SREDNICKI CAPÍTULO 2

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Sumário

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1. Exercício 1

$$
\begin{aligned}
g_{\mu\nu}\left(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \delta\omega^{\mu}{}_{\sigma}\right) &= g_{\rho\sigma} \\
g_{\mu\nu}\left(\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} + \delta^{\mu}{}_{\rho}\delta\omega^{\nu}{}_{\sigma} + \delta^{\nu}{}_{\sigma}\delta\omega^{\mu}{}_{\rho}\right) &= g_{\rho\sigma} \\
g_{\rho\sigma} + g_{\rho\nu}\delta\omega^{\nu}{}_{\sigma} + g_{\mu\sigma}\delta\omega^{\mu}{}_{\rho} &= g_{\rho\sigma} \\
\delta\omega_{\rho\sigma} &= -\delta\omega_{\sigma\rho}
\end{aligned}
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2. Exercício 2

$$
\begin{aligned}
U^{-1}(\Lambda)U(\Lambda')U(\Lambda) &= U\big(\Lambda^{-1}\Lambda'\Lambda\big)\\
U^{-1}\left(\mathbb{1} + \frac{\mathrm{i}}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) &= U\big(\Lambda^{-1}(\mathbb{1}+\delta\omega')\Lambda\big)\\
\mathbb{1} + \frac{\mathrm{i}}{2\hbar}\delta\omega'_{\mu\nu}U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) &= \mathbb{1} + \frac{\mathrm{i}}{2\hbar}\big(\Lambda^{-1}\big)_{\alpha}{}^{\mu}\delta\omega'_{\mu\nu}\Lambda^{\nu}{}_{\beta}M^{\alpha\beta}\\
\delta\omega'_{\mu\nu}U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) &= \delta\omega'_{\mu\nu}\Lambda^{\nu}{}_{\beta}\Lambda^{\mu}{}_{\alpha}M^{\alpha\beta}\\
U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) &= \Lambda^{\nu}{}_{\beta}\Lambda^{\mu}{}_{\alpha}M^{\alpha\beta}
\end{aligned}
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3. Exercício 3

$$\begin{aligned}
\left(\mathbb{1} - \frac{\mathrm{i}}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)M^{\mu\nu}\left(\mathbb{1} - \frac{\mathrm{i}}{\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) &= \left(\delta^{\nu}{}_{\beta} + \delta\omega^{\nu}{}_{\beta}\right)(\delta^{\mu}{}_{\alpha} + \delta\omega^{\mu}{}_{\alpha})M^{\alpha\beta} \\
M^{\mu\nu} + \frac{\mathrm{i}}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] &= M^{\mu\nu} + \delta\omega^{\nu}{}_{\beta}M^{\mu\beta} + \delta\omega^{\mu}{}_{\alpha}M^{\alpha\nu} \\
\frac{\mathrm{i}}{2\hbar}\delta\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] &= \frac{\delta\omega_{\rho\sigma}}{2}(g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\nu\sigma}M^{\mu\rho}) \\
&\quad + \frac{\delta\omega_{\rho\sigma}}{2}(g^{\mu\rho}M^{\sigma\nu} + g^{\mu\sigma}M^{\rho\nu} + g^{\mu\rho}M^{\sigma\nu} - g^{\mu\sigma}M^{\rho\nu}) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= \mathrm{i}\hbar(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho})
\end{aligned}$$

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4. Exercício 4

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\begin{aligned}
[J_i, J_j] &= \frac{1}{4}\epsilon_{iab}\epsilon_{jcd}\big[M^{ab}, M^{cd}\big] \\
&= \frac{\mathrm{i}\hbar}{4}\{\delta_{ij}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \delta_{ic}(\delta_{ad}\delta_{bj} - \delta_{aj}\delta_{bd}) + \delta_{id}(\delta_{aj}\delta_{bc} - \delta_{ac}\delta_{bj})\}(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho}) \\
&= \frac{\mathrm{i}\hbar}{2}\big(-g^{a}{}_{a}M_{ji} - g^{b}{}_{b}M_{ji} + g_{ja}M^{a}{}_{i} + g_{jb}M^{b}{}_{i} + g^{a}{}_{i}M_{ja} + g^{b}{}_{i}M_{jb}\big) \\
&= \mathrm{i}\hbar M_{ij} = \frac{\mathrm{i}\hbar}{2}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})M^{ab} \\
&= \mathrm{i}\hbar\epsilon_{ij}{}^{k}J_k
\end{aligned}
$$

