GATF

Lecture 13 : Lie Groups and their ~~A~~ Lie algebras.

Chapter 4: Lie ~~Groups~~ Theory.

· 4.1 Lie groups.

Def. A lie group is a pair $(G, \cdot)$ where G is a set and $\cdot : G \times G \to G$ such:

(i) $\cdot$ is associative, neutral element, inverse (may or not be commutative (abelian))

(ii) G is a smooth manifold,

and $\mu : G \times G \to G$, ~~continuous~~ smooth!

$(g, h) \mapsto g \cdot h$

→ smooth manifold with the inherited smooth atlas from G.

and $i : G \to G$ smooth!

$g \mapsto g^{-1}$ such $g^{-1} \cdot g = g \cdot g^{-1} = e$.

Examples: (a) $G = \mathbb{R}^n$, $\cdot = +_{\mathbb{R}^n}$, $\mathcal{O}_{\mathbb{R}^n}$ is a commutative lie group, called the n-dimensional translation group.

(b): $G = S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$

$\cdot = \cdot_{\mathbb{C}}$

$\mathcal{O}_{S^1}$,

is a commutative lie group. Usually called $U(1)$.

(c): $G = \{\phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \mid \det \phi \neq 0\}$

$\cdot = \circ$

$\mathcal{O}$, This is a non-commutative Lie group, often called the general linear group of n-dimension over $\mathbb{R}$

or, for short: $GL(n,\mathbb{R})$.

(d) Let V be a n-dimensional $\mathbb{R}$, vector space equipped with a pseudo-inner product

$$(\cdot,\cdot): V \times V \xrightarrow{\sim} \mathbb{R}$$

(i) bilinear
(ii) symmetric
(iii) non-degenerate: $\forall w \in V: (v,w)=0 \Rightarrow v=0$.

* Remark: Stronger requirement than ~~(iii)~~ non-degeneracy is positive definiteness (iii') $\forall v \in V: (v,v) \geq 0$ and $(v,v)=0 \Rightarrow v=0$.

That would make $(\cdot,\cdot)$ a inner-product.

Theorem: There are (up to isomorphisms) only as many pseudo-inner products on V as there are different signatures

$$\boxed{\text{Basis of } V}$$

Signature $(p,q) \Rightarrow (e_a, e_b) = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ 0 & & & & & -1 \end{bmatrix}_{ab}$

(positive definiteness is the requirement ~~that~~ the signature of basis $(n,0)$.

Define: $G = \{ \phi : V \xrightarrow{\sim} V \mid \forall v,w \in V : (\phi(v),\phi(w)) = (v,w)\}$

with $\bullet = 0$

is called $O(p,q) \subseteq GL(p+q,\mathbb{R})$

the orthogonal group (Lie group)

with respect to the pseudo-inner product.

4.2 The Lie Algebra of a Lie group:

Def: $(G, \cdot)$ Lie Group. Then, for any $g \in G$, $\exists\, l_g : G \to G$

$$h \mapsto l_g(h) := g \cdot h \quad \text{(Called the left translation)}$$

Observation: Each l_g is an isomorphism between G, that is, is bijective but does not preserves the group multiplication:

$$l_g(h_1 \cdot h_2) = l_g(h_1) \cdot l_g(h_2)$$

$$\boxed{g h_1 h_2 \neq g h_1 g h_2}$$ so l_g fails to be a group isomorphism.

but it is a diffeomorphism on G.

* Remark: We can extend the pull back of covector fields to the pull back of covector fields. But the push forward of a vector ~~field~~ cannot be extended to vector fields, unless the underlying smooth map is a diffeomorphism.

$h: M \to N$ Smooth map.

as l_g is a diffeomorphism, we can consider Push forward!

* Upshot: Can push forward any vector field X on G to another vector field

$$\underbrace{(l_g * (X))_{gh}}_{\in \Gamma(TG)} := l_g * \underbrace{(X_h)}_{\in T_h G}$$

Def: Let $(G, \circ)$ Lie group and X vector field over G then X is called left-invariant if for any $g \in G$.

$$l_{g*} X = X \quad \text{, alternatively:}$$

$\longrightarrow$ vector field equality

$$\forall h \in G \quad l_{g*}(X_h) = X_{gh} \quad (\text{point-wise})$$

$\longrightarrow$ vector equality

$$\forall h \in G \; (l_{g*} X_h) f = X_{gh} f \quad \forall f \in C^\infty(G)$$

$$X_h (f \circ l_g) = (X f)_{gh}$$
$$= [(X f) \circ l_g](h)$$

$$[X (f \circ l_g)](h) = [(X f) \circ l_g](h) \quad \forall h \in G.$$

$$X(f \circ l_g) = (Xf) \circ l_g \quad \forall f \in C^\infty(G).$$

Def: The set of all left-invariant vector fields on a Lie group G denoted:

$$L(G) \subseteq \underbrace{\Gamma(TG)}_{C^\infty(G)\text{-module.}}$$

Is $L(G)$ a $C^\infty(G)$-module? No!

