Exercícios :

3.1 —

$$[a(\vec{k}), a(\vec{k}')] = i^2 \int d^3\vec{x} \int d^3\vec{y} \left[e^{-ikx} \overleftrightarrow{\partial_0} \phi(x), e^{-ik'x} \overleftrightarrow{\partial_0} \phi(y) \right]$$

$$= -\int d^3\vec{x}\, d^3\vec{y} \left[e^{-ikx} \Pi(\vec{x},t) + i\omega\, e^{-ikx} \phi(\vec{x},t) , e^{-ik'y} \Pi(\vec{y},t) + i\omega' e^{-ik'y} \phi(\vec{y},t) \right]$$

$$= -\int d^3\vec{x}\, d^3\vec{y}\, \exp(-ikx - ik'y) \Big([\Pi(\vec{x}), \Pi(\vec{y})] + i\omega [\phi(\vec{x}), \Pi(\vec{y})] + i\omega' [\Pi(\vec{x}), \phi(\vec{y})] - \omega\omega' [\phi(\vec{x}), \phi(\vec{y})] \Big)$$

$$= \int d^3\vec{x}\, d^3\vec{y}\, \exp(-ikx - ik'x) \left(\omega\, \delta(\vec{x}-\vec{y}) - \omega' \delta(\vec{x}-\vec{y}) \right)$$

$$= \int d^3\vec{y}\, \exp(-yi(k+k'))(\omega - \omega')$$

$$= (2\pi)^3 \delta^3(\vec{k}+\vec{k}')(\omega - \omega')\, e^{-it(-\omega-\omega')}$$

$$= (2\pi)^3 \delta^3(\vec{k}+\vec{k}')(\omega-\omega')\, e^{it(\omega+\omega')} = 0 .$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = \int d^3\vec{x} \int d^3\vec{y}\, e^{ikx + ik'y} \left[-i\Pi(\vec{x}) + \omega\, \phi(\vec{x}), -i\Pi(\vec{y}) + \omega' \phi(\vec{y}) \right]$$

$$= \int d^3\vec{x} \int d^3\vec{y}\, e^{ikx+ik'y} \Big(-[\Pi(\vec{x}), \Pi(\vec{y})] - i\omega' [\Pi(\vec{x}), \phi(\vec{y})] - i\omega [\phi(\vec{x}), \Pi(\vec{y})] + \omega\omega' [\phi(\vec{x}), \phi(\vec{y})] \Big)$$

$$= \int d^3\vec{x} \int d^3\vec{y}\ \exp\left(ikx + ik'y\right)\left(-\omega'\delta(\vec{x}-\vec{y}) + \omega\,\delta(\vec{x}+\vec{y})\right)$$

$$= (2\pi)^3\, \delta(\vec{k}+\vec{k}')(\omega-\omega')\, e^{-it(\omega+\omega')} = 0.$$

$$[a(\vec{k}),\, a^\dagger(\vec{k}')] = \int d^3\vec{x}\, d^3\vec{y}\ \exp\left(ik'y - ikx\right)\Big[i\Pi(\vec{x}) + \omega\,\phi(\vec{x}),\ -i\Pi(\vec{y}) + \omega'\phi(\vec{y})\Big]$$

$$= \int d^3\vec{x}\, d^3\vec{y}\ \exp(ik'y - ikx)\Big([\Pi(\vec{x}),\Pi(\vec{y})] + i\omega'[\Pi(\vec{x}),\phi(\vec{y})] - i\omega[\phi(\vec{x}),\Pi(\vec{y})] + \omega\omega'[\phi(\vec{x}),\phi(\vec{y})]\Big)$$

$$= \int d^3\vec{x}\, d^3\vec{y}\ \exp(ik'y - ikx)\left(\omega'\delta(\vec{x}-\vec{y}) + \omega\,\delta(\vec{x}-\vec{y})\right)$$

$$= (2\pi)^3\,\delta(\vec{k}-\vec{k}')(\omega'+\omega)\, e^{it(\omega-\omega')}$$

$$= (2\pi)^3\, 2\omega\, \delta(\vec{k}-\vec{k}').$$

3.2 –

para $n = 1$:

$$H|\vec{k}_1\rangle = \int \tilde{dk}\ \omega\, a^\dagger(\vec{k})\, a(\vec{k})\, a^\dagger(\vec{k}_1)|0\rangle$$

$$= \int \tilde{dk}\ \omega\, a^\dagger(\vec{k})\left(a^\dagger(\vec{k}_1)\,a(\vec{k}) + (2\pi)^3 2\omega\,\delta(\vec{k}-\vec{k}_1)\right)|0\rangle$$

$$= \int \tilde{dk}\ 2\omega^2 (2\pi)^3\,\delta(\vec{k}-\vec{k}_1)\, a^\dagger(\vec{k})|0\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega}\, 2\omega\cdot\omega\,(2\pi)^3\,\delta(\vec{k}-\vec{k}_1)\, a^\dagger(\vec{k})|0\rangle = \omega_1\, a^\dagger(\vec{k}_1)|0\rangle = \omega_1|\vec{k}_1\rangle$$