$$
\begin{aligned}
[J_i, K_j] &= \frac{1}{2}\epsilon_{iab}\big[M^{ab}, M^{j0}\big] \\
&= \frac{\mathrm{i}\hbar}{2}\epsilon_{iab}\big(g^{aj}M^{b0} - g^{bj}M^{a0} - g^{a0}M^{bj} + g^{b0}M^{aj}\big) \\
&= \frac{\mathrm{i}\hbar}{2}\epsilon_{iab}g^{aj}M^{b0} + \frac{\mathrm{i}\hbar}{2}\epsilon_{iab}g^{bj}M^{a0} \\
&= \mathrm{i}\hbar\epsilon_{ijb}M^{b0} \\
&= \mathrm{i}\hbar\epsilon_{ij}{}^{k}K_k
\end{aligned}
$$

$$
\begin{aligned}
[K_i, K_j] &= \big[M^{i0}, M^{j0}\big] \\
&= \mathrm{i}\hbar\big(g^{ij}M^{00} - g^{0j}M^{i0} - g^{i0}M^{0j} + g^{00}M^{ij}\big) \\
&= -\frac{\mathrm{i}\hbar}{2}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})M^{ab} \\
&= -\mathrm{i}\hbar\epsilon_{ij}{}^{k}J_k
\end{aligned}
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5. Exercício 5

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\begin{aligned}
U^{-1}(\Lambda)P^\mu U(\Lambda) &= \Lambda^\mu{}_\nu P^\nu \\
\left(\mathbb{1} - \frac{\mathrm{i}}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)P^\mu\left(\mathbb{1} + \frac{\mathrm{i}}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right) &= (\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu)P^\nu \\
P^\mu + \frac{\mathrm{i}}{2\hbar}\delta\omega_{\rho\sigma}[P^\mu, M^{\rho\sigma}] &= P^\mu + \delta\omega^\mu{}_\nu P^\nu \\
\delta\omega_{\rho\sigma}[P^\mu, M^{\rho\sigma}] &= -\mathrm{i}\hbar\delta\omega_{\rho\sigma}(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho) \\
[P^\mu, M^{\rho\sigma}] &= \mathrm{i}\hbar(g^{\mu\sigma}P^\rho - g^{\mu\rho}P^\sigma)
\end{aligned}
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6. Exercício 6

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\begin{aligned}
[J_i, H] &= \left[J_i, P^0\right] \\
&= \frac{1}{2}\epsilon_{ijk}\left[M^{jk}, P^0\right] \\
&= -\frac{1}{2}\epsilon_{ijk}\left(g^{0k}P^j - g^{0j}P^k\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[J_i, P_j] &= \frac{1}{2}\epsilon_{iab}\left[M^{ab}, P_j\right] \\
&= -\frac{\mathrm{i}\hbar}{2}\epsilon_{iab}\left(g^{ib}P^a - g^{ja}P^b\right) \\
&= \mathrm{i}\hbar\epsilon_{ijk}P^k
\end{aligned}
$$

$$
\begin{aligned}
[K_i, H] &= -\left[P^0, M^{i0}\right] \\
&= -\mathrm{i}\hbar\left(g^{00}P^i - g^{0i}P^0\right) \\
&= \mathrm{i}\hbar P^i
\end{aligned}
$$

$$
\begin{aligned}
[K_i, P_j] &= \left[M^{i0}, P^j\right] \\
&= -\mathrm{i}\hbar\left(g^{j0}P^i - g^{ji}P^0\right) \\
&= \mathrm{i}\hbar\delta_{ij}H
\end{aligned}
$$

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7. Exercício 7

A propriedade é $T(a)T(b) = T(a+b)$,

$$\begin{aligned}
T^{-1}(\delta a)T(b)T(\delta a) &= T(b) \\
\left(\mathbb{1} + \frac{\mathrm{i}\hbar}{\delta a_{\mu}} P^{\mu}\right) T(b) \left(\mathbb{1} - \frac{\mathrm{i}}{\hbar}\delta a_{\mu} P^{\mu}\right) &= T(b) \\
T(b) + \frac{\mathrm{i}}{\hbar}[P^{\mu}, T(b)]\delta a_{\mu} &= T(b) \\
[P^{\mu}, T(b)] &= 0 \\
\left[P^{\mu}, \mathbb{1} - \frac{\mathrm{i}}{\hbar}\delta b_{\nu} P^{\nu}\right] &= 0 \\
[P^{\mu}, P^{\nu}] &= 0
\end{aligned}$$

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8. Exercício 8

a).

