GATF

Lecture 14: Classification of Lie algebras and Dynkin diagrams

4.3 - Classification of complex Lie algebras
consider $(L, [\cdot,\cdot])$ $\hookrightarrow$ a $\mathbb{C}$-vectorspace.

Example: G a complex lie group
$\hookrightarrow$ complex manifold (C^ω)

Theorem (Levi): Every finite-dim complex lie algebra $(L, [\cdot,\cdot])$ can be decomposed as:

$$L = R \oplus_s (L_1 \oplus \cdots \oplus L_n).$$

where ⓐ R is a lie subalgebra of L solvable:

$$\{[r_1, r_2] \mid r_1, r_2 \in R\} =: [R,R] \supseteq$$

$$\hookrightarrow [[R,R],[R,R]] \supseteq [\cdots,\cdots] \supseteq \cdots \supseteq \{0\}$$

if this is true R is solvable!

ⓑ $L_1, \dots, L_n$ are simple lie algebras
- L_i is non-abelian
- L_i contains ~~no~~ non-trivial ideal.

(sub-vector space).

Ideals ($I \subseteq L$ such that : $[I, L] \subseteq I$)

Trivial ideals : $\{0\}, L$.

ⓒ direct sum between Lie algebras :

$L_1 \oplus L_2 :=$ direct sum of vector space plus: $[L_1, L_2] = 0$.

ⓓ Semi-direct sum :

$R \oplus_s L :=$ direct sum between vector spaces , plus : $[R, L] \subseteq R$

* Remarks : ⓐ hard to classify solvable Lie algebras.

ⓑ A Lie algebra has no solvable part is called <u>semi-simple</u>.

ⓒ Simple Lie algebras are the independent building blocks we will classify .

Preparation of the classification .

Def : $(L, [\![\cdot, \cdot]\!])$ a ~~com~~ $\mathbb{C}$-Lie algebra .
$h \in L$,

$$\begin{aligned} ad(h) : L &\xrightarrow{\sim} L \\ l &\mapsto ad(h)\, l := [\![h, l]\!] \end{aligned}$$

that is: $\mathrm{ad}_h := [h, \cdot\,]$. Called the adjoint map with respect to h.

Def: ~~Killing~~ The bilinear map:

$$K : L \times L \xrightarrow{\sim} \mathbb{C}$$
$$(a,b) \longmapsto K(a,b) := \mathrm{Tr}\left[\underline{\mathrm{ad}_a \circ \mathrm{ad}_b}\right]$$

is called the Killing "form" — not a true form,

Endomorphism! can be taken ~~trace~~.

* Remark: Because L is finite-dim vec-space: Tr is cyclic!

* Remark: L is semi-simple if and only if, K is non-degenerate.

$$\forall a \in L : K(a,b) = 0 \Rightarrow b = 0.$$

consider a basis: $e_1, \dots, e_n$ L
$\epsilon^1, \dots, \epsilon^n$ L^*

$$K(e_i,e_j) = ad(e_j)^m{}_k\, ad(e_i)^n{}_m\, \epsilon^k(e_n)$$

$$= ad(e_j)^m{}_n\, ad(e_i)^n{}_m$$

$$= C^m{}_{jn}\, C^n{}_{im}$$

$$= C^n{}_{im}\, C^m{}_{jn}$$

again: L is semi-simple if, and only if, K is a pseudo-inner product.

*Remark: $ad_h : L \xrightarrow{\sim} L$
$K : L \times L \xrightarrow{\sim} L$

recall: $\phi : V \to V$ is called (anti-)symmetric ~~non-degenerate~~ wrt non-degenerate symmetric bilinear form B if:

$$B(\phi(v), w) = (\pm) B(v, \phi(w))$$

Fact: ad_h is anti symmetric wrt the Killing form.

(Simple $\Rightarrow$ Semi-simple).

Def: $(L, [\cdot,\cdot])$ Lie algebra, then a Cartan subalgebra H is:

- $H \subseteq L$ vector subspace.
- maximal subalgebra of L such that exists a basis $h_1, \dots, h_m$ of H that can be

that can be extended to a basis

$h_1, \dots, h_m, e_1, \dots, e_{n-m}$ of L.

such ~~that~~

$e_1, \dots, e_{n-m}$ are eigenvector for any

ad_h, $h \in H$.

$$ad_h e_\alpha = \underset{\underset{\mathbb{C}}{\cap}}{\lambda_\alpha(h)} e_\alpha$$

Theorem: Any finite dimensional possesses a Cartan subalgebra.

