3.1 $$[a(\vec{k}), a(\vec{p})] = i^2 \int d^3\vec{x} \int d^3\vec{y} \left[e^{-ikx} \overleftrightarrow{\partial_0} \phi(x), e^{-ipy} \overleftrightarrow{\partial_0} \phi(y) \right]$$

$$= -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx-ipy} \left[\pi(\vec{x},t) + i\omega_{\vec{k}} \phi(\vec{x},t), \pi(\vec{y},t) + i\omega_{\vec{p}} \phi(\vec{y},t) \right]$$

$$= \int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx-ipy} \Big([\pi(\vec{y},t), \pi(\vec{x},t)] - i\omega_{\vec{k}} [\phi(\vec{x},t), \pi(\vec{y},t)]$$
$$+ i\omega_{\vec{p}} [\phi(\vec{y},t), \pi(\vec{x},t)] + \omega_{\vec{k}}\omega_{\vec{p}} [\phi(\vec{x},t), \phi(\vec{y},t)] \Big)$$

$$= \int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx-ipy} \left(\omega_{\vec{k}} \delta^3(\vec{x}-\vec{y}) - \omega_{\vec{p}} \delta^3(\vec{x}-\vec{y}) \right)$$

$$= \int d^3\vec{x}\, e^{-i\vec{x}(k+p)} (\omega_{\vec{k}} - \omega_{\vec{p}}) = (2\pi)^3 \delta^3(\vec{k}+\vec{p})(\omega_{\vec{k}} - \omega_{\vec{p}}) e^{it(\omega_{\vec{k}}+\omega_{\vec{p}})}$$

$$= 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{p})] = i^2 \int d^3\vec{x}\, d^3\vec{y}\, e^{ikx+ipy} \Big([\pi(\vec{y},t), \pi(\vec{x},t)]$$
$$+ i\omega_{\vec{p}} [\pi(\vec{x},t), \phi(\vec{y},t)] + i\omega_{\vec{k}} [\phi(\vec{x},t), \pi(\vec{y},t)]$$
$$- \omega_{\vec{k}}\omega_{\vec{p}} [\phi(\vec{x},t), \phi(\vec{y},t)] \Big)$$

$$= (2\pi)^3 \delta^3(\vec{k}+\vec{p})(\omega_{\vec{k}} - \omega_{\vec{p}}) e^{-it(\omega_{\vec{k}}+\omega_{\vec{p}})} = 0$$

$$[a(\vec{k}), a^\dagger(\vec{p})] = \int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx+ipy} \Big([\pi(\vec{x},t), \pi(\vec{y},t)] - i\omega_{\vec{p}} [\pi(\vec{x},t), \phi(\vec{y},t)]$$
$$- i\omega_{\vec{k}} [\phi(\vec{x},t), \pi(\vec{y},t)] + \omega_{\vec{k}}\omega_{\vec{p}} [\phi(\vec{x},t), \phi(\vec{y},t)] \Big)$$

$$= \int d^3\vec{x}\, d^3\vec{y}\, e^{ipy-ikx} \left(\omega_{\vec{p}} \delta^3(\vec{x}-\vec{y}) + \omega_{\vec{k}} \delta^3(\vec{x}-\vec{y}) \right)$$

$$= (2\pi)^3 \delta^3(\vec{k}-\vec{p})(\omega_{\vec{k}}+\omega_{\vec{p}}) e^{it(\omega_{\vec{k}}-\omega_{\vec{p}})} = (2\pi)^3 \delta^3(\vec{k}-\vec{p})\, 2\omega_{\vec{k}}$$

