Imãs girando

14 de outubro de 2022

1 Equações do movimento do caso exato

O campo magnético gerado por um momento magnético $\mathbf{m}$ girando com velocidade angular ω no plano xy é, sendo a posição do imã livre, $\mathbf{r}(r,\theta,\phi)$, e a orientação de seu momento magnético, $\mathbf{m}'(\theta',\phi',\psi')$,

$$\mathbf{m} = m(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t),\ \mathbf{B} = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left[3\sin\theta\cos(\phi-\omega t)\begin{pmatrix}\hat{\mathbf{x}}\sin\theta\cos\phi\\ +\hat{\mathbf{y}}\sin\theta\sin\phi\\ +\hat{\mathbf{z}}\cos\theta\end{pmatrix} - \hat{\mathbf{x}}\cos\omega t - \hat{\mathbf{y}}\sin\omega t\right]$$

$$U = -\mathbf{m}'\cdot\mathbf{B} - zMg,\quad \mathbf{m}' = m'(-\hat{\mathbf{x}}\sin\theta'\sin\phi' + \hat{\mathbf{y}}\sin\theta'\cos\phi' + \hat{\mathbf{z}}\cos\theta')$$

$$U = -\frac{\mu_0 m m'}{4\pi r^3}\begin{bmatrix}-3\sin^2\theta\cos(\phi-\omega t)\cos\phi\sin\theta'\sin\phi' + \cos\omega t\sin\theta'\sin\phi'\\ +3\sin^2\theta\cos(\phi-\omega t)\sin\phi\sin\theta'\cos\phi' - \sin\omega t\sin\theta'\cos\phi'\\ +3\sin\theta\cos\theta\cos(\phi-\omega t)\cos\theta'\end{bmatrix} - r\cos\theta Mg$$

$$U = -\frac{\mu_0 m m'}{4\pi r^3}\begin{bmatrix}3\sin^2\theta\sin\theta'\cos(\phi-\omega t)\sin(\phi-\phi') + \sin\theta'\sin(\phi'-\omega t)\\ +3\sin\theta\cos\theta\cos(\phi-\omega t)\cos\theta'\end{bmatrix} - r\cos\theta Mg$$

A Lagrangiana é,

$$L = \frac{M}{2}\begin{bmatrix}\dot{r}^2\\ +r^2\sin^2\theta\dot{\phi}^2\\ +r^2\dot{\theta}^2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}I_1\left(\dot{\phi}'\sin\theta'\sin\psi' + \dot{\theta}'\cos\psi'\right)^2\\ +I_1\left(\dot{\phi}'\sin\theta'\cos\psi' - \dot{\theta}'\sin\psi'\right)^2\\ +I_3\left(\dot{\psi}' + \dot{\phi}'\cos\theta'\right)^2\end{bmatrix} - U$$

1.1 Equações do movimento para ψ'

$$
\begin{aligned}
&\frac{\partial}{\partial \psi'} L = 0 \\
&\frac{\partial}{\partial \dot{\psi}'} L = I_3 \Big(\dot{\psi}' + \dot{\phi}' \cos\theta \Big) = p_{\psi'} \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0
\end{aligned}
\tag{1}
$$

1.2 Equações do movimento para ϕ'

$$
\begin{aligned}
\frac{\partial}{\partial \phi'} L &= -\frac{\mu_0 m m' \sin\theta'}{4\pi r^3} \big[3\sin^2\theta \cos(\phi - \omega t)\cos(\phi - \phi') - \cos(\phi' - \omega t) \big] \\
\frac{\partial}{\partial \dot{\phi}'} L &= I_1 \dot{\phi}' \sin^2\theta' + p_{\psi'} \cos\theta' \\
I_1 \ddot{\phi}' \sin^2\theta' + 2 I_1 \dot{\phi}' \dot{\theta}' \sin\theta' \cos\theta' - p_{\psi'} \sin\theta' \dot{\theta}' &= \\
- \frac{\mu_0 m m' \sin\theta'}{4\pi r^3} &\big[3\sin^2\theta \cos(\phi - \omega t)\cos(\phi - \phi') - \cos(\phi' - \omega t) \big]
\end{aligned}
\tag{2}
$$

