NOTAS DE TEORIA TÉRMICA DE CAMPOS

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Sumário

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1. Introdução

Vamos primeiramente motivar o tratamento via integrais de caminho, para isto, vamos tomar o resultado conhecido de,

$$\langle q''|e^{-\frac{i}{\hbar}\left(t''-t'\right)\hat{H}}|q'\rangle = \mathcal{N}\int\limits_{\substack{q\left(t'\right)=q' \\ q\left(t''\right)=q''}} \mathcal{D}q\mathcal{D}p\exp\left\{\frac{i}{\hbar}\int\limits_{t'}^{t''}\mathrm{d}t\,[p\dot{q}-H]\right\} \tag{1.1}$$

Assim, note que podemos calcular,

$$\begin{aligned}
\operatorname{Tr} e^{-\beta\hat{H}} &= \mathcal{N}\int \mathrm{d}q'\ \langle q'|e^{-\frac{i}{\hbar}\left(-\frac{i}{2}\beta\hbar-\frac{i}{2}\beta\hbar\right)\hat{H}}|q'\rangle & (1.2)\\
&= \mathcal{N}\int\limits_{q\left(i\frac{1}{2}\beta\right)=q\left(-i\frac{1}{2}\beta\right)} \mathcal{D}q\mathcal{D}p\exp\left\{\frac{i}{\hbar}\int\limits_{\frac{i}{2}\beta\hbar}^{-\frac{i}{2}\beta\hbar}[p\dot{q}-H]\right\} & (1.3)
\end{aligned}$$

Fazendo $t=-i\tau$,

$$\begin{aligned}
\operatorname{Tr} e^{-\beta\hat{H}} &= \mathcal{N}\int \mathcal{D}q\mathcal{D}p\exp\left\{\frac{i}{\hbar}(-i)\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\,[ip\dot{q}-H]\right\} & (1.4)\\
&= \mathcal{N}\int \mathcal{D}q\mathcal{D}p\exp\left\{\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\,[ip\dot{q}-H]\right\} & (1.5)
\end{aligned}$$

Vamos tomar o Hamiltoniano como,

$$H = \frac{p^2}{2m}+V(q)-E_0 \tag{1.6}$$

Assim,

$$\begin{aligned}
\operatorname{Tr} e^{-\beta\hat{H}} &= \mathcal{N}\int \mathcal{D}q\mathcal{D}p\exp\left\{\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\left[ip\dot{q}-\frac{p^2}{2m}-V(q)+E_0\right]\right\} & (1.7)\\
&= \mathcal{N}\int \mathcal{D}q\mathcal{D}p\exp\left\{\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\beta\hbar}\mathrm{d}\tau\left[-\frac{1}{2m}\Big(p^2-2mip\dot{q}+(mi\dot{q})^2-(mi\dot{q})^2\Big)-V(q)+E_0\right]\right\} & (1.8)\\
&= \mathcal{N}\int \mathcal{D}q\mathcal{D}p\exp\left\{\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\left[-\frac{1}{2m}(p-mi\dot{q})^2-\frac{m}{2}\dot{q}^2-V(q)+E_0\right]\right\} & (1.9)\\
&= \mathcal{N}\int \mathcal{D}q\mathcal{D}p\exp\left\{-\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\,\mathcal{E}\right\}\exp\left\{-\frac{1}{2m\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\,p^2\right\} & (1.10)\\
&= \mathcal{N}\int \mathcal{D}q\exp\left\{-\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\,\mathcal{E}\right\} & (1.11)
\end{aligned}$$

Vamos especificar agora $V=\frac{m\omega^2}{2}q^2$, logo,

$$\mathrm{Tr}\, e^{-\beta\hat{H}} = \mathcal{N}\int \mathcal{D}q \exp\left\{-\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar} \mathrm{d}\tau \left[\frac{m}{2}\dot{q}^2 + \frac{m\omega^2}{2}q^2 - \frac{\hbar\omega}{2}\right]\right\} \tag{1.12}$$

$$= \mathcal{N} e^{\beta\frac{\hbar\omega}{2}}\int \mathcal{D}q \exp\left\{-\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar} \mathrm{d}\tau \left[\frac{m}{2}\dot{q}^2 + \frac{m\omega^2}{2}q^2\right]\right\} \tag{1.13}$$

$$= \mathcal{N} e^{\beta\frac{\hbar\omega}{2}}\int \mathcal{D}q \exp\left\{-\frac{1}{\hbar}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar} \mathrm{d}\tau \left[\frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(q\dot{q}) - \frac{m}{2}q\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}q + \frac{m\omega^2}{2}q^2\right]\right\} \tag{1.14}$$

$$= \mathcal{N} e^{\beta\frac{\hbar\omega}{2}}\int \mathcal{D}q \exp\left\{-\frac{1}{2}\int\limits_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar} \mathrm{d}\tau\, q\left[-\frac{m}{\hbar}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \frac{m\omega^2}{\hbar}\right]q\right\} \tag{1.15}$$

Precisamos então resolver um problema de auto-valores,

$$\left[-\frac{m}{\hbar}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \frac{m\omega^2}{\hbar}\right]q = \lambda q,\ q\left(-\frac{1}{2}\beta\hbar\right) = q\left(\frac{1}{2}\beta\hbar\right) \tag{1.16}$$

Fazemos como tentativa,

$$q(\tau) = e^{i\tau\kappa}a + e^{-i\tau\kappa}a^* \tag{1.17}$$

Aplicando as condições de contorno,

$$e^{-\frac{1}{2}i\beta\hbar\kappa}a + e^{\frac{1}{2}i\beta\hbar\kappa}a^* = e^{\frac{1}{2}i\beta\hbar\kappa}a + e^{-\frac{1}{2}i\beta\hbar\kappa}a^* \tag{1.18}$$

$$\left(e^{\frac{1}{2}i\beta\hbar\kappa} - e^{-\frac{1}{2}i\beta\hbar\kappa}\right)a = \left(e^{\frac{1}{2}i\beta\hbar\kappa} - e^{-\frac{1}{2}i\beta\hbar\kappa}\right)a^* \tag{1.19}$$

$$\sin\left(\frac{1}{2}\beta\hbar\kappa\right) = 0,\ \kappa_n = \frac{2\pi n}{\beta\hbar},\ n \in \mathbb{Z} \tag{1.20}$$

Logo os auto-valores são,

$$\left[-\frac{m}{\hbar}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + \frac{m\omega^2}{\hbar}\right]q = \lambda q \tag{1.21}$$

$$\left[\frac{m4\pi^2n^2}{\beta^2\hbar^3} + \frac{m\omega^2}{\hbar}\right]q = \lambda q \tag{1.22}$$

$$\lambda_n = \frac{m4\pi^2n^2}{\beta^2\hbar^3} + \frac{m\omega^2}{\hbar},\ n \in \mathbb{Z} \tag{1.23}$$

O determinante pode ser calculado como,

$$\mathrm{Det}[D] = \prod_{n=1}^{\infty}\lambda_n \tag{1.24}$$

$$= \exp\left\{\ln\left[\prod_{n=1}^{\infty}\lambda_n\right]\right\} \tag{1.25}$$

$$= \exp\left\{\sum_{n=1}^{\infty}\ln\left[\lambda_n\right]\right\} \tag{1.26}$$

$$= \exp\left\{\sum_{n=1}^{\infty}\frac{\ln\left[\lambda_n\right]}{\lambda_n^s}\bigg|_{s=0}\right\} \tag{1.27}$$

$$= \exp\left\{-\frac{\mathrm{d}}{\mathrm{d}s}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^s}\bigg|_{s=0}\right\} \tag{1.28}$$

Definimos então a função *zeta espectral* como,

$$\zeta_D(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \tag{1.29}$$

De forma que,

$$\mathrm{Det}[D] = \exp\left\{ -\frac{\mathrm{d}}{\mathrm{d}s}\zeta_D \Big|_{s=0} \right\} \tag{1.30}$$

Para o nosso caso,

$$\begin{aligned}
\mathrm{Det}[D] &= \prod_{n=1}^{\infty} \left(\frac{\pi^2 m}{\beta^2 \hbar^3} n^2 + \frac{m\omega^2}{\hbar} \right) && (1.31)\\
&= \prod_{n=1}^{\infty} \left(\frac{\pi^2 m}{\beta^2 \hbar^3} n^2 \right) \prod_{j=1}^{\infty} \left(1 + \frac{\omega^2 \hbar^2 \beta^2}{\pi^2 j^2} \right) && (1.32)\\
&= \prod_{n=1}^{\infty} \left(\frac{\pi^2 m}{\beta^2 \hbar^3} n^2 \right) \frac{\pi}{\beta\hbar\omega} \sinh\left(\beta\hbar\omega\right) && (1.33)\\
&= \frac{\pi}{\beta\hbar\omega} \sinh\left(\beta\hbar\omega\right) \exp\left\{ -\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{D'} \Big|_{s=0} \right\} && (1.34)
\end{aligned}$$

Onde,

$$\begin{aligned}
\zeta_{D'}(s) &= \sum_{n=1}^{\infty} \frac{1}{\left(\frac{\pi^2 m}{\beta^2 \hbar^3} n^2 \right)^s} && (1.35)\\
&= \left(\frac{\beta^2 \hbar^3}{\pi^2 m} \right)^s \zeta(2s) && (1.36)
\end{aligned}$$

