Lecture 10

4.4 Cotangent Spaces & gradient:

Def: Let M be a smooth manifold, then the cotangent ~~bundle~~ space $T_p^*M := (T_pM)^*$ (dual).

* Remark: If $\dim M < \infty \Rightarrow \dim(T_pM) < \infty$
$\Rightarrow T_pM \cong_{vec} T_p^*M$ (not canonical).

As T_pM and T_p^*M are constructed at p, we can construct the tensor space:

$$T^r_s(T_pM) := \{ t : T_p^*M^{\times r} \times T_pM^{\times s} \xrightarrow{\sim} \mathbb{R} \}$$

Def: Let $f \in C^\infty(M)$. Then at every point $p \in M$ we have a linear map $d_p : C^\infty(M) \longrightarrow T_p^*M$
$$f \longmapsto d_pf$$

defined by $X \in T_pM$: $(d_pf)(X) := Xf$, called the gradient operator at p. Then d_pf is the gradient of the function f at the point p.

* Remark: d_pf is a covector, not a vector.
* Remark: Let X be ~~tangent~~ to a level set

$$N_c(f) := \{ p \in M \mid f(p) = c \}, \text{ ~~then~~ } (d_pf)(X) = 0$$

Gradient operator can be used in:

<u>Def</u>: Let $p \in M$ smooth manifold, and $(\mathcal{U}, x)$ a chart, $p \in \mathcal{U}$.

~~Theo~~ Then: $\underset{T_p^*M}{\underset{\cap}{d_p x^1}}, \underset{T_p^*M}{\underset{\cap}{d_p x^2}}, \ldots, \underset{T_p^*M}{\underset{\cap}{d_p x^n}}$ is called

the chart induced covector basis at the point $p \in M$.

$$(d_p x^a)\Big(\underbrace{\Big(\frac{\partial}{\partial x^b}\Big)_p}_{\text{chart induced basis of } T_pM}\Big) = \Big(\Big(\frac{\partial}{\partial x^b}\Big)(x^a)\Big)_p = \partial_b\Big(\underbrace{\underset{M \to \mathbb{R}}{x^a} \circ \underset{\mathbb{R}^{dn} \to M}{(x)^{-1}}}_{\mathbb{R}^{dn} \to \mathbb{R}}\Big)(x(p))$$

but $x^a \circ x^{-1} = \text{proj}_a : \mathbb{R}^n \to \mathbb{R}$, $\partial_b \text{proj}_a = \delta^a{}_b$

$(d_p x^a)\Big(\Big(\frac{\partial}{\partial x^b}\Big)_p\Big) = \delta^a{}_b$, Then, $d_p x^a$ is a linear independent set in T_p^*M, and therefore besides being a basis, it is the dual basis to $\Big(\frac{\partial}{\partial x^b}\Big)_p$

4.5 Push-forward & pull-back.

<u>Def</u>: $\phi : M \to N$ be a smooth map between smooth manifolds. Then, the push-forward: ϕ_{*_p} of the map ϕ at $p \in M$, is the linear map:

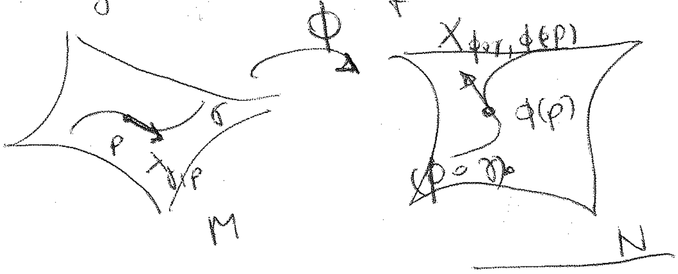
$$\phi_{*_p} : T_pM \xrightarrow{\sim} T_{\phi(p)}N$$
$$X \longmapsto \phi_{*_p}(X)$$

Let $f : N \to \mathbb{R}$ smooth.

