Lecture 9

4.3 Tangent Spaces to a manifold

Let M be a smooth manifold. Then we can construct the vector space over $\mathbb{R}$:

$$(C^\infty(M), +, \cdot) \to (\lambda g)(p) := \lambda \cdot g(p)$$
$$\| $$
$$\{f : M \to \mathbb{R} \mid \text{smooth}\} \to (f+g)(p) := f(p) + g(p)$$

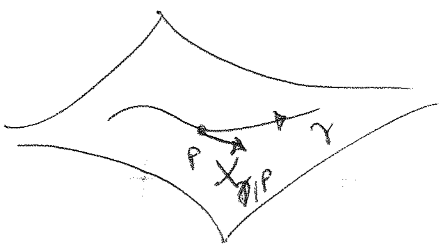
Def: Let $\gamma : \mathbb{R} \to M$ be a smooth curve through a point $p \in M$, without loss of generality, $\gamma(0) = p$. Then the directional derivative operator at $p \in M$ along the curve γ. Is the linear map:

$$X_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$$
$$f \mapsto (f \circ \gamma)'(0)$$
$$f \circ \gamma : \mathbb{R} \to \mathbb{R}$$

Terminology: In differential geometry, $X_{\gamma,p}$ is usually called the tangent vector to the curve γ at $p \in M$.

A precise intuition: $X_{\gamma,p}$ is the velocity of the curve γ at p. Consider: $\delta(\lambda) := \gamma(2\lambda)$, then:

$$X_{\delta,p}(f) = (f \circ \delta)'(0) = 2(f \circ \gamma)'(0) = 2X_{\gamma,p}f$$



Def: The ~~tangent~~ vector space T_pM is the set:

$$T_pM := \{ X_{\gamma,p} \mid \gamma \text{ smooth curve through } p \}$$

equipped with

$$\oplus : T_pM \times T_pM \to ?$$

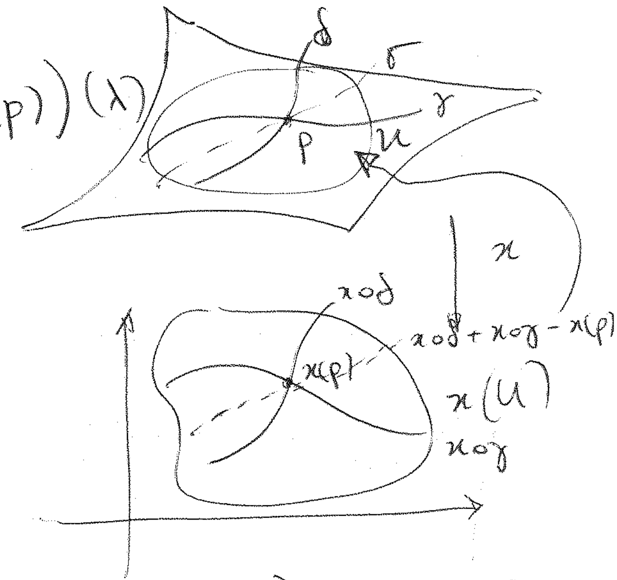
$$\odot : \mathbb{R} \times T_pM \to ?$$

Defined pointwise: $(X_{\gamma,p} \oplus X_{\delta,p})(f) := X_{\gamma,p}(f) + X_{\delta,p}(f)$

$$(\lambda \odot X_{\gamma,p})(f) := \lambda (X_{\gamma,p} f).$$

Is $(X_{\gamma,p} \oplus X_{\delta,p}) \in T_pM$? Construct σ as, given $p \in \mathcal{U}$, $(\mathcal{U}, x)$ a chart.

$$\sigma(\lambda) := x^{-1} \circ (x \circ \gamma + x \circ \delta - x(p))(\lambda)$$



$$\begin{tikzcd}
\mathrm{preim}_\gamma(\mathcal{U}) \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & \mathcal{U} \arrow[d, "x"] \\
 & x(\mathcal{U}) \subseteq \mathbb{R}^n
\end{tikzcd}$$

Check: $\sigma(0) = x^{-1} \circ (x(p) + x(p) - x(p)) = p$ ✓

$$X_{\sigma,p} f = (f \circ \sigma)'(0)$$

$$= [\underbrace{f \circ x^{-1}}_{\mathbb{R}^n \to \mathbb{R}} \circ \underbrace{(x \circ \delta + x \circ \gamma - x(p))}_{\mathbb{R} \to \mathbb{R}^n}]'(0)$$

$$= \partial_a (f \circ x^{-1})(x(p)) \cdot (x^a \circ \gamma + x^a \circ \delta - x^a(p))'(0)$$

$$= (f \circ x^{-1} \circ x \circ \gamma)'(0) + (f \circ x^{-1} \circ x \circ \delta)'(0)$$

$$= (f \circ \gamma)'(0) + (f \circ \delta)'(0) = (X_{\gamma,p} \oplus X_{\delta,p})(f).$$

Aside: algebras & derivations:

Def: A vector space $(V,+,\cdot)$ equipped with a "product", bilinear map, $\bullet : V \times V \xrightarrow{\sim} V$ is called an algebra.

$(V, +, \cdot, \bullet)$.

Example: $(C^\infty(M), +, \cdot)$ $\mathbb{R}$-vector space, with:

$$\bullet : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$
$$(f,g) \mapsto f \bullet g \quad , \quad (f\bullet g)(p) := f(p) \cdot g(p).$$

Then $(C^\infty(M), +, \cdot, \bullet)$ is a algebra.

Def: A derivation is a linear map
can be a different algebra B if B is a bimodule over A.

