(2.1) $$g_{\mu\nu}(\delta^\mu{}_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \delta\omega^\nu{}_\sigma) = g_{\rho\sigma}$$

$$g_{\mu\nu}(\delta^\mu{}_\rho\delta^\nu{}_\sigma + \delta^\mu{}_\rho\,\delta\omega^\nu{}_\sigma + \delta^\nu{}_\sigma\,\delta\omega^\mu{}_\rho) = g_{\rho\sigma}$$

$$g_{\rho\sigma} + g_{\rho\nu}\delta\omega^\nu{}_\sigma + g_{\mu\sigma}\delta\omega^\mu{}_\rho = g_{\rho\sigma}$$

$$\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$$

(2.2) $$U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$$

$$U(\Lambda)^{-1}\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) = U\left(\Lambda^{-1}(\mathbb{1}+\delta\omega')\Lambda\right)$$

$$\mathbb{1} + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \mathbb{1} + \frac{i}{2\hbar}(\Lambda^{-1})_\alpha{}^\mu\,\delta\omega_{\mu\nu}\Lambda^\nu{}_\beta M^{\alpha\beta}$$

$$\delta\omega'_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \delta\omega_{\mu\nu}\Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha M^{\alpha\beta}$$

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha M^{\alpha\beta}$$

(2.3) $$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)M^{\mu\nu}\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) =$$

$$= (\delta^\nu{}_\beta + \delta\omega^\nu{}_\beta)(\delta^\mu{}_\alpha + \delta\omega^\mu{}_\alpha)M^{\alpha\beta}$$

$$M^{\mu\nu} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] = M^{\mu\nu} + \delta\omega^\nu{}_\beta M^{\mu\beta} + \delta\omega^\mu{}_\alpha M^{\alpha\nu}$$

$$\frac{i}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] = \frac{\delta\omega_{\rho\sigma}}{2}\Big(g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} + g^{\nu\rho}M^{\mu\sigma}$$
$$- g^{\nu\sigma}M^{\mu\rho}\Big)$$
$$+ \frac{\delta\omega_{\rho\sigma}}{2}\Big(g^{\mu\rho}M^{\sigma\nu} + g^{\mu\sigma}M^{\rho\nu} + g^{\mu\rho}M^{\sigma\nu}$$
$$- g^{\mu\sigma}M^{\rho\nu}\Big)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\Big(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}$$
$$- g^{\mu\sigma}M^{\nu\rho}\Big)$$

(2.4) $[J_i, J_j] = \frac{1}{4}\epsilon_{iab}\epsilon_{jcd}[M^{ab}, M^{cd}]$

$$= \frac{1}{4}\Big[\delta_{ij}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \delta_{ic}(\delta_{ad}\delta_{bj} - \delta_{aj}\delta_{bd})$$
$$+ \delta_{id}(\delta_{aj}\delta_{bc} - \delta_{ac}\delta_{bj})\Big] \times$$
$$\times i\hbar\big(g^{ac}M^{bd} - g^{bc}M^{ad} - g^{ad}M^{bc} + g^{bd}M^{ac}\big)$$

$$= \frac{\hbar i}{2}\big(-g^{a}{}_{a}M_{ji} - g^{b}{}_{b}M_{ji} + g_{ja}M^{a}{}_{i} + g_{jb}M^{b}{}_{i}$$
$$+ g^{a}{}_{i}M_{ja} + g^{b}{}_{i}M_{jb}\big)$$

$$= i\hbar M_{ij} = \frac{i\hbar}{2}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})M^{ab}$$

$$= i\hbar\,\epsilon_{ij}{}^{k}J_k.$$

$$[J_i, K_j] = \frac{1}{2}\epsilon_{iab}[M^{ab}, M^{j0}]$$

$$= \frac{i\hbar}{2}\epsilon_{iab}\left(g^{aj}M^{b0} - g^{bj}M^{a0} - g^{a0}M^{bj} + g^{b0}M^{aj}\right)$$

$$= \frac{i\hbar}{2}\epsilon_{iab}g^{aj}M^{b0} + i\frac{\hbar}{2}\epsilon_{iab}g^{bj}M^{a0}$$

$$= i\hbar\,\epsilon_{iab}g^{aj}M^{b0} = i\hbar\,\epsilon_{ijb}M^{b0}$$

$$= i\hbar\,\epsilon_{ij}{}^{k}K_k$$

$$[K_i, K_j] = [M^{i0}, M^{j0}]$$

$$= i\hbar\left(g^{ij}M^{00} - g^{0j}M^{i0} - g^{i0}M^{0j} + g^{00}M^{ij}\right)$$

$$= -\frac{i\hbar}{2}\left(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right)M^{ab}$$

$$= -i\hbar\,\epsilon_{ij}{}^{k}J_k$$

2.5 $$U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$$

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)P^\mu\left(\mathbb{1} + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) = \left(\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu\right)P^\nu$$

$$P^\mu + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}[P^\mu, M^{\rho\sigma}] = P^\mu + \delta\omega^\mu{}_\nu P^\nu$$

$$\delta_{\rho\sigma}[P^\mu, M^{\rho\sigma}] = -i\hbar\,\delta\omega_{\rho\sigma}\left(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho\right)$$

$$[P^\mu, M^{\rho\sigma}] = i\hbar\left(g^{\mu\sigma}P^\rho - g^{\mu\rho}P^\sigma\right)$$

