Scaling and Singularities in the Entrainment of Globally Coupled Oscillators

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The onset of collective behavior among oscillators with random frequencies is studied for globally coupled phase dynamical models. A Fokker-Planck equation for the phase distribution describes the dynamics including diffusion due to the noise in the frequencies. We analyze instabilities of the phase-incoherent state using amplitude equations for the unstable modes. In terms of the diffusion coefficient D, the linear growth rate γ, and the mode number l, the nonlinearly saturated mode amplitude typically scales like $|\alpha_\infty| \sim \sqrt{\gamma(\gamma + l^2 D)}$. The unusual $\gamma + l^2 D$ factor arises from a singularity in the cubic term of the amplitude equation.

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The onset of collective oscillations is a multifaceted phenomenon of interest in physics, chemistry, biology, and most recently neuroscience [1–5]. One important class of models describes a collection of N dissipative (limit cycle) oscillators that have some weak mutual interaction. For sufficiently weak couplings, the basic form of the uncoupled cycle persists, and the fundamental effect of the interaction is to alter the frequencies and evolving phases of the oscillators. When the coupling between two oscillators, $f(\theta_j - \theta_i)$, is uniform for all oscillator pairs, then the evolution of the phases is given by

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j - \theta_i) + \xi_i(t), \tag{1}$$

for $i = 1, \ldots, N$. If the oscillators are uncoupled ($K = 0$), then the phases simply evolve according to the unperturbed frequencies ω_i whose distribution is described by a density $g(\omega)$ characterizing the population. This normalized density is taken to have zero mean ($\overline{\omega} = 0$); this can always be achieved by changing variables, $\theta_i \to \theta_i - \overline{\omega} t$, if necessary. For both physical and mathematical reasons, it is interesting to include in (1) the effect of extrinsic white noise $\xi_i(t)$ perturbing the deterministic phase dynamics; this perturbation is defined by the ensemble averages $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(s)\, \xi_j(t) \rangle = 2D\delta_{ij}\delta(s - t)$.

The form of the coupling function $f(\phi)$ depends on the description of the underlying limit cycles and their mutual interaction, and will vary from one setting to another [2]. Mathematically, since the coupling $f(\phi)$ is necessarily 2π periodic, we describe the general form by its Fourier expansion

$$f(\phi) = \sum_{n=-\infty}^{\infty} f_n\, e^{in\phi}. \tag{2}$$

The simplest nontrivial possibility arises when the coupling is dominated by a single Fourier component, and the theoretical literature is largely focused on the case $f(\phi) = \sin\phi$ since this describes a strictly attractive interaction between oscillators with different phases. Early work by Kuramoto showed that, in the absence of noise [$\xi_i(t) = 0$], there was a critical coupling strength K_c above which a population $g(\omega)$ would begin to show frequency locking and partial ordering in the phases θ_j. This transition was analyzed by an order parameter R defined as the time average of $R(t)$,

$$R(t)\, e^{i\psi(t)} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}. \tag{3}$$

For large N if $K < K_c$, then $R \approx 0$, and for $K > K_c$, R scaled like $R \sim (K - K_c)^{1/2}$. In the limit $N \to \infty$, this numerical result was also obtained analytically from a self-consistent calculation of R. For a population of identical oscillators [$g(\omega) = \delta(\omega)$] in the presence of noise, Kuramoto analyzed the system of stochastic equations for the phases via the resulting Fokker-Planck equation which described the phase distribution. In this theory, the solution with $R = 0$ becomes unstable for $K > K_c$ and a new state with $R \sim (K - K_c)^{1/2}$ emerges. The value for K_c depends on the noise strength D in this case. Subsequently, work has generalized the Fokker-Planck approach to treat the phase dynamics for $f(\phi) = \sin\phi$ in populations with nontrivial frequency distributions $g(\omega)$. These studies also show a bifurcation to phase ordered states with the same scaling $R \sim (K - K_c)^{1/2}$.