Theorem: If L is simple, H is abelian, $\underline{[H, H] = 0}$.

Observation: $[h, e_\alpha] = \lambda_\alpha(h) e_\alpha$

is linear in h thus:

$$\lambda_\alpha : H \xrightarrow{\sim} \mathbb{C} \quad , \quad \lambda_\alpha \in H^*$$

Def: The $\lambda_1, \dots, \lambda_{n-m} \in H^*$ are called the roots of the lie algebra L

Call $\Phi := \{\lambda_i\}$ the root set, $\Phi \subseteq H^*$

* Remark: • ad_x anti-symmetrical wrt (K)
$\Rightarrow (\lambda \in \Phi \Rightarrow -\lambda \in \Phi)$.

$\Rightarrow$ in matrix elements:
$$\begin{pmatrix} \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & & 0 \\ & \boxed{\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}} & & \\ & & \boxed{\begin{smallmatrix} 0 & \ddots \end{smallmatrix}} & \\ 0 & & & \ddots \end{pmatrix}$$

• Φ is not linearly independent.

Def: A set of fundamental roots
finite $= \pi_1, \dots, \pi_f$.
$\Pi \subseteq \Phi$ such ~~that~~

(a) Π is linearly independent
(b) $\forall \lambda \in \Phi : \exists n_1, \dots, n_f \in \mathbb{N} :$ $\exists \varepsilon \in \{1, -1\}$

$$\lambda = \varepsilon \sum_{i=1}^{f} n_i \pi_i$$

Fact: such $\Pi \subseteq \Phi$ can always be found.

Theorem: $\mathrm{Span}_{\mathbb{C}}(\Pi) = H^*$

* Remark: π is not unique!

Def: (1) $H^*_{\mathbb{R}} := \mathrm{Span}_{\mathbb{R}}(\Pi)$, Thus:

$$\Pi \subseteq \Phi \subseteq \mathrm{Span}_{\pm\mathbb{N}}(\Pi) \subseteq H^*_{\mathbb{R}} \subseteq H^*$$

(2) Have: $K : L \times L \xrightarrow{\sim} \mathbb{C}$, we can look to

$K|_H : H \times H \to \mathbb{C}$, we want:

$K^*|_{H^*}$, $K^* : H^* \times H^* \xrightarrow{\sim} \mathbb{C}$

$K^*(\mu, \nu) := K(i^{-1}(\mu), i^{-1}(\nu))$

def: $i : H \xrightarrow{\sim} H^*$

$h \longmapsto i(h) := K(h, \cdot)$ i^{-1} exists if K is non-degenerated (L (semi-simple)).

Theorem: $K^*_{\mathbb{R}} : H^*_{\mathbb{R}} \times H^*_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}$

and $K^*(\alpha, \alpha) \geq 0$ and $= 0$ iff $\underline{\alpha = 0}$.

$K^*_{\mathbb{R}}$ is a inner product.

$H^*_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}(\pi)$ we can calculate lengths and angles, in particular, one can calculate lenght and angles of (fundamental) roots.

π is of extremelly importance when characterizing Lie groups. Can we recover Φ from π? Yes!

Def: For any $\lambda \in \Phi$ define $S_\lambda : H^*_{\mathbb{R}} \xrightarrow{\sim} H^*_{\mathbb{R}}$

$$S_\lambda(\mu) := \mu - 2 \frac{(\lambda, \mu)}{(\lambda, \lambda)} \lambda$$

certainly linear in μ and non-linear in λ.

Any such S_λ is called a Weyl transformation, and

$W := \{ S_\lambda \mid \lambda \in \Phi \}$ called the Weyl group with the group operation being $\circ$.

Theorem:

① "The Weyl group is generated by the fundamental roots in π"

$$\forall w \in W : \exists \pi_1, \dots, \pi_n \in \pi : w = S_{\pi_1} \circ S_{\pi_2} \circ \dots \circ S_{\pi_n}.$$

② "Every root can be ~~rotated~~ produced from a fundamental root $\pi_1 \in \pi$ by action of the Weyl group"

$$\forall \lambda \in \Phi : \exists w \in W : \exists \pi_1 \in \pi : \lambda = w(\pi)$$

③ "The Weyl group merely permutes the roots"

$$\forall w \in W : \forall \lambda \in \Phi : w(\lambda) \in \Phi.$$

Now, showdown:

Consider: for any $\pi_i, \pi_j \in \pi$.

$$\Phi \ni S_{\pi_i}(\pi_j) = \pi_j - 2 \frac{K^*(\pi_i, \pi_j)}{K^*(\pi_i, \pi_i)} \pi_i$$

but:

$$S_{\pi_i}(\pi_j) = \varepsilon \sum_k n_k \pi_k$$

$$\Rightarrow \quad 1 \cdot \pi_j + (-2) \underbrace{\frac{K^*(\pi_i,\pi_j)}{K^*(\pi_i,\pi_i)}}_{\in \mathbb{N}} \pi_i$$

$\dfrac{2K^*(\pi_i,\pi_j)}{K^*(\pi_i,\pi_i)}$ is a non-positive integer for $(i\neq j)$.

