph  $G_k$  is mapped to the empirical graphon dilated by  $v(G_k)/\sqrt{e(G_k)}$ . The same paper also establishes that  $e(G_k)/s_k^2 \rightarrow \|W\|_1$  a.s.. Thus, in the  $\|W\|_1 = 1$  case, these dilated empirical graphons, considered as pixel pictures, will look asymptotically identical to the  $v(G_k)/s_k$ -dilated empirical graphons that we use as estimators in the known sizes case. This suggests that a close connection between consistent estimation and convergence in the cut distance. Indeed, in the dense graph setting these notions of convergence are known to be equivalent (in the dense setting, the convergence  $W_k \rightarrow_{\text{GP}} W$  as  $k \rightarrow \infty$  is equivalent to left convergence [DJ08], and



left convergence is equivalence to convergence in the cut norm [BCLS+08]). An analogous result in the sparse graph setting would allow for a very different approach to proving our convergence result in the known size setting, restricted to the special case that the generating graphex is an integrable graphon.

The paper is organized as follows: In Section 2 we give formal definitions for the basic tools of the paper. The sampling result is derived in Section 3. In Section 4 we prove the estimation result for the setting where observation sizes are included as part of the observation. We build on this in Section 5 to prove the estimation result for the setting where the true underlying observation sizes are not observed.

## 2. PRELIMINARIES

The basic object of interest in this paper is point processes on  $\mathbb{R}_+^2$ , interpreted as the edge sets of random graphs with vertices labeled in  $\mathbb{R}_+$ .

**Definition 2.1.** An *adjacency measure* is a purely atomic, symmetric, simple, locally finite measure on  $\mathbb{R}_+^2$ .

If  $\xi = \sum_{i,j} \delta_{(\theta_i, \theta_j)}$  is an adjacency measure then the associated graph with labels in  $\mathbb{R}_+$  is one with edge set  $\{(\theta_i, \theta_j)\}$ , where  $\theta_i \leq \theta_j$ ; the vertex set is deduced from the edge set.

The defining property of graphex processes is that, intuitively speaking, the labels of the vertices of the graph are uninformative about the graph structure. This is formalized by requiring that the associated adjacency measure is jointly exchangeable, where

**Definition 2.2.** A random measure  $\xi$  on  $\mathbb{R}_+^2$  is *jointly exchangeable* if  $\xi \circ (\phi \otimes \phi) \stackrel{d}{=} \xi$  for any measure-preserving transformation  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ .

A representation theorem for jointly exchangeable random measures on  $\mathbb{R}_+^2$  was given by Kallenberg [Kal05; Kal90]. This result was translated to the setting of random graphs in [VR15]. Writing  $\Lambda$  for Lebesgue measure and  $\mu_W(\cdot) = \int_{\mathbb{R}_+} W(x, \cdot) dx$ , the defining object of the representation theorem is:

**Definition 2.3.** A *graphex* is a triple  $(I, S, W)$ , where  $I \geq 0$  is a non-negative real,  $S : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is integrable, and the *graphon*  $W : \mathbb{R}_+^2 \rightarrow [0, 1]$  is symmetric, and satisfies

- (1)  $\Lambda\{\mu_W = \infty\} = 0$  and  $\Lambda\{\mu_W > 1\} < \infty$ ,
- (2)  $\Lambda^2[W; \mu_W \vee \mu_W \leq 1] = \int_{\mathbb{R}_+^2} W(x, y) 1[\mu_W(x) \leq 1] 1[\mu_W(y) \leq 1] dx dy < \infty$ ,
- (3)  $\int_{\mathbb{R}_+} W(x, x) dx < \infty$ .

We say that a graphex is non-trivial if  $I + \|S\|_1 + \|W\|_1 > 0$ , i.e. if it is not the case that the graphex is 0 a.e.

The representation theorem is:



**Theorem 2.4.** *Let  $\xi$  be a random adjacency measure.  $\xi$  is jointly exchangeable iff there exists a (possibly random) graphex  $\mathcal{W} = (I, S, W)$  such that, almost surely,*

$$\begin{aligned} \xi = & \sum_{i,j} 1[W(\vartheta_i, \vartheta_j) \leq \zeta_{\{i,j\}}] \delta_{\theta_i, \theta_j} \\ & + \sum_{j,k} 1[\chi_{jk} \leq S(\vartheta_j)] (\delta_{\theta_j, \sigma_{jk}} + \delta_{\sigma_{jk}, \theta_j}) \\ & + \sum_k 1[\eta_k \leq I] (\delta_{\rho_k, \rho'_k} + \delta_{\rho'_k, \rho_k}), \end{aligned} \quad (2.1)$$

for some collection of independent uniformly distributed random variables  $(\zeta_{\{i,j\}})$  in  $[0, 1]$ ; some independent unit-rate Poisson processes  $\{(\theta_j, \vartheta_j)\}$  and  $\{(\sigma_{ij}, \chi_{ij})\}_j$ , for  $i \in \mathbb{N}$ , on  $\mathbb{R}_+^2$  and  $\{(\rho_j, \rho'_j, \eta_j)\}$  on  $\mathbb{R}_+^3$ .

**Definition 2.5.** A graphex process associated with graphex  $(I, S, W)$  is the random adjacency measure  $\Gamma$  of the form given in Eq. (2.1). The graphex process model is the family  $(\Gamma_s)_{s \in \mathbb{R}_+}$ , where  $\Gamma_s(\cdot) = \Gamma(\cdot \cap [0, s]^2)$ .

*Remark 2.6.* In [VR15] the Kallenberg exchangeable graph was defined as the random graph with vertex labels in  $\mathbb{R}_+$  associated with  $\Gamma$ . The definition of the graphex process differs slightly, motivated by the use of techniques from the theory of distributional convergence of point processes, which makes explicit appeal to the point process structure desirable. It will sometimes be useful in exposition to conflate the graphex process with the associated labeled graph, so statements such as “the number of edges of  $\Gamma_s$ ” are sensible.  $\triangleleft$

We will often have occasion to refer to the unlabeled finite graph associated with a finite adjacency measure.

**Definition 2.7.** Let  $\xi$  be a finite adjacency measure. The *unlabelled graph associated with  $\xi$*  is  $\mathcal{G}(\xi)$ .

A particularly important case is the graph associated to the size- $s$  graphex process  $\Gamma_s$ , which is almost surely finite. We will have frequent occasion to refer to the distributions of both the labeled and unlabeled graphs:

**Definition 2.8.** Let  $(\Gamma_s)_{s \in \mathbb{R}_+}$  be a graphex process generated by  $\mathcal{W}$ . The *finite graphex process distribution* with parameters  $\mathcal{W}$  and  $s$  is  $\text{KEG}(\mathcal{W}, s) = \text{P}(\Gamma_s \in \cdot \mid \mathcal{W}, s)$ , and  $\text{KEG}(\mathcal{W}) = \text{KEG}(\mathcal{W}, \infty)$ . The *finite unlabeled graphex process distribution* with parameters  $\mathcal{W}$  and  $s$  is  $\text{uKEG}(\mathcal{W}, s) = \text{P}(\mathcal{G}(\Gamma_s) \in \cdot \mid \mathcal{W}, s)$ .

In order to pass from  $\mathcal{G}(\xi)$  back to some adjacency measure  $\xi'$  such that  $\mathcal{G}(\xi') = \mathcal{G}(\xi)$ , we must reintroduce labels. A simple scheme is to produce labels independently and uniformly in some range:

**Definition 2.9.** Let  $G$  be an unlabeled graph with edge set  $E$ , and let  $s > 0$ . A *random labeling of  $G$  into  $[0, s]$* ,  $\text{Lbl}_s(G, \{U_i\})$ , is a random adjacency measure  $\text{Lbl}_s(G, \{U_i\}) = \sum_{(i,j) \in E} \delta_{(U_i, U_j)}$ , where  $U_i \stackrel{iid}{\sim} \text{Uni}[0, s]$ , for  $i \in \mathbb{N}$ . Where there is no risk of confusion, we will write  $\text{Lbl}_s(G)$  for  $\text{Lbl}_s(G, \{U_i\})$  where  $U_i \stackrel{iid}{\sim} \text{Uni}[0, t]$ , for  $i \in \mathbb{N}$ , independently of everything else.

Because our notion of consistent estimation is a requirement of distributional convergence, the distributions of these random labelings will play a large role. Clearly, the distribution of  $\text{Lbl}_s(G)$  is a measurable function of  $G$  and  $s$ .

**Definition 2.10.** We write  $\text{embed}(G, s)(\cdot) = \mathbb{P}(\text{Lbl}_s(G) \in \cdot)$  for the distribution of  $\text{Lbl}_s(G)$ . When  $G$  is itself random, a random embedding of  $G$  into  $[0, s]$  is defined by  $\text{embed}(G, s) = \mathbb{P}[\text{Lbl}_s(G)|G]$ .

We typically think of graphex processes as defining a nested collection of  $\mathbb{R}_+$ -labeled graph valued random variables  $(\Gamma_s)_{s \in \mathbb{R}_+}$ . In modeling situations where the labeling is irrelevant, it is natural to instead look at the (countable) collection of all distinct graph structures taken on by  $(\Gamma_s)_{s \in \mathbb{R}_+}$ ; this is the graph sequence associated with  $\Gamma$ . We now turn to formally defining the graph sequence associated with an arbitrary adjacency measure  $\xi$ . To that end, define  $E : \mathbb{R}_+ \rightarrow \mathbb{N}$  by

$$E(s) = \frac{1}{2} \xi[0, s]^2 \quad \text{for } s \in \mathbb{R}_+. \quad (2.2)$$

In the absence of self loops,  $E(s)$  is the number of edges present between vertices with labels in  $[0, s]$ . In general, the jumps of  $E$  correspond with the appearance of edges.

**Definition 2.11.** Let  $\xi$  be an adjacency measure. The jump times of  $\xi$ , written as  $\tau(\xi)$ , is the sequence  $\tau_1, \tau_2, \dots$  of jumps of  $E$  in order of appearance.