Sabendo que $\frac{\mathrm{d}}{\mathrm{d}s}\zeta \Big|_{s=0} = -\frac{1}{2}\ln\left(2\pi\right)$ e $\zeta(0) = -\frac{1}{2}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{D'} \Big|_{s=0} &= \ln\left(\frac{\beta^2 \hbar^3}{\pi^2 m} \right) \left(\frac{\beta^2 \hbar^3}{\pi^2 m} \right)^s \zeta(2s) \Big|_{s=0} + \left(\frac{\beta^2 \hbar^3}{\pi^2 m} \right)^s 2 \frac{\mathrm{d}}{\mathrm{d}s}\zeta \Big|_{s=0} && (1.37)\\
&= -\frac{1}{2} \ln\left(\frac{\beta^2 \hbar^3}{\pi^2 m} \right) - \ln\left(2\pi\right) && (1.38)\\
&= \ln\left(\frac{\pi m^{\frac{1}{2}}}{\beta \hbar^{\frac{3}{2}}} \right) + \ln\left(\frac{1}{2\pi} \right) && (1.39)\\
&= \ln\left(\frac{m^{\frac{1}{2}}}{2\beta \hbar^{\frac{3}{2}}} \right) && (1.40)
\end{aligned}$$

Logo,

$$\begin{aligned}
\mathrm{Det}[D] &= \frac{\pi}{\beta\hbar\omega} \sinh\left(\beta\hbar\omega\right) \exp\left\{ \ln\left(\frac{2\beta\hbar^{\frac{3}{2}}}{m^{\frac{1}{2}}} \right) \right\} && (1.41)\\
&= \frac{\pi}{\beta\hbar\omega} \sinh\left(\beta\hbar\omega\right) \frac{2\beta\hbar^{\frac{3}{2}}}{m^{\frac{1}{2}}} && (1.42)\\
&= \frac{2\pi\hbar^{\frac{1}{2}}}{\omega m^{\frac{1}{2}}} \sinh\left(\beta\hbar\omega\right) && (1.43)
\end{aligned}$$

Finalmente,

$$\mathrm{Tr}\, e^{-\beta \hat{H}} \tag{1.44}$$

$$= \mathcal{N}\left(\frac{m\omega}{\pi\hbar}\tanh\left(\frac{1}{2}\beta\hbar\omega\right)\right)^{-\frac{1}{2}} e^{\beta\frac{\hbar\omega}{2}} \left(\mathrm{Det}[D]\right)^{-\frac{1}{2}} \tag{1.45}$$

$$= \mathcal{N}\left(\frac{m\omega}{\pi\hbar}\tanh\left(\frac{1}{2}\beta\hbar\omega\right)\right)^{-\frac{1}{2}} e^{\beta\frac{\hbar\omega}{2}} \left(\frac{2\pi\hbar^{\frac{1}{2}}}{\omega m^{\frac{1}{2}}}\sinh\left(\beta\hbar\omega\right)\right)^{-\frac{1}{2}} \tag{1.46}$$

$$= \mathcal{N}\left(\frac{4m^{\frac{1}{2}}}{\hbar^{\frac{1}{2}}}\sinh^2\left(\frac{1}{2}\beta\hbar\omega\right)\right)^{-\frac{1}{2}} e^{\beta\frac{\hbar\omega}{2}} \tag{1.47}$$

$$= \mathcal{N}\frac{\hbar^{\frac{1}{4}}}{2m^{\frac{1}{4}}}\frac{e^{\beta\frac{\hbar\omega}{2}}}{\sinh\left(\frac{1}{2}\beta\hbar\omega\right)} \tag{1.48}$$

$$= \mathcal{N}\frac{1}{1-e^{-\beta\hbar\omega}} \tag{1.49}$$

2. Campo Escalar Real

Vamos agora generalizar o cálculo para um campo escalar real. A motivação é calcular propriedades termodinâmicas de um grande número de partículas descritas por um campo escalar real. A função de partição é,

$$\operatorname{Tr} e^{-\beta\hat{H}} = \mathcal{N}\int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \mathcal{E}_0\right]\right\} \tag{2.1}$$

$$= \mathcal{N}\int \mathcal{D}\phi \exp\left\{i\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 + \mathcal{E}_0\right]\right\} \tag{2.2}$$

Fazendo $t = -i\tau$,

$$\operatorname{Tr} e^{-\beta\hat{H}} = \mathcal{N}\int \mathcal{D}\phi \exp\left\{i(-i)\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\left[-\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 + \mathcal{E}_0\right]\right\} \tag{2.3}$$

$$= \mathcal{N}\int \mathcal{D}\phi \exp\left\{-\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 - \mathcal{E}_0\right]\right\} \tag{2.4}$$

$$= \mathcal{N}e^{V\beta\mathcal{E}_0}\int \mathcal{D}\phi \exp\left\{-\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right]\right\} \tag{2.5}$$

$$= \mathcal{N}e^{V\beta\mathcal{E}_0}\int \mathcal{D}\phi \exp\left\{-\frac{1}{2}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\,\phi\left[-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right]\phi\right\} \tag{2.6}$$

Para realizar o cálculo da integral funcional vamos calcular o determinante do operador,

$$\left[-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right]\varphi = \lambda\varphi,\ \varphi\left(-\frac{1}{2}\beta,\mathbf{x}\right) = \varphi\left(\frac{1}{2}\beta,\mathbf{x}\right) \tag{2.7}$$

As auto-funções são,

$$e^{i\omega\tau + i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k}) \tag{2.8}$$

Então, as auto-funções reais são,

$$\varphi_{\omega,\mathbf{k}}(\tau,\mathbf{k}) = e^{i\omega\tau + i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k}) + e^{-i\omega\tau - i\mathbf{k}\cdot\mathbf{x}}a^*(\omega,\mathbf{k}) \tag{2.9}$$

$$\left[-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right]\varphi_{\omega,\mathbf{k}} = \left[\omega^2 + \mathbf{k}^2 + m^2\right]\varphi_{\omega,\mathbf{k}} \tag{2.10}$$

Aplicando as condições de contorno,

$$\varphi_{\omega,\mathbf{k}}\left(-\frac{1}{2}\beta,\mathbf{k}\right) = \varphi_{\omega,\mathbf{k}}\left(\frac{1}{2}\beta,\mathbf{k}\right) \tag{2.11}$$

$$e^{-i\frac{1}{2}\beta\tau + i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k}) + e^{i\frac{1}{2}\beta\tau - i\mathbf{k}\cdot\mathbf{x}}a^*(\omega,\mathbf{k}) = e^{i\frac{1}{2}\beta\tau + i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k}) + e^{-i\frac{1}{2}\beta\tau - i\mathbf{k}\cdot\mathbf{x}}a^*(\omega,\mathbf{k}) \tag{2.12}$$

$$e^{-i\mathbf{k}\cdot\mathbf{x}}a^*(\omega,\mathbf{k})\left[e^{i\omega\frac{1}{2}\beta} - e^{-i\omega\frac{1}{2}\beta}\right] = e^{i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k})\left[e^{i\omega\frac{1}{2}\beta} - e^{-i\omega\frac{1}{2}\beta}\right] \tag{2.13}$$

$$\sin\left(\frac{1}{2}\beta\omega\right) = 0 \Rightarrow\ \omega_n = \frac{2\pi n}{\beta},\ n\in\mathbb{Z} \tag{2.14}$$

Isto é, as auto-funções com as condições de contorno são,

$$\varphi_{n,\mathbf{k}}(\tau,\mathbf{k}) = e^{i\omega_n\tau + i\mathbf{k}\cdot\mathbf{x}}a_n(\mathbf{k}) + e^{-i\omega_n\tau - i\mathbf{k}\cdot\mathbf{x}}a_n^*(\mathbf{k}),\ \omega_n = \frac{2\pi n}{\beta},\ n\in\mathbb{Z} \tag{2.15}$$

Que implica nos auto-valores serem,

$$\lambda_n(\mathbf{k}) = \omega_n^2 + \omega_{\mathbf{k}}^2,\ n \in \mathbb{Z},\ \mathbf{k} \in \mathbb{R}^3 \tag{2.16}$$

Então,

$$\mathrm{Det}\left[-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right] = \prod_{\mathbf{k}\in\mathbb{R}^3}\left\{\prod_{n\in\mathbb{Z}}\left(\frac{4\pi^2 n^2}{\beta^2} + \omega_{\mathbf{k}}^2\right)\right\} \tag{2.17}$$

$$= \prod_{\mathbf{k}\in\mathbb{R}^3}\left\{\omega_{\mathbf{k}}^2 \prod_{n\in\mathbb{Z}^*}\left(\frac{4\pi^2 n^2}{\beta^2} + \omega_{\mathbf{k}}^2\right)\right\} \tag{2.18}$$

$$= \prod_{\mathbf{k}\in\mathbb{R}^3}\left\{\left[\omega_{\mathbf{k}} \prod_{n=1}^{\infty}\left(\frac{4\pi^2 n^2}{\beta^2} + \omega_{\mathbf{k}}^2\right)\right]^2\right\} \tag{2.19}$$

$$= \prod_{\mathbf{k}\in\mathbb{R}^3}\left\{\left[\omega_{\mathbf{k}}\left(\prod_{n=1}^{\infty}\frac{4\pi^2 n^2}{\beta^2}\right)\left(\prod_{p=1}^{\infty}\left(1 + \frac{\beta^2\omega_{\mathbf{k}}^2}{4\pi^2 p^2}\right)\right)\right]^2\right\} \tag{2.20}$$

$$= \prod_{\mathbf{k}}\left\{\left[\omega_{\mathbf{k}}\beta\frac{2\pi}{\beta\omega_{\mathbf{k}}}\sinh\left(\frac{1}{2}\beta\omega_{\mathbf{k}}\right)\right]^2\right\} \tag{2.21}$$

$$= \exp\left\{2V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[2\pi\sinh\left(\frac{1}{2}\beta\omega_{\mathbf{k}}\right)\right]\right\} \tag{2.22}$$

