Capítulo II:

Transformação de Lorentz: Transformação linear que preserva produto interno:

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu \quad , \quad g_{\mu\nu} \Lambda^\nu{}_\sigma \Lambda^\mu{}_\rho = g_{\rho\sigma}$$

As transformações de Lorentz formam um grupo. Consideramos, especialmente aquelas que:

$$\det \Lambda = +1 \quad e \quad \Lambda^0{}_0 \geq 1 .$$

São ditas próprias ortócronas.

Uma representação é possível de ser construída dos geradores:

$$U(1 + \delta\omega) = 1 + \frac{i}{2\hbar} \delta\omega_{\mu\nu} M^{\mu\nu}$$

Onde $M^{\mu\nu}$ são os geradores e $\delta\omega + 1$ é uma transformação infinitesimal.

$\delta\omega_{\mu\nu}$ e $M^{\mu\nu}$ são anti simétricos.

Se consideramos:

$$U(\Lambda)^{-1} U(\Lambda') U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$$

impondo $\Lambda' = 1 + \delta\omega'$. Obtemos:

$$U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}$$

isto é, cada índice vetorial transforma com uma Transformação de Lorentz. Logo, esperamos que:

$$U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu .$$

se expandimos a relação anterior para $\Lambda = 1 + \delta\omega$, obtemos:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right)$$
$$- i\hbar\left(g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho}\right)$$

o que define a álgebra de Lie satisfeita por $M^{\mu\nu}$.
Indentificamos o momento angular com:

$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ e os boosts como:

$K_i = M^{i0}$.

caso façamos $\Lambda = 1 + \delta\omega$ na equação com P^μ:

$$[P^\mu, M^{\rho\sigma}] = i\hbar\left(g^{\mu\sigma}P^\rho - g^{\mu\rho}P^\sigma\right).$$

que implica:

$$[J_i, H] = 0$$
$$[J_i, P_j] = 0$$
$$[K_i, H] = i\hbar P_i$$
$$[K_i, P_j] = i\hbar\,\delta_{ij}H.$$

e claro que P^μ satisfaz:

$$[P_i, P_j] = 0 \quad , \quad [P_i, H] = 0.$$

Todas estas relações de comutação estabelecem a álgebra de Lie de Poincaré.

para um campo escalar, a evolução temporal é:

$$\exp(iHt/\hbar)\,\phi(x,0)\,\exp(-iHt/\hbar) \equiv \phi(x,t)$$

generalizando para:

$$\exp\left(-\frac{iP^\mu x_\mu}{\hbar}\right)\phi(0)\exp\left(\frac{iP^\mu x_\mu}{\hbar}\right) = \phi(x).$$

podemos definir o operador de translação no espaço-tempo como:

$$T(a) \equiv \exp(-iPa/\hbar)$$

então:

$$T(a)^{-1}\phi(x)\,T(a) = \phi(x-a)$$

infinitesimalmente:

$$T(\delta a) = \mathbb{1} - \frac{i}{\hbar}\delta a_\mu P^\mu$$

a relação anterior nos leva a ver que:

$$U(\Lambda)^{-1}\phi(x)\,U(\Lambda) = \phi(\Lambda^{-1}x)$$

e se ϕ carregar índices:

$$U(\Lambda)^{-1}\partial^\mu\phi(x)\,U(\Lambda) = \bar{\partial}^\nu\phi(\Lambda^{-1}x)\,\Lambda^\mu{}_\nu.$$

isso implica que $\partial_\mu\partial^\mu\phi$ é invariante, pois:

$$U(\Lambda)^{-1}\partial_\mu\partial^\mu\phi(x)\,U(\Lambda) = \bar{\partial}_\mu\bar{\partial}^\mu\phi(\Lambda^{-1}x)$$

e como $U(\Lambda)^{-1}\phi(x)\,U(\Lambda) = \phi(\Lambda^{-1}x)$

segue que $\left(\partial_\mu\partial^\mu - \frac{m^2}{\hbar^2c^2}\right)\phi(x) = 0$ é Lorentz invariante.

