Imãs girando

14 de outubro de 2022

1 Equações do movimento do caso exato

O campo magnético gerado por um momento magnético $\mathbf{m}$ girando com velocidade angular ω no plano xy é, sendo a posição do imã livre, $\mathbf{r}(r,\theta,\phi)$, e a orientação de seu momento magnético, $\mathbf{m}'(\theta',\phi',\psi')$,

$$\mathbf{m} = m(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t),\ \ \mathbf{B} = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left[3\sin\theta\cos(\phi-\omega t)\begin{pmatrix}\hat{\mathbf{x}}\sin\theta\cos\phi \\ +\hat{\mathbf{y}}\sin\theta\sin\phi \\ +\hat{\mathbf{z}}\cos\theta\end{pmatrix} - \hat{\mathbf{x}}\cos\omega t - \hat{\mathbf{y}}\sin\omega t\right]$$

$$U = -\mathbf{m}'\cdot\mathbf{B} - zMg,\quad \mathbf{m}' = m'(-\hat{\mathbf{x}}\sin\theta'\sin\phi' + \hat{\mathbf{y}}\sin\theta'\cos\phi' + \hat{\mathbf{z}}\cos\theta')$$

$$U = -\frac{\mu_0 m m'}{4\pi r^3}\begin{bmatrix}-3\sin^2\theta\cos(\phi-\omega t)\cos\phi\sin\theta'\sin\phi' + \cos\omega t\sin\theta'\sin\phi' \\ +3\sin^2\theta\cos(\phi-\omega t)\sin\phi\sin\theta'\cos\phi' - \sin\omega t\sin\theta'\cos\phi' \\ +3\sin\theta\cos\theta\cos(\phi-\omega t)\cos\theta'\end{bmatrix} - r\cos\theta Mg$$

$$U = -\frac{\mu_0 m m'}{4\pi r^3}\begin{bmatrix}3\sin^2\theta\sin\theta'\cos(\phi-\omega t)\sin(\phi-\phi') + \sin\theta'\sin(\phi'-\omega t) \\ +3\sin\theta\cos\theta\cos(\phi-\omega t)\cos\theta'\end{bmatrix} - r\cos\theta Mg$$

A Lagrangiana é,

$$L = \frac{M}{2}\begin{bmatrix}\dot{r}^2 \\ +r^2\sin^2\theta\dot{\phi}^2 \\ +r^2\dot{\theta}^2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}I_1\left(\dot{\phi}'\sin\theta'\sin\psi' + \dot{\theta}'\cos\psi'\right)^2 \\ +I_1\left(\dot{\phi}'\sin\theta'\cos\psi' - \dot{\theta}'\sin\psi'\right)^2 \\ +I_3\left(\dot{\psi}' + \dot{\phi}'\cos\theta'\right)^2\end{bmatrix} - U$$

1.1 Equações do movimento para ψ'

$$\begin{aligned}
&\frac{\partial}{\partial \psi'} L = 0 \\
&\frac{\partial}{\partial \dot{\psi}'} L = I_3\left(\dot{\psi}' + \dot{\phi}' \cos\theta\right) = p_{\psi'} \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0
\end{aligned} \tag{1}$$

1.2 Equações do movimento para ϕ'

$$\begin{aligned}
&\frac{\partial}{\partial \phi'} L = -\frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\left[3\sin^2\theta\cos(\phi-\omega t)\cos(\phi-\phi') - \cos(\phi'-\omega t)\right] \\
&\frac{\partial}{\partial \dot{\phi}'} L = I_1 \dot{\phi}' \sin^2\theta' + p_{\psi'}\cos\theta' \\
I_1\ddot{\phi}'\sin^2\theta' + 2I_1\dot{\phi}'\dot{\theta}'\sin\theta'\cos\theta' - p_{\psi'}\sin\theta'\dot{\theta}' = \\
&- \frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\left[3\sin^2\theta\cos(\phi-\omega t)\cos(\phi-\phi') - \cos(\phi'-\omega t)\right]
\end{aligned} \tag{2}$$