For couplings more general than $f(\phi) = \sin\phi$, the properties of (1) are not understood, and this is an interesting subject for several reasons. First, the couplings that are *derived* when a reduction to phase dynamics is actually carried out can easily have a more complicated structure [2,6]. Second, recent results by Daido indicate that as the form of $f(\phi)$ is modified, the nature of the scaling exponent $R \sim (K - K_c)^\beta$ can change from the value 1/2 [7]. Thus different forms of $f(\phi)$ will correspond to different universality classes. Daido specifically considers (1) without noise and applies his "order function" formalism devised as a generalization of Kuramoto's self-consistent calculation of R in the limit $N \to \infty$ [8]. His treatment assumes the transition is triggered by the $f_{\pm 1}$ components of the coupling and analyzes the self-consistent equation perturbatively to leading nonlinear order. The nature of the solution depends on a certain expression "$\mathrm{Im}(f_{-2}\hat{C})$"

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where $\hat{C}$ is a complicated function of the Fourier components of the order function. If this expression vanishes then the solution scales in the usual manner with $\beta = 1/2$, but if this expression is nonzero then the solution scales with $\beta = 1$. Thus a coupling with $f_1 f_2 \neq 0$ is predicted to produce transitions with weaker phase ordering near onset than the transitions associated with the $\sin\phi$ coupling. A third motivation for analyzing the transitions in (1) due to couplings of general form is the unusual character of the bifurcations found in the Fokker-Planck description of these phase-ordering transitions: In the absence of noise, the unstable modes correspond to eigenvalues emerging from a neutral continuum at onset [9–11]. This same feature has been noted in instabilities in other systems such as collisionless plasmas [12,13], ideal shear flows [14–16], solitary waves [17,18], and bubble clouds [19]. It is known that in some of these systems the nonlinear interactions between the unstable modes and the continuum can be singular in the sense that the amplitude equations for the modes become singular as the eigenvalue approaches the continuum [12,13,16]. In these cases the singularities serve to alter the "expected" scaling behavior of the unstable modes. This is known not to occur for (1) when $f(\phi) = \sin\phi$ [11], but Daido's results suggest this conclusion may depend crucially on the form of the coupling.

In this Letter, I analyze the bifurcation from the phase-incoherent state with $R = 0$, in the limit $N \to \infty$, within the framework of the Fokker-Planck equation [9–11,20]

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial \theta} = D \frac{\partial^2 \rho}{\partial \theta^2}. \tag{4}$$

The density $\rho(\theta, \omega, t)$ is defined so that $Ng(\omega) \times \rho(\theta, \omega, t)\, d\theta\, d\omega$ describes the number of oscillators with natural frequencies in $[\omega, \omega + d\omega]$ and phases in $[\theta, \theta + d\theta]$. Thus $\rho(\theta, \omega, t)\, d\theta$ denotes the fraction of oscillators with natural frequency ω and phase in $[\theta, \theta + d\theta]$ and must satisfy the normalization $\int_0^{2\pi} d\theta \times \rho(\theta, \omega, t) = 1$ when $g(\omega) \neq 0$. In the limit $N \to \infty$, the deterministic part of the phase velocity (1) is expressed as an integral over the population

$$v(\theta, \omega, t) = \omega + K \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} d\omega' f(\theta' - \theta) \times \rho(\theta', \omega', t) g(\omega'), \tag{5}$$

and the coupling $f(\phi)$ is described by its Fourier expansion (2).

In this framework, the appearance of collective behavior is described by autonomous equations for the amplitudes of the unstable modes (18). The time-asymptotic solutions of the amplitude equations in turn determine the scaling of any order parameter used to measure the strength of the collective behavior. In this study, the amplitude equation is calculated to third order (19) where the coefficient of the cubic term (20) exhibits a singular behavior as $D \to 0$ that alters the scaling of the time-asymptotic state. At $D = 0$, for the unstable modes previously considered by Daido, this analysis predicts the same shift in scaling when the coupling contains both f_1 and f_2 components, and generalizes his result to the case of an arbitrary coupling with an unstable mode of general form. In addition, the cubic term describes the regularizing effect of the noise.

Equations (4) and (5) provide a continuum description of the oscillator population for which issues of stability and bifurcation can be analyzed in some detail. In this description, Kuramoto's order parameter (3) is given by

$$R(t)\, e^{i\psi(t)} = \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} d\omega' \rho(\theta', \omega', t)\, g(\omega') e^{i\theta'}, \tag{6}$$

and the incoherent state ($R = 0$) is described by the uniform distribution $\rho_0 = 1/2\pi$; this distribution is an equilibrium for (4) since $v(\theta, \omega, t) = \omega + K f_0$ at $\rho = \rho_0$. By setting $\rho(\theta, \omega, t) = \rho_0 + \eta(\theta, \omega, t)$, the dynamics can be rewritten for η,

$$\frac{\partial \eta}{\partial t} = \mathcal{L}\eta + \mathcal{N}(\eta), \tag{7}$$

in terms of the linear operator

$$\mathcal{L}\eta \equiv D \frac{\partial^2 \eta}{\partial \theta^2} - (\omega + K f_0) \frac{\partial \eta}{\partial \theta} + \frac{K}{2\pi} \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} d\omega' f'(\theta' - \theta) \eta(\theta', \omega', t) g(\omega') \tag{8}$$

and nonlinear terms

$$\mathcal{N}(\eta) = K \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} d\omega' \eta(\theta', \omega', t)\, g(\omega') \left[\eta(\theta, \omega, t) f'(\theta' - \theta) - \frac{\partial \eta}{\partial \theta}(\theta, \omega, t) f(\theta' - \theta) \right]. \tag{9}$$

In (8) and (9), $f'(\phi) \equiv df/d\phi$, and note that the normalization of ρ implies $\int_0^{2\pi} d\theta\, \eta(\theta, \omega, t) = 0$.

