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Super Riemann Surfaces

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Abstract: Abstract...

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1 Introduction/Motivation

The Bosonic String Theory (BST) is known to achieve several desirable properties which up to present date haven't been done in usual Quantum Field Theory, the most prominent one is it being a perturbatively renormalizable theory which contains in its spectrum a massless spin-2 particle, this perturbative computation of amplitudes in BST is almost only possible due to the heavy simplifications the anomaly free gauge group Diff$(M)\times$Weyl allows[1]. This means, as in the path integral we're integrating over metrics, the gauge redundancies permits us to forget about the metrics and to integrate over only the different kinds of topologies of two dimensional manifolds, so that in a generic string scattering situation, what would be a non-compact generic two dimensional manifold turns into a compact two dimensional manifold — a choice over the equivalence class created by the gauge group: sphere, torus, ... —, and what was the asymptotic states — the *non-compact part* of the original manifold — turns into *punctures* in the new compact two dimensional manifold. The advantages is, this process is nicely described by complex coordinates in the two dimensional (real) manifold, where the gauge transformations amounts to holomorphic change of complex coordinates, and the study of such objects, complex coordinates in two dimensional (real) manifolds, or better, one dimensional complex manifolds, has already lots of years of development in mathematics which we can borrow, these are called Riemann Surfaces[1] (RS).

Despite being a astonishing success in some points, BST still fails, at least perturbatively, to give any room to accommodate the particle zoo present at our world, principally, there are no means of introducing fermions in the target space theory, this, among other reasons, is the motif of pursuing other types of theories. A natural guess to overcome the

[1]There is actually a distinction of a Riemann Surface and a two dimensional (real) manifold, every Riemann Surface is a two dimensional (real) manifold, but the converse is not true.

fermion problem is to introduce world-sheet fermions ψ^μ[2, 3][2],

$$S \sim \int_M \mathrm{d}^2 z \left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + \text{ghosts}\right) \tag{1.1}$$

which under quantization gives an analogous problem with the one present in BST[3],

$$\begin{aligned} \left[X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\right] &= \mathrm{i}\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\ \left[\psi^\mu(\tau,\sigma), \psi^\nu(\tau,\sigma')\right] &= \left[\tilde{\psi}^\mu(\tau,\sigma), \tilde{\psi}^\nu(\tau,\sigma')\right] = \pi\eta^{\mu\nu}\delta(\sigma-\sigma') \end{aligned}$$

that is, time-like fields $X^0, \psi^0, \tilde{\psi}^0$ have wrong sign commutator, which implies they will create ghost states in the theory, the resolution in BST is to use the gauge group — a.k.a. the Virasoro constrains —, to remove these non-physical states, but here, the best we could do is to use again the Virasoro constrains to get rid of the bosonic wrong sign states, and we would still had the fermionic wrong sign states. Here the only possible resolution is to find an other gauge redundancy of this theory, such that we can use it to eliminate the non-physical states. Luckily, this new action provides a possible candidate of gauge redundancy, as it has a $\mathcal{N}=1$ global supersymmetry (SUSY),

$$\delta_\epsilon X^\mu = -\epsilon\psi^\mu - \epsilon^* \tilde{\psi}^\mu \tag{1.2a}$$

$$\delta_\epsilon \psi^\mu = \epsilon \partial X^\mu, \quad \delta_\epsilon \tilde{\psi}^\mu = \epsilon^* \bar{\partial} X^\mu \tag{1.2b}$$

Sadly enough, this supersymmetry algebra only closes on-shell and is global instead of local, despite this, one by one these issues can be unveiled. The uplift from a global symmetry to a local redundancy can be done by means of introducing a new field in the action, the world-sheet gravitino, and the promotion of the algebra closing off-shell can also be addressed by the inclusion of an auxiliary field in the action. Both these constructions are essential to Superstring Theory (SST), and the resulting theory enjoys a superconformal gauge group, which is given by our familiar algebra[4],

$$\begin{aligned} T(z)T(w) &\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \quad T(z) \sim \partial X \partial X + \cdots \\ T(z)G(w) &\sim \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}, \quad G(z) \sim \psi \partial X + \cdots \\ G(z)G(w) &\sim \frac{2T(w)}{(z-w)^2} \end{aligned}$$

The downside here is: in going from a conformal theory — which we could benefit from developments in RS —, to a superconformal theory, there seems to be a loss of geometrical visualization — as due to $G(z)$ being fermionic is not clear how it's action on the coordinates z should be interpreted — that could affect our, before mentioned, *ease* of computing scattering amplitudes. To maintain the geometric interpretation and the off-shell supersymmetry is the role of the Super Riemann Surfaces (SRS).

[2]We'll ignore multiplicative factors and set $\alpha' = 2$ which can be restored by dimensional analysis.
[3]We're using the graded commutator notation.
[4]With the inclusion of ghosts.