1.3 Equações do movimento para θ'

$$\begin{aligned}
\frac{\partial}{\partial \theta'} L = \frac{\mu_0 m m'}{4\pi r^3}&\left[\begin{matrix} 3\sin^2\theta\cos\theta'\cos(\phi-\omega t)\sin(\phi-\phi') + \cos\theta'\sin(\phi'-\omega t) \\ -3\sin\theta\cos\theta\sin\theta'\cos(\phi-\omega t)\end{matrix}\right] \\
&+ I_1 {\dot{\phi}'}^2 \sin^2\theta' - \dot{\phi}'\sin\theta' p_{\psi'} \\
\frac{\partial}{\partial \dot{\theta}'} L = I_1 \dot{\theta}'& \\
I_1\ddot{\theta}' = \frac{\partial}{\partial \psi'} L&
\end{aligned} \tag{3}$$

1.4 Equações do movimento para r

$$\begin{aligned}
&\frac{\partial}{\partial r} L = M r \dot{\theta}^2 + M r \sin^2\theta \dot{\phi}^2 + M g \cos\theta - 3\frac{\mu_0 m m'}{4\pi r^4}[\cdots] \\
&\frac{\partial}{\partial \dot{r}} L = M\dot{r} \\
&M\ddot{r} = \frac{\partial}{\partial r} L
\end{aligned} \tag{4}$$

1.5 Equações do movimento para ϕ

$$\frac{\partial}{\partial \phi}L = \frac{3\mu_0 m m' \sin\theta}{4\pi r^3}[\sin\theta\sin\theta'\cos(2\phi-\omega t-\phi') - \cos\theta\cos\theta'\sin(\phi-\omega t)]$$
$$\frac{\partial}{\partial \dot\phi}L = Mr^2\sin^2\theta\dot\phi$$

1.6 Equações do movimento para θ

$$\begin{aligned}\frac{\partial}{\partial \theta}L &= \frac{3\mu_0 m m' \cos(\phi-\omega t)}{4\pi r^3}[\sin(2\theta)\sin\theta'\sin(\phi-\phi') + \cos(2\theta)\cos\theta']\\ &\quad - r\sin\theta Mg + Mr^2\sin\theta\cos\theta\dot\phi^2\\ \frac{\partial}{\partial \dot\theta}L &= Mr^2\dot\theta\end{aligned}$$

2 Primeiras aproximações

Para eliminar a dependência temporal explícita supomos que,

$$\phi = \omega t + \alpha \tag{5}$$
$$\phi' = \omega t + \alpha' \tag{6}$$

2.1 Equações do movimento para ψ'

$$\begin{aligned}&\frac{\partial}{\partial \psi'}L = 0\\ &\frac{\partial}{\partial \dot\psi'}L = I_3\left(\dot\psi' + \omega\cos\theta\right) = p_{\psi'}\\ &\frac{\mathrm{d}}{\mathrm{d}t}p_{\psi'} = 0\end{aligned} \tag{7}$$

2.2 Equações do movimento para ϕ'

$$
\begin{aligned}
\frac{\partial}{\partial\phi'}L &= -\frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\left[3\sin^2\theta\cos(\alpha)\cos(\alpha-\alpha') - \cos(\alpha')\right] \\
\frac{\partial}{\partial\dot{\phi}'}L &= I_1\omega\sin^2\theta' + p_{\psi'}\cos\theta' \\
2I_1\omega\dot{\theta}'\sin\theta'\cos\theta' - p_{\psi'}\sin\theta'\dot{\theta}' &= \\
&- \frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\left[3\sin^2\theta\cos(\alpha)\cos(\alpha-\alpha') - \cos(\alpha')\right]
\end{aligned} \tag{8}
$$

2.3 Equações do movimento para θ'

$$
\begin{aligned}
\frac{\partial}{\partial\theta'}L &= \frac{\mu_0 m m'}{4\pi r^3}\left[\begin{matrix} 3\sin^2\theta\cos\theta'\cos(\alpha)\sin(\alpha-\alpha') + \cos\theta'\sin(\alpha') \\ -3\sin\theta\cos\theta\sin\theta'\cos(\alpha)\end{matrix}\right] \\
&+ I_1\omega^2\sin^2\theta' - \omega\sin\theta' p_{\psi'} \\
\frac{\partial}{\partial\dot{\theta}'}L &= I_1\dot{\theta}' \\
I_1\ddot{\theta}' &= \frac{\partial}{\partial\theta'}L
\end{aligned} \tag{9}
$$