(2.6)

$$[J_i, H] = [J_i, P^0]$$
$$= \frac{1}{2}\epsilon_{ijk}[M^{jk}, P^0]$$
$$= -\frac{1}{2}\epsilon_{ijk}[P^0, M^{jk}]$$
$$= -\frac{i\hbar}{2}\epsilon_{ijk}\left(g^{0k}P^j - g^{0j}P^k\right)$$
$$= 0$$

$$[J_i, P_j] = \frac{1}{2}\epsilon_{iab}[M^{ab}, P_j]$$
$$= -\frac{1}{2}\epsilon_{iab}[P_j, M^{ab}]$$
$$= \frac{-i\hbar}{2}\epsilon_{iab}\left(g^{jb}P^a - g^{ja}P^b\right)$$
$$= i\hbar\,\epsilon_{ijk}P^k$$

$$[K_i, H] = [K_i, P^0]$$
$$= [M^{i0}, P^0]$$
$$= -[P^0, M^{i0}]$$
$$= -i\hbar(g^{00}P^i - g^{0i}P^0)$$
$$= i\hbar P^i$$

$$[K_i, P_j] = [M^{i0}, P^j]$$
$$= -[P^j, M^{i0}]$$
$$= -i\hbar(g^{j0}P^i - g^{ji}P^0)$$
$$= i\hbar\delta_{ij}H$$

(2.7) A propriedade é $T(a)T(b) = T(a+b)$:

$$T(\delta a)^{-1}\, T(b)\, T(\delta a) = T(b)$$

$$\left(\mathbb{1} + \frac{i}{\hbar}\delta a_\mu P^\mu\right) T(b) \left(\mathbb{1} - \frac{i}{\hbar}\delta a_\mu P^\mu\right) = T(b)$$

$$T(b) + \frac{i}{\hbar}[P^\mu, T(b)]\,\delta a_\mu = T(b) \Rightarrow [P^\mu, T(b)] = 0$$

$$[P^\mu, T(\delta b)] = 0$$

$$\left[P^\mu, \mathbb{1} - \frac{i}{\hbar}\delta b_\nu P^\nu\right] = 0$$

$$[P^\mu, P^\nu] = 0$$

(2.8) (a) $$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$$

$$\left(\mathbb{1} - \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\phi(x)\left(\mathbb{1} + \frac{i\,\delta\omega_{\mu\nu}M^{\mu\nu}}{2\hbar}\right) = \phi(x - \delta\omega x)$$

$$\phi(x) + \frac{i}{2\hbar}[\phi(x), M^{\mu\nu}]\delta\omega_{\mu\nu} = \phi(x) - (\delta\omega\, x)^\rho\partial_\rho\phi(x)$$

$$\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] = 2i\hbar\,\delta\omega_{\mu\nu}\, g^{\rho\mu}x^\nu\partial_\rho\phi(x)$$

$$\overset{!}{=} i\hbar\,\delta\omega_{\mu\nu}(g^{\rho\mu}x^\nu - g^{\rho\nu}x^\mu)\partial_\rho\phi(x)$$

$$\overset{!}{=} \frac{\hbar}{i}\delta\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x)$$

$$[\phi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\phi(x)$$

(b) $$[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] = [\mathcal{L}^{\mu\nu}\phi(x), M^{\rho\sigma}]$$

$$= \mathcal{L}^{\mu\nu}[\phi(x), M^{\rho\sigma}] = \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x).$$

