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Super Riemann Surfaces

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Abstract: Abstract...

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1 Introduction/Motivation

The Bosonic String Theory (BST) is known to achieve several desirable properties which up to present date haven't been done in usual Quantum Field Theory, the most prominent one is it being a perturbatively renormalizable theory which contains in its spectrum a massless spin-2 particle, this perturbative computation of amplitudes in BST is almost only possible due to the heavy simplifications the anomaly free gauge group Diff(M)$\times$Weyl allows[1]. This means, as in the path integral we're integrating over metrics, the gauge redundancies permits us to forget about the metrics and to integrate over only the different kinds of topologies of two dimensional manifolds, so that in a generic string scattering situation, what would be a non-compact generic two dimensional manifold turns into a compact two dimensional manifold — a choice over the equivalence class created by the gauge group: sphere, torus, ... —, and what was the asymptotic states — the *non-compact part* of the original manifold — turns into *punctures* in the new compact two dimensional manifold. The advantages is, this process is nicely described by complex coordinates in the two dimensional (real) manifold, where the gauge transformations amounts to holomorphic change of complex coordinates, and the study of such objects, complex coordinates in two dimensional (real) manifolds, or better, one dimensional complex manifolds, has already lots of years of development in mathematics which we can borrow, these are called Riemann Surfaces[1] (RS).

Despite being a astonishing success in some points, BST still fails, at least perturbatively, to give any room to accommodate the particle zoo present at our world, principally, there are no means of introducing fermions in the target space theory, this, among other

[1]There is actually a distinction of a Riemann Surface and a two dimensional (real) manifold, every Riemann Surface is a two dimensional (real) manifold, but the converse is not true.

reasons, is the motif of pursuing other types of theories. A natural guess to overcome the fermion problem is to introduce world-sheet fermions ψ^μ[2, 3][2],

$$S \sim \int\limits_M \mathrm{d}^2 z \left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + \text{ghosts}\right) \tag{1.1}$$

which under quantization gives an analogous problem with the one present in BST[3],

$$\begin{aligned}
\left[X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\right] &= \mathrm{i}\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\
\left[\psi^\mu(\tau,\sigma), \psi^\nu(\tau,\sigma')\right] = \left[\tilde{\psi}^\mu(\tau,\sigma), \tilde{\psi}^\nu(\tau,\sigma')\right] &= \pi\eta^{\mu\nu}\delta(\sigma-\sigma')
\end{aligned}$$

that is, time-like fields $X^0, \psi^0, \tilde{\psi}^0$ have wrong sign commutator, which implies they will create ghost states in the theory, the resolution in BST is to use the gauge group — a.k.a. the Virasoro constrains —, to remove these non-physical states, but here, the best we could do is to use again the Virasoro constrains to get rid of the bosonic wrong sign states, and we would still had the fermionic wrong sign states. Here the only possible resolution is to find an other gauge redundancy of this theory, such that we can use it to eliminate the non-physical states. Luckily, this new action provides a possible candidate of gauge redundancy, as it has a $\mathcal{N}=1$ global supersymmetry (SUSY),

$$\delta_\epsilon X^\mu = -\epsilon\psi^\mu - \epsilon^*\tilde{\psi}^\mu \tag{1.2a}$$

$$\delta_\epsilon \psi^\mu = \epsilon\partial X^\mu, \quad \delta_\epsilon\tilde{\psi}^\mu = \epsilon^*\bar{\partial}X^\mu \tag{1.2b}$$

Sadly enough, this supersymmetry algebra only closes on-shell and is global instead of local, despite this, one by one these issues can be unveiled. The uplift from a global symmetry to a local redundancy can be done by means of introducing a new field in the action, the world-sheet gravitino, and the promotion of the algebra closing off-shell can also be addressed by the inclusion of an auxiliary field in the action. Both these constructions are essential to Superstring Theory (SST), and the resulting theory enjoys a superconformal gauge group, which is given by our familiar super Virasoro algebra[4],

$$\begin{cases} T(z) & \sim \partial X\partial X + \cdots \\ G(z) & \sim \psi\partial X + \cdots \end{cases} \Rightarrow \begin{cases} T(z)T(w) & \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\ T(z)G(w) & \sim \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \\ G(z)G(w) & \sim \frac{2T(w)}{(z-w)^2} \end{cases} \Rightarrow \begin{cases} [L_m, L_n] & = (m-n)L_{m+n} \\ [G_r, G_s] & = 2L_{r+s} \\ [L_m, G_r] & = \left(\frac{m}{2}-r\right)G_{m+r} \end{cases}$$

The downside here is: in going from a conformal theory — which we could benefit from developments in RS —, to a superconformal theory, there seems to be a loss of geometrical visualization — as due to $G(z)$ being fermionic is not clear how it's action on the coordinates z should be interpreted — that could affect our, before mentioned, *ease* of computing scattering amplitudes. To maintain the geometric interpretation and the off-shell supersymmetry is the role of the Super Riemann Surfaces (SRS).

[2] We'll ignore multiplicative factors and set $\alpha' = 2$ which can be restored by dimensional analysis.
[3] We're using the graded commutator notation.
[4] With the inclusion of ghosts.

2 Super Riemann Surfaces and Consequences

2.1 Definition of SRS

As we foretold, to obtain a geometrical description of the superconformal structure we need a geometrical description of the generators $T(z), G(z)$, in the bosonic case we have such a description as differential operators such $T(z) \to L_n \sim z^{n+1}\partial_z$. But, to do so for $G(z)$ — which is grassmann odd — would require to have a grassmann odd coordinate. This is the start of the history of SRS. They're a special type of **Complex supermanifolds**, so, in order to understand them, we need first to understand what is a complex supermanifold.

Definition 2.1. *A **complex supermanifold** Σ of dimension $1|1$ is a space locally isomorphic to $\mathbb{C}^{1|1}$, that is, it's locally covered by coordinate charts $z|\theta : U \subset \Sigma \to \mathbb{C}^{1|1}$ such that z is a complex grassmann even coordinate, and θ is a complex grassmann odd coordinate.*

Of course we have to say something about the transition functions between overlapping charts. Due to the complex nature, we really just have two kinds of choice, either we impose that all transition functions are just continuous, or that they are holomorphic. It's clear that for our context the latter is way more useful, so we'll choose it. Maybe it's needed clarification on what we mean by a transition function $(z_i|\theta_i)^{-1} \circ z_j|\theta_j : z_j|\theta_j(U_j) \subset \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ to be holomorphic.

Definition 2.2. *A function $f(z, \theta) : U \subset \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ is said to be holomorphic iff the expansion in powers of θ — which is finite due to the oddness —,*

$$f(z, \theta) = f_0(z) + \theta f_1(z)$$

has both $f_0, f_1 : U \cap \mathbb{C} \subset \mathbb{C} \to \mathbb{C}^{1|1}$ holomorphic.

