GATF

Lecture 8

4.2 Vector spaces:

Def: An (algebraic) field $(K,+,\cdot)$ is a set K and two maps $\cdot, + : K\times K \to K$, that satisfy:

$\overbrace{C\ A\ N\ I}^{+}$ and $\overbrace{C\ A\ N\ I}^{\cdot}$ $K\setminus\{0\}$

C: commutative, A: associative, N: neutral (0), I: identity; for $\cdot$, N: 1

Remark: A weaker notion, to become important later, is a ring $(R,+,\cdot)$

$\overbrace{C\ A\ N\ I}^{+}$ $\overbrace{\cancel{C}\ A\ N\ \cancel{I}}^{\cdot}$

Ex: $(\mathbb{Z},+,\cdot)$ commutative ring, $M_{m\times m}(\mathbb{R})$ ring.

$(\mathbb{R},+,\cdot)$ field,

Def: A K-vector space $(V,\oplus,\odot)$, $\oplus : V\times V \to V$
$(K,+,\cdot)$ $\odot : K\times V \to V$

which satisfy: $\overbrace{C\ A\ N\ I}^{\oplus}$ $\overset{\odot}{A}\ \overset{\odot}{D}\ \overset{\oplus}{D}\ \overset{\odot}{U}$

Def: $U \subseteq V$ is a vector subspace if, $\forall u_1,u_2\in U$: $u_1\oplus u_2 \in U$ and $\forall \lambda\in K$, $\lambda\odot u_1\in U$.

from now on, $\oplus \to +$ and $\odot \to \cdot$

Def: Linear map: $f : \underset{\oplus,\odot}{V} \to \underset{\boxplus,\boxdot}{W}$ if:

(i) $\forall v_1, v_2 \in V$: $f(v_1\oplus v_2) = f(v_1)\boxplus f(v_2)$

(ii) $\forall \lambda\in K, v\in V$: $f(\lambda\odot v) = \lambda\boxdot f(v)$.

if f is a bijection, f is said a vector space isomorphism.

$V \cong_{vec} W$.

Def: $\mathrm{Hom}(V,W) := \{ f : V \xrightarrow[\sim]{\text{linear}} W \}$ can be made into a vector space by defining $\oplus : \mathrm{Hom}(V,W) \times \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$

$$(f,g) \mapsto f \oplus g.$$

where $(f+g) : V \longrightarrow W$

$$v \longmapsto f(v) +_W g(v) =: (f \oplus g)(v) \quad (\text{linear map})$$

Similarly for $\odot$, $\lambda \odot g := \lambda \cdot_W (g(v))$ not linear if K is a ring!

For fields everything is well defined.

Terminology: $\mathrm{End}(V) := \mathrm{Hom}(V,V)$ "endomorphisms"

$\mathrm{Aut}(V) := \{ f : V \xrightarrow{\sim} V \mid \text{invertible} \}$ "automorphisms".

$\mathrm{Aut}(V) \subseteq_{vec} \mathrm{End}(V)$.

dual vector space. $V^* := \mathrm{Hom}(V,K)$; K considered as a vector space.

Def: A (p,q) tensor T is a multilinear map

$$T : \underbrace{V^* \times \dots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \dots \times V}_{q \text{ copies}} \xrightarrow{\sim} K$$

Def: $\underbrace{\underbrace{V \otimes \dots \otimes V}_{p} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{q}}_{T^p_q V} := \{ T \mid T \text{ is a } (p,q) \text{ tensor} \}$

Def: Tensor product : $\otimes : T^p_q V \times T^r_s V \longrightarrow T^{p+r}_{q+s} V$.

$$(T\otimes S)(v_1,\dots,v_p,v_{p+1},\dots,v_{p+r},\omega_1,\dots,\omega_q,\omega_{q+1},\dots,\omega_{q+s})$$

$$= T(v_1,\dots,v_p,\omega_1,\dots,\omega_q)\cdot S(v_{p+1},\dots,v_{p+r},\omega_{q+1},\dots,\omega_{q+s}).$$

Example: (i) $T^0_1 V \equiv V^* :\cong \{T : V \xrightarrow{\sim} K\}$

(ii) $T^1_1 V \equiv V\otimes V^* := \{T : V^*\times V \xrightarrow{\sim} K\} \cong_{vec} End(V^*)$

given : $T\in V\otimes V^*$, construct $\hat{T}\in End(V^*)$ as :

$$\hat{T} : V^* \xrightarrow{\sim} V^*$$
$$\omega \longmapsto T(\omega, \cdot), \quad \text{but},$$

$$T(\omega, v) = (\hat{T}(\omega))(v).$$

(iii) $T^1_0 V \cong_{vec} V$? No.

