SCALAR PROXY

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1. INTRODUCTION

We will work most with the scalar proxy given by the lagrangian,

$$(1.1) \qquad \mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2M^2}\Box\phi\Box\phi - \frac{\kappa}{2}\Box\phi\phi^2$$

The idea here is reintegrate the higher derivative term, in order to obtain a lower derivative term, but in terms of additional fields. This is easily done by,

$$(1.2) \qquad \mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \Box\phi\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box\phi\phi^2$$

The new lagrangian has mixed propagator terms, to diagonalize it is also easy, we just open in terms of $\phi = h - \eta$,

$$(1.3) \qquad \mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 + \eta\Box(h-\eta) + \frac{M^2}{2}\eta^2$$

$$(1.4) \qquad \mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 - \partial_\mu\eta\partial^\mu(h-\eta) + \frac{M^2}{2}\eta^2$$

$$(1.5) \qquad \mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2$$

The Feynman rules are easily red as,

- $h \dashdash h \quad = \frac{1}{i}\frac{1}{p^2}$

- $\eta \dashdash \eta \quad = -\frac{1}{i}\frac{1}{p^2+M^2}$

- $= i\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$

- $= -i\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$

- $= i\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$

- $= -i\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$

Which can also be seen directly from the Feynman rules of the ϕ field,

- $\phi \dashdash \phi \quad = \frac{1}{i}\frac{1}{p^2 + \frac{p^4}{M^2}}$

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$$= i\kappa\left(p_1^2 + p_2^2 + p_3^2\right) = i\kappa(p_1 + p_2 + p_3)^2 - 2i\kappa(p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1) = -i\kappa(\langle12\rangle[12] + \langle23\rangle[23] + \langle31\rangle[31])$$

So that the four point amplitude can be computed by,

(1.6) $$= \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle12\rangle[12] + \langle2P\rangle[2P] + \langle P1\rangle[P1])(\langle34\rangle[34] - \langle4P\rangle[4P] - \langle P3\rangle[P3])$$

(1.7) $$= \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle12\rangle[12] - \langle P2\rangle[2P] - \langle P1\rangle[1P])(\langle34\rangle[34] + \langle P4\rangle[4P] + \langle P3\rangle[3P])$$

(1.8) $$= \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle12\rangle[12] + \langle P|1+2|P])(\langle34\rangle[34] - \langle P|3+4|P])$$

(1.9) $$= \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle12\rangle[12] - \langle P|P|P])(\langle34\rangle[34] - \langle P|P|P])$$

(1.10) $$= \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}\left(\langle12\rangle[12] - 2P^2\right)\left(\langle34\rangle[34] - 2P^2\right)$$

(1.11) $$= -i\frac{(\kappa M)^2}{s(M^2 - s)}(\langle12\rangle[12] + 2s)(\langle34\rangle[34] + 2s)$$

It's trivial to read the t and u channels from this expression,

(1.12)

$$= -i\frac{(\kappa M)^2}{t(M^2 - t)}(\langle23\rangle[23] + 2t)(\langle41\rangle[41] + 2t)$$

(1.13)

$$= -i\frac{(\kappa M)^2}{u(M^2 - u)}(\langle24\rangle[24] + 2u)(\langle31\rangle[31] + 2u)$$

So that the full 4–point amplitude is,

(1.14)

$$= -i\frac{(\kappa M)^2}{stu(M^2 - s)(M^2 - t)(M^2 - u)}\left[(\langle12\rangle[12] + 2s)(\langle34\rangle[34] + 2s)tu\left(M^2 - t\right)\left(M^2 - u\right) + (\langle23\rangle[23] + 2t)(\langle41\rangle[41] + 2t\right.$$

Let us specialize when $1, 2$ are massless and $3, 4$ are massive, then,

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(1.15)

$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\Big[s\big(-s+2M^2+2s\big)tu\big(M^2-t\big)\big(M^2-u\big)+\big(-t+M^2+2t\big)\big(-t+M^2+2t\big)su\big(M^2$$

(1.16)

$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\Big[stu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+su\big(M^2+t\big)\big(M^2+t\big)\big(M^2-s\big)\big(M^2-u\big)+st$$

(1.17)

$$= -\mathrm{i}\frac{(\kappa M)^2}{tu(M^2-s)(M^2-t)(M^2-u)}\Big[tu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+u\big(M^2+t\big)^2\big(M^2-s\big)\big(M^2-u\big)+t\big(M^2+u\big)^2\big($$

2. Conformal Toy Model

Consider the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\Box\phi\Box\phi - \frac{g}{2}\phi^2\Box\phi - \frac{g^2}{8}\phi^4 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

Notice the form of the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\left(\Box\phi + \frac{g}{2}\phi^2\right)^2 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

