Exercícios:

2.1 -

$$g_{\mu\nu}(\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \delta\omega^\nu{}_\sigma) = g_{\rho\sigma}$$

$$g_{\mu\nu}(\delta^\mu{}_\rho \delta^\nu{}_\sigma + \delta^\mu{}_\rho \delta\omega^\nu{}_\sigma + \delta^\nu{}_\sigma \delta\omega^\mu{}_\sigma) = g_{\rho\sigma}$$

$$g_{\rho\sigma} + g_{\rho\nu}\delta\omega^\nu{}_\sigma + g_{\mu\sigma}\delta\omega^\mu{}_\rho = g_{\rho\sigma}$$

$$\Rightarrow \delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}.$$

2.2 -

$$U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$$

$$U(\Lambda)^{-1}\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) = U\left(\Lambda^{-1}(\mathbb{1}+\delta\omega')\Lambda\right)$$

$$\mathbb{1} + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}\,U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \mathbb{1} + \frac{i}{2\hbar}(\Lambda^{-1})_\alpha{}^\mu\,\delta\omega'_{\mu\nu}\,\Lambda^\nu{}_\beta M^{\alpha\beta}$$

$$\delta\omega'_{\mu\nu}\,U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \delta\omega'_{\mu\nu}\Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha M^{\alpha\beta}$$

2.3 -

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)M^{\mu\nu}\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)$$

$$= (\delta^\nu{}_\beta + \delta\omega^\nu{}_\beta)(\delta^\mu{}_\alpha + \delta\omega^\mu{}_\alpha)M^{\alpha\beta}$$

$$M^{\mu\nu} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] = M^{\mu\nu} + \delta\omega^\nu{}_\beta M^{\mu\beta} + \delta\omega^\mu{}_\alpha M^{\alpha\nu}$$

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[M^{\mu\nu},M^{\rho\sigma}\right] = \frac{\delta\omega_{\rho\sigma}}{2}\left(g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

$$+ \frac{\delta\omega_{\rho\sigma}}{2}\left(g^{\mu\rho}M^{\sigma\nu} + g^{\mu\sigma}M^{\rho\nu} + g^{\mu\rho}M^{\sigma\nu} - g^{\mu\sigma}M^{\rho\nu}\right)$$

$$\left[M^{\mu\nu},M^{\rho\sigma}\right] = i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}\right)$$

2.4 —

$$\left[J_i, J_j\right] = \frac{1}{4}\epsilon_{iab}\epsilon_{jcd}\left[M^{ab},M^{cd}\right]$$

$$= \frac{1}{4}\Big(\delta_{ij}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \delta_{ic}(\delta_{ad}\delta_{bj} - \delta_{aj}\delta_{bd})$$

$$+ \delta_{id}(\delta_{aj}\delta_{bc} - \delta_{ac}\delta_{bj})\Big) \cdot$$

$$\cdot\, i\hbar\left(g^{ac}M^{bd} - g^{bc}M^{ad} - g^{ad}M^{bc} + g^{bd}M^{ac}\right)$$

$$= \frac{i\hbar}{4}\Big(\delta_{ij}\big(g^a{}_a M^b{}_b - g^b{}_a M^a{}_b - g^a{}_b M^b{}_a + g^b{}_b M^a{}_a$$

$$- g^a{}_b M^b{}_a + g^b{}_b M^a{}_a + g^a{}_a M^b{}_b - g^b{}_a M^a{}_b\big)$$

$$+ \delta_{ic}\big(g^{ac}M_{ja} - g_j{}^c M^a{}_a - g^a{}_a M_j{}^c + g_{ja}M^{ac}$$

$$- g_j{}^c M^b{}_b + g^{bc}M_{jb} + g_{jb}M^{bc} - g^b{}_b M_j{}^c\big)$$

$$+ \delta_{id}\big(g_{jb}M^{bd} - g^b{}_b M_j{}^d - g_j{}^d M^b{}_b + g^{bd}M_{jb}$$