Another definition which will be of use for us is the notion of non-zero objects in a supermanifold.

Definition 2.3. *Any **non-zero** object — function, vector, ... — is one such that is non-zero up to grassmann odd variables.*

With this we have all the ingredients to state the definition of a SRS, the point is: we want for the SRS to have a notion of superconformal transformations — which is compatible with the super Virasoro algebra from SST — as the RS had. The good thing about RS is, they don't need actually more structure than being a complex manifold, as any transformation $z \to f(z)$ is indeed a conformal transformation. Two distinct characteristic of conformal transformations in SR are: they preserve the tangent space, that is $\partial_z \to \partial_z z' \partial_{z'}$, and they preserve angles. These properties cannot be used immediately to give a prescription of super conformal transformation. If we try to set them as being all the set of transformations which preserve the tangent space of a complex supermanifold we would get actually an algebra that is way bigger than the super Virasoro algebra, and to define them as being *angle preserving* would require to introduce a metric, which is too much of additional structure.

The idea is to look at what is really essential for us to have, if we're hoping to get any non-trivial theory with super conformal transformations, we need some kind of derivative operator which transforms covariantly under such. This is not the end, as we need to guarantee — in a coordinate invariant way — that covariant transformations of this derivative operator — which we'll call super conformal transformations — do in fact *mix* the grassmann odd and even coordinates. While seem kind of arbitrary, this requirement already excludes lot's of possible choices as ∂_z and ∂_θ. Let's sum up what we concluded, to set up a SRS we need a notion of super conformal transformation, which we concluded is equivalent of choosing a covariant derivative — what can be seen as choosing a subspace of the tangent space $T\Sigma$ —, but no just any covariant derivative, one such that guarantees that we're in fact *coupling* both coordinates. There are just two possible choices for the dimension of the subspace we're choosing, $0|1$ or $1|0$. We argue that it's impossible to choose a $1|0$ dimensional subspace of $T\Sigma$ and guarantee that in some sense the covariance under it *couples* both coordinates, this is so because no operation done with only grassmann even variables can return a grassmann odd variable. But, the opposite is true! It's possible to compose two grassmann odd variables to give a grassmann even one. Hence, we conclude that to define a notion of super conformal transformation is needed to choose a $0|1$ dimensional subspace of $T\Sigma$, what about the condition to ensure *mixing* of coordinates? Let's save for the definition[4–6],

Definition 2.4. *A **Super Riemann Surface** Σ is a complex supermanifold of dimension $1|1$ that possesses a distinct subbundle/subspace $\mathcal{D} \subset T\Sigma$ of dimension $0|1$ that satisfy the completely non-integrability condition,*

$$\forall D \in \mathcal{D} : D \text{ is non-zero} \Rightarrow [D, D] \notin \mathcal{D}$$

$\mathcal{D}$ is said to be a complex structure of this SRS.

Here we see the beauty of this definition, as we discussed, we have the distinguished structure as a $0|1$ dimensional subspace of $T\Sigma$, and the condition that it must satisfy is the so called *completely non-integrability*, which is dependent on the graded Lie Bracket operation defined for vector fields $[\cdot, \cdot]$, what is important here is that $[D, D]$ is only non-zero if D is odd, and in this case, by statistics, it returns a even vector field. Here the hypothesis of D being non-zero is crucial, otherwise, $[D, D]$ could be zero and so, despite being even, could still belong to $\mathcal{D}$. Hence, the completely non-integrability condition provides a notion of *coupling* both coordinates without having to evoke new structure in our manifold. It's instructive to see what kind of elements belongs to this subbundle,

Lemma 2.1. *Given a SRS with distinct subbundle $\mathcal{D}$, it's always possible to construct a coordinate system $z|\theta$, such that locally, any element $D \in \mathcal{D}$ is of the form,*

$$D_\theta \coloneqq D_{U_{z|\theta}} = \partial_\theta + \theta\partial_z, \quad D_\theta^2 \coloneqq \frac{1}{2}[D_\theta, D_\theta] = \partial_z \tag{2.1}$$

we call this coordinate system a superconformal one.

Proof. In a given coordinate system/chart $z|\theta$ we always can decompose any element of the tangent space as,

$$D_{U_{z|\theta}} = a(z,\theta)\partial_\theta + b(z,\theta)\partial_z$$

the condition of $D_{U_{z|\theta}}$ being non-zero, as stated in definition 2.4, is equivalent to $a(z,\theta) \neq 0$, hence, it's possible to scale $\theta \to a\theta$, doing this and also expanding, due to the grassmann odd character, $b(z,\theta) = b_0(z) + \theta b_1(z)$,

$$D_{U_{z|\theta}} = \partial_\theta + (b_0 + \theta b_1)\partial_z$$

which now we compute the graded Lie Bracket,

$$\begin{aligned}
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= [\partial_\theta + (b_0 + \theta b_1)\partial_z, \partial_\theta + (b_0 + \theta b_1)\partial_z] \\
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= 2[\partial_\theta, (b_0 + \theta b_1)\partial_z] + [(b_0 + \theta b_1)\partial_z, (b_0 + \theta b_1)\partial_z] \\
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= 2\partial_\theta(b_0 + \theta b_1)\partial_z + 2(b_0 + \theta b_1)[\partial_\theta, \partial_z] + 2(b_0 + \theta b_1)\big(\partial_z(b_0 + \theta b_1)\partial_z + (b_0 + \theta b_1)\partial_z^2\big) \\
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= 2b_1\partial_z + 2\theta b_1\partial_z b_0\partial_z - 2\theta b_0\partial_z b_1\partial_z \qquad (2.2)
\end{aligned}$$

so, the only way $\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] \in \mathcal{D}$ could possibly be true is if the above expression is identically zero, which is only possible if $b_1 = 0$. Thus, the completely non-integrability condition gives the requirement $b_1 \neq 0$, hence, we can perform a further change of coordinates $\theta \to -b_1^{-1}b_0 + \theta$, $z \to b_1 z$,

$$D_\theta \coloneqq D_{U_{z|\theta}} = \partial_\theta + \theta\partial_z$$

And lastly, setting $b_1 = 1, b_0 = 0$ in eq. (2.2) we get, $D_\theta^2 = \partial_z$. ∎

The existence of such structure is what settles apart a SRS from a generic complex supermanifold. To grasp a better understanding of this non-integrability condition we have to take a look at the second equation in (2.1), for a usual coordinate basis of the tangent space we always have $[\partial_I, \partial_J] = 0$, which kind of induces a splitting of the tangent space as $\mathbb{C}^{1|0} \times \cdots \times \mathbb{C}^{0|1} \times \cdots$, but of course, as this is a coordinate basis, this splitting is not some kind of inner property of the manifold, it's a coordinate dependent gimmick. But, with our definition being coordinate independent, we're saying that existis a global splitting of the tangent space $T\Sigma \cong \mathbb{C}^{1|0} \times \mathbb{C}^{0|1} \cong T\Sigma/\mathcal{D} \times \mathcal{D} \cong \mathcal{D}^2 \times \mathcal{D}$.