3.2 Para $n = 1$:

$$H|\vec{k}_1\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\, \omega_{\vec{k}}\, a^\dagger(\vec{k})\, a(\vec{k})\, a^\dagger(\vec{k}_1)|0\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3\, 2}\, a^\dagger(\vec{k}) \left[a^\dagger(\vec{k}_1) a(\vec{k}) + (2\pi)^3 2\omega_{\vec{k}}\, \delta^3(\vec{k}-\vec{k}_1)\right]|0\rangle$$

$$= \int \frac{d^3\vec{k}\,(2\pi)^3 2\omega_{\vec{k}}\, \delta(\vec{k}-\vec{k}_1)}{(2\pi)^3\, 2} a^\dagger(\vec{k})|0\rangle$$

$$= \omega_{\vec{k}_1}|\vec{k}_1\rangle$$

Se é válido para n :

$$H|\vec{k}_{n+1}\,\vec{k}_n \dots \vec{k}_1\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\, \omega_{\vec{k}}\, a^\dagger(\vec{k}) a(\vec{k})\, a^\dagger(\vec{k}_{n+1})\, a^\dagger(\vec{k}_n) \dots a^\dagger(\vec{k}_1)|0\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3\, 2}\, a^\dagger(\vec{k}_{n+1})\, a^\dagger(\vec{k})\, a(\vec{k})\, a^\dagger(\vec{k}_n) \dots a^\dagger(\vec{k}_1)|0\rangle$$

$$+ \int \frac{d^3\vec{k}}{(2\pi)^3\, 2}\, \delta^3(\vec{k}-\vec{k}_{n+1})\, a^\dagger(\vec{k})\, a^\dagger(\vec{k}_n) \dots a^\dagger(\vec{k}_1)|0\rangle$$

$$= a^\dagger(\vec{k}_{n+1})\, H\, |\vec{k}_n \dots \vec{k}_1\rangle + \omega_{\vec{k}_{n+1}}|\vec{k}_{n+1} \dots \vec{k}_1\rangle$$

$$= a^\dagger(\vec{k}_{n+1}) \left(\omega_{\vec{k}_n} + \dots + \omega_{\vec{k}_1}\right)|\vec{k}_n \dots \vec{k}_1\rangle + \omega_{\vec{k}_{n+1}}|\vec{k}_{n+1} \dots \vec{k}_1\rangle$$

$$= \left(\omega_{\vec{k}_1} + \dots + \omega_{\vec{k}_{n+1}}\right)|\vec{k}_{n+1} \dots \vec{k}_1\rangle$$

Logo vale para todo n.

(3.3) Defino: $\tilde{\phi}(k) = \int d^4x \, e^{-ikx} \phi(x)$; $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\phi}(k)$

$$U(\Lambda)^{-1} \tilde{\phi}(k) U(\Lambda) = \int d^4x \, e^{-ikx} U(\Lambda)^{-1} \phi(x) U(\Lambda)$$

$$= \int d^4x \, e^{-ikx} \phi(\Lambda^{-1}x)$$

$$= \int d^4y \, e^{-ik(\Lambda y)} \phi(y)$$

$$= \int d^4y \, e^{-i(\Lambda^{-1}k)y} \phi(y)$$

$$= \tilde{\phi}(\Lambda^{-1}k)$$

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \left(a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx} \right)$$

$$\int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\phi}(k) = \int \frac{d^4k (2\pi) \delta(k^2+m^2)}{(2\pi)^4} \left(\theta(k^0) a(\vec{k}) e^{ikx} + \theta(-k^0) a^\dagger(-\vec{k}) e^{ikx} \right)$$

$$\tilde{\phi}(k) = 2\pi \, \delta(k^2+m^2) \left(a(\vec{k}) \theta(k^0) + a^\dagger(-\vec{k}) \theta(-k^0) \right)$$

como lorentz n troca sinal de k^0, suponho $k^0 \geq 0$.

$$U(\Lambda)^{-1} \tilde{\phi}(k) U(\Lambda) = 2\pi \, \delta((\Lambda k)^2 + m^2) \theta(\Lambda k^0) \, U(\Lambda)^{-1} a(\vec{k}) U(\Lambda)$$

$$\tilde{\phi}(\Lambda^{-1}k) = 2\pi \, \delta(k^2+m^2) \theta(k^0) \, U(\Lambda)^{-1} a(\vec{k}) U(\Lambda)$$

$$= 2\pi \, \delta((\Lambda^{-1}k)^2 + m^2) \theta((\Lambda^{-1}k^0)) \, a(\Lambda^{-1}\vec{k})$$

$$\Rightarrow \quad U(\Lambda)^{-1} a(\vec{k}) U(\Lambda) = a(\Lambda^{-1}\vec{k})$$

analogamente: $a^\dagger(\Lambda^{-1}\vec{k}) = U(\Lambda)^{-1} a^\dagger(\vec{k}) U(\Lambda)$.