$$
\begin{aligned}
U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \phi\big(\Lambda^{-1}x\big) \\
\left(\mathbb{1} - \frac{\mathrm{i}}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\phi(x)\left(\mathbb{1} + \frac{\mathrm{i}}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) &= \phi(x - \delta\omega x) \\
\phi(x) + \frac{\mathrm{i}}{2\hbar}[\phi(x), M^{\mu\nu}]\delta\omega_{\mu\nu} &= \phi(x) - (\delta\omega x)^{\rho}\partial_{\rho}\phi(x) \\
\delta\omega_{\mu\nu}[\phi(x), M^{\mu\nu}] &= 2\mathrm{i}\hbar\delta\omega_{\mu\nu}g^{\rho\mu}x^{\nu}\partial_{\rho}\phi(x) \\
&= \mathrm{i}\hbar\delta\omega_{\mu\nu}(g^{\rho\mu}x^{\nu} - g^{\rho\nu}x^{\mu})\partial_{\rho}\phi(x) \\
&= \frac{\hbar}{\mathrm{i}}\delta\omega_{\mu\nu}(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\phi(x) \\
[\phi(x), M^{\mu\nu}] &= \mathcal{L}^{\mu\nu}\phi(x)
\end{aligned}
$$

b).

$$
\begin{aligned}
[[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] &= [\mathcal{L}^{\mu\nu}\phi(x), M^{\rho\sigma}] \\
&= \mathcal{L}^{\mu\nu}[\phi(x), M^{\rho\sigma}] \\
&= \mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma}\phi(x)
\end{aligned}
$$

c).

$$
\begin{aligned}
&[[A, B], C] + [[B, C], A] + [[C, A], B] \\
&= ABC - BAC - CAB + CBA + BCA - CBA - ABC + ACB + CAB - ACB - BCA + BAC \\
&= 0
\end{aligned}
$$

d).

$$
\begin{aligned}
[\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]] &= -[M^{\mu\nu}, [M^{\rho\sigma}, \phi(x)]] - [M^{\rho\sigma}, [\phi(x), M^{\mu\nu}]] \\
&= -[[\phi(x), M^{\rho\sigma}], M^{\mu\nu}] + [[\phi(x), M^{\mu\nu}], M^{\rho\sigma}] \\
&= (\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\phi(x)
\end{aligned}
$$

e).

$$
\begin{aligned}
(\mathcal{L}^{\mu\nu}\mathcal{L}^{\rho\sigma} - \mathcal{L}^{\rho\sigma}\mathcal{L}^{\mu\nu})\phi(x) &= [\mathcal{L}^{\mu\nu}, \mathcal{L}^{\rho\sigma}]\phi(x) \\
&= -\hbar^{2}[x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}, x^{\rho}\partial^{\sigma} - x^{\sigma}\partial^{\rho}]\phi(x) \\
&= -\mathrm{i}\hbar(g^{\nu\rho}L^{\mu\sigma} + g^{\mu\rho}L^{\sigma\nu} + g^{\mu\sigma}L^{\nu\rho} - g^{\nu\sigma}L^{\rho\mu}) \quad = [\phi(x), -\mathrm{i}\hbar(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho})]
\end{aligned}
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f). Segue diretamente do item anterior, pois avaliamos $[\phi(x), [M^{\mu\nu}, M^{\rho\sigma}]]$, a menos de uma carga central a relação de comutação é válida.

a).

