Capítulo I :

equação de Klein Gordon:

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\phi = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right)\phi$$

$$-\hbar^2 c^2 \partial_0^2 \phi = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right)\phi$$

$$\hbar^2 c^2 \underbrace{\left(-\partial_0^2 + \nabla^2\right)}_{\partial_\mu \partial^\mu}\phi = m^2 c^4 \phi .$$

$$\hbar^2 c^2 \partial_\mu \partial^\mu \phi = m^2 c^4 \phi$$

$$\left(\partial_\mu \partial^\mu - \frac{m^2 c^2}{\hbar^2}\right)\phi = 0$$

$\int d^3x\, \phi^* \phi$ ñ é independente temporalmente!!

Tentativa de Dirac :

$$i\hbar \partial_t \phi_a = \left(-i\hbar c (\alpha^j)_{ab} \partial_j + mc^2 (\beta)_{ab}\right)\phi_b$$

$$(H)_{ab} = c P_j (\alpha^j)_{ab} + mc^2 (\beta)_{ab}$$

$$P_j = -i\hbar \partial_j$$

note que :

$$(H^2)_{ab} = c^2 P_j P_k (\alpha^j \alpha^k)_{ab} + mc^3 P_j (\alpha^j \beta + \beta \alpha^j)_{ab} + m^2 c^4 (\beta^2)_{ab}$$

o que implica nas relações de anti-comutações:

$$(\{\alpha^j, \alpha^k\})_{ab} = 2\delta^{jk}\delta_{ab}$$

$$(\{\alpha^j, \beta\})_{ab} = 0 \quad , \quad (\beta^2)_{ab} = \delta_{ab}.$$

$$\hookrightarrow (H^2)_{ab} = (P^2c^2 + m^2c^4)\delta_{ab} \quad \leftarrow \text{Hamiltoniano relativístico.}$$

Exercício (I):

$$(H)_{ab}(H)_{bc} = (cP_j(\alpha^j)_{ab} + mc^2(\beta)_{ab})(cP_k(\alpha^k)_{bc} + mc^2(\beta)_{bc})$$

$$= c^2 P_j(\alpha^j)_{ab} P_k(\alpha^k)_{bc} + mc^3 P_j(\alpha^k)_{ab}(\beta)_{bc}$$

$$+ mc^3 P_k(\beta)_{ab}(\alpha^k)_{bc} + m^2c^4(\beta)_{ab}(\beta)_{bc}$$

$$= c^2 P_j P_k(\alpha^j\alpha^k)_{ac} + mc^3 P_j(\alpha^j\beta + \beta\alpha^j)_{ac}$$

$$+ m^2c^4(\beta^2)_{ac}$$

$$\Rightarrow \left(\frac{\alpha^j\alpha^k + \alpha^k\alpha^j}{2}\right)_{ac} = \delta^{jk}\delta_{ac}$$

$$(\alpha^j\beta + \beta\alpha^j)_{ac} = 0$$

$$(\beta^2)_{ac} = \delta_{ac}$$

$$\Rightarrow \{\alpha^j, \alpha^k\}_{ab} = 2\delta^{jk}\delta_{ab}$$

$$\{\alpha^j, \beta\}_{ab} = 0 \quad , \quad (\beta^2)_{ab} = \delta_{ab}.$$

Os autovalores de β são:

$$\beta v = \lambda v \Rightarrow \beta^2 v = \lambda \beta v = \lambda^2 v$$
$$v = \lambda^2 v \Rightarrow \lambda = \pm 1.$$

mas como $\quad \alpha^j \beta = -\beta \alpha^j.$

$$\beta = -\alpha^j \beta \alpha^j$$

assim:

$$Tr(\beta) = Tr(-\alpha^j \beta \alpha^j) = -Tr(\beta \alpha^j \alpha^j)$$
$$= -Tr(\beta) \Rightarrow Tr(\beta) = 0.$$

como $\quad Tr(\beta) = \sum_\lambda \lambda.$

isso implica na dimensão de β ser par.

para α^j:

$$\alpha^j v = \lambda v \Rightarrow {\alpha^j}^2 v = \lambda \alpha^j v = \lambda^2 v$$
$$v = \lambda^2 v \Rightarrow \lambda = \pm 1.$$

mas: $\quad \alpha^j \beta = -\beta \alpha^j$

$$\alpha^j = -\beta \alpha^j \beta$$

$$Tr(\alpha^j) = -Tr(\beta \alpha^j \beta) = -Tr(\alpha^j \beta^2)$$
$$= -Tr(\alpha^j) \Rightarrow Tr(\alpha^j) = 0$$

que também implica na dimensão de α^j ser par. ∎

Exercício (II):

o caso $n=1$:

$$H|\phi,t\rangle = \int d^3x\, a^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right)a(x)\int d^3x_1\, \phi(x_1;t)\, a^\dagger(x_1)|0\rangle$$

$$+\frac{1}{2}\int d^3x\, d^3y\, V(x-y)\, a^\dagger(x)a^\dagger(y)\, a(y)\, a(x)\int d^3x_1\, \phi(x_1;t)\, a^\dagger(x_1)|0\rangle$$

$$=\int d^3x \int d^3x_1\, a^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right)\phi(x_1;t)\left(a^\dagger(x_1)a(x) + \delta^3(x_1-x)\right)|0\rangle$$

$$+\frac{1}{2}\int d^3x\, d^3y\, d^3x_1\, V(x-y)\, \phi(x_1;t)\, a^\dagger(x)a^\dagger(y)a(y)\left(a^\dagger(x_1)a(x) + \delta^3(x_1-x)\right)|0\rangle$$

$$=\int d^3x\, a^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right)\phi(x;t)|0\rangle$$

$$=\int d^3x\, a^\dagger(x)\, i\hbar\frac{\partial}{\partial t}\phi(x;t)|0\rangle$$

$$= i\hbar\frac{\partial}{\partial t}\int d^3x\, \phi(x;t)a^\dagger(x)|0\rangle = i\hbar\frac{\partial}{\partial t}|\phi,t\rangle$$

logo: $\quad H|\phi,t\rangle = i\hbar\frac{\partial}{\partial t}|\phi,t\rangle$

suponha agora:

$$H|\phi,t\rangle = i\hbar\frac{\partial}{\partial t}|\phi,t\rangle:$$

$$\int d^3x\, a^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right)a(x)\int d^3x_1\, \phi(x_1;t)\, a^\dagger(x_1)|0\rangle$$

$$= i\hbar\frac{\partial}{\partial t}\int d^3x\, \phi(x;t)\, a^\dagger(x)|0\rangle$$

$$\Rightarrow \int d^3x\int d^3x_1\, a^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right)\phi(x_1;t)\left(a^\dagger(x_1)a(x) + \delta^3(x_1-x)\right)|0\rangle$$

$$= i\hbar\frac{\partial}{\partial t}\int d^3x\, \phi(x;t)\, a^\dagger(x)|0\rangle$$

$$\Rightarrow \int d^3x \; a^\dagger(x) \left(-\frac{\hbar^2\nabla^2}{2m} + U(x)\right) \phi(x;t) \,|0\rangle$$

$$= \int d^3x \; a^\dagger(x) \, i\hbar \frac{\partial}{\partial t} \phi(x;t) \,|0\rangle$$

$$\Rightarrow \left(-\frac{\hbar^2\nabla^2}{2m} + U(x)\right) \phi(x,t) = i\hbar \frac{\partial}{\partial t} \phi(x;t).$$

O mesmo raciocínio se aplica para $n > 1$. ■

Exercício (III):

$$[N, H] = \left[\int d^3x \, a^\dagger(x) a(x) \, , \int d^3y \, a^\dagger(y) \left(-\frac{\hbar^2\nabla^2}{2m} + U(y)\right) a(y) \right]$$

$$= \int d^3x \int d^3y \Big(a^\dagger(x) a(x) a^\dagger(y) \left(-\frac{\hbar^2\nabla^2}{2m} + U(y)\right) a(y)$$

$$- a^\dagger(y) \left(-\frac{\hbar^2\nabla^2}{2m} + U(y)\right) a(y) a^\dagger(x) a(x) \Big).$$

$$= \int d^3x \, d^3y \Big[a^\dagger(x) a(x) a^\dagger(y) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(y)\right) a(y)$$

$$- a^\dagger(y) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(y)\right)\left(a^\dagger(x) a(y) + \delta^3(x-y)\right) a(x) \Big]$$

$$= \int d^3x \, d^3y \Big[a^\dagger(x) a(x) a^\dagger(y) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(y)\right) a(y)$$

$$- a^\dagger(x) a^\dagger(y) a(x) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(y)\right) a(y) \Big]$$

$$- \int d^3y \, a^\dagger(y) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(y)\right) a(y)$$

$$= \int d^3x \, d^3y \Big[a^\dagger(x) a(x) a^\dagger(y) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(y)\right) a(y)$$

$$- a^\dagger(x) \left(a(x) a^\dagger(y) - \delta^3(y-x)\right)\left(-\frac{\hbar^2\nabla^2}{2m} + U(y)\right) a(y) \Big]$$

$$- \int d^3y \, a^\dagger(y) \left(-\frac{\hbar^2}{2m}\nabla^2 + U(y)\right) a(y)$$

$$= 0$$ ■