GATF

Lecture 15: The Lie group $SL(2,\mathbb{C})$ and its Lie algebra $sl(2,\mathbb{C})$

4.4 The relativistic spin group $SL(2,\mathbb{C})$

ⓐ Set $SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1 \right\}$

ⓑ Group $(SL(2,\mathbb{C}), \bullet)$, $\bullet : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & h \end{pmatrix} := \begin{pmatrix} ae+bg & af+hb \\ ce+dg & cf+dh \end{pmatrix}$$

associative

neutral element: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

inverse: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

ⓒ Topological Space $(SL(2,\mathbb{C}), \mathcal{O})$.

(c.1) defining a topology on $\mathbb{C}$. Can be made by the open balls:

$$B_r(z) := \{ y \in \mathbb{C} \mid |y - z| < r \}.$$

(c.2): $\mathcal{O}_{\mathbb{C}\times\mathbb{C}\times\cdots\times\mathbb{C}}$.

(c.3) $SL(2,\mathbb{C}) \subseteq \mathbb{C}^4$:

$$\mathcal{O}_{SL(2,\mathbb{C})} := \mathcal{O}_{\mathbb{C}^4}\big|_{SL(2,\mathbb{C})}$$

$\left(SL(2,\mathbb{C}),\ \mathcal{O}_{\mathbb{C}^4}\big|_{SL(2,\mathbb{C})}\right)$ ~~topological~~ space !!

(d) Check ~~that~~ $(SL(2,\mathbb{C}), \mathcal{O})$ is topological manifold !

Construct charts.

$$\mathcal{U} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \,\middle|\, a \neq 0 \right\}$$

$$x : \mathcal{U} \to x(\mathcal{U}) \subseteq \mathbb{C}^3$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := (a, b, c) \quad \text{(continuous!)}$$

$$x^{-1} : x(\mathcal{U}) \to \mathcal{U}$$
$$(a,b,c) \mapsto \begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}. \quad \text{(continuous!)}$$

homeomorphism. ~~(x(u))~~
$(\mathcal{U}, x)$ is a chart.

Another chart:

$$V := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \,\middle|\, d \neq 0 \right\}$$

$$y : V \to y(V) \subseteq \mathbb{C}^3$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (b, c, d)$$

$$y^{-1} : y(V) \to V$$
$$(b, c, d) \mapsto \begin{pmatrix} \frac{1+bc}{d} & b \\ c & d \end{pmatrix}.$$

(V, y) is a chart!

need third char:

$$W := \{ \ldots \mid b \neq 0 \}$$

$$z : W \to z(W)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, d)$$

$$z^{-1} : z(W) \to W$$
$$(a, b, d) \mapsto \begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix}$$

(W, z) a chart.

$\mathcal{U} \cup W = SL(2, \mathbb{C})$. $(SL(2,\mathbb{C}), \mathcal{O})$ is a complex 3-manifold.

e) $(SL(2,\mathbb{C}), \mathcal{A})$ $\mathbb{C}$-differentiable manifold!

$$\{ \underset{a \neq 0}{(\mathcal{U}, x)}, \underset{b \neq 0}{(W, z)} \} = \mathcal{A}_{top}.$$

Consider:

$$z \circ x^{-1} : x(\mathcal{U} \cap W) \to z(\mathcal{U} \cap W)$$
$$(a, b, c) \mapsto z(x^{-1}(a, b, c))$$
$$= z\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix} = \left(a, b, \frac{1+bc}{a}\right)$$

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$$z\circ x^{-1}(a,b,c) = \left(a, b, \frac{1+bc}{a}\right) \qquad \boxed{\begin{array}{c} a\neq 0 \\ \text{and } b\neq 0\end{array}}$$

is $\boxed{C^\infty}$ (C^ω). also $x\circ z^{-1}$ is.

Thus we can define $\mathcal{A}$ as the maximal atlas containing $\mathcal{A}_{top}$.

$(SL(2,\mathbb{C}), \mathcal{A})$ is $\mathbb{C}$-differentiable 3-manifold !!

f) Is $(SL(2,\mathbb{C}), \cdot)$ a lie group?

$\mu : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$

$i : SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ are $\mathbb{C}$-differentiable

$$\begin{tikzcd}
SL(2,\mathbb{C}) \arrow[r, "i"] & SL(2,\mathbb{C}) \\
U \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, "x"] & V \arrow[u, phantom, "\cup"] \arrow[d, "y"] \\
x(U) \arrow[r, "y\circ i\circ x^{-1}"] \arrow[d, phantom, "\text{in}"] & y(V) \arrow[d, phantom, "\text{in}"] \\
\mathbb{C}^*\times\mathbb{C}\times\mathbb{C} & \mathbb{C}\times\mathbb{C}\times\mathbb{C}^*
\end{tikzcd}$$

$$y\circ i\circ x^{-1}(a,bc) = y\circ i\begin{pmatrix} a & b \\ c & \frac{1+bc}{a}\end{pmatrix}$$

$$-b, -c, a$$

$$= y\begin{pmatrix} \frac{1+bc}{a} & -b \\ -c & a\end{pmatrix} = (\quad)$$

then i is differentiable ? check for:

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "z"'] & W \arrow[d, "z"] \\ z(W) \arrow[r, "z\circ i\circ z^{-1}"] & z(W) \end{tikzcd} \Rightarrow \text{differentiable!!!}$$

C^ω

Similarly, using the atlas we can show ~~that~~ μ is C^ω.

