ADTGR SEMINAR

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1. Introduction

Einstein-Hilbert action:

$$S_{\mathrm{EH}} = \frac{1}{\kappa^2}\int\limits_M \mathrm{d}^D x\,\sqrt{|g|}g^{ab}R_{cb}{}^{c}{}_{a}$$

The Vierbein/Tetrad formalism:

$$\begin{aligned}
\eta_{\mu\nu} &= \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) \\
g_{ab} &= e^{\mu}{}_{a} e^{\nu}{}_{b} \eta_{\mu\nu} \\
\eta_{\mu\nu} &= e_{\mu}{}^{a} e_{\nu}{}^{b} g_{ab}
\end{aligned}$$

A connection is defined with respect to a vector basis:

$$\begin{aligned}
\nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= \boldsymbol{\omega}(\mathbf{X})^{\mu}{}_{\nu}\mathbf{e}_\mu \\
\nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= X^a \omega_{a}{}^{\mu}{}_{\nu}\mathbf{e}_\mu \\
\mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) &= X^a \omega_{a}{}^{\mu}{}_{\nu}\mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\mu) \\
\mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) + \mathbf{g}(\mathbf{e}_\nu, \nabla_{\mathbf{X}}(\mathbf{e}_\alpha)) &= X^a \omega_{a}{}^{\mu}{}_{\nu}\mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\mu) + X^a \omega_{a}{}^{\mu}{}_{\alpha}\mathbf{g}(\mathbf{e}_\nu, \mathbf{e}_\mu) \\
\mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) + \mathbf{g}(\nabla_{\mathbf{X}}(\mathbf{e}_\alpha), \mathbf{e}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha} \\
e_{\alpha}{}^{c} g_{cb} \nabla_a \mathbf{e}_\nu &= e_{\alpha}{}^{c} g_{cb} \omega_{a}{}^{\mu}{}_{\nu}\mathbf{e}_\mu \\
e_{\alpha}{}^{c} g_{cb} \nabla_a \mathbf{e}_\nu + e_{\alpha}{}^{c}\mathbf{e}_\nu \nabla_a g_{cb} &= \omega_{a\alpha\nu} \\
e_{\alpha}{}^{c}\nabla_a\big(g_{cb} e_{\nu}{}^{b}\big) &= \omega_{a\alpha\nu} \\
\nabla_a\big(e_{\alpha}{}^{c} g_{cb} e_{\nu}{}^{b}\big) - g_{cb} e_{\nu}{}^{b}\nabla_a e_{\alpha}{}^{c} &= \omega_{a\alpha\nu} \\
\nabla_a(\eta_{\alpha\nu}) - g_{cb} e_{\nu}{}^{b}\nabla_a e_{\alpha}{}^{c} &= \omega_{a\alpha\nu} \\
-g_{cb} e_{\nu}{}^{b}\nabla_a e_{\alpha}{}^{c} &= \omega_{a\alpha\nu} \\
-\omega_{a\nu\alpha} &= \omega_{a\alpha\nu}, \quad \text{Metric compatibility}
\end{aligned} \tag{1.1}$$

Riemann curvature tensor, $\mathbf{Riem}(\mathbf{X}, \mathbf{Y}) : \mathfrak{X} \to \mathfrak{X}$:

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$$
\begin{aligned}
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(\nabla_\mathbf{X}\nabla_\mathbf{Y} - \nabla_\mathbf{Y}\nabla_\mathbf{X} - \nabla_{[\mathbf{X},\mathbf{Y}]}\big)\mathbf{e}_\mu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \Big(\nabla_\mathbf{X}\big(Y^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - \nabla_\mathbf{Y}\big(X^a\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \Big(\nabla_\mathbf{X}\big(Y^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - \nabla_\mathbf{Y}\big(X^a\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(Y^b\nabla_\mathbf{X}\big(\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - X^a\nabla_\mathbf{Y}\big(\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) \\
&\qquad + \nabla_\mathbf{X}\big(Y^b\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - \nabla_\mathbf{Y}\big(X^b\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) \\
&\qquad + \big(\nabla_\mathbf{X}\big(Y^b\big) - \nabla_\mathbf{Y}\big(X^b\big)\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\big)\mathbf{e}_\nu + X^aY^b\omega_b{}^\nu{}_\mu\omega_a{}^\alpha{}_\nu\mathbf{e}_\alpha - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\big)\mathbf{e}_\nu - X^aY^b\omega_a{}^\nu{}_\mu\omega_b{}^\alpha{}_\nu\mathbf{e}_\alpha\big) \\
&\qquad + \big(X^a\partial_aY^b + X^a\Gamma_a{}^b{}_cY^c - Y^a\partial_aX^b - Y^a\Gamma_a{}^b{}_cX^c\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - \big(X^a\partial_aY^b - Y^a\partial_aX^b\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^b\left(\nabla_a\big(\omega_b{}^\nu{}_\mu\big) + \omega_b{}^\alpha{}_\mu\omega_a{}^\nu{}_\alpha - \nabla_b\big(\omega_a{}^\nu{}_\mu\big) - \omega_a{}^\alpha{}_\mu\omega_b{}^\nu{}_\alpha + \Gamma_a{}^c{}_b\omega_c{}^\nu{}_\mu - \Gamma_b{}^c{}_a\omega_c{}^\nu{}_\mu\right)\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^b\left(\partial_a\omega_b{}^\nu{}_\mu - \partial_b\omega_a{}^\nu{}_\mu + \omega_a{}^\nu{}_\alpha\omega_b{}^\alpha{}_\mu - \omega_b{}^\nu{}_\alpha\omega_a{}^\alpha{}_\mu\right)\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^b\left(\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}\right)_{ab}{}^\nu{}_\mu\,\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^bR_{ab}{}^\nu{}_\mu\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\big(e_\mu{}^e\boldsymbol{\partial}_e\big) &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e_\mu{}^e\mathbf{Riem}(\mathbf{X},\mathbf{Y})\boldsymbol{\partial}_e &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e_\mu{}^eX^aY^bR_{ab}{}^c{}_e\boldsymbol{\partial}_c &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e^\mu{}_de_\mu{}^eR_{ab}{}^c{}_e &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c \\
R_{ab}{}^c{}_d &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c
\end{aligned}
$$

