GATF

Lecture 12

4.7 Differential Forms & DeRham cohomology.

Def: M smooth m-manifold, a differential n-form $\boxed{0 \leq n \leq m}$ is a $\binom{0}{n}$-tensor field ω that is totally anti-symmetric.

$$\omega(X_1, \dots, X_n) = \operatorname{sgn}(\pi)\, \omega(X_{\pi(1)}, \dots, X_{\pi(n)})$$

with π being a permutation, $X_i \in \Gamma(TM)$.

Examples:

(a) If M is orientable then there exists a nowhere vanishing top-form $(n = m)$ ω providing a volume.

(b) Electromagnetic field strength F is a 2-form

(c) Q smooth manifold (Configuration Space)

T^*Q (Phase space), $\exists\, \omega$ 2-form (symplectic form)

Notation: Set of all n-forms in M: $\Omega^n(M)$.

which naturally is a $C^\infty(M)$-module.

However, taking tensor products of forms does not yield a form.

Def: $\wedge : \Omega^n(M) \times \Omega^m(M) \xrightarrow{\sim} \Omega^{m+n}(M)$

$$(\omega, \sigma) \longmapsto \omega \wedge \sigma$$

$$(\omega \wedge \sigma)(X_1, \dots, X_{n+m}) = \frac{1}{n!} \cdot \frac{1}{m!} \sum_{\pi \in \mathrm{Per}(m+n)} \mathrm{sgn}(\pi)(\omega \otimes \sigma)(X_{\pi(1)}, \dots, X_{\pi(n+m)})$$

Example: $\omega, \sigma \in \Omega^1(M)$:

$$\omega \wedge \sigma := \omega \otimes \sigma - \sigma \otimes \omega$$

* Recall: $h : M \to N$ smooth, induces:

$h_p^* : T^*_{h(p)} N \to T^*_p M$, which can be used to define: $h^* : \Omega^1(N) \xrightarrow{\sim} \Omega^1(M)$

$$\Omega^1(N) = \Gamma(T^*N)$$

Def: Let $\omega \in \Omega^n(N)$, then, we want to define the pullback of ω by $h : M \to N$

$h^* \omega \in \Omega^n(M)$.

push-forward of vector field

$$(h^* \omega)(X_1, \dots, X_n) := \omega(h_*(X_1), \dots, h_*(X_n))$$

$X_i \in \Gamma(TM)$, $h_*(X_1) \in \Gamma(TN)$

$(h_*(X)(p) f) = X_{(h^{-1}(p))}(f \circ h)$ (see better definition).

Theorem: The pullback distributes over the wedge product.

$$h^*(\omega \wedge \sigma) = (h^* \omega) \wedge (h^* \sigma)$$

Proof: ?

There is a space where $\wedge$ is a closed operation?

Def: The $C^\infty(M)$-module $|\dim M = m|$ $\Omega(M) : \Omega^0(M) \oplus \dots \oplus \Omega^m(M)$

$$\Omega^0(M) = C^\infty(M)$$

$\Omega(M)$ is often denoted by $Gr(M)$ and is called the grassmann algebra.

We have to equip $(\underbrace{\Omega(M), +, \odot}_{C^\infty(M)\text{-module}}, \wedge)$ grassmann algebra

$\odot$: $C^\infty(M)$ multiplication

$\wedge$: $\Omega(M)$ "multiplication"

where: $\wedge : \Omega(M) \times \Omega(M) \to \Omega(M)$ defined by linear continuation.

Example: $\psi = \underset{\Omega^1(M)}{\underset{\cap}{\omega}} + \underset{\Omega^3(M)}{\underset{\cap}{\sigma}} \in \Omega(M)$

$\varphi \in \Omega^n(M)$, $\varphi \wedge \psi = \varphi \wedge (\omega + \sigma)$

$= \varphi \wedge \omega + \varphi \wedge \sigma$

Example: Grassmann numbers:

Theorem. $\omega \in \Omega^n(M)$, $\sigma \in \Omega^m(M)$.

$$\omega \wedge \sigma = (-1)^{n \cdot m} \sigma \wedge \omega$$

* Remark: If: $\psi, \varphi \in \Omega(M)$, $\psi \wedge \varphi = (\overset{?}{\cdots}) \varphi \wedge \psi$

no such relation!!