$\Rightarrow$ $SL(2,\mathbb{C})$ is a Lie group (complex) 3-dimensional!

(g) Lie algebra $L(SL(2,\mathbb{C}))$ of the Lie group $SL(2,\mathbb{C})$

Notation:

$sl(2,\mathbb{C}) \equiv L(SL(2,\mathbb{C}))$

recall: $L(G) := \{ X \in \Gamma(TG) \mid l_{g*}(X_h) = X_{gh} \}$

recall: $T_e G \cong L(G)$.

recall: $[\![\cdot,\cdot]\!] : T_e G \times T_e G \to T_e G$.

$[\![A, B]\!] := [\, l_{g*} A,\ l_{g*} B \,]\big|_e$

If: $f \in C^\infty(G)$

$$\left[l_{g*}\left(\left(\frac{\partial}{\partial x^i}\right)_e\right)\right]_g f = \left(\frac{\partial}{\partial x^i}\right)_e \left(f \circ l_g\right)$$

$$= \partial_i \left(\underbrace{f \circ l_g \circ x^{-1}}_{\mathbb{C}^* \times \mathbb{C} \times \mathbb{C} \supseteq x(\mathcal{U}) \longrightarrow \mathbb{C}} \right) \Big|_{x(e)}$$

$$= \partial_i \left(\underbrace{\left(f \circ x^{-1}\right)}_{\mathbb{C}^3 \to \mathbb{C}} \circ \underbrace{\left(x \circ l_g \circ x^{-1}\right)}_{\mathbb{C}^3 \to \mathbb{C}^3} \right)$$

$$= \partial_m \left(f \circ x^{-1}\right) \Big|_{\underbrace{(x \circ l_g \circ x^{-1})(x(e))}_{\underbrace{e}_{\underbrace{g}_{x(g)}}}} \cdot \partial_i \left(x^m \circ l_g \circ x^{-1}\right) \Big|_{x(e)} .$$

$$= \partial_m \left(f \circ x^{-1}\right) \Big|_{x(g)} \cdot \partial_i \left(x^m \circ l_g \circ x^{-1}\right) \Big|_{x(e)} .$$

if we consider $SL(2,\mathbb{C})$, the map

$$= \left(\frac{\partial}{\partial x^m}\right)_g f$$

$$\left(x^m \circ l_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} x^{-1}\right)(e, f, g)$$

$$= \left(x^m \circ l_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right) \begin{pmatrix} e & f \\ g & \frac{1+fg}{e} \end{pmatrix}$$

⑥

$$x^m \begin{pmatrix} ae + bg & af + \frac{b+bfg}{e} \\ ce + dg & cf + \frac{d+fdg}{e} \end{pmatrix}$$

$$= \left(ae+bg \,,\; af + \frac{b+bfg}{e} \,,\; ce+dg \right)^m .$$

$$\Rightarrow \partial_i \left(x^m \circ l_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \circ x^{-1} \right) \Big|_{(1,0,0)}$$

$$= \left. \begin{bmatrix} a & 0 & b \\ -\frac{b(1+fg)}{e^2} & a + \frac{bg}{e} & \frac{bf}{e} \\ c & 0 & d \end{bmatrix}^m_{\;i} \right|_{(1,0,0)} = \begin{bmatrix} a & 0 & b \\ -b & a & 0 \\ c & 0 & d \end{bmatrix}^m_{\;i}$$

So :

$$\left[l_{g*}\left(\left(\frac{\partial}{\partial x^i} \right)_e \right) \right]_g f = \left(\left(\frac{\partial}{\partial x^m} \right)_g f \right) \left[\;\cdots\; \right]^m_{\;i}$$

$\boxed{i=1}$

$$= a \left(\frac{\partial}{\partial x^1} \right)_g - b \left(\frac{\partial}{\partial x^2} \right)_g + c \left(\frac{\partial}{\partial x^3} \right)_g$$

$\boxed{i=2}$

$$= a \left(\frac{\partial}{\partial x^2} \right)_g$$

$\boxed{i=3}$

$$= b \left(\frac{\partial}{\partial x^1} \right)_g + d \left(\frac{\partial}{\partial x^3} \right)_g .$$

$$\left[\!\left[\left(\frac{\partial}{\partial x^1}\right)_e , \left(\frac{\partial}{\partial x^2}\right)_e \right]\!\right] f := \left[l_{g*}\left(\frac{\partial}{\partial x^1}\right)_e , l_{g*}\left(\frac{\partial}{\partial x^2}\right)_e \right] f \Big|_e$$

$$= \left[a\left(\frac{\partial}{\partial x^1}\right)_g - b\left(\frac{\partial}{\partial x^2}\right)_g + c\left(\frac{\partial}{\partial x^3}\right)_g , a\left(\frac{\partial}{\partial x^2}\right)_g \right] f \Big|_e$$

$$= \;???$$

$$\Downarrow$$

$$= C^m{}_{12} \left(\frac{\partial}{\partial x^m}\right)_g . \qquad C \equiv \text{structure constants of } sl(2,\mathbb{C}).$$

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