Finalmente então,

$$\mathrm{Tr}\, e^{-\beta\hat{H}} \tag{2.23}$$

$$= \mathcal{N} e^{V\beta\mathcal{E}_0}\exp\left\{-\frac{V}{(2\pi)^3}\int\mathrm{d}^3\mathbf{k}\ln\left[\sinh\left(\frac{1}{2}\beta\omega_{\mathbf{k}}\right)\right]\right\} \tag{2.24}$$

$$= \mathcal{N} e^{V\beta\mathcal{E}_0}\exp\left\{-V\beta\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\omega_{\mathbf{k}}}{2}\right\}\exp\left\{-4\pi\frac{V}{(2\pi)^3}\int_0^{\infty}\mathrm{d}k\, k^2\ln\left[1 - e^{-\beta\omega_{\mathbf{k}}}\right]\right\} \tag{2.25}$$

Onde podemos fixar, $\mathcal{E}_0 = \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\omega}{2}$, dessa forma,

$$\mathrm{Tr}\, e^{-\beta\hat{H}} = \mathcal{N}\exp\left\{-4\pi\frac{V}{(2\pi)^3}\int_0^{\infty}\mathrm{d}k\, k^2\ln\left[1 - e^{-\beta\omega_{\mathbf{k}}}\right]\right\} \tag{2.26}$$

$$F = -\frac{1}{\beta}\ln\left[\mathrm{Tr}\, e^{-\beta\hat{H}}\right] = \frac{4\pi}{\beta}\frac{V}{(2\pi)^3}\int_0^{\infty}\mathrm{d}k\, k^2\ln\left[1 - e^{-\beta\omega_{\mathbf{k}}}\right] \tag{2.27}$$

$$F = \frac{V}{2\pi^2\beta}\int_0^{\infty}\mathrm{d}k\, k^2\ln\left[1 - \exp\left\{-\beta\sqrt{k^2 + m^2}\right\}\right] \tag{2.28}$$

$$F = \frac{V}{2\pi^2\hbar^3\beta}\int_0^{\infty}\mathrm{d}k\, k^2\ln\left[1 - \exp\left\{-\beta\sqrt{c^2k^2 + c^4m^2}\right\}\right] \tag{2.29}$$

Tudo dito aqui é suficiente para analisar as propriedades termostáticas de um campo em equilíbrio térmico, porém, e caso queiramos saber o valor esperado de algum observável no equilíbrio térmico do campo? Para observáveis $\mathcal{O}_1, \mathcal{O}_2, \cdots$,

$$\langle\mathcal{O}_1\mathcal{O}_2\cdots\rangle_\beta = \frac{1}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]}\mathrm{Tr}\left[e^{-\beta\hat{H}}\mathrm{T}\{\mathcal{O}_1\mathcal{O}_2\cdots\}\right] \tag{2.30}$$

Claramente,

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\mathrm{T}\{\phi_1\phi_2\cdots\}\right] = \mathcal{N}\int\mathcal{D}\phi\ \phi_1\phi_2\cdots\exp\left\{i\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta}\mathrm{d}^4x\,\mathcal{L}\right\} \tag{2.31}$$

Para calcular esta quantidade fazemos o procedimento padrão de adicionar uma corrente na lagrangiana,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \mathcal{E}_0 + J\phi \tag{2.32}$$

Assim,

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\mathrm{T}\{\phi_1\phi_2\cdots\}\right] = \frac{1}{i}\frac{\delta}{\delta J_1}\frac{1}{i}\frac{\delta}{\delta J_2}\cdots\mathrm{Tr}\left[e^{-\beta\hat{H}(J)}\right]\Bigg|_{J=0} \tag{2.33}$$

Para isso então, calculemos a quantidade,

(2.34)
$$\mathrm{Tr}\left[e^{-\beta\hat{H}(J)}\right] = \mathcal{N}\int\mathcal{D}\phi\exp\left\{i\int\limits_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta}\mathrm{d}t\,\mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t} - \frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi - \frac{1}{2}m^2\phi^2 + \mathcal{E}_0 + J\phi\right]\right\}$$

(2.35)
$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\phi\exp\left\{\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[-\frac{1}{2}\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial\tau} - \frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi - \frac{1}{2}m^2\phi^2 + J\phi\right]\right\}$$

(2.36)
$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\phi\exp\left\{-\frac{1}{2}\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[\phi\left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right)\phi - J\phi - \phi J\right]\right\}$$

(2.37)
$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\phi\exp\left\{-\frac{1}{2}\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4x\,\mathrm{d}^4y\left[\phi(y)\left(-\frac{\partial^2}{\partial\tau_y^2} - \nabla_y^2 + m^2\right)\delta^{(4)}(y-x)\phi(x) - J(y)\delta^{(4)}(y-x)\phi(x) - \phi(y)\delta^{(4)}(y-x)J(x)\right]\right\}$$

(2.38)
$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\phi\exp\left\{-\frac{1}{2}\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4x\,\mathrm{d}^4y\left[\phi(y)\Delta^{-1}(y,x)\phi(x) - \int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4z\,J(z)\Delta(z,y)\Delta^{-1}(y,x)\phi(x) - \phi(y)\Delta^{-1}(y,x)\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4z\,\Delta(x,z)J(z)\right]\right\}$$

(2.39)
$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\phi\exp\left\{-\frac{1}{2}\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4x\,\mathrm{d}^4y\left[\left(\phi(y) - \int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4z\,J(z)\Delta(z,y)\right)\Delta^{-1}(y,x)\left(\phi(x) - \int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4z\,\Delta(x,z)J(z)\right)\right]\right\}$$
$$\times\exp\left\{\frac{1}{2}\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4x\,\mathrm{d}^4y\,J(y)\Delta(y,x)J(x)\right\}$$

(2.40)
$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\phi\exp\left\{-\frac{1}{2}\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4x\,\mathrm{d}^4y\,\phi(y)\Delta^{-1}(y,x)\phi(x)\right\}\exp\left\{\frac{1}{2}\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}^4x\,\mathrm{d}^4y\,J(y)\Delta(y,x)J(x)\right\}$$

Isto é,

$$Z_0[\beta;J] = Z_0[\beta]\exp\left\{\frac{1}{2}\int\limits_\beta\mathrm{d}^4x\,\mathrm{d}^4y\,J(y)\Delta(y,x)J(x)\right\} \tag{2.41}$$

$$Z_0[\beta;J] = Z_0[\beta]\exp\left\{-\frac{1}{2}\int\mathrm{d}^4x\,\mathrm{d}^4y\,J(y)\Delta(y,x)J(x)\right\} \tag{2.42}$$

Para calcularmos $\Delta(y,x)$ é necessário inverter o operador $\Delta^{-1}(y,x)$, note que,

$$\Delta^{-1}(y,x) = \left[-\frac{\partial^2}{\partial \tau_y^2} - \nabla_y^2 + m^2\right]\delta^{(4)}(y-x) \tag{2.43}$$

$$= \sum_{n\in\mathbb{Z}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}\left[-\frac{\partial^2}{\partial \tau_y^2} - \nabla_y^2 + m^2\right] e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} e^{i\frac{2\pi n}{\beta}(\tau_y-\tau_x)} \tag{2.44}$$

$$= \sum_{n\in\mathbb{Z}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta} e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} e^{i\frac{2\pi n}{\beta}(\tau_y-\tau_x)} \left[\omega_n^2 + \omega_{\mathbf{k}}^2\right] \tag{2.45}$$

Logo, seu inverso é,

$$\Delta(y,x) = \sum_{n\in\mathbb{Z}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta} e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} e^{i\omega_n(\tau_y-\tau_x)} \frac{1}{\omega_n^2+\omega_{\mathbf{k}}^2},\ \omega_n = \frac{2\pi n}{\beta} \tag{2.46}$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} \left(e^{i\omega_{\mathbf{k}}|t_y-t_x|} + \frac{e^{i\omega_{\mathbf{k}}(t_y-t_x)} + e^{-i\omega_{\mathbf{k}}(t_y-t_x)}}{e^{\beta\omega_{\mathbf{k}}}-1}\right) \tag{2.47}$$

$$= \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{e^{ik\cdot(y-x)}}{k^2+m^2} + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} \frac{\cos(\omega_{\mathbf{k}}(t_y-t_x))}{e^{\beta\omega_{\mathbf{k}}}-1} \tag{2.48}$$

Suponha que desejamos analisar uma teoria com interação, supomos,

$$\mathcal{L} = \frac{1}{2}\phi(\partial_\mu\partial^\mu - m^2)\phi - \frac{\lambda}{4!}\phi^4 + \mathcal{E}_0 \tag{2.49}$$