In the EH action,

$$S_{\mathrm{EH}} = \frac{1}{\kappa^2} \int\limits_M \mathrm{d}^D x \, \sqrt{|g|} g^{ab} R_{cb}{}^{c}{}_{a}$$

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$$S_{\mathrm{EH}} = \frac{1}{\kappa^2} \int\limits_M \mathrm{d}^D x \, \sqrt{|g|} g^{ab} g^{cd} e^{\mu}{}_{a} R_{cb\nu\mu} e^{\nu}{}_{d}$$

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2} \int\limits_M \mathrm{d}^D x \, \sqrt{|g|} \big(g^{ab} g^{dc} - g^{ac} g^{db} \big) e^{\mu}{}_{a} R_{cb\nu\mu} e^{\nu}{}_{d}$$

$$S_{\mathrm{EH}} = \frac{1}{2(D-2)!\kappa^2} \int\limits_M \mathrm{d}^D x \, \sqrt{|g|} \epsilon^{a_1 \cdots a_{D-2} ad} \epsilon_{a_1 \cdots a_{D-2}}{}^{bc} e^{\mu}{}_{a} R_{cb\nu\mu} e^{\nu}{}_{d}$$

$$S_{\mathrm{EH}} = -\frac{1}{2(D-2)!\kappa^2} \int\limits_M \mathrm{d}^D x \, \sqrt{|g|} \epsilon^{ada_1 \cdots a_{D-2}} \epsilon^{bc}{}_{a_1 \cdots a_{D-2}} e^{\nu}{}_{a} R_{bc\nu\mu} e^{\mu}{}_{d}$$

$$S_{\mathrm{EH}} = -\frac{1}{2\kappa^2} \int\limits_M \mathrm{d}^D x \, \epsilon^{ada_1 \cdots a_{D-2}} e^{\nu}{}_{a} \star R_{a_1 \cdots a_{D-2}\nu\mu} e^{\mu}{}_{d}$$

$$S_{\mathrm{EH}} = -\frac{1}{2\kappa^2} \int\limits_M \frac{1}{D!} \epsilon_{b_1 \cdots b_D} \, \mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_D} \, \epsilon^{ada_1 \cdots a_{D-2}} e^{\nu}{}_{a} \star R_{a_1 \cdots a_{D-2}\nu\mu} e^{\mu}{}_{d}$$

$$S_{\mathrm{EH}} = -\frac{1}{2\kappa^2} \int\limits_M \mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_D} \, g_{b_1}{}^{[a} g_{b_2}{}^{d} g_{b_3}{}^{a_1} \cdots g_{b_D}{}^{a_{D-2}]} e^{\nu}{}_{a} \star R_{a_1 \cdots a_{D-2}\nu\mu} e^{\mu}{}_{d}$$

$$S_{\mathrm{EH}} = -\frac{1}{2\kappa^2} \int\limits_M \mathrm{d}x^{a} \wedge \mathrm{d}x^{d} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_{D-2}} \, e^{\nu}{}_{a} e^{\mu}{}_{d} \star R_{a_1 \cdots a_{D-2}\nu\mu}$$

$$S_{\mathrm{EH}} = -\frac{1}{2\kappa^2} \int\limits_M e^{\nu}{}_{a} \, \mathrm{d}x^{a} \wedge e^{\mu}{}_{d} \, \mathrm{d}x^{d} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_{D-2}} \star R_{a_1 \cdots a_{D-2}\nu\mu}$$

$$S_{\mathrm{EH}} = -\frac{1}{2\kappa^2} \int\limits_M \tilde{\mathbf{e}}^{\nu} \wedge \tilde{\mathbf{e}}^{\mu} \wedge \star \mathbf{R}_{\nu\mu}$$

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