The linear stability of ρ_0, the onset of linear instability, and the subsequent nonlinear bifurcation have been previously analyzed in detail for the specific case $f(\phi) = \sin\phi$ [9–11]. For an arbitrary coupling $f(\phi)$, the generalization of this stability theory is summarized here as a prerequisite for the bifurcation analysis. The operator $\mathcal{L}$ acts independently on each Fourier subspace $\exp(in\theta)$; consequently, the spectrum can be described by analyzing $\mathcal{L}\Psi = \lambda\Psi$ for functions $\Psi(\theta, \omega) = \psi(\omega) \exp(in\theta)$ with $n > 0$. In general this spectrum has both eigenvalues and a continuous component: For each mode number $n = 1, 2, \ldots$, there is a line of continuous spectrum at $\mathrm{Re}\,\lambda = -n^2 D$; in addition, $\mathcal{L}$ has eigenvalues when the function

$$\Lambda_n(z) \equiv 1 + Kf_n^* \int_{-\infty}^{\infty} d\omega \frac{g(\omega)}{\omega + Kf_0 - z - inD} \tag{10}$$

has roots. More precisely, if for a particular mode number $n = l$, one finds $\Lambda_l(z_0) = 0$, then $\lambda = -ilz_0$ is an eigenvalue of $\mathcal{L}$ and $\Psi(\theta, \omega) = \psi(\omega)\exp(il\theta)$ is the eigenvector for λ with

$$\psi(\omega) = \frac{-Kf_l^*}{\omega + Kf_0 - z_0 - ilD}. \tag{11}$$

The adjoint operator $(\mathcal{L}^\dagger A, B) = (A, \mathcal{L}\, B)$, defined via the inner product $(A, B) \equiv \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega\, A(\theta, \omega)^* B(\theta, \omega)$, has a corresponding eigenfunction $\tilde{\Psi} = \tilde{\psi}(\omega)\exp(il\theta)/2\pi$ where

$$\tilde{\psi}(\omega) = \frac{-g(\omega)}{\Lambda_l'(z_0)^*(\omega + Kf_0 - z_0^* + ilD)} \tag{12}$$

and $\mathcal{L}^\dagger\tilde{\Psi} = \lambda^*\tilde{\Psi}$. The normalization in (12) assumes the root z_0 under consideration is simple; this assumption can be relaxed if necessary but is characteristic of the codimension one bifurcations [11]. This adjoint eigenfunction satisfies $(\tilde{\Psi}, \Psi) = 1$ and defines the projection, $\eta \rightarrow (\tilde{\Psi}, \eta)\, \Psi$, from η onto the Ψ component of η.

The example of a Lorentzian population,

$$g(\omega) = \frac{\epsilon}{\pi}\left[\frac{1}{\omega^2 + \epsilon^2}\right], \tag{13}$$

provides an instructive illustration. For given values of (l, K, ϵ), the solutions to $\Lambda_l(z_0) = 0$ are easily located and one finds that $\mathcal{L}$ has an eigenvalue

$$\lambda = -l[lD + \epsilon + K\,\mathrm{Im}(f_l)] - ilK[\mathrm{Re}(f_l) + f_0] \tag{14}$$

whenever $K\,\mathrm{Im}(f_l) < -\epsilon$. Since $\epsilon > 0$ and $K \geq 0$, the occurrence of these modes requires a coupling such that $\mathrm{Im}(f_l) < 0$. For $f(\phi) = \sin\phi$, this is only possible for $l = 1$, but in general the mode number is not constrained. These modes are linearly stable when $lD + \epsilon + K \times \mathrm{Im}(f_l) > 0$ and become linearly unstable for $K > K_c$ where $K_c = -(lD + \epsilon)/\mathrm{Im}(f_l)$. For $f(\phi) = \sin\phi$ and $l = 1$ this reduces to the familiar result $K_c = 2(\epsilon + D)$ [9].