Exercício (I):

$$g_{\mu\nu}(\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \delta\omega^\nu{}_\sigma) = g_{\rho\sigma}$$

$$g_{\mu\nu}(\delta^\mu{}_\rho \delta^\nu{}_\sigma + \delta^\mu{}_\rho\, \delta\omega^\nu{}_\sigma + \delta^\nu{}_\sigma\, \delta\omega^\mu{}_\rho) = g_{\rho\sigma}$$

$$g_{\rho\sigma} + g_{\rho\nu}\delta\omega^\nu{}_\sigma + g_{\mu\sigma}\,\delta\omega^\mu{}_\rho = g_{\rho\sigma}$$

$$\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$$ ■

Exercício (II):

$$U(\Lambda)^{-1}\, U(\Lambda')\, U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$$

$$U(\Lambda)^{-1}\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega'_{\mu\nu} M^{\mu\nu}\right) U(\Lambda) = U\left(\Lambda^{-1}(\mathbb{1} + \delta\omega')\Lambda\right)$$

$$\mathbb{1} + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}\, U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = U(\mathbb{1} + \Lambda^{-1}\delta\omega'\Lambda)$$

$$\mathbb{1} + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}\, U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \mathbb{1} + \frac{i}{2\hbar}(\Lambda^{-1})_\alpha{}^\mu \delta\omega'_{\mu\nu}\Lambda^\nu{}_\beta M^{\alpha\beta}$$

$$\delta\omega'_{\mu\nu}\, U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \delta\omega'_{\mu\nu}\Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha M^{\alpha\beta}$$ ■

Exercício (III):

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) M^{\mu\nu}\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) = (\delta^\nu{}_\beta + \delta\omega^\nu{}_\beta)(\delta^\mu{}_\alpha + \delta\omega^\mu{}_\alpha)M^{\alpha\beta}$$

$$M^{\mu\nu} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\mu\nu}M^{\rho\sigma} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}M^{\mu\nu}$$

$$= M^{\mu\nu} + \delta\omega^\nu{}_\beta M^{\mu\beta} + \delta\omega^\mu{}_\alpha M^{\alpha\nu}$$

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] = \delta\omega_{\rho\beta}\, g^{\nu\rho} M^{\mu\beta} + \delta\omega_{\rho\alpha} g^{\mu\rho} M^{\alpha\nu}$$

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] = \delta\omega_{\rho\sigma}\, g^{\nu\rho} M^{\mu\sigma} + \delta\omega_{\rho\sigma}\, g^{\mu\rho} M^{\sigma\nu}$$

$$= \frac{\delta\omega_{\rho\sigma}}{2}\left(\underbrace{g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}}_{=0} + g^{\nu\rho}M^{\mu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

$$+\frac{\delta\omega_{\rho\sigma}}{2}\left(\underbrace{g^{\mu\rho}M^{\sigma\nu} + g^{\mu\sigma}M^{\rho\nu}}_{=0} + g^{\mu\rho}M^{\sigma\nu} - g^{\mu\sigma}M^{\rho\nu}\right)$$

$$= \frac{\delta\omega_{\rho\sigma}}{2}\left(g^{\nu\rho}M^{\mu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\rho}M^{\sigma\nu} - g^{\mu\sigma}M^{\rho\nu}\right).$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left[g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right]$$

■

Exercício IV:

$$[J_i, J_j] = \frac{1}{4}\epsilon_{iab}\epsilon_{jcd}[M^{ab}, M^{cd}]$$

$$= \frac{1}{4}\Big[\delta_{ij}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \delta_{ic}(\delta_{ad}\delta_{bj} - \delta_{aj}\delta_{bd})$$

$$+ \delta_{id}(\delta_{aj}\delta_{bc} - \delta_{ac}\delta_{bj})\Big] i\hbar\Big(g^{ac}M^{bd} - g^{bc}M^{ad}$$

$$- g^{ad}M^{bc} + g^{bd}M^{ac}\Big)$$

$$= \frac{i\hbar}{4}\Big[\delta_{ij}\Big(g^a{}_a\cancel{M^b{}_b} - g^b{}_a M^a{}_b - g^a{}_b M^b{}_a + g^b{}_b\cancel{M^a{}_a}$$

$$- g^a{}_b M^b{}_a + g^b{}_b\cancel{M^a{}_a} + g^a{}_a\cancel{M^b{}_b} - g^b{}_a M^a{}_b\Big)$$

$$+\delta_{ic}\Big(g^{ac}M_{ja} - \cancel{g_j{}^c M^a{}_a} - g^a{}_a M_j{}^c + g_{ja}M^{ac}$$

$$- \cancel{g_j{}^c M^b{}_b} + g^{bc}M_{jb} + g_{jb}M^{bc} - g^b{}_b M_j{}^c\Big)$$

