GATF

Lecture 11

4.7 Tensor fields and modules:

Def: M smooth manifold, and (TM, π, M) it's tangent bundle. A vector field is a smooth section of TM:

$$\sigma : M \longrightarrow TM \quad , \quad \pi \circ \sigma = id_M .$$

Call the set of all vector fields $\Gamma(TM)$, and equip it with:

$$\oplus : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(\sigma, \tau) \longmapsto \sigma \oplus \tau$$

$$\forall p \in M : \quad \sigma \oplus \tau (p) := \underset{T_pM}{\underset{\cap}{\sigma(p)}} + \underset{T_pM}{\underset{\cap}{\tau(p)}}$$

$$\odot : C^\infty(M) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(f, \sigma) \longmapsto f \odot \sigma$$

$$\forall p \in M : (f \odot \sigma)(p) := f(p) \cdot \sigma(p)$$

Recall ~~that~~ $(C^\infty(M), +, \cdot)$ is a Ring!

No inverse!

Thus: $\Gamma(TM)$ is a vector space over a Ring! This is different from a vector space over a field. Over a Ring this is called a Module.

Recall: Ring: $(R, +, \circ)$: $+ : R \times R \to R$

$\circ : R \times R \to R$

$\overset{+}{C}\overset{+}{A}\overset{+}{N}\overset{+}{I}$ $(\overset{\circ}{C})$ $\overset{\circ}{A}(\overset{\circ}{N})(\overset{\circ}{I})$ $D^{+/\circ}$

commutative unitary division rings.

Example: $(C^\infty(M), +, \circ)$ is commutative, unitary, but not division ring.

Def: $(M, \oplus, \odot)$ is called an R-module if : ring (left-multiplication)

$\oplus : M \times M \to M$

$\odot : R \times M \to M$ could be defined as $\odot : M \times R \to M$.

Satisfy: $\overset{\oplus}{C}\overset{\oplus}{A}\overset{\oplus}{N}\overset{\oplus}{I}$ ADDU

("vector space" over a ring)

Example: $(\Gamma(TM), \oplus, \odot)$ is a $C^\infty(M)$ - module

Key fact: Unlike a vector space, a module, generically, does not have a basis (unless R is a division ring).

Theorem: If D is a division ring, then a D-module V has a basis.

Corollary: Every vector space has a basis, since a field is a division ring.

Geometrical example ~~...~~ for R not a division ring.

(a) $M = \mathbb{R}^2$, $v \in \Gamma(TM)$, thus, as

$\Gamma(TM)$ is a $C^\infty(M)$ module, and has a basis!

If R is not division ring a R-module can ~~...~~ not has a basis, but if R is division ring, a R-module is garanteed to have basis.

In this example: $\exists e_1, e_2 \in \Gamma(TM)$ such that: $\forall v \in \Gamma(TM)$: $\exists v^1, v^2 \in C^\infty(M)$, $v = v^a e_a$.

unique

ⓑ $M = S^2$, $v \in \Gamma(TM)$, now $\Gamma(TM)$ has no basis

Due to a famous result: Hairy Ball theorem !!

Technically: $\nexists$ everywhere non-zero and smooth vector field on S^2. Thus if exists a basis of $\Gamma(TS^2)$ it is zero on some point, contradiction!

Proof: Requires the axiom of choice in the version of Zorn's lemma.

$\star$ Zorn's lemma: A partially ordered set P whose every totally ordered subset T has an upper bound in P contains a maximum element.

→ A set P and a relation $\leq$, such:

(i) $\forall x \in P$: $x \leq x$ (reflexivity).

(ii) $\forall x, y \in P$: $(x \leq y \wedge y \leq x) \Rightarrow x = y$ (anti-symmetry)

(iii) $\forall x, y, z \in P$: $(x \leq y \wedge y \leq z) \Rightarrow x \leq z$ (transitivity).

→ Totally ordered if:

(ii) anti-symmetry

(iii) transitivity (iv) totality: $\forall a, b \in T$: $a \leq b \vee b \leq a$.

* ~~Upper bound: is a upper bound to $T \subseteq P$ if:~~

~~$\forall t \in T: t \leq u$ (T must be totally ordered).~~

* Maximal element: m is maximal element of P if:

$\nexists x \in P : m \leq x$ ($x \neq m$) (As P is partially ordered the maximal element is not unique !!)

