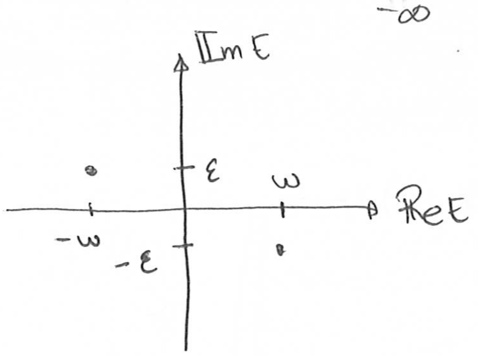
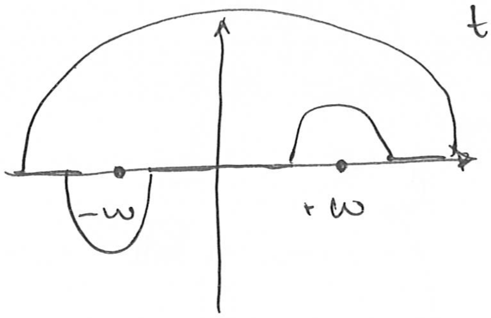
7.1

$$G(t-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2 - E^2}$$

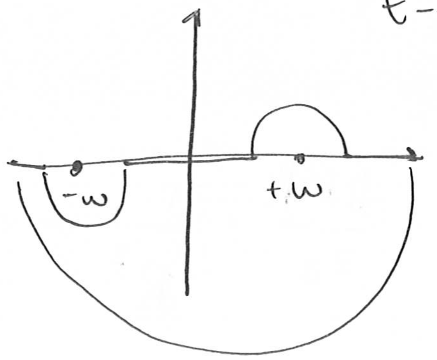


logo o contorno é:

$t-t'<0$



$t-t'>0$.



$$G(t'-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2 - E^2}$$, mas, se $t-t'>0$:

$$-\oint \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2 - E^2} = \mathrm{Res}\left[\frac{1}{2\pi}\frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2-E^2}\right]_{E=\omega-i\epsilon}$$

$$= \lim_{E\to\omega-i\epsilon} 2\pi i \frac{[E-(\omega-i\epsilon)]\, e^{-iE(t-t')}}{2\pi\left[(\omega-i\epsilon-E)(\omega-i\epsilon+E)\right]}$$

$$= -i\frac{e^{-i(\omega-i\epsilon)(t-t')}}{(\omega-i\epsilon+\omega-i\epsilon)} = \frac{-i\, e^{-i\omega(t-t')}}{2\omega}$$

$$-\oint \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2 - E^2} = -\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2 - E^2} - i\int_0^{-\pi} \frac{d\theta}{2\pi} \frac{R e^{i\theta} e^{-ie^{i\theta}R(t-t')}}{(\omega-i\epsilon)^2 - e^{2i\theta}R^2}$$

$$-\oint \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2-E^2} = -\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2-E^2} + \frac{iR}{2\pi}\int_{-\pi}^{0} d\theta \frac{e^{i\theta} e^{-Ri(t-t')(\cos\theta+i\sin\theta)}}{(\omega-i\epsilon)^2 - R^2 e^{2i\theta}}$$

$$\left\| \frac{iR}{2\pi}\int_{-\pi}^{0} d\theta \frac{e^{i\theta} e^{-iR(t-t')\cos\theta}}{(\omega-i\epsilon)^2-R^2 e^{(2i\theta)}} e^{R(t-t')\sin\theta} \right\| <$$

$$< \frac{\|R\|}{2\pi} \cdot \frac{\pi}{R^2} \cdot \| e^{R(t-t')\sin\theta} \| < \frac{1}{\|R\|}$$

mas $\sin\theta \leq 0$ para $\theta \in (-\pi, 0)$

logo $\lim_{R\to 0}$ temos: $-\oint \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2-E^2} = -\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(\omega-i\epsilon)^2-E^2} =$

$$= \frac{-i}{2\omega} e^{-i\omega(t-t')}$$

$$\Rightarrow G(t-t') = \frac{i}{2\omega} e^{-i\omega(t-t')} \quad ; \quad t-t' > 0.$$

mesmo argumento pode ser feito para $t-t' < 0$, chegando em:

$$G(t-t') = \frac{i}{2\omega} e^{-i\omega(t-t')} \qquad \blacksquare$$

(7.2)

$$G(t-t') = \frac{i}{2w}\, exp\left(-iw|t-t'|\right)$$

$$\frac{\partial}{\partial t} G(t-t') = \frac{\partial G}{\partial |t-t'|}\, \underbrace{\frac{\partial |t-t'|}{\partial t}}_{2\theta(t-t')-1}$$