Whats the derivative of forms?

* Recall: $f \in C^\infty(M)$, $\boxed{\begin{aligned} d : C^\infty(M) &\to \Omega^1(M) \\ f &\mapsto df \end{aligned}}$ $\quad \Omega^0(M) = C^\infty(M)$

clearly: $(df)(p) := d_p f \in T^*_p M$

Extend d to be a map between $\Omega^n(M) \to \Omega^{n+1}(M)$.

Why do not extend to general tensors? That is not possible without adding more structure to the manifold.

Def: The exterior derivative operator $d: \Omega^n(M) \to \Omega^{n+1}(M)$

where: $(d\omega)(X_1, \dots, X_{n+1}) := \sum_{i=1}^{n+1}$ $(-1)^{i+1} X_i\left(\omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1})\right)$

$\Gamma(TM) \ni X_1, \dots, X_{n+1}$

$$+ \sum_{i<j} (-1)^{i+j}\, \omega\left([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1}\right)$$

Proof: $d\omega$ is totally antisymmetric, is $C^\infty(M)$ - multilinear

- Commutator of vector fields: $X, Y \in \Gamma(TM)$,

notice ~~that~~, if $p \in M$, $f \in C^\infty(M)$. $X(p) \in T_pM$

and $X(f): M \to \mathbb{R}$

$p \mapsto X(p) f$

So: $X(Yf) - Y(Xf) =: [X, Y] f$

is well defined: $[\cdot, \cdot]: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$.

Example: $(d\omega)(X, Y) = X(\underbrace{\omega(Y)}_{\in C^\infty(M)}) - \underbrace{Y(\omega(X))}_{\in C^\infty(M)}$

$$- \omega([X, Y])$$

$C^\infty(M)$ scaling: $d\omega(fX, Y) = fX(\omega(Y)) - Y(\omega(fX))$

$$- \omega([fX, Y])$$

$= fX(\omega(Y)) - Y(f\omega(X)) - \omega((fX) \circ Y - Y \circ (fX))$

$= fX(\omega(Y)) - (Yf) \cdot \omega(X) - fY(\omega(X)) - \omega\left(fXY - (Yf)X - fYX\right)$

$= fX(\omega(Y)) - \cancel{(Yf)\,\omega(X)} - fY(\omega(X)) + \cancel{(Yf)\,\omega(X)}$

$- f\omega([X, Y])$

$= \boxed{f\, d\omega(X, Y)}$

Theorem: $\omega \in \Omega^n(M)$, $\varphi \in \Omega^m(M)$:

$$d(\omega \wedge \varphi) = (d\omega) \wedge \varphi + (-1)^n \omega \wedge d\varphi .$$

Theorem: Exterior differentiation "commutes" with the pullback.

$$h^*(d\omega) = d(h^*\omega)$$

Similarly the action of d expand to the grassmann algebra:

$$d : \Omega(M) \to \Omega(M).$$

Physical Examples:

(a) (Maxwell) electrodynamics F (2-form), $dF = 0 \in \Omega^3(M)$.

Homogenous equation.

$\Rightarrow$ $\boxed{F = dA}$ electromagnetic potential

$\hookrightarrow$ This is true if $\boxed{M = \mathbb{R}^4}$.

(b) Classical mechanics: "Symplectic form" (i) $\omega \in \Omega^2(T^*Q)$

(ii) $d\omega = 0$

On T^*Q: $\theta := p_i dq^i$ (symplectic potential)

$$(q^1, \dots, q^{\dim Q}, p_1, \dots, p_{\dim Q})$$

$$\omega := d\theta \Rightarrow d(d\theta) = 0$$

de Rham cohomology!

Based heavily on the fact ~~that~~:

Theorem: $d^2 = 0$, $d(d\omega) = 0 \;\forall \omega$.