1.3 Equações do movimento para θ'

$$
\begin{aligned}
\frac{\partial}{\partial \theta'} L &= \frac{\mu_0 m m'}{4\pi r^3} \left[\begin{array}{c} 3\sin^2\theta \cos\theta' \cos(\phi - \omega t) \sin(\phi - \phi') + \cos\theta' \sin(\phi' - \omega t) \\ -3 \sin\theta \cos\theta \sin\theta' \cos(\phi - \omega t) \end{array} \right] \\
&\quad + I_1 {\dot{\phi}'}^2 \sin^2\theta' - \dot{\phi}' \sin\theta' p_{\psi'} \\
\frac{\partial}{\partial \dot{\theta}'} L &= I_1 \dot{\theta}' \\
I_1 \ddot{\theta}' &= \frac{\partial}{\partial \psi'} L
\end{aligned}
\tag{3}
$$

1.4 Equações do movimento para r

$$
\begin{aligned}
\frac{\partial}{\partial r} L &= M r \dot{\theta}^2 + M r \sin^2\theta \dot{\phi}^2 + M g \cos\theta - 3 \frac{\mu_0 m m'}{4\pi r^4} [\cdots] \\
\frac{\partial}{\partial \dot{r}} L &= M \dot{r} \\
M \ddot{r} &= \frac{\partial}{\partial r} L
\end{aligned}
\tag{4}
$$

1.5 Equações do movimento para ϕ

$$\frac{\partial}{\partial \phi} L = \frac{3\mu_0 m m' \sin\theta}{4\pi r^3}[\sin\theta \sin\theta' \cos(2\phi - \omega t - \phi') - \cos\theta \cos\theta' \sin(\phi - \omega t)]$$
$$\frac{\partial}{\partial \dot{\phi}} L = M r^2 \sin^2\theta \dot{\phi}$$

1.6 Equações do movimento para θ

$$\frac{\partial}{\partial \theta} L = \frac{3\mu_0 m m' \cos(\phi - \omega t)}{4\pi r^3}[\sin(2\theta) \sin\theta' \sin(\phi - \phi') + \cos(2\theta) \cos\theta']$$
$$- r \sin\theta M g + M r^2 \sin\theta \cos\theta \dot{\phi}^2$$
$$\frac{\partial}{\partial \dot{\theta}} L = M r^2 \dot{\theta}$$

2 Primeiras aproximações

Para eliminar a dependência temporal explícita supomos que,

$$\phi = \omega t + \alpha \tag{5}$$
$$\phi' = \omega t + \alpha' \tag{6}$$

2.1 Equações do movimento para ψ'

$$\frac{\partial}{\partial \psi'} L = 0$$
$$\frac{\partial}{\partial \dot{\psi}'} L = I_3\left(\dot{\psi}' + \omega \cos\theta\right) = p_{\psi'}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0 \tag{7}$$

2.2 Equações do movimento para ϕ'

$$
\begin{aligned}
\frac{\partial}{\partial \phi'} L &= -\frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\big[3\sin^2\theta\cos(\alpha)\cos(\alpha-\alpha') - \cos(\alpha')\big] \\
\frac{\partial}{\partial \dot{\phi}'} L &= I_1\omega\sin^2\theta' + p_{\psi'}\cos\theta'
\end{aligned}
$$

$$
\begin{aligned}
2I_1\omega\dot{\theta}'\sin\theta'\cos\theta' - p_{\psi'}\sin\theta'\dot{\theta}' = \\
-\frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\big[3\sin^2\theta\cos(\alpha)\cos(\alpha-\alpha') - \cos(\alpha')\big]
\end{aligned} \tag{8}
$$

2.3 Equações do movimento para θ'

$$
\begin{aligned}
\frac{\partial}{\partial \theta'} L &= \frac{\mu_0 m m'}{4\pi r^3}\left[\begin{array}{c} 3\sin^2\theta\cos\theta'\cos(\alpha)\sin(\alpha-\alpha') + \cos\theta'\sin(\alpha') \\ -3\sin\theta\cos\theta\sin\theta'\cos(\alpha) \end{array}\right] \\
&+ I_1\omega^2\sin^2\theta' - \omega\sin\theta' p_{\psi'} \\
\frac{\partial}{\partial \dot{\theta}'} L &= I_1\dot{\theta}' \\
I_1\ddot{\theta}' &= \frac{\partial}{\partial \theta'} L
\end{aligned} \tag{9}
$$