(iv) $T^1_1 V \cong_{vec} End(V)$? No

(v) $(V^*)^* \cong_{vec} V$? No.

} only for $\dim V < \infty$.

Def: Given V vector space, then a subset $B\subseteq V$ is called a (Hamel) basis if : (i) Every finite subset $\{b_1,\dots,b_N\} \subseteq B$ is linearly independent $:\Longleftrightarrow \left(\sum_{i=1}^N \lambda^i b_i = 0 \Rightarrow \lambda^1 = \dots = \lambda^N = 0\right)$

(ii) $\forall v\in V; \exists v^1,\dots,v^M\in K : \exists b_1,\dots,b_M\in B : v = \sum_{i=1}^M v^i b_i$.

Def: Dimension of V, $\dim V := |B|$.

Theorem: If $\dim V<\infty$, then $(V^*)^* \cong_{vec} V$.

$V \equiv$ vector space, $V^* \equiv$ covector space.

Def: $T \in T^p_q V$, $\dim V < \infty$. Let $e_1, \dots, e_{\dim V}$ be a basis of V and $\epsilon^1, \dots, \epsilon^{\dim V}$ dual basis of V^*. The components of T are:

$$T^{a_1 \dots a_p}{}_{b_1 \dots b_q} := T(\epsilon^{a_1}, \dots, \epsilon^{a_p}, e_{b_1}, \dots, b_q)$$

Remark: Usually, one chooses on V^* a basis induced from e_i, by the condition: $\epsilon^a(e_b) = \delta^a{}_b$.

Reconstruction of T from the components:

$$T = T^{a_1 \dots a_p}{}_{b_1 \dots b_q}\, e_{a_1} \otimes \dots \otimes e_{a_p} \otimes \epsilon^{b_1} \otimes \dots \otimes \epsilon^{b_q}$$

Change of basis: Let $\dim V = n < \infty$. e_i basis of V

new basis: $\tilde{e}_j = \sum_{i=1}^{n} A^i{}_j e_i$ for this to be a basis it must be possible to: $e_i = \sum_{j=1}^{n} B^j{}_i \tilde{e}_j$

Notation: $T = T^{a_1 \dots a_p}{}_{b_1 \dots b_q}\, e_{a_1} \otimes \dots \otimes e_{a_p} \otimes \epsilon^{b_1} \otimes \dots \otimes \epsilon^{b_q}$

Implicity summation. $\sum_{a_1=1}^{n} \dots \sum_{b_1=1}^{n}$.

Basis of V are writed if down index

Basis of V^* with up index.

Remark: Having choosen basis in V, it's tempting

$$V^* \ni \omega = \omega_a \epsilon^a \leftrightsquigarrow \omega \hat{=} (\omega_1, \dots, \omega_n)$$

$$V \ni v = v^a e_a \leftrightsquigarrow v \hat{=} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\mathrm{End}(V) \overset{v}{\cong}_{vec} T^1_1 V \ni \phi = \phi^a{}_b\, e_a \otimes \epsilon^b \leftrightsquigarrow \phi \hat{=} \begin{pmatrix} \phi^1{}_1 & \dots & \phi^1{}_n \\ \vdots & \ddots & \\ \phi^n{}_1 & \dots & \phi^n{}_n \end{pmatrix}$$

Consider $\phi, \psi \in$ ~~Ha~~ $\mathrm{End}(V)$, thus $\phi \circ \psi \in \mathrm{End}(V)$.

