ORIGEM DA CONSTANTE DE PLANCK

VICENTE V. FIGUEIRA

Sumário

Date: 11 de novembro de 2023.

1. Introdução

Introdução

2. Cálculo

Da teoria da informação sabemos que precisamos maximizar a quantidade

$$\mathcal{I}[p] = -\int \mathcal{D}x\, p[x] \ln\left(\frac{p[x]}{m[x]}\right)$$

Com as devidas condições de contorno. No nosso caso, a condição de contorno é da **ação média ser a ação clássica**, isto é,

$$\int \mathcal{D}x\, p[x] S[x] = S[x_{\text{cl}}]$$

Com é claro $x_{\text{cl}}(t)$ sendo definido por,

$$\frac{\delta S}{\delta x(t)}[x_{\text{cl}}] = 0$$

Logo, o que devemos maximizar é,

$$\mathcal{I}[p] = -\int \mathcal{D}x\, \ln\left(\frac{p[x]}{m[x]}\right) - \alpha\left(\int \mathcal{D}x\, p[x]S[x] - S[x_{\text{cl}}]\right)$$
$$\begin{cases} \frac{\delta \mathcal{I}}{\delta p[x]} & = 0 = -\ln\left(\frac{p[x]}{m[x]}\right) - 1 - \alpha S[x] \\ \frac{\partial \mathcal{I}}{\partial \alpha} & = 0 = \int \mathcal{D}x\, p[x]S[x] - S[x_{\text{cl}}] \end{cases}$$
$$p[x] = m[x] \exp\left(-\alpha S[x] - 1\right)$$
$$S[x_{\text{cl}}] = \int \mathcal{D}x\, m[x] \exp\left(-\alpha S[x] - 1\right) S[x]$$

Definimos assim

$$m[x] = \left(\int \mathcal{D}x\, \exp\left(-\alpha S[x] - 1\right)\right)^{-1}$$

Dessa forma, podemos escrever,

$$Z(\alpha) = \int \mathcal{D}x\, \exp\left(-\alpha S[x]\right)$$

Onde a condição se torna

$$-\frac{\partial}{\partial \alpha} \ln\left[Z(\alpha)\right] = S[x_{\text{cl}}]$$

Resume-se o cálculo à calcular $Z(\alpha)$. Impomos os vínculos do caminho como sendo,

$$x(t_{\text{i}}) = x_{\text{i}},\ x(t_{\text{f}}) = x_{\text{f}}$$

Dividimos em N partes com $x_0 = x_{\text{i}},\ x_N = x_{\text{f}},\ \Delta t = \frac{t_{\text{f}} - t_{\text{i}}}{N}$,

$$\begin{aligned} Z(\alpha) &= \lim_{N\to\infty} \int \prod_{j=1}^{N-1} \mathrm{d}x_j \exp\left\{-\frac{\alpha m}{2} \sum_{k=1}^{N} \frac{(x_k - x_{k-1})^2}{\Delta t^2} \Delta t\right\} \\ &= \lim_{N\to\infty} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N-1}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}y_j \exp\left\{-\sum_{k=1}^{N} (y_k - y_{k-1})^2\right\} \\ &= \lim_{N\to\infty} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N-1}{2}} \frac{\pi^{\frac{N-1}{2}}}{N^{\frac{1}{2}}} \exp\left\{-\frac{(y_N - y_0)^2}{N}\right\} \\ &= \lim_{N\to\infty} \left(\frac{2\Delta t\pi}{\alpha m}\right)^{\frac{N-1}{2}} \frac{1}{\sqrt{N}} \exp\left\{-\frac{\alpha m}{2} \frac{(x_{\text{f}} - x_{\text{i}})^2}{t_{\text{f}} - t_{\text{i}}}\right\} \\ &= \exp\left\{-\alpha S[x_{\text{cl}}]\right\} \lim_{N\to\infty} \left(\frac{2\pi(t_{\text{f}} - t_{\text{i}})}{\alpha m}\right)^{\frac{N-1}{2}} N^{-\frac{N}{2}} \end{aligned}$$

O que é claramente divergente, pois é necessário uma normalização da integral funcional, a normalização tem origem no fato de que,

$$\int_{\mathbb{R}^D} \mathrm{d}^D\mathbf{x} \exp\left(-\frac{\pi}{a}\mathbf{x}\cdot\mathbf{x}\right) = a^{\frac{D}{2}}$$