2.4 Equações do movimento para r

$$
\begin{aligned}
\frac{\partial}{\partial r}L &= Mr\dot{\theta}^2 + Mr\sin^2\theta\omega^2 + Mg\cos\theta - 3\frac{\mu_0 m m'}{4\pi r^4}[\cdots] \\
\frac{\partial}{\partial\dot{r}}L &= M\dot{r} \\
M\ddot{r} &= \frac{\partial}{\partial r}L
\end{aligned} \tag{10}
$$

2.5 Equações do movimento para ϕ

$$
\begin{aligned}
\frac{\partial}{\partial\phi}L &= \frac{3\mu_0 m m' \sin\theta}{4\pi r^3}[\sin\theta\sin\theta'\cos(2\alpha-\alpha') - \cos\theta\cos\theta'\sin(\alpha)] \\
\frac{\partial}{\partial\dot{\phi}}L &= Mr^2\sin^2\theta\omega
\end{aligned}
$$

2.6 Equações do movimento para θ

$$\begin{aligned}
\frac{\partial}{\partial \theta} L &= \frac{3\mu_0 m m' \cos(\alpha)}{4\pi r^3}[\sin(2\theta)\sin\theta'\sin(\alpha-\alpha') + \cos(2\theta)\cos\theta'] \\
&\quad - r\sin\theta M g + M r^2 \sin\theta\cos\theta\omega^2 \\
\frac{\partial}{\partial \dot{\theta}} L &= M r^2 \dot{\theta}
\end{aligned}$$

3 Aproximações seguintes

Supomos que o segundo imã oscila quase exatamente abaixo do imã acima deste, isto é,

$$\sin\theta \approx \theta \tag{11}$$

3.1 Equações do movimento para ψ'

$$\begin{aligned}
&\frac{\partial}{\partial \psi'} L = 0 \\
&\frac{\partial}{\partial \dot{\psi}'} L = I_3\left(\dot{\psi}' + \omega\right) = p_{\psi'} \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0
\end{aligned} \tag{12}$$

3.2 Equações do movimento para ϕ'

$$\begin{aligned}
\frac{\partial}{\partial \phi'} L &= -\frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\left[3\theta^2\cos(\alpha)\cos(\alpha-\alpha') - \cos(\alpha')\right] \\
\frac{\partial}{\partial \dot{\phi}'} L &= I_1 \omega \sin^2\theta' + p_{\psi'}\cos\theta'
\end{aligned}$$

$$\begin{aligned}
2I_1\omega\dot{\theta}'\sin\theta'\cos\theta' - p_{\psi'}\sin\theta'\dot{\theta}' = \\
-\frac{\mu_0 m m' \sin\theta'}{4\pi r^3}\left[3\theta^2\cos(\alpha)\cos(\alpha-\alpha') - \cos(\alpha')\right]
\end{aligned} \tag{13}$$

3.3 Equações do movimento para θ'

$$\frac{\partial}{\partial\theta'}L = \frac{\mu_0 m m'}{4\pi r^3}\begin{bmatrix} 3\theta^2\cos\theta'\cos(\alpha)\sin(\alpha-\alpha')+\cos\theta'\sin(\alpha') \\ -3\theta\sin\theta'\cos(\alpha)\end{bmatrix}$$
$$+ I_1\omega^2\sin^2\theta' - \omega\sin\theta' p_{\psi'}$$
$$\frac{\partial}{\partial\dot{\theta}'}L = I_1\dot{\theta}'$$
$$I_1\ddot{\theta}' = \frac{\partial}{\partial\theta'}L \tag{14}$$

3.4 Equações do movimento para r

$$\frac{\partial}{\partial r}L = Mr\dot{\theta}^2 + Mr\theta^2\omega^2 + Mg - 3\frac{\mu_0 m m'}{4\pi r^4}[\cdots]$$
$$\frac{\partial}{\partial\dot{r}}L = M\dot{r}$$
$$M\ddot{r} = \frac{\partial}{\partial r}L \tag{15}$$