Note that the map  $\xi \mapsto \tau(\xi)$  is measureable. Intuitively,  $\tau_1, \tau_2, \dots$  are the sample sizes at which edges are added to the unlabeled graph associated with the adjacency measure.

Let  $\chi_s$  denote the operation of restricting an adjacency measure to those vertices with labels in  $[0, s]$ , in the sense that  $\chi_s \xi(\cdot) = \xi(\cdot \cap [0, s]^2)$ . We now formalize the sequence of all distinct unlabeled graphs associated with  $(\chi_s \xi)_{s \in \mathbb{R}_+}$ :

**Definition 2.12.** The *graph sequence associated with  $\xi$* , written  $\mathcal{G}(\xi)$ , is the sequence  $\mathcal{G}(\chi_{\tau_1} \xi), \mathcal{G}(\chi_{\tau_2} \xi), \dots$ , where  $\tau_1, \tau_2, \dots$  are the jump times of  $\xi$ .

### 3. SAMPLING

$\Gamma_r$ , a graphex process of size  $r$ , may be generated from  $\Gamma_s$ , a graphex process of size  $s > r$ , by restricting  $\Gamma_s$  to  $[0, r]^2$ . In this section we show that this restriction has a natural relation to  $p$ -sampling:  $\mathcal{G}(\Gamma_r)$  may be generated as an  $r/s$ -sampling of  $\mathcal{G}(\Gamma_s)$ .

The first result we need is that random labelings preserve the law of exchangeable adjacency measures. Intuitively, the labels of the size- $s$  graphex process can be invented by labeling each vertex i.i.d.  $\text{Uni}[0, s]$ .

**Lemma 3.1.** *Let  $s > 0$  and let  $\Gamma_s$  be a size- $s$  graphex process generated by  $\mathcal{W}$ . Then,  $\text{KEG}(\mathcal{W}, s) = \mathbb{E}[\text{embed}(\mathcal{G}(\Gamma_s), s)]$ .*

*Proof.* It suffices to show that  $\text{Lbl}_s(\mathcal{G}(\Gamma_s)) \stackrel{d}{=} \Gamma_s$ .

Suppose  $\Gamma_s$  is generated as in Eq. (2.1). For simplicity of exposition, suppose that the generating graphex is  $(0, 0, W)$ , and the associated latent Poisson process is  $\Pi_s$ . Let  $\{\theta'_i\}_{i \in \mathbb{N}} \stackrel{iid}{\sim} \text{Uni}[0, s]$ , and let  $\Pi'_s = \{(\theta'_i, \vartheta_i) : (\theta_i, \vartheta_i) \in \Pi_s\}$ . By a property of the Poisson process,  $\Pi'_s \stackrel{d}{=} \Pi_s$ . Let  $\Gamma'_s$  be a size- $s$  graphex process generated using the same latent variables as  $\Gamma_s$ , but with  $\Pi'_s$  replacing  $\Pi_s$ . Then, by construction,  $\Gamma'_s \stackrel{d}{=} \text{Lbl}_s(\mathcal{G}(\Gamma_s))$ . Moreover,  $\Gamma'_s$  is distributed as a size- $s$  graphex process, so  $\Gamma'_s \stackrel{d}{=} \Gamma_s$ .

An essentially identical argument proves the result for a graphex process generated by the full graphex.  $\square$

The main sampling result is:

**Theorem 3.2.** *Let  $\mathcal{W}$  be a graphex, let  $s > 0$  and  $r \in [0, s]$ , let  $G_s \sim \text{uKEG}(\mathcal{W}, s)$ , and let  $G_r$  be an  $r/s$ -sampling of  $G_s$ . Then,  $G_r \sim \text{uKEG}(\mathcal{W}, s)$ .*

*Proof.* Let  $\xi_s = \text{Lbl}_s(G_s)$ . It is an obvious consequence of Lemma 3.1 that  $\xi_s$  is equal in distribution to a size- $s$  graphex process generated by  $\mathcal{W}$ . Let  $\xi_r$  be the restriction of  $\xi_s$  to  $[0, r]^2$ , so  $\mathcal{G}(\xi_r) \sim \text{uKEG}(\mathcal{W}, r)$ . Each vertex of  $\xi_s$  has a label in  $[0, r]$  independently with probability  $r/s$ ; thus,  $\mathcal{G}(\xi_r) \stackrel{d}{=} G_r$ .  $\square$

#### 4. ESTIMATION WITH KNOWN SIZES

This section explains our estimation results for the case where the observations are  $(G_1, s_1), (G_2, s_2), \dots$ , where  $G_k = \mathcal{G}(\Gamma_{s_k})$  for some graphex process  $\Gamma$  generated according to a graphex  $\mathcal{W}$  and some sequence  $s_k \uparrow \infty$  in  $\mathbb{R}_+$ . We consider both the case of an arbitrary non-random divergent sequence and the case where the sizes are taken to be the jumps of the graphex process (that is, the sizes at which new edges enter the graph), in which case we denote the sequence as  $\tau_1, \tau_2, \dots$ . As motivated in the introduction, our notion of estimation is formalized as:

**Definition 4.1.** Let  $\mathcal{W}_1, \mathcal{W}_2, \dots$  be a sequence of graphexes. Write  $\mathcal{W}_n \rightarrow_{\text{GP}} \mathcal{W}$  as  $n \rightarrow \infty$  when, for all  $s \in \mathbb{R}_+$ , it holds that  $\text{uKEG}(\mathcal{W}_n, s) \rightarrow \text{uKEG}(\mathcal{W}, s)$  weakly as  $n \rightarrow \infty$ .

The goal of estimation is: given a sequence of observations  $(G_1, s_1), (G_2, s_2), \dots$ , produce

$$\hat{W}_{(G_k, s_k)} : \mathbb{R}_+^2 \rightarrow [0, 1] \quad (4.1)$$

such that  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$ , where the convergence may be almost sure or merely in probability.

The main result of this section is that the dilated empirical graphons  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} \mathcal{W}$  for  $(G_1, s_1), (G_2, s_2), \dots$  generated by a graphex  $\mathcal{W}$ ; i.e. the dilated empirical graphon is a consistent estimator for  $\mathcal{W}$ .

We now turn to an intuitive description of the broad structure of the argument. Conditional on  $G_k$ , let  $\xi^k = \text{Lbl}_{s_k}(G_k)$  and let  $\text{embed}(G_k, s_k)$  be the distribution of  $\xi^k$  conditional on  $G_k$ . The first convergence result, Theorem 4.3, is that, almost surely, the random distributions  $\text{embed}(G_k, s_k)$  converge weakly to  $\mathcal{L}(\Gamma) = \text{KEG}(\mathcal{W})$ . That is, for almost every realization of a graphex process, the point processes defined by randomly labeling the observed finite graphs converge in distribution to the original graphex process. The analogous statement in the i.i.d. sequence setting is that, given some  $(X_1, X_2, \dots)$  where  $X_k \stackrel{iid}{\sim} P$ , and  $\sigma_n$  a random permutation on  $[1, \dots, n]$ , the random distributions  $P(X_{\sigma_n(1)}, \dots, X_{\sigma_n(n)} \in \cdot \mid X_1, \dots, X_n)$  converge weakly almost surely to  $P((X_1, X_2, \dots) \in \cdot)$  as  $n \rightarrow \infty$ .

The convergence in distribution of the point processes on  $\mathbb{R}_+^2$  is equivalent to convergence in distribution of the point processes restricted to  $[0, r]^2$  for every finite  $r \in \mathbb{R}_+$ . This perspective lends itself naturally to the interpretation of the limit result as a qualitative approximation theorem: intuitively,  $P(\xi^k([0, r]^2 \cap \cdot) \in \cdot \mid G_k)$  approximates  $\text{KEG}(\mathcal{W}, r)$ , with the approximation becoming exact in the

limit  $r/s_k \rightarrow 0$ . This perspective also makes clear the first critical connection between estimation and sampling: conditional on  $G_k$ ,  $\mathcal{G}(\xi^k([0, r]^2 \cap \cdot))$  has the same distribution as an  $r/s_k$ -sampling of  $G_k$ .

The second key observation is that, conditional on  $G_k$ , a sample from  $\text{uKEG}(\hat{W}_{(G_k, s_k)}, r)$  may be generated by sampling  $\text{Poi}(r/s_k v(G_k))$  vertices with replacement from  $G_k$  and returning the induced edge set. The second step in the proof is to show that this sampling scheme is asymptotically equivalent to  $r/s_k$ -sampling in the limit of  $s_k \uparrow \infty$ ; this is the role of Lemmas 4.5 and 4.7.

Theorem 4.8 then puts together these results to conclude that, almost surely,  $\text{KEG}(\hat{W}_{(G_k, s_k)}) \rightarrow \text{KEG}(\mathcal{W})$  weakly as  $k \rightarrow \infty$ . Some additional technical rigmarole is required to show that this also gives convergence of the (unlabeled) random graphs. This later convergence is the main result of this section, and is established in Theorem 4.12.

**4.1. Convergence in Distribution of Random Embeddings.** This subsection uses results from the theory of distributional convergence of point processes to show that, almost surely,  $\text{embed}(G_k, s_k) \rightarrow \text{uKEG}(\mathcal{W}, \infty)$  weakly as  $k \rightarrow \infty$ .

We will need the following definition and technical lemma: A separating class for a locally compact second countable Hausdorff space  $S$  is a class  $\mathcal{U} \subset S$  such that for any compact open sets with  $K \subset G$  there is some  $U \in \mathcal{U}$  with  $K \subset U \subset G$ .