We will now pursue why does this structure is able to reconstruct the off-shell super Virasoro algebra in SST.

2.2 Superconformality in SRS

As we foretold, having a distinct subbundle $\mathcal{D}$, is possible for us to define super conformal coordinate changes. The idea is, vector fields naturally introduce coordinate changes by integral curves, hence,

Definition 2.5. *A vector field $W \in T\Sigma$ is said to generate a super conformal coordinate transformation on a SRS Σ, if it preserves the subbundle $\mathcal{D}$, that is,*

$$W \in T\Sigma \text{ generates superconformal transformation} \Leftrightarrow \forall D \in \mathcal{D} : [W, D] \in \mathcal{D}.$$

As good as having a formal definition may be, our true interest here is to obtain these transformations in a given basis, in particular, a superconformal basis,

Lemma 2.2. *The set of all vector fields that generate superconformal transformation can be decomposed in a basis, with a superconformal coordinate system, of one even and one odd vector fields such,*

$$G_f = f(z)(\partial_\theta - \theta\partial_z), \quad L_g = g(z)\partial_z + \frac{1}{2}g'(z)\theta\partial_\theta \tag{2.3}$$

Proof. We will compute it using D_θ and a decomposition of $W = a\partial_\theta + b\partial_z$

$$[W, D_\theta] = [a\partial_\theta + b\partial_z, D_\theta] = a[\partial_\theta, D_\theta] \mp D_\theta a\partial_\theta + b[\partial_z, D_\theta] \mp D_\theta b\partial_z$$
$$[W, D_\theta] = a\partial_z \mp D_\theta a\partial_\theta \mp D_\theta b\partial_z = \mp D_\theta a\partial_\theta \mp (D_\theta b \mp a)\partial_z$$

Where the signs reefer to W being even, top sign, or odd, bottom sign. To say that $[W, D_\theta] \in \mathcal{D}$ is the same to say that $[W, D_\theta] \propto D_\theta$, hence, the condition is,

$$\mp(D_\theta b \mp a) = \mp D_\theta a\theta$$
$$D_\theta b = D_\theta a\theta \pm a$$

In this form it may seem difficult to solve it, but, we will use our virtue of foresight to propose an ansatz, $b = -a\theta$,

$$D_\theta(-a\theta) = D_\theta a\theta \pm a \Rightarrow -D_\theta a\theta \pm a = D_\theta a\theta \pm a \Rightarrow D_\theta a\theta = 0 \Rightarrow \partial_\theta a\theta = 0 \Rightarrow \begin{cases} a & = a(z) \\ b & = -a(z)\theta \end{cases}$$

With this ansatz we got exactly an solution, that is, one family of odd vector fields that generate superconformal transformations are $G_f = f(z)(\partial_\theta - \theta\partial_z)$. But, this is not the end, as we have two dimensions, one odd and one even, we know that there is one more solution to this equation,

$$D_\theta b = D_\theta a\theta \pm a$$
$$D_\theta^2 b = D_\theta^2 a\theta \pm D_\theta a \pm D_\theta a$$
$$\partial_z b = \partial_z a\theta \pm 2\partial_\theta a \pm 2\theta\partial_z a = -\partial_z a\theta \pm 2\partial_\theta a$$

it's clear that the condition $\partial_\theta a = 0$ will return our already found solution, hence, we try to solve this equation for $\partial_z a\theta = 0 \Rightarrow \partial_z b = \pm 2\partial_\theta a$, substituting back in the original equation,

$$\partial_\theta b + \theta\partial_z b = \partial_\theta a\theta \pm a$$
$$\partial_\theta b \pm 2\theta\partial_\theta a = \partial_\theta a\theta \pm a\partial_\theta\theta$$
$$\partial_\theta b + 2\partial_\theta a\theta = \partial_\theta a\theta \pm a\partial_\theta\theta$$
$$\partial_\theta b = -\partial_\theta a\theta \pm a\partial_\theta\theta = -\partial_\theta(a\theta)$$

As we already solve for $b = -a\theta$, the only other possible solution that can be obtained from here is $\partial_\theta b = \partial_\theta(a\theta) = 0 \Rightarrow a = c(z)\theta,\ b = b(z)$, and the last consistency condition gives,

$$\partial_z b = \pm 2\partial_\theta a \Rightarrow \partial_z b = 2c(z) \Rightarrow \begin{cases} a & = \frac{1}{2}\partial_z b(z)\theta \\ b & = b(z) \end{cases}$$

which give to us the second and last linear independent solution, a family of even vector fields that generate superconformal transformations $L_g = g(z)\partial_z + \frac{1}{2}g'(z)\theta\partial_\theta$. ■

It's no coincidence the naming we utilized in eq. (2.3), as those two are the generators of the superconformal transformations, they should also be related to the SST super Virasoro generators L_n, G_r, and actually this is true, they indeed furnish a differential representation of the super Virasoro,

Lemma 2.3. *The vector field basis, eq. (2.3), of superconformal transformations for* $f(z) = z^{r+\frac{1}{2}}, g(z) = -z^{n+1}$,

$$L_n = -z^{n+1}\partial_z - \frac{n+1}{2}z^n\theta\partial_\theta, \quad G_r = z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z), \quad n \in \mathbb{Z},\ r \in \mathbb{Z} + \frac{1}{2} \tag{2.4}$$

furnishes a representation of the super Virasoro algebra for the NS sector,

$$[L_m, L_n] = (m-n)L_{m+n} \tag{2.5a}$$

$$[G_r, G_s] = 2L_{r+s} \tag{2.5b}$$

$$[L_m, G_r] = \Big(\frac{m}{2} - r\Big)G_{m+r} \tag{2.5c}$$

Proof. The computation is very straightforward, we start with $L - L$,

$$\begin{aligned}
[L_m, L_n] &= \left[z^{m+1}\partial_z + \frac{m+1}{2}z^m\theta\partial_\theta, z^{n+1}\partial_z + \frac{n+1}{2}z^n\theta\partial_\theta\right]\\
[L_m, L_n] &= \left[z^{m+1}\partial_z, z^{n+1}\partial_z\right] + \left[z^{m+1}\partial_z, \frac{n+1}{2}z^n\theta\partial_\theta\right] + \left[\frac{m+1}{2}z^m\theta\partial_\theta, z^{n+1}\partial_z\right]\\
[L_m, L_n] &= z^{m+1}\partial_z z^{n+1}\partial_z - z^{n+1}\partial_z z^{m+1}\partial_z + \frac{n+1}{2}z^{m+1}\theta\partial_z z^n\partial_\theta - \frac{m+1}{2}z^{n+1}\theta\partial_z z^m\partial_\theta\\
[L_m, L_n] &= (n+1)z^{m+n+1}\partial_z - (m+1)z^{n+m+1}\partial_z + \frac{(n+1)n}{2}z^{m+n}\theta\partial_\theta - \frac{(m+1)m}{2}z^{n+m}\theta\partial_\theta\\
[L_m, L_n] &= -(m-n)z^{m+n+1}\partial_z + \big(n^2 + n - m^2 - m\big)\frac{z^{n+m}}{2}\theta\partial_\theta\\
[L_m, L_n] &= -(m-n)z^{m+n+1}\partial_z - (m-n)(n+m+1)\frac{z^{n+m}}{2}\theta\partial_\theta = (m-n)L_{m+n}
\end{aligned}$$