(3.4) (a) $T(\delta a)^{-1}\phi(x) T(\delta a) = \phi(x-\delta a)$

$$\left(\mathbb{1} + \frac{i}{\hbar}\delta a_\mu P^\mu\right)\phi(x)\left(\mathbb{1} - \frac{i}{\hbar}\delta a_\mu P^\mu\right) = \phi(x) - \delta a_\nu \partial^\nu \phi(x)$$

$$[\phi(x), P^\mu] = -i\hbar\, \partial^\mu \phi(x).$$

(b) $[P^0, \phi(x)] = -i\hbar\dot{\phi}(x) \Rightarrow \frac{1}{i\hbar}[\phi(x), H] = \dot{\phi}(x)$

(c) $H = \frac{1}{2}\int d^3\vec{y}\left(\Pi^2(\vec{y},t) + (\vec{\nabla}\phi)^2(\vec{y},t) + m^2\phi^2(\vec{y},t)\right)$

$$[\phi(\vec{x},t), H] = \frac{1}{2}\int d^3\vec{y}\left(\Pi(\vec{y},t)[\phi(\vec{x},t), \Pi(\vec{y},t)] + [\phi(\vec{x},t), \Pi(\vec{y},t)]\Pi(\vec{y},t)\right)$$

$$= i\hbar\,\Pi(\vec{x},t) \Rightarrow \underline{\Pi(x) = \dot{\phi}(x)}$$

$$[\Pi(\vec{x},t), H] = \frac{1}{2}\int d^3\vec{y}\left(\partial^j_{\vec{y}}[\Pi(\vec{x},t), \phi(\vec{y},t)]\,\partial_j\phi(\vec{y},t)\right.$$

$$+ \partial^j\phi(\vec{y},t)\partial_j[\Pi(\vec{x},t), \phi(\vec{y},t)] + m^2[\Pi(\vec{x},t), \phi(\vec{y},t)]\phi(\vec{y},t)$$

$$\left. + m^2\phi(\vec{y},t)[\Pi(\vec{x},t), \phi(\vec{y},t)]\right)$$

$$= \frac{\hbar}{2}\int d^3\vec{y}\left(-2i\,\partial_j\phi(\vec{y},t)\,\partial^j\delta^3(\vec{x}-\vec{y}) - 2im^2\delta^3(\vec{x}-\vec{y})\phi(\vec{y},t)\right)$$

$$[\pi(\vec{x},t), H] = i\hbar \int d^3\vec{y} \left(\partial_j \partial^j \phi(\vec{y},t)\, \delta^3(\vec{x}-\vec{y}) - m^2 \delta^3(\vec{x}-\vec{y})\, \phi(\vec{y},t) \right)$$

$$= i\hbar \vec{\nabla}^2 \phi(\vec{x},t) - i\hbar m^2 \phi(\vec{x},t)$$

$$\frac{1}{i\hbar}[\pi(\vec{x},t), H] = \dot{\pi} = \ddot{\phi} = -\partial_0 \partial^0 \phi$$

$$-\partial_0\partial^0 \phi = \partial_j \partial^j \phi - m^2 \phi$$

$$\Rightarrow (\partial^2 \phi - m^2 \phi) = 0 \,.$$ Klein Gordon !

