



GATF

Lecture 13: Lie Groups and their Lie algebras.

Chapter 4: Lie Groups Theory.

4.1 Lie groups.

Def: A lie group is a pair (G, \cdot) where G is a set and $\cdot: G \times G \rightarrow G$ such:

- (i) \cdot is associative, neutral element, inverse (may or not be commutative)
- (ii) G is a smooth manifold, and $\mu: G \times G \rightarrow G$, $(g, h) \mapsto g \cdot h$ is smooth! (oblation)

smooth manifold with the inherited smooth atlas from G .

and $i: G \rightarrow G$ smooth!

$$g \mapsto g^{-1} \text{ such } g^{-1} \cdot g = g \cdot g^{-1} = e.$$

Examples: (a) $G = \mathbb{R}^n$, $\cdot = +_{\mathbb{R}^n}$, $\Theta_{\mathbb{R}^n}$ is a commutative lie group, called the n -dimensional translation group.

(b) $G = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

$$\cdot = \cdot_{\mathbb{C}}$$

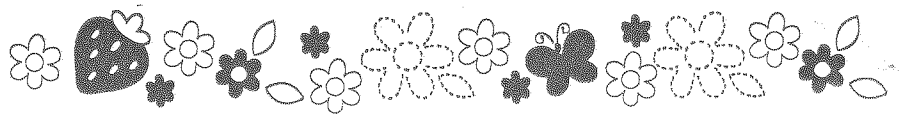
$$\Theta_{S^1}$$

is a commutative lie group. Usually called $U(1)$.

(c) $G = \{ \phi: \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \mid \det \phi \neq 0 \}$

$$\cdot = \circ$$

(d) This is a non-commutative Lie group, often called the general linear group of n -dimension over \mathbb{R}



or, for short: $GL(n, \mathbb{R})$.

(d) Let V be a n -dimensional vector space \mathbb{R}_1 equipped with a pseudo-inner product.

$$(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$$

(i) bilinear

(ii) symmetric

(iii) non-degenerate:

$$\forall v \in V \quad (v, w) = 0 \Rightarrow v = 0.$$

* Remark: Stronger requirement than non-degeneracy is positive definiteness (iii') $\forall v \in V : (v, v) \geq 0$ and

$$(v, v) = 0 \Rightarrow v = 0.$$

That would make (\cdot, \cdot) an inner-product.

Theorem: There are (up to isomorphism) only as many pseudo-inner products on V as there are different signatures.

Basis of V

Signature $(p, q) \Rightarrow (e_a, e_b) = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots & \\ & & & & -1 & \\ & & & & & \ddots & \\ & & & & & & -1 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix}_{ab}$

(positive definiteness is the requirement that the signature of basis $(n, 0)$).

Define: $O(p, q) = \{ \phi : V \rightarrow V \mid \forall v, w \in V : (\phi(v), \phi(w)) = (v, w) \}$

with $\cdot = 0$

is called $O(p, q) \subseteq GL(p+q, \mathbb{R})$

the orthogonal group (Lie group)

with respect to the pseudo-inner product.



4.2 The Lie Algebra of a Lie group:

Def: (G, \cdot) Lie Group. Then, for any $g \in G$,
 $\exists l_g : G \rightarrow G$
 $h \mapsto l_g(h) := g \cdot h$ (Called the left translation)

Observation: Each l_g is an isomorphism between G ,
 that is, is bijective ^{but does not} and preserves the group multiplication:

$$l_g(h_1 \cdot h_2) = l_g(h_1) \cdot l_g(h_2)$$

$[g h_1 h_2 \neq g h_1 g h_2]$ so l_g fails to be a group isomorphism.

but it is a diffeomorphism on G .

* Remark: We can extend the pull back of covector fields to the pull back of covector fields. But the push forward of a vector ~~field~~ cannot be extended to vector fields, unless the underlying smooth map is a diffeomorphism.