now $G - G$,

$$\begin{aligned}
[G_r, G_s] &= \left[z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z), z^{s+\frac{1}{2}}(\partial_\theta - \theta\partial_z)\right]\\
[G_r, G_s] &= -\left(z^{r+\frac{1}{2}}\theta\partial_z z^{s+\frac{1}{2}} + z^{s+\frac{1}{2}}\theta\partial_z z^{r+\frac{1}{2}}\right)(\partial_\theta - \theta\partial_z) + z^{r+s+1}[(\partial_\theta - \theta\partial_z), (\partial_\theta - \theta\partial_z)]\\
[G_r, G_s] &= -\left(s + \frac{1}{2} + r + \frac{1}{2}\right)z^{r+s}\theta\partial_\theta - 2z^{r+s+1}\partial_z = 2\left(-z^{r+s+1}\partial_z - \frac{r+s+1}{2}z^{r+s}\theta\partial_\theta\right)\\
[G_r, G_s] &= 2L_{r+s}
\end{aligned}$$

and lastly $L - G$,

$$
\begin{aligned}
[L_m, G_r] &= \left[-z^{m+1}\partial_z - \frac{m+1}{2} z^m \theta\partial_\theta, z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z)\right] \\
[L_m, G_r] &= -z^{m+1}\partial_z z^{r+\frac{1}{2}}\partial_\theta + \frac{m+1}{2} z^{m+r+\frac{1}{2}}\partial_\theta + z^{m+1}\partial_z z^{r+\frac{1}{2}}\theta\partial_z - z^{r+\frac{1}{2}}\theta\partial_z z^{m+1}\partial_z \\
&\quad + \frac{m+1}{2} z^{m+r+\frac{1}{2}}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} + \frac{1}{2} - r - \frac{1}{2}\right) z^{m+r+\frac{1}{2}}\partial_\theta + \left(r + \frac{1}{2} - \frac{m}{2} - \frac{1}{2}\right) z^{m+r+\frac{1}{2}}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) z^{m+r+\frac{1}{2}}(\partial_\theta - \theta\partial_z) = \left(\frac{m}{2} - r\right) G_{m+r}
\end{aligned}
$$

■

As we have mentioned before, this last result is one of the main reasons why SRS are useful, they make the superconformal algebra on-shell, it's instructive to see explicitly from eq. (2.5) the realization of SUSY, $\left[G_{-\frac{1}{2}}, G_{-\frac{1}{2}}\right] = 2L_{-1} = -2\partial_z$. Nevertheless, there are still many points left to explanation, as stated in lemma 2.3, the resulting super Virasoro algebra from SRS are in the NS sector, that is, they only work for states with NS boundary conditions. As a reminder, NS boundary conditions are imposed in the world-sheet fermions (1.1) as $\psi^\mu(w + 2\pi\mathrm{i}) = -\psi^\mu(w)$, with $w = \ln z$ the cylinder coordinate. We will not prove here, but, it's impossible to reconstruct the super Virasoro algebra in the R sector — $\psi^\mu(w + 2\pi\mathrm{i}) = \psi^\mu(w) \Rightarrow G_r,\ r \in \mathbb{Z}$ — with (2.3). While this may seem a big problem, there is an interesting interpretation of why this is the case, and in fact this is no obstruction to the introduction of R sector fermions, for now we just cite what is the form of the super Virasoro generators for the R sector, and later on there will be a explanation on why this should be true,

Lemma 2.4. *The following vector fields,*

$$
L_n = -z^{n+1}\partial_z - \frac{n}{2} z^n \theta\partial_\theta, \quad G_r = z^r(\partial_\theta - \theta z\partial_z), \quad n, r \in \mathbb{Z} \tag{2.6}
$$

furnishes a differential representation of the super Virasoro algebra for the R sector,

$$
[L_m, L_n] = (m - n)L_{m+n} \tag{2.7a}
$$

$$
[G_r, G_s] = 2L_{r+s} \tag{2.7b}
$$

$$
[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r} \tag{2.7c}
$$

Proof. Again, the calculation is very straightforward, we start with $L - L$,

$$
\begin{aligned}
[L_m, L_n] &= \left[z^{m+1}\partial_z + \frac{m}{2} z^m \theta\partial_\theta, z^{n+1}\partial_z + \frac{n}{2} z^n \theta\partial_\theta\right] \\
[L_m, L_n] &= \left[z^{m+1}\partial_z, z^{n+1}\partial_z\right] + \left[z^{m+1}\partial_z, \frac{n}{2} z^n \theta\partial_\theta\right] + \left[\frac{m}{2} z^m \theta\partial_\theta, z^{n+1}\partial_z\right] \\
[L_m, L_n] &= z^{m+1}\partial_z z^{n+1}\partial_z - z^{n+1}\partial_z z^{m+1}\partial_z + \frac{n}{2} z^{m+1}\theta\partial_z z^n \partial_\theta - \frac{m}{2} z^{n+1}\theta\partial_z z^m \partial_\theta \\
[L_m, L_n] &= (n+1) z^{m+n+1}\partial_z - (m+1) z^{n+m+1}\partial_z + \frac{n^2}{2} z^{m+n}\theta\partial_\theta - \frac{m^2}{2} z^{n+m}\theta\partial_\theta \\
[L_m, L_n] &= -(m-n) z^{m+n+1}\partial_z + \left(n^2 - m^2\right)\frac{z^{n+m}}{2}\theta\partial_\theta \\
[L_m, L_n] &= -(m-n) z^{m+n+1}\partial_z - (m-n)(n+m)\frac{z^{n+m}}{2}\theta\partial_\theta = (m-n) L_{m+n}
\end{aligned}
$$

now $G - G$,

$$
\begin{aligned}
[G_r, G_s] &= [z^r(\partial_\theta - \theta z\partial_z), z^s(\partial_\theta - \theta z\partial_z)] \\
[G_r, G_s] &= -\left(z^{r+1}\theta\partial_z z^s + z^{s+1}\theta\partial_z z^r\right)(\partial_\theta - \theta z\partial_z) + z^{r+s}[(\partial_\theta - \theta z\partial_z), (\partial_\theta - \theta z\partial_z)] \\
[G_r, G_s] &= -(s+r) z^{r+s}\theta\partial_\theta - 2 z^{r+s} z\partial_z = 2\left(-z^{r+s+1}\partial_z - \frac{r+s}{2} z^{r+s}\theta\partial_\theta\right) \\
[G_r, G_s] &= 2 L_{r+s}
\end{aligned}
$$