**Lemma 4.2.** *Let  $\phi, \phi_1, \phi_2, \dots$  be simple point processes on a locally compact second countable Hausdorff space  $S$ . If*

$$\phi_n(U) \xrightarrow{d} \phi(U), \quad n \rightarrow \infty \quad (4.2)$$

*weakly for all  $U$  in some separating class for  $S$  then*

$$\phi_n \xrightarrow{d} \phi, \quad n \rightarrow \infty \quad (4.3)$$

*weakly.*

*Proof.* By [Kal01, Thms. 16.28 and 16.29], it suffices to check that  $\text{P}(\phi_n(U) = 0) \rightarrow \text{P}(\phi(U) = 0)$  and that  $\limsup_n \text{P}(\phi_n(U) > 1) \leq \text{P}(\phi(U) > 1)$ . Because  $\phi_n(U)$  is a non-negative integer a.s., both conditions are implied by  $\phi_n(U) \xrightarrow{d} \phi(U)$ .  $\square$

**Theorem 4.3.** *Let  $\Gamma$  be a graphex process generated by a non-trivial graphex  $\mathcal{W}$ , let  $s_1, s_2, \dots$  be some sequence in  $\mathbb{R}_+$  such that  $s_k \uparrow \infty$  as  $k \rightarrow \infty$  and let  $G_k = \mathcal{G}(\Gamma_{s_k})$  for all  $k$ . Then  $\text{embed}(G_k, s_k) \rightarrow \text{KEG}(\mathcal{W})$  weakly almost surely.*

*Proof.* For each  $k \in \mathbb{N}$ , conditional on  $G_k$ , let  $\xi^k$  be a point process with law  $\text{embed}(G_k, s_k)$ . Note that  $\Gamma \sim \text{KEG}(\mathcal{W})$ . Observe that the collection  $\mathcal{U}$  of finite unions of rectangles with rational end points is a separating class for  $\mathbb{R}_+^2$ . Further,  $\xi^k$  is simple for all  $k \in \mathbb{N}$ , as is  $\Gamma$ . Thus by Lemma 4.2, to show the claimed result it will suffice to show that, for all  $U \in \mathcal{U}$ ,  $\text{P}(\xi^k(U) \in \cdot \mid G_k) \rightarrow \text{P}(\Gamma(U) \in \cdot)$  weakly as  $k \rightarrow \infty$ .

Fix  $U$ . To establish this condition we first show that for all bounded continuous functions  $f$ , it holds that  $\lim_{k \rightarrow \infty} \mathbb{E}[f(\xi^k(U)) \mid G_k] = \mathbb{E}[f(\Gamma(U))]$  a.s. Let  $\mathcal{F}_{-s}$  be the partially labelled graph derived from  $\Gamma$  by forgetting the labels of all nodes with label  $\theta_i < s$ . Take  $r \in \mathbb{R}_+$  large enough so that  $U \subset [0, r]^2$ . Then for  $s_k > r$ ,

$$\mathbb{E}[f(\Gamma(U)) \mid \mathcal{F}_{-s_k}] = \mathbb{E}[f(\xi^k(U)) \mid G_k]. \quad (4.4)$$

Define  $U_t = U + (t, t)$  for  $t \in \mathbb{R}_+$  and let

$$X_s^{(r)} = \frac{1}{s-r} \int_0^{s-r} f(\Gamma(U_t)) dt. \quad (4.5)$$

Observe that for  $s$  such that  $t \leq s-r$ , the joint exchangeability of  $\Gamma$  implies

$$\mathbb{E}[f(\Gamma(U_t)) \mid \mathcal{F}_{-s}] = \mathbb{E}[f(\Gamma(U)) \mid \mathcal{F}_{-s}]. \quad (4.6)$$

Moreover, by the linearity of conditional expectation, for  $s > r$ , it holds that  $\mathbb{E}[X_s^{(r)} \mid \mathcal{F}_{-s}] = \mathbb{E}[f(\Gamma(U)) \mid \mathcal{F}_{-s}]$ .

A standard result [Dur10, Ex. 5.6.2] shows that  $\lim_{k \rightarrow \infty} \mathbb{E}[X_{s_k}^{(r)} \mid \mathcal{F}_{-s_k}] = \mathbb{E}[X_\infty^{(r)} \mid \mathcal{F}_{-\infty}]$  a.s. if  $X_{s_k}^{(r)} \rightarrow X_\infty^{(r)}$  a.s. and there is some integrable random variable that dominates  $X_{s_k}^{(r)}$  for all  $k$ ; the second condition holds because  $f$  is bounded. Notice that  $Y_t = f(\Gamma(B_t))$  is a stationary stochastic process. Moreover, it's easy to see from the graphex process construction that  $Y_t$  and  $Y_{t'}$  are independent whenever  $|t-t'| > r$ , so  $(Y_t)$  is mixing. The ergodic theorem then gives  $\lim_{k \rightarrow \infty} X_{s_k}^{(r)} = \mathbb{E}[f(\Gamma(U))]$  a.s. This means

$$\lim_{k \rightarrow \infty} \mathbb{E}[f(\xi^k(U)) \mid G_k] \rightarrow \mathbb{E}[f(\Gamma(U))] \text{ a.s.}, \quad (4.7)$$

as promised.

For  $l \in \mathbb{Z}_+$ , let  $f_l(\cdot) = 1[\cdot \leq l]$ , let  $A_l^{(U)}$ , for each  $U \in \mathcal{U}$ , be the set on which

$$\lim_{k \rightarrow \infty} \mathbb{E}[f_l(\xi^k(U)) \mid G_k] = \mathbb{E}[f_l(\Gamma(U))] \quad (4.8)$$

and let  $A_U = \bigcap_l A_l^{(U)}$ . We have shown that  $P(A_l^{(U)}) = 1$ , and so  $P(A_U) = 1$  and on  $A_U$  it holds that  $\lim_{k \rightarrow \infty} P(\xi^k(U) \in \cdot \mid G_k) = P(\Gamma(U) \in \cdot)$  weakly. Let  $A = \bigcap_{U \in \mathcal{U}} A_U$ , then  $P(A) = 1$  and on  $A$  it holds that

$$\lim_{k \rightarrow \infty} P(\xi^k(U) \in \cdot \mid G_k) = P(\Gamma(U) \in \cdot) \quad (4.9)$$

weakly for all  $U \in \mathcal{U}$ , completing the proof.  $\square$

We need to do a little bit more work to show convergence in the case where the observations are taken at the jumps of the graphex process.

**Theorem 4.4.** *Let  $\Gamma$  be a graphex process generated by a non-trivial graphex  $\mathcal{W}$ , and let  $\tau_1, \tau_2, \dots$  be the jump times of  $\Gamma$ . Let  $G_k = \mathcal{G}(\Gamma_{\tau_k})$  for each  $k \in \mathbb{N}$ . Then  $\text{embed}(G_k, \tau_k) \rightarrow \text{uKEG}(\mathcal{W}, \infty)$  weakly almost surely as  $k \rightarrow \infty$ .*

*Proof.* For each  $k \in \mathbb{N}$ , let  $\xi^k$  be a point process with law  $\text{embed}(G_k, \tau_k)$ .

As in the proof of Theorem 4.3, to establish the claim it suffices to show that, for all bounded continuous functions  $f$  and all rectangles  $U$ , it holds that

$$\lim_{k \rightarrow \infty} \mathbb{E}[f(\xi^k(U)) \mid G_k, \tau_k] = \mathbb{E}[f(\Gamma(U))] \text{ a.s.} \quad (4.10)$$

Let  $\mathcal{F}_{-s}$  be as in proof of Theorem 4.3. It is clear that  $\mathcal{F}_{-\tau_k} \subset \mathcal{F}_{-\tau_{k-1}}$  for all  $k$ . Because  $U \subset [0, r]^2$  for some finite  $r$  and  $\tau_k \uparrow \infty$  a.s. as  $k \rightarrow \infty$  it holds that

$$\lim_{k \rightarrow \infty} \mathbb{E}[f(\Gamma(U)) \mid \mathcal{F}_{-\tau_k}] = \lim_{k \rightarrow \infty} \mathbb{E}[f(\xi^k(U)) \mid G_k, \tau_k] \text{ a.s.} \quad (4.11)$$

Applying reverse martingale convergence to the l.h.s. we conclude the r.h.s. exists a.s.

It remains to identify the limit. To that end, we will define a coupling between the counts on test set  $U$  at a subsequence of the jump times and the counts on  $U$  at some deterministic sequence, which is known to converge to the desired limit. Let  $s_k = \sum_{n=1}^k \frac{1}{n}$ , let  $\{\tau_{k_j}\}$  be a subsequence of the jump times defined such that at most one point in  $\{\tau_{k_j}\}$  lies in  $[s_l, s_{l+1})$  for all  $l$  and define  $s_{k_j}$  to be the subsequence of  $\{s_k\}$  such that  $s_{k_j}$  is the largest value in  $\{s_k\}$  that is smaller than  $\tau_{k_j}$ . Intuitively, this gives a random subsequence of the jump times and a random subsequence of  $\{s_k\}$  such that the points  $s_{k_j}$  and  $\tau_{k_j}$  become arbitrarily close as  $j \rightarrow \infty$ . For each  $j \in \mathbb{N}$ , let  $G_j^s = \mathcal{G}(\Gamma_{s_{k_j}})$ , and let  $G_j^\tau = \mathcal{G}(\Gamma_{\tau_{k_j}})$ .

