6.1 ⓐ $H = \frac{p^2}{2m} + V(q)$

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \mathcal{D}p \exp\left[i \int_{t'}^{t''} dt \left[p\dot{q} - p^2/_{2m} - V(q) \right]\right]$$

$$= \int \mathcal{D}q \mathcal{D}p \exp\left[i \int_{t'}^{t''} dt \left\{ -\frac{1}{2m}\left(p - \dot{q}\frac{m}{2}\right)^2 + \dot{q}^2 \frac{m}{2} - V(q) \right\}\right]$$

$$= \int \mathcal{D}q \, e^{iS} \int \prod_{j=0}^{N} \left[\frac{dp_j}{2\pi} \exp\left(-i p_j^2 \frac{\delta t}{2m}\right)\right] \qquad \delta t = \frac{\Delta t}{N+1}$$

$$= \int \mathcal{D}q \, e^{iS} \prod_{j=0}^{N} \left[\frac{1}{2\pi} \sqrt{\frac{\pi 2m}{\delta t \, i}} \right]$$

$$= \prod_{j=1}^{N} dq_j \left(\frac{m}{2\pi \delta t \, i} \right)^{\frac{N+1}{2}} e^{iS}$$

$$C = \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N+1}{2}}$$

ⓑ
$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \exp\left[i \int_{t'}^{t''} dt \, \frac{m\dot{q}^2}{2} \right]$$

$$= \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N+1}{2}} \int \prod_{j=1}^{N} \left[dq_j \exp\left(-\frac{m}{2\delta t i} (q_{j+1} - q_j)^2 \right) \right]$$

caso $N = 1$:

$$\int dq_1\, e^{-\frac{m}{2\delta t i}(q''-q_1)^2 - \frac{m}{2\delta t i}(q_1-q')^2}$$

$$= \int dq_1 \exp\left[-\frac{m}{\delta t i} q_1^2 + \frac{m q_1}{\delta t i}(q''+q') - \frac{m}{2\delta t i}(q''^2 + q'^2) \right]$$

$$= \int dq_1 \exp\left[\frac{-m}{\delta t i}\left(q_1^2 - 2q_1\frac{(q'+q'')}{2} + \frac{(q''+q')^2}{4} \right) + \frac{m}{\delta t i}\frac{(q''+q')^2}{4} - \frac{m}{2\delta t i}(q''^2+q'^2) \right]$$

$$= \int dq_1 \exp\left[\frac{-m}{\delta t i}\left(q_1 - \frac{(q''+q')}{2}\right)^2 - \frac{m}{4\delta t i}(q''-q')^2 \right]$$

$$= \sqrt{\frac{\pi \delta t i}{m}} \exp\left(\frac{-m}{4\delta t i}(q''-q')^2 \right)$$

Para $N=2$:

$$\int dq_1 dq_2\, e^{-\frac{m}{2\delta t i}(q''-q_2)^2 - \frac{m}{2\delta t i}(q_2-q_1)^2 - \frac{m}{2\delta t i}(q_1-q')^2}$$

$$= \int dq_2 \sqrt{\frac{\pi \delta t i}{m}}\, e^{-\frac{m}{2\delta t i}(q''-q_2)^2}\, e^{-\frac{m}{4\delta t i}(q_2-q')^2}$$

$$= \sqrt{\frac{\pi \delta t i}{m}} \int dq_2 \exp\left[-\frac{m q_2^2}{2\delta t i} - \frac{m q_2^2}{4\delta t i} + \frac{2m q'' q_2}{2\delta t i} + \frac{2m q' q_2}{4\delta t i} - \frac{m}{2\delta t i} q''^2 - \frac{m}{4\delta t i} q'^2 \right]$$

$$= \sqrt{\frac{\pi \delta t i}{m}} \int dq_2 \exp\left[\frac{-3m}{4\delta t i} q_2^2 + \frac{2m q_2}{\delta t i}\left(\frac{q''}{2} + \frac{q'}{4} \right)\frac{3}{3}\frac{4}{4} - \frac{m}{2\delta t i}\left(q''^2 + q'^2/2\right) \right]$$

$$= \sqrt{\frac{\pi \delta t i}{m}} \int dq_2 \exp\left[\frac{-3m}{4\delta t i} q_2^2 + \frac{3m}{4\delta t i} 2q_2 \frac{1}{3}(2q_f'' + q_f') - \frac{3m}{4\delta t i} \frac{1}{9}(2q_f''+q_f')^2 \right.$$