$$
\begin{aligned}
U^{-1}(\Lambda)\partial^\rho\phi(x)U(\Lambda) &= \Lambda^\rho{}_\sigma\bar{\partial}^\sigma\phi\big(\Lambda^{-1}x\big)\\
\left(\mathbb{1}-\frac{\mathrm{i}}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\partial^\rho\phi(x)\left(\mathbb{1}+\frac{\mathrm{i}}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right) &= \left(\delta^\rho{}_\sigma+\frac{\mathrm{i}}{2\hbar}\delta\omega_{\alpha\beta}\Big(S_{\mathrm{V}}^{\alpha\beta}\Big)^\rho{}_\sigma\right)\bar{\partial}^\sigma\phi\big(\Lambda^{-1}x\big)\\
\partial^\sigma\phi(x)+\frac{\mathrm{i}}{2\hbar}\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] &= \bar{\partial}^\rho(x-\delta\omega x)+\frac{\mathrm{i}}{2\hbar}\delta\omega_{\alpha\beta}\Big(S_{\mathrm{V}}^{\alpha\beta}\Big)^\rho{}_\sigma\bar{\partial}^\sigma\phi(x-\delta\omega x)\\
\frac{\mathrm{i}}{2\hbar}\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] &= -(\delta\omega x)_\gamma\partial^\gamma\partial^\rho\phi(x)+\frac{\mathrm{i}}{2\hbar}\delta\omega_{\alpha\beta}\Big(S_{\mathrm{V}}^{\alpha\beta}\Big)^\rho{}_\sigma\big(\partial^\sigma\phi(x)-(\delta\omega)_\delta\partial^\delta\partial^\sigma\phi(x)\big)\\
\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] &= 2\mathrm{i}\hbar\delta\omega_{\mu\nu}x^\nu\partial^\mu\partial^\rho\phi(x)+\delta\omega_{\mu\nu}(S_{\mathrm{V}}^{\mu\nu})^\rho{}_\sigma\partial^\sigma\phi(x)\\
\delta\omega_{\mu\nu}[\partial^\rho\phi(x),M^{\mu\nu}] &= -\frac{\hbar}{\mathrm{i}}\delta\omega_{\mu\nu}(x^\nu\partial^\mu-x^\mu\partial^\nu)\partial^\rho\phi(x)+\delta\omega_{\mu\nu}(S_{\mathrm{V}}^{\mu\nu})^\rho{}_\sigma\partial^\sigma\phi(x)\\
[\partial^\rho\phi(x),M^{\mu\nu}] &= \mathcal{L}^{\mu\nu}\partial^\rho\phi(x)+(S_{\mathrm{V}}^{\mu\nu})^\rho{}_\sigma\partial^\sigma\phi(x)
\end{aligned}
$$

b).

$$
\begin{aligned}
[S_{\mathrm{V}}^{\mu\nu},S_{\mathrm{V}}^{\rho\sigma}]^\alpha{}_\beta &= (S_{\mathrm{V}}^{\mu\nu})^\alpha{}_\tau(S_{\mathrm{V}}^{\rho\sigma})^\tau{}_\beta-(S_{\mathrm{V}}^{\rho\sigma})^\alpha{}_\tau(S_{\mathrm{V}}^{\mu\nu})^\tau{}_\beta\\
&= -\hbar^2(g^{\mu\alpha}\delta^\nu{}_\tau-g^{\nu\alpha}\delta^\mu{}_\tau)\Big(g^{\rho\tau}\delta^\sigma{}_\beta-g^{\sigma\tau}\delta^\rho{}_\beta\Big)+\hbar^2(g^{\rho\alpha}\delta^\sigma{}_\tau-g^{\sigma\alpha}\delta^\rho{}_\tau)\Big(g^{\mu\tau}\delta^\nu{}_\beta-g^{\nu\tau}\delta^\mu{}_\beta\Big)\\
&= -\hbar^2\Big\{g^{\mu\rho}\big(g^{\sigma\alpha}\delta^\nu{}_\beta-g^{\nu\alpha}\delta^\sigma{}_\beta\big)+g^{\nu\rho}\Big(g^{\mu\alpha}\delta^\sigma{}_\beta-g^{\sigma\alpha}\delta^\mu{}_\beta\Big)\Big\}-\hbar^2\Big\{g^{\mu\sigma}\Big(g^{\nu\alpha}\delta^\rho{}_\beta-g^{\rho\alpha}\delta^\nu{}_\beta\Big)+g^{\nu\sigma}\Big(g^{\rho\alpha}\delta^\mu{}_\beta-g^{\mu\alpha}\delta^\rho{}_\beta\Big)\Big\}\\
&= \mathrm{i}\hbar\Big\{g^{\mu\rho}(S_{\mathrm{V}}^{\nu\sigma})^\alpha{}_\beta-g^{\nu\rho}(S_{\mathrm{V}}^{\mu\sigma})^\alpha{}_\beta-g^{\mu\sigma}(S_{\mathrm{V}}^{\nu\rho})^\alpha{}_\beta+g^{\nu\sigma}(S_{\mathrm{V}}^{\mu\rho})^\alpha{}_\beta\Big\}
\end{aligned}
$$

c).