(c) $[[A,B],C] + [[B,C],A] + [[C,A],B] =$

$$= ABC - BAC - CAB + CBA + BCA - CBA - ABC$$
$$+ ACB + CAB - ACB - BCA + BAC$$

$$= 0$$

(d) $[\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = -[M^{\mu\nu}, [M^{\rho\sigma}, \phi(x)]]$
$\qquad - [M^{\rho\sigma}, [\phi(x), M^{\mu\nu}]]$

$$= -[[\phi, M^{\rho\sigma}], M^{\mu\nu}] + [[\phi(x), M^{\mu\nu}], M^{\rho\sigma}]$$

$$= (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\,\phi(x)$$

(e) $(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\,\phi(x) = [\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\,\phi(x)$

$$= -\hbar^2 [x^\mu \partial^\nu - x^\nu \partial^\mu, x^\rho \partial^\sigma - x^\sigma \partial^\rho]\,\phi(x)$$

$$= -i\hbar\,(g^{\nu\rho}\mathcal{L}^{\mu\sigma} + g^{\mu\rho}\mathcal{L}^{\sigma\nu} + g^{\mu\sigma}\mathcal{L}^{\nu\rho} + g^{\nu\sigma}\mathcal{L}^{\rho\mu})\,\phi(x)$$

$$= [\phi(x), -i\hbar\,(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho})]$$

(f): Segue diretamente do item (e) pois avaliamos $[\phi,[M^{\mu\nu},M^{\rho\sigma}]]$, a menos de uma carga central a relação de comutação é válida

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(2,9) (a) $U^{-1}(\Lambda)\,\partial^\rho\phi(x)\,U(\Lambda) = \Lambda^\rho{}_\sigma\,\bar\partial^\sigma\phi(\Lambda^{-1}x)$

$$\left(\mathbb{1}-\frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\partial^\rho\phi(x)\left(\mathbb{1}+\frac{i\,\delta\omega_{\mu\nu}}{2\hbar}M^{\mu\nu}\right) =$$

$$\left(\delta^\rho{}_\sigma+\frac{i\,\delta\omega_{\alpha\beta}}{2\hbar}(S_V^{\alpha\beta})^\rho{}_\sigma\right)\bar\partial\phi(\Lambda^{-1}x)$$

$$\partial^\rho\phi(x)+\frac{i}{2\hbar}\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] = \bar\partial^\rho\phi(x-\delta\omega x)$$
$$+\frac{i}{2\hbar}\delta\omega_{\alpha\beta}(S_V^{\alpha\beta})^\rho{}_\sigma\,\bar\partial^\sigma\phi(x-\delta\omega x)$$

$$\frac{i}{2\hbar}\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] = -(\delta\omega x)_\gamma\,\partial^\gamma\partial^\rho\phi(x)$$
$$+\frac{i}{2\hbar}\delta\omega_{\alpha\beta}(S_V^{\alpha\beta})^\rho{}_\sigma\left(\partial^\sigma\phi(x)-(\delta\omega x)_\delta\,\partial^\delta\partial^\sigma\phi(x)\right)$$

$$\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] = 2i\hbar\,\delta\omega_{\mu\nu}x^\nu\partial^\mu\partial^\rho\phi(x)+\delta\omega_{\mu\nu}(S_V^{\mu\nu})^\rho{}_\sigma\,\partial^\sigma\phi(x)$$

$$\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] = -\frac{\hbar}{i}\delta\omega_{\mu\nu}(x^\nu\partial^\mu-x^\mu\partial^\nu)\partial^\rho\phi(x)$$
$$+\delta\omega_{\mu\nu}(S_V^{\mu\nu})^\rho{}_\sigma\,\partial^\sigma\phi(x)$$

$$[\partial^\rho \phi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^\rho\phi(x) + (S_V{}^{\mu\nu})^\rho{}_\sigma\, \partial^\sigma \phi(x)$$ ▣

ⓑ $$[S_V{}^{\mu\nu}, S_V{}^{\rho\sigma}]^\alpha{}_\beta = (S_V{}^{\mu\nu})^\alpha{}_\tau (S_V{}^{\rho\sigma})^\tau{}_\beta - (S_V{}^{\rho\sigma})^\alpha{}_\tau (S_V{}^{\mu\nu})^\tau{}_\beta$$

$$= -\hbar^2 (g^{\mu\alpha}\delta^\nu{}_\tau - g^{\nu\alpha}\delta^\mu{}_\tau)(g^{\rho\tau}\delta^\sigma{}_\beta - g^{\sigma\tau}\delta^\rho{}_\beta)$$

$$+ \hbar^2 (g^{\rho\alpha}\delta^\sigma{}_\tau - g^{\sigma\alpha}\delta^\rho{}_\tau)(g^{\mu\tau}\delta^\nu{}_\beta - g^{\nu\tau}\delta^\mu{}_\beta)$$

$$= -\hbar^2 \left(g^{\mu\rho}(g^{\sigma\alpha}\delta^\nu{}_\beta - g^{\nu\alpha}\delta^\sigma{}_\beta) + g^{\nu\rho}(g^{\mu\alpha}\delta^\sigma{}_\beta - g^{\sigma\alpha}\delta^\mu{}_\beta)\right)$$