It possesses the Feynman rules,

- $\phi \dashrightarrow \phi = \frac{1}{\mathrm{i}}\frac{1}{m^2 p^2 + p^4}$

- $= \mathrm{i}g\big(p_1^2 + p_2^2 + p_3^2 + m^2\big)$

- $= -3\mathrm{i}g^2$

Let's compute the self energy,

(2.1)
$$\mathrm{i}\Pi\big(p^2\big) = $$

$$(2.2) \qquad i\Pi^{(1)} = -\frac{3}{2}ig^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{i} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2}$$

$$(2.3) \qquad i\Pi^{(1)} = -\frac{3}{2}g^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2}$$

$$(2.4) \qquad i\Pi^{(1)} = -\frac{3}{2}g^2 \frac{i}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \left(m^2\right)^{\frac{D}{2}-2} \frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(1)}$$

$$(2.5) \qquad i\Pi^{(1)} = -\frac{3}{2}ig^2 \frac{\left(m^2\right)^{-\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-\epsilon)}$$

$$(2.6) \qquad i\Pi^{(2)} = \frac{1}{2}(ig)^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{i^2} \frac{1}{\ell^2(\ell+p)^2} \frac{\left(m^2 + \ell^2 + p^2 + (\ell+p)^2\right)^2}{\ell^2 + m^2} \frac{1}{(\ell+p)^2 + m^2}$$

For the mass renormalization we can take $p=0$,

$$(2.7) \qquad i\Pi^{(2)} = \frac{1}{2}g^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{\left(m^2 + 2\ell^2\right)^2}{\ell^4(\ell^2 + m^2)^2}$$

Let's compute the four point amplitude for this theory,

$$(2.8) \qquad \begin{aligned}&\phi_2 \quad \xrightarrow{P} \quad \phi_3 \\ &\phi_1 \qquad\qquad \phi_4\end{aligned} = (ig)^2 \frac{\left(p_1^2 + p_2^2 + (p_1+p_2)^2 + m^2\right)\left(p_3^2 + p_4^2 + (p_3+p_4)^2 + m^2\right)}{i(p_1+p_2)^2\left((p_1+p_2)^2 + m^2\right)}$$

First let's consider all legs massless,

$$(2.9) \qquad \begin{aligned}&\phi_2 \quad \xrightarrow{P} \quad \phi_3 \\ &\phi_1 \qquad\qquad \phi_4\end{aligned} = ig^2 \frac{\left(-s+m^2\right)\left(-s+m^2\right)}{(-s)(-s+m^2)} = -ig^2 \frac{\left(-s+m^2\right)}{s}$$

So,

$$(2.10) \qquad \begin{aligned}&\phi_2 \qquad\qquad \phi_3 \\ &\phi_1 \qquad\qquad \phi_4\end{aligned} = -ig^2 \frac{\left(-s+m^2\right)}{s} - ig^2 \frac{\left(-t+m^2\right)}{t} - ig^2 \frac{\left(-u+m^2\right)}{u} - 3ig^2$$

$$(2.11) \qquad \begin{aligned}&\phi_2 \qquad\qquad \phi_3 \\ &\phi_1 \qquad\qquad \phi_4\end{aligned} = -ig^2 \frac{\left(-s+m^2\right)}{s} - ig^2 \frac{\left(-t+m^2\right)}{t} - ig^2 \frac{\left(-u+m^2\right)}{u} - ig^2\frac{s}{s} - ig^2\frac{t}{t} - ig^2\frac{u}{u}$$

$$(2.12) \qquad \begin{aligned}&\phi_2 \qquad\qquad \phi_3 \\ &\phi_1 \qquad\qquad \phi_4\end{aligned} = -ig^2 m^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right) = ig^2 m^2 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 14\rangle[14]} + \frac{1}{\langle 13\rangle[13]}\right)$$

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Para uma perna massiva, ϕ_4,

(2.13)

$$= (\mathrm{i}g)^2 \frac{(-s+m^2)(-s)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2$$

So,

(2.14)

$$= \mathrm{i}g^2 + \mathrm{i}g^2 + \mathrm{i}g^2 - 3\mathrm{i}g^2 = 0$$

Para duas pernas massivas, $\phi_{3,4}$,

(2.15)

$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2\frac{s+m^2}{s}$$

(2.16)

$$= (\mathrm{i}g)^2 \frac{(-t)(-t)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2\frac{t}{-t+m^2}$$

(2.17)

$$= -\mathrm{i}g^2\frac{u}{-u+m^2}$$

So,

(2.18)

$$= \mathrm{i}g^2\frac{s+m^2}{s} - \mathrm{i}g^2\frac{t}{-t+m^2} - \mathrm{i}g^2\frac{u}{-u+m^2} - 3\mathrm{i}g^2$$

(2.19)

$$= \mathrm{i}g^2\frac{s+m^2}{s} - \mathrm{i}g^2\frac{t}{-t+m^2} - \mathrm{i}g^2\frac{u}{-u+m^2} - \mathrm{i}g^2\frac{s}{s} - \mathrm{i}g^2\frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2\frac{-u+m^2}{-u+m^2}$$

(2.20)

$$= -\mathrm{i}g^2m^2\left(-\frac{1}{s} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2}\right) = -\mathrm{i}g^2m^2\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 14\rangle[14]} + \frac{1}{\langle 13\rangle[13]}\right)$$

Para uma perna sem massa ϕ_1,

(2.21)

$$= (ig)^2 \frac{(-s)(-s-m^2)}{i(-s)(-s+m^2)} = -ig^2 \frac{s+m^2}{-s+m^2}$$

(2.22)

$$= (ig)^2 \frac{(-t)(-t-m^2)}{i(-t)(-t+m^2)} = -ig^2 \frac{t+m^2}{-t+m^2}$$

(2.23)

$$= (ig)^2 \frac{(-u)(-u-m^2)}{i(-u)(-u+m^2)} = -ig^2 \frac{u+m^2}{-u+m^2}$$

(2.24)