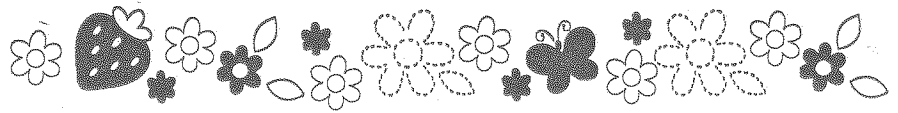
$h: M \rightarrow N$ Smooth map.

or l_g is a diffeomorphism, we can consider Push forward!

* Upshot: Can push forward any vector field X on G to another vector field

$$(l_g)_* (X)_{g \cdot h} := l_g_* (X)_h$$

$\downarrow \in \Gamma(TG) \quad \quad \downarrow \in T_h G$



Def: Let (G, \circ) Lie group and X vector field over G
then X is called left-invariant if for any $g \in G$:

$$L_{g*} X = X, \text{ alternatively:}$$

→ vector field equality

$$\forall h \in G \quad L_{g*}(X_h) = X_{gh} \quad (\text{point-wise})$$

→ vector equality

$$\forall h \in G \quad (L_{g*} X_h) f = X_{gh} f \quad f \in C^\infty(G)$$

$$\begin{aligned} X_h(f \circ L_g) &= (Xf)_{gh} \\ &= [(Xf) \circ L_g](h) \end{aligned}$$

$$[X(f \circ L_g)](h) = [(Xf) \circ L_g](h) \quad \forall h \in G$$

$$X(f \circ L_g) = (Xf) \circ L_g \quad \forall f \in C^\infty(G)$$

Def: The set of all left-invariant vector fields on a Lie group G denoted:

$$L(G) \subseteq \Gamma(TG)$$

$\overbrace{C^\infty(G)\text{-module}}$

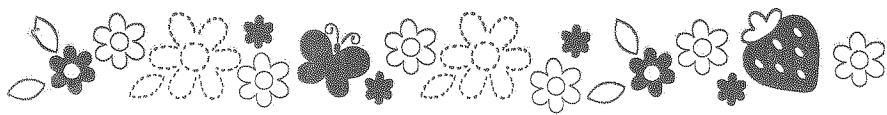
In $L(G)$ a $C^\infty(G)$ -module? No!

Let $f \in C^\infty(G)$:

$$L_{g*}(f(X_h)) = f(h) L_{g*}(X_h) = f(h) X_{gh}$$

but:

$$L_{g*}(Y(h)) = Y(gh) = f(gh) X_{gh} \neq f(h) X_{gh}$$



but it is indeed a \mathbb{R} -vector space!

Def: An ^{scalar} Lie algebra, is $(L, +, \cdot, [\cdot, \cdot])$ is a \mathbb{K} -vector space, $(L, +, \cdot)$ equipped with a Lie bracket $[\cdot, \cdot]: L \times L \rightarrow L$ such that:

- (i) bilinear:
- (ii) anti-symmetrical
- (iii) Jacobi-Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Example: $L = \Gamma(TM)$, \mathbb{R} -vector space, with: $[\cdot, \cdot] := [\cdot, \cdot]$ (bracket of vector fields).
(\mathbb{R} -bilinear).

$(\Gamma(TM), +, \cdot, [\cdot, \cdot])$ Infinite dimensional ~~vector space~~ Lie algebra.

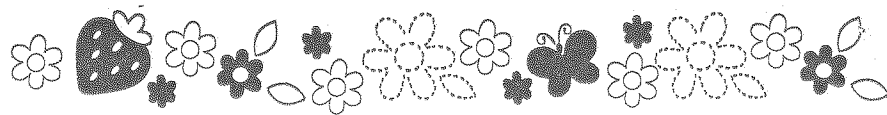
Theorem: $(L(G), [\cdot, \cdot])$ is a Lie subalgebra of $(\Gamma(TG), [\cdot, \cdot])$.