3.5 Equações do movimento para ϕ

$$\frac{\partial}{\partial\phi}L = \frac{3\mu_0 m m'\theta}{4\pi r^3}[\theta\sin\theta'\cos(2\alpha-\alpha') - \cos\theta'\sin(\alpha)]$$
$$\frac{\partial}{\partial\dot{\phi}}L = Mr^2\theta^2\omega$$

3.6 Equações do movimento para θ

$$\frac{\partial}{\partial\theta}L = \frac{3\mu_0 m m'\cos(\alpha)}{4\pi r^3}[2\theta\sin\theta'\sin(\alpha-\alpha') + \cos\theta']$$
$$- r\theta Mg + Mr^2\theta\omega^2$$
$$\frac{\partial}{\partial\dot{\theta}}L = Mr^2\dot{\theta}$$

4 Aproximações seguintes

Supomos que o segundo imã oscila quase exatamente em uma posição com valor $r = a$, isto é,

$$r = a(1+\epsilon), \quad \epsilon \ll 1 \tag{16}$$

4.1 Equações do movimento para ψ'

$$\begin{aligned}
&\frac{\partial}{\partial \psi'} L = 0 \\
&\frac{\partial}{\partial \dot{\psi}'} L = I_3 \left(\dot{\psi}' + \omega \right) = p_{\psi'} \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0
\end{aligned} \tag{17}$$

4.2 Equações do movimento para ϕ'

$$\begin{aligned}
&\frac{\partial}{\partial \phi'} L = -\frac{3\mu_0 m m' \sin\theta'}{4\pi a^3} \left[\theta^2 \cos(\alpha) \cos(\alpha - \alpha') + \epsilon \cos(\alpha') - \frac{1}{3} \cos(\alpha') \right] \\
&\frac{\partial}{\partial \dot{\phi}'} L = I_1 \omega \sin^2\theta' + p_{\psi'} \cos\theta' \\
2I_1 \omega \dot{\theta}' \sin\theta' \cos\theta' - p_{\psi'} \sin\theta' \dot{\theta}' = & \\
& - \frac{3\mu_0 m m' \sin\theta'}{4\pi a^3} \left[\theta^2 \cos(\alpha) \cos(\alpha - \alpha') + \epsilon \cos(\alpha') - \frac{1}{3} \cos(\alpha') \right]
\end{aligned} \tag{18}$$

4.3 Equações do movimento para θ'

$$\begin{aligned}
\frac{\partial}{\partial \theta'} L = & \frac{3\mu_0 m m'}{4\pi a^3} \begin{bmatrix} \theta^2 \cos\theta' \cos(\alpha) \sin(\alpha - \alpha') - \frac{1}{3} \cos\theta' \sin(\alpha') \\ -\epsilon \cos\theta' \sin(\alpha') - \theta \sin\theta' \cos(\alpha) \end{bmatrix} \\
& + I_1 \omega^2 \sin^2\theta' - \omega \sin\theta' p_{\psi'} \\
\frac{\partial}{\partial \dot{\theta}'} L = & I_1 \dot{\theta}' \\
I_1 \ddot{\theta}' = & \frac{\partial}{\partial \theta'} L
\end{aligned} \tag{19}$$

4.4 Equações do movimento para r

$$\begin{aligned}
&\frac{\partial}{\partial r}L = Ma\dot{\theta}^2 + Ma\theta^2\omega^2 + Mg - 3\frac{\mu_0 mm'}{4\pi a^4}(1-4\epsilon)[\cdots] \\
&\frac{\partial}{\partial \dot{r}}L = Ma\dot{\epsilon} \\
&Ma\ddot{\epsilon} = \frac{\partial}{\partial r}L
\end{aligned} \tag{20}$$

4.5 Equações do movimento para ϕ

$$\begin{aligned}
\frac{\partial}{\partial \phi}L &= \frac{3\mu_0 mm'\theta}{4\pi a^3}\left[\begin{matrix}\theta\sin\theta'\cos(2\alpha-\alpha') - \cos\theta'\sin(\alpha) \\ +3\epsilon\cos\theta'\sin(\alpha)\end{matrix}\right] \\
\frac{\partial}{\partial \dot{\phi}}L &= Ma^2\theta^2\omega
\end{aligned}$$