(d) $$[\phi(x), P^j] = -\int d^3\vec{y}\, [\phi(\vec{x},t), \pi(\vec{y},t)\partial^j \phi(\vec{y},t)]$$

$$= -\int d^3\vec{y} \Big(\pi(\vec{y},t)[\phi(\vec{x},t), \partial^j \phi(\vec{y},t)]$$

$$+ [\phi(\vec{x},t), \pi(\vec{y},t)]\partial^j \phi(\vec{y},t) \Big)$$

$$= -i\hbar \partial^j \phi(x)$$

(e) $$\vec{P} = -\int d^3\vec{y}\, \pi(\vec{y},t) \vec{\nabla}\phi(\vec{y},t)$$

$$= -\int d^3\vec{y} \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left(-i\omega_{\vec{p}}\, a(\vec{p}) e^{ipy} + i\omega_{\vec{p}}\, a^\dagger(\vec{p}) e^{-ipy} \right) \times$$

$$\times \left(i\vec{k}\, a(\vec{k}) e^{iky} - i\vec{k}\, a^\dagger(\vec{k}) e^{-iky} \right)$$

$$= \int \frac{d^3\vec{y}\, d^3\vec{k}\, d^3\vec{p}\, \vec{k}}{(2\pi)^6\, 4\omega_{\vec{k}}} \Big(a^\dagger(\vec{p}) a(\vec{k}) e^{iy(k-p)} - a^\dagger(\vec{p}) a^\dagger(\vec{k}) e^{-iy(p+k)} - a(\vec{p}) a(\vec{k}) e^{iy(p+k)}$$

$$+ a(\vec{p}) a^\dagger(\vec{k}) e^{iy(p-k)} \Big)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 4\omega_{\vec{k}}} \vec{k} \Big(a^\dagger(\vec{k}) a(\vec{k}) - a^\dagger(-\vec{k}) a(\vec{k}) e^{it2\omega_{\vec{k}}} - a(-\vec{k}) a(\vec{k}) e^{-it2\omega_{\vec{k}}}$$

$$+ a(\vec{k}) a^\dagger(\vec{k}) \Big)$$

$$= \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \vec{k} \left(a^\dagger(\vec{k}) a(\vec{k}) + a(\vec{k}) a^\dagger(\vec{k}) \right)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \vec{k} \left(a^\dagger(\vec{k}) a(\vec{k}) + (2\pi)^3 2\omega_{\vec{k}} \delta^3(0) \right)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \vec{k}\, a^\dagger(\vec{k}) a(\vec{k}) + \delta^3(0) \int d^3\vec{k}\, \vec{k}$$

(3.5) (a) $$\frac{\partial \mathcal{L}}{\partial \phi^\dagger} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger}$$

$$-m^2 \phi = \partial_\mu(-\partial^\mu \phi) \Rightarrow (\partial^2 - m^2)\phi = 0$$

(b) $$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\partial^0 \phi^\dagger = \dot{\phi}^\dagger$$

$$\Pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = -\partial^0 \phi = \dot{\phi}$$

$$\mathcal{H} = \Pi \dot{\phi} + \Pi^\dagger \dot{\phi}^\dagger - \mathcal{L}$$

$$= \Pi \Pi^\dagger + \Pi^\dagger \Pi + \partial^\mu \phi^\dagger \partial_\mu \phi + m^2 \phi^\dagger \phi - \Omega_0$$

$$= \Pi^\dagger \Pi + (\vec{\nabla}\phi)^\dagger (\vec{\nabla}\phi) + m^2 \phi^\dagger \phi - \Omega_0$$

(c)

$$\int d^3\vec{x}\, e^{-ikx}\phi(x) = \int \frac{d^3\vec{p}\, d^3\vec{x}}{(2\pi)^3 2\omega_{\vec{p}}}\left(a(\vec{p})e^{ix(p-k)} + b^\dagger(\vec{p})e^{-ix(p+k)}\right)$$

$$= \frac{1}{2\omega_{\vec{k}}}\left(a(\vec{k}) + b^\dagger(-\vec{k})e^{it2\omega_{\vec{k}}}\right)$$

$$\int d^3\vec{x}\, e^{-ikx}\pi^\dagger(x) = \int \frac{d^3\vec{p}\, d^3\vec{x}}{(2\pi)^3 2\omega_{\vec{p}}}\left(-i\omega_{\vec{p}}\, a(\vec{p})e^{i(p-k)x} + i\omega_{\vec{p}}\, b^\dagger(\vec{p})e^{-ix(p+k)}\right)$$