2.4 Equações do movimento para r

$$
\begin{aligned}
\frac{\partial}{\partial r} L &= Mr\dot{\theta}^2 + Mr\sin^2\theta\omega^2 + Mg\cos\theta - 3\frac{\mu_0 m m'}{4\pi r^4}[\cdots] \\
\frac{\partial}{\partial \dot{r}} L &= M\dot{r} \\
M\ddot{r} &= \frac{\partial}{\partial r} L
\end{aligned} \tag{10}
$$

2.5 Equações do movimento para ϕ

$$
\begin{aligned}
\frac{\partial}{\partial \phi} L &= \frac{3\mu_0 m m' \sin\theta}{4\pi r^3}[\sin\theta\sin\theta'\cos(2\alpha-\alpha') - \cos\theta\cos\theta'\sin(\alpha)] \\
\frac{\partial}{\partial \dot{\phi}} L &= Mr^2\sin^2\theta\omega
\end{aligned}
$$

2.6 Equações do movimento para θ

$$\begin{aligned}
\frac{\partial}{\partial\theta}L &= \frac{3\mu_0 mm'\cos(\alpha)}{4\pi r^3}[\sin(2\theta)\sin\theta'\sin(\alpha-\alpha')+\cos(2\theta)\cos\theta'] \\
&\quad - r\sin\theta Mg + Mr^2\sin\theta\cos\theta\omega^2 \\
\frac{\partial}{\partial\dot{\theta}}L &= Mr^2\dot{\theta}
\end{aligned}$$

3 Aproximações seguintes

Supomos que o segundo imã oscila quase exatamente abaixo do imã acima deste, isto é,

$$\sin\theta \approx \theta \tag{11}$$

3.1 Equações do movimento para ψ'

$$\begin{aligned}
\frac{\partial}{\partial\psi'}L &= 0 \\
\frac{\partial}{\partial\dot{\psi}'}L &= I_3\left(\dot{\psi}'+\omega\right) = p_{\psi'} \\
\frac{\mathrm{d}}{\mathrm{d}t}p_{\psi'} &= 0
\end{aligned} \tag{12}$$

3.2 Equações do movimento para ϕ'

$$\begin{aligned}
\frac{\partial}{\partial\phi'}L &= -\frac{\mu_0 mm'\sin\theta'}{4\pi r^3}\left[3\theta^2\cos(\alpha)\cos(\alpha-\alpha')-\cos(\alpha')\right] \\
\frac{\partial}{\partial\dot{\phi}'}L &= I_1\omega\sin^2\theta' + p_{\psi'}\cos\theta'
\end{aligned}$$

$$\begin{aligned}
2I_1\omega\dot{\theta}'\sin\theta'\cos\theta' - p_{\psi'}\sin\theta'\dot{\theta}' &= \\
&- \frac{\mu_0 mm'\sin\theta'}{4\pi r^3}\left[3\theta^2\cos(\alpha)\cos(\alpha-\alpha')-\cos(\alpha')\right]
\end{aligned} \tag{13}$$

3.3 Equações do movimento para θ'

$$\begin{aligned}
\frac{\partial}{\partial\theta'}L &= \frac{\mu_0 m m'}{4\pi r^3}\begin{bmatrix} 3\theta^2\cos\theta'\cos(\alpha)\sin(\alpha-\alpha') + \cos\theta'\sin(\alpha') \\ -3\theta\sin\theta'\cos(\alpha)\end{bmatrix} \\
&\quad + I_1\omega^2\sin^2\theta' - \omega\sin\theta' p_{\psi'} \\
\frac{\partial}{\partial\dot{\theta}'}L &= I_1\dot{\theta}' \\
I_1\ddot{\theta}' &= \frac{\partial}{\partial\theta'}L
\end{aligned} \tag{14}$$

3.4 Equações do movimento para r

$$\begin{aligned}
\frac{\partial}{\partial r}L &= Mr\dot{\theta}^2 + Mr\theta^2\omega^2 + Mg - 3\frac{\mu_0 m m'}{4\pi r^4}[\cdots] \\
\frac{\partial}{\partial\dot{r}}L &= M\dot{r} \\
M\ddot{r} &= \frac{\partial}{\partial r}L
\end{aligned} \tag{15}$$