Para o limite $D \to \infty$,

$$a^\infty = \begin{cases} 0 & : \quad 0 < a < 1 \\ 1 & : \quad a = 1 \\ \infty & : \quad a > 1 \end{cases}$$

Que não é contínua no parâmetro a. Uma das possibilidades é tomar a medida,

$$\int \mathcal{D}x \, \exp\left(-\frac{\pi}{a}x^2\right) = \lim_{D\to\infty} \int_{\mathbb{R}^D} \prod_{j=1}^{D} \left(a^{-\frac{1}{2}} \, \mathrm{d}x_j\right) \exp\left(-\frac{\pi}{a}\mathbf{x}\cdot\mathbf{x}\right) = 1$$

Logo para cada $\mathrm{d}x_j$ é necessário adicionar um fator de normalização R_N dependente da quantidade de intervalos de divisão. Como R_N deve ter dimensão de inverso de comprimento, a dependência em $\alpha, m, t_\mathrm{f} - t_\mathrm{i}$ é fixada, a dependência adicional em N é tal que previne a divergência da integral, a definir temos uma função arbitrária adimensional da variável

$$\lambda = \frac{\alpha m (x_\mathrm{f} - x_\mathrm{i})^2}{2(t_\mathrm{f} - t_\mathrm{i})} = \alpha S[x_\mathrm{cl}]$$

$$\begin{aligned}
Z(\alpha) &= \exp\left\{-\alpha S[x_\mathrm{cl}]\right\} \lim_{N\to\infty} \left(\frac{2\pi(t_\mathrm{f}-t_\mathrm{i})}{\alpha m}\right)^{\frac{N-1}{2}} N^{-\frac{N}{2}} R_N^{N-1} \\
Z(\alpha) &= \exp\left\{-\alpha S[x_\mathrm{cl}]\right\} \lim_{N\to\infty} \left(\frac{2\pi(t_\mathrm{f}-t_\mathrm{i})}{\alpha m}\right)^{\frac{N-1}{2}} N^{-\frac{N}{2}} \left(\sqrt{\frac{\alpha m}{2\pi(t_\mathrm{f}-t_\mathrm{i})}}\right)^{N-1} \left(N^{\frac{N}{2(N-1)}}\right)^{N-1} F(\alpha S[x_\mathrm{cl}]) \\
Z(\alpha) &= \exp\left\{-\alpha S[x_\mathrm{cl}]\right\} F(\alpha S[x_\mathrm{cl}]) \\
\ln Z &= -\alpha S[x_\mathrm{cl}] + \ln F \\
-\frac{\partial}{\partial\alpha}\ln Z &= S[x_\mathrm{cl}] - \frac{1}{F}\frac{\partial}{\partial\alpha}F \overset{!}{=} S[x_\mathrm{cl}] \Rightarrow F \equiv 1
\end{aligned}$$

Portanto a densidade de probabilidade é,

$$p[x] = \exp\left(-\alpha S[x] + \alpha S[x_\mathrm{cl}]\right)$$

Outro cálculo possível de ser feito é o da posição média em um tempo $t_\mathrm{i} < t < t_\mathrm{f}$,