For $D > 0$, the resulting bifurcation for $K > K_c$ can be analyzed by a center manifold reduction which yields an amplitude equation describing the time-asymptotic behavior of the unstable mode. We introduce this amplitude by writing η in terms of the critical linear modes Ψ and the remaining degrees of freedom S: $\eta(\theta, \omega, t) = [\alpha(t)\Psi(\theta, \omega) + \text{c.c.}] + S(\theta, \omega, t)$ where $(\tilde{\Psi}, S) = 0$. In terms of (α, A) the evolution equation (7) becomes

$$\dot{\alpha} = \lambda\,\alpha + (\tilde{\Psi}, \mathcal{N}(\eta)), \tag{15}$$

$$\frac{\partial S}{\partial t} = \mathcal{L}\,S + \mathcal{N}(\eta) - [(\tilde{\Psi}, \mathcal{N}(\eta))\,\Psi + \text{c.c.}]. \tag{16}$$

The center manifold theorem asserts that any new solutions created by the bifurcation can be found by assuming S is a function of α and α^*,

$$S(\theta, \omega, t) = H(\theta, \omega, \alpha(t), \alpha(t)^*) = \sum_{n=-\infty}^{\infty} H_n(\omega, \alpha(t), \alpha(t)^*)\, e^{in\theta}. \tag{17}$$

For these solutions η^c we have $\eta^c(\theta, \omega, t) = [\alpha(t) \times \Psi(\theta, \omega) + \text{c.c.}] + H(\theta, \omega, \alpha(t), \alpha(t)^*)$, and their dynamics is described by the two-dimensional flow

$$\dot{\alpha} = \lambda\,\alpha + (\tilde{\Psi}, \mathcal{N}(\eta^c)). \tag{18}$$

The center manifold dynamics (18) and the function H in (17) are both constrained by the symmetries of the problem. The group O(2) is generated by rotations $\beta \cdot (\theta, \omega) = (\theta + \beta, \omega)$ and reflections $\kappa \cdot (\theta, \omega) = -(\theta, \omega)$ which act on functions $\eta(\theta, \omega)$ in the usual way: for any transformation $\chi \in \mathrm{O}(2)$, $(\chi \cdot \eta)(\theta, \omega) \equiv \eta[\chi^{-1} \cdot (\theta, \omega)]$. The operators $\mathcal{L}$ and $\mathcal{N}$ commute with rotations for arbitrary choices of $g(\omega)$ and $f(\phi)$; in addition if $g(\omega) = g(-\omega)$ *and* $f(\phi) = -f(-\phi)$, then $\mathcal{L}$ and $\mathcal{N}$ commute with the reflection κ. In the latter circumstance the bifurcation problem has O(2) symmetry, otherwise the rotational symmetry alone corresponds to SO(2).

The Fourier coefficients of H for $n > 0$ are zero unless n is a multiple of l, the mode number of the instability. Rotational symmetry implies that the nonzero coefficients have the form $H_l = \alpha\,\sigma h_l(\omega, \sigma)$ for $n = l$ and $H_n(\omega, \alpha, \alpha^*) = \alpha^m h_n(\omega, \sigma)\,\delta_{n,ml}$ for $m = 2, \ldots$ where $\sigma = |\alpha|^2$ denotes the basic SO(2) invariant. The functions $h_n(\omega, \sigma)$ are unconstrained by the rotations, but must satisfy $h_n(-\omega, \sigma)^* = h_n(\omega, \sigma)$ when reflection symmetry holds. Similarly, rotational symmetry implies the amplitude equation (18) must have the general form $\dot{\alpha} = p(\sigma)\alpha$ where $p(\sigma)$ is a real-valued function if the reflection symmetry holds; otherwise $p(\sigma)$ is generically complex valued.

For K near K_c, we expand $p(\sigma) = p_0 + p_1\sigma + \cdots$ and $h_n(\omega, \sigma) = h_{n,0}(\omega) + h_{n,1}(\omega)\sigma + \ldots$ and seek the leading (and presumably dominant) nonlinear terms in the amplitude equation (18)

$$\dot{\alpha} = \alpha[\lambda + p_1|\alpha|^2 + \cdots]. \tag{19}$$

The calculation of p_1 from (18) yields

$$p_1 = -2\pi iKl[f_l\langle\tilde{\psi}, h_{2l,0}\rangle + f_{2l}^*\langle\tilde{\psi}, \psi^*\rangle\langle g, h_{2l,0}\rangle], \tag{20}$$

where the brackets denote an integration over ω, $\langle A, B\rangle \equiv \int_{-\infty}^{\infty} d\omega\, A^* B$, and the function $h_{2l,0}$ is determined self-consistently to be [21]

$$h_{2l,0}(\omega) = \frac{-4\pi lKf_l^*}{2l(\omega + Kf_0 - z_0 - i2lD)} \times \left[\psi(\omega) - \frac{iKf_{2l}^*}{2l\Lambda_{2l}(z_0)} \times \int_{-\infty}^{\infty} d\omega' \frac{g(\omega')\psi(\omega')}{\omega' + Kf_0 - z_0 - i2lD}\right]. \tag{21}$$