$$= \frac{-i}{2}\left(a(\vec{k}) - b^\dagger(-\vec{k})e^{it2\omega_{\vec{k}}}\right)$$

$$a(\vec{k}) = i\int d^3\vec{x}\, e^{-ikx}\left(\pi^\dagger(x) - i\omega_{\vec{k}}\phi(x)\right)$$

$$\int d^3\vec{x}\, e^{-ikx}\phi^\dagger(x) = \int \frac{d^3\vec{p}\, d^3\vec{x}}{(2\pi)^3 2\omega_{\vec{p}}}\left(b(\vec{p})e^{ix(p-k)} + a^\dagger(\vec{p})e^{-ix(p+k)}\right)$$

$$= \frac{1}{2\omega_{\vec{k}}}\left(b(\vec{k}) + a^\dagger(-\vec{k})e^{it2\omega_{\vec{k}}}\right)$$

$$\int d^3\vec{x}\, e^{-ikx}\pi(x) = \int \frac{d^3\vec{p}\, d^3\vec{x}}{(2\pi)^3 2\omega_{\vec{p}}}\left(-i\omega_{\vec{p}}\, b(\vec{p})e^{ix(p-k)} + i\omega_{\vec{p}}\, a^\dagger(\vec{p})e^{-ix(p+k)}\right)$$

$$= \frac{-i}{2}\left(b(\vec{k}) - a^\dagger(-\vec{k})e^{it2\omega_{\vec{k}}}\right)$$

$$b(\vec{k}) = i\int d^3\vec{x}\, e^{-ikx}\left(\pi(x) - i\omega_{\vec{k}}\phi^\dagger(x)\right)$$

(d) $$[a(\vec{k}), a(\vec{p})] = -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx-ipy} \left[\pi^\dagger(\vec{x},t) - i\omega_{\vec{k}}\phi(\vec{x},t), \pi^\dagger(\vec{y},t) - i\omega_{\vec{p}}\phi(\vec{y},t)\right]$$

$$= 0 = [a^\dagger(\vec{p}), a^\dagger(\vec{k})]$$

$$[b(\vec{k}), b(\vec{p})] = -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx-ipy} \left[\pi(\vec{x},t) - i\omega_{\vec{k}}\phi^\dagger(\vec{x},t), \pi(\vec{y},t) - i\omega_{\vec{p}}\phi^\dagger(\vec{y},t)\right]$$

$$= 0 = [b^\dagger(\vec{p}), b^\dagger(\vec{k})]$$

$$[a(\vec{k}), b(\vec{p})] = -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx-ipy} \left[\pi^\dagger(\vec{x},t) - i\omega_{\vec{k}}\phi(\vec{x},t), \pi(\vec{y},t) - i\omega_{\vec{p}}\phi^\dagger(\vec{y},t)\right]$$

$$= 0 = [b^\dagger(\vec{p}), a^\dagger(\vec{k})]$$

$$[a(\vec{k}), b^\dagger(\vec{p})] = -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx+ipy} \left[\pi^\dagger(\vec{x},t) - i\omega_{\vec{k}}\phi(\vec{x},t), \pi^\dagger(\vec{y},t) + i\omega_{\vec{p}}\phi(\vec{y},t)\right]$$

$$= 0 = [b(\vec{p}), a^\dagger(\vec{k})]$$

$$[a(\vec{k}), a^\dagger(\vec{p})] = -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx+ipy} \left[\pi^\dagger(\vec{x},t) - i\omega_{\vec{k}}\phi(\vec{x},t), \pi(\vec{y},t) + i\omega_{\vec{p}}\phi^\dagger(\vec{y},t)\right]$$

$$= -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx+ipy}\, \omega i\left(i\delta^3(\vec{x}-\vec{y}) + i\delta^3(\vec{x}-\vec{y})\right)$$

$$= (2\pi)^3\, 2\omega_{\vec{k}}\, \delta^3(\vec{k}-\vec{p})$$

$$[b(\vec{k}), b^\dagger(\vec{p})] = -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx+ipy} \left[\pi(\vec{x},t) - i\omega_{\vec{k}}\phi^\dagger(\vec{x},t), \pi^\dagger(\vec{y},t) + i\omega_{\vec{p}}\phi(\vec{y},t)\right]$$