$\quad \shortparallel$

C_{ij} (not symmetric)
cartan matrix.

Clearly: $C_{ii} = 2$.

now define; bond number:

$$n_{ij} := C_{ij} C_{ji} = 4 \frac{K^*(\pi_i \pi_j)}{K^*(\pi_i,\pi_i)} \frac{K^*(\pi_j,\pi_i)}{K^*(\pi_j,\pi_j)}$$

$$\underline{n_{ij} = 4\cos^2(\measuredangle(\pi_i,\pi_j))}$$

$$\boxed{0 \leq n_{ij} \leq 4} \qquad 0 \leq n_{ij} < 4 \quad i \neq j$$

as n_{ij} must be integer:

$$n_{ij} = 0, 1, 2, 3$$

C_{ij}	C_{ji}	n_{ij}
0	0	0
-1	-1	1
-1	-2	2
-2	-1	2
-1	-3	3
-3	-1	3

<u>see</u> notes!

$(\pi_i, \pi_j) < (\pi_j, \pi_j)$ or $>$

$C_{ij} < C_{ji} \Rightarrow K^*(\pi_i,\pi_i) > K^*(\pi_j,\pi_j)$

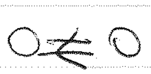
$$C_{ij} = \frac{2K^*(\pi_i,\pi_j)}{K^*(\pi_i,\pi_i)}$$

Dynkin diagrams:

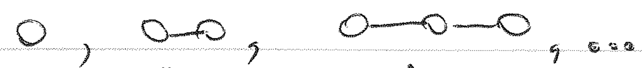
① for every fundamental root draw a circle: o

② if o o represent $\pi_i, \pi_j \in \Pi$, draw n_{ij} lines between them.

③ if there are 2 or 3 lines between two roots, use $<$ sign to relate

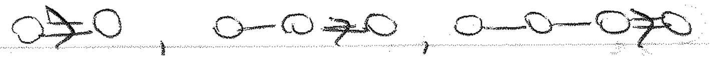


Theorem: (Killing, Cartan): Any finite-dim, simple $\mathbb{C}$-Lie algebra, can be reconstructed from it's set of fundamental roots and the latter only come in the following form.

A_l: 

A_1 A_2 A_3

$l \geq 2$

B_l :  , at the end

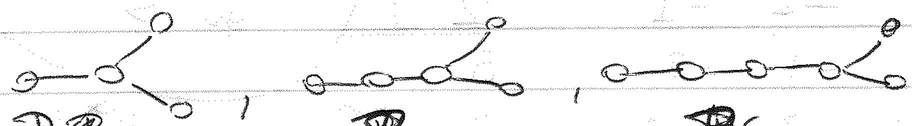
B_2 , B_3 , B_4

$l \geq 3$

C :  , . . .

C_3 C_4

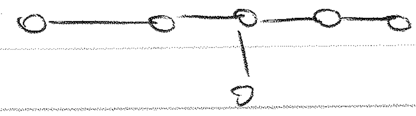
$l \geq 4$

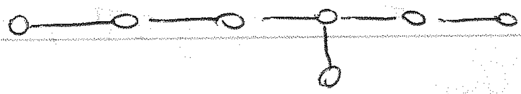
D_l : 

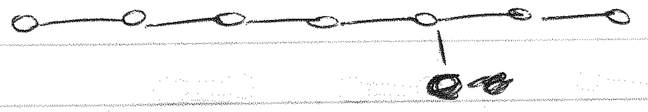
D_4 D_5 D_6

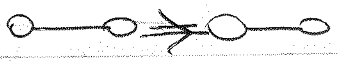
four classical lie algebras series!

There are more 5 exceptional lie algebras

 E_6

 E_7

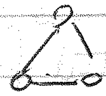
 E_8

 F_4

 G_2

These are all the simple lie algebras!

Example

 is not a simple lie algebra!