$\phi_{*p}(X) f := X(\underbrace{f \circ \phi}_{\in C^\infty(M)})$

$M \xrightarrow{\phi} N \xrightarrow{f} \mathbb{R}$

* Remark: This is the only linear map that can be constructed from ϕ, X, f.
* Remark: ϕ_{*p} is often called the derivative of the function f at p.
* Remark: The ~~tangent~~ vector $X_{\gamma,p}$ is pushed forward to the ~~tangent~~ vector of the smooth curve $\phi\circ\gamma$ at $\phi(p)$.



$$X_{\phi\circ\gamma,\phi(p)} = \phi_{*p}(X_{\gamma,p}).$$

Let $f \in C^\infty(N)$, $\boxed{\phi_{*p}(X_{\gamma,p})} f = X_{\gamma,p}(f\circ\phi)$

$$= (\underbrace{f}_{N\to\mathbb{R}}\circ\underbrace{\phi}_{M\to N}\circ\underbrace{\gamma}_{\mathbb{R}\to M})'(0) = (\underbrace{f}_{N\to\mathbb{R}}\circ(\underbrace{\phi\circ\gamma}_{\mathbb{R}\to N}))'(0) = \boxed{X_{\phi\circ\gamma,\phi(p)}} f$$

Thus: $X_{\phi\circ\gamma,\phi(p)} = \phi_{*p}(X_{\gamma,p})$.

Def: Let $\phi: M \to N$ smooth. Then, the pullback ϕ^* of the map ϕ at $\phi(p)$ is the linear map:

$$\phi^*_p : T^*_{\phi(p)}N \xrightarrow{\sim} T^*_p M$$
$$\omega \longmapsto \phi^*_p(\omega) \text{ defined by:}$$

$$\underset{\substack{\Uparrow \\ T_pM}}{\phi^*_p(\omega)(X)} := \omega\big(\underbrace{\phi_{*_p}(X)}_{\in\, T_{\phi(p)}N}\big)$$

$$\phi^*_p(\omega)(X) := \omega\left(\phi_{*_p}(X)\right)$$

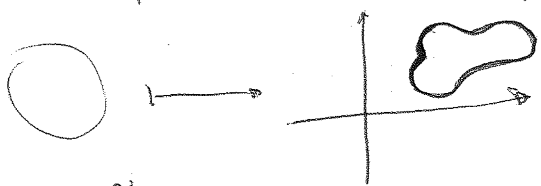
Vectors are pushed forward. Covectors are pull back.

Application: Decide the question under which circumstances some smooth manifold M can "sit" in some $\mathbb{R}^n$.

can be immersion or embedding. (embedding $\Rightarrow$ immersion)

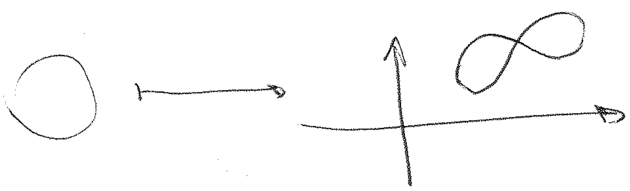
Def: A smooth map $\phi: M \to \mathbb{R}^n$ is called an <u>immersion</u> of M into $\mathbb{R}^n$ if the derivative of ϕ: $\phi_{*_p}: T_pM \xrightarrow{\sim} T_{\phi(p)}\mathbb{R}^n$ is injective for any $p \in M$.

Example: $M = S^1$, $n=2$. (i) $\phi: S^1 \to \mathbb{R}^2$



Immersion.

(ii) $\tilde{\phi}: S^1 \to \mathbb{R}^2$.