$$- \hbar^2 \left(g^{\mu\sigma}(g^{\nu\alpha}\delta^\rho{}_\beta - g^{\rho\alpha}\delta^\nu{}_\beta) + g^{\nu\sigma}(g^{\rho\alpha}\delta^\mu{}_\beta - g^{\mu\alpha}\delta^\rho{}_\beta)\right)$$

$$= i\hbar \Big[g^{\mu\rho}(S_V{}^{\nu\sigma})^\alpha{}_\beta - g^{\nu\rho}(S_V{}^{\mu\sigma})^\alpha{}_\beta - g^{\mu\sigma}(S_V{}^{\nu\rho})^\alpha{}_\beta$$

$$+ g^{\nu\sigma}(S_V{}^{\mu\rho})^\alpha{}_\beta \Big]$$ ▣

ⓒ $$(S_V{}^{12})^\mu{}_\nu = \frac{\hbar}{i}(g^{1\mu}\delta^2{}_\nu - g^{2\mu}\delta^1{}_\nu)$$

$$= \frac{\hbar}{i}(\delta_{1\mu}\delta_{2\nu} - \delta_{2\mu}\delta_{1\nu})$$

$$S_V^{12} = \frac{\hbar}{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (S_V^{12})^2 = \frac{\hbar^2}{i^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(S_V^{12})^3 = \frac{\hbar^3}{i^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\frac{\hbar^2}{i^2}\left(S_V^{12}\right).$$

$$\exp\left(\frac{-i\theta S_V^{12}}{\hbar}\right) = \sum_{n=0}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^n \frac{\left(S_V^{12}\right)^n}{n!}$$

$$= \mathbb{1} + \sum_{n=0}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^{2n+1}\frac{\left(S_V^{12}\right)^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^{2n}\frac{\left(S_V^{12}\right)^{2n}}{(2n)!}$$

$$= \mathbb{1} + \sum_{n=0}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^{2n+1}\left(-\frac{\hbar^2}{i^2}\right)^n\frac{S_V^{12}}{(2n+1)!} + \sum_{n=1}^{\infty}\left(\frac{-i\theta}{\hbar}\right)^{2n}\frac{\left(S_V^{12}\right)^2}{(2n)!}\left(\frac{-\hbar^2}{i^2}\right)^{n-1}$$

$$= \mathbb{1} + \frac{\left(S_V^{12}\right)^2}{\hbar^2}\left(-\theta^2/2 + \theta^4/4! - \dots\right) + \frac{i}{\hbar}S_V^{12}\left(-\theta + \frac{\theta^3}{3!} - \dots\right)$$

$$= \mathbb{1} + \frac{\left(S_V^{12}\right)^2}{\hbar^2}(\cos\theta - 1) - \frac{i}{\hbar}S_V^{12}\,\sin\theta$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) $$(S_V^{30})^\mu{}_\nu = \frac{\hbar}{i}\left(g^{3\mu}\delta^0{}_\nu - g^{0\mu}\delta^3{}_\nu\right)$$

$$= \frac{\hbar}{i}\left(\delta_{3\mu}\delta_{0\nu} - \delta_{3\nu}\right)$$

$$S_v^{30} = \frac{\hbar}{i}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} \qquad (S_v^{30})^2 = \frac{\hbar^2}{i^2}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$(S_v^{30})^3 = \frac{\hbar^2}{i^2} S_v^{30}$$

$$\exp\left(\frac{i\eta}{\hbar} S_v^{30}\right) = \sum_{n=0}^{\infty} \left(\frac{i\eta}{\hbar}\right)^n \frac{(S_v^{30})^n}{n!}$$

$$= \mathbb{1} + \sum_{n=0}^{\infty} \left(\frac{i\eta}{\hbar}\right)^{2n+1} \left(\frac{\hbar^2}{i^2}\right)^n \frac{S_v^{30}}{(2n+1)!} + \sum_{n=1}^{\infty}\left(\frac{i\eta}{\hbar}\right)^{2n} \frac{(S_v^{30})^2}{(2n)!}\left(\frac{\hbar^2}{i^2}\right)^{n-1}$$

$$= \mathbb{1} + \frac{(S_v^{30})^2}{\hbar^2}\left(-\frac{\eta^2}{2} - \frac{\eta^4}{4!} - \dots\right) + \frac{S_v^{30}}{\hbar} i \left(\eta + \frac{\eta^3}{3!} + \dots\right)$$

$$= \mathbb{1} - \frac{(S_v^{30})^2}{\hbar^2}(\cosh\eta - 1) + \frac{(S_v^{30})}{\hbar} i \sinh\eta$$

$$= \begin{pmatrix}\cosh\eta & 0 & 0 & \sinh\eta\\ 0&1&0&0\\ 0&0&1&0\\ \sinh\eta & 0 & 0 & \cosh\eta\end{pmatrix}$$