$$- g^a{}_a M_j{}^d + g_{ja}M^{ad} + g^{ad}M_{ja} - g_j{}^d M^a{}_a\big)\Big)$$

$$= \frac{i\hbar}{2}\Big(-\delta_{ij}(g^b{}_a M^a{}_b + g^a{}_b M^b{}_a)\Big)$$

$$+ \frac{i\hbar}{4}\Big(g^a{}_i M_{ja} - g^a{}_a M_{ji} + g_{ja} M^a{}_i + g^b{}_i M_{jb} + g_{jb} M^b{}_i$$

$$- g^b{}_b M_{ji} + g_{jb} M^b{}_i - g^b{}_b M_{ji} + g^b{}_i M_{jb} - g^a{}_a M_{ji}$$

$$+ g_{ja} M^a{}_i + g^a{}_i M_{ja} \Big)$$

$$= \frac{i\hbar}{2}\Big(-\delta_{ij}(g^b{}_a M^a{}_b + g^a{}_b M^b{}_a) - g^a{}_a M_{ji} - g^b{}_b M_{ji} + g_{ja} M^a{}_i$$

$$+ g_{jb} M^b{}_i + g^a{}_i M_{ja} + g^b{}_i M_{jb} \Big)$$

$$= \frac{i\hbar}{2}\Big(-\delta_{ij}(M_{bb} + M_{aa}) - g^a{}_a M_{ji} - g^b{}_b M_{ji}$$

$$+ M_{ji} + M_{ji} + M_{ji} + M_{ji} \Big)$$

$$= -i\hbar M_{ji}$$

$$= \frac{i\hbar}{2} 2 M_{ij}$$

$$= \frac{i\hbar}{2}(M_{ij} - M_{ji})$$

$$= \frac{i\hbar}{2}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}) M^{ab}$$

$$= \frac{i\hbar}{2} \epsilon^k{}_{ij} \epsilon_{kab} M^{ab}$$

$$= i\hbar \epsilon_{ij}{}^k J_k$$

$$[J_i, K_j] = \frac{1}{2}\epsilon_{iab}\,[M^{ab}, M^{j0}]$$

$$= \frac{i\hbar}{2}\epsilon_{iab}\left(g^{aj}M^{b0} - g^{bj}M^{a0} - g^{a0}M^{bj} + g^{b0}M^{aj}\right)$$

$$= \frac{i\hbar}{2}\epsilon_{iab}\,g^{aj}M^{b0} + \frac{i\hbar}{2}\epsilon_{iba}\,g^{bj}M^{a0}$$

$$= \frac{i\hbar}{2}\epsilon_{iab}\,g^{aj}M^{b0}$$

$$= \frac{i\hbar}{2}\epsilon_{ijb}M^{b0} = i\hbar\,\epsilon_{ij}{}^{k}K_k$$

$$[K_i, K_j] = [M^{i0}, M^{j0}]$$

$$= i\hbar\left[g^{ij}M^{00} - g^{0j}M^{i0} - g^{i0}M^{0j} + g^{00}M^{ij}\right]$$

$$= -i\hbar\,M^{ij}$$

$$= -\frac{i\hbar}{2}\left(M^{ij} - M^{ji}\right)$$

$$= -\frac{i\hbar}{2}\left(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right)M^{ab}$$

$$= -\frac{i\hbar}{2}\epsilon^{k}{}_{ij}\,\epsilon_{kab}M^{ab}$$

$$= -i\hbar\,\epsilon_{ij}{}^{k}J_k$$

2.5 —

$$U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu \quad , \quad \Lambda = \mathbb{1} + \delta\omega$$

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma} M^{\rho\sigma}\right) P^\mu \left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma} M^{\rho\sigma}\right) = \left(\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu\right) P^\nu$$

$$P^\mu + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[P^\mu, M^{\rho\sigma}\right] = P^\mu + \delta\omega^\mu{}_\nu P^\nu$$

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}\left[P^\mu, M^{\rho\sigma}\right] = \delta\omega_{\rho\sigma}\, g^{\mu\rho} P^\sigma$$

$$\delta\omega_{\rho\sigma}\left[P^\mu, M^{\rho\sigma}\right] = -i\hbar\left(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho\right)$$

$$\left[P^\mu, M^{\rho\sigma}\right] = i\hbar\left(g^{\mu\sigma}P^\rho - g^{\mu\rho}P^\sigma\right)$$