$$\begin{aligned}
\langle x(t)\rangle &= \int \mathcal{D}x \, x(t) \exp\left(-\frac{\alpha m}{2}\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t' \, \dot{x}^2(t')\right) \exp\left(\alpha S[x_\mathrm{cl}]\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \int \prod_{j=1}^{N-1} \mathrm{d}x_j \, x_l \exp\left(-\frac{\alpha m}{2\Delta t}\sum_{k=1}^{N}(x_k - x_{k-1})^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}y_j \, y_l \exp\left(-\sum_{k=1}^{N}(y_k - y_{k-1})^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N}{2}} \frac{\pi^{\frac{l-1}{2}}}{l^{\frac{1}{2}}} \frac{\pi^{\frac{N-l-1}{2}}}{(N-l)^{\frac{1}{2}}} \int \mathrm{d}y_l \, y_l \exp\left(-\frac{1}{l}(y_l - y_0)^2 - \frac{1}{N-l}(y_N - y_l)^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N}{2}} \frac{\pi^{\frac{N-2}{2}}}{l^{\frac{1}{2}}(N-l)^{\frac{1}{2}}} \times \\
&\quad \times \int \mathrm{d}y_l \, y_l \exp\left(-\frac{N}{l(N-l)}\left(y_l - \frac{(l y_N + (N-l)y_0)}{N}\right)^2 + \frac{(l y_N + (N-l)y_0)^2}{Nl(N-l)} - \frac{y_0^2}{l} - \frac{y_N^2}{N-l}\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N}{2}} \frac{\pi^{\frac{N-2}{2}}}{l^{\frac{1}{2}}(N-l)^{\frac{1}{2}}} \sqrt{\frac{\pi l(N-l)}{N}} \frac{(N-l)y_0 + l y_N}{N} \exp\left(-\frac{1}{N}(y_N - y_0)^2\right) \\
&= \lim_{N\to\infty} \left(\sqrt{\frac{\alpha m}{2\pi(t_\mathrm{f}-t_\mathrm{i})}}\right)^{N-1} \left(N^{\frac{N}{2(N-1)}}\right)^{N-1} \left(\frac{2(t_\mathrm{f}-t_\mathrm{i})}{\alpha m N}\right)^{\frac{N}{2}} \frac{\pi^{\frac{N-1}{2}}}{\sqrt{N}} \sqrt{\frac{\alpha m N}{2(t_\mathrm{f}-t_\mathrm{i})}} \frac{(t_\mathrm{f}-t)x_\mathrm{i} + (t-t_\mathrm{i})x_\mathrm{f}}{t_\mathrm{f}-t_\mathrm{i}} \\
&= \frac{(t_\mathrm{f}-t)x_\mathrm{i} + (t-t_\mathrm{i})x_\mathrm{f}}{t_\mathrm{f}-t_\mathrm{i}}
\end{aligned}$$

Como esperado para o caminho clássico. Podemos calcular agora a variância da posição,

$$\begin{aligned}
\left\langle x^2(t)\right\rangle &= \int \mathcal{D}x\, x^2(t) \exp\left(-\frac{\alpha m}{2}\int\limits_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t'\, \dot{x}^2(t')\right) \exp\left(\alpha S[x_\mathrm{cl}]\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N+1}{2}} \frac{\pi^{\frac{N-2}{2}}}{l^{\frac{1}{2}}(N-l)^{\frac{1}{2}}} \times \\
&\quad\times \int \mathrm{d}y_l\, y_l^2 \exp\left(-\frac{N}{l(N-l)}\left(y_l - \frac{(ly_N + (N-l)y_0)}{N}\right)^2 - \frac{1}{N}(y_N - y_0)^2\right) \\
&= \lim_{N\to\infty} \left(\sqrt{\frac{\alpha m}{2\pi(t_\mathrm{f}-t_\mathrm{i})}}\right)^{N-1} N^{\frac{N}{2}} \left(\frac{2(t_\mathrm{f}-t_\mathrm{i})}{N\alpha m}\right)^{\frac{N+1}{2}} \frac{\pi^{\frac{N-2}{2}}}{l^{\frac{1}{2}}(N-l)^{\frac{1}{2}}} \sqrt{\frac{\pi l(N-l)}{N}} \left(\frac{l(N-l)}{2N} + \frac{(ly_N + (N-l)y_0)^2}{N^2}\right) \\
&= \lim_{N\to\infty} \frac{2(t_\mathrm{f}-t_\mathrm{i})}{\alpha m} \frac{1}{N} \left(N \frac{(t-t_\mathrm{i})(t_\mathrm{f}-t)}{2(t_\mathrm{f}-t_\mathrm{i})^2} + \frac{\alpha m N}{2(t_\mathrm{f}-t_\mathrm{i})} \frac{((t-t_\mathrm{i})x_\mathrm{f} + (t_\mathrm{f}-t)x_\mathrm{i})^2}{(t_\mathrm{f}-t_\mathrm{i})^2}\right) \\
&= \left\langle x(t)\right\rangle^2 + \frac{(t-t_\mathrm{i})(t_\mathrm{f}-t)}{\alpha m(t_\mathrm{f}-t_\mathrm{i})}
\end{aligned}$$