$$
\begin{aligned}
\big(S_{\mathrm{V}}^{12}\big)^\mu{}_\nu &= \frac{\hbar}{\mathrm{i}}\big(g^{1\mu}\delta^2{}_\nu-g^{2\mu}\delta^1{}_\nu\big)\\
&= \frac{\hbar}{\mathrm{i}}(\delta_{1\mu}\delta_{2\nu}-\delta_{2\mu}\delta_{1\nu})
\end{aligned}
$$

$$
S_{\mathrm{V}}^{12}=\frac{\hbar}{\mathrm{i}}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix},\ \big(S_{\mathrm{V}}^{12}\big)^2=\frac{\hbar^2}{\mathrm{i}^2}\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}
$$

$$
\big(S_{\mathrm{V}}^{12}\big)^3=\frac{\hbar^3}{\mathrm{i}^3}\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}=-\frac{\hbar^2}{\mathrm{i}^2}S_{\mathrm{V}}^{12}
$$

$$
\begin{aligned}
\exp\left(-\frac{\mathrm{i}\theta}{\hbar}S_{\mathrm{V}}^{12}\right) &= \sum_{n=0}^{\infty}\left(-\frac{\mathrm{i}\theta}{\hbar}\right)^n\frac{\big(S_{\mathrm{V}}^{12}\big)^n}{n!}\\
&= \mathbb{1}+\sum_{n=0}^{\infty}\left(-\frac{\mathrm{i}\theta}{\hbar}\right)^{2n+1}\frac{\big(S_{\mathrm{V}}^{12}\big)^{2n+1}}{(2n+1)!}+\sum_{n=1}^{\infty}\left(-\frac{\mathrm{i}\theta}{\hbar}\right)^{2n}\frac{\big(S_{\mathrm{V}}^{12}\big)^{2n}}{(2n)!}\\
&= \mathbb{1}+\sum_{n=0}^{\infty}\left(-\frac{\mathrm{i}\theta}{\hbar}\right)^{2n+1}\left(-\frac{\hbar^2}{\mathrm{i}^2}\right)^n\frac{S_{\mathrm{V}}^{12}}{(2n+1)!}+\sum_{n=1}^{\infty}\left(-\frac{\mathrm{i}\theta}{\hbar}\right)^{2n}\left(-\frac{\hbar^2}{\mathrm{i}^2}\right)^{n-1}\frac{\big(S_{\mathrm{V}}^{12}\big)^2}{(2n)!}\\
&= \mathbb{1}+\frac{1}{\hbar^2}\big(S_{\mathrm{V}}^{12}\big)^2(\cos\theta-1)-\frac{\mathrm{i}}{\hbar}S_{\mathrm{V}}^{12}\sin\theta\\
&= \begin{pmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}
\end{aligned}
$$

d).

$$
\begin{aligned}
\big(S_{\mathrm{V}}^{30}\big)^\mu{}_\nu &= \frac{\hbar}{\mathrm{i}}\big(g^{3\mu}\delta^0{}_\nu-g^{0\mu}\delta^3{}_\nu\big)\\
&= \frac{\hbar}{\mathrm{i}}(\delta_{3\mu}\delta_{0\nu}-\delta_{0\mu}\delta_{3\nu})
\end{aligned}
$$

$$S_{\mathrm{V}}^{30} = \frac{\hbar}{\mathrm{i}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \left(S_{\mathrm{V}}^{30}\right)^2 = \frac{\hbar^2}{\mathrm{i}^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(S_{\mathrm{V}}^{30}\right)^3 = \frac{\hbar^3}{\mathrm{i}^3} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{\hbar^2}{\mathrm{i}^2} S_{\mathrm{V}}^{30}$$

$$\begin{aligned}
\exp\left(\frac{\mathrm{i}\eta}{\hbar} S_{\mathrm{V}}^{30}\right) &= \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}\eta}{\hbar}\right)^n \frac{\left(S_{\mathrm{V}}^{30}\right)^n}{n!} \\
&= \mathbb{1} + \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}\eta}{\hbar}\right)^{2n+1} \frac{\left(S_{\mathrm{V}}^{30}\right)^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}\eta}{\hbar}\right)^{2n} \frac{\left(S_{\mathrm{V}}^{30}\right)^{2n}}{(2n)!} \\
&= \mathbb{1} + \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}\eta}{\hbar}\right)^{2n+1} \left(\frac{\hbar^2}{\mathrm{i}^2}\right)^n \frac{S_{\mathrm{V}}^{30}}{(2n+1)!} + \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}\eta}{\hbar}\right)^{2n} \left(\frac{\hbar^2}{\mathrm{i}^2}\right)^{n-1} \frac{\left(S_{\mathrm{V}}^{30}\right)^2}{(2n)!} \\
&= \mathbb{1} + \frac{1}{\hbar^2}\left(S_{\mathrm{V}}^{30}\right)^2 (1 - \cosh\eta) + \frac{\mathrm{i}}{\hbar} S_{\mathrm{V}}^{30} \sinh\eta \\
&= \begin{pmatrix} \cosh\eta & 0 & 0 & \sin\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\eta & 0 & 0 & \cos\eta \end{pmatrix}
\end{aligned}$$

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