2.6 —

$$[J_i, H] = [J_i, P^0]$$

$$= \frac{1}{2}\epsilon_{ijk}\left[M^{jk}, P^0\right]$$

$$= -\frac{1}{2}\epsilon_{ijk}\left[P^0, M^{jk}\right]$$

$$= -\frac{i\hbar}{2}\epsilon_{ijk}\left(g^{0k}P^j - g^{0j}P^k\right)$$

$$= 0$$

$$[J_i, P_j] = \frac{1}{2}\epsilon_{iab}\left[M^{ab}, P_j\right]$$

$$= -\frac{1}{2}\epsilon_{iab}\left[P_j, M^{ab}\right]$$

$$= -\frac{i\hbar}{2}\epsilon_{iab}\left(g^{jb}P^a - g^{ja}P^b\right)$$

$$= -\frac{i\hbar}{2}\left(\epsilon_{iaj}P^a - \epsilon_{ijb}P^b\right)$$

$$= i\hbar\,\epsilon_{ijk}P^k .$$

$$[K_i, H] = [K_i, P^0]$$

$$= [M^{i0}, P^0]$$

$$= -[P^0, M^{i0}]$$

$$= -i\hbar\left(g^{00}P^i - g^{0i}P^0\right)$$

$$= i\hbar P^i$$

$$[K_i, P_j] = [M^{i0}, P^j]$$

$$= -[P^j, M^{i0}]$$

$$= -i\hbar\left(g^{j0}P^i - g^{ji}P^0\right)$$

$$= i\hbar\,\delta_{ij}H .$$

2.7 —

O operador de translação deve satisfazer:

$$T(a)\,T(b) = T(a+b) .$$

$$T(\delta a)^{-1}\, T(b)\, T(\delta a) = T(b) ,$$

$$\left(\mathbb{1} + \frac{i}{\hbar}\delta a_\mu P^\mu\right) T(b) \left(\mathbb{1} - \frac{i}{\hbar}\delta a_\mu P^\mu\right) = T(b)$$

$$T(b) + \frac{i}{\hbar}\delta a_\mu [P^\mu, T(b)] = T(b)$$

$$[P^\mu, T(b)] = 0$$

$$[P^\mu, T(\delta b)] = 0$$

$$[P^\mu, \mathbb{1} - \frac{i}{\hbar}\delta a_\nu P^\nu] = 0$$

$$[P^\mu, P^\nu] = 0 .$$

■

2.6 —

ⓐ $$U(\Lambda)^{-1}\phi(x)\, U(\Lambda) = \phi(\Lambda^{-1} x)$$

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\mu\nu} M^{\mu\nu}\right)\phi(x)\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega_{\mu\nu} M^{\mu\nu}\right) = \phi(x - \delta\omega x)$$

$$\phi(x) + \frac{i}{2\hbar}[\phi(x), M^{\mu\nu}]\,\delta\omega_{\mu\nu} = \phi(x) - (\delta\omega)^\rho \partial_\rho \phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = 2\hbar i\, \delta\omega_{\mu\nu}\, g^{\rho\mu} x^\nu \partial_\rho \phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = \delta\omega_{\mu\nu}\, i\hbar\left(g^{\rho\mu}x^\nu - g^{\rho\nu}x^\mu\right)\partial_\rho\phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = \delta\omega_{\mu\nu}\frac{\hbar}{i}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\phi(x)$$

$$[\phi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\phi(x) .$$

(b)

$$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\phi(x), M^{\rho\sigma}]$$

$$= \mathcal{L}^{\mu\nu}[\phi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x).$$

