Soluções para Yu-Tin Scattering Amplitudes

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Exercício 2.1

Seja o momento,

$$p^\mu = E\begin{pmatrix} 1 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

Sabemos que,

$$\sigma^\mu_{a\dot{b}} = \left(\mathbb{1}, \boldsymbol{\sigma}\right)$$

Assim,

$$\begin{aligned}
p_{a\dot{b}} &= p_\mu \sigma^\mu_{a\dot{b}} = -p^0\mathbb{1} + \mathbf{p}\cdot\boldsymbol{\sigma} \\
&= \begin{pmatrix} -p^0+p^3 & p^1-\mathrm{i}p^2 \\ p^1+\mathrm{i}p^2 & -p^0-p^3 \end{pmatrix} = -\begin{pmatrix} p^0-p^3 & -p^1+\mathrm{i}p^2 \\ -p^1-\mathrm{i}p^2 & p^0+p^3 \end{pmatrix} \\
&= -\begin{pmatrix} \pm\sqrt{p^0-p^3}(\pm)\sqrt{p^0-p^3} & \pm\sqrt{p^0-p^3}(\pm)\frac{-p^1+\mathrm{i}p^2}{\sqrt{p^0-p^3}} \\ \pm\frac{-p^1-\mathrm{i}p^2}{\sqrt{p^0-p^3}}(\pm)\sqrt{p^0-p^3} & \frac{\left(p^0\right)^2-\left(p^3\right)^2}{\pm\sqrt{p^0-p^3}(\pm)\sqrt{p^0-p^3}} \end{pmatrix} \\
&= -\begin{pmatrix} \pm\sqrt{p^0-p^3}(\pm)\sqrt{p^0-p^3} & \pm\sqrt{p^0-p^3}(\pm)\frac{-p^1+\mathrm{i}p^2}{\sqrt{p^0-p^3}} \\ \pm\frac{-p^1-\mathrm{i}p^2}{\sqrt{p^0-p^3}}(\pm)\sqrt{p^0-p^3} & \frac{\left(p^1\right)^2+\left(p^2\right)^2}{\pm\sqrt{p^0-p^3}(\pm)\sqrt{p^0-p^3}} \end{pmatrix} \\
&= -\begin{pmatrix} \pm\sqrt{p^0-p^3}(\pm)\sqrt{p^0-p^3} & \pm\sqrt{p^0-p^3}(\pm)\frac{-p^1+\mathrm{i}p^2}{\sqrt{p^0-p^3}} \\ \pm\frac{-p^1-\mathrm{i}p^2}{\sqrt{p^0-p^3}}(\pm)\sqrt{p^0-p^3} & \frac{-p^1-\mathrm{i}p^2}{\pm\sqrt{p^0-p^3}}\frac{-p^1+\mathrm{i}p^2}{(\pm)\sqrt{p^0-p^3}} \end{pmatrix} \\
&= -(\pm)t\begin{pmatrix} \sqrt{p^0-p^3} \\ \frac{-p^1-\mathrm{i}p^2}{\sqrt{p^0-p^3}} \end{pmatrix}(\pm)t^{-1}\begin{pmatrix} \sqrt{p^0-p^3} & \frac{-p^1+\mathrm{i}p^2}{\sqrt{p^0-p^3}} \end{pmatrix}
\end{aligned}$$

Para qualquer t, de fato podemos absorver o sinal da raiz quadrada nele,

$$\begin{aligned}
p_{a\dot{b}} &= -t\begin{pmatrix} \sqrt{p^0-p^3} \\ \frac{-p^1-\mathrm{i}p^2}{\sqrt{p^0-p^3}} \end{pmatrix}t^{-1}\begin{pmatrix} \sqrt{p^0-p^3} & \frac{-p^1+\mathrm{i}p^2}{\sqrt{p^0-p^3}} \end{pmatrix} \\
p_{a\dot{b}} &= -|p]_a\langle p|_{\dot{b}}
\end{aligned}$$

Vamos agora ignorar t, que está relacionado com a transformação pelo Little-Group. Assim obtemos,

$$\begin{aligned}
|p]_a &= \begin{pmatrix} \sqrt{p^0-p^3} \\ \frac{-p^1-\mathrm{i}p^2}{\sqrt{p^0-p^3}} \end{pmatrix} \\
&= \sqrt{E}\begin{pmatrix} \sqrt{1-\cos\theta} \\ \frac{-\sin\theta\cos\phi-\mathrm{i}\sin\theta\sin\phi}{\sqrt{1-\cos\theta}} \end{pmatrix} \\
&= \sqrt{E}\begin{pmatrix} \sqrt{2\sin^2\left(\frac{\theta}{2}\right)} \\ \frac{-\sin\theta}{\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}e^{\mathrm{i}\phi} \end{pmatrix} \\
&= \sqrt{2E}\begin{pmatrix} \sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi} \end{pmatrix} \sim \sqrt{2E}\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)e^{-\mathrm{i}\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}
\end{aligned}$$