2 Formulation of SRS

2.1 Intuitive Description of SRS

As mentioned in the last section, for the action[5] (1.1) to be on-shell invariant under (1.2), it's needed to add an auxiliary field in it. We'll try to understand why, and at the same time introduce the superspace formalism. We know that the supersymmetry algebra contains a commutation relation such,

$$[Q, Q] \propto P$$

so that they have to be interpreted as a space-time — in this case world-sheet — symmetry (redundancy), instead of a internal symmetry. This is interesting, because we can understand the world-sheet as being the quotient group,

$$\mathbb{R}^2 \cong \mathbb{C} \cong ISO(1,1)/SO(1,1)$$

what implies that we can also understand our space-time (world-sheet) with SUSY — Super-Space —, as being the quotient group of the Super Poincaré group with respect to the Lorentz group,

$$\text{Super-Space} \cong \mathbb{C}^{1|1} \cong ISO(1,1|1)/SO(1,1)$$

Why does this is of relevance? Due to we being able to write a generic element of the group $ISO(1,1|1)$ as,

$$ISO(1,1|1) \ni g(\sigma, \theta, \omega) = \exp\left(-\mathrm{i}\sigma_a P^a - \mathrm{i}\theta_A Q^A + \frac{\mathrm{i}}{2}\omega_{ab} J^{ab}\right)$$

if we factor out the Lorentz group, we obtain an expression for the elements of the Super-Space, they're parametrized by two bosonic coordinates σ^a, and two fermionic coordinates θ^A — the fermionic nature is guarantee by the fermionic nature of the SUSY generators Q^A —. As we're ultimately interested in the complex structure, we switch to $z, \bar{z}, \theta, \bar{\theta}$ by the usual substitutions, $z = \sigma^1 - \sigma^0$ and similarly for θ. These coordinates we introduced are useful because they allow for a differential representation of the SUSY algebra, which is analogous to the differential representation of the translations which we're accustomed $P \sim L_{-1} \sim \partial_z$. Inspection shows that the right choice is $Q_\theta = \partial_\theta - \theta\partial_z$, with the analogous anti-holomorphic one,

$$[Q_\theta, Q_\theta] = 2\partial_\theta\partial_\theta - 2\theta\partial_z\partial_\theta - 2\partial_\theta(\theta\partial_z) + 2\theta\partial_z(\theta\partial_z) = -2\partial_z$$

what neatly satisfy the correct algebra. The good thing about this kind of differential representation is that possesses a natural action on functions of the Super-Space — which we'll shown in a bit —, instead of the mysterious action in (1.2). Together with this differential representation of the SUSY generator, it's useful to introduce a *covariant derivative* D_θ, in the sense that it preserves the supersymmetry transformation of the object it's acting on,

$$D_\theta = \partial_\theta + \theta\partial_z, \quad [D_\theta, Q_\theta] = 0, \quad [D_\theta, D_\theta] = 2\partial_z$$

[5] We're going to forget about the world-sheet gravitino.

while we'll only be able to show that this is the right choice in the next subsection, there are a few motifs behind this definition. Remember, our main goal here is to obtain a geometric visualization of this supersymmetry, in other words, a geometric visualization of the Super Conformal group. As we know, a conformal transformation can be defined as being a coordinate change such that ∂_z is changed to a multiple of itself, as we already argued here, in the Super Conformal group, we have not only z, but also θ, so we need a differential operator such: (i) It commutes with the SUSY generator. (ii) A Super Conformal transformation can be defined as a coordinate change $z, \theta \to z'(z, \bar{z}, \theta, \bar{\theta}), \theta'(z, \bar{z}, \theta, \bar{\theta})$ that preserves this differential operator. Notice that ∂_z is consistent with condition (i), but, if we try to impose condition (ii) we gain only the usual bosonic conformal transformations. Our claim is, the most general differential operator that satisfy both conditions is a multiple of D_θ. We'll not prove here, but under such a super conformal transformation this differential operator transforms as $D_\theta = (D_\theta \theta') D_{\theta'}$.