By construction,  $G_j^s \subset G_j^\tau$ . Label the vertices of  $G_j^\tau$  as  $1, \dots, v(G_j^\tau)$  such that  $1, \dots, v(G_j^s)$  is the vertex set of  $G_j^s$ . Let  $\xi^{(s,j)} = \text{Lbl}_{s_{k_j}}(G_j^s)$ , and let  $\xi^{(\tau,j)} = \text{Lbl}_{\tau_{k_j}}(G_j^\tau)$ . The occupancy counts of the test may then sampled according to:

- (1)  $V_1, \dots, V_{v(G_j^\tau)} \stackrel{iid}{\sim} \text{Uni}[0, 1]$
- (2)  $\xi^{(\tau,j)}(U) = |\{(v_i, v_j) \in e(G_j^\tau) : (V_i \tau_{k_j}, V_j \tau_{k_j}) \in U\}|$
- (3)  $\xi^{(s,j)}(U) = |\{(v_i, v_j) \in e(G_j^s) : (V_i s_{k_j}, V_j s_{k_j}) \in U\}|$

By construction,  $G_j^\tau \setminus G_j^s$  is a star; call the center of this star  $c$ . Choosing  $r$  such that  $U \subset [0, r]^2$ , it is clear that if  $V_c \tau_{k_j} \notin [0, r]$  then  $\xi^{(\tau,j)}(U) \leq \xi^{(s,j)}(U)$  under this coupling. The occupancy counts are the number of edges in random induced subgraphs given by including each vertex with probability  $\frac{r}{\tau_{k_j}}$  and  $\frac{r}{s_{k_j}}$  respectively. This perspective makes it clear that, conditional on  $c$  not being included when sampling from  $G_j^\tau$ , the counts will be equal as long as no vertices of the induced subgraph of  $G_j^s$  are “forgotten” when the inclusion probability is reduced to  $\frac{r}{\tau_{k_j}}$ . The probability that  $V_i s_{k_j} \in [0, r]$  but  $V_i \tau_{k_j} \notin [0, r]$  is  $\frac{r}{\tau_{k_j} s_{k_j}}(\tau_{k_j} - s_{k_j})$ . Moreover, there are at most  $\xi^{(s,j)}(U)$  vertices in the subgraph sampled from  $G_j^s$  so, in particular,

$$\mathbb{E}[\xi^{(s,j)}(U) - \xi^{(\tau,j)}(U) \mid E_{\bar{c}}, \Gamma] \leq \frac{r}{\tau_{k_j} s_{k_j}}(\tau_{k_j} - s_{k_j})\xi^{(s,j)}(U), \quad (4.12)$$

where  $E_{\bar{c}}$  denotes the event that  $c$  is not included in the subgraph sampled from  $G_j^\tau$ . Then, denoting the event  $\{\xi^{(s,j)}(U) = \xi^{(\tau,j)}(U)\}$  as  $E_U$ ,

$$\mathbb{P}(E_U \mid \Gamma) \geq (1 - \mathbb{P}(E_{\bar{c}}))(1 - \mathbb{P}(\bar{E}_U \mid \Gamma, E_{\bar{c}})) \quad (4.13)$$

$$\geq (1 - \frac{r}{\tau_{k_j}})(1 - \frac{r}{\tau_{k_j} s_{k_j}}(\tau_{k_j} - s_{k_j})\xi^{(s,j)}(U)). \quad (4.14)$$

By construction,  $\tau_{k_j} - s_{k_j} \leq \frac{1}{k_j}$ , so  $\lim_{j \rightarrow \infty} \frac{\tau_{k_j} - s_{k_j}}{s_{k_j}} = 0$ . In combination with  $\lim_{j \rightarrow \infty} \xi^{(s,j)}(U) = \Gamma(U)$  a.s. and the fact that  $\Gamma(U)$  is almost surely finite, the inequality we have just derived then implies that

$$\lim_{j \rightarrow \infty} \mathbb{P}(\xi^{(s,j)}(U) \neq \xi^{(\tau,j)}(U) \mid \Gamma) = 0 \text{ a.s.} \quad (4.15)$$

In view of Theorem 4.3, we thus have that

$$\lim_{k \rightarrow \infty} \mathbb{E}[f(\xi^k(U)) \mid G_k, \tau_k] = \mathbb{E}[f(\Gamma(U))] \text{ a.s.}, \quad (4.16)$$

as required. □

**4.2. Asymptotic Equivalence of Sampling Schemes.** As alluded to above, a key insight for showing that  $\hat{W}_{(G_k, s_k)}$  is a valid estimator is that, conditional on  $G_k$ , a graph generated according to  $\text{uKEG}(\hat{W}_{(G_k, s_k)}, r)$  may be viewed as a random subgraph of  $G_k$  induced by sampling  $\text{Poi}(\frac{r}{s_k}v(G_k))$  vertices from  $G_k$  with replacement and returning the edge set of the vertex-induced subgraph. The correctness of this scheme can be seen as follows:

- (1) Let  $\Pi$  be the latent Poisson process used to generate a sample from  $\text{uKEG}(\hat{W}_{(G_k, s_k)}, r)$ , as in Theorem 2.4, and let  $\Pi_r = \Pi(\cdot \cap [0, r]^2)$ . Because  $\hat{W}_{(G_k, s_k)}$  has compact support  $[0, v(G_k)/s_k]^2$ , only  $\Pi_r$  restricted to  $[0, r] \times [0, v(G_k)/s_k]$  can participate in the graph.
- (2)  $\Pi_r$  restricted to  $[0, r] \times [0, v(G_k)/s_k]$  may be generated by producing  $J_{s_k, r} \sim \text{Poi}(rv(G_k)/s_k)$  points  $(\theta_i, \vartheta_j)$  where, conditional on  $J_{s_k, r}$ ,  $\theta_i \stackrel{iid}{\sim} \text{Uni}[0, r]$  and  $\vartheta_i \stackrel{iid}{\sim} \text{Uni}[0, v(G_k)/s_k]$ , also independently of each other.
- (3) The  $\{0, 1\}$ -valued structure of  $\hat{W}_{(G_k, s_k)}$  means that choosing latent values  $\vartheta_i \stackrel{iid}{\sim} \text{Uni}[0, v(G_k)/s_k]$  is equivalent to choosing vertices of  $G_k$  uniformly at random with replacement.

Our task is to show that the sampling scheme just described is asymptotically equivalent to  $r/s_k$ -sampling of  $G_k$ . To that end, we observe that  $r/s_k$ -sampling is the same as sampling  $\text{Bin}(v(G_k), r/s_k)$  vertices of  $G_k$  without replacement and returning the induced edge set. This makes it clear that there are two main distinctions between the sampling schemes: Binomial vs. Poisson number of vertices sampled, and with vs. without replacement sampling. This motivates defining three distinct random subgraphs of  $G_k$ :

- (1)  $X_r^{(k)}$ : Sample  $\text{Bin}(v(G_k), \frac{r}{s_k})$  vertices without replacement and return the induced edge set
- (2)  $H_r^{(k)}$ : Sample  $\text{Bin}(v(G_k), \frac{r}{s_k})$  vertices with replacement and return the induced edge set
- (3)  $M_r^{(k)}$ : Sample  $\text{Poi}(\frac{r}{s_k}v(G_k))$  vertices with replacement and return the induced edge set

The observation that, conditional on  $G_k$ ,  $\xi_r^k \stackrel{d}{=} \text{Lbl}_r(X_r^{(k)})$  makes the connection with the previous subsection clear.

Our aim is to show that when  $r/s_k$  is small the different random subgraphs are all close in distribution. A natural way to encode this is the total variation distance between their distributions. However, because the distributions are themselves random ( $G_k$  measurable) variables this is rather awkward. It is instead convenient to work with couplings of the random subgraphs conditional on  $G_k$ ; this gives a natural notion of conditional total variation distance. See [Hol12] for an introduction to coupling arguments.

Although we only need the sampling equivalence for sequences of graphs corresponding to a graphex process, we state the theorems for generic random graphs where possible.

The following result, which plays a similar role in the estimation theory of graphons in the dense setting, is simply the asymptotic equivalence of sampling with and without replacement.



**Lemma 4.5.** *Let  $G$  be an almost surely finite random graph, with  $e$  edges and  $v$  vertices. let  $X_r$  be a random subgraph of  $G$  given by sampling  $\text{Bin}(v(G), \frac{r}{s})$  vertices without replacement and returning the induced edge set, and let  $H_r$  be a random subgraph of  $G$  given by sampling  $\text{Bin}(v(G), \frac{r}{s})$  vertices with replacement and returning the induced edge set. Then there is a coupling such that*

$$\mathbb{P}(H_r \neq X_r \mid G) \leq 2e \left( \frac{r^3}{s^3} + 2 \frac{r^3}{s^3 v^2} + 3 \frac{r^2}{s^2 v} + \frac{r}{s v^2} \right) \quad (4.17)$$

Moreover, specializing to the graphex process case, with  $H_r^{(k)}$  and  $X_r^{(k)}$  defined as above, under the same coupling,

$$\mathbb{P}(H_r^{(k)} \neq X_r^{(k)} \mid G_k) \xrightarrow{p} 0, \quad (4.18)$$

as  $k \rightarrow \infty$ . Further, if  $\tau_1, \tau_2, \dots$  are the jump times of  $\Gamma$  then taking  $s_k = \tau_k$  for all  $k \in \mathbb{N}$ , it holds that under this coupling

$$\mathbb{P}(H_r^{(k)} \neq X_r^{(k)} \mid G_k, \tau_k) \xrightarrow{p} 0, \quad (4.19)$$

as  $k \rightarrow \infty$ .

*Proof.* Given  $G$ , we may sample  $X_r$  according to the following scheme:

- (1) Sample  $K_{s,r} \sim \text{Bin}(v, \frac{r}{s})$
- (2) Sample a list  $L = (L_1, L_2, \dots, L_{K_{s,r}})$  of vertices from  $G$  without replacement
- (3) Return the edge set of the induced subgraph given by restricting  $G$  to  $L$

Given  $G$ , we may sample  $H_r$  similarly, except we use a list sampled with replacement; we couple  $H_r$  and  $X_r$  by coupling with and without replacement sampling of the vertex list. The following sampling scheme for a list  $\tilde{L}$  returns a list that, given  $G$ , has the distribution of a length  $K_{s,r}$  list of vertices sampled with replacement from  $G$ . Given  $G$  we sample  $\tilde{L}$  according to:

- (1) Sample  $L$  as above
- (2)  $\tilde{L}_1 = L_1$
- (3) For  $j = 1 \dots K_{s,r}$ , set  $\tilde{L}_j = L_j$  with probability  $1 - \frac{j-1}{v}$ . Otherwise, sample  $\tilde{L}_j$  uniformly at random from  $\{L_1, \dots, L_{j-1}\}$ .