3.5 Equações do movimento para ϕ

$$\begin{aligned}
\frac{\partial}{\partial\phi}L &= \frac{3\mu_0 m m'\theta}{4\pi r^3}[\theta\sin\theta'\cos(2\alpha-\alpha') - \cos\theta'\sin(\alpha)] \\
\frac{\partial}{\partial\dot{\phi}}L &= Mr^2\theta^2\omega
\end{aligned}$$

3.6 Equações do movimento para θ

$$\begin{aligned}
\frac{\partial}{\partial\theta}L &= \frac{3\mu_0 m m'\cos(\alpha)}{4\pi r^3}[2\theta\sin\theta'\sin(\alpha-\alpha') + \cos\theta'] \\
&\quad - r\theta Mg + Mr^2\theta\omega^2 \\
\frac{\partial}{\partial\dot{\theta}}L &= Mr^2\dot{\theta}
\end{aligned}$$

4 Aproximações seguintes

Supomos que o segundo imã oscila quase exatamente em uma posição com valor $r = a$, isto é,

$$r = a(1+\epsilon), \quad \epsilon \ll 1 \tag{16}$$

4.1 Equações do movimento para ψ'

$$\begin{aligned}
&\frac{\partial}{\partial \psi'} L = 0 \\
&\frac{\partial}{\partial \dot{\psi}'} L = I_3\left(\dot{\psi}' + \omega\right) = p_{\psi'} \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0
\end{aligned} \tag{17}$$

4.2 Equações do movimento para ϕ'

$$\begin{aligned}
&\frac{\partial}{\partial \phi'} L = -\frac{3\mu_0 m m' \sin\theta'}{4\pi a^3}\left[\theta^2 \cos(\alpha)\cos(\alpha-\alpha') + \epsilon\cos(\alpha') - \frac{1}{3}\cos(\alpha')\right] \\
&\frac{\partial}{\partial \dot{\phi}'} L = I_1 \omega \sin^2\theta' + p_{\psi'} \cos\theta' \\
&2I_1\omega\dot{\theta}'\sin\theta'\cos\theta' - p_{\psi'}\sin\theta'\dot{\theta}' = \\
&\qquad -\frac{3\mu_0 m m' \sin\theta'}{4\pi a^3}\left[\theta^2 \cos(\alpha)\cos(\alpha-\alpha') + \epsilon\cos(\alpha') - \frac{1}{3}\cos(\alpha')\right]
\end{aligned} \tag{18}$$

4.3 Equações do movimento para θ'

$$\begin{aligned}
\frac{\partial}{\partial \theta'} L &= \frac{3\mu_0 m m'}{4\pi a^3}\begin{bmatrix} \theta^2\cos\theta'\cos(\alpha)\sin(\alpha-\alpha') - \frac{1}{3}\cos\theta'\sin(\alpha') \\ -\epsilon\cos\theta'\sin(\alpha') - \theta\sin\theta'\cos(\alpha) \end{bmatrix} \\
&\quad + I_1\omega^2\sin^2\theta' - \omega\sin\theta' p_{\psi'} \\
\frac{\partial}{\partial \dot{\theta}'} L &= I_1\dot{\theta}' \\
I_1\ddot{\theta}' &= \frac{\partial}{\partial \theta'} L
\end{aligned} \tag{19}$$

4.4 Equações do movimento para r

$$
\begin{aligned}
&\frac{\partial}{\partial r}L = Ma\dot{\theta}^2 + Ma\theta^2\omega^2 + Mg - 3\frac{\mu_0 mm'}{4\pi a^4}(1-4\epsilon)[\cdots] \\
&\frac{\partial}{\partial \dot{r}}L = Ma\dot{\epsilon} \\
&Ma\ddot{\epsilon} = \frac{\partial}{\partial r}L
\end{aligned} \tag{20}
$$

4.5 Equações do movimento para ϕ

$$
\begin{aligned}
&\frac{\partial}{\partial \phi}L = \frac{3\mu_0 mm'\theta}{4\pi a^3}\left[\begin{matrix}\theta\sin\theta'\cos(2\alpha-\alpha') - \cos\theta'\sin(\alpha) \\ +3\epsilon\cos\theta'\sin(\alpha)\end{matrix}\right] \\
&\frac{\partial}{\partial \dot{\phi}}L = Ma^2\theta^2\omega
\end{aligned}
$$