$$+\delta_{id}\Big(g_{jb}M^{bd} - g^b{}_b M_j{}^d - \cancel{g_j{}^d M^b{}_b} + g^{bd}M_{jb}$$

$$- g^a{}_a M_j{}^d + g_{ja}M^{ad} + g^{ad}M_{ja} - \cancel{g_j{}^d M^a{}_a}\Big)\Big]$$

$$= \frac{i\hbar}{2}\Big[-\delta_{ij}\left(g^b{}_a M^a{}_b + g^a{}_b M^b{}_a\right)\Big]$$

$$+ \frac{i\hbar}{4}\Big[g^a{}_i M_{ja} - g^a{}_a M_{ji} + g_{ja} M^a{}_i + g^b{}_i M_{jb} + g_{jb} M^b{}_i$$

$$- g^b{}_b M_{ji} + g_{jb} M^b{}_i - g^b{}_b M_{ji} + g^b{}_i M_{jb}$$

$$- g^a{}_a M_{ji} + g_{ja} M^a{}_i + g^a{}_i M_{ja} \Big]$$

$$= \frac{i\hbar}{2}\Big[-\delta_{ij}\left(g^b{}_a M^a{}_b + g^a{}_b M^b{}_a\right) - g^a{}_a M_{ji} - g^b{}_b M_{ji}$$

$$+ g_{ja} M^a{}_i + g_{jb} M^b{}_i + g^a{}_i M_{ja}$$

$$+ g^b{}_i M_{jb} \Big]$$

$$= \frac{i\hbar}{2}\Big[-\delta_{ij}\left(\cancel{M}^b{}_b + \cancel{M}^a{}_a\right) - \overbrace{g^a{}_a}^{=3} M_{ji} - g^b{}_b M_{ji}$$

$$+ M_{ji} + M_{ji} + M_{ji} + M_{ji} \Big]$$

$$= -i\hbar M_{ji}$$

$$= \frac{i\hbar}{2} \cdot 2 M_{ij}$$

$$= \frac{i\hbar}{2}\left(M_{ij} - M_{ji}\right)$$

$$= \frac{i\hbar}{2}\left(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right) M^{ab}$$

$$= \frac{i\hbar}{2} \epsilon^k{}_{ij}\, \epsilon_{kab} M^{ab} = i\hbar\, \epsilon_{ij}{}^k \left(\frac{\epsilon_{kab} M^{ab}}{2}\right)$$

$$= i\hbar\, \epsilon_{ij}{}^k J_k$$

$$[J_i, K_j] = \frac{1}{2}\epsilon_{iab}\left[M^{ab}, M^{j0}\right]$$

$$= \frac{i\hbar}{2}\epsilon_{iab}\left[g^{aj}M^{b0} - g^{bj}M^{a0} - \cancel{g^{a0}}M^{bj} + \cancel{g^{b0}}M^{aj}\right]$$

$$= \frac{i\hbar}{2}\epsilon_{iab}\, g^{aj}M^{b0} + \frac{i\hbar\,\epsilon_{iba}}{2}\, g^{bj}M^{a0}$$

$$= i\hbar\,\epsilon_{iab}\, g^{aj}M^{b0} = i\hbar\,\epsilon_{ijb}M^{b0}$$

$$= i\hbar\,\epsilon_{ij}{}^{k}K_k .$$

$$[K_i, K_j] = [M^{i0}, M^{j0}]$$

$$= i\hbar\left[\cancel{g^{ij}}M^{00} - \cancel{g^{0j}}M^{i0} - \cancel{g^{i0}}M^{0j} + g^{00}M^{ij}\right]$$

$$= i\hbar(-1)M^{ij}$$

$$= \frac{-i\hbar}{2}\left(M^{ij} - M^{ji}\right) = \frac{-i\hbar}{2}\left(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right)M^{ab}$$

$$= \frac{-i\hbar}{2}\epsilon^{k}{}_{ij}\,\epsilon_{kab}M^{ab}$$

$$= -i\hbar\,\epsilon_{ij}{}^{k}\left(\frac{\epsilon_{kab}M^{ab}}{2}\right)$$

$$= -i\hbar\,\epsilon_{ij}{}^{k}J_k \qquad \blacksquare$$

Exercício V :

$$U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$$

$$\left(1 - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)P^\mu\left(1 + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) = \left(\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu\right)P^\nu$$

$$P^\mu + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}P^\mu M^{\rho\sigma} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}P^\mu = P^\mu + \delta\omega^\mu{}_\nu P^\nu$$