$H_r$  is then sampled by returning the edge set of the induced subgraph given by taking  $\tilde{L}$  as the vertex set.

Evidently, under this coupling,  $X_r = H_r$  as long as

- (1) Every entry of  $L$  where  $L \neq \tilde{L}$  does not participate in an edge in  $X_r$
- (2) Every entry of  $\tilde{L}$  where  $L \neq \tilde{L}$  does not participate in an edge in  $X_r$

Call the number of entries violating the first condition  $F_1$  and the number of entries violating the second condition  $F_2$ , and let  $N$  be the total number of entries where  $L, \tilde{L}$  differ. Observe that when  $K_{s,r} > 0$ , almost surely,

$$\mathbb{E}[F_1 \mid v(H_r), N, K_{s,r}, G] = \frac{v(X_r)}{K_{s,r}} N \quad (4.20)$$

$$\mathbb{E}[F_2 \mid v(H_r), N, K_{s,r}, G] = \frac{v(X_r)}{K_{s,r}} N. \quad (4.21)$$

Further observe that because the sites where the lists disagree are chosen without reference to the graph structure it holds that  $v(X_r)$  and  $N$  are independent given

$G$  and  $K_{s,r}$ , so

$$\mathbb{E}[F_1 + F_2 \mid v(X_r), K_{s,r}, G] = 2 \frac{v(X_r)}{K_{s,r}} \mathbb{E}[N \mid K_{s,r}, G]. \quad (4.22)$$

Moreover, almost surely,

$$\mathbb{E}[N \mid K_{s,r}, G] = \sum_{j=2}^{K_{s,r}} \frac{j-1}{v} \quad (4.23)$$

$$= \frac{1}{2v} (K_{s,r}^2 - K_{s,r}). \quad (4.24)$$

Using Markov's inequality along with the observation that

$$\mathbb{P}(X_r \neq H_r \mid K_{s,r} < 2) = 0, \quad (4.25)$$

and  $K_{s,r}^2 - K_{s,r} \leq K_{s,r}^2$  on  $K_{s,r} \geq 2$ , Eq. (4.24) implies that, almost surely,

$$\mathbb{P}(X_r \neq H_r \mid v(X_r), K_{s,r}, G) \leq \mathbb{E}[F_1 + F_2 \mid v(X_r), K_{s,r}, G] \quad (4.26)$$

$$\leq \frac{K_{s,r}}{v} v(X_r). \quad (4.27)$$

To prove the first assertion of the theorem statement, we now observe that  $v(X_r) \leq 2e(X_r)$  and  $\mathbb{E}[e(X_r) \mid s, K_{s,r}, G] \leq e \frac{K_{s,r}^2}{v^2}$  (since each edge is included with marginal probability at most  $\frac{K_{s,r}^2}{v^2}$ ), so it holds almost surely that

$$\mathbb{P}(X_r \neq H_r \mid s, G) \leq e \frac{2}{v^3} \mathbb{E}[K_{s,r}^3 \mid G] \quad (4.28)$$

$$= 2e \left( \frac{r^3}{s^3} - 3 \frac{r^3}{s^3 v} + 2 \frac{r^3}{s^3 v^2} + 3 \frac{r^2}{s^2 v} - 3 \frac{r^2}{s^2 v^2} + \frac{r}{s v^2} \right). \quad (4.29)$$

To prove the second assertion of the theorem statement we apply Eq. (4.27) to the graph  $G_k$  sampled at rate  $r/s_k$ , so

$$\mathbb{P}(X_r^{(k)} \neq H_r^{(k)} \mid v(X_r^{(k)}), K_{s_k,r}, G_k) \leq \frac{K_{s_k,r}}{v} v(X_r^{(k)}). \quad (4.30)$$

Markov's inequality with  $\mathbb{E}[\frac{K_{s_k,r}}{v(G_k)} \mid G_k] = r/s_k$  implies that, given  $G_k$ ,  $\frac{K_{s_k,r}}{v(G_k)} \xrightarrow{p} 0$  as  $k \rightarrow \infty$ . Further, by Theorem 4.3 and the observation that  $X_r^{(k)} \stackrel{d}{=} \mathcal{G}(\xi^k(\cdot \cap [0, r]^2))$  where  $\xi^k \sim \text{embed}(G_k, s_k)$ , it holds that  $v(X_r^{(k)}) \xrightarrow{d} v(\Gamma_r)$  a.s. as  $k \rightarrow \infty$ . Since the integrability conditions on graphexes guarantee that  $v(\Gamma_r)$  is almost surely finite, we have

$$\frac{K_{s_k,r}}{v(G_k)} v(X_r^{(k)}) \xrightarrow{p} 0, \quad (4.31)$$

as  $k \rightarrow \infty$  and this implies,

$$\mathbb{P}(X_r^{(k)} \neq H_r^{(k)} \mid v(X_r^{(k)}), K_{s_k,r}, G_k) \xrightarrow{p} 0, \quad (4.32)$$

as  $k \rightarrow \infty$ . Now,

$$\mathbb{P}(X_r^{(k)} \neq H_r^{(k)} \mid G_k) = \mathbb{E}[\mathbb{P}(X_r^{(k)} \neq H_r^{(k)} \mid v(X_r^{(k)}), K_{s_k,r}, G_k) \mid G_k], \quad (4.33)$$

and  $\mathbb{P}(X_r^{(k)} \neq H_r^{(k)} \mid G_k)$  is bounded by 1 for all  $k$ , so the second claim follows by the dominated convergence theorem for conditional expectations, [Dur10, Thm. 5.9].

The proof of the final claim goes through mutatis mutandis as the proof of the second assertion, subject to the observations that  $\tau_k \uparrow \infty$  a.s., that we must condition on  $\tau_k$  for each  $k$ , and that Theorem 4.4 should be used in place of Theorem 4.3.  $\square$

*Remark 4.6.* In the case that  $\mathcal{W} = (0, 0, W)$  and  $W$  is integrable, it holds that  $v(G_k) = \Omega(s_k)$  a.s. and  $e(G_k) = \Theta(s_k^2)$  a.s. [BCCH16, Props. 2.18 and 5.2], in which case the rate from the first part of the above lemma is  $O(r^3/s_k)$ . Note that in this case, the convergence in probability may be replaced by convergence almost surely. This lemma is in fact the only component of the proof where a weakening of almost sure convergence is necessary, so (as remarked below), whenever almost sure convergence holds for the equivalence of with and without replacement sampling, almost sure convergence holds for the main estimation result.  $\triangleleft$

It remains to show that the  $\text{Poi}(r/s_k v(G_k))$  and  $\text{Bin}(v(G_k), r/s_k)$  samplings are asymptotically equivalent. Note that the rate  $(v(G_k)/s_k)$  at which the empirical graphon is dilated guarantees that the expected number of vertices sampled according to each scheme is equal; this is the reason that this rate was chosen.

**Lemma 4.7.** *Let  $G$  be an almost surely finite random graph with  $v$  vertices. Let  $H_r$  be a random subgraph of  $G$  given by sampling  $\text{Bin}(v, \frac{r}{s})$  vertices with replacement and returning the induced edge set, and let  $M_r$  be a random subgraph of  $G$  given by sampling  $\text{Poi}(v \frac{r}{s})$  vertices with replacement and returning the induced edge set. Then there is a coupling such that*

$$\mathbb{P}(H_r \neq M_r \mid G) \leq \frac{r}{s} \text{ a.s.} \quad (4.34)$$

*Proof.* Conditional on  $G$ ,  $H_r$  may be sampled by:

- (1) sample  $K_{s,r} \sim \text{Bin}(v, r/s)$  vertices with replacement from  $G$ ;
- (2) return the edge set of the induced subgraph.

Conditional  $G$ ,  $M_r$  may be sampled by:

- (1) sample  $J_{s,r} \sim \text{Poi}(r \frac{v}{s})$  vertices with replacement from  $G$ .
- (2) return the edge set of the induced subgraph.

Comparing the two sampling schemes, it is immediate that there is a coupling such that

$$\mathbb{P}(H_r \neq M_r \mid G) \leq \mathbb{P}(K_{s,r} \neq J_{s,r} \mid G). \quad (4.35)$$

Note that  $\mathbb{E}[K_{s,r} \mid G] = \mathbb{E}[J_{s,r} \mid G]$ . The approximation of a sum of Bernoulli random variables by a Poisson with the same expectation as the sum is well studied: if  $X_1, \dots, X_l$  are independent random variables with  $\text{Bern}(p_i)$  distributions such that  $\lambda = \sum_{i=1}^l p_i$  and  $T \sim \text{Poi}(\lambda)$  then there is a coupling [Hol12, Sec. 5.3] such that  $\mathbb{P}(T \neq \sum_{i=1}^l X_i) \leq \frac{1}{\lambda} \sum_{i=1}^l p_i^2$ . This implies that there is a coupling of  $K_{s,r}$  and  $J_{s,r}$  such that

$$\mathbb{P}(K_{s,r} \neq J_{s,r} \mid G) \leq \frac{r}{s}, \quad (4.36)$$

completing the proof.  $\square$

**4.3. Estimating  $\mathcal{W}$ .** We now combine our results to show that the law of the graphex process generated by the empirical graphex converges to the law of a graphex process generated by the underlying  $\mathcal{W}$ .