Proof: $[\cdot, \cdot]$ is closed on $\mathfrak{L}(G)$

let $X, Y \in L(G)$, ~~for~~ $f \in C^\infty(G)$
 $\ell_g: G \rightarrow G$.

$$\begin{aligned} [X, Y](f \circ \ell_g) &= X(Y(f \circ \ell_g)) - Y(X(f \circ \ell_g)) \\ &= X((Yf) \circ \ell_g) - Y((Xf) \circ \ell_g) \\ &= X(Yf) \circ \ell_g - Y(Xf) \circ \ell_g \\ &= ([X, Y]f) \circ \ell_g \end{aligned}$$

Theorem: The Lie algebra $L(G)$ is isomorphic (\mathbb{R} -vector space) to $T_e G$. $L(G) \cong_{\text{vec}} T_e G$.



~~Proof~~ Corollary: $\dim(L(G)) = \dim(T_e G)$
 $= \dim(G)$

Proof: Need to construct a linear isomorphism

$$j: T_e G \xrightarrow{\sim} L(G)$$
$$A \mapsto j(A) \quad , \quad \text{where } j(A)_g$$

$$j(A)_g = l_g^* A \in \frac{T_{ge} G}{T_g G}$$

(i) $j(A)$ is left invariant?

$$l_h^* (j(A)_g) = l_h^* (l_g^* A)$$

$$\stackrel{?}{=} l_{hg}^* (A) = j(A)_{hg} \quad \text{Yes.}$$

$$\rightarrow (l_h^* (l_g^* X)) f = (l_g^* X) (f \circ l_h)$$

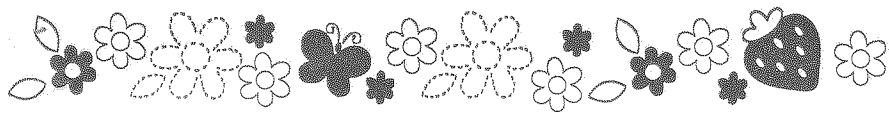
$$= X (f \circ l_h \circ l_g)$$

$$= X (f \circ l_{hg}) = (l_{hg}^* X) f$$

(ii) Linear? Yes because $l_g^*(A)$ is linear!

(iii) Injective? $j(A) = j(B) \Rightarrow \forall g \in G: l_g^* A = l_g^* B$

~~Then if $l_g^* A = l_g^* B$ for $g = e \Rightarrow l_e^* A = l_e^* B$~~
 $\Rightarrow A = B$



(iv) γ is surjective! Let $X \in L(G)$:

define $A^X := X e^{\frac{\cdot}{\gamma} \in T_e G}$, $\gamma(A^X)_g = \log_*(X_e) = \overline{X_g}$

$$\Rightarrow \gamma(A^X) = X$$

Thus we can study $L(G)$ by studying $T_e G$, which is a much simpler space. Can we define a Lie bracket on $T_e G$ to make it a Lie algebra?

$$(L(G), [\cdot, \cdot]) \subseteq (\Gamma(TG), [\cdot, \cdot])$$

but we can $(T_e G, \llbracket \cdot, \cdot \rrbracket) \cong_{\text{Lie}} (L(G), [\cdot, \cdot])$?

$$\text{we want } \gamma(\llbracket A, B \rrbracket) = [\gamma(A), \gamma(B)]$$

$$A, B \in T_e G, \gamma(A), \gamma(B) \in L(G)$$

Define $\llbracket \cdot, \cdot \rrbracket$ accordingly!!!

$$\llbracket A, B \rrbracket := \gamma^{-1}([\gamma(A), \gamma(B)])$$

$$\llbracket A, B \rrbracket := \gamma^{-1}([\gamma(A), \gamma(B)])$$

$(T_e G, \llbracket \cdot, \cdot \rrbracket)$ is a Lie algebra isomorphic to $(L(G), [\cdot, \cdot])$