So,

(2.25)

$$= -ig^2 \frac{s+m^2}{-s+m^2} - ig^2 \frac{t+m^2}{-t+m^2} - ig^2 \frac{u+m^2}{-u+m^2} - 3ig^2$$

(2.26)

$$= -ig^2 \frac{s+m^2}{-s+m^2} - ig^2 \frac{t+m^2}{-t+m^2} - ig^2 \frac{u+m^2}{-u+m^2} - ig^2 \frac{-s+m^2}{-s+m^2} - ig^2 \frac{-t+m^2}{-t+m^2} - ig^2 \frac{-u+m^2}{-u+m^2}$$

(2.27)

$$= -ig^2 m^2 \left(\frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -ig^2 m^2 \left(\frac{1}{\langle 12 \rangle [12]} + \frac{1}{\langle 14 \rangle [14]} + \frac{1}{\langle 13 \rangle [13]} \right)$$

Cut comparison, only massless legs

(2.28)

$$= \frac{ig^2 m^2 (igm^2)^2}{(im^2)^3} \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{\langle 1l \rangle [1l]} - \frac{1}{\langle 2l \rangle [2l]} \right)$$

to solve for the cuts, $l^2 = (l+3+4)^2 = (l+4)^2 = 0$,

(2.29)
$$l^2 = 0 \Rightarrow l = -|l\rangle [l|$$

(2.30)
$$0 = (l+4)^2 = \langle l4 \rangle [l4] = 0 \Rightarrow |l] = |4]$$

(2.31)
$$0 = (l+3+4)^2 = \langle lP_{34} \rangle [lP_{34}] + (3+4)^2 = \langle l|3+4|l] + \langle 34 \rangle [34] = \langle l|3+4|4] + \langle 34 \rangle [34]$$

(2.32)
$$\langle 43 \rangle [34] = -\langle l3 \rangle [34] \Rightarrow |l\rangle = -|4\rangle + z|3\rangle$$

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(2.33)
$$l = -(-|4\rangle + z|3\rangle)[4|$$

The cuts are solved by this. Hence,

$$= g^4 \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right)$$

$$= g^4 \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{(-\langle 14\rangle + z\langle 13\rangle)[14]} - \frac{1}{(-\langle 24\rangle + z\langle 23\rangle)[24]} \right)$$

Now for internal massive lines,

(2.34)

$$= \frac{-ig^2m^2\left(-igm^2\right)^2}{\left(-im^2\right)^3} \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right)$$

With the cuts being, $l^2 = (l+3+4)^2 = (l+4)^2 = -m^2$,

(2.35)
$$0 = (l+4)^2 - l^2 = 2l \cdot p_4$$

(2.36)
$$0 = (l+4+3)^2 - l^2 = 2l \cdot (4+3) + (4+3)^2 = 2l \cdot p_3 + (4+3)^2$$

As ansatz, $l = |4\rangle[4| + \alpha|4\rangle[3| + \beta|3\rangle[4|$ satisfy both conditions above. The remaining condition is,

(2.37)
$$l^2 = -m^2$$

(2.38)
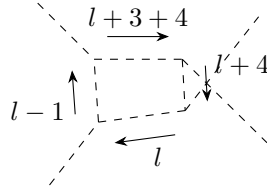$$-\alpha\beta[43]\langle 43\rangle = -m^2 \Rightarrow \alpha = \frac{m^2}{\beta\langle 34\rangle[34]}$$

Setting now $-\beta = z$,

(2.39)
$$l = |4\rangle[4| - \frac{m^2}{z\langle 34\rangle[34]}|4\rangle[3| - z|3\rangle[4|$$

The value of the diagram is,

$$= g^4 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1l\rangle[l1]} + \frac{1}{\langle 2l\rangle[l2]} \right)$$

$$= g^4 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{-\langle 1|l|1]} + \frac{1}{-\langle 2|l|2]} \right)$$

$$= g^4 \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 14\rangle[41] - \frac{m^2}{z\langle 34\rangle[34]}\langle 14\rangle[31] - z\langle 13\rangle[41]} - \frac{1}{\langle 2|l|2]} \right)$$

The explicit cut loop amplitude is,

Triple cut has no improvement, what about a double cut,

(2.40)

$$= \frac{\left(ig^2m^2\right)^2}{\left(im^2\right)^2} \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1\ell\rangle[1\ell]} - \frac{1}{\langle 2\ell\rangle[2\ell]} \right) \left(\frac{1}{\langle 34\rangle[34]} + \frac{1}{\langle 3\ell\rangle[3\ell]} + \frac{1}{\langle 4\ell\rangle[4\ell]} \right)$$

Five point amplitude,
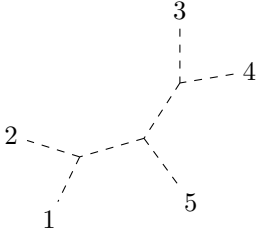
$$= \frac{(\mathrm{i}g)^3 \left(m^2 + p_1^2 + p_2^2 + (p_1+p_2)^2\right)\left(m^2 + p_5^2 + (p_3+p_4)^2 + (p_1+p_2)^2\right)\left(m^2 + p_3^2 + p_4^2 + (p_3+p_4)^2\right)}{i^2(p_1+p_2)^2\left((p_1+p_2)^2 + m^2\right)(p_3+p_4)^2\left((p_3+p_4)^2 + m^2\right)}$$