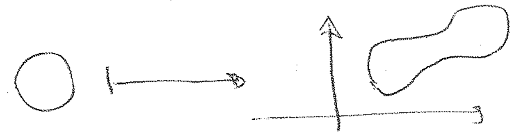


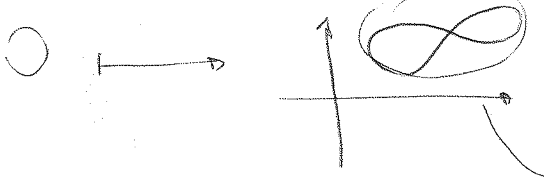
Immersion, $\tilde{\phi}$ is not injective but $\tilde{\phi}_*$ is injective.

Def: A smooth map $\phi: M \to \cancel{\mathbb{R}}\, N$ is an embedding if

(i) ϕ is an immersion

(ii) $\phi(M) \cong_{top} M$ $\left(\Rightarrow \phi(M) \cong_{diff} M\right)$

Again:  is an embedding.

but  is not an embedding, because is not a manifold.

Theorem (Whitney): Any smooth manifold M can be:

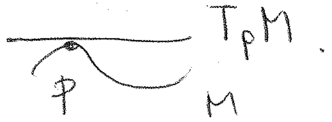
i) embedded in $\mathbb{R}^{2\dim M}$

ii) immersed in $\mathbb{R}^{2\dim M - 1}$

Theorem (Stronger): Any smooth manifold M can be immersed into $\mathbb{R}^{2\dim M - a(\dim M)}$, where $a(m)$ is the number of ones in a binary expansion of m.

Ex: $\dim M = 3 = 1.2^1 + 1.2^0$, $a(3) = 2 \Rightarrow \dim M = 3$ can be immersed into $\mathbb{R}^4$.

4.6 Tangent bundles & vectors fields.

So far: 

Def: Let M be smooth manifold. Then, the tangent bundle is the set:

$$TM := \dot{\bigcup_{p\in M}} T_pM \quad (T_pM \text{ are disjoint pairwise}).$$

and further we define the bundle projection:

$$\pi : TM \longrightarrow M$$
$$X \longmapsto p, \quad p \text{ is the point for which } X \in T_pM.$$

So far (TM, π, M) is a set bundle. We want to turn TM into a smooth manifold, construct a smooth atlas of TM from the atlas of M. The topology on TM is given as: Take (U_α, x_α) the set of charts in M (diffeomorphism). Thinking ~~$TM = \{(p,X) \in \ldots\}$~~

$TM = \{(p, X) \mid X \in T_pM, p \in M\}$, We define the map: $\tilde{x}_\alpha : \text{preim}_\pi(U_\alpha) \to \mathbb{R}^{2n}$:

$$\left(p, X^i \left(\frac{\partial}{\partial x_\alpha^i}\right)_p\right) \longmapsto (\underbrace{x_\alpha(p)}_{\in \mathbb{R}^n}, \underbrace{X^i}_{\in \mathbb{R}^n})$$

A subset $A \subseteq TM$ is open iff: $\tilde{x}_\alpha(A \cap \text{preim}_\pi(U_\alpha))$ is open in $\mathbb{R}^{2n}$ $\boxed{\forall \alpha}$ with the standard topology.

Let $\mathcal{A}_M$ smooth atlas in M. Take some $(U, x) \in \mathcal{A}_M$ and construct: $(\text{preim}_\pi(U), \xi)$

$$\xi : \text{preim}_\pi(U) \longrightarrow \xi(\text{preim}_\pi(U)) \subseteq \mathbb{R}^{2n}. \qquad \text{as: } X = X^a \left(\frac{\partial}{\partial x^a}\right)_{\pi(X)}$$

$$X \longmapsto (x^1(\pi(X)), \ldots, x^n(\pi(X)), \underbrace{X^a}_{\in \mathbb{R}^n}).$$

These $(\text{preim}_\pi(U), \xi)$ are the charts of TM (whose are C^∞-compatible)

That makes TM a smooth manifold and therefore (TM, π, M) a <u>smooth bundle</u>.