There is an immediate subtlety to address: Section 4.1 deals with convergence in distribution of point processes (i.e., labeled graphs), and Section 4.2 deals with convergence in distribution of unlabeled graphs. We first give the main convergence result for the point process case. In order to state this result compactly it is convenient to metrize weak convergence. To this end, we recall that the space of boundedly finite measures may be equipped with a metric such that it is a complete separable metric space [DVJ03, Eqn. A.2.6]. Let  $d_p(\cdot, \cdot)$  be the Prokhorov metric on the space of probability measures over boundedly finite measures induced by the aforementioned metric. Then  $d_p(\cdot, \cdot)$  metrizes weak convergence: i.e., for a sequence of boundedly finite random measures  $\{\Pi_n\}$  it holds that  $\Pi_n \xrightarrow{d} \Pi$  as  $n \rightarrow \infty$  if and only if  $d_p(\mathcal{L}(\Pi_n), \mathcal{L}(\Pi)) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Theorem 4.8.** *Let  $\Gamma$  be a graphex process generated by non-trivial graphex  $\mathcal{W}$  and let  $s_1, s_2, \dots$  be a (possibly random) sequence in  $\mathbb{R}_+$  such that  $s_k \uparrow \infty$  almost surely as  $k \rightarrow \infty$ . Let  $G_k = \mathcal{G}(\Gamma_{s_k})$  for  $k \in \mathbb{N}$ . Suppose that either*

- (1)  $(s_k)$  is independent of  $\Gamma_k$ , or
- (2)  $s_k = \tau_k$  for all  $k \in \mathbb{N}$ , where  $\tau_1, \tau_2, \dots$  are the jump times of  $\Gamma$ .

Then

$$d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}), \text{KEG}(\mathcal{W})) \xrightarrow{p} 0, \quad (4.37)$$

as  $k \rightarrow \infty$ .

*Proof.* For notational simplicity, we treat the deterministic index case first.

For  $r \in \mathbb{R}_+$ , let  $\text{embed}(G_k, s_k)|_r$  denote the probability measure over point processes on  $[0, r]^2$  induced by generating a point process according to  $\text{embed}(G_k, s_k)$  and restricting to  $[0, r]^2$ .

By the triangle inequality,

$$d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}, r), \text{KEG}(\mathcal{W}, r)) \leq d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}, r), \text{embed}(G_k, s_k)|_r) \quad (4.38)$$

$$+ d_p(\text{embed}(G_k, s_k)|_r, \text{KEG}(\mathcal{W}, r)). \quad (4.39)$$

Conditional on  $G_k$  and  $s_k$ , let  $X_r^k$  be an  $r/s_k$ -sampling of  $G_k$  and let  $M_r^k$  be a random subgraph of  $G_k$  given by sampling  $\text{Poi}(v(G_k)r/s_k)$  vertices with replacement and returning the edge set of the vertex-induced subgraph. By Lemmas 4.5 and 4.7 it holds that there is a sequence of couplings such that

$$\mathbb{P}(M_r^k \neq X_r^k \mid G_k, s_k) \xrightarrow{p} 0, \quad k \rightarrow \infty. \quad (4.40)$$

Observe that  $\Gamma_r^k \stackrel{d}{=} \text{Lbl}_r(M_r^k, \{U_i\})$  and  $\xi_r^k \stackrel{d}{=} \text{Lbl}_r(X_r^k, \{U_i\})$ , where  $U_i \stackrel{iid}{\sim} \text{Uni}[0, r]$  for  $i \in \mathbb{N}$ . Here  $\xi_r^k$  is a random labeling of  $G_k$ , as in Theorem 4.3. Thus, the couplings of the unlabeled graphs lift to couplings of the point processes such that

$$\mathbb{P}(\Gamma_r^k \neq \xi_r^k \mid G_k, s_k) \xrightarrow{p} 0, \quad k \rightarrow \infty. \quad (4.41)$$

The relationship between couplings and total variation distance then implies

$$\|\text{KEG}(\hat{W}_{(G_k, s_k)}, r) - \text{embed}(G_k, s_k)|_r\|_{\text{TV}} \xrightarrow{p} 0, \quad k \rightarrow \infty, \quad (4.42)$$

so also,

$$d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}, r), \text{embed}(G_k, s_k)|_r) \xrightarrow{p} 0, \quad k \rightarrow \infty. \quad (4.43)$$

Second, by Theorem 4.3,

$$d_p(\text{embed}(G_k, s_k)|_r, \text{KEG}(\mathcal{W})) \xrightarrow{p} 0, \quad k \rightarrow \infty. \quad (4.44)$$

Thus,

$$d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}, r), \text{KEG}(\mathcal{W}, r)) \xrightarrow{p} 0, \quad k \rightarrow \infty. \quad (4.45)$$

By [Kal01, Lem. 4.4], convergence in probability for each element of a sequence lifts to convergence in probability of the entire sequence:

$$(d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}, 1), \text{KEG}(\mathcal{W}, 1)), d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}, 2), \text{KEG}(\mathcal{W}, 2)), \dots) \xrightarrow{p} 0, \quad k \rightarrow \infty. \quad (4.46)$$

As the space of boundedly finite measures on  $\mathbb{R}_+^2$  is homeomorphic to the space of sequences of restrictions of boundedly finite measures to  $[0, r]^2$ , for  $r \in \mathbb{N}$ , it follows that

$$d_p(\text{KEG}(\hat{W}_{(G_k, s_k)}), \text{KEG}(\mathcal{W})) \xrightarrow{p} 0, \quad k \rightarrow \infty. \quad (4.47)$$

The same proof mutatis mutandis applies for convergence along the jump times. The main substitution is the use of Theorem 4.4 in place of Theorem 4.3.  $\square$

*Remark 4.9.* For graphexes such that  $e(G_k)/s_k^3 \rightarrow 0$  a.s. and  $v(G_k) = \Omega(s_k)$  the convergence in probability above can be replaced by almost sure convergence by replacing all the convergence in probability statements in the body of the proof by almost sure statements. This class of such graphexes includes all integrable  $(0, 0, W)$ .  $\triangleleft$

We now turn to the analogous result for the case of unlabeled graphs generated by the dilated empirical graphon. We begin with a technical lemma that allows us to deduce convergence in distribution of unlabeled graphs from convergence in distribution of the associated adjacency measures. Note that the map taking an adjacency measure to its associated graph is measurable, but not continuous, and so this result does not follow from a naive application of the continuous mapping theorem.

**Lemma 4.10.** *Let  $S$  be a discrete space,  $T$  a metric space,  $Q_1, Q_2, \dots$  a tight sequence of probability measures on  $S$ , and  $K$  a probability kernel from  $S$  to  $T$ , such that  $K$  is injective when considered as a map from probability measures on  $S$  to probability measures on  $T$ . If  $Q_1 K, Q_2 K, \dots$  converge weakly to  $QK$  then  $Q_1, Q_2, \dots$  converges weakly to  $Q$ .*

*Proof.* Assume otherwise. Case 1:  $Q_n \rightarrow Q' \neq Q$  weakly. By [Kal01, Lem. 16.24] and the discreteness of  $S$ ,  $Q_n K \rightarrow Q' K$  weakly. Since  $K$  is injective  $Q' K \neq QK$ , a contradiction.

Case 2:  $Q_n$  does not converge weakly. Since the sequence  $Q_n$  is tight it does converge subsequentially. Choose two infinite subsequences  $Q_{i_1}, Q_{i_2}, \dots$  and  $Q_{j_1}, Q_{j_2}, \dots$  with respective limits  $Q', Q''$  with  $Q' \neq Q''$ . But then, by [Kal01, Lem. 16.24] and the discreteness of  $S$ ,  $Q'_{i_k} K \rightarrow Q' K$  and  $Q''_{j_k} K \rightarrow Q'' K$ , hence  $Q' K = QK = Q'' K$ , but  $K$  is injective, hence  $Q' = Q = Q''$ , a contradiction.  $\square$

The motivating application of this last lemma is showing that a sequence of graphs  $G_1, G_2, \dots$  converge in distribution if and only if their random labelings into  $[0, s]$  for some  $s$  also converge in distribution. To parse the following theorem, note that when

$G$  is a finite random graph, and  $s \in \mathbb{R}_+$ , then  $P(G \in \cdot) \text{embed}(\cdot, s) = P(\text{Lbl}_s(G) \in \cdot)$ .

**Lemma 4.11.** *Let  $K_s(\cdot) = \text{embed}(\cdot, s)$  for  $s \in \mathbb{R}_+$ , let  $Q, Q_1, Q_2, \dots$  be probability measures on the space of almost surely finite random graphs, let  $\zeta_k = Q_k K_s$  and let  $\zeta = Q K_s$ . Then,  $Q_k \rightarrow Q$  weakly as  $k \rightarrow \infty$  if and only if  $\zeta_k \rightarrow \zeta$  weakly as  $k \rightarrow \infty$ .*

*Proof.* The forward direction (convergence in distribution of the random graphs implies convergence in distribution of the random adjacency measures) follows immediately from the discreteness of the space of finite graphs and [Kal01, Lem. 16.24].

Conversely, suppose that  $\zeta_k \rightarrow \zeta$  weakly as  $k \rightarrow \infty$ , and, for every  $n \in \mathbb{N}$ , let  $E_n$  be the set of adjacency measures  $\xi$  such that  $\xi([0, s]^2) \leq n$ , i.e.,  $E_n$  is the event that the graph has fewer than  $n$  edges. Note that  $E_n$  is a  $\zeta$ -continuity set by the definition of  $K_s$ , and therefore, by weak convergence,  $\zeta_k(E_n) \rightarrow \zeta(E_n)$  as  $k \rightarrow \infty$  for every  $n \in \mathbb{N}$ . Let  $E'_n$  be the set of graphs with fewer than  $n$  edges. By definition,  $Q_k(E'_n) = \zeta_k(E_n)$  and  $Q(E'_n) = \zeta(E_n)$ , hence  $Q_k(E'_n) \rightarrow Q(E'_n)$ . But  $E'_n$  is a finite (hence, compact) set, hence  $\{Q_k\}_{k \in \mathbb{N}}$  is tight. Noting in addition that  $K_s$  is injective, the result follows from Lemma 4.10.  $\square$

The following theorem is a formalization of  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$  in probability:

**Theorem 4.12.** *Let  $\Gamma$  be a graphex process generated by non-trivial graphex  $\mathcal{W}$  and let  $s_1, s_2, \dots$  be a (possibly random) sequence in  $\mathbb{R}_+$  such that  $s_k \uparrow \infty$  almost surely as  $k \rightarrow \infty$ . Let  $G_k = \mathcal{G}(\Gamma_{s_k})$  for  $k \in \mathbb{N}$ . Suppose that either*

- (1)  $(s_k)$  is independent of  $\Gamma_k$ , or
- (2)  $s_k = \tau_k$  for all  $k \in \mathbb{N}$ , where  $\tau_1, \tau_2, \dots$  are the jump times of  $\Gamma$ .