Let's consider the special case of all massless,
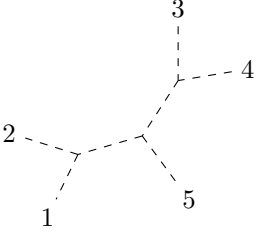
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\left(m^2 + (p_3+p_4)^2 + (p_1+p_2)^2\right)\left(m^2 + (p_3+p_4)^2\right)}{(p_3+p_4)^2\left((p_3+p_4)^2 + m^2\right)} = \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\left(m^2 + (p_3+p_4)^2 + (p_1+p_2)^2\right)}{(p_3+p_4)^2}$$

Combining this graph with,

$$= \frac{-\mathrm{i}3g^2\mathrm{i}g\left(m^2 + (p_1+p_2)^2\right)}{\mathrm{i}(p_1+p_2)^2\left((p_1+p_2)^2 + m^2\right)} = \frac{-3\mathrm{i}g^3}{(p_1+p_2)^2}$$

We get,
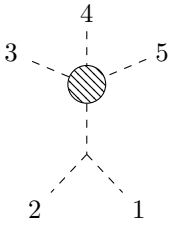
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\left[\frac{\left(m^2 + (p_3+p_4)^2 + (p_1+p_2)^2\right)}{(p_3+p_4)^2} - \frac{(p_3+p_4)^2}{(p_3+p_4)^2} + \frac{\left(m^2 + (p_3+p_5)^2 + (p_1+p_2)^2\right)}{(p_3+p_5)^2} - \frac{(p_3+p_5)^2}{(p_3+p_5)^2} + \frac{\left(m^2 + (p_5\right.}{}\right.$$

$$= \frac{\mathrm{i}g^3\left(m^2 + (p_1+p_2)^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_3+p_5)^2} + \frac{1}{(p_5+p_4)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,

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+ perm

Which will be,

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$$\frac{ig^3}{(p_2 + p_3)^2} \frac{m^2 + (p_2 + p_3)^2 + (p_4 + p_5)^2}{(p_4 + p_5)^2} - \frac{3ig^3}{(p_2 + p_3)^2} - \frac{3ig^3}{(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5)$$

Summing all the contributions we have,

$$= \frac{ig^3 m^2}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} \right] + (2 \leftrightarrow 3, 4, 5)$$

$$+ ig^3 \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_2 + p_4)^2} + \frac{1}{(p_2 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_2)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_2)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_2)^2} \right.$$

$$+ \frac{ig^3 m^2}{(p_2 + p_3)^2 (p_4 + p_5)^2} - \frac{2ig^3}{(p_2 + p_3)^2} - \frac{2ig^3}{(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5)$$

$$= \frac{ig^3 m^2}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} \right] + (2 \leftrightarrow 3, 4, 5)$$

$$+ \frac{ig^3 m^2}{(p_2 + p_3)^2 (p_4 + p_5)^2} + \frac{ig^3 m^2}{(p_2 + p_4)^2 (p_3 + p_5)^2} + \frac{ig^3 m^2}{(p_2 + p_5)^2 (p_4 + p_3)^2}$$

By residue, any amplitude with just one massive external on-shell leg is zero. For two massive external on-shell legs, let's take as massive $1, 2$,
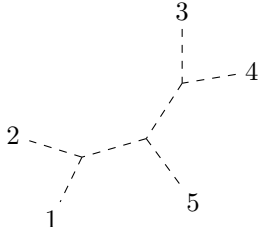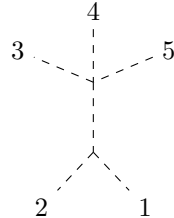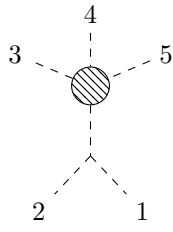

$$= ig^3 \frac{\left((p_1 + p_2)^2 - m^2 \right)}{(p_1 + p_2)^2 \left(m^2 + (p_1 + p_2)^2 \right)} \frac{\left(m^2 + (p_1 + p_2)^2 + (p_3 + p_4)^2 \right)}{(p_3 + p_4)^2}$$

Combining this graph with,

$$= -3ig^3 \frac{(p_1 + p_2)^2 - m^2}{(p_1 + p_2)^2 \left(m^2 + (p_1 + p_2)^2 \right)}$$

so,
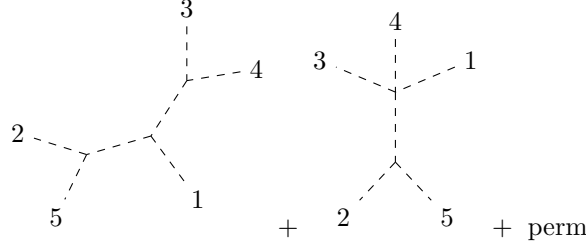
$$= ig^3 \frac{\left((p_1 + p_2)^2 - m^2 \right)}{(p_1 + p_2)^2 \left(m^2 + (p_1 + p_2)^2 \right)} \left[\frac{\left(m^2 + (p_1 + p_2)^2 + (p_3 + p_4)^2 \right)}{(p_3 + p_4)^2} - \frac{(p_3 + p_4)^2}{(p_3 + p_4)^2} + (5 \leftrightarrow 3, 4) \right]$$

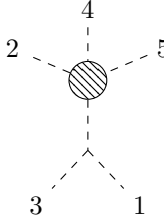$$= ig^3 \frac{\left((p_1 + p_2)^2 - m^2 \right)}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right]$$