Partindo deste podemos obter os outros via conjugação,

$$(|p]_a)^* = \langle p|_{\dot{a}} = \sqrt{2E}\begin{pmatrix}-\sin\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi} & \cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$\epsilon^{\dot{b}\dot{a}}\langle p|_{\dot{a}} = |p\rangle^{\dot{b}} = \sqrt{2E}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-\sin\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi}\\ \cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$|p\rangle^{\dot{b}} = \sqrt{2E}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi}\end{pmatrix}$$

$$\left(|p\rangle^{\dot{b}}\right)^* = [p|^b = \sqrt{2E}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)e^{-\mathrm{i}\phi}\end{pmatrix}$$

Para verificar que '$|p\rangle^{\dot{a}}$' satisfaz a equação de Weyl basta verificar que,

$$\langle p|_{\dot{a}}|p\rangle^{\dot{a}} = 0$$

O que de fato é verdade, pois,

$$\begin{aligned}\langle p|_{\dot{a}}|p\rangle^{\dot{a}} &= 2E\begin{pmatrix}-\sin\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi} & \cos\left(\frac{\theta}{2}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi}\end{pmatrix}\\ &= 2Ee^{\mathrm{i}\phi}\left(-\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) = 0\end{aligned}$$

Resta-nos checar que $p^{\dot{a}b} = -|p\rangle^{\dot{a}}[p|^b$,

$$\begin{aligned}-|p\rangle^{\dot{a}}[p|^b &= -2E\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi}\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)e^{-\mathrm{i}\phi}\end{pmatrix}\\ &= E\begin{pmatrix}-2\cos^2\left(\frac{\theta}{2}\right) & -2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-\mathrm{i}\phi}\\ -2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{\mathrm{i}\phi} & -2\sin^2\left(\frac{\theta}{2}\right)\end{pmatrix}\\ &= E\begin{pmatrix}-1-\cos\theta & -\sin\theta e^{-\mathrm{i}\phi}\\ -\sin\theta e^{\mathrm{i}\phi} & -1+\cos\theta\end{pmatrix}\\ &= E\begin{pmatrix}-1-\cos\theta & -\sin\theta\cos\phi+\mathrm{i}\sin\theta\sin\phi\\ -\sin\theta\cos\phi-\mathrm{i}\sin\theta\sin\phi & -1+\cos\theta\end{pmatrix}\\ &= \begin{pmatrix}-p^0-p^3 & -p^1+\mathrm{i}p^2\\ -p^1-\mathrm{i}p^2 & -p^0+p^3\end{pmatrix} = -p^0\mathbb{1}-\mathbf{p}\cdot\boldsymbol{\sigma} = p_\mu\bar{\sigma}^{\mu\dot{a}b}\end{aligned}$$

Exercício 2.2

O operador de helicidade é,

$$\Sigma = \frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\frac{1}{2}\epsilon_{0\alpha\mu\nu}\frac{p^\alpha}{\|p^0\|}$$

Escolhendo um referencial como '$p^\alpha = \begin{pmatrix}E & 0 & 0 & E\end{pmatrix}$',

$$\begin{aligned}\Sigma &= \frac{\mathrm{i}}{8}\epsilon_{03\mu\nu}[\gamma^\mu,\gamma^\nu]\\ \Sigma &= \frac{\mathrm{i}}{4}\epsilon_{0312}\left[\gamma^1,\gamma^2\right]\\ \Sigma &= \frac{\mathrm{i}}{4}\begin{pmatrix}\sigma^1\bar{\sigma}^2-\sigma^2\bar{\sigma}^1 & \mathbf{0}\\ \mathbf{0} & \bar{\sigma}^1\sigma^2-\bar{\sigma}^2\sigma^1\end{pmatrix}\\ \Sigma &= \frac{\mathrm{i}}{4}(-2\mathrm{i})\begin{pmatrix}\sigma^3{}_a{}^b & \mathbf{0}\\ \mathbf{0} & \sigma^{3\dot{a}}{}_{\dot{b}}\end{pmatrix}\\ \Sigma &= \frac{1}{2}\begin{pmatrix}\sigma^3{}_a{}^b & \mathbf{0}\\ \mathbf{0} & \sigma^{3\dot{a}}{}_{\dot{b}}\end{pmatrix}\end{aligned}$$

Agora com as identificações,

$$v_+ = |p]_b = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_- = |p\rangle^{\dot{b}} = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Temos,

$$\begin{aligned}
\Sigma v_+ &= \frac{1}{2}\sigma^3{}_a{}^b |p]_b \\
\Sigma v_+ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\Sigma v_+ &= -\frac{1}{2}\sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}v_+
\end{aligned}$$

E,

$$\begin{aligned}
\Sigma v_- &= \frac{1}{2}\sigma^{3\dot{a}}{}_{\dot{b}} |p\rangle^{\dot{b}} \\
\Sigma v_- &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\Sigma v_- &= \frac{1}{2}\sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}v_-
\end{aligned}$$

Seção 2.3