Suponha que seja válido para $n=k$;

$$H\,|\vec{k}_{k+1}\,\vec{k}_k \cdots \vec{k}_1\rangle = \int \tilde{dk}\,\omega\, a^\dagger(\vec{k})\, a(\vec{k}) \cdots a^\dagger(\vec{k}_2)\, a^\dagger(\vec{k}_1)\,|0\rangle$$

$$= \int \tilde{dk}\,\omega\, a^\dagger(\vec{k})\left(a^\dagger(\vec{k}_{k+1})\, a(\vec{k}) + (2\pi)^3 2\omega\, \delta(\vec{k}-\vec{k}_{k+1})\right)\cdots|0\rangle$$

$$= \int \tilde{dk}\,\omega\, a^\dagger(\vec{k}_{k+1})\, a^\dagger(\vec{k})\, a(\vec{k})\, a^\dagger(\vec{k}_k) \cdots |0\rangle$$

$$+ \int d^3\vec{k}\,\omega\, \delta(\vec{k}-\vec{k}_{k+1})\, a^\dagger(\vec{k})\, a^\dagger(\vec{k}_k) \cdots |0\rangle$$

$$= a^\dagger(\vec{k}_{k+1})\, H\,|\vec{k}_k \cdots \vec{k}_1\rangle + \omega_{k+1}\,|\vec{k}_{k+1}\cdots\vec{k}_1\rangle$$

$$= a^\dagger(\vec{k}_{k+1})(\omega_1 + \cdots + \omega_k)\,|\vec{k}_k \cdots \vec{k}_1\rangle$$

$$+ \omega_{k+1}\,|\vec{k}_{k+1} \cdots \vec{k}_1\rangle$$

$$= (\omega_{k+1} + \cdots + \omega_1)\,|\vec{k}_{k+1} \cdots \vec{k}_1\rangle$$

por indução vale para todo n.

3.3 —

Defina: $$\tilde{\phi}(k) = \int d^4x\, \exp(-ikx)\, \phi(x)$$ e

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} \exp(ikx)\, \tilde{\phi}(k).$$

$$U(\Lambda)^{-1}\, \tilde{\phi}(k)\, U(\Lambda) = \int d^4x\, \exp(-ikx)\, U(\Lambda)^{-1}\phi(x)\, U(\Lambda)$$

$$= \int d^4x\, \exp(-ikx)\, \phi(\Lambda^{-1}x)$$

$$= \int d^4x' \, \exp\left(ik(\Lambda x')\right) \phi(x')$$

$$= \int d^4x' \, \exp\left(i(\Lambda^{-1}k)\, x'\right) \phi(x')$$

$$= \tilde{\phi}(\Lambda^{-1}k) \,, \quad \text{mas:}$$

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2w} \left(a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx} \right)$$

$$= \int \frac{d^4k}{(2\pi)^4} 2\pi \, \delta(k^2+m^2) \left(a(\vec{k})\theta(k^0) e^{ikx} + a^\dagger(-\vec{k})\theta(-k^0) e^{ikx} \right)$$

$$\int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\phi}(k) = \int \frac{d^4k'}{(2\pi)^4} (2\pi)\, \delta(k'^2+m^2) \left(a(\vec{k}')\theta(k'^0) e^{ik'x} + a^\dagger(-\vec{k}')\theta(-k'^0) e^{ik'x} \right)$$

$$\tilde{\phi}(k) = (2\pi)\, \delta(k^2+m^2) \left(a(\vec{k})\theta(k^0) + a^\dagger(-\vec{k})\theta(-k^0) \right)$$

se $k^0 > 0$ transformação de Lorentz não alteram o sinal,

$$U(\Lambda)^{-1} \tilde{\phi}(k) U(\Lambda) = 2\pi\, \delta(k^2+m^2)\, U(\Lambda)^{-1} a(\vec{k})\, U(\Lambda)$$

$$\tilde{\phi}(\Lambda^{-1}k) = 2\pi\, \delta(k^2+m^2)\, U(\Lambda)^{-1} a(\vec{k})\, U(\Lambda) .$$

$$\Rightarrow \quad a(\Lambda^{-1}\vec{k}) = U(\Lambda)^{-1} a(\vec{k})\, U(\Lambda)$$

analogamente: $\quad a^\dagger(\Lambda^{-1}\vec{k}) = U(\Lambda)^{-1} a^\dagger(\vec{k})\, U(\Lambda)$. ∎