The other contributions are,

$$= \frac{ig^3(p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{(p_4+p_5)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,

which are,

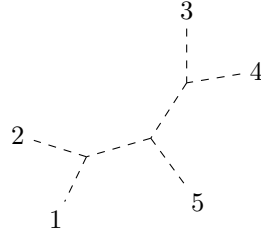$$= ig^3\frac{(p_2+p_3)^2+(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2}-\frac{3ig^3}{m^2+(p_2+p_3)^2}-\frac{3ig^3}{(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

So, summing all the contributions,

$$= ig^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

$$+\frac{ig^3(p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$+ig^3\frac{(p_2+p_3)^2+(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2}-\frac{3ig^3}{m^2+(p_2+p_3)^2}-\frac{3ig^3}{(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

$$= -ig^3\frac{m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

$$+ig^3\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

$$+ig^3\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$-ig^3\frac{m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$+ig^3\frac{-m^2+2p_2\cdot p_3+(p_4+p_5)^2}{(2p_2\cdot p_3)(p_4+p_5)^2}-\frac{3ig^3}{2p_2\cdot p_3}-\frac{3ig^3}{(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

$$= -\frac{ig^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

$$+ig^3\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

$$+ig^3\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}+\frac{1}{2p_2\cdot p_3}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_3+p_5)^2}+\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_3}+\frac{1}{(p_4+p_3)^2}\right]$$

$$-\frac{ig^3m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

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$$-\frac{ig^3m^2}{(2p_2\cdot p_3)(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

$$-\frac{2ig^3}{2p_2\cdot p_3}-\frac{2ig^3}{(p_4+p_5)^2}-\frac{2ig^3}{2p_2\cdot p_4}-\frac{2ig^3}{(p_3+p_5)^2}-\frac{2ig^3}{2p_2\cdot p_5}-\frac{2ig^3}{(p_4+p_3)^2}$$

$$=-\frac{ig^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

$$-\frac{ig^3m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$
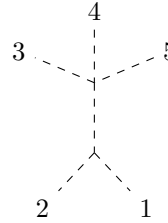
$$-\frac{ig^3m^2}{(2p_2\cdot p_3)(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

Is almost the same of the all massless, but with a different denominator in the $1,2$ channel. Now with three massive legs, being $3,4,5$,
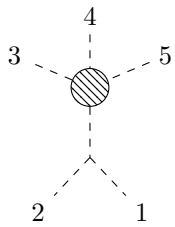
$$=ig^3\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_1+p_2)^2(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}$$

With,

$$=\frac{-i3g^2ig\left(m^2+(p_1+p_2)^2\right)}{i(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)}=\frac{-3ig^3}{(p_1+p_2)^2}$$

So,

$$=ig^3\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_1+p_2)^2(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}-\frac{3ig^3}{(p_1+p_2)^2}$$

$$=\frac{ig^3}{(p_1+p_2)^2}\left[\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}-\frac{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}\right]+(5\leftrightarrow 3,4)$$

$$=\frac{ig^3}{(p_1+p_2)^2}\left[\frac{(p_1+p_2)^2\left(-m^2+(p_3+p_4)^2\right)-2m^2(p_3+p_4)^2}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}\right]+(5\leftrightarrow 3,4)$$

The other topology is,
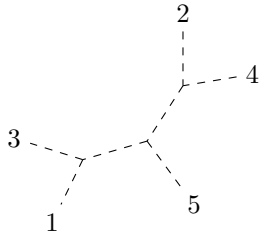
$$= \mathrm{i}g^3 \frac{\left(m^2 + p_3^2 + (p_1+p_3)^2\right)\left(m^2 + p_5^2 + (p_1+p_3)^2 + (p_2+p_4)^2\right)\left(m^2 + p_4^2 + (p_2+p_4)^2\right)}{(p_1+p_3)^2\left(m^2 + (p_1+p_3)^2\right)(p_2+p_4)^2\left(m^2 + (p_2+p_4)^2\right)}$$

$$= \mathrm{i}g^3 \frac{\left((p_1+p_3)^2 + (p_2+p_4)^2\right)}{\left(m^2 + (p_1+p_3)^2\right)\left(m^2 + (p_2+p_4)^2\right)}$$
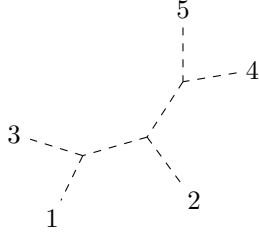
$$= \mathrm{i}g^3 \frac{\left(m^2 + p_3^2 + (p_1+p_3)^2\right)\left(m^2 + (p_1+p_3)^2 + (p_5+p_4)^2\right)\left(m^2 + p_5^2 + p_4^2 + (p_5+p_4)^2\right)}{(p_1+p_3)^2\left(m^2 + (p_1+p_3)^2\right)(p_5+p_4)^2\left(m^2 + (p_5+p_4)^2\right)}$$

$$= \mathrm{i}g^3 \frac{\left(m^2 + (p_1+p_3)^2 + (p_5+p_4)^2\right)\left(-m^2 + (p_5+p_4)^2\right)}{\left(m^2 + (p_1+p_3)^2\right)(p_5+p_4)^2\left(m^2 + (p_5+p_4)^2\right)}$$