Certamente a nova função de partição é,

$$Z[\beta;J] = \exp\left\{-\frac{\lambda}{4!}\int \mathrm{d}^4 z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right\} Z_0[\beta;J] \tag{2.50}$$

$$Z[\beta;J] = Z_0[\beta]\exp\left\{-\frac{\lambda}{4!}\int \mathrm{d}^4 z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right\} \exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} \tag{2.51}$$

Em primeira ordem,

$$Z[\beta;J] \tag{2.52}$$

$$= Z_0[\beta]\left(1 - \frac{\lambda}{4!}\int \mathrm{d}^4 z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right)\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} \tag{2.53}$$

$$= Z_0[\beta]\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} - Z_0[\beta]\frac{\lambda}{4!}\left(\frac{\delta}{\delta J_z}\right)^3\left[-\Delta_{za}J_a\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\}\right] \tag{2.54}$$

$$= Z_0[\beta]\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} - Z_0[\beta]\frac{\lambda}{4!}\left(\frac{\delta}{\delta J_z}\right)^2\left[\left(-\Delta_{zz} + (\Delta_{za}J_a)^2\right)\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\}\right] \tag{2.55}$$

$$= Z_0[\beta]\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} - Z_0[\beta]\frac{\lambda}{4!}\left(\frac{\delta}{\delta J_z}\right)\left[\left(3\Delta_{zz}\Delta_{za}J_a - (\Delta_{za}J_a)^3\right)\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\}\right] \tag{2.56}$$

$$= Z_0[\beta]\left[1 - \frac{\lambda}{4!}\left(3\Delta_{zz}^2 - 6\Delta_{zz}(\Delta_{za}J_a)^2 + (\Delta_{za}J_a)^4\right)\right]\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} \tag{2.57}$$

Isso já é suficiente para obter a primeira correção para a função de partição,

$$Z[\beta] = Z_0[\beta]\left[1 - \frac{\lambda}{8}\Delta^2(0)V\beta\right] \tag{2.58}$$

Mas claramente $\Delta(0)$ é divergente, isto é, precisamos adicionar contra-termos para renormalizar a teoria,

$$\mathcal{L} = \frac{1}{2}\phi(\partial_\mu\partial^\mu - m^2)\phi - \frac{\lambda}{4!}\phi^4 + \mathcal{E}_0 - \delta_{m^2}\frac{1}{2}\phi^2 + \delta_{\phi^2}\frac{1}{2}\phi\partial_\mu\partial^\mu\phi - \delta_\lambda\frac{1}{4!}\phi^4 \tag{2.59}$$

Assim a função de partição renormalizada é, esperando que δ_{ϕ^2} e δ_λ sejam de ordem superior em λ,

$$Z[\beta;J] \tag{2.60}$$

$$= Z_0[\beta]\left(1 - \frac{\lambda}{4!}\int \mathrm{d}^4z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 - \frac{\delta_{m^2}}{2}\int \mathrm{d}^4z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^2\right)\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} \tag{2.61}$$

$$= Z_0[\beta]\left(1 - \frac{\lambda}{4!}\int \mathrm{d}^4z \left(\frac{\delta}{\delta J(z)}\right)^4 + \frac{\delta_{m^2}}{2}\int \mathrm{d}^4z \left(\frac{\delta}{\delta J(z)}\right)^2\right)\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} \tag{2.62}$$

$$= Z_0[\beta]\left(1 - \frac{\lambda}{4!}\left(3\Delta_{zz}^2 - 6\Delta_{zz}(\Delta_{za}J_a)^2 + (\Delta_{za}J_a)^4\right) + \frac{\delta_{m^2}}{2}\left(-\Delta_{zz} + (\Delta_{za}J_a)^2\right)\right)\exp\left\{-\frac{1}{2}J_x\Delta_{xy}J_y\right\} \tag{2.63}$$

Isto é,

$$Z[\beta] = Z_0[\beta]\left(1 - \frac{\lambda}{8}\Delta_{zz}^2 - \frac{\delta_{m^2}}{2}\Delta_{zz}\right) \tag{2.64}$$

Para calcular a função de dois pontos,

$$\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}Z[\beta;J]\bigg|_{J=0} \tag{2.65}$$

$$= -Z_0[\beta]\left(-\Delta_{x_1x_2} - \frac{\lambda}{4!}\left(-3\Delta_{zz}^2\Delta_{x_1x_2} - 12\Delta_{zz}\Delta_{zx_1}\Delta_{zx_2}\right) + \frac{\delta_{m^2}}{2}\left(\Delta_{zz}\Delta_{x_1x_2} + 2\Delta_{zx_1}\Delta_{zx_2}\right)\right) \tag{2.66}$$

$$= Z_0[\beta]\left(\Delta_{x_1x_2} - \frac{\lambda}{8}\Delta_{zz}^2\Delta_{x_1x_2} - \frac{\lambda}{2}\Delta_{zz}\Delta_{zx_1}\Delta_{zx_2} - \frac{\delta_{m^2}}{2}\Delta_{zz}\Delta_{x_1x_2} - \delta_{m^2}\Delta_{zx_1}\Delta_{zx_2}\right) \tag{2.67}$$

$$= \Delta_{x_1x_2}Z_0[\beta]\left(1 - \frac{\lambda}{8}\Delta_{zz}^2 - \frac{\delta_{m^2}}{2}\Delta_{zz}\right) - \Delta_{zx_1}\Delta_{zx_2}Z_0[\beta]\left(\frac{\lambda}{2}\Delta_{zz} + \delta_{m^2}\right) \tag{2.68}$$

Então,

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{Z[\beta]}\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}Z[\beta;J]\bigg|_{J=0} \tag{2.69}$$

$$= \Delta_{x_1x_2} - \Delta_{zx_1}\Delta_{zx_2}\frac{Z_0[\beta]\left(\frac{\lambda}{2}\Delta_{zz} + \delta_{m^2}\right)}{Z_0[\beta]\left(1 - \frac{\lambda}{8}\Delta_{zz}^2 - \frac{\delta_{m^2}}{2}\Delta_{zz}\right)} \tag{2.70}$$

$$= \Delta_{x_1x_2} - \Delta_{zx_1}\Delta_{zx_2}\left(\frac{\lambda}{2}\Delta_{zz} + \delta_{m^2}\right) \tag{2.71}$$

$\Delta(0)$ é,

$$\Delta(0) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left(1 + \frac{2}{e^{\beta\omega_{\mathbf{k}}} - 1}\right) = \Delta^{T=0}(0) + \Delta^{T}(0) \tag{2.72}$$

Assim, a divergência é eliminada tomando-se,

$$\delta_{m^2} = -\frac{\lambda}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} = -\frac{\lambda}{2}\Delta^{T=0}(0) \tag{2.73}$$

Note que assim, a massa que entra no propagador exato será,

$$\langle\phi(x_1)\phi(x_2)\rangle = \Delta_{x_1x_2} - \Delta_{x_1z}\Delta_{zx_2}\left(\frac{\lambda}{2}\Delta_{zz} + \delta_{m^2}\right) \tag{2.74}$$

$$\sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)-\omega_n(t_1-t_2)}\frac{1}{\omega_n^2+\mathbf{k}^2+m_\beta^2} = \sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)-\omega_n(t_1-t_2)}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2} \tag{2.75}$$

$$-\int\mathrm{d}^4z\sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{z})-\omega_n(t_1-t_z)}\sum_{m\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3\beta}e^{i\mathbf{q}\cdot(\mathbf{z}-\mathbf{x}_2)-\omega_m(t_z-t_2)}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\frac{1}{\omega_m^2+\mathbf{q}^2+m^2} \tag{2.76}$$

$$\times\left(\frac{\lambda}{2}\sum_{l\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{q}'}{(2\pi)^3\beta}\frac{1}{\omega_l^2+\mathbf{q}'^2+m^2} + \delta_{m^2}\right) \tag{2.77}$$

$$\sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)-\omega_n(t_1-t_2)}\frac{1}{\omega_n^2+\mathbf{k}^2+m_\beta^2} = \sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)-\omega_n(t_1-t_2)}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2} \tag{2.78}$$

$$-\sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)-\omega_n(t_1-t_2)}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\lambda\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1} \tag{2.79}$$

$$\frac{1}{\omega_n^2+\mathbf{k}^2+m_\beta^2} = \frac{1}{\omega_n^2+\mathbf{k}^2+m^2} - \frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\lambda\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1} \tag{2.80}$$

$$\frac{1}{\omega_n^2+\mathbf{k}^2+m_\beta^2} = \frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\left(1 - \frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\lambda\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1}\right) \tag{2.81}$$

$$\frac{1}{\omega_n^2+\mathbf{k}^2+m_\beta^2} = \frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\left(1 + \frac{1}{\omega_n^2+\mathbf{k}^2+m^2}\lambda\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1}\right)^{-1} \tag{2.82}$$

$$\omega_n^2+\mathbf{k}^2+m_\beta^2 = \omega_n^2+\mathbf{k}^2+m^2+\lambda\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1} \tag{2.83}$$

Isto é, temos a correção térmica da massa,

$$m_\beta^2 = m^2 + \lambda\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}}\frac{1}{e^{\beta\omega_\mathbf{k}}-1} \tag{2.84}$$

O caso $m^2 = 0$ pode ser avaliado analiticamente, com,

$$m_\beta^2 = \frac{\lambda}{24\beta^2} \tag{2.85}$$

E assim a nova função de partição é,

$$Z[\beta] = Z_0[\beta]\left(1 - V\beta\frac{\lambda}{8}\Delta^2(0) - V\beta\frac{\delta_{m^2}}{2}\Delta(0)\right) \tag{2.86}$$