In local coordinates: $d\omega = \partial_b \omega_{a_1, \dots, a_n} \, dx^b \wedge dx^{a_1} \wedge \dots \wedge dx^{a_n}$

$$d(d\omega) = (\partial_c \partial_b \omega_{a_1 \dots a_n}) \, dx^c \wedge dx^b \wedge dx^{a_1} \wedge \dots \wedge dx^{a_n}$$

exchange

* $\theta_{[m_1,\dots m_f]} := \frac{1}{f!}\sum_{\pi\in Perm(f)} sgn(\pi)\, \theta_{\pi(m_1)\dots\pi(m_f)}$ (anti-symmetrization bracket)

$\theta_{(m_1,\dots,m_f)} := \frac{1}{f!}\sum_{\pi\in Perm(f)} \theta_{\pi(m_1)\dots\pi(m_f)}$ (symmetrization bracket)

* $A_{ab}B^{[ab]} = A_{[ab]}B^{ab}$, $A_{(ab)}B^{[a,b]} = 0$.

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as $dx\dots$ is anti symmetric, is the same as: $dx^{[c}\dots$

Then: $d(d\omega) = (\partial_{[c}\partial_b\, \omega_{a_1\dots a_n]})\, dx^{c}\wedge dx^b\wedge dx^{a_1}\wedge\dots\wedge dx^{a_n]}$

$= 0$.

$d^2 = 0 \in \Omega^{n+2}(M)$.

This implies that we have a sequence of maps

$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{\dim M}(M) \xrightarrow{d} 0$

$d \uparrow$

0

Def: $\phi : P \xrightarrow{\sim} Q$ P, Q; $C^\infty(M)$-modules.

$Q \supseteq \operatorname{im}\phi := \{\phi(p)\in Q \mid p\in P\}$, and:

$P \supseteq \ker\phi := \{p\in P \mid \phi(p) = 0\}$

$\ker(d) \subseteq \Omega^n(M)$ $\quad d : \Omega^n(M) \to \Omega^{n+1}(M)$

$Im(d) \subseteq \Omega^n(M)$ $\quad d : \Omega^{n-1}(M) \to \Omega^n(M)$.

$d^2 = 0 \Leftrightarrow Im(d) \subseteq \ker(d)$.

Terminology: $\Omega^{n-1}(M) \xrightarrow{d_1} \Omega^n(M) \xrightarrow{d_2} \Omega^{n+1}(M)$.

$\omega \in \Omega^n(M)$: called exact if: $\omega \in Im(d_1)$ $\exists \alpha: \omega = d\alpha$

called closed if: $\omega \in Ker(d_2)$ $d\omega = 0$

$d^2 = 0 \Leftrightarrow$ (exact $\Rightarrow$ closed).

5

If $A \in \Omega^1(M) \Rightarrow F = dA \Rightarrow dF = 0$ Obviously

But

$dF = 0 \Rightarrow dA = F$? F necessarily is closed but is it exact??

further needless terminology:

$B^n := \mathrm{Im}(d) \subseteq \Omega^n(M)$ Exact

$Z^n := \mathrm{Ker}(d) \subseteq \Omega^n(M)$ Closed

It is true ~~that~~ $B^n \subseteq Z^n$. Question: Are they equal?

No But Poincaré lemma:

$$B^n = Z^n \text{ if } M = \mathbb{R}^m.$$

Q2: If not, how to quantify by how much do closed forms fails to be exact?

Def: The n-th de Rham cohomology group is the quotient vector space

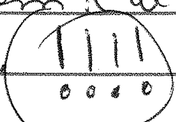
$$H^n(M) := Z^n/B^n$$

On Z^n we have a equivalence relation.

$$\omega \sim \sigma :\Leftrightarrow \boxed{\omega - \sigma \in B^n}$$

$$Z^n/B^n \equiv Z^n/\sim$$

Theorem (de Rham): $H^n(M)$ only depend on the topology of M.



$$H^0(M) \cong \mathbb{R}^{\#(\text{connect components of } M)}$$

• $H^0(\mathbb{R}) = \mathbb{R}$, but $H^{n\geq 1}(\mathbb{R}) = \{0\}$.