4.6 Equações do movimento para θ

$$
\begin{aligned}
\frac{\partial}{\partial \theta}L = &\frac{3\mu_0 mm'\cos(\alpha)}{4\pi a^3}(1-3\epsilon)[2\theta\sin\theta'\sin(\alpha-\alpha') + \cos\theta'] \\
&- a(1+\epsilon)\theta Mg + Ma^2(1+2\epsilon)\theta\omega^2 \\
\frac{\partial}{\partial \dot{\theta}}L = &\, Ma^2(1+2\epsilon)\dot{\theta}
\end{aligned}
$$

5 Aproximações seguintes

Supomos que o segundo imã oscila com $\theta' \ll 1$, isto é,

$$\sin\theta' \approx \theta' \tag{21}$$

5.1 Equações do movimento para ψ'

$$\begin{aligned}
&\frac{\partial}{\partial \psi'} L = 0 \\
&\frac{\partial}{\partial \dot{\psi}'} L = I_3\left(\dot{\psi}' + \omega\right) = p_{\psi'} \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0
\end{aligned} \tag{22}$$

5.2 Equações do movimento para ϕ'

$$\begin{aligned}
\frac{\partial}{\partial \phi'} L &= \frac{\mu_0 m m' \theta' \cos\alpha'}{4\pi a^3}[1 - 3\epsilon] \\
\frac{\partial}{\partial \dot{\phi}'} L &= I_1 \omega \sin^2\theta' + p_{\psi'} \cos\theta' \\
2 I_1 \omega \dot{\theta}' \theta' - p_{\psi'} \theta' \dot{\theta}' &= \frac{\mu_0 m m' \theta' \cos\alpha'}{4\pi a^3}[1 - 3\epsilon]
\end{aligned} \tag{23}$$

5.3 Equações do movimento para θ'

$$\begin{aligned}
\frac{\partial}{\partial \theta'} L &= \frac{3\mu_0 m m'}{4\pi a^3} \begin{bmatrix} \theta^2 \cos(\alpha) \sin(\alpha - \alpha') - \frac{1}{3}\sin(\alpha') \\ -\epsilon \sin(\alpha') - \theta\theta' \cos(\alpha) \end{bmatrix} \\
&\quad + I_1 \omega^2 {\theta'}^2 - \omega \theta' p_{\psi'} \\
\frac{\partial}{\partial \dot{\theta}'} L &= I_1 \dot{\theta}' \\
I_1 \ddot{\theta}' &= \frac{\partial}{\partial \theta'} L
\end{aligned} \tag{24}$$

5.4 Equações do movimento para r

$$\begin{aligned}
&\frac{\partial}{\partial r} L = M a \dot{\theta}^2 + M a \theta^2 \omega^2 + M g - 3\frac{\mu_0 m m'}{4\pi a^4}(1 - 4\epsilon)[\theta' \sin(\alpha') + 3\theta \cos\alpha] \\
&\frac{\partial}{\partial \dot{r}} L = M a \dot{\epsilon} \\
&M a \ddot{\epsilon} = \frac{\partial}{\partial r} L
\end{aligned} \tag{25}$$

5.5 Equações do movimento para ϕ

$$\begin{aligned}
\frac{\partial}{\partial\phi}L &= \frac{3\mu_0 m m'\theta\sin\alpha}{4\pi a^3}[3\epsilon - 1]\\
\frac{\partial}{\partial\dot{\phi}}L &= Ma^2\theta^2\omega
\end{aligned}$$

5.6 Equações do movimento para θ

$$\begin{aligned}
\frac{\partial}{\partial\theta}L = &\frac{3\mu_0 m m'\cos(\alpha)}{4\pi a^3}[2\theta\theta'\sin(\alpha-\alpha') + \cos\theta' - 3\epsilon\cos\theta']\\
&- a(1+\epsilon)\theta Mg + Ma^2(1+2\epsilon)\theta\omega^2\\
\frac{\partial}{\partial\dot{\theta}}L = &Ma^2(1+2\epsilon)\dot{\theta}
\end{aligned}$$