(c)

$$[[A,B],C] + [[B,C],A] + [[C,A],B] =$$

$$= ABC - BAC - CAB + CBA + BCA - CBA$$

$$- ABC + ACB + CAB - ACB - BCA + BAC$$

$$= 0$$

(d)

$$[\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]] =$$

$$= -[M^{\mu\nu}, [M^{\rho\sigma}, \phi(x)]] - [M^{\rho\sigma}, [\phi(x), M^{\mu\nu}]]$$

$$= -[[\phi(x), M^{\rho\sigma}], M^{\mu\nu}] + [[\phi(x), M^{\mu\nu}], M^{\rho\sigma}]$$

$$= (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\phi(x).$$

(e) $$(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\phi(x) = [\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\phi(x) =$$

$$= -\hbar^2 [x^\mu\partial^\nu - x^\nu\partial^\mu, x^\rho\partial^\sigma - x^\sigma\partial^\rho]\phi(x)$$

$$= -\hbar^2 (x^\mu[\partial^\nu, x^\rho\partial^\sigma] + [x^\mu, x^\rho\partial^\sigma]\partial^\nu -$$

$$- x^\mu [\partial^\nu, x^\sigma \partial^\rho] - [x^\mu, x^\sigma \partial^\rho]\partial^\nu - x^\nu [\partial^\mu, x^\rho \partial^\sigma]$$

$$-[x^\nu, x^\rho\partial^\sigma]\partial^\mu + x^\nu[\partial^\mu, x^\sigma\partial^\rho] + [x^\nu, x^\sigma\partial^\rho]\partial^\mu\Big)\phi(x)$$

como $[\partial^\mu, x^\nu] = g^{\mu\nu}$,

$$= -\hbar^2 \Big(x^\mu g^{\nu\rho}\partial^\sigma - x^\rho g^{\mu\sigma}\partial^\nu - x^\mu g^{\nu\sigma}\partial^\rho + x^\sigma g^{\mu\rho}\partial^\nu$$

$$- x^\nu g^{\mu\rho}\partial^\sigma + x^\rho g^{\nu\sigma}\partial^\mu + x^\nu g^{\mu\sigma}\partial^\rho - x^\sigma g^{\nu\rho}\partial^\mu\Big)\phi(x).$$

$$= \frac{\hbar}{i}\left(g^{\nu\rho}L^{\mu\sigma} + g^{\mu\rho}L^{\sigma\nu} + g^{\mu\sigma}L^{\nu\rho} + g^{\nu\sigma}L^{\rho\mu}\right)\phi(x).$$

$$= -i\hbar\Big(g^{\nu\rho}[\phi(x), M^{\mu\sigma}] + g^{\mu\rho}[\phi(x),M^{\sigma\nu}] + g^{\mu\sigma}[\phi(x), M^{\nu\rho}]$$

$$+ g^{\nu\sigma}[\phi(x), M^{\rho\mu}]\Big)$$

$$= \left[\phi(x), -i\hbar\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho} + g^{\nu\rho}M^{\mu\sigma}\right)\right]$$

(f)

Segue diretamente do exercício anterior pois estávamos calculando $[\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]]$.

2.9 —

(a) $$U(\Lambda)^{-1}\partial^\rho\phi(x)\,U(\Lambda) = \Lambda^\rho{}_\sigma\, \bar\partial^\sigma \phi(\Lambda^{-1}x)$$

$$\left(1 - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\partial^\rho\phi(x)\left(1+\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) = \left(\delta^\rho{}_\sigma + \frac{i\,\delta\omega_{\alpha\beta}}{2\hbar}(S_V^{\alpha\beta})^\rho{}_\sigma\right)\bar\partial^\sigma\phi(\Lambda^{-1}x$$

$$\partial^\rho \phi(x) + \frac{i}{2\hbar} \delta\omega_{\mu\nu} [\partial^\rho \phi(x), M^{\mu\nu}] = \bar{\partial}^\rho \phi(x - \delta\omega x) + \frac{i}{2\hbar} \delta\omega_{\alpha\beta} (S_v^{\alpha\beta})^\rho{}_\sigma \bar{\partial}^\sigma \phi(x - \delta\omega x)$$

$$\frac{i}{2\hbar} \delta\omega_{\mu\nu} [\partial^\rho \phi(x), M^{\mu\nu}] = -(\delta\omega\, x)_\gamma \partial^\gamma \bar{\partial}^\rho \phi(x)$$