$$(\phi\circ\psi)(\epsilon^a, e_b) = \epsilon^a\left(\left[(\phi\circ\psi)^c{}_d\, e_c \otimes \epsilon^d\right](e_b)\right)$$

$$= \epsilon^a\left((\phi\circ\psi)^c{}_b\, e_c\right) = \underline{\underline{(\phi\circ\psi)^a{}_b}}$$

$$= \epsilon^a\left((\phi\circ\psi)(e_b)\right) = \epsilon^a\left(\phi(\psi(e_b))\right)$$

$$= \epsilon^a\left(\phi(\psi^m{}_b\, e_m)\right) = \psi^m{}_b\, \epsilon^a(\phi(e_m))$$

$$= \psi^m{}_b\, \epsilon^a(\phi^n{}_m\, e_n)$$

$$= \psi^m{}_b\, \phi^n{}_m\, \epsilon^a(e_n)$$

$$= \underline{\underline{\psi^m{}_b\, \phi^a{}_m}}$$

Then: $\underline{\underline{(\phi\circ\psi)^a{}_b = \phi^a{}_m \psi^m{}_b}}$. Of course.

$$\omega(v) = \omega_a\, \epsilon^a(v^b e_b) = \omega_a v^b\, \epsilon^a(e_b)$$

$$= \underline{\omega_a v^a}.$$

Change of Basis: $\tilde{e}_a = A^b{}_a\, e_b$ It must be so $\exists B^j{}_i$

$e_i = B^j{}_i\, \tilde{e}_j$. How vector change?

(i) Covector ω: $\omega_a := \omega(e_a) = \omega(B^b{}_a\, \tilde{e}_b)$

$$= B^b{}_a\, \omega(\tilde{e}_b) = B^b{}_a\, \tilde{\omega}_b$$

(ii) vector : $v^a := v(\epsilon^a) = \epsilon^a(v) = \epsilon^a(v^b e_b)$

$$= \epsilon^a(\tilde{v}^b A^m{}_b \tilde{e}_m) = \tilde{v}^b A^m{}_b \epsilon^a(\tilde{e}_m)$$

$\boxed{v^a = A^a{}_b \tilde{v}^b}$ vector

$\boxed{\omega_a = \tilde{\omega}_b B^b{}_a}$ covector.

(iii) Tensor : $T^{a\dots}{}_{b\dots} = A^a{}_m B^n{}_b \cdots \tilde{T}^{m\dots}{}_{n\dots}$

Of course : $A : V \xrightarrow{\sim} V$ and $B = A^{-1} : V \xrightarrow{\sim} V$.

$$\Downarrow \quad A^a{}_m B^m{}_b = \delta^a{}_b$$
$$B^a{}_m A^m{}_b = \delta^a{}_b.$$

Determinants : If $\phi \in T^1_1 V$:

$$\phi^a{}_b \sim \begin{pmatrix} \phi^1{}_1 & \cdots & \phi^1{}_n \\ \vdots & \ddots & \vdots \\ \phi^n{}_1 & \cdots & \phi^n{}_n \end{pmatrix} : g \in T^0_2 V \qquad g_{ab} \sim \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$

$$\phi \to A^{-1} \phi A \qquad\qquad g \to A^T g A.$$

Def : On V, $\dim V = n$, vector space, an m-form is a $T^0_m V$ tensor ω $(0 \le m \le n)$ ~~that~~ is totally anti-symmetrical

$$\omega(v_1, \dots, v_m) = \mathrm{sgn}(\pi)\, \omega(v_{\pi(1)}, \dots, v_{\pi(m)}) \, ; \quad \pi \in S_m.$$

If $m = 0$, $\underline{\omega \in K}$.

If $m > n$, $\omega = 0$

Special case : Top forms $n \equiv m$. If ω, ω' are top forms both non-vanishing, $\exists c \in K$. $\omega' = c\omega$.

Def: Choice of one top form ω is called a choice of Volume in V.

Def: Let $v_1, \dots, v_n \in V$, then the volume spanned by ~~them~~ is: $\mathrm{vol}(v_1, \dots, v_n) = \omega(v_1, \dots, v_n)$

Def: Let $\phi \in \mathrm{End}(V)$, define:

$$\det \phi := \frac{\omega(\phi(e_1), \dots, \phi(e_n))}{\omega(e_1, \dots, e_n)}$$ for some ω, and some basis e_i

Is this independent of choice of ω and e_i? As ω is top form, $\frac{\omega}{\omega}$ is independent of ω. And it is in fact basis-free.

Determinants are only defined to endomorphisms!