$$\left. + \frac{3m}{4\delta t i} \frac{1}{9} (2q_f''+q_f')^2 - \frac{m}{2\delta t i}(q_f''^2 + q_f'^2/2) \right]$$

$$= \sqrt{\frac{\pi \delta t i}{m}} \int dq_2 \exp\left[\frac{-3m}{4\delta t i} \left(q_2 - \frac{1}{3}(2q_f''+q_f')\right)^2 + \frac{m}{3\delta t i} (q_f''+q_f'/2)^2 \right.$$

$$\left. - \frac{m}{2\delta t i}(q_f''^2 + q_f'^2/2) \right]$$

$$= \sqrt{\frac{\pi \delta t i}{m}} \sqrt{\frac{4\delta t \pi i}{3m}} \exp\left[-\frac{m}{6\delta t i} \left(3q_f''^2 + \frac{3}{2} q_f'^2 - 2q_f''^2 - \frac{q_f'^2}{2} \right.\right.$$

$$\left.\left. - 2q_f'' q_f' \right) \right]$$

$$= \sqrt{\frac{4}{3}} \left(\frac{\delta t \pi i}{m} \right)^{2/2} \exp\left[-\frac{m}{6\delta t i} (q_f'' - q_f')^2 \right]$$

$$= \frac{1}{\sqrt{2+1}} \left(\frac{2\delta t \pi i}{m} \right)^{2/2} \exp\left[\frac{-m}{2\delta t i (2+1)} (q_f'' - q_f')^2 \right]$$

suponho que vale para $N = k$:

$$\int dq_{f1} \dots dq_{fk}\, e^{(\dots)} = \frac{1}{\sqrt{k+1}} \left(\frac{2\delta t \pi i}{m} \right)^{k/2} \exp\left(\frac{-m}{2\delta t i (k+1)} (q_f'' - q_f')^2 \right)$$

para $k+1$:

$$\int dq_f\, e^{-\frac{m}{2\delta t i}(q_f''-q_f)^2} \left(\frac{2\delta t \pi i}{m} \right)^{k/2} \frac{1}{\sqrt{k+1}} e^{\frac{-m}{2\delta t i (k+1)} (q_f - q_f')^2}$$

$$= \frac{1}{\sqrt{k+1}}\left(\frac{2\pi\delta t i}{m}\right)^{k/2}\int dq \exp\left[-\frac{q^2 m}{2\delta t i}-\frac{q^2 m}{2\delta t i(k+1)}+\frac{q q' m}{\delta t i(k+1)}\right.$$

$$\left.+\frac{q q'' m}{\delta t i}-\frac{q'^2 m}{2\delta t i(k+1)}-\frac{q''^2 m}{\delta t i 2}\right]$$

$$= \frac{1}{\sqrt{k+1}}\left(\frac{2\pi\delta t i}{m}\right)^{k/2}\int dq \exp\left[-\frac{m(k+2)}{2\delta t i(k+1)}q^2+\frac{2q\, m(k+2)}{2\delta t i(k+1)}\left[\frac{q'+(k+1)q''}{(k+2)}\right]\right.$$

$$-\frac{m(k+2)}{2\delta t i(k+1)}\left[\frac{q'+(k+1)q''}{(k+2)}\right]^2$$

$$\left.+\frac{m(k+2)}{2\delta t i(k+1)}\left[\frac{q'+(k+1)q''}{(k+2)}\right]^2-\frac{m}{2\delta t i(k+1)}\left[q'^2+q''^2(k+1)\right]\right]$$

$$= \frac{1}{\sqrt{k+1}}\left(\frac{2\pi\delta t i}{m}\right)^{k/2}\int dq \exp\left[-\frac{m(k+2)}{2\delta t i(k+1)}\left(q-\frac{(q'+(k+1)q'')}{k+2}\right)^2\right.$$

$$+\frac{m}{2\delta t i(k+1)(k+2)}\left(q'^2+2q'q''(k+1)+q''^2(k+1)^2\right.$$

$$\left.\left.-\left(q'^2+(k+1)q''^2\right)(k+2)\right)\right]$$

$$= \frac{1}{\sqrt{k+1}}\left(\frac{2\pi\delta t i}{m}\right)^{k/2}\sqrt{\frac{\pi 2\delta t i(k+1)}{m(k+2)}}\exp\left\{\frac{-m}{2\delta t i(k+1)(k+2)}\left[q'^2(k+1)\right.\right.$$

$$\left.\left.-2q'q''(k+1)+q''^2(k+1)\right]\right\}$$

$$= \frac{1}{\sqrt{k+2}}\left(\frac{2\pi\delta t i}{m}\right)^{\frac{k+1}{2}}\exp\left\{\frac{-m}{2\delta t i(k+2)}(q''-q')^2\right\}$$