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[P^\mu, M^{\rho\sigma}\right] = \delta\omega^\mu{}_\nu P^\nu$$

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}[P^\mu, M^{\rho\sigma}] = \delta\omega_{\rho\sigma}\, g^{\mu\rho}P^\sigma$$

$$\delta\omega_{\rho\sigma}[P^\mu, M^{\rho\sigma}] = -i\frac{2\hbar}{2}\left(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho\right)$$

$$[P^\mu, M^{\rho\sigma}] = i\hbar\left(g^{\mu\sigma}P^\rho - g^{\mu\rho}P^\sigma\right).$$

Exercício VI :

$$[J_i, H] = [J_i, P^0]$$

$$= \frac{1}{2}\epsilon_{ijk}[M^{jk}, P^0]$$

$$= -\frac{1}{2}\epsilon_{ijk}[P^0, M^{jk}]$$

$$= -\frac{1}{2}\epsilon_{ijk}\, i\hbar\left(g^{0k}P^j - g^{0j}P^k\right)$$

$$= 0$$

$$[J_i, P_j] = \frac{1}{2}\epsilon_{iab}[M^{ab}, P_j]$$

$$= -\frac{1}{2}\epsilon_{iab}[P_j, M^{ab}]$$

$$= -\frac{1}{2}\epsilon_{iab}\, i\hbar\left(g^{jb}P^a - g^{ja}P^b\right)$$

$$= -\frac{i\hbar\,\epsilon_{iab}}{2}\left(\delta^{jb}P^a - \delta^{ja}P^b\right)$$

$$= \frac{-i\hbar}{2}\left(\epsilon_{iaj}P^a - \epsilon_{ijb}P^b\right)$$

$$= i\hbar\,\epsilon_{ijk}P^k$$

$$[K_i, H] = [K_i, P^0]$$
$$= [M^{i0}, P^0]$$
$$= -[P^0, M^{i0}]$$
$$= -i\hbar\left(g^{00}P^i - g^{0i}P^0\right)$$
$$= i\hbar P^i$$

$$[K_i, P_j] = [M^{i0}, P^j]$$
$$= -[P^j, M^{i0}]$$
$$= -i\hbar\left(g^{j0}P^i - g^{ji}P^0\right)$$
$$= i\hbar\,\delta_{ij} H .$$ ■

Exercício VII :

O operador de translação deve satisfazer a relação:

$$T(a)\,T(b) = T(a+b).$$

$$\Rightarrow T(\delta a)^{-1}\,T(b)\,T(\delta a) = T(b)$$

$$\left(\mathbb{1} + \frac{i}{\hbar}\delta a_\mu P^\mu\right) T(b) \left(\mathbb{1} - \frac{i}{\hbar}\delta a_\mu P^\mu\right) = T(b)$$

$$T(b) - \frac{i}{\hbar}\delta a_\mu T(b) P^\mu + \frac{i}{\hbar}\delta a_\mu P^\mu T(b) = T(b)$$

$$\frac{i}{\hbar}\delta a_\mu [P^\mu, T(b)] = 0$$

$$[P^\mu, T(b)] = 0$$

$$[P^\mu, T(\delta b)] = 0$$

$$[P^\mu, \mathbb{1} - \tfrac{i}{\hbar}\delta b_\nu P^\nu] = 0$$

$$[P^\mu, P^\nu] = 0 \quad \blacksquare$$

Exercício VIII:

(a) $$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\phi(x)\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)$$

$$= \phi((\mathbb{1} - \delta\omega)x)$$

$$\phi(x) + \frac{i}{2\hbar}\delta\omega_{\mu\nu}\phi(x)M^{\mu\nu} - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\phi(x)$$

$$= \phi(x - \delta\omega x)$$

$$\phi(x) + \frac{i}{2\hbar}\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}]$$

$$= \phi(x) - (\delta\omega x)^\rho \partial_\rho \phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = 2\hbar i\, \delta\omega^\rho{}_\sigma x^\sigma \partial_\rho \phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = 2\hbar i\, \delta\omega_{\mu\nu} g^{\rho\mu} x^\nu \partial_\rho \phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = \delta\omega_{\mu\nu}\hbar i\left(g^{\rho\mu}x^\nu - g^{\rho\nu}x^\mu\right)\partial_\rho\phi(x)$$