Also,

$$= -3\mathrm{i}g^2 g \frac{\left(m^2 + p_3^2 + (p_3+p_1)^2\right)}{(p_3+p_1)^2\left(m^2 + (p_3+p_1)^2\right)}$$

$$= -3\mathrm{i}g^3 \frac{1}{\left(m^2 + (p_3+p_1)^2\right)}$$

So,

$$= \frac{\mathrm{i}g^3}{m^2 + (p_1+p_3)^2}\left[\frac{\left((p_1+p_3)^2 + (p_2+p_4)^2\right) - m^2 - (p_2+p_4)^2}{\left(m^2 + (p_2+p_4)^2\right)} + \frac{\left((p_1+p_3)^2 + (p_2+p_5)^2\right) - m^2 - (p_2+p_5)^2}{\left(m^2 + (p_2+p_5)^2\right)}\right]$$

$$+ \frac{\mathrm{i}g^3}{m^2 + (p_1+p_3)^2}\left[\frac{\left(m^2 + (p_1+p_3)^2 + (p_4+p_5)^2\right)\left(-m^2 + (p_4+p_5)^2\right) - (p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)}{(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)}\right]$$

$$= \frac{\mathrm{i}g^3\left((p_1+p_3)^2 - m^2\right)}{m^2 + (p_1+p_3)^2}\left[\frac{1}{m^2 + (p_2+p_4)^2} + \frac{1}{m^2 + (p_2+p_5)^2}\right]$$

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$$+ \frac{ig^3}{m^2 + (p_1 + p_3)^2}\left[\frac{-m^4 - m^2(p_1 + p_3)^2 + (p_1 + p_3)^2(p_4 + p_5)^2 - m^2(p_4 + p_5)^2}{(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)}\right]$$

$$= \frac{ig^3\left((p_1 + p_3)^2 - m^2\right)}{m^2 + (p_1 + p_3)^2}\left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2}\right] + (3 \leftrightarrow 4, 5)$$

$$- \frac{ig^3 m^2}{(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} + (3 \leftrightarrow 4, 5)$$
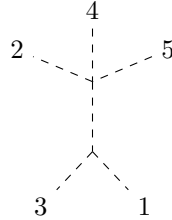
Now, with 1 in the middle,
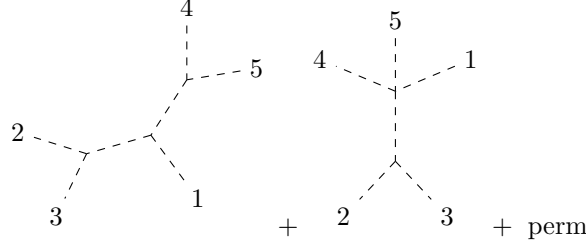
Which is,
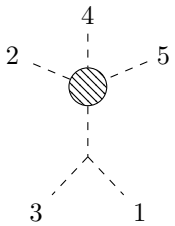
$$= \frac{ig^3\left(m^2 + p_3^2 + (p_2 + p_3)^2\right)\left(m^2 + p_1^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2\right)\left(m^2 + p_5^2 + p_4^2 + (p_5 + p_4)^2\right)}{(p_2 + p_3)^2\left(m^2 + (p_2 + p_3)^2\right)(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3\left(-m^2 + (p_4 + \right.}{(p_4 + p_5)^2\left(m^2 + (p_4\right.}$$

$$= \frac{ig^3\left(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2\right)\left(-m^2 + (p_5 + p_4)^2\right)}{\left(m^2 + (p_2 + p_3)^2\right)(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3\left(-m^2 + (p_4 + p_5)^2\right)}{(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} + (3 \leftrightarrow 4, 5)$$

Summing all the contributions we have,

$$ig^3\frac{\left(-m^2 + (p_5 + p_4)^2\right)}{(p_5 + p_4)^2\left(m^2 + (p_5 + p_4)^2\right)} + (3 \leftrightarrow 4, 5)$$

$$- \frac{2ig^3 m^2}{(p_1 + p_2)^2}\frac{1}{\left(m^2 + (p_5 + p_4)^2\right)} + (3 \leftrightarrow 4, 5)$$

$$+ \frac{ig^3\left((p_1 + p_3)^2 - m^2\right)}{m^2 + (p_1 + p_3)^2}\left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2}\right] + (3 \leftrightarrow 4, 5)$$

$$- \frac{ig^3 m^2}{(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} + (3 \leftrightarrow 4, 5)$$

$$+ \frac{ig^3\left(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2\right)\left(-m^2 + (p_5 + p_4)^2\right)}{\left(m^2 + (p_2 + p_3)^2\right)(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} + (3 \leftrightarrow 4, 5)$$

$$- \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3\left(-m^2 + (p_4 + p_5)^2\right)}{(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} + (3 \leftrightarrow 4, 5)$$

$$- \frac{2ig^3 m^2}{(p_1 + p_2)^2}\frac{1}{\left(m^2 + (p_5 + p_4)^2\right)} + (3 \leftrightarrow 4, 5)$$

$$+ \frac{ig^3\left((p_1 + p_3)^2 - m^2\right)}{m^2 + (p_1 + p_3)^2}\left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2}\right] + (3 \leftrightarrow 4, 5)$$

$$- \frac{ig^3 m^2}{(p_4 + p_5)^2\left(m^2 + (p_4 + p_5)^2\right)} + (3 \leftrightarrow 4, 5)$$