and lastly $L - G$,

$$
\begin{aligned}
[L_m, G_r] &= \left[-z^{m+1}\partial_z - \frac{m}{2} z^m \theta\partial_\theta, z^r(\partial_\theta - \theta z\partial_z)\right] \\
[L_m, G_r] &= -z^{m+1}\partial_z z^r \partial_\theta + \frac{m}{2} z^{m+r}\partial_\theta + z^{m+1}\partial_z z^{r+1}\theta\partial_z - z^{r+1}\theta\partial_z z^{m+1}\partial_z \\
&\qquad + \frac{m}{2} z^{m+r+1}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) z^{m+r}\partial_\theta + \left(r + 1 - \frac{m}{2} - 1\right) z^{m+r+1}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) z^{m+r}(\partial_\theta - \theta z\partial_z) = \left(\frac{m}{2} - r\right) G_{m+r}
\end{aligned}
$$

■

While might seem taken out of a hat, this shows two main points: (i) that is possible to somewhat *embed* the R sector in SRS, but, (ii) the fact that (2.6) cannot be written as (2.3), together with lemma 2.2, shows that the super Virasoro R sector cannot possibly preserve our distinct subspace $\mathcal{D}$. There is a small flaw in this argument that may shed some hope in the interpretation of R fermions, which we are going to postpone. For now we follow working with just the NS sector.

2.3 Super fields and weights

Having now constructed SRS, and also shown that they do satisfy an on-shell super Virasoro algebra, the next step is to do a matching of how a theory formulated in SRS can give the action in eq. (1.1). The main ideia here is to follow analogous to BST, first to describe objects that transform covariantly under superconformal transformations, and later use them to construct a superconformal invariant action. It's instructive to first understand how θ behaves under dilations, which for now is the closest we have to a conformal weight. As we know, dilations are the integral curves of, eq. (2.4), $L_0 = -z\partial_z - \frac{1}{2}\theta\partial_\theta$, which are trivially $z|\theta \to \lambda z|\lambda^{\frac{1}{2}}\theta$. This kind of is a reminiscent of z having conformal weight -1 and θ having conformal weight $-\frac{1}{2}$, this is not really true, as we used global transformations, to get a real definition of what may be superconformal weights we have to look at general transformations. By now we are fully aware that superconformal transformations preserve $\mathcal{D}$, other way to pose it is, superconformal transformations of a superconformal coordinate system $z|\theta$ to another superconformal coordinate system $\hat{z}|\hat{\theta}$ changes D_θ to $F(z,\theta)D_{\hat{\theta}}$, hence, it should be true that,

$$\begin{aligned} D_\theta = F(z,\theta)D_{\hat{\theta}} \Rightarrow D_\theta\hat{\theta} &= F(z,\theta)D_{\hat{\theta}}\hat{\theta} \\ D_\theta\hat{\theta} &= F(z,\theta) \Rightarrow D_\theta = D_\theta\hat{\theta}D_{\hat{\theta}} \end{aligned} \tag{2.8}$$

which is the definition we will be taking of a superconformal transformation. Plenty other consistency conditions can be obtained from eq. (2.8) by applying it to $z, \hat{z}, \theta$. Those are specially useful if one want to show what is the most general form of a superconformal transformation explicitly, but, there isn't so much of insight coming from this, and thus we won't attempt it here. What indeed is of interest to us is how eq. (2.8) is similar to the conformal transformation of the BST counterpart, $\partial_z = \partial_z z' \partial_{z'}$, which is used to define the conformal weights scaling,

$$\left(\partial_z z'\right)^h \left(\partial_{\bar{z}} \bar{z}'\right)^{\tilde{h}} \phi'\left(z', \bar{z}'\right) = \phi(z, \bar{z})$$

but, as $D_\theta^2 = \partial_z$, this suggest that we should consider the following as a definition of the superconformal scaling weights,

$$\left(D_\theta \theta'\right)^{2h} \phi'\left(z', \theta'\right) = \phi(z, \theta)$$

notice the $2h$ instead of h. But, still, there is something off with this definition, the lack of conjugated variables $\bar{z}, \bar{\theta}$, there is a reason for this. In the presence of both complex even and odd variables, z, θ, it's in fact impossible to define a notion of complex conjugation $z|\theta \to \bar{z}|\bar{\theta}$, hence, we cannot talk about $z, \bar{z}|\theta, \bar{\theta}$ in a single SRS. This is no problem, as we can always set the full world-sheet Σ as being a product of two SRS, Σ_L, Σ_R, one with local holomorphic coordinates $z|\theta$, and the other with local anti-holomorphic coordinates $\bar{z}|\bar{\theta}$. Notice, here not necessarily $\bar{z} = z^*$. Now, in the full world-sheet $\Sigma \cong \Sigma_L \times \Sigma_R$, it's possible to define superconformal weights as,

$$\left(D_\theta \theta'\right)^{2h} \left(D_{\bar{\theta}} \bar{\theta}'\right)^{2h} \phi'\left(z', \bar{z}', \theta', \bar{\theta}'\right) = \phi\left(z, \bar{z}, \theta, \bar{\theta}\right)$$

One last matter that we haven't touched yet is the integration measure. In possess of superconformal fields, we have to decide over what we should integrate them to obtain a superconformally invariant action. The answer is kind of trivial, we should integrate over top forms! Sadly, there are lots of subtleties involving forms and integration in supermanifolds which we won't deal with here. We're just to cite a result that we will need.

Lemma 2.5. *In a SRS Σ_L there is a natural way to define a 2–form with dimensions $1|1$ called the Berezinian, Ber(Σ_L), which in a coordinate system $z|\theta$ is denoted as $[\mathrm{d}z\,|\,\mathrm{d}\theta]$. It provides a natural way of defining line integrals in Σ_L,*

$$\int\limits_{\gamma\subset\Sigma_L}[\mathrm{d}z\,|\,\mathrm{d}\theta](f_0(z)+\theta f_1(z))=\int\limits_{\gamma\cap\mathbb{C}^{1|0}\subset\Sigma_L\cap\mathbb{C}^{1|0}}\mathrm{d}z\,f_1(z)$$