For the special case $f(\phi) = \sin\phi$, the instability arises for $l = 1$ and the f_{2l} component is zero. Then the second terms in (20) and (21) are absent and the results of Ref. [11] are recovered.

When $f_{2l} \neq 0$, the new terms change the appearance of the bifurcation significantly due to the factor $\langle \tilde{\psi}, \psi^* \rangle$ in (20). From (11) and (12) one sees that the integrand has poles $\omega_\pm = (\Omega/l - Kf_0) \pm i(l^2D + \gamma)/l$ above and below the contour along the real axis; here $z_0 = (\Omega + i\gamma)/l$ is the root and γ the linear growth rate from $\lambda = -ilz_0$. The $\gamma \to 0^+$ limit of $\langle \tilde{\psi}, \psi^* \rangle$ in p_1 produces a pinching singularity when D is small or zero, and this singularity contributes a factor $(\gamma + l^2D)$ to the denominator when the integral is evaluated,

$$\langle \tilde{\psi}, \psi^* \rangle = -\frac{l f_l \,\mathrm{Im}(f_l)}{(\gamma + l^2D)\,|f_l|^2\,\Lambda_l'(z_0)}\,. \tag{22}$$

The remaining integrals $\langle \tilde{\psi}, h_{2l,0} \rangle$ and $\langle g, h_{2l,0} \rangle$ in (20) are well behaved as $\gamma \to 0^+$ since all poles lie in the same half plane.

The effect of the singularity $(\gamma + l^2D)$ in (22) is clarified by scaling the amplitude

$$\alpha(t) \equiv \sqrt{\gamma(\gamma + l^2D)}\, r(\gamma t) e^{-i\varphi(t)} \tag{23}$$

so that the equations for $r(\tau)$ and $\varphi(t)$ from (19) are nonsingular,

$$\frac{dr}{d\tau} = r(\tau)[1 + \mathrm{Re}(p_1)(\gamma + l^2D)r^2 + \cdots], \tag{24}$$

$$\frac{d\varphi}{dt} = \Omega - \mathrm{Im}(p_1)\,\gamma(\gamma + l^2D)r^2 + \cdots. \tag{25}$$

Here $\tau \equiv \gamma t$ is the slow time scale determined by the linear instability, and the coefficients in (24) and (25) are now finite as $\gamma \to 0^+$ even when $D = 0$. Assuming that as $t \to \infty$ the instability saturates with the mode amplitude tending to a nonzero limit $r(\tau) \to r_\infty$, then the magnitude of this mode $|\alpha_\infty| = \sqrt{\gamma(\gamma + l^2D)}\, r_\infty$ determines the scaling exhibited by the entrained state. For $\gamma + l^2D$ sufficiently small, the amplitude equation (24) becomes independent of γ and D and the scaling behavior of the entrained state follows the explicit dependence shown in (23). For $D > 0$, there is a crossover from $|\alpha_\infty| \sim \gamma$ for $\gamma > l^2D$ to $|\alpha_\infty| \sim \sqrt{\gamma}$ for $\gamma < l^2D$. In the noise-free limit, this crossover does not occur and the $|\alpha_\infty| \sim \gamma$ scaling persists as the true asymptotic behavior. Since $\gamma \sim K - K_c$ near onset, these results determine the scaling exponent $|\alpha_\infty| \sim (K - K_c)^\beta$. In the special case $D = 0$ and $l = 1$, these conclusions support Daido's findings: if $f_2 = 0$, then the singularity is suppressed and the $\beta = 1/2$ scaling occurs; if $f_2 \neq 0$, then the $l = 1$ instability leads to an entrained state characterized by $\beta = 1$.

Several important and related questions remain: When $f_{2l} \neq 0$ and the cubic singularity occurs, what are the singularities of the higher order terms in the amplitude equation? Does the amplitude scaling in (23) suffice to control the higher order singularities if they occur? Finally, when $f_{2l} = 0$ are there higher order singularities that alter β from the value 1/2 predicted by the cubic analysis? These issues will be addressed in a subsequent paper [22].

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