In ZF, A.C. $\Leftrightarrow$ Z.L.

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* Upper bound: $u \in P$ is an upper bound to $T \subseteq P$ (totally ordered subset) if:

$\forall t \in T : t \leq u$.

Back to the proof:

ⓐ Let S be a generating system of V (D-module, $D \equiv$ division ring)

$\forall v \in V ; \exists e_1, \ldots, e_N \in S : \exists v^1, \ldots, v^N \in D : \boxed{v = v^a e_a}$

(S always exists, since, $S = V$).

ⓑ Define a partially ordered set $(P, \leq)$ by:

$P := \{ \mathcal{U} \in 2^S \mid \mathcal{U}$ is $\underbrace{\text{linear independent}}_{\text{every finite subset of } \mathcal{U} \text{ is linear independent}} \}$

Partial order $\subseteq$:

ⓒ Let T be any totally ordered subset of P, thus,

$\bigcup T$ is an upper bound to T, and is a linear independent subset of S, as T is totally ordered.

$\Rightarrow$ P has maximal element & call one of them (B).
Z.L.

By construction, B is a maximal linear independent subset of S

(d) Claim: $S = \mathrm{Span}(B)$, $\mathrm{Span}(A) = \{$ any finite linear combination of elements of $A \}$

Let $v \in S$, Since B is maximal, $B \cup \{v\}$ is either B or linearly dependent, ~~that~~ means ~~that~~.

$\exists e_1, \dots, e_N \in B \cdot \exists a^1, \dots, a^N \in D : \exists a \in D$:

$a^i e_i + a v = 0$, and not all $a^1 \dots, a^N, a$ vanish, clearly $a \neq 0$, as $e_1, \dots, e_N$ are linear independent.

Then: $a^i e_i = -a v$, as $(-a) \in D$ (division ring)

$\exists (-a)^{-1} \in D$, $(-a)^{-1} \cdot a = 1_D$ Thus:

$\boxed{(-a)^{-1} a^i e_i = v}$ Thus $S = \mathrm{Span}(B)$.

(e) As consequence: $V = \mathrm{Span}(B)$. That is, B is a basis of V.

Every D-module has a Hammel basis!

* Remind: $C^\infty(M)$ is not division ring, then $\Gamma(TM)$ is not garanteed to have a basis.

Terminology:

(a) A module over a ring is called free if it has a basis

(b) A module Γ over a ring is called projective if it is a direct summed of a free module F

$$\Gamma \oplus \underset{\text{module}}{Q} = F$$

* Remark: Free $\Rightarrow$ Projective.

* Remark: A finitely generated R-module F is free, then

$$F \cong \underbrace{R \oplus \cdots \oplus R}_{n}$$

Theorem (Sorre, Swan, others).

$\{$smooth sections of a vector fibre bundle over a smooth manifold $M\}$ is finitely generated projective $C^\infty(M)$-module.

$$\Gamma(E) \oplus \underset{C^\infty(M)\text{-module}}{Q} = F \to \text{free module}.$$

* Corollary: Q quantifies how much $\Gamma(E)$ fails to have a basis

Theorem: P, Q are finitely generated (projective) modules over a commutative ring R, then: $\mathrm{Hom}_R(P,Q) := \{\phi : P \xrightarrow{\sim} Q \mid R\text{-linear}\}$ is again finitely generated (projective) module.

Proof: $$a \odot \phi(r \cdot v) = a \odot \big(r \odot \phi(v)\big) = (a \cdot r) \overset{a}{\odot} \phi(v)$$

$$= r \underset{a}{\cdot} (a \odot \phi)(v) = r \cdot (a \cdot \phi(v))$$

In particular: $\mathrm{Hom}_{C^\infty(M)}(\Gamma(TM), C^\infty(M)) =: \Gamma(TM)^* \cong \Gamma(T^*M)$

Now prepared for the standard text book definition:

Def: A $\binom{r}{s}$ tensor field t on a smooth manifold M is a $C^\infty(M)$ - multilinear map

$$t : \underbrace{\Gamma(T^*M)\times\cdots\times\Gamma(T^*M)}_{r} \times \underbrace{\Gamma(TM)\times\cdots\times\Gamma(TM)}_{s} \xrightarrow{\sim} C^\infty(M)$$