It also transform covariantly under superconformal transformations,

$$\left[z'|\theta'\right]=[z|\theta]D_\theta\theta' \tag{2.9}$$

Proof. See [6]. ∎

Might seem that that the Berezinian has a downside, it only sets up a procedure for doing line integrals, not volume ones — the ones we're interested in —. This is not really of a problem, as we mentioned before, it's impossible to set up $\bar{z},\bar{\theta}$ with just a single holomorphic SRS, this is also a reason that we cannot construct a volume form in a holomorphic SRS, hence, the resolution of this problem is what we already done, it is to work with $\Sigma=\Sigma_L\times\Sigma_R$, and thus set up the Berezinian as $\mathrm{Ber}(\Sigma)=\mathrm{Ber}(\Sigma_L)\otimes\mathrm{Ber}(\Sigma_R)$. Now, $\mathrm{Ber}(\Sigma)$ in local coordinates $z,\bar{z}|\theta,\bar{\theta}$ is $\left[\mathrm{d}z\,,\mathrm{d}\bar{z}\,|\,\mathrm{d}\theta\,,\mathrm{d}\bar{\theta}\right]=[\mathrm{d}z\,|\,\mathrm{d}\theta]\otimes\left[\mathrm{d}\bar{z}\,|\,\mathrm{d}\bar{\theta}\right]$ So that now we can pose our action as being an integral such,

$$S=\int\limits_{\Sigma}\left[\mathrm{d}z\,,\mathrm{d}\bar{z}\,|\,\mathrm{d}\theta\,,\mathrm{d}\bar{\theta}\right]\phi\left(z,\bar{z},\theta,\bar{\theta}\right)$$

By the transformation properties, eq. (2.9), we need ϕ to be a $\left(\frac{1}{2},\frac{1}{2}\right)$ weight superfield in order to the action to be invariant. Notice, if we want a non-trivial theory, we better have derivatives, and the only covariant ones at our disposal are $D_\theta,D_{\bar{\theta}}$, which by eq. (2.8) rises the weights by $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$. Hence, the easiest way to set up an action is from a complex $(0,0)$ superfield $\mathbb{X}^\mu\left(z,\bar{z},\theta,\bar{\theta}\right)$ using the derivatives $D_\theta,D_{\bar{\theta}}$,

$$S=-\int\limits_{\Sigma}\left[\mathrm{d}z\,,\mathrm{d}\bar{z}\,|\,\mathrm{d}\theta\,,\mathrm{d}\bar{\theta}\right]D_\theta\mathbb{X}^\mu D_{\bar{\theta}}\mathbb{X}_\mu \tag{2.10}$$

This is manifestly superconformally invariant, and, by now doesn't seem a lot similar to eq. (1.1), but, we have a trick up in our sleave, due to the oddness of $\theta,\bar{\theta}$ we can expand $\mathbb{X}^\mu$ in powers of them,

$$\mathbb{X}^\mu\left(z,\bar{z},\theta,\bar{\theta}\right)=X^\mu(z,\bar{z})+\mathrm{i}\theta\psi^\mu(z,\bar{z})+\mathrm{i}\bar{\theta}\tilde{\psi}^\mu(z,\bar{z})+\bar{\theta}\theta F^\mu(z,\bar{z}) \tag{2.11}$$

Here we used our virtue of foresight to name each component of the superfield as the ones in eq. (1.1), this is possible because as we saw, $\theta,\bar{\theta}$ kind of transforms — under global transformations — as would a field of conformal weights $\left(-\frac{1}{2},0\right),\left(0,-\frac{1}{2}\right)$. This forces $\psi^\mu,\tilde{\psi}^\mu$ to transforms as $\left(\frac{1}{2},0\right),\left(0,\frac{1}{2}\right)$ — as they do naturally in SST — to keep every term in the expansion a scalar. The F^μ field doesn't have a counterpart in SST because it is a auxiliary field. We shown that SRS makes the superconformal/super Virasoro algebra/redundancy off-shell, this is partially the role of the F^μ, despite having a trivial equation of motion — which we'll show — it has a non-trivial transformation under the superconformal transformations. Hence, when integrated out, we get exactly the SST action, but at the price of losing one of the fields, turning the superconformal redundancy valid only on-shell. Let us obtain the SST action from our try (2.10),

Lemma 2.6. *With the identification* (2.11),

$$S = -\int_\Sigma [\mathrm{d}z\,,\mathrm{d}\bar{z}\,|\,\mathrm{d}\theta\,,\mathrm{d}\bar{\theta}]\, D_\theta \mathbb{X}^\mu D_{\bar{\theta}} \mathbb{X}_\mu = \int_{\Sigma_{\text{red}}} \mathrm{d}z\,\mathrm{d}\bar{z}\left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + F^\mu F_\mu\right) \tag{2.12}$$

Proof. The idea here is simple, as stated in lemma 2.5, the integration measure will pick up just terms proportional to $\bar{\theta}\theta$, hence, we just need to worry about them, we start computing what is $D_\theta \mathbb{X}^\mu$,

$$\begin{aligned} D_\theta \mathbb{X}^\mu &= (\partial_\theta + \theta\partial_z)\Big(X^\mu + \mathrm{i}\theta\psi^\mu + \mathrm{i}\bar{\theta}\tilde{\psi}^\mu + \bar{\theta}\theta F^\mu\Big) \\ D_\theta \mathbb{X}^\mu &= \mathrm{i}\psi^\mu - \bar{\theta}F^\mu + \theta\partial\Big(X^\mu + \mathrm{i}\bar{\theta}\tilde{\psi}^\mu\Big) \end{aligned}$$

analogously,

$$\begin{aligned} D_{\bar{\theta}} \mathbb{X}^\mu &= \big(\partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}\big)\Big(X^\mu + \mathrm{i}\theta\psi^\mu + \mathrm{i}\bar{\theta}\tilde{\psi}^\mu + \bar{\theta}\theta F^\mu\Big) \\ D_{\bar{\theta}} \mathbb{X}^\mu &= \mathrm{i}\tilde{\psi}^\mu + \theta F^\mu + \bar{\theta}\bar{\partial}(X^\mu + \mathrm{i}\theta\psi^\mu) \end{aligned}$$

so that,

$$\begin{aligned} D_\theta \mathbb{X}^\mu D_{\bar{\theta}} \mathbb{X}_\mu &= \Big(\mathrm{i}\psi_\mu - \bar{\theta}F_\mu + \theta\partial X_\mu + \mathrm{i}\theta\bar{\theta}\partial\tilde{\psi}_\mu\Big)\Big(\mathrm{i}\tilde{\psi}_\mu + \theta F_\mu + \bar{\theta}\bar{\partial}X_\mu + \mathrm{i}\bar{\theta}\theta\bar{\partial}\psi_\mu\Big) \\ D_\theta \mathbb{X}^\mu D_{\bar{\theta}} \mathbb{X}_\mu &= -\bar{\theta}\theta\psi^\mu\bar{\partial}\psi_\mu - \bar{\theta}\theta F^\mu F_\mu - \bar{\theta}\theta\partial X^\mu\bar{\partial}X_\mu - \bar{\theta}\theta\tilde{\psi}^\mu\partial\tilde{\psi}_\mu + \text{non } \bar{\theta}\theta \text{ proportional terms} \end{aligned}$$

from which we can readily integrate keeping only the $\bar{\theta}\theta$ terms,

$$S = -\int_\Sigma [\mathrm{d}z\,,\mathrm{d}\bar{z}\,|\,\mathrm{d}\theta\,,\mathrm{d}\bar{\theta}]\, D_\theta \mathbb{X}^\mu D_{\bar{\theta}} \mathbb{X}_\mu = \int_{\Sigma_{\text{red}}} \mathrm{d}z\,\mathrm{d}\bar{z}\left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + F^\mu F_\mu\right)$$