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$$+ \frac{ig^3\left(m^2 + (p_2+p_3)^2 + (p_5+p_4)^2\right)\left(-m^2 + (p_5+p_4)^2\right)}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{3ig^3}{m^2 + (p_2+p_3)^2} + \frac{2ig^3 m^2}{(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} - \frac{2ig^3}{\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{(p_1+p_2)^2} \frac{1}{\left(m^2 + (p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{m^2 + (p_1+p_3)^2}\left[\frac{1}{m^2 + (p_2+p_4)^2} + \frac{1}{m^2 + (p_2+p_5)^2} + \frac{1}{m^2 + (p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)$$

$$+ ig^3\left[\frac{1}{m^2 + (p_2+p_4)^2} + \frac{1}{m^2 + (p_2+p_5)^2} + \frac{1}{m^2 + (p_4+p_5)^2}\right]$$

$$+ ig^3\left[\frac{1}{m^2 + (p_2+p_3)^2} + \frac{1}{m^2 + (p_2+p_5)^2} + \frac{1}{m^2 + (p_3+p_5)^2}\right]$$

$$+ ig^3\left[\frac{1}{m^2 + (p_2+p_4)^2} + \frac{1}{m^2 + (p_2+p_3)^2} + \frac{1}{m^2 + (p_4+p_3)^2}\right]$$

$$+ \frac{ig^3\left(m^2 + (p_2+p_3)^2 + (p_5+p_4)^2\right)\left(-m^2 + (p_5+p_4)^2\right)}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{3ig^3}{m^2 + (p_2+p_3)^2} + \frac{ig^3 m^2}{(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} - \frac{2ig^3}{\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{(p_1+p_2)^2} \frac{1}{\left(m^2 + (p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{m^2 + (p_1+p_3)^2}\left[\frac{1}{m^2 + (p_2+p_4)^2} + \frac{1}{m^2 + (p_2+p_5)^2} + \frac{1}{m^2 + (p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)$$

$$+ \frac{ig^3\left(m^2 + (p_2+p_3)^2 + (p_5+p_4)^2\right)\left(-m^2 + (p_5+p_4)^2\right)}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{ig^3}{m^2 + (p_2+p_3)^2} + \frac{ig^3 m^2}{(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} - \frac{ig^3}{\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{(p_1+p_2)^2} \frac{1}{\left(m^2 + (p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{m^2 + (p_1+p_3)^2}\left[\frac{1}{m^2 + (p_2+p_4)^2} + \frac{1}{m^2 + (p_2+p_5)^2} + \frac{1}{m^2 + (p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)$$

$$+ \frac{ig^3\left(m^2 + (p_2+p_3)^2 + (p_5+p_4)^2\right)\left(-m^2 + (p_5+p_4)^2\right)}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$+ ig^3 \frac{-(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right) + m^2\left(m^2 + (p_2+p_3)^2\right) - (p_4+p_5)^2\left(m^2 + (p_2+p_3)^2\right)}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{(p_1+p_2)^2} \frac{1}{\left(m^2 + (p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{m^2 + (p_1+p_3)^2}\left[\frac{1}{m^2 + (p_2+p_4)^2} + \frac{1}{m^2 + (p_2+p_5)^2} + \frac{1}{m^2 + (p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)$$

$$+ ig^3 \frac{-m^4 + m^2(p_4+p_5)^2 - m^2(p_2+p_3)^2 + (p_2+p_3)^2(p_4+p_5)^2 - m^2(p_5+p_4)^2 + (p_4+p_5)^4}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2 + (p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$+ ig^3 \frac{-(p_4+p_5)^2 m^2 - (p_4+p_5)^4 + m^4 + m^2(p_2+p_3)^2 - (p_4+p_5)^2 m^2 - (p_2+p_3)^2(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{(p_1+p_2)^2} \frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5)$$

$$- \frac{2ig^3 m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2} + \frac{1}{m^2+(p_2+p_5)^2} + \frac{1}{m^2+(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)$$

$$- 2ig^3 m^2 \frac{1}{\left(m^2+(p_2+p_3)^2\right)\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

Now, let's do the cuts, consider a two loop four point amplitude with five cuts,
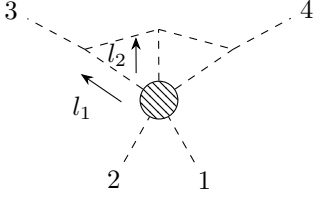
$$= -\frac{g^3}{m^4}[\mathcal{A}_5(1,2,l_{1h},l_{2h},l_{3h}) - \mathcal{A}_5(1,2,l_{1h},l_{2\eta},l_{3\eta}) - \mathcal{A}_5(1,2,l_{1\eta},l_{2h},l_{3\eta}) - \mathcal{A}_5(1,2,l_{1\eta},l_{2\eta},l_{3h}) + 2\mathcal{A}_5(1,2,l_{1\eta}$$

3. CUT SOLUTIONS

Of course in each amplitude we have different cut solutions. Now let us solve them,