$$\left[\phi(x), M^{\mu\nu}\right] = \frac{\hbar}{i}\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)\phi(x)$$

$$\left[\phi(x), M^{\mu\nu}\right] = \mathcal{L}^{\mu\nu}\phi(x)$$

(b):

$$\left[\left[\phi(x), M^{\mu\nu}\right], M^{\rho\sigma}\right] =$$

$$= \left[\mathcal{L}^{\mu\nu}\phi(x), M^{\rho\sigma}\right]$$

$$= \mathcal{L}^{\mu\nu}\left[\phi(x), M^{\rho\sigma}\right] + \underbrace{\left[\mathcal{L}^{\mu\nu}, M^{\rho\sigma}\right]}_{=0}\phi(x)$$

$$= \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x)$$

(c):

$$\left[[A,B],C\right] + \left[[B,C],A\right] + \left[[C,A],B\right]$$

$$= [A,B]C - C[A,B] + [B,C]A - A[B,C] + [C,A]B - B[C,A]$$

$$= \cancel{ABC} - \cancel{BAC} - \cancel{CAB} + \cancel{CBA} + \cancel{BCA} - \cancel{CBA} - \cancel{ABC} + \cancel{ACB} + \cancel{CAB} - \cancel{ACB} - \cancel{BCA} + \cancel{BAC}$$

$$= 0.$$

ⓓ :

$$\left[\phi(x),\left[M^{\mu\nu},M^{\rho\sigma}\right]\right] =$$

$$= -\left[M^{\mu\nu},\left[M^{\rho\sigma},\phi(x)\right]\right]-\left[M^{\rho\sigma},\left[\phi(x),M^{\mu\nu}\right]\right]$$

$$= -\left[\left[\phi(x),M^{\rho\sigma}\right],M^{\mu\nu}\right]+\left[\left[\phi(x),M^{\mu\nu}\right],M^{\rho\sigma}\right]$$

$$= \left(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}-\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\right)\phi(x).$$

ⓔ :

$$\left(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}-\mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu}\right)\phi(x) =$$

$$= \left[\mathcal{L}^{\mu\nu},\mathcal{L}^{\rho\sigma}\right]\phi(x)$$

$$= -\hbar^2\left[x^\mu\partial^\nu-x^\nu\partial^\mu,\ x^\rho\partial^\sigma-x^\sigma\partial^\rho\right]\phi(x)$$

$$= -\hbar^2\Big(x^\mu\left[\partial^\nu,x^\rho\partial^\sigma\right]+\left[x^\mu,x^\rho\partial^\sigma\right]\partial^\nu$$
$$-x^\mu\left[\partial^\nu,x^\sigma\partial^\rho\right]-\left[x^\mu,x^\sigma\partial^\rho\right]\partial^\nu$$
$$-x^\nu\left[\partial^\mu,x^\rho\partial^\sigma\right]-\left[x^\nu,x^\rho\partial^\sigma\right]\partial^\mu$$
$$+x^\nu\left[\partial^\mu,x^\sigma\partial^\rho\right]+\left[x^\nu,x^\sigma\partial^\rho\right]\partial^\mu\Big)\phi(x)$$

mas : $\left[\partial^\mu,x^\nu\right]=g^{\mu\nu}$

então :

$$= -\hbar^2\Big(x^\mu\left[\partial^\nu,x^\rho\right]\partial^\sigma+x^\rho\left[x^\mu,\partial^\sigma\right]\partial^\nu$$
$$-x^\mu\left[\partial^\nu,x^\sigma\right]\partial^\rho-x^\sigma\left[x^\mu,\partial^\rho\right]\partial^\nu$$

$$- x^\nu [\partial^\mu, x^\rho] \partial^\sigma - x^\rho [x^\nu, \partial^\sigma] \partial^\mu$$

$$+ x^\nu [\partial^\mu, x^\sigma] \partial^\rho + x^\sigma [x^\nu, \partial^\rho] \partial^\mu \Big) \phi(x)$$