2.1. **Re-derivação da Função de Partição.** Vamos agora reobter o resultado da Função de Partição seguindo um argumento um pouco mais intuitivo e menos rigoroso, note que

$$Z(\beta) = \mathrm{Tr}\left[\exp\left(-\beta\hat{H}\right)\right] \tag{2.87}$$

Para nosso hamiltoniano, sabemos que o mesmo pode ser escrito como,

$$\hat{H} = V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\omega_\mathbf{k}\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) \tag{2.88}$$

$$= V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\omega_\mathbf{k}\hat{n}(\mathbf{k}) \tag{2.89}$$

Logo, o traço pode ser reescrito como,

$$
\begin{aligned}
Z(\beta) &= \mathrm{Tr}\left[\exp\left(-\beta\hat{H}\right)\right] && (2.90)\\
&= \sum_{\{n(\mathbf{k})\}} \exp\left(-\beta V \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\omega_{\mathbf{k}} n(\mathbf{k})\right) && (2.91)\\
&= \sum_{\{n(\mathbf{k})\}} \exp\left(-\beta \sum_{\mathbf{k}} \omega_{\mathbf{k}} n(\mathbf{k})\right) && (2.92)\\
&= \sum_{\{n(\mathbf{k})\}} \prod_{\mathbf{k}} \exp\left(-\beta\omega_{\mathbf{k}} n(\mathbf{k})\right) && (2.93)\\
&= \prod_{\mathbf{k}} \sum_{n=0}^{\infty} \exp\left(-\beta\omega_{\mathbf{k}} n\right) && (2.94)\\
&= \prod_{\mathbf{k}} \frac{1}{1-\exp\left(-\beta\omega_{\mathbf{k}}\right)} && (2.95)\\
&= \prod_{\mathbf{k}} \exp\left\{-\ln\left[1-e^{-\beta\omega_{\mathbf{k}}}\right]\right\} && (2.96)\\
&= \exp\left\{-\sum_{\mathbf{k}} \ln\left[1-e^{-\beta\omega_{\mathbf{k}}}\right]\right\} && (2.97)\\
&= \exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left[1-e^{-\beta\omega_{\mathbf{k}}}\right]\right\} && (2.98)
\end{aligned}
$$

3. Campo Escalar Complexo

Nossa lagrangiana é,

$$\mathcal{L} = -\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + \mathcal{E}_0 \tag{3.1}$$

A grande diferença agora é que temos uma carga conservada associada a transformação global $\phi \to e^{i\theta}\phi$, dada por

$$\mathcal{L}' = (\phi + i\theta\phi)^\dagger \partial_\mu \partial^\mu (\phi + i\theta\phi) - m^2 (\phi + i\theta\phi)^\dagger (\phi + i\theta\phi) + \mathcal{E}_0 \tag{3.2}$$

$$\mathcal{L}' = \phi^\dagger \partial_\mu \partial^\mu \phi - i\phi^\dagger \theta \partial_\mu \partial^\mu \phi + i\phi^\dagger \partial_\mu \partial^\mu i\theta\phi - m^2 \phi^\dagger \phi + im^2 \phi^\dagger \theta \phi - im^2 \phi^\dagger \theta \phi + \mathcal{E}_0 \tag{3.3}$$

$$\mathcal{L}' = \phi^\dagger \partial_\mu \partial^\mu \phi - m^2 \phi^\dagger \phi + \mathcal{E}_0 = \mathcal{L} \tag{3.4}$$

Logo, o Teorema de Noether garante,

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} i\phi - i\phi^\dagger \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger} \tag{3.5}$$

$$j^\mu = -i\partial^\mu \phi^\dagger \phi + i\phi^\dagger \partial^\mu \phi, \ \partial_\mu j^\mu = 0 \tag{3.6}$$

$$j^\mu = i\big(\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger\big) \tag{3.7}$$

Ou seja,

$$-Q = \int \mathrm{d}^3\mathbf{x}\, j^0 = i \int \mathrm{d}^3\mathbf{x} \left(\phi^\dagger \partial^0 \phi - \phi \partial^0 \phi^\dagger\right) \tag{3.8}$$

$$Q = i \int \mathrm{d}^3\mathbf{x} \left(\phi^\dagger \partial_0 \phi - \phi \partial_0 \phi^\dagger\right) \tag{3.9}$$

$$Q = i \int \mathrm{d}^3\mathbf{x} \left(\pi^\dagger \phi^\dagger - \phi \pi\right) \tag{3.10}$$

Com é claro o Hamiltoniano dado por,

$$H = \int \mathrm{d}^3\mathbf{x} \left(\pi^\dagger \pi + \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi + m^2 \phi^\dagger \phi\right) - V\mathcal{E}_0 \tag{3.11}$$

Isso quer dizer que temos que adicionar um potencial químico a esta quantidade conservada na função de partição, isto é, devemos avaliar,

$$\operatorname{Tr}\left[e^{-\beta\left(\hat{H} - \mu\hat{Q}\right)}\right] \tag{3.12}$$

$$= \mathcal{N} \int \mathcal{D}\big(\pi^\dagger, \pi\big)\mathcal{D}\big(\phi^\dagger, \phi\big) \exp\left\{ \int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\, \mathrm{d}^3\mathbf{x} \left[i\pi^\dagger \frac{\partial}{\partial\tau}\phi^\dagger + i\pi \frac{\partial}{\partial\tau}\phi - \mathcal{H} + i\mu\big(\pi^\dagger\phi^\dagger - \pi\phi\big) \right] \right\} \tag{3.13}$$

$$= \mathcal{N} e^{\beta V \mathcal{E}_0} \int \mathcal{D}\big(\pi^\dagger, \pi\big)\mathcal{D}\big(\phi^\dagger, \phi\big) \exp\left\{ \int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\, \mathrm{d}^3\mathbf{x} \left[i\pi^\dagger \frac{\partial}{\partial\tau}\phi^\dagger + i\pi \frac{\partial}{\partial\tau}\phi - \pi^\dagger\pi - \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi - m^2\phi^\dagger\phi + i\mu\big(\pi^\dagger\phi^\dagger - \pi\phi\big) \right] \right\} \tag{3.14}$$

$$= \mathcal{N} e^{\beta V \mathcal{E}_0} \int \mathcal{D}\big(\pi^\dagger, \pi\big)\mathcal{D}\big(\phi^\dagger, \phi\big) \exp\left\{ \int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\, \mathrm{d}^3\mathbf{x} \left[-\left(\pi^\dagger - i\frac{\partial}{\partial\tau}\phi + i\mu\phi\right)\left(\pi - i\frac{\partial}{\partial\tau}\phi^\dagger - i\mu\phi^\dagger\right) \right] \right\} \tag{3.15}$$

$$\times \exp\left\{ \int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\, \mathrm{d}^3\mathbf{x} \left[-\frac{\partial}{\partial\tau}\phi \frac{\partial}{\partial\tau}\phi^\dagger - \mu \frac{\partial}{\partial\tau}\phi\phi^\dagger + \mu\phi\frac{\partial}{\partial\tau}\phi^\dagger + \mu^2\phi^\dagger\phi - \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi - m^2\phi^\dagger\phi \right] \right\}$$

$$= \mathcal{N} e^{\beta V \mathcal{E}_0} \int \mathcal{D}\big(\phi^\dagger, \phi\big) \exp\left\{ -\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\, \mathrm{d}^3\mathbf{x} \left[\frac{\partial}{\partial\tau}\phi \frac{\partial}{\partial\tau}\phi^\dagger + \mu \frac{\partial}{\partial\tau}\phi\phi^\dagger - \mu\phi\frac{\partial}{\partial\tau}\phi^\dagger - \mu^2\phi^\dagger\phi + \boldsymbol{\nabla}\phi^\dagger \cdot \boldsymbol{\nabla}\phi + m^2\phi^\dagger\phi \right] \right\} \tag{3.16}$$

$$= \mathcal{N} e^{\beta V \mathcal{E}_0} \int \mathcal{D}\big(\phi^\dagger, \phi\big) \exp\left\{ -\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\, \mathrm{d}^3\mathbf{x}\, \phi^\dagger \left[-\frac{\partial^2}{\partial\tau^2} + 2\mu\frac{\partial}{\partial\tau} - \nabla^2 + m^2 - \mu^2 \right] \phi \right\} \tag{3.17}$$

Basta agora calcular os autovalores do operador diferencial,

$$\left[-\frac{\partial^2}{\partial\tau^2}+2\mu\frac{\partial}{\partial\tau}-\nabla^2+m^2-\mu^2\right]\phi=\lambda\phi,\ \phi\left(-\frac{1}{2}\beta,\mathbf{x}\right)=\phi\left(\frac{1}{2}\beta,\mathbf{x}\right) \tag{3.18}$$

Como tentativa de auto-função,

$$\varphi_{\omega,\mathbf{k}}(\tau,\mathbf{x})=e^{i\omega\tau+i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k}) \tag{3.19}$$

$$\left[-\frac{\partial^2}{\partial\tau^2}+2\mu\frac{\partial}{\partial\tau}-\nabla^2+m^2-\mu^2\right]\varphi_{\omega,\mathbf{k}}(\tau,\mathbf{x})=\left[\omega^2+2\mu i\omega-\mu^2+\omega_{\mathbf{k}}^2\right]\varphi_{\omega,\mathbf{k}}(\tau,\mathbf{x}) \tag{3.20}$$