$$+ \frac{i}{2\hbar} \delta\omega_{\alpha\beta} (S_v^{\alpha\beta})^\rho{}_\sigma \left(\bar{\partial}^\sigma \phi(x) - (\delta\omega x)_\epsilon \partial^\epsilon \bar{\partial}^\sigma \phi(x) \right)$$

$$\delta\omega_{\mu\nu} [\partial^\rho \phi(x), M^{\mu\nu}] = 2\hbar i\, \delta\omega_{\mu\nu} x^\nu \partial^\mu \partial^\rho \phi(x)$$

$$+ \delta\omega_{\mu\nu} (S_v{}^{\mu\nu})^\rho{}_\sigma \partial^\sigma \phi(x)$$

$$\delta\omega_{\mu\nu} [\partial^\rho \phi(x), M^{\mu\nu}] = -\frac{\hbar}{i} \delta\omega_{\mu\nu} (x^\nu \partial^\mu - x^\mu \partial^\nu) \partial^\rho \phi(x)$$

$$+ \delta\omega_{\mu\nu} (S_v{}^{\mu\nu})^\rho{}_\sigma$$

$$[\partial^\rho \phi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu} \partial^\rho \phi(x) + (S_v{}^{\mu\nu})^\rho{}_\sigma \partial^\sigma \phi(x).$$

(b)

$$[S_v{}^{\mu\nu}, S_v{}^{\rho\sigma}]^\alpha{}_\beta = (S_v{}^{\mu\nu})^\alpha{}_\tau (S_v{}^{\rho\sigma})^\tau{}_\beta - (S_v{}^{\rho\sigma})^\alpha{}_\tau (S_v{}^{\mu\nu})^\tau{}_\beta$$

$$= -\hbar^2 (g^{\mu\alpha} \delta^\nu{}_\tau - g^{\nu\alpha} \delta^\mu{}_\tau)(g^{\rho\tau} \delta^\sigma{}_\beta - g^{\sigma\tau} \delta^\rho{}_\beta)$$

$$+ \hbar^2 (g^{\rho\alpha} \delta^\sigma{}_\tau - g^{\sigma\alpha} \delta^\rho{}_\tau)(g^{\mu\tau} \delta^\nu{}_\beta - g^{\nu\tau} \delta^\mu{}_\beta)$$

$$= -\hbar^2 (g^{\mu\alpha} g^{\rho\nu} \delta^\sigma{}_\beta - g^{\mu\alpha} g^{\sigma\nu} \delta^\rho{}_\beta - g^{\nu\alpha} g^{\rho\mu} \delta^\sigma{}_\beta$$

$$+ g^{\nu\alpha} g^{\sigma\mu} \delta^\rho{}_\beta - g^{\rho\alpha} g^{\mu\alpha} \delta^\nu{}_\beta + g^{\rho\alpha} g^{\nu\sigma} \delta^\mu{}_\beta$$

$$+ g^{\sigma\alpha} g^{\mu\rho} \delta^\nu{}_\beta - g^{\sigma\alpha} g^{\nu\rho} \delta^\mu{}_\beta)$$

$$= \frac{\hbar}{i}\frac{\hbar}{i}\Big(g^{\mu\rho}\left(g^{\sigma\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\sigma}{}_{\beta}\right) + g^{\nu\rho}\left(g^{\mu\alpha}\delta^{\sigma}{}_{\beta} - g^{\sigma\alpha}\delta^{\mu}{}_{\beta}\right)$$

$$+ g^{\mu\sigma}\left(g^{\nu\alpha}\delta^{\rho}{}_{\beta} - g^{\rho\alpha}\delta^{\nu}{}_{\beta}\right) + g^{\nu\alpha}\left(g^{\rho\alpha}\delta^{\mu}{}_{\beta} - g^{\mu\alpha}\delta^{\rho}{}_{\beta}\right)\Big)$$

$$= -i\hbar\left(g^{\mu\rho}(S_v^{\sigma\nu})^{\alpha}{}_{\beta} + g^{\nu\rho}(S_v^{\mu\sigma})^{\alpha}{}_{\beta} + g^{\mu\sigma}(S_v^{\nu\rho})^{\alpha}{}_{\beta} + g^{\nu\sigma}(S_v^{\rho\mu})^{\alpha}{}_{\beta}\right)$$