$$= -\hbar^2 \Big(x^\mu g^{\nu\rho} \partial^\sigma - x^\rho g^{\mu\sigma} \partial^\nu - x^\mu g^{\nu\sigma} \partial^\rho + x^\sigma g^{\mu\rho} \partial^\nu$$
$$- x^\nu g^{\mu\rho} \partial^\sigma + x^\rho g^{\nu\sigma} \partial^\mu + x^\nu g^{\mu\sigma} \partial^\rho - x^\sigma g^{\nu\rho} \partial^\mu \Big) \phi(x)$$

$$= \frac{\hbar}{i} \Big(g^{\nu\rho} \mathcal{L}^{\mu\sigma} + g^{\mu\rho} \mathcal{L}^{\sigma\nu} + g^{\mu\sigma} \mathcal{L}^{\nu\rho} + g^{\nu\sigma} \mathcal{L}^{\rho\mu} \Big) \phi(x)$$

$$= -i\hbar \Big(g^{\nu\rho} [\phi(x), M^{\mu\sigma}] + g^{\mu\rho} [\phi(x), M^{\sigma\nu}] + g^{\mu\sigma} [\phi(x), M^{\nu\rho}]$$
$$+ g^{\nu\sigma} [\phi(x), M^{\rho\mu}] \Big)$$

$$= \Big[\phi(x), -i\hbar \big(g^{\nu\rho} M^{\mu\sigma} + g^{\mu\rho} M^{\sigma\nu} + g^{\mu\sigma} M^{\nu\rho} + g^{\nu\sigma} M^{\rho\mu} \big) \Big]$$

(f) :

como :

$$[\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]]$$

$$= \Big[\phi(x), i\hbar \big(g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\nu\rho} + g^{\nu\sigma} M^{\mu\rho} \big) \Big]$$

logo :

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar \big(g^{\mu\rho} M^{\nu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\mu\sigma} M^{\nu\rho} + g^{\nu\sigma} M^{\mu\rho} \big) .$$

$$+ C$$

$$[C, \phi(x)] = 0 \quad e \quad [C, \partial^\alpha \cdots \partial^\beta \phi(x)] = 0 . \qquad \blacksquare$$

Exercício (IX):

(a):

$$U(\Lambda)^{-1}\partial^\rho \phi(x) U(\Lambda) = \Lambda^\rho{}_\sigma \bar{\partial}^\sigma \phi(\Lambda^{-1}x)$$

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\partial^\rho \phi(x)\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)$$
$$= \left(\delta^\rho{}_\sigma + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}(S_V^{\alpha\beta})^\rho{}_\sigma\right)\bar{\partial}^\sigma \phi(\Lambda^{-1}x)$$

$$\partial^\rho\phi(x) + \frac{i}{2\hbar}\delta\omega_{\mu\nu}\left(\partial^\rho\phi(x)M^{\mu\nu} - M^{\mu\nu}\partial^\rho\phi(x)\right)$$
$$= \bar{\partial}^\rho\phi(x - \delta\omega\, x) + \frac{i}{2\hbar}\delta\omega_{\alpha\beta}(S_V^{\alpha\beta})^\rho{}_\sigma\bar{\partial}^\sigma\phi(x-\delta\omega x)$$

$$\partial^\rho\phi(x) + \frac{i}{2\hbar}\delta\omega_{\mu\nu}\left[\partial^\rho\phi(x), M^{\mu\nu}\right]$$
$$= \bar{\partial}^\rho\phi(x) - (\delta\omega\, x)_\gamma\partial^\gamma\bar{\partial}^\rho\phi(x)$$
$$+ \frac{i}{2\hbar}\delta\omega_{\alpha\beta}(S_V^{\alpha\beta})^\rho{}_\sigma\left(\bar{\partial}^\sigma\phi(x) - (\delta\omega\, x)_\varepsilon\partial^\varepsilon\bar{\partial}^\sigma\phi(x)\right)$$

$$\delta\omega_{\mu\nu}\left[\partial^\rho\phi(x), M^{\mu\nu}\right]$$
$$= -\frac{2\hbar}{i}\delta\omega_{\mu\nu}x^\nu\partial^\mu\partial^\rho\phi(x)$$
$$+ \delta\omega_{\mu\nu}(S_V^{\mu\nu})^\rho{}_\sigma\partial^\sigma\phi(x)$$

$$\delta\omega_{\mu\nu}\left[\partial^\rho\phi(x), M^{\mu\nu}\right]$$
$$= -\frac{\hbar}{i}\delta\omega_{\mu\nu}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\partial^\rho\phi(x)$$
$$+ \delta\omega_{\mu\nu}(S_V^{\mu\nu})^\rho{}_\sigma\partial^\sigma\phi(x)$$

$$\left[\partial^\rho \phi(x), M^{\mu\nu}\right] = \mathcal{L}^{\mu\nu}\partial^\rho \phi(x) + (S_V^{\mu\nu})^\rho{}_\sigma \partial^\sigma \phi(x) \quad \blacksquare$$