De modo que a variância é,

$$\left\langle x^2(t)\right\rangle - \left\langle x(t)\right\rangle^2 = \frac{(t-t_\mathrm{i})(t_\mathrm{f}-t)}{\alpha m(t_\mathrm{f}-t_\mathrm{i})}$$

Para a velocidade,

$$\left\langle \dot{x}(t)\right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle x(t)\right\rangle = \frac{x_\mathrm{f}-x_\mathrm{i}}{t_\mathrm{f}-t_\mathrm{i}}$$

Para a variância da velocidade,

$$\begin{aligned}
\left\langle \dot{x}^2(t)\right\rangle &= \int \mathcal{D}x\, \dot{x}^2(t) \exp\left(-\frac{\alpha m}{2}\int\limits_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t'\, \dot{x}^2(t')\right) \exp\left(\alpha S[x_\mathrm{cl}]\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \int \prod_{j=1}^{N-1} \mathrm{d}x_j \left(\frac{x_l - x_{l-1}}{t_l - t_{l-1}}\right)^2 \exp\left(-\frac{\alpha m}{2\Delta t}\sum_{k=1}^{N}(x_k - x_{k-1})^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N+1}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}y_j \left(\frac{y_l - y_{l-1}}{t_l - t_{l-1}}\right)^2 \exp\left(-\sum_{k=1}^{N}(y_k - y_{k-1})^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} \left(\frac{\alpha m}{2\Delta t}\right)^{\frac{N-1}{2}} \frac{N^{\frac{1}{2}}}{\pi^{\frac{N-1}{2}}} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N+1}{2}} \frac{\pi^{\frac{l-2}{2}}}{(l-1)^{\frac{1}{2}}} \frac{\pi^{\frac{N-l-1}{2}}}{(N-l)^{\frac{1}{2}}} \times \\
&\quad\times \int \mathrm{d}y_l\, \mathrm{d}y_{l-1} \left(\frac{y_l - y_{l-1}}{t_l - t_{l-1}}\right)^2 \exp\left(-(y_l - y_{l-1})^2 - \frac{1}{l-1}(y_{l-1} - y_0)^2 - \frac{1}{N-l}(y_N - y_l)^2\right)
\end{aligned}$$

$$\begin{aligned}
\left\langle \dot{x}^2(t)\right\rangle &= \int \mathcal{D}x\, \dot{x}^2(t) \exp\left(-\frac{\alpha m}{2}\int\limits_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t'\, \dot{x}^2(t')\right) \exp\left(\alpha S[x_\mathrm{cl}]\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \int \prod_{j=1}^{N-1} \mathrm{d}x_j \left(\frac{x_l - x_{l-1}}{t_l - t_{l-1}}\right)^2 \exp\left(-\frac{\alpha m}{2\Delta t}\sum_{k=1}^{N}(x_k - x_{k-1})^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} R_N^{N-1} \left(\frac{2\Delta t}{\alpha m}\right)^{\frac{N+1}{2}} \int \prod_{j=1}^{N-1} \mathrm{d}y_j \left(\frac{y_l - y_{l-1}}{t_l - t_{l-1}}\right)^2 \exp\left(-\sum_{k=1}^{N}(y_k - y_{k-1})^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} \frac{N^{\frac{1}{2}}}{\pi^{\frac{N-1}{2}}} \frac{2\Delta t}{\alpha m(t_l - t_{l-1})^2} \int \prod_{j=1}^{N-1} \mathrm{d}y_j \left(y_l^2 - 2y_l y_{l-1} + y_{l-1}^2\right) \exp\left(-\sum_{k=1}^{N}(y_k - y_{k-1})^2\right) \\
&= \exp\left(\alpha S[x_\mathrm{cl}]\right) \lim_{N\to\infty} \frac{N^{\frac{1}{2}}}{\pi^{\frac{N-1}{2}}} \frac{2\Delta t}{\alpha m(t_l - t_{l-1})^2} \int \prod_{j=1}^{N-1} \mathrm{d}y_j \left(y_l^2 - 2y_l y_{l-1} + y_{l-1}^2\right) \exp\left(-\sum_{k=1}^{N}(y_k - y_{k-1})^2\right)
\end{aligned}$$