Logo vale para todo n. Portanto:

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \, \exp\left[i \int_{t'}^{t''} dt \, \frac{m\dot{q}^2}{2} \right]$$

$$= \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N+1}{2}} \int \prod_{j=1}^{N} \left[dq_j \, \exp\left(\frac{-m}{2\delta t i} (q_{j+1} - q_j)^2 \right) \right]$$

$$= \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N+1}{2}} \cdot \frac{1}{\sqrt{N+1}} \left(\frac{2\pi \delta t i}{m} \right)^{N/2} \exp\left(\frac{-m}{2\delta t i (N+1)} (q'' - q')^2 \right)$$

$$= \left(\frac{m}{2\pi \delta t \, i} \right)^{1/2} \cdot \frac{1}{\sqrt{N+1}} \exp\left(\frac{-m}{2 i \delta t (N+1)} (q'' - q')^2 \right)$$

como $\delta t = \dfrac{\Delta T}{N+1} = \dfrac{t'' - t'}{N+1}$:

$$\langle q'', t'' | q', t' \rangle = \left(\frac{m}{\underbrace{2\pi (t'' - t') i}} \right)^{1/2} \exp\left(\underbrace{\frac{-m}{2 i (t'' - t')}} (q'' - q')^2 \right)$$

$\text{comp}^{-2} \times [\hbar] = \text{massa} \times \text{tempo}^{-1}$ $\quad [\hbar] = \dfrac{\text{massa} \times \text{comprimento}^2}{\text{tempo}}$

$$\langle q'', t'' | q', t' \rangle = \left(\frac{m}{2\pi \hbar (t'' - t') i} \right)^{1/2} \exp\left(\frac{-m}{2(t'' - t') \hbar i} (q'' - q')^2 \right)$$

(c) $\langle q'', t'' | q', t' \rangle = \langle q'' | \exp\left(-i H (t'' - t') \right) | q' \rangle \; ; \; H = \dfrac{p^2}{2m}$

$$= \int dp \int dp' \, \langle q'' | p'' \rangle \langle p'' | e^{\frac{-i p^2}{2m}(t'' - t')} | p' \rangle \langle p' | q' \rangle$$

$$\langle q'', t'' | q', t' \rangle = \int dp' dp'' \frac{e^{iq''p''} e^{-iq'p'}}{2\pi} e^{-i\frac{p'^2}{2m}(t''-t')} \langle p'' | p' \rangle$$

$$= \int dp' dp'' \frac{\delta(p'-p'')}{2\pi} \exp\left[i\left(q''p'' - q'p' - \frac{p'^2}{2m}(t''-t') \right)\right]$$

$$= \int \frac{dp'}{2\pi} \exp\left[i\left(q''p' - q'p' - \frac{p'^2}{2m}(t''-t') \right)\right]$$

$$= \int \frac{dp'}{2\pi} \exp\left[-\frac{i(t''-t')}{2m} \left\{ p'^2 - 2p' \frac{(q''-q')}{} \frac{m}{(t''-t')} + \frac{(q''-q')^2 m^2}{(t''-t')^2} \right\} + \frac{i(t''-t')}{2m} \frac{(q''-q')^2 m^2}{(t''-t')^2} \right]$$

$$= \int \frac{dp'}{2\pi} \exp\left[-\frac{i(t''-t')}{2m} \left(p' - \frac{m}{(t''-t')}(q''-q') \right)^2 \right] \times$$

$$\times \exp\left(\frac{-m}{2(t''-t')\,i} (q''-q')^2 \right)$$

$$= \sqrt{\frac{2\pi m}{i(t''-t')}} \cdot \frac{1}{2\pi} \exp\left(\frac{-m}{2i(t''-t')}(q''-q')^2 \right)$$, inserindo $\hbar$.

$$= \sqrt{\frac{m}{2\pi i \hbar (t''-t')}} \exp\left(\frac{-m}{2\hbar i (t''-t')}(q''-q')^2 \right)$$

mesmo resultado.