(b):

$$\left[S_V{}^{\mu\nu}, S_V{}^{\rho\sigma}\right]^\alpha{}_\beta$$

$$= (S_V{}^{\mu\nu})^\alpha{}_\tau (S_V{}^{\rho\sigma})^\tau{}_\beta - (S_V{}^{\rho\sigma})^\alpha{}_\tau (S_V{}^{\mu\nu})^\tau{}_\beta$$

$$= -\hbar^2 (g^{\mu\alpha}\delta^\nu{}_\tau - g^{\nu\alpha}\delta^\mu{}_\tau)(g^{\rho\tau}\delta^\sigma{}_\beta - g^{\sigma\tau}\delta^\rho{}_\beta)$$
$$+\hbar^2 (g^{\rho\alpha}\delta^\sigma{}_\tau - g^{\sigma\alpha}\delta^\rho{}_\tau)(g^{\mu\tau}\delta^\nu{}_\beta - g^{\nu\tau}\delta^\mu{}_\beta)$$

$$= -\hbar^2 \Big(g^{\mu\alpha} g^{\rho\nu}\delta^\sigma{}_\beta - g^{\mu\alpha} g^{\sigma\nu}\delta^\rho{}_\beta - g^{\nu\alpha} g^{\rho\mu}\delta^\sigma{}_\beta$$
$$+ g^{\nu\alpha} g^{\sigma\mu}\delta^\rho{}_\beta - g^{\rho\alpha} g^{\mu\sigma}\delta^\nu{}_\beta + g^{\rho\alpha} g^{\nu\sigma}\delta^\mu{}_\beta$$
$$+ g^{\sigma\alpha} g^{\mu\rho}\delta^\nu{}_\beta - g^{\sigma\alpha} g^{\nu\rho}\delta^\mu{}_\beta \Big)$$

$$= \frac{\hbar}{i}\frac{\hbar}{i}\Big(g^{\mu\rho}(g^{\sigma\alpha}\delta^\nu{}_\beta - g^{\nu\alpha}\delta^\sigma{}_\beta)$$
$$+ g^{\nu\rho}(g^{\mu\alpha}\delta^\sigma{}_\beta - g^{\sigma\alpha}\delta^\mu{}_\beta)$$
$$+ g^{\mu\sigma}(g^{\nu\alpha}\delta^\rho{}_\beta - g^{\rho\alpha}\delta^\nu{}_\beta)$$
$$+ g^{\nu\sigma}(g^{\rho\alpha}\delta^\mu{}_\beta - g^{\mu\alpha}\delta^\rho{}_\beta)\Big)$$

$$= -i\hbar \Big(g^{\mu\rho}(S_V^{\sigma\nu})^\alpha{}_\beta + g^{\nu\rho}(S_V{}^{\mu\sigma})^\alpha{}_\beta + g^{\mu\sigma}(S_V{}^{\nu\rho})^\alpha{}_\beta$$
$$+ g^{\nu\sigma}(S_V{}^{\rho\mu})^\alpha{}_\beta \Big)$$

$$= i\hbar \Big(g^{\mu\rho}(S_V{}^{\nu\sigma})^\alpha{}_\beta - g^{\nu\rho}(S_V{}^{\mu\sigma})^\alpha{}_\beta$$
$$- g^{\mu\sigma}(S_V{}^{\nu\rho})^\alpha{}_\beta + g^{\nu\sigma}(S_V{}^{\mu\rho})^\alpha{}_\beta \Big)$$