Vamos então impor as condições de contorno,

$$\phi_{\omega,\mathbf{k}}=e^{i\omega\tau+i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k})+e^{-i\omega\tau-i\mathbf{k}\cdot\mathbf{x}}b^\dagger(\omega,\mathbf{k}),\ \phi_{\omega,\mathbf{k}}\left(-\frac{1}{2}\beta,\mathbf{x}\right)=\phi_{\omega,\mathbf{k}}\left(\frac{1}{2}\beta,\mathbf{x}\right) \tag{3.21}$$

$$e^{-\frac{1}{2}i\omega\beta+i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k})+e^{\frac{1}{2}i\omega\beta-i\mathbf{k}\cdot\mathbf{x}}b^\dagger(\omega,\mathbf{k})=e^{\frac{1}{2}i\omega\beta+i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k})+e^{-\frac{1}{2}i\omega\beta-i\mathbf{k}\cdot\mathbf{x}}b^\dagger(\omega,\mathbf{k}) \tag{3.22}$$

$$\left[e^{\frac{1}{2}i\omega\beta}-e^{-\frac{1}{2}i\omega\beta}\right]e^{-i\mathbf{k}\cdot\mathbf{x}}b^\dagger(\omega,\mathbf{k})=\left[e^{\frac{1}{2}i\omega\beta}-e^{-\frac{1}{2}i\omega\beta}\right]e^{i\mathbf{k}\cdot\mathbf{x}}a(\omega,\mathbf{k}) \tag{3.23}$$

$$\sin\left(\frac{1}{2}\beta\omega\right)=0\Rightarrow\omega_n=\frac{2\pi n}{\beta},\ n\in\mathbb{Z} \tag{3.24}$$

Isto é, as auto-funções com as condições de contorno são,

$$\phi_{n,\mathbf{k}}(\tau,\mathbf{k})=e^{i\omega_n\tau+i\mathbf{k}\cdot\mathbf{x}}a_n(\mathbf{k})+e^{-i\omega_n\tau-i\mathbf{k}\cdot\mathbf{x}}b_n^*(\mathbf{k}),\ \omega_n=\frac{2\pi n}{\beta},\ n\in\mathbb{Z} \tag{3.25}$$

Que implica nos auto-valores serem,

$$\lambda_n(\mathbf{k})=\omega_n^2+\omega_{\mathbf{k}}^2-\mu^2+2i\mu\omega_n,\ n\in\mathbb{Z},\ \mathbf{k}\in\mathbb{R}^3 \tag{3.26}$$

Então,

$$\operatorname{Det}\left[-\frac{\partial^2}{\partial\tau^2}+2\mu\frac{\partial}{\partial\tau}-\nabla^2+m^2-\mu^2\right]=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{\prod_{n\in\mathbb{Z}}\left(\frac{4\pi^2n^2}{\beta^2}+\omega_{\mathbf{k}}^2-\mu^2+2i\mu\frac{2\pi n}{\beta}\right)\right\} \tag{3.27}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{(\omega_{\mathbf{k}}^2-\mu^2)\prod_{n\in\mathbb{Z}^*}\left(\frac{4\pi^2n^2}{\beta^2}+\omega_{\mathbf{k}}^2-\mu^2+2i\mu\frac{2\pi n}{\beta}\right)\right\} \tag{3.28}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{(\omega_{\mathbf{k}}^2-\mu^2)\prod_{n=1}^{\infty}\left[\left(\frac{4\pi^2n^2}{\beta^2}+\omega_{\mathbf{k}}^2-\mu^2\right)^2+4\mu^2\frac{4\pi^2n^2}{\beta^2}\right]\right\} \tag{3.29}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{(\omega_{\mathbf{k}}^2-\mu^2)\prod_{n=1}^{\infty}\left[\frac{16\pi^4n^4}{\beta^4}+(\omega_{\mathbf{k}}^2+\mu^2)8\frac{\pi^2n^2}{\beta^2}+(\omega_{\mathbf{k}}^2-\mu^2)^2\right]\right\} \tag{3.30}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{(\omega_{\mathbf{k}}^2-\mu^2)\prod_{n=1}^{\infty}\left[\left(\frac{4\pi^2n^2}{\beta^2}+\omega_{\mathbf{k}}^2+\mu^2\right)^2-4\omega_{\mathbf{k}}^2\mu^2\right]\right\} \tag{3.31}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{(\omega_{\mathbf{k}}^2-\mu^2)\prod_{n=1}^{\infty}\left[\frac{4\pi^2n^2}{\beta^2}+(\omega_{\mathbf{k}}-\mu)^2\right]\prod_{p=1}^{\infty}\left[\frac{4\pi^2p^2}{\beta^2}+(\omega_{\mathbf{k}}+\mu)^2\right]\right\} \tag{3.32}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{(\omega_{\mathbf{k}}^2-\mu^2)\left(\prod_{m=1}^{\infty}\frac{4\pi^2m^2}{\beta^2}\right)^2\prod_{n=1}^{\infty}\left[1+\frac{\beta^2(\omega_{\mathbf{k}}-\mu)^2}{4\pi^2n^2}\right]\prod_{p=1}^{\infty}\left[1+\frac{\beta^2(\omega_{\mathbf{k}}+\mu)^2}{4\pi^2p^2}\right]\right\} \tag{3.33}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{(\omega_{\mathbf{k}}^2-\mu^2)\beta^2\frac{4\pi^2}{\beta^2(\omega_{\mathbf{k}}-\mu)(\omega_{\mathbf{k}}+\mu)}\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)\right)\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)\right)\right\} \tag{3.34}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left\{4\pi^2\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)\right)\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)\right)\right\} \tag{3.35}$$

$$=\exp\left\{V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[4\pi^2\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)\right)\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)\right)\right]\right\} \tag{3.36}$$

(3.37)

Então,

$$\operatorname{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right] \tag{3.38}$$

$$= \mathcal{N} e^{\beta V \mathcal{E}_0} \exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left[\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)\right)\sinh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)\right)\right]\right\} \tag{3.39}$$

Usamos agora o fato de que $\mathcal{E}_0 = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\omega_{\mathbf{k}}$

$$\operatorname{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right] \tag{3.40}$$

$$= \mathcal{N}\exp\left\{V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[e^{\beta\omega}\right]\right\}\exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left[\sinh\left(\frac{1}{2}\beta(\omega+\mu)\right)\sinh\left(\frac{1}{2}\beta(\omega-\mu)\right)\right]\right\} \tag{3.41}$$

$$= \mathcal{N}\exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[e^{-\frac{1}{2}\beta\omega}\sinh\left(\frac{1}{2}\beta(\omega+\mu)\right)\right]\right\}\exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[e^{-\frac{1}{2}\beta\omega}\sinh\left(\frac{1}{2}\beta(\omega-\mu)\right)\right]\right\} \tag{3.42}$$

$$= \mathcal{N}\exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[e^{\frac{1}{2}\beta\mu}-e^{-\beta\omega-\frac{1}{2}\beta\mu}\right]\right\}\exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[e^{-\frac{1}{2}\beta\mu}-e^{-\beta\omega+\frac{1}{2}\beta\mu}\right]\right\} \tag{3.43}$$

$$= \mathcal{N}\exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1-e^{-\beta(\omega+\mu)}\right]\right\}\exp\left\{-V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1-e^{-\beta(\omega-\mu)}\right]\right\} \tag{3.44}$$

Podemos então calcular a energia livre de Helmholtz,

$$F = -\frac{1}{\beta}\ln\left[\operatorname{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right]\right] \tag{3.45}$$

$$= \frac{1}{\beta}V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1-e^{-\beta(\omega+\mu)}\right] + \frac{1}{\beta}V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1-e^{-\beta(\omega-\mu)}\right] \tag{3.46}$$

Queremos agora olhar para os campos fermiônicos, dados pela lagrangiana,

$$\mathcal{L} = \bar{\Psi}\big(i\partial\!\!\!/ - m\big)\Psi + \mathcal{E}_0 \tag{4.1}$$

Note que esta possui uma simetria de $\Psi \to e^{i\theta}\Psi$, cuja está associada uma quantidade conservada,

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}i\Psi - i\bar{\Psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\Psi}} \tag{4.2}$$

$$j^\mu = i\bar{\Psi}\gamma^\mu i\Psi \tag{4.3}$$

$$j^\mu = -\bar{\Psi}\gamma^\mu\Psi \tag{4.4}$$

$$Q = \int \mathrm{d}^3\mathbf{x}\, j_0 = \int \mathrm{d}^3\mathbf{x}\,\Psi^\dagger\Psi \tag{4.5}$$