*Then, for every infinite sequence  $N \subseteq \mathbb{N}$ , there exists an infinite subsequence  $N' \subseteq N$ , such that*

$$\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} \mathcal{W} \text{ a.s.} \quad (4.48)$$

*along  $N'$ .*

*Proof.* We first treat the case (1) where the times  $(s_k)$  are independent of  $\Gamma$ .

Let  $N \subseteq \mathbb{N}$  be an infinite sequence. Theorem 4.8 implies that there is some infinite subsequence  $N' \subseteq N$  such that, for all  $r \in \mathbb{R}_+$ ,  $\text{KEG}(\hat{W}_{(G_k, s_k)}, r) \rightarrow \text{KEG}(\mathcal{W}, r)$  weakly almost surely along  $N'$ .

Let  $r \in \mathbb{R}_+$  and  $K_r(\cdot) = \text{embed}(\cdot, r)$ . For all  $k \in N'$ ,

$$\text{uKEG}(\hat{W}_{(G_k, s_k)}, r) K_r = \text{KEG}(\hat{W}_{(G_k, s_k)}, r) \text{ a.s.}, \quad (4.49)$$

and  $\text{uKEG}(\mathcal{W}, r) K_r = \text{KEG}(\mathcal{W}, r)$ . Moreover, the graph corresponding to a size- $r$  graphex process is almost surely finite. Thus Lemma 4.11 applies and we have that  $\text{uKEG}(\hat{W}_{(G_k, s_k)}, r) \rightarrow \text{uKEG}(\mathcal{W}, r)$  weakly a.s. along  $N'$ . This holds for all  $r \in \mathbb{R}_+$ , so we have even that  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} \mathcal{W}$  a.s. along  $N'$ .

The same proof mutatis mutandis applies for convergence along the jump times.  $\square$

**Remark 4.13.** For graphexes such that  $e(G_k)/s_k^3 \rightarrow 0$  a.s. and  $v(G_k) = \Omega(s_k)$ , Theorem 4.8 implies that  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$  almost surely and not merely in probability. The class of graphexes with these two properties includes all graphexes of the form  $(0, 0, W)$  for integrable  $W$ .  $\triangleleft$

## 5. ESTIMATION FOR UNKNOWN SIZES

We now turn to the case where only the graph structure of the graphex process is observed, rather than the graph structure and the sizes of the observation.

We first show how distinct adjacency measures can give rise to the same graph sequence. For a measurable map  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and adjacency measure  $\xi$ , define  $\xi^\phi$  to be the measure given by  $\xi^\phi(A \times B) = \xi(\phi^{-1}(A) \times \phi^{-1}(B))$ , for every measurable  $A, B \subseteq \mathbb{R}_+$ . The graph sequences underlying an adjacency measure  $\xi$  is invariant to the action  $\phi \mapsto \xi^\phi$  of every strictly monotonic and increasing function  $\phi$ .

**Proposition 5.1.** *Let  $\xi$  be an adjacency measure and let  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be strictly monotonic and increasing. Then  $\mathcal{G}(\xi) = \mathcal{G}(\xi^\phi)$ .*

*Proof.* Let  $\{\tau_k\}$  and  $\{\tau_k^\phi\}$  be the stopping sizes of  $\xi$  and  $\xi^\phi$ , respectively.

Since  $\phi$  is strictly monotonic it is also invertible. From this observation it is easily seen that  $(\theta_i, \theta_j)$  is an atom of  $\xi$  if and only if  $(\phi(\theta_i), \phi(\theta_j))$  is an atom of  $\xi^\phi$ . It is then clear that, for all  $k \in \mathbb{N}$ ,  $\phi(\tau_k) = \tau_k^\phi$  and, moreover, the graph structure of  $\{(x_i, \tau_k) : (x_i, \tau_k) \in \xi\}$  is equal to the graph structure of  $(y_i, \tau_k^\phi) : (y_i, \tau_k^\phi) \in \xi^\phi$ . That is, the subgraph of all edges added at the  $k$ th step is equal for both graph sequences, for all  $k \in \mathbb{N}$ . Moreover, the first entry of each graph sequence is (obviously) equal to the subgraph of all edges added at the first step. The proof is then completed by induction.  $\square$

If  $\phi$  is an arbitrary strictly monotonic mapping and  $\xi$  is an exchangeable adjacency measure, it will not generally be the case that  $\xi^\phi$  is exchangeable. One family of mappings that preserves exchangeability is  $\phi(x) = cx$ , for  $c \in \mathbb{R}_+$ . We define the  $c$ -dilation of an adjacency measure  $\xi$  to be the adjacency measure  $\xi^\phi$  for this map. Because  $\xi^\phi$  is exchangeable there is some graphex  $\mathcal{W}'$  that generates it: the next result shows that the  $\frac{1}{c}$ -dilation of a graphex process corresponds to a  $c$ -dilation of its graphex.

**Lemma 5.2.** *Let  $\Gamma$  be a graphex process with graphex  $\mathcal{W} = (I, S, W)$ . Then the  $\frac{1}{c}$ -dilation of  $\Gamma$  is a graphex process  $\Gamma'$  with generating graphex  $\mathcal{W}' = (I', S', W')$  where  $I' = c^2 I$ ,  $S'(x) = cS(x/c)$ , and  $W'(x, y) = W(x/c, y/c)$ .*

*Proof.* Let  $\Gamma$  be a graphex process generated by  $W$  with latent Poisson processes  $\Pi, \Pi_1, \Pi_2, \dots$ , and  $\Pi^i$  on  $\mathbb{R}_+^2$  and  $\mathbb{R}_+^3$ , respectively. Define  $f(\Pi) = \{(\frac{1}{c}\theta, c\vartheta) : (\theta, \vartheta) \in \Pi\}$ , define  $f(\Pi_n) = \{(\frac{1}{c}\sigma, c\chi) : (\sigma, \chi) \in \Pi_n\}$ , for  $n \in \mathbb{N}$ , and define  $f(\Pi^i) = \{(\frac{1}{c}\rho, \frac{1}{c}\rho', c^2\eta) : (\rho, \rho', \eta) \in \Pi^i\}$ . Note that  $f(\Pi)$  and  $f(\Pi_n)$ , for  $n \in \mathbb{N}$ , are unit-rate Poisson processes on  $\mathbb{R}_+^2$ , and  $f(\Pi^i)$  is a unit-rate Poisson process on  $\mathbb{R}_+^3$ . Indeed, the joint law of  $(\Pi, \Pi_1, \Pi_2, \dots, \Pi^i)$  is the same as that of  $(f(\Pi), f(\Pi_1), f(\Pi_2), \dots, f(\Pi^i))$ .

Then  $\Gamma'$ , the  $\frac{1}{c}$ -dilation of  $\Gamma$ , is the graphex process generated by  $\mathcal{W}'$  with latent Poisson processes  $f(\Pi), f(\Pi_1), f(\Pi_2), \dots, f(\Pi^i)$  reusing the same i.i.d. collection  $(\zeta_{\{i,j\}})$  in  $[0, 1]$  as was used to generate  $\Gamma$ . To see this, note that  $\Gamma'$  includes edge  $(\frac{1}{c}\theta_i, \frac{1}{c}\theta_j)$  if and only if  $\zeta_{\{i,j\}} \leq W'(c\vartheta_i, c\vartheta_j)$  if and only if  $\zeta_{\{i,j\}} \leq W(\vartheta_i, \vartheta_j)$  if and only if  $\Gamma$  includes edge  $(\theta_i, \theta_j)$ . Similarly,  $\Gamma'$  includes edge  $(\frac{1}{c}\theta_i, \frac{1}{c}\sigma_{ij})$  if and only if  $c\chi_{ij} \leq S'(c\vartheta) = cS(\vartheta)$  if and only if  $\chi_{ij} \leq S(\vartheta)$  if and only if  $\Gamma$  includes edge  $(\theta_i, \sigma_{ij})$ . Finally,  $\Gamma'$  includes edge  $(\frac{1}{c}\rho, \frac{1}{c}\rho')$  if and only if  $c^2\eta \leq I' = c^2 I$  if and only if  $\Gamma$  includes edge  $(\rho, \rho')$ . Thus  $\Gamma'$  is a  $\frac{1}{c}$ -dilation of  $\Gamma$ , as was to be shown.  $\square$

Define the  $c$ -dilation of a graphex  $\mathcal{W}$  to be the graphex  $\mathcal{W}'$  defined in the statement of Lemma 5.2. We have the following consequence:



**Theorem 5.3.** *Let  $\mathcal{W}$  be a graphex, let  $\mathcal{W}'$  be the  $c$ -dilation of  $\mathcal{W}$  for some  $c > 0$ , and let  $\Gamma$  and  $\Gamma'$  be graphex processes with graphexes  $\mathcal{W}$  and  $\mathcal{W}'$ , respectively. Then  $\mathcal{G}(\Gamma) \stackrel{d}{=} \mathcal{G}(\Gamma')$ .*

*Proof.* Follows immediately from Lemma 5.2 and Proposition 5.1.  $\square$

As a consequence of this result, if the observed data is the graph sequence—that is, if the size  $s$  is unknown—then the dilation of the generating graphex is not identifiable. Therefore, the notion of estimation that we used in the known-size setting is not appropriate, because it requires  $\mathcal{G}_r(\mathcal{W}_n) \xrightarrow{d} \mathcal{G}_r(\mathcal{W})$  as  $n \rightarrow \infty$  for all sizes  $r \in \mathbb{R}_+$ .