Let $f \in C^\infty(G)$:

$$l_{g*}(f(h) X_h) = f(h)\, l_{g*}(X_h) = f(h) X_{gh}$$

but:

$$l_{g*}(Y(h)) = Y(gh) = f(gh) X_{gh}$$

but it is indeed a $\mathbb{R}$-vector space!

Def: An abstract Lie algebra, is $(L, +, \overset{\text{scalar}}{\cdot}, [\![\cdot,\cdot]\!])$ is a $\mathbb{K}$-vector space $(L, +, \cdot)$ equipped with a Lie bracket $[\![\cdot,\cdot]\!] : L \times L \to L$ such that:

(i) bilinear

(ii) anti-symmetrical

(iii) Jacobi-Identity: $[\![X,[\![Y,Z]\!]]\!] + [\![Y,[\![Z,X]\!]]\!] + [\![Z,[\![X,Y]\!]]\!] = 0$

Example: $L = \Gamma(TM)$, $\mathbb{R}$-vector space, with: $[\![\cdot,\cdot]\!] := [\cdot,\cdot]$ (bracket of vector fields) ($\mathbb{R}$-bilinear).

$(\Gamma(TM), +, \cdot, [\cdot,\cdot])$ Infinite dimensional ~~vector space~~ lie algebra.

Theorem: $(L(G), [\cdot,\cdot])$ is a lie subalgebra of $(\Gamma(TG), [\cdot,\cdot])$

Proof: $[\cdot,\cdot]$ is closed on $L(G)$

Let $X, Y \in L(G)$, $f \in C^\infty(G)$

$l_g : G \to G$.

$$[X,Y](f \circ l_g) = X(Y(f \circ l_g)) - Y(X(f \circ l_g))$$
$$= X((Yf) \circ l_g) - Y((Xf) \circ l_g)$$
$$= X(Yf) \circ l_g - Y(Xf) \circ l_g$$
$$= ([X,Y]f) \circ l_g$$

Theorem: The lie algebra $L(G)$ is isomorphic ($\mathbb{R}$-vector space) to T_eG. $L(G) \cong_{vec} T_eG$.

~~Proof~~ Corollary: $\dim(L(G)) = \dim(T_eG)$
$= \dim(G)$.

Proof: Need to construct a linear isomorphism

$$j : T_eG \xrightarrow{\sim} L(G)$$
$$A \longmapsto j(A) \quad , \text{ where } \cancel{j(A)_g}$$

$$j(A)_g = l_{g*} A \in \underset{T_gG}{\underline{T_{ge}G}}$$

(i) $j(A)$ is left invariant?

$$l_{h*}\left(j(A)_g\right) = l_{h*}\left(l_{g*}A\right)$$
$$\overset{?}{=} l_{hg*}(A) = j(A)_{hg} \text{ Yes.}$$

$$\left(l_{h*}\left(l_{g*}X\right)\right)f = \left(l_{g*}X\right)\left(f \circ l_h\right)$$
$$= X\left(f \circ l_h \circ l_g\right)$$
$$= X\left(f \circ l_{hg}\right) = \left(l_{hg*}X\right)f$$

(ii) Linear? Yes because $l_{g*}(A)$ is linear!

(iii) Injective? $j(A) = j(B) \Rightarrow \forall g \in G : l_{g*}A = l_{g*}B$

$\cancel{\forall g \in G\ \forall f \in C^\infty(G)\ (l_{g*}A)f = (l_{g*}B)f}$ $g = e \Rightarrow l_{e*}A = l_{e*}B$
$\Rightarrow \boxed{A = B}$

(iv) j is surjective? Let $X \in L(G)$:

define $A^X := X_e \underline{\in T_eG}$, $j(A^X)_g = l_{g*}(X_e) = \boxed{X_g}$

$\boxed{\Rightarrow j(A^X) = X}$.

Thus we can study $L(G)$ by studying T_eG, which is a much simpler space. Can be defined a lie bracket on T_eG to make it a Lie algebra?

$(L(G), [\cdot,\cdot]) \subseteq (\Gamma(TG), [\cdot,\cdot])$

but ~~to~~ can $(T_eG, [\![\cdot,\cdot]\!]) \cong_{lie} (L(G), [\cdot,\cdot])$?

we want $j([\![A,B]\!]) = [j(A), j(B)]$

$A, B \in T_eG$, $j(A), j(B) \in L(G)$.

Define $[\![\cdot,\cdot]\!]$ accordingly!!

~~$[\![\cdot,\cdot]\!]$~~ $[\![A,B]\!] := j^{-1}([j(A), j(B)])$

$[\![A,B]\!] := j^{-1}([j(A), j(B)])$

$(T_eG, [\![\cdot,\cdot]\!])$ is a lie algebra isomorphic to $(L(G), [\cdot,\cdot])$