Portanto, a função de partição é dada por,

$$\mathrm{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right] = \mathcal{N}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left\{i\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta}\mathrm{d}t\,\mathrm{d}^3\mathbf{x}\left[i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi + \mu\bar{\Psi}\gamma^0\Psi + \mathcal{E}_0\right]\right\} \tag{4.6}$$

$$\mathrm{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right] = \mathcal{N}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left\{i\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta}\mathrm{d}t\,\mathrm{d}^3\mathbf{x}\left[i\bar{\Psi}\gamma^0\partial_0\Psi + i\bar{\Psi}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\Psi - m\bar{\Psi}\Psi + \mu\bar{\Psi}\gamma^0\Psi + \mathcal{E}_0\right]\right\} \tag{4.7}$$

$$\mathrm{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right] = \mathcal{N}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left\{i(-i)\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[-\bar{\Psi}\gamma^0\frac{\partial}{\partial\tau}\Psi + i\bar{\Psi}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\Psi - m\bar{\Psi}\Psi + \mu\bar{\Psi}\gamma^0\Psi + \mathcal{E}_0\right]\right\} \tag{4.8}$$

$$\mathrm{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right] = \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left\{-\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[\bar{\Psi}\gamma^0\frac{\partial}{\partial\tau}\Psi - i\bar{\Psi}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\Psi + m\bar{\Psi}\Psi - \mu\bar{\Psi}\gamma^0\Psi\right]\right\} \tag{4.9}$$

$$\mathrm{Tr}\left[e^{-\beta\left(\hat{H}-\mu\hat{Q}\right)}\right] = \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left\{-\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\,\bar{\Psi}\left[\gamma^0\frac{\partial}{\partial\tau} - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\mathbb{1} - \mu\gamma^0\right]\Psi\right\} \tag{4.10}$$

A equação de autovalores é,

$$\left[\gamma^0\frac{\partial}{\partial\tau} - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\mathbb{1} - \mu\gamma^0\right]\Psi = \lambda\Psi,\ \Psi\left(-\frac{1}{2}\beta,\mathbf{x}\right) = -\Psi\left(\frac{1}{2}\beta,\mathbf{x}\right) \tag{4.11}$$

$$\sum_{n=-\infty}^{+\infty}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}\left[\gamma^0\frac{\partial}{\partial\tau} - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\mathbb{1} - \mu\gamma^0\right]e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\frac{2\pi\tau}{\beta}\left(n+\frac{1}{2}\right)}\psi_n(\mathbf{k}) = \sum_{n=-\infty}^{+\infty}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\beta}e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\frac{2\pi\tau}{\beta}\left(n+\frac{1}{2}\right)}\lambda\psi_n(\mathbf{k}) \tag{4.12}$$

$$\left[\gamma^0 i\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right) + \boldsymbol{\gamma}\cdot\mathbf{k} + m\mathbb{1} - \gamma^0\mu\right]\psi_n(\mathbf{k}) = \lambda\psi_n(\mathbf{k}) \tag{4.13}$$

$$\left[\gamma^0\mathrm{i}\omega_n + \boldsymbol{\gamma}\cdot\mathbf{k} + m\mathbb{1} - \gamma^0\mu\right]\psi_n(\mathbf{k}) = \lambda\psi_n(\mathbf{k}) \tag{4.14}$$

$$\begin{pmatrix} m-\lambda+\mathrm{i}\omega_n-\mu & 0 & k_z & k_x-ik_y \\ 0 & m-\lambda+\mathrm{i}\omega_n-\mu & k_x+ik_y & -k_z \\ -k_z & -k_x+ik_y & m-\lambda+\mu-\mathrm{i}\omega_n & 0 \\ -k_x-ik_y & k_z & 0 & m-\lambda+\mu-\mathrm{i}\omega_n \end{pmatrix}\begin{pmatrix} a_n(\mathbf{k}) \\ b_n(\mathbf{k}) \\ c_n(\mathbf{k}) \\ d_n(\mathbf{k}) \end{pmatrix} = 0 \tag{4.15}$$

Que pode ser escrita como,

$$\begin{cases} a(m-\lambda+\mathrm{i}\omega_n-\mu)+k_zc+d(k_x-ik_y) & =0 \\ b(m-\lambda+\mathrm{i}\omega_n-\mu)+c(k_x+ik_y)-dk_z & =0 \\ -ak_z+b(-k_x+ik_y)+c(m-\lambda+\mu-\mathrm{i}\omega_n) & =0 \\ -a(k_x+ik_y)+bk_z+d(m-\lambda+\mu-\mathrm{i}\omega_n) & =0 \end{cases} \tag{4.16}$$

$$\begin{cases} a & =\frac{-k_zc-d(k_x-ik_y)}{\mathrm{i}\omega_n-\mu+m-\lambda} \\ b & =\frac{-c(k_x+ik_y)+dk_z}{\mathrm{i}\omega_n-\mu+m-\lambda} \end{cases} \tag{4.17}$$

$$\begin{cases} \frac{k_z^2c+k_zd(k_x-ik_y)}{\mathrm{i}\omega_n-\mu+m-\lambda}+(-k_x+ik_y)\frac{dk_z-c(k_x+ik_y)}{\mathrm{i}\omega_n-\mu+m-\lambda}+c(\mu-\mathrm{i}\omega_n+m-\lambda) & =0 \end{cases} \tag{4.18}$$

$$\begin{cases} c\mathbf{k}^2 & =c\Big((\mathrm{i}\omega_n-\mu)^2-(m-\lambda)^2\Big) \\ d\mathbf{k}^2 & =d\Big((\mathrm{i}\omega_n-\mu)^2-(m-\lambda)^2\Big) \end{cases} \tag{4.19}$$

Portanto temos que,

$$a=\frac{-k_zc-d(k_x-ik_y)}{\mathrm{i}\omega_n-\mu+m-\lambda} \tag{4.20}$$

$$b=\frac{-c(k_x+ik_y)+dk_z}{\mathrm{i}\omega_n-\mu+m-\lambda} \tag{4.21}$$

$$\lambda_n^{\pm}(\mathbf{k})=m\pm\sqrt{(\mathrm{i}\omega_n-\mu)^2-\mathbf{k}^2} \tag{4.22}$$

Assim o determinante que queremos calcular é,

$$\mathrm{Det}\left[\gamma^0\frac{\partial}{\partial\tau}-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m\mathbb{1}-\mu\gamma^0\right] \tag{4.23}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\prod_{n\in\mathbb{Z}}\left(m+\sqrt{(\mathrm{i}\omega_n-\mu)^2-\mathbf{k}^2}\right)\left(m-\sqrt{(\mathrm{i}\omega_n-\mu)^2-\mathbf{k}^2}\right) \tag{4.24}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left[\prod_{n\in\mathbb{Z}}\left(\omega_{\mathbf{k}}^2-(\mathrm{i}\omega_n-\mu)^2\right)\right] \tag{4.25}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left[\prod_{n\in\mathbb{Z}}(\omega_{\mathbf{k}}-\mathrm{i}\omega_n+\mu)(\omega_{\mathbf{k}}+\mathrm{i}\omega_n-\mu)\right] \tag{4.26}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left[\prod_{n=0}^{\infty}\left(\omega_n^2+(\omega_{\mathbf{k}}+\mu)^2\right)\left(\omega_n^2+(\omega_{\mathbf{k}}-\mu)^2\right)\right] \tag{4.27}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left[\prod_{n=0}^{\infty}\left(\frac{4\pi^2}{\beta^2}\left(n+\frac{1}{2}\right)^2\right)^2\prod_{p=0}^{\infty}\left(1+\frac{\beta^2(\omega_{\mathbf{k}}+\mu)^2}{4\pi^2\left(p+\frac{1}{2}\right)^2}\right)\prod_{q=0}^{\infty}\left(1+\frac{\beta^2(\omega_{\mathbf{k}}-\mu)^2}{4\pi^2\left(q+\frac{1}{2}\right)^2}\right)\right] \tag{4.28}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left[\prod_{n=0}^{\infty}\left(\frac{4\pi^2}{\beta^2}\left(n+\frac{1}{2}\right)^2\right)^2\cosh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)\right)\cosh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)\right)\right] \tag{4.29}$$

$$=\prod_{\mathbf{k}\in\mathbb{R}^3}\left[4\cosh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)\right)\cosh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)\right)\right] \tag{4.30}$$

Logo a função de partição é,

$$\mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \mathcal{N}e^{\beta V\mathcal{E}_0}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left\{-\int\limits_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta}\mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\,\bar{\Psi}\left[\gamma^0\frac{\partial}{\partial\tau} - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\mathbb{1} - \mu\gamma^0\right]\Psi\right\} \tag{4.31}$$

$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\exp\left\{V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[\cosh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)\right)\cosh\left(\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)\right)\right]\right\} \tag{4.32}$$

$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\exp\left\{V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[\left(e^{\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)} + e^{-\frac{1}{2}\beta(\omega_{\mathbf{k}}+\mu)}\right)\left(e^{\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)} + e^{-\frac{1}{2}\beta(\omega_{\mathbf{k}}-\mu)}\right)\right]\right\} \tag{4.33}$$

$$= \mathcal{N}e^{\beta V\mathcal{E}_0}\exp\left\{V\beta\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\omega_{\mathbf{k}}\right\}\exp\left\{V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1+e^{-\beta(\omega_{\mathbf{k}}+\mu)}\right]\right\}\exp\left\{V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1+e^{-\beta(\omega_{\mathbf{k}}-\mu)}\right]\right\} \tag{4.34}$$

Isto é,

$$Z(\beta) = \exp\left\{V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1+e^{-\beta(\omega_{\mathbf{k}}+\mu)}\right]\right\}\exp\left\{V\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1+e^{-\beta(\omega_{\mathbf{k}}-\mu)}\right]\right\} \tag{4.35}$$

$$F = -\frac{V}{\beta}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1+e^{-\beta(\omega_{\mathbf{k}}+\mu)}\right] - \frac{V}{\beta}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left[1+e^{-\beta(\omega_{\mathbf{k}}-\mu)}\right] \tag{4.36}$$