The appropriate notion of estimation in this setting is then:

**Definition 5.4.** Let  $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2, \dots$  be a sequence of graphexes, and let  $\Gamma, \Gamma^1, \Gamma^2, \dots$  be graphex processes generated by each graphex. Write  $\mathcal{W}_k \rightarrow_{\text{GS}} \mathcal{W}$  as  $k \rightarrow \infty$  when  $\mathcal{G}(\Gamma^k) \xrightarrow{d} \mathcal{G}(\Gamma)$  as  $k \rightarrow \infty$ .

Note that this is equivalent to requiring convergence in distribution of the length- $l$  prefixes of the graph sequences, for all  $l \in \mathbb{N}$ . Intuitively, a length- $l$  graph sequence generated by the estimator is close in distribution to a length- $l$  graph sequence generated by the true graphex, provided the observed graph is large enough. This perspective explains how a sequence of compactly supported graphexes can estimate a graphex that is not itself compactly supported. The following is immediate from Theorem 5.3.

**Corollary 5.5.** *Let  $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2, \dots$  be a sequence of graphexes, let  $c, c_1, c_2, \dots > 0$ , and let  $\mathcal{W}^c, \mathcal{W}_1^{c_1}, \mathcal{W}_2^{c_2}, \dots$  be the corresponding dilations. Then  $\mathcal{W}_k \rightarrow_{\text{GS}} \mathcal{W}$  as  $k \rightarrow \infty$  if and only if  $\mathcal{W}_k^{c_k} \rightarrow_{\text{GS}} \mathcal{W}^c$  as  $k \rightarrow \infty$ .*

Intuitively speaking,  $\mathcal{W}_k \rightarrow_{\text{GS}} \mathcal{W}$  as  $k \rightarrow \infty$  demands less than  $\mathcal{W}_k \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$ , because in the former case we don't need to find a correct rate of dilation for the graphex. The intuition that convergence in distribution of the graph sequence is weaker than convergence in distribution of  $(\mathcal{G}(\Gamma_s))_{s \in \mathbb{R}_+}$  is borne out by the next lemma:

**Lemma 5.6.** *Let  $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2, \dots$  be graphexes where  $\mathcal{W}$  is non-trivial and  $\mathcal{W}_k \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$ . Then  $\mathcal{W}_k \rightarrow_{\text{GS}} \mathcal{W}$  as  $k \rightarrow \infty$ .*

*Proof.* Let  $\Gamma^k$  be graphex processes generated by  $\mathcal{W}_k$ , and let  $\Gamma$  be generated by  $\mathcal{W}$ . For  $n \in \mathbb{N}$ , let  $G_n^k = \mathcal{G}(\Gamma_n^k)$ , and let  $G_n = \mathcal{G}(\Gamma_n)$ .

Consider the sequence  $H_n^k = (\mathcal{G}(\Gamma_1^k), \mathcal{G}(\Gamma_2^k), \dots, \mathcal{G}(\Gamma_n^k))$ , where each entry is itself an a.s. finite graph sequence and entry  $j$  is a prefix of entry  $j+1$ . Let  $\eta_n^k = P(H_n^k \in \cdot)$ , and let  $\eta_n = P((\mathcal{G}(\Gamma_1), \mathcal{G}(\Gamma_2), \dots, \mathcal{G}(\Gamma_n)) \in \cdot)$ . Intuitively speaking, we are breaking up the graph sequence of the entire graphex process into the graph sequences up to size 1, 2,  $\dots$  and  $\eta_n$  is the joint distribution of the first  $n$  of these partial graph sequences. Our short term goal is to show that  $\eta_n^k \rightarrow \eta_n$  weakly as  $k \rightarrow \infty$ .

To that end, let  $G$  be a finite graph and consider the random variable

$$L_n(G) = (\mathcal{G}(\text{Lbl}_n(G)([0, j]^2 \cap \cdot)))_{j=1, \dots, n}. \quad (5.1)$$

This is a nested sequence of graph sequences given by mapping  $G$  to an adjacency measure on  $[0, n]^2$  and then returning the sequence of graph sequences corresponding

to this adjacency matrix at sizes  $1, \dots, n$ . The significance of this construction is that we may use it to define a probability kernel,

$$K_n(G, \cdot) = P(L_n(G) \in \cdot), \quad (5.2)$$

such that  $P(G_n^k \in \cdot)K_n = \mathbb{E}K_n(G_n^k, \cdot) = \eta_n^k$  and  $P(G_n \in \cdot)K_n = \mathbb{E}K_n(G_n, \cdot) = \eta_n$ . By assumption, we have  $\mathcal{W}_k \rightarrow_{\text{GP}} \mathcal{W}$  as  $k \rightarrow \infty$ , whence  $G_n^k \xrightarrow{d} G_n$  as  $k \rightarrow \infty$ . By the discreteness of the space of finite graphs and [Kal01, Lem. 16.24] it then holds that,

$$P(G_n^k \in \cdot)K_n \rightarrow P(G_n \in \cdot)K_n, \quad (5.3)$$

weakly as  $k \rightarrow \infty$ . It thus holds by the construction of  $K_n$  that

$$\eta_n^k \rightarrow \eta_n, \quad (5.4)$$

weakly as  $k \rightarrow \infty$ .

We now have that an arbitrary length prefix of the graph sequence converges in distribution, when the notion of length is given by the latent sizes. It remains to argue that this convergence holds for arbitrary prefixes in the usual sequence sense. To that end, we observe that because Eq. (5.4) holds for all  $n \in \mathbb{N}$ , by [Kal01, Thm. 4.29] it further holds that

$$(\mathcal{G}(\Gamma_1^k), \mathcal{G}(\Gamma_2^k), \dots) \xrightarrow{d} (\mathcal{G}(\Gamma_1), \mathcal{G}(\Gamma_2), \dots), \quad k \rightarrow \infty. \quad (5.5)$$

There is function  $f$  such that, for every locally finite but infinite adjacency measure  $\xi$  on  $\mathbb{R}_+^2$ , with restrictions  $\xi_j$  to  $[0, j]^2$ ,

$$f(\mathcal{G}(\xi_1), \mathcal{G}(\xi_2), \dots) = \mathcal{G}(\xi) \quad (5.6)$$

and  $f$  is even continuous because every finite prefix of  $\mathcal{G}(\xi)$  is determined by some finite prefix of the left hand side. Extend  $f$  to the space of all nested graph sequences arbitrarily. Note that  $f$  is continuous at  $\Gamma$  a.s. because  $\Gamma$  is a.s. locally finite and  $\|W\|_1 > 0$  implies  $\Gamma$  is a.s. infinite. Hence, the result follows by the continuous mapping theorem [Kal01, Thm. 4.27].  $\square$

We now turn to establishing the main estimation result for the setting where the sizes are not included as part of the observation. In this setting, the observations are increasing sequences of graphs  $G_1, G_2, \dots$ . There are two natural models for the observations: In one model,  $G_k = \mathcal{G}(\Gamma_{s_k})$  for some graphex process  $\Gamma$  and (possibly random, independent, and a.s.) increasing and diverging sequence of sizes  $s_1, s_2, \dots$ . Alternatively, in the other model, the sequence  $G_1, G_2, \dots$  is the graph sequence  $\mathcal{G}(\Gamma)$  of some graphex process  $\Gamma$ .

A natural estimator is the empirical graphon,  $\tilde{W}_{G_k}$ , reflecting the intuition that the dilation necessary in the previous section for convergence of the generated graphex process is irrelevant for convergence in distribution of the associated graph sequence. Somewhat more precisely, we view the empirical graphon as the canonical representative of the equivalence class of graphons given by equating graphons that induce the same distribution on graph sequences. The main result of this section is that  $\tilde{W}_{G_k} \rightarrow_{\text{GS}} \mathcal{W}$  as  $k \rightarrow \infty$  in probability, for either of the natural models for the observed sequence  $G_1, G_2, \dots$ .

**Theorem 5.7.** *Let  $\Gamma$  be a graphex process generated by some non-trivial graphex  $\mathcal{W}$  and let  $G_1, G_2, \dots$  be some sequence of graphs such that either*

- (1) *There is some random sequence  $(s_k)$ , independent from  $\Gamma$ , such that  $s_k \uparrow \infty$  a.s. and  $G_k = \mathcal{G}(\Gamma_{s_k})$  for all  $k \in \mathbb{N}$ , or*  
 (2)  $(G_1, G_2, \dots) = \mathcal{G}(\Gamma)$ .

Then, for every infinite sequence  $N \subseteq \mathbb{N}$ , there is an infinite subsequence  $N' \subseteq N$ , such that

$$\tilde{W}_{G_k} \rightarrow_{\text{GS}} \mathcal{W} \text{ a.s.}, \quad (5.7)$$

along  $N'$ .

*Proof.* We prove case (1). Case (2) follows mutatis mutandis, substituting  $\tau_k$  for  $s_k$ .

Let  $\hat{W}_{(G_k, s_k)}$  denote the dilated empirical graphon of  $G_k$  with observation size  $s_k$ . By Theorem 4.12, for every sequence  $N \subseteq \mathbb{N}$ , there is an infinite subsequence  $N' \subseteq N$ , such that  $\hat{W}_{(G_k, s_k)} \rightarrow_{\text{GP}} \mathcal{W}$  along  $N'$ . By Lemma 5.6 and  $\mathcal{W}$  being non-trivial, this implies that  $\tilde{W}_{(G_k, s_k)} \rightarrow_{\text{GS}} \mathcal{W}$  along  $N'$ . For every  $k$ ,  $\tilde{W}_{G_k}$  is some dilation of  $\hat{W}_{(G_k, s_k)}$ , hence, the result follows by Corollary 5.5.  $\square$

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