$$= i\hbar\left(g^{\mu\rho}(S_v^{\nu\sigma})^{\alpha}{}_{\beta} - g^{\nu\rho}(S_v^{\mu\sigma})^{\alpha}{}_{\beta} - g^{\mu\sigma}(S_v^{\nu\rho})^{\alpha}{}_{\beta} + g^{\nu\sigma}(S_v^{\mu\rho})^{\alpha}{}_{\beta}\right)$$

c) $$(S_v^{12})^{\mu}{}_{\nu} = \frac{\hbar}{i}\left(g^{1\mu}\delta^{2}{}_{\nu} - g^{2\mu}\delta^{1}{}_{\nu}\right)$$

$$= \frac{\hbar}{i}\left(\delta_{1\mu}\delta_{2\nu} - \delta_{2\mu}\delta_{1\nu}\right)$$

$$S_v^{12} = \frac{\hbar}{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (S_v^{12})^2 \cdot \left(\frac{-i}{\hbar}\right)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(S_v^{12})^3\left(\frac{-i}{\hbar}\right)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\left(S_v^{12}\left(-\frac{i}{\hbar}\right)\right).$$

$$\exp\left(\frac{-i\theta}{\hbar}S_v^{12}\right) = \sum_{n=0}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^{n}(S_v^{12})^{n}\cdot\frac{1}{n!}$$

$$= \mathbb{1} + \sum_{n=1}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^{2n-1} S_v^{12}\,\frac{(\hbar)^{2n-2}}{(2n-1)!} + \sum_{n=1}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^{2n}\frac{(S_v^{12})^2(\hbar)^{2n-2}}{(2n)!}$$

$$= \mathbb{1} + \frac{(S_v^{12})^2}{\hbar^2}\left(-\frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots\right)$$

$$+ \frac{i}{\hbar}(S_v^{12})\left(-\theta + \frac{\theta^3}{3!} - \cdots\right)$$

$$= \mathbb{1} + \frac{(S_v^{12})^2}{\hbar^2}(\cos\theta - 1) - \frac{i}{\hbar} S_v^{12} \operatorname{sen}\theta$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \operatorname{sen}\theta & 0 \\ 0 & \operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^\mu{}_\nu$$

(d) $$(S_v^{30})^\mu{}_\nu = \frac{\hbar}{i}\left(g^{3\mu}\delta^0{}_\nu - g^{0\mu}\delta^3{}_\nu\right)$$

$$= \frac{\hbar}{i}\left(\delta_{3\mu}\delta_{0\nu} - \delta_{0\mu}\delta_{3\nu}\right)$$

$$= \frac{\hbar}{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$(S_v^{30})^2 = \frac{\hbar^2}{i^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (S_v^{30})^3 = (S_v^{30})\,\hbar^2/i^2$$

$$\exp\left(i\eta \frac{S_v^{30}}{\hbar}\right) = \sum_{n=0}^{\infty} \left(\frac{i\eta}{\hbar}\right)^n \left(S_v^{30}\right)^n \cdot \frac{1}{n!}$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \left(\frac{i\eta}{\hbar}\right)^{2n} \frac{1}{(2n)!} \left(S_v^{30}\right)^2 (\hbar)^{2n-2} (-1)^{n-1}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{i\eta}{\hbar}\right)^{2n-1} \frac{1}{(2n-1)!} \left(S_v^{30}\right) \hbar^{2n-2} (-1)^{n-1}$$

$$= \mathbb{1} + \frac{\left(S_v^{30}\right)^2}{\hbar^2}\left(-\eta^2/2 - \eta^4/4! - \dots\right)$$

$$+ \frac{\left(S_v^{30}\right)}{i\hbar}\left(-\eta - \frac{\eta^3}{3!} - \dots\right)$$

$$= \mathbb{1} - \frac{\left(S_v^{30}\right)^2}{\hbar^2}\left(\cosh\eta - 1\right) - \frac{S_v^{30}}{i\hbar}\sinh\eta$$

$$= \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$$