$$= -\int d^3\vec{x}\, d^3\vec{y}\, e^{-ikx+ipy}\, \omega i\left(i\delta^3(\vec{x}-\vec{y}) + i\delta^3(\vec{x}-\vec{y})\right)$$

$$= (2\pi)^3\, 2\omega_{\vec{k}}\, \delta^3(\vec{k}-\vec{p})$$

ⓔ $H = \int d^3\vec{x}\, \mathcal{H}$

$$= \int d^3\vec{x} \left(\Pi^\dagger \Pi + (\vec{\nabla}\phi)^\dagger (\vec{\nabla}\phi) + m^2 \phi^\dagger \phi - \Omega_0 \right)$$

$$= \int \frac{d^3\vec{x}\, d^3\vec{k}\, d^3\vec{p}}{(2\pi)^6 4 w_{\vec{k}} w_{\vec{p}}} \Big[\left(-iw_{\vec{k}} a(\vec{k}) e^{ikx} + iw_{\vec{k}} b^\dagger(\vec{k}) e^{-ikx}\right)\left(-iw_{\vec{p}} b(\vec{p}) e^{ipx} + iw_{\vec{p}} a^\dagger(\vec{p}) e^{-ipx}\right)$$

$$+ \left(i\vec{k}\, b(\vec{k}) e^{ikx} - i\vec{k}\, a^\dagger(\vec{k}) e^{-ikx}\right)\left(i\vec{p}\, a(\vec{p}) e^{ipx} - i\vec{p}\, b^\dagger(\vec{p}) e^{-ipx}\right)$$

$$+ m^2 \left(b(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx}\right)\left(a(\vec{p}) e^{ipx} + b^\dagger(\vec{p}) e^{-ipx}\right) \Big]$$

$$- \int d^3\vec{x}\, \Omega_0 .$$

$$= -\int \frac{d^3\vec{k}}{(2\pi)^3 4 w_{\vec{k}}^2} \Big[w_{\vec{k}}^2 a(\vec{k}) b(-\vec{k}) - w_{\vec{k}}^2 a(\vec{k}) a^\dagger(\vec{k}) - w_{\vec{k}}^2 b^\dagger(\vec{k}) b(\vec{k}) + w_{\vec{k}}^2 b^\dagger(\vec{k}) a^\dagger(-\vec{k})$$

$$+ \vec{k}^2 b(\vec{k}) a(-\vec{k}) - \vec{k}^2 b(\vec{k}) b^\dagger(\vec{k}) - \vec{k}^2 a^\dagger(\vec{k}) a(\vec{k})$$

$$+ \vec{k}^2 a^\dagger(\vec{k}) b^\dagger(-\vec{k}) - m^2 b(\vec{k}) a(-\vec{k}) - m^2 b(\vec{k}) b^\dagger(\vec{k})$$

$$- m^2 a^\dagger(\vec{k}) a(\vec{k}) - m^2 a^\dagger(\vec{k}) b(-\vec{k}) \Big] - \int d^3\vec{x}\, \Omega_0$$

$$= \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3 2 w_{\vec{k}}} w_{\vec{k}} \left(a^\dagger(\vec{k}) a(\vec{k}) + a(\vec{k}) a^\dagger(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) + b(\vec{k}) b^\dagger(\vec{k}) \right)$$

$$- \int d^3\vec{x}\, \Omega_0$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2 w_{\vec{k}}} w_{\vec{k}} \left(a^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) + 2 (2\pi)^3 2 w_{\vec{k}} \delta^3(0) \right) - \int d^3\vec{x}\, \Omega_0$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2 w_{\vec{k}}} w_{\vec{k}} \left(a^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) \right) + 2 \int d^3\vec{k}\, w_{\vec{k}} \delta^3(0) - \int d^3\vec{x}\, \Omega_0$$

$$\hookrightarrow \Omega_0 = 2 \int d^3\vec{k}\, w_{\vec{k}} .$$

$\boxtimes$