$= 1$ se $t-t'>0$

$= -1$ se $t-t'<0$

$$\frac{\partial}{\partial t} G(t-t') = \left(2\theta(t-t')-1\right)\frac{\partial G}{\partial |t-t'|}$$

$$= \left(2\theta(t-t')-1\right)\left[\frac{i}{2w}\cdot(-iw)\, e^{-iw|t-t'|}\right]$$

$$= \left(2\theta(t-t')-1\right)(-iw)\, G(t-t')$$

$$\frac{\partial^2}{\partial t^2} G(t-t') = -iw\left(2\theta'(t-t')\right) G(t-t')$$

$$-iw\left(2\theta(t-t')-1\right)\left(2\theta(t-t')-1\right)(-iw)\, G(t-t')$$

$$= -iw2\,\delta(t-t')\, G(t-t')$$

$$-w^2\left(4\cancel{\theta^2(t-t')} - 2\cancel{\theta(t-t')} - 2\theta(t-t') + 1\right) G(t-t')$$

$$= -iw2\,\delta(t-t')\, \frac{i e^{-iw|t-t'|}}{2w} - w^2\, G(t-t')$$

$$\frac{\partial^2}{\partial t^2} G(t-t') + w^2 G(t-t') = \delta(t-t')\, e^{-iw|t-t'|}$$

$$= \delta(t-t')$$

∎

7.3) (a) $H = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2$

$$\dot{Q} = i\left[\frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2 ; Q\right]$$

$$= -\frac{i}{2m}[Q, P^2] = -\frac{i}{2m}P[Q,P] - \frac{i}{2m}[Q,P]P$$

$$\dot{Q} = \frac{P}{m}$$

$$\dot{P} = i\left[\frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2 ; P\right]$$

$$= \frac{i m\omega^2}{2}[Q^2, P] = \frac{i m\omega^2}{2} Q[Q,P] + \frac{i m\omega^2}{2}[Q,P]Q$$

$$= -m\omega^2 Q = \dot{P}$$

$$\ddot{P} = -m\omega^2 \dot{Q} = -m\omega^2 \frac{P}{m}$$

$$= -\omega^2 P \Rightarrow P(t) = \cos(\omega t)P(0)$$

$$Q = \frac{-1}{m\omega^2}\dot{P} = \frac{-1}{m\omega^2}(-\omega)\sin(\omega t)P(0)$$

(7.3) (a) $H = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2$

$$\dot{Q} = i \left[\frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2 , Q \right]$$

$$= -\frac{i}{2m} [Q, P^2] = -\frac{i}{2m} P [Q, P] - \frac{i}{2m} [Q, P] P$$

$$\Rightarrow \dot{Q} = \frac{P}{m}.$$

$$\dot{P} = i \left[\frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2 , P \right]$$

$$= \frac{i m\omega^2}{2} [Q^2, P] = \frac{i m\omega^2 Q}{2} [Q, P] + \frac{i m\omega^2}{2} [Q, P] Q$$

$$\Rightarrow \dot{P} = -m\omega^2 Q$$

$$\ddot{P} = -m\omega^2 \dot{Q} = -m\omega^2 \frac{P}{m} = -\omega^2 P$$

$$\Rightarrow P = A e^{i\omega t} + B e^{-i\omega t}$$

$$Q = -\frac{1}{m\omega^2} \dot{P} = -\frac{i}{m\omega} A e^{i\omega t} + \frac{i}{m\omega} B e^{-i\omega t}$$

$P(0) = A + B$ $\qquad A = \frac{P(0) + i m\omega Q(0)}{2}$

$m\omega i Q(0) = A - B$ $\qquad B = \frac{P(0) - i m\omega Q(0)}{2}$

A solução é:

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$$P(t) = \frac{e^{i\omega t}}{2}\left(P + im\omega Q\right) + \frac{e^{-i\omega t}}{2}\left(P - im\omega Q\right)$$

$$Q(t) = \frac{e^{i\omega t}}{2}\left(Q - \frac{iP}{m\omega}\right) + \frac{e^{-i\omega t}}{2}\left(Q + \frac{iP}{m\omega}\right)$$