$D : A \xrightarrow{\sim} A$ which additionally satisfies the Leibnitz rule: $D(f \bullet g) = (Df) \bullet g + f \bullet (Dg)$

Example: (i) $A = (C^\infty(M), +, \cdot, \bullet)$, $D = X_p \in T_pM$.

$$X_p : C^\infty(M) \to \mathbb{R}.$$

$$X_p(fg) = X_p(f)\, g(p) + f(p) \cdot X_p(g) \ ?$$

(ii) Let $A = End(V)$. Define $\bullet : A \times A \to A$

$$(\phi, \psi) \mapsto \phi \circ \psi - \psi \circ \phi$$

$(End(V), +, \cdot, [\cdot,\cdot])$ is a algebra (Lie algebra)

$$\phi \bullet \psi := [\phi, \psi].$$

can be defined a derivation: $D_H = [H, \cdot] : A \xrightarrow{\sim} A$

$D([A,B]) = [D(A), B] + [A, D(B)]$ Jacobi!

Leibnitz!

So it's a derivation.

Theorem: $\underbrace{\dim(T_pM)}_{\text{as vector space}} = \underbrace{\dim(M)}_{\text{as topological manifold}}$.

- Idea: Construct a vector space basis from a chart

Choose $(\mathcal{U}, x)$, $p \in \mathcal{U}$. Consider $(\dim M)$-many curves

$\gamma_{(a)}: \mathbb{R} \to \mathcal{U}$, with $(x^b \circ \gamma_{(a)})(\lambda) := \delta^b_a \lambda$

Calculate the ~~tangent~~ vectors: $\gamma_{(a)}(0) = p$, asuming a chart

$x(p) = 0_{\mathbb{R}^{\dim M}}$

$e_a := X_{\gamma_{(a)}, p}$, $e_a f = X_{\gamma_{(a)}, p} f = (f \circ \gamma_{(a)})'(0)$

$$e_a f = (\underbrace{f \circ x^{-1}}_{\mathbb{R}^{\dim M} \to \mathbb{R}} \circ \underbrace{x \circ \gamma_{(a)}}_{\mathbb{R} \to \mathbb{R}^{\dim M}})'(0) = \partial_b (f \circ x^{-1})(x(\gamma_{(a)}(0))) \cdot (x^b \circ \gamma_{(a)})'(0)$$

$$= \partial_b (f \circ x^{-1})(x(p)) \cdot \delta^b{}_a = \partial_a (f \circ x^{-1})(x(p))$$

$$=: \left(\frac{\partial}{\partial x^a} f\right)_p, \quad f: M \to \mathbb{R} \; !$$

$(\partial_a)_{x(p)}: C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$

$\left(\frac{\partial}{\partial x^a}\right)_p : C^\infty(M, \mathbb{R}) \to \mathbb{R}$

In summary: $e_a = \left(\frac{\partial}{\partial x^a}\right)_p$. Claim: Any vector at T_pM can be written as:

$X = \underset{\in \mathbb{R}}{X^a} \left(\frac{\partial}{\partial x^a}\right)$, that is, $\frac{\partial}{\partial x^a}$; $1 \le a \le \dim M$ is a generating system to T_pM.

$\exists\, \mu : \mathbb{R} \to M$ smooth curve ~~through~~ p : $X = X_{\mu,p}$.

$$X_{\mu,p} f = (f\circ\mu)'(0) = (f\circ x^{-1}\circ x\circ\mu)'(0)$$

$$= \partial_b (f\circ x^{-1})(x(p)) \cdot (x^b\circ\mu)'(0)$$

$$= \left(\frac{\partial}{\partial x^b} f\right)_p \cdot \underbrace{(x^b\circ\mu)'(0)}_{\mathbb{R}\to\mathbb{R}} = \underbrace{(x^b\circ\mu)'(0)}_{\in\,\mathbb{R}} \left(\frac{\partial}{\partial x^b} f\right)_p .$$

$$= X^b \left(\frac{\partial}{\partial x^b} f\right)_p \quad \text{with} \quad X^b = (x^b\circ\mu)'(0) .$$

Thus $\left(\frac{\partial}{\partial x^a}\right)_p$ generate T_pM , but are they linearly independent? Given : $\lambda^a \left(\frac{\partial}{\partial x^a}\right)_p = 0$, we act on $x^b : \mathcal{U} \to \mathbb{R}$ b-th component of x.

$$\begin{tikzcd} \mathcal{U} \arrow[r, "x^b"] \arrow[d, "x"] & \mathbb{R} \\ x(\mathcal{U}) \subseteq \mathbb{R}^n \arrow[ur, "x^b\circ x^{-1}"'] & \end{tikzcd}$$

x^b is smooth if $x^b\circ x^{-1}$ is :

$$(x^b\circ x^{-1})(a^1, \dots, a^n) = a^b$$

$$(x^b\circ x^{-1}) = \mathrm{proj}_b$$

Then : $\lambda^a \left(\frac{\partial}{\partial x^a}\right)_p x^b = 0$

$$\lambda^a \underbrace{\partial_a (x^b\circ x^{-1})(x(p))}_{\delta^b{}_a} = 0 .$$

$\lambda^a \delta^b{}_a = 0 \Rightarrow \lambda^b = 0$ Then $\frac{\partial}{\partial x^a}$ are linearly independent and Hence , a basis.

Terminology : $X \in T_pM$, $X = X^a \left(\frac{\partial}{\partial x^a}\right)_p$, the real numbers X^a are called the ~~real~~ ~~coordinates~~ components of the vector X with respect to the ~~tangent~~ space basis induced by the chart.

In a change of charts : $y^b = y^b(x^1, \dots, x^n)$

$A^a{}_b = \left(\frac{\partial y^a}{\partial x^b}\right)\Big|_p$ is how ~~tangent~~ vectors transform.

This concludes as not considering position "vectors" as vectors.