5. Campo de Calibre

O objetivo é refazer os cálculos anteriores para a teoria descrita pela Lagrangiana,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{E}_0 \tag{5.1}$$

Porém esta possui uma liberdade de calibre da forma,

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\theta(x) \tag{5.2}$$

Portanto não podemos apenas calcular $\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]$, pois estaremos sobre-contando estados e contando estados não físicos, para isso, é necessário fazer $\mathrm{Tr}\left[\hat{\mathbb{P}}e^{-\beta\hat{H}}\right]$, no qual $\hat{\mathbb{P}}$ é um projetor sobre estados físicos, isto é, estamos fazendo o Traço sobre todas as configurações de campo que são equivalentes por mudança de calibre. Podemos fazer isto integrando por todas as configurações inequivalentes e posteriormente integrando por todos os calibres, fazendo estes iguais a zero.

$$\mathrm{Tr}\left[\hat{\mathbb{P}}e^{-\beta\hat{H}}\right] = \int \mathcal{D}\theta \int \mathcal{D}A\delta(\theta(x)) \exp\left\{ i\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta} \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\right\} \tag{5.3}$$

Porém, podemos querer utilizar outra fixação de calibre que não $\theta = 0$, uma implícita, da forma $G(A) = g(A) - \alpha(x)$, podemos então novamente voltar a integrar sobre todas as configurações de campo com,

$$\mathrm{Tr}\left[\hat{\mathbb{P}}e^{-\beta\hat{H}}\right]_\alpha = \int \mathcal{D}A\delta(G(A))\mathrm{Det}\left[\frac{\delta G(A)}{\delta\theta}\right] \exp\left\{ i\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta} \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\right\} \tag{5.4}$$

O determinante pode ser escrito como uma integral funcional sobre campos fermiônicos ditos *fantasmas*,

$$\mathrm{Tr}\left[\hat{\mathbb{P}}e^{-\beta\hat{H}}\right]_\alpha = \int \mathcal{D}A \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\delta(G(A)) \exp\left\{ -\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\,\bar{\eta}\frac{\delta G(A)}{\delta\theta}\eta\right\} \exp\left\{ i\int_{i\frac{1}{2}\beta}^{-i\frac{1}{2}\beta} \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\right\} \tag{5.5}$$

Que ao fazer todas as manipulações necessárias pode ser escrita como,

$$\mathrm{Tr}\left[\hat{\mathbb{P}}e^{-\beta\hat{H}}\right] = \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}\bar{\eta} \exp\left\{ -\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + \bar{\eta}\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\eta\right]\right\} \tag{5.6}$$

Escolhemos o calibre de Feynman, $\xi = 1$,

$$\mathrm{Tr}\left[\hat{\mathbb{P}}e^{-\beta\hat{H}}\right] = \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}\bar{\eta} \exp\left\{ -\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[-\frac{1}{2}A_\mu\partial_\mu\partial_\mu A_\mu + \bar{\eta}\partial_\mu\partial_\mu\eta\right]\right\} \tag{5.7}$$

Como já calculamos o resultado de,

$$\int \mathcal{D}\phi \exp\left\{ -\frac{1}{2}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[-\phi\left(\partial_\mu\partial_\mu + m^2\right)\phi\right]\right\} = \exp\left\{ -V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left[\sinh\left(\frac{1}{2}\beta\sqrt{k^2+m^2}\right)\right]\right\} \tag{5.8}$$

Segue naturalmente que,

$$\int \mathcal{D}A \exp\left\{ -\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[-\frac{1}{2}A_\mu\partial_\mu\partial_\mu A_\mu\right]\right\} = \exp\left\{ -4V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left[\sinh\left(\frac{1}{2}\beta\|\mathbf{k}\|\right)\right]\right\} \tag{5.9}$$

$$\int \mathcal{D}\eta\mathcal{D}\bar{\eta} \exp\left\{ -\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \mathrm{d}\tau\,\mathrm{d}^3\mathbf{x}\left[\bar{\eta}\partial_\mu\partial_\mu\eta\right]\right\} = \exp\left\{ 2V\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left[\sinh\left(\frac{1}{2}\beta\|\mathbf{k}\|\right)\right]\right\} \tag{5.10}$$

Suponhamos que desejamos calcular o seguinte produtório,

$$\prod_{n=1}^{\infty} \lambda_n \tag{A.1}$$

Para isso reescrevemos,

$$\prod_{n=1}^{\infty} \lambda_n = \exp\left\{ \ln\left[\prod_{n=1}^{\infty} \lambda_n\right]\right\} \tag{A.2}$$

$$= \exp\left\{\sum_{n=1}^{\infty} \ln \lambda_n\right\} \tag{A.3}$$

$$= \exp\left\{\sum_{n=1}^{\infty} \frac{\ln \lambda_n}{\lambda_n^s}\bigg|_{s=0}\right\} \tag{A.4}$$

$$= \exp\left\{-\frac{\mathrm{d}}{\mathrm{d}s}\sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}\bigg|_{s=0}\right\} \tag{A.5}$$

Definimos então a função *zeta espectral* como,

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \tag{A.6}$$

De forma que,

$$\prod_{n=1}^{\infty} \lambda_n = \exp\left\{-\frac{\mathrm{d}}{\mathrm{d}s}\zeta_\lambda\bigg|_{s=0}\right\} \tag{A.7}$$

Para realizar esse cálculo é necessário relacionar $\zeta_\lambda(s)$ com a função zeta de Riemann,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{A.8}$$

O caso de principal interesse é $\lambda_n = \frac{4\pi^2 n^2}{a^2}$, assim,

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{\left(\frac{4\pi^2 n^2}{a^2}\right)^s} \tag{A.9}$$

$$= \left(\frac{a^2}{4\pi^2}\right)^s \zeta(2s) \tag{A.10}$$

Sabendo que $\frac{\mathrm{d}}{\mathrm{d}s}\zeta\big|_{s=0} = -\frac{1}{2}\ln(2\pi)$ e $\zeta(0) = -\frac{1}{2}$

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta_\lambda\bigg|_{s=0} = \ln\left(\frac{a^2}{4\pi^2}\right)\left(\frac{a^2}{4\pi^2}\right)^s \zeta(2s)\bigg|_{s=0} + \left(\frac{a^2}{4\pi^2}\right)^s 2\frac{\mathrm{d}}{\mathrm{d}s}\zeta\bigg|_{s=0} \tag{A.11}$$

$$= -\frac{1}{2}\ln\left(\frac{a^2}{4\pi^2}\right) - \ln(2\pi) \tag{A.12}$$

$$= -\ln\left(\frac{a}{2\pi}\right) - \ln(2\pi) \tag{A.13}$$

$$= -\ln a \tag{A.14}$$

(A.15)

Portanto,

$$\prod_{n=1}^{\infty} \frac{4\pi^2 n^2}{a^2} = \exp\{\ln a\} \tag{A.16}$$

$$\prod_{n=1}^{\infty} \frac{4\pi^2 n^2}{a^2} = a \tag{A.17}$$

Outro caso de interesse é da forma de,

$$\lambda_n = \frac{4\pi^2}{a^2}\left(n - \frac{1}{2}\right)^2 \tag{A.18}$$

Precisaremos agora utilizar a função zeta de Hurwitz,

$$\zeta(s,b) = \sum_{n=0}^{\infty} \frac{1}{(n+b)^s} \tag{A.19}$$

Temos que,

$$\begin{aligned}
\zeta_\lambda(s) &= \sum_{n=1}^{\infty} \frac{1}{\left(\frac{4\pi^2}{a^2}\left(n-\frac{1}{2}\right)^2\right)^s} && \text{(A.20)}\\
&= \sum_{n=0}^{\infty} \frac{1}{\left(\frac{4\pi^2}{a^2}\left(n+\frac{1}{2}\right)^2\right)^s} && \text{(A.21)}\\
&= \left(\frac{a^2}{4\pi^2}\right)^s \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^{2s}} && \text{(A.22)}\\
&= \left(\frac{a^2}{4\pi^2}\right)^s \zeta\left(2s, \frac{1}{2}\right) && \text{(A.23)}
\end{aligned}$$

Sabendo de, $\zeta\left(0, \frac{1}{2}\right) = 0$ e $\zeta'\left(0, \frac{1}{2}\right) = -\frac{1}{2}\ln 2$, temos que,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\zeta_\lambda(s)\bigg|_{s=0} &= \ln\left(\frac{a^2}{4\pi^2}\right)\left(\frac{a^2}{4\pi^2}\right)^s \zeta\left(2s, \frac{1}{2}\right)\bigg|_{s=0} + \left(\frac{a^2}{4\pi^2}\right)^s \frac{\mathrm{d}}{\mathrm{d}s}\zeta\left(2s, \frac{1}{2}\right)\bigg|_{s=0} && \text{(A.24)}\\
&= -\ln 2 && \text{(A.25)}
\end{aligned}$$

$$\prod_{n=1}^{\infty} \frac{4\pi^2}{a^2}\left(n - \frac{1}{2}\right)^2 = \exp\{\ln 2\} \tag{A.26}$$

$$\prod_{n=1}^{\infty} \frac{4\pi^2}{a^2}\left(n - \frac{1}{2}\right)^2 = 2 \tag{A.27}$$