(b) $$H = \frac{P^2}{m2} + \frac{m\omega^2 Q^2}{2}$$

$$= \frac{1}{2m}\left[P^2 + m^2\omega^2 Q^2\right]$$

$$= \frac{1}{2m}\left[P.P - Pm\omega iQ + iQm\omega P + (m\omega Q)^2 + PQ\, im\omega - QP\, im\omega\right]$$

$$= \frac{1}{2m}\left[P + im\omega Q\right]\left[P - im\omega Q\right] + \frac{im\omega}{2m}\left[P, Q\right]$$

$$= \frac{1}{2m}(P + im\omega Q)(P - im\omega Q) + \frac{\omega}{2}$$

$$= \omega\left(\frac{P}{\sqrt{2m\omega}} + i\sqrt{\frac{m\omega}{2}}Q\right)\left(\frac{P}{\sqrt{2m\omega}} - i\sqrt{\frac{m\omega}{2}}Q\right) + \frac{\omega}{2}$$

ou adicionando $\hbar$:

$$= \hbar\omega\underbrace{\left(\frac{P}{\sqrt{2m\omega\hbar}} + i\sqrt{\frac{m\omega}{2\hbar}}Q\right)}_{a^+}\underbrace{\left(\frac{P}{\sqrt{2m\omega\hbar}} - i\sqrt{\frac{m\omega}{2\hbar}}Q\right)}_{a} + \frac{\hbar\omega}{2}$$

$$= \hbar\omega\left(a^+ a + 1/2\right)$$

então temos:

$$P(t) = e^{i\omega t}\sqrt{\frac{m\omega\hbar}{2}}\, a^{\dagger} + e^{-i\omega t}\sqrt{\frac{m\omega\hbar}{2}}\, a$$

$$Q(t) = -ie^{i\omega t}\sqrt{\frac{\hbar}{2m\omega}}\, a^{\dagger} + ie^{-i\omega t}\sqrt{\frac{\hbar}{2m\omega}}\, a$$

ⓒ $\langle 0|T\{Q(t_1)Q(t_2)\}|0\rangle \quad ; \quad t_1 > t_2.$

$$\langle 0|\left[ie^{-i\omega t_1}\sqrt{\frac{\hbar}{2m\omega}}\, a - ie^{i\omega t_1}\sqrt{\frac{\hbar}{2m\omega}}\, a^{\dagger}\right]\left[ie^{-i\omega t_2}\sqrt{\frac{\hbar}{2m\omega}}\, a - ie^{i\omega t_2}\sqrt{\frac{\hbar}{2m\omega}}\, a^{\dagger}\right]|0\rangle$$

$$i^2\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 \langle 0|\cancel{a^2 e^{-i\omega(t_1+t_2)}} - aa^{\dagger}e^{-i\omega t_1 + i\omega t_2} - \cancel{a^{\dagger}a\, e^{i\omega t_1 - i\omega t_2}} + \cancel{a^{\dagger 2} e^{i\omega(t_1+t_2)}}|0\rangle$$

$$= \frac{\hbar}{2m\omega}\langle 0|a a^{\dagger}|0\rangle e^{i\omega(t_2-t_1)} \qquad [a, a^{\dagger}] = 1$$

$$aa^{\dagger} = 1 + a^{\dagger}a$$

$$= \frac{\hbar}{m}\cdot\frac{1}{2\omega}e^{-i\omega(t_1-t_2)} \qquad t_1 - t_2 > 0.$$

logo, $\langle 0|T\{Q(t_1)Q(t_2)\}|0\rangle = \dfrac{\hbar}{im}\dfrac{i}{2\omega}e^{-i\omega|t_1-t_2|}$

$$= \frac{\hbar}{mi}G(t_1-t_2)$$

supondo agora: $t_1 > t_2 > t_3 > t_4$:

$\langle 0|T\{Q(t_1)Q(t_2)Q(t_3)Q(t_4)\}|0\rangle$

$$= \left(\frac{\hbar}{2m\omega}\right)^{4/2} i^4\langle 0|\left(e^{-i\omega t_1}a - e^{i\omega t_1}a^{\dagger}\right)\left(e^{-i\omega t_2}a - e^{i\omega t_2}a^{\dagger}\right)\left(e^{-i\omega t_3}a - e^{i\omega t_3}a^{\dagger}\right) \times$$

$$\times \left(e^{-i\omega t_4} a - e^{i\omega t_4} a^\dagger \right) \} |0\rangle \quad :=$$

$$\left(i^4 \frac{\hbar^2}{4m^2\omega^2} \right) \langle 0| \Big(a^2 a^{\dagger 2} e^{-i\omega t_1 - i\omega t_2 + i\omega t_3 + i\omega t_4} + a a^\dagger a a^\dagger e^{-i\omega t_1 + i\omega t_2 - i\omega t_3 + i\omega t_4}$$