The analogy proposes us to define Super Fields. A Super Field is a function of the Super-Space $\mathbb{A}\big(z, \bar{z}, \theta, \bar{\theta}\big)$, it's said to have weights $(h, \tilde{h})$ if it changes as,

$$\big(D_\theta \theta'\big)^{2h} \big(D_{\bar{\theta}} \bar{\theta}'\big)^{2\tilde{h}} \mathbb{A}'\big(z', \bar{z}', \theta', \bar{\theta}'\big) = \mathbb{A}\big(z, \bar{z}, \theta, \bar{\theta}\big)$$

a super field turns out to be a useful construction due to the natural action of a SUSY transformation,

$$[\mathbb{A}(z, \theta), Q] = -\mathrm{i} Q_\theta \mathbb{A}(z, \theta)$$

and the natural transformation of the measure,

$$\mathrm{d}^2 z' \, \mathrm{d}^2 \theta' = \mathrm{d}^2 z \, \mathrm{d}^2 \theta \, D_\theta \theta' D_{\bar{\theta}} \bar{\theta}'$$

which allows for a easy construction of an action, as the measure transforms as a weight $(-\frac{1}{2}, -\frac{1}{2})$ super field, we just need to integrate a $(\frac{1}{2}, \frac{1}{2})$ super field, and as each covariant derivative transforms as $(\frac{1}{2}, 0)$, a natural candidate is the derivative of a $(0, 0)$ super field $\mathbb{X}^\mu$. Due to the fermionic nature of the θ, this can be expanded as, using a little of foresight,

$$\mathbb{X}^\mu\big(z, \bar{z}, \theta, \bar{\theta}\big) = X^\mu(z, \bar{z}) + \mathrm{i}\theta \psi^\mu(z) + \mathrm{i}\bar{\theta} \tilde{\psi}^\mu(\bar{z}) + \bar{\theta}\theta F^\mu(z, \bar{z})$$

here we see already our familiar fields X^μ, ψ^μ, and the presence of an additional field F^μ, which transforms as $(\frac{1}{2}, \frac{1}{2})$, and also non trivially under the SUSY, as hinted by $Q_\theta \mathbb{X}^\mu$, this field is necessary to ensure the off-shell SUSY invariance, and it modifies the (1.2). As we mentioned before, a super conformal invariant action can be build as,

$$S = \frac{1}{4\pi} \int\limits_\Sigma \mathrm{d}^2 z \, \mathrm{d}^2 \theta \, D_{\bar{\theta}} \mathbb{X}^\mu D_\theta \mathbb{X}_\mu \tag{2.1}$$

where the fermionic integration will of course only extract the term proportional to $\bar{\theta}\theta$, this computation is straightforward, giving,

$$S = \frac{1}{4\pi} \int\limits_\Sigma \mathrm{d}^2 z \left(\partial_z X^\mu \partial_{\bar{z}} X^\mu + \psi^\mu \partial_{\bar{z}} \psi_\mu + \tilde{\psi}^\mu \partial_z \tilde{\psi}_\mu + F^\mu F_\mu \right)$$

exactly our starting action! With of course the auxiliary field F^μ, which has the trivial equation of motion $F^\mu = 0$. This implies that the action (1.1) is just (2.1) with the auxiliary field integrated out.

2.2 Formal Definition of SRS

Now we go on to formalize the description made before, a Super Riemann Surface is a special kind of a complex supermanifold, we're particularly interested in SRSs of dimension $1|1$, let's start step by step.

Definition 2.1 *A* ***complex supermanifold*** Σ *of dimension* $1|1$ *is a space locally isomorphic to* $\mathbb{C}^{1|1}$*, that is, it's locally covered by coordinate charts* $z|\theta : U \subset \Sigma \to \mathbb{C}^{1|1}$ *such that* z *is a complex even coordinate, and* θ *is a complex odd coordinate.*

The definition of a SRS is build on top of this one,

Definition 2.2 *A* ***Super Riemann Surface*** Σ *is a complex supermanifold of dimension* $1|1$ *that possesses a completely non-integrable dimension* $0|1$ *holomorphic subbundle* $\mathcal{D} \subset T\Sigma$.

This definition might hide what it's trying to convey through other definitions, so, we're going to break it in pieces. $T\Sigma$ is the tangent bundle/space of the supermanifold Σ, which locally is spanned by a coordinate basis $\partial_\theta, \partial_z$, this is exactly the same notion that we have in real manifolds. Now, by *dimension* $0|1$ *holomorphic subbundle* $\mathcal{D}$ we mean we have a closed subspace of the tangent vector field space $T\Sigma$ in which every element is odd. Lastly, to say $\mathcal{D}$ is *completely non-integrable* is to say[6],

$$\forall D \in \mathcal{D} \, : \, D \text{ non zero } \Rightarrow D^2 := \frac{1}{2}[D, D] \notin \mathcal{D}$$

By D being non-zero it means that when it's written in a given basis $z, \theta : U \subset \Sigma \to \mathbb{C}^{1|1}$,

$$D \text{ is non zero } \Leftrightarrow D\Big|_U = a(z,\theta)\partial_\theta + b(z,\theta)\partial_z, \quad a(z,\theta) \neq 0$$

3 Punctures in SRS

A Mathematical Toolkit

References

[1] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[2] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[3] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory Vol. 1: 25th Anniversary Edition.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 11, 2012.

[6] Here $[\cdot, \cdot]$ should be interpreted as a graded vector field Lie Bracket.