4.6 Equações do movimento para θ

$$\begin{aligned}
\frac{\partial}{\partial \theta}L = {}& \frac{3\mu_0 mm'\cos(\alpha)}{4\pi a^3}(1-3\epsilon)[2\theta\sin\theta'\sin(\alpha-\alpha') + \cos\theta'] \\
&- a(1+\epsilon)\theta Mg + Ma^2(1+2\epsilon)\theta\omega^2 \\
\frac{\partial}{\partial \dot{\theta}}L = {}& Ma^2(1+2\epsilon)\dot{\theta}
\end{aligned}$$

5 Aproximações seguintes

Supomos que o segundo imã oscila com $\theta' \ll 1$, isto é,

$$\sin\theta' \approx \theta' \tag{21}$$

5.1 Equações do movimento para ψ'

$$
\begin{aligned}
&\frac{\partial}{\partial \psi'} L = 0 \\
&\frac{\partial}{\partial \dot{\psi}'} L = I_3\left(\dot{\psi}' + \omega\right) = p_{\psi'} \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{\psi'} = 0
\end{aligned} \tag{22}
$$

5.2 Equações do movimento para ϕ'

$$
\begin{aligned}
\frac{\partial}{\partial \phi'} L &= \frac{\mu_0 m m' \theta' \cos\alpha'}{4\pi a^3}[1 - 3\epsilon] \\
\frac{\partial}{\partial \dot{\phi}'} L &= I_1 \omega \sin^2\theta' + p_{\psi'} \cos\theta' \\
2 I_1 \omega \dot{\theta}' \theta' - p_{\psi'} \theta' \dot{\theta}' &= \frac{\mu_0 m m' \theta' \cos\alpha'}{4\pi a^3}[1 - 3\epsilon]
\end{aligned} \tag{23}
$$

5.3 Equações do movimento para θ'

$$
\begin{aligned}
\frac{\partial}{\partial \theta'} L &= \frac{3\mu_0 m m'}{4\pi a^3} \begin{bmatrix} \theta^2 \cos(\alpha) \sin(\alpha - \alpha') - \frac{1}{3}\sin(\alpha') \\ -\epsilon \sin(\alpha') - \theta\theta' \cos(\alpha) \end{bmatrix} \\
&\quad + I_1 \omega^2 {\theta'}^2 - \omega \theta' p_{\psi'} \\
\frac{\partial}{\partial \dot{\theta}'} L &= I_1 \dot{\theta}' \\
I_1 \ddot{\theta}' &= \frac{\partial}{\partial \theta'} L
\end{aligned} \tag{24}
$$

5.4 Equações do movimento para r

$$
\begin{aligned}
&\frac{\partial}{\partial r} L = M a \dot{\theta}^2 + M a \theta^2 \omega^2 + M g - 3\frac{\mu_0 m m'}{4\pi a^4}(1 - 4\epsilon)[\theta' \sin(\alpha') + 3\theta \cos\alpha] \\
&\frac{\partial}{\partial \dot{r}} L = M a \dot{\epsilon} \\
&M a \ddot{\epsilon} = \frac{\partial}{\partial r} L
\end{aligned} \tag{25}
$$

5.5 Equações do movimento para ϕ

$$\frac{\partial}{\partial\phi}L = \frac{3\mu_0 m m'\theta\sin\alpha}{4\pi a^3}[3\epsilon - 1]$$
$$\frac{\partial}{\partial\dot{\phi}}L = Ma^2\theta^2\omega$$

5.6 Equações do movimento para θ

$$\begin{aligned}\frac{\partial}{\partial\theta}L = {} & \frac{3\mu_0 m m'\cos(\alpha)}{4\pi a^3}[2\theta\theta'\sin(\alpha - \alpha') + \cos\theta' - 3\epsilon\cos\theta'] \\ & - a(1+\epsilon)\theta M g + Ma^2(1+2\epsilon)\theta\omega^2 \\ \frac{\partial}{\partial\dot{\theta}}L = {} & Ma^2(1+2\epsilon)\dot{\theta}\end{aligned}$$