$$+ a a^\dagger a^\dagger a \, e^{-i\omega t_1 + i\omega t_2 + i\omega t_3 - i\omega t_4}$$

$$+ a^{\dagger 2} a^2 \, e^{i\omega t_1 + i\omega t_2 - i\omega t_3 - i\omega t_4}$$

$$+ a^\dagger a a^\dagger a \, e^{i\omega t_1 - i\omega t_2 + i\omega t_3 - i\omega t_4}$$

$$+ a^\dagger a^2 a^\dagger \, e^{i\omega t_1 - i\omega t_2 - i\omega t_3 + i\omega t_4} \Big) |0\rangle$$

$$= \frac{\hbar^2}{4m^2\omega^2} \langle 0| \Big(a(a^\dagger a + 1) a^\dagger e^{-i\omega[t_1 + t_2 - t_3 - t_4]} + (a^\dagger a + 1) a a^\dagger e^{-i\omega[t_1 - t_2 + t_3 - t_4]}$$

$$+ (a^\dagger a + 1)(a^\dagger a) e^{-i\omega[t_1 - t_2 - t_3 + t_4]} \Big) |0\rangle$$

$$= \frac{\hbar^2}{4m^2\omega^2} \langle 0| \Big(\big((a^\dagger a + 1) a a^\dagger + a a^\dagger \big) e^{-i\omega[t_1 + t_2 - t_3 - t_4]} + a a^\dagger e^{-i\omega[t_1 - t_2 + t_3 - t_4]}$$

$$\cancel{\ldots} \Big) |0\rangle$$

$$= \frac{\hbar^2}{4m^2\omega^2} \langle 0| \, a a^\dagger e^{-i\omega(t_1 - t_3)} e^{-i\omega(t_2 - t_4)} + a a^\dagger e^{-i\omega(t_1 - t_4)} e^{-i\omega(t_2 - t_3)}$$

$$+ a a^\dagger e^{-i\omega(t_1 - t_2)} e^{-i\omega(t_3 - t_4)} \quad |0\rangle$$

$$= \frac{\hbar^2}{4m^2\omega^2} e^{-i\omega(t_1 - t_3)} e^{-i\omega(t_2 - t_4)} + \frac{\hbar^2}{4m^2\omega^2} e^{-i\omega(t_1 - t_4)} e^{-i\omega(t_2 - t_3)}$$

$$+ \frac{\hbar^2}{4m^2\omega^2} e^{-i\omega(t_1 - t_2)} e^{-i\omega(t_3 - t_4)}$$

$$= \frac{\hbar^2}{m^2\omega^2} G(t_1-t_3)G(t_2-t_4) + \frac{\hbar^2}{m^2\omega^2} G(t_1-t_4)G(t_2-t_3)$$

$$+ \frac{\hbar^2}{m^2\omega^2} G(t_1-t_2)\, G(t_3-t_4) \qquad \blacksquare$$

(7.4) $\langle 0|0\rangle_f = \exp\left[\frac{-1}{4w} \int_{-\infty}^{+\infty} dt\,dt'\, f(t) f(t')\, e^{-iw|t-t'|} \right]$

$$\|\langle 0|0\rangle_f\|^2 = \exp\left[\frac{-1}{4w} \int_{-\infty}^{+\infty} dt\,dt'\, f(t) f(t') \left(e^{-iw|t-t'|} + e^{iw|t-t'|} \right) \right]$$

$$\equiv e^{-iw(t-t')} + e^{iw(t-t')}$$

$$= \exp\left[-\frac{1}{4w} \int_{-\infty}^{+\infty} dt\,dt'\, f(t) e^{-iwt} f(t') e^{iwt'} \right.$$

$$\left. - \frac{1}{4w} \int_{-\infty}^{+\infty} dt\,dt'\, f(t) e^{iwt} f(t') e^{-iwt'} \right]$$

$$= \exp\left[\frac{-1}{4w} \tilde{f}(w) \tilde{f}(-w) - \frac{1}{4w} \tilde{f}(w) \tilde{f}(-w) \right]$$

$$= \exp\left(-\frac{1}{2w} \tilde{f}(w) \tilde{f}(-w) \right) \quad ; \quad \tilde{f}(-w) = \tilde{f}(w)^*$$

$$\|\langle 0|0\rangle_f\|^2 = \exp\left(-\frac{1}{2w} \| \tilde{f}(w) \|^2 \right) \qquad \blacksquare$$

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