4)

3.4 -

(a) $T(\delta a)^{-1}\,\phi(x)\,T(\delta a) = \phi(x-\delta a)$

$$\left(\mathbb{1} + \frac{i}{\hbar}\delta a_\mu P^\mu\right)\phi(x)\left(\mathbb{1} - \frac{i}{\hbar}\delta a_\mu P^\mu\right) = \phi(x) - \delta a_\nu \partial^\nu \phi(x)$$

$$\frac{i}{\hbar}\left[P^\mu, \phi(x)\right] = -\partial^\mu \phi(x).$$

$$\left[P^\mu, \phi(x)\right] = i\hbar\, \partial^\mu \phi(x).$$

(b) $\left[P^0, \phi(x)\right] = -i\hbar\, \dot{\phi}(x)$

$$\frac{i}{\hbar}\left[H, \phi(x)\right] = \dot{\phi}(x)$$

(c) $H = \frac{1}{2}\int d^3\vec{x}\left(\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$

$$\left[H, \phi(x)\right] = \frac{1}{2}\int d^3\vec{y}\left(\left[\Pi^2(\vec{y}), \phi(\vec{x})\right] + \left[(\vec{\nabla}\phi)^2(\vec{y}), \phi(\vec{x})\right] + m^2\left[\phi^2(\vec{y}), \phi(\vec{x})\right]\right)$$

$$= \frac{1}{2}\int d^3\vec{y}\left(\Pi(\vec{y})\left[\Pi(\vec{y}), \phi(\vec{x})\right] + \left[\Pi(\vec{y}), \phi(\vec{x})\right]\Pi(\vec{y})\right)$$

$$= -i\,\Pi(\vec{x})$$, mas:

$$\left[H, \phi(x)\right] = -i\,\dot{\phi}(\vec{x}) \Rightarrow \Pi(x) = \dot{\phi}(x)$$

e :

$$[\Pi(\vec{x}), H] = \frac{1}{2}\int d^3\vec{y}\left[\Pi(\vec{x}), \partial^i\phi(\vec{y})\partial_i\phi(\vec{y}) + m^2\phi^2(\vec{y})\right]$$

$$= \frac{1}{2}\int d^3\vec{y}\left(\partial^i[\Pi(\vec{x}),\phi(\vec{y})]\partial_i\phi(\vec{y}) + \partial^i\phi(\vec{y})\partial_i[\Pi(\vec{x}),\phi(\vec{y})]\right.$$

$$\left. + m^2[\Pi(\vec{x}),\phi(\vec{y})]\phi(\vec{y}) + m^2\phi(\vec{y})[\Pi(\vec{x}),\phi(\vec{y})]\right)$$

$$= \frac{1}{2}\int d^3\vec{y}\left(-2i\,\partial_i\phi(\vec{y})\partial^i\delta(\vec{x}-\vec{y}) - 2i\,m^2\delta(\vec{x}-\vec{y})\,\phi(\vec{y})\right)$$

$$= -i\int d^3\vec{y}\left(-\partial_i\partial^i\phi(\vec{y})\,\delta(\vec{x}-\vec{y}) + m^2\delta(\vec{x}-\vec{y})\phi(\vec{y})\right)$$

$$= i\nabla^2\phi(\vec{x}) - i\,m^2\phi(\vec{x})$$

$[\Pi(\vec{x}), H] = i\dot{\Pi}(\vec{x})$. assim:

$$\dot{\Pi}(\vec{x}) = \ddot{\phi}(\vec{x}) = \nabla^2\phi(\vec{x}) - m^2\phi(\vec{x}).$$

$$\partial_\mu\partial^\mu\phi(x) - m^2\phi(x) = 0.$$

(d) $$[P^i, \phi(x)] = -\left[\int d^3\vec{y}\,\Pi(\vec{y},t)\partial^i\phi(\vec{y},t), \phi(\vec{x},t)\right]$$

$$= -\int d^3\vec{y}\left(\Pi(\vec{y},t)[\partial^i\phi(\vec{y},t),\phi(\vec{x},t)] + [\Pi(\vec{y},t),\phi(\vec{x},t)]\partial^i\phi(\vec{y},t)\right)$$

$$= -\int d^3\vec{y}\left(\Pi(\vec{y},t)\partial^i[\phi(\vec{y},t),\phi(\vec{x},t)] - i\delta^3(\vec{y}-\vec{x})\partial^i\phi(\vec{y},t)\right)$$

$$= i\partial^i\phi(x).$$