■

This last result makes the connection between theories formulated in SRS, and SST natural. It's clear from eq. (2.12) that the equation of motion for F^μ is just $F^\mu = 0$, hence it can be trivially integrated out, giving exactly eq. (1.1). One might worry about the ghosts, but, these can be also naturally incorporated in the superfield formalism, we won't do this here. What is interesting to see is what are the SUSY transformations, eq. (1.2), in this superfield formalism. For this is necessary to remember that the SUSY generator is $G_{-\frac{1}{2}}$ — at least for the NS sector —, which has an action as,

$$\begin{aligned} \delta_\epsilon \mathbb{X}^\mu &= \Big(\epsilon G_{-\frac{1}{2}} + \epsilon^* \bar{G}_{-\frac{1}{2}}\Big)\mathbb{X}^\mu = \big(\epsilon(\partial_\theta - \theta\partial_z) + \epsilon^*\big(\partial_{\bar{\theta}} - \bar{\theta}\partial_{\bar{z}}\big)\big)\Big(X^\mu + \mathrm{i}\theta\psi^\mu + \mathrm{i}\bar{\theta}\tilde{\psi}^\mu + \bar{\theta}\theta F^\mu\Big) \\ \delta_\epsilon \mathbb{X}^\mu &= \epsilon\big(\mathrm{i}\psi^\mu - \bar{\theta}F^\mu\big) - \epsilon\theta\partial\Big(X^\mu + \mathrm{i}\bar{\theta}\tilde{\psi}^\mu\Big) + \epsilon^*\Big(\mathrm{i}\tilde{\psi}^\mu + \theta F^\mu\Big) - \epsilon^*\bar{\theta}\bar{\partial}(X^\mu + \mathrm{i}\theta\psi^\mu) \end{aligned}$$

If we set up $\delta_\epsilon \mathbb{X}^\mu = \delta_\epsilon X^\mu + \mathrm{i}\theta\delta_\epsilon\psi^\mu + \mathrm{i}\bar{\theta}\delta_\epsilon\tilde{\psi}^\mu + \bar{\theta}\theta\delta_\epsilon F^\mu$, then we can do the matching of the last expression to give the following transformation laws,

$$\begin{cases} \delta_\epsilon X^\mu & = \mathrm{i}\epsilon\psi^\mu + \mathrm{i}\epsilon^*\tilde{\psi}^\mu \\ \delta_\epsilon \psi^\mu & = -\mathrm{i}\epsilon\partial X^\mu + \mathrm{i}\epsilon^* F^\mu \\ \delta_\epsilon \tilde{\psi}^\mu & = -\mathrm{i}\epsilon^*\bar{\partial}X^\mu - \mathrm{i}\epsilon F^\mu \\ \delta_\epsilon F^\mu & = -\mathrm{i}\epsilon^*\bar{\partial}\psi^\mu + \mathrm{i}\epsilon\partial\tilde{\psi}^\mu \end{cases} \Rightarrow \text{on-shell } F^\mu = 0 \Rightarrow \begin{cases} \delta_\epsilon X^\mu & = \mathrm{i}\epsilon\psi^\mu + \mathrm{i}\epsilon^*\tilde{\psi}^\mu \\ \delta_\epsilon \psi^\mu & = -\mathrm{i}\epsilon\partial X^\mu \\ \delta_\epsilon \tilde{\psi}^\mu & = -\mathrm{i}\epsilon^*\bar{\partial}X^\mu \end{cases}$$

Which is exactly eq. (1.2), with just extra factors of $-\mathrm{i}$ due to conventions.

We tried in this section to show that SRS can provide a natural environment for superconformal theories, there are a lot of ways we could continue our discussion, we choose here to close with a few remarks about vertex operators, insertions and punctures.

3 Punctures in SRS

We will dwell with insertions of vertex operators in this last section, mainly because they're the arguments of a correlation function related to scattering, and also due to there being a slight sparkle of insight about why the R sector super Virasoro algebra don't preserve the structure $\mathcal{D}$ of the SRS. First, let us recall what we know of vertex operators insertions in BST, naively, given a vertex operator $V(z)$ we *insert* it at a point $z = z_0$, this is also called a puncture because is as if the chosen point $z = z_0$ was removed from the manifold — weather we should just *insert* V at a chosen point $z = z_0$, or we integrate V over the whole manifold is a matter of moduli space, we will not linger much about the moduli space of SRS, but still is a needed issue to deal with perturbative SST —, but remember, the vertex operator cannot be anything, it has to be physical, that is, it has to satisfy the physical condition

$$Q_{\mathrm{BRST}}V(z_0) = 0$$

this imposes various constrains — in particular with respect to the ghost number —, the one we are particularly interested in is,

$$L_n V(z_0) = 0, \quad n \geq 0 \tag{3.1}$$

which comes from $Q_{\mathrm{BRST}} \sim \sum_n c_{-n}L_n + \cdots$, despite the constrains imposed in the form of the vertex operator, there is a kind of overlooked constraint imposed into the insertion point z_0. The constraint is: the point z_0 has to be preserved by the BRST transformations, in particular, by eq. (3.1), the point z_0 has to be preserved by the actions of L_n. For the BST this is a trivial constraint, as the differential form of L_n expanded at $z = z_0$ is $L_n = -(z - z_0)^{n+1}\partial_z$, which trivially preserve the point $z = z_0$ for $n \geq 0$.