3.1. **all massless.** The cut condition is,

$$k_1^2 = k_2^2 = (3-k_1)^2 = (3-k_1-k_2)^2 = (3+4-k_1-k_2)^2 = 0$$

The first and third condition enforces $k_1 = -|k_1]\langle 3|$. But the fourth and fifth conditions enforces $3-k_1-k_2 = n$, with $n \cdot 4 = 0$ & $n^2 = 0$. Lastly, the second condition imposes $(3-k_1-n)^2 = -23 \cdot n + 2k_1 \cdot n = 0$, that is,

$$[3n]\langle n3 \rangle = [k_1 n]\langle n3 \rangle$$

which has two solutions, $|n] = |k_1] - |3]$ & $|n\rangle = z|4\rangle$ or $|n\rangle = |3\rangle$ & $|n] = z|4]$. When working with scalar particles it's better to choose the first solution, as this avoids singularities in denominators such as $(k_1 \cdot k_2)^{-1}$. Hence, the solution we're going to choose is,

$$\begin{cases} k_1 &= -|k_1]\langle 3| \\ k_2 &= -|3]\langle 3| + |k_1]\langle 3| + z(|k_1] - |3])\langle 4| \end{cases}$$

3.2. **massive legs first topology.** Our approach to massive legs is to shift the solution with massless, in order to obtain a well behaved solution in the $m^2 \to 0$ limit. For this topology the cut constrains are,

$$l_1^2 = l_2^2 = (3-l_1)^2 = -m^2 \ \& \ (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = 0$$

The ideia here is to define, $l_i = k_i + \alpha_i q_i$ (no sum), with $q_i^2 = 0$ and $\alpha_i = -m^2(2k_i \cdot q_i)^{-1}$, then, q_i, k_i are not allowed to have any dependence on m^2. The first and second constrains are already satisfied. The third one gives,

$$-23 \cdot l_1 = 0 \to 3 \cdot (k_1 + \alpha_1 q_1) = 0 \to 3 \cdot q_1 = 0$$

As $|q_1\rangle = |3\rangle$ is forbidden, $|q_1] = |3]$. The fourth and fifth constrains imposes,

$$\begin{cases} -n \cdot (\alpha_1 q_1 + \alpha_2 q_2) + \alpha_1 \alpha_2 q_1 \cdot q_2 &= 0 \\ 4 \cdot (\alpha_1 q_1 + \alpha_2 q_2) &= 0 \end{cases}$$

This imposes actually $q_1 \cdot q_2 = 0$, for this to be true we have to options, either $|q_2] = |3]$, or $|q_2\rangle = |q_1\rangle$. If we choose the first, we can shift k_1 by 3 such to make $|q_1\rangle = |4\rangle$, this imposes further $|q_2\rangle = |4\rangle$. Hence, a possible solution is,

$$q_1 = q_2 = -|3]\langle 4|$$

3.3. **massive legs second topology.** The constrains now are slightly different,

$$l_1^2 = (3-l_1)^2 = 0 \ \& \ l_2^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

Which has as solution $q_2 = -|4]\langle 3|$

3.4. massive legs third topology.

Now the constrain is difficult to solve,

$$l_2^2 = 0 \ \& \ l_1^2 = (3 - l_1)^2 = (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = -m^2$$

The second and third constrains give, $l_1 = -|k_1]\langle 3| - \alpha|3]\langle l_1|$. Now, the fourth and fifth constrains gives,

$$3 - l_1 - l_2 = -z(|k_1] - |3])\langle 4| + \beta|4]\langle n|$$

With of course $\beta = -\frac{m^2}{z\langle 4n\rangle[4|(|k_1] - |3])}$. At last the second constrain gives,

$$l_2 = -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n|$$

$$l_2^2 = 0 = -z\langle 34\rangle[k_13] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1] - |3])$$

$$0 = \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43] + m^2$$

$$-2m^2 = \beta\langle 3n\rangle[4|(|3] - |k_1]) + \alpha z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43]$$

$$-2m^2 = -\frac{m^2}{z\langle 4n\rangle[4|(|k_1] - |3])}\langle 3n\rangle[4|(|3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]}z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43]$$

For the quadratic term in m^2 to vanish is necessary $\langle l_1n\rangle = 0 \to |l_1\rangle \propto |n\rangle$, thus,

$$-2m^2 = \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle}z\langle 4l_1\rangle$$

$$-2 = \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle}z\langle 4n\rangle \to \langle 3n\rangle = -z\langle 4n\rangle$$

The best parametrization is $|n\rangle = |l_1\rangle = |4\rangle - \frac{1}{z}|3\rangle$.

3.5. massive legs fourth topology.

Now the constrain is the hardest to solve,

$$l_1^2 = l_2^2 = (3 - l_1)^2 = (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = -m^2$$

Happily, most of the work was done already in the last solution, we just have to change:

$$l_2 = -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n|$$

$$l_2^2 = -m^2 \to m^2 = -z\langle 34\rangle[k_13] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1] - |3])$$

$$m^2 = \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43] + m^2$$

$$-m^2 = \beta\langle 3n\rangle[4|(|3] - |k_1]) + \alpha z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43]$$

$$-m^2 = -\frac{m^2}{z\langle 4n\rangle[4|(|k_1] - |3])}\langle 3n\rangle[4|(|3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]}z\langle l_14\rangle[k_13] - \alpha\beta\langle l_1n\rangle[43]$$

Again we fix $|l_1\rangle \propto |n\rangle$. The solution then is given by,

$$-m^2 = \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle}z\langle 4l_1\rangle$$

$$-1 = \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle}z\langle 4n\rangle \to \langle 3n\rangle = -z\left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)\langle 4n\rangle = -ze^{\frac{1}{3}\pi i}\langle 4n\rangle$$