(c) :

$$(S^{12}_{V})^{\mu}{}_{\nu} = \frac{\hbar}{i}\left(g^{1\mu}\delta^{2}{}_{\nu} - g^{2\mu}\delta^{1}{}_{\nu}\right)$$

$$= \frac{\hbar}{i}\left(\delta_{1\mu}\delta_{2\nu} - \delta_{2\mu}\delta_{1\nu}\right)$$

$$S^{12}_{V} = \frac{\hbar}{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(S^{12}_{V}\,\frac{-i}{\hbar}\right)^{2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(S^{12}_{V}\,\frac{-i}{\hbar}\right)^{3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\left(S^{12}_{V}\,\frac{-i}{\hbar}\right)$$

$$S_v^{12\,3} \frac{(-1)^3}{\hbar^3} i^3 = S_v^{12} \frac{i}{\hbar} \Rightarrow S_v^{12\,3} = \hbar^2 S_v^{12}$$

logo: $\exp\left(\frac{-i\theta}{\hbar} S_v^{12}\right) = \sum_{n=0}^{\infty} \left(\frac{-i\theta}{\hbar}\right)^n \left(S_v^{12}\right)^n \cdot \frac{1}{n!}$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \left(\frac{-i\theta}{\hbar}\right)^{2n-1} \cdot \frac{(\hbar)^{2n-2}}{(2n-1)!} S_v^{12} + \sum_{n=1}^{\infty} \left(\frac{-i\theta}{\hbar}\right)^{2n} \frac{(S_v^{12})^2 (\hbar)^{2n-2}}{(2n)!}$$

$$= \mathbb{1} + \frac{i}{\hbar} \sum_{n=1}^{\infty} \frac{(-i)^{2n}\theta^{2n-1}}{(2n-1)!} S_v^{12} + \frac{1}{\hbar^2} \sum_{n=1}^{\infty} \frac{(-i\theta)^{2n}}{(2n)!} (S_v^{12})^2$$

$$= \mathbb{1} + \frac{(S_v^{12})^2}{\hbar^2} \cdot \left(-\frac{\theta^2}{2} + \frac{\theta^4}{4!} - \dots \right)$$

$$+ \frac{i}{\hbar} S_v^{12} \cdot \left(-\theta + \frac{\theta^3}{3!} - \dots \right)$$

$$= \mathbb{1} + \frac{(S_v^{12})^2}{\hbar^2} (\cos\theta - 1) - \frac{i}{\hbar} S_v^{12} \operatorname{sen}\theta$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \cos\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \operatorname{sen}\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\operatorname{sen}\theta & 0 \\ 0 & \operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^\mu{}_v .$$

(d):

$$(S^{30}{}_v)^\mu{}_v = \frac{\hbar}{i}\left(g^{3\mu} \delta^0{}_v - g^{0\mu} \delta^3{}_v \right)$$

$$= \frac{\hbar}{i}\left(\delta_{3\mu}\delta_{0v} + \delta_{0\mu}\delta_{3v} \right)$$

$$\left(S^{30}_{\ \ v}\right)^{\mu}{}_{\nu} = \frac{\hbar}{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(S^{30}_{\ \ v}\right)^{2} = \frac{\hbar^2}{i^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(S^{30}_{\ \ v}\right)^{3} = \frac{\hbar^3}{i^3}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \left(S^{30}_{\ \ v}\right)\cdot\frac{\hbar^2}{i^2}$$

$$\exp\left(\frac{i\eta S^{30}_{\ \ v}}{\hbar}\right) = \sum_{n=0}^{\infty}\left(\frac{i\eta}{\hbar}\right)^n \left(S^{30}_{\ \ v}\right)^n\cdot\frac{1}{n!}$$

$$= \mathbb{1} + \sum_{n=1}^{\infty}\left(\frac{i\eta}{\hbar}\right)^{2n}\frac{1}{(2n)!}\left(S^{30}_{\ \ v}\right)^2(\hbar)^{2n-2}(-1)^{n-1}$$

$$+ \sum_{n=1}^{\infty}\left(\frac{i\eta}{\hbar}\right)^{2n-1}\frac{1}{(2n-1)!}\left(S^{30}_{\ \ v}\right)\hbar^{2n-2}(-1)^{n-1}$$

$$= \mathbb{1} + \frac{\left(S^{30}_{\ \ v}\right)^2}{\hbar^2}\left(-\frac{\eta^2}{2} - \frac{\eta^4}{4!} - \dots\right)$$

$$+ \frac{\left(S^{30}_{\ \ v}\right)}{i\hbar}\left(-\eta - \frac{\eta^3}{3!} - \dots\right)$$

$$= \mathbb{1} - \frac{\left(S^{30}_{\ \ v}\right)^2}{\hbar^2}(\cosh\eta - 1) - \frac{S^{30}_{\ \ v}}{i\hbar}\sinh\eta$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \cosh\eta\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sinh\eta\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$$