If we would try to naively extend this definition to SRS, we get that we should insert a vertex operator $V(z|\theta)$ at a chosen point $z_0|\theta_0$, we shouldn't forget that the vertex operator is subjected to the physicality conditions,

$$Q_{\mathrm{BRST}}V(z_0|\theta_0) = 0 \tag{3.2}$$

here already we have to make some assumptions. To open the BRST operator in terms of the modes we have to say weather we're in the NS sector, or in the R sector. Let's start with our more familiar NS sector:

3.1 NS sector

The physicality condition, eq. (3.2), open in the NS sector gives the following constraint, neglecting the ghost ones,

$$\begin{aligned} L_n V(z_0|\theta_0) &= 0, \quad n \geq 0 \\ G_r V(z_0|\theta_0) &= 0, \quad r \geq \frac{1}{2} \end{aligned}$$

again, those non-trivially restricts the form of the vertex operator, but, also, they impose conditions on the insertion point, as was shown in the BST case. To better understand what those conditions on the insertion point are let's go to the differential form of those operators, eq. (2.4), which we rewrite here not expanded around $z = 0, \theta = 0$, but around z_0, θ_0,

$$\begin{aligned} L_n &= -(z - z_0)^{n+1}\partial_z - \frac{(n+1)}{2}(z - z_0)^n(\theta - \theta_0)\partial_\theta \\ G_r &= (z - z_0)^{r+\frac{1}{2}}(\partial_\theta - (\theta - \theta_0)\partial_z) \end{aligned}$$

It is clear that both these operators, for $n \geq 0, r \geq \frac{1}{2}$, when evaluated at the point z_0, θ_0 are identically zero, hence, they preserve the insertion point. In other words, for the NS sector it actually makes sense to *insert* a vertex operator at a point of the SRS. Another point about the NS sector that is elucidative is the boundary conditions phrased in the superfield formalism. The usual way they are imposed is in the complex cylinder, by $\psi^\mu(w + 2\pi) = -\psi^\mu(w)$, but notice, from eq. (2.11) there is a nice description of the NS boundary condition as,

$$\mathbb{X}^\mu\left(z\mathrm{e}^{2\pi\mathrm{i}}, \bar{z}\mathrm{e}^{-2\pi\mathrm{i}}, -\theta, -\bar{\theta}\right) = \mathbb{X}^\mu\left(z, \bar{z}, \theta, \bar{\theta}\right)$$

This is kind of suggestive for us to go on with our identification $z \sim z\mathrm{e}^{2\pi\mathrm{i}}$, and also identify $\theta \sim -\theta$. Actually, this seems to be the case, as for the case of zero genus the SRS is $\mathbb{CP}^{1|1}$.

3.2 R sector

We know by our previous experience that something will go wrong with R fermions, let's start by supposing a vertex operator is inserted at a fixed point $z_0|\theta_0$ and analyzing the physicality condition, eq. (3.2),

$$\begin{aligned} L_n V(z_0|\theta_0) &= 0, \quad n \geq 0 \\ G_r V(z_0|\theta_0) &= 0, \quad r \geq 0 \end{aligned}$$

as we saw before, the super Virasoro generators in the R sector are different from the NS sector. Let's rewrite them here, eq. (2.6), but expanded around the point $z_0|\theta_0$,

$$\begin{aligned} L_n &= -(z - z_0)^{n+1}\partial_z - \frac{n}{2}(z - z_0)^n(\theta - \theta_0)\partial_\theta \\ G_r &= (z - z_0)^r(\partial_\theta - (\theta - \theta_0)(z - z_0)\partial_z) \end{aligned}$$

for $L_n, G_r n \geq 0, r \geq 1$ acting on the point $z_0|\theta_0$ there is no problem, they're identically zero, hence they preserve the insertion point. The only problematically generator is G_0, which when evaluated at the insertion point gives,

$$G_0\Big|_{z_0|\theta_0} = \partial_\theta$$

This is not zero, neither preserves the insertion point, it generate a translation in the odd variable, $z_0|\theta_0 \to z_0|\theta_0 + \eta$. Hence, it is not possible for an insertion of a R fermion to be associated with a given marked point $z_0|\theta_0$ on the SRS. In fact, by this argument we get that an insertion of a R sector vertex operator has to be associated with a divisor, that is, a whole $0|1$ dimensional submanifold defined by the choice of a bosonic coordinate $z = z_0$. That is equivalent, in some sense, to say that a puncture of a R fermion in the SRS is associated with a singularity of the manifold at that point. This is not the correct statement. We can use our discoveries about the particularities of the R fermions to give a more precise meaning of what we mean by a singularity in the manifold. We've seen before that the generators of the super Virasoro algebra in the R sector, eq. (2.6), cannot possibly preserve the structure the structure $\mathcal{D}$, lemmas 2.2 and 2.3, but also, during the proof of lemma 2.1, in particular at eq. (2.2), we saw that the only way to have $[\mathcal{D}, \mathcal{D}] \notin \mathcal{D}$ is to have $[\mathcal{D}, \mathcal{D}] = 0$ at some point. This also can be interpreted, in some sense, as being a singularity of the complex structure $\mathcal{D}$. All the hints points that the singularity happening in the degeneracy of the $0|1$ dimensional submanifold when a vertex operator is inserted, is the same happening when $[\mathcal{D}, \mathcal{D}] = 0$. We will now try to reconcile these two aspects.

By the proof of lemma 2.1, in particular with eq. (2.2), we get that if at some point $z = z_0$ we have $\left[D_{U_{z_0}}, D_{U_{z_0}}\right] = 0$, then $b_1 \propto z - z_0$, and also exists a coordinate basis $z|\theta$ such that,

$$D_\theta^* =: D_{U_{z_0}} = \partial_\theta + \theta(z - z_0)\partial_z$$

Notice, if we sit at an open set located away from $z = z_0$, it's always possible to make the coordinate change $\hat{z} = \ln(z - z_0)$, this coordinate change induces the following,

$$\begin{aligned} D_\theta^* &= \partial_\theta + \theta(z - z_0)\partial_z \\ D_\theta^* &= \partial_\theta + \theta(z - z_0)\partial_z \hat{z} \partial_{\hat{z}} = \partial_\theta + \theta\partial_{\hat{z}} = D_\theta \end{aligned}$$

Hence, away from the degeneracy point $z = z_0$, the lemma 2.2 still holds for the complex structure $D_\theta^* = \partial_\theta + \theta(z - z_0)\partial_z$, that is, the superconformal symmetries induced by D_θ^* away from $z = z_0$ are exactly the ones given by the NS sector super Virasoro algebra. The question is, what happens at $z = z_0$?

Lemma 3.1. *Given a degenerated conformal structure $D_\theta^* = \partial_\theta + \theta(z - z_0)\partial_z$, the set of all vector fields W that generate superconformal transformations $[W, D_\theta^*] \in \mathcal{D}$, can be decomposed in a basis of even and odd vector fields such,* *Proof.* We'll proceed similar to lemma 2.2, $W = a\partial_\theta + b\partial_z$,

$$[W, D_\theta^*] = [a\partial_\theta + b\partial_z, D_\theta^*] = a[\partial_\theta, D_\theta^*] \mp D_\theta^* a\partial_\theta + b[\partial_z, D_\theta^*] \mp D_\theta^* b\partial_z$$
$$[W, D_\theta^*] = a(z - z_0)\partial_z \mp D_\theta^* a\partial_\theta + b\theta\partial_z \mp D_\theta^* b\partial_z$$
$$[W, D_\theta^*] = \mp D_\theta^* a\partial_\theta \mp (D_\theta^* b \mp b\theta \mp a(z - z_0))\partial_z$$

Here the condition $[W, D_\theta^*] \propto D_\theta^*$ is,

$$D_\theta^* b \mp b\theta \mp a(z - z_0) = D_\theta^* a\theta(z - z_0)$$

∎

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