Homework III

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A Faddeev-Popov Gauge Fixing

Our Action functional is,

$$S_X + \lambda\chi = \frac{1}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}R + \frac{\lambda}{2\pi}\int\limits_{\partial M} \mathrm{d}s\,K \tag{A.1}$$

we would like to define the quantum theory by means of the path integral, that is, we expect that,

$$Z \stackrel{?}{=} \int \mathcal{D}X\mathcal{D}h\exp\left(-S_X[X,h] - \lambda\chi\right) \tag{A.2}$$

should give a well defined theory, but, already from A.2 there're several problems that arise, one of them is: *What should be interpreted from the path integral itself? We haven't defined any manifold to our metric h and scalar fields X to live in, also, even if we had defined such, the path integral relies on explicit coordinate points, $\mathcal{D}h = \prod\limits_\sigma \mathrm{d}h_{ab}(\sigma)$, which are highly dependent on charts.*

This is a valid claim, our way to avoid it is to *define $\mathcal{D}h$* to mean: *Sum over all **allowed** two dimensional Riemannian manifolds, and all possible metric structures in these.* Here, ***allowed*** requires a prescription, which manifolds are or aren't allowed impacts the obtained string theory. Happily, every two dimensional manifold has a definite value for the Euler Characteristic χ, hence, we can sort them out by it,

$$\begin{aligned} Z &\stackrel{?}{=} \sum_{\{M\}} \int\limits_{\mathrm{Met}(M)} \mathcal{D}h \int \mathcal{D}X\exp\left(-S_X[X,h] - \lambda\chi\right) \\ Z &\stackrel{?}{=} \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int\limits_{\mathrm{Met}(M_\chi)} \mathcal{D}h \int \mathcal{D}X\exp\left(-S_X[X,h]\right) \end{aligned} \tag{A.3}$$

Where M is to be understood as a two dimensional Riemannian manifold and M_χ is one with Euler Characteristic χ, $\mathrm{Met}(M_\chi)$ is the space of all metrics which can be assigned to M_χ, we have written $\sum\limits_{\{M_\chi\}}$ in the special case of there being more than one manifold with same Euler Characteristic[1], also, the functional integral over X should be read as integrating over all maps from M_χ to $\mathbb{R}^{1,D-1}$. While this is better defined than before, i.e. not coordinate dependent, we still have a few problems, first, it's know that A.1 has a Gauge Group of $\mathrm{Diff}(M)\times\mathrm{Weyl}(M)$, but, in our second try of a definition of the path integral, we're integrating the metrics over $\mathrm{Met}(M_\chi)$, it's clear that may happen of two elements of $\mathrm{Met}(M_\chi)$ be equivalent under a $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ transformation, to put in more clear terms, we're worried if exists $h', h \in \mathrm{Met}(M_\chi)$ such,

$$h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

the existence of those kinds of elements is troublesome, as $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ is a infinite dimensional group of redundancies, this means we're over-counting physical configurations by a infinite amount. The solution is to look for an equivalence class of metrics under this Gauge Group action,

$$\mathscr{M}_\chi = \mathrm{Met}(M_\chi)/\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$$

[1] As we're interested only in Differentiable Manifolds, more than manifold should read: More than one equivalence class of Differentiable Manifolds.

the equivalence class is to be understood as[2],

$$h' \sim h \Leftrightarrow h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

that is, two metrics are the same representative of an element of $\mathscr{M}_\chi$ iff they differ by a composition of a Diffeomorphism and Weyl transformation. We'll denote a given composition of a Diffeomorphism followed by a Weyl transformation by ζ,

$$h' = \zeta \circ h$$

Notice that the set of equivalence class of metrics, or, the set of inequivalent $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ metrics $\mathscr{M}_\chi$ is highly dependent on the topology of M_χ, for example, for $M_\chi \cong \mathbb{R}^2 \cong \mathbb{C}$, it's trivial, there is just one point in the set $\mathscr{M}_\chi$, in other words, every metric is equivalent, which isn't true for more complex topologies.

Thus, it's possible for us to set up a well defined version of the path integral, just replace $\mathrm{Met}(M_\chi)$ by $\mathscr{M}_\chi$,

$$Z = \sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathcal{D}h\int\mathcal{D}X\exp\left(-S_X[X,h]\right) \tag{A.4}$$

where the integration is to be understood as by choosing for each equivalence class in $\mathscr{M}_\chi$ a representative element in $\mathrm{Met}(M_g)$. While this is a satisfactory definition for the path integral, it feels a little clunky, we rather have an path integral over all the possible metrics — in the sense defined before —, well, this is achievable. First, for each equivalence class of $\mathscr{M}_\chi$ elect one representative element of $\mathrm{Met}(M_\chi)$, we'll denote these elements as $\hat{h}(\mathbf{t})$ — here $\mathbf{t}$ is a parametrization of the correspondent equivalence class in $\mathscr{M}_\chi$, we haven't proved here, and won't, but $\mathscr{M}_\chi$ is a finite N dimensional manifold, hence, $\mathbf{t}$ is a N-tuple of real numbers —, by construction, these representatives are inequivalent under $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, hence,

$$\zeta_1 \circ \hat{h}(\mathbf{t}_1) = \zeta_2 \circ \hat{h}(\mathbf{t}_2) \Leftrightarrow \mathbf{t}_1 = \mathbf{t}_2 \text{ and } \zeta_1 = \zeta_2$$

so that every element in $\mathrm{Met}(M_g)$ can be written as a unique[3] composition of a given ζ into a given $\hat{h}(\mathbf{t})$. Now, we rewrite the pictorial integral over $\mathscr{M}_\chi$ is a more formal way, using the parametrization we just described,

$$Z = \sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int\mathcal{D}X\exp\left(-S_X\left[X,\hat{h}(\mathbf{t})\right]\right)$$
$$Z = \sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)}\mathcal{D}\zeta\delta(\zeta)\int\mathcal{D}X\exp\left(-S_X\left[X,\hat{h}(\mathbf{t})\right]\right)$$

in the last line we introduced a one by integrating[4] over the delta functional, as this integral picks only $\zeta = 0$, what should be understood as $\zeta = \mathrm{id}$ in the group, we can deform a little the

[2]In all charts.

[3]The uniqueness or not depends on a few factors, here we'll always, unless specified otherwise, interpret $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ as the group *generated by* all possible compositions of Diffeomorphisms and Weyl transformations, but a element of it, ζ, is not to be interpreted as a unique composition of Diffeomorphism and Weyl factors, as there might be some Diffeomorphism which are equivalent to Weyl transformations, what is indeed true is that every element ζ of the Gauge Group is a unique combination of an element of $\mathrm{Diff}(M_\chi)/\mathrm{Weyl}(M_\chi)$ and an element of $\mathrm{Weyl}(M_\chi)$.

[4]Again, following the same remarks made before, the integral over $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ should not be interpreted as integrating over the whole of $\mathrm{Diff}(M_\chi)$ and after integrating over the whole $\mathrm{Weyl}(M_\chi)$, this would for sure be an over-counting, but rather should be interpreted as integrating over the whole group *generated by* compositions of $\mathrm{Diff}(M_\chi)$ and $\mathrm{Weyl}(M_\chi)$, which is equivalent of integrating over the whole $\mathrm{Diff}(M_\chi)/\mathrm{Weyl}(M_\chi)$, and after integrating over the whole $\mathrm{Weyl}(M_\chi)$.

integration to,

$$Z=\sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)}\mathcal{D}\zeta\,\delta(\zeta)\int\mathcal{D}X\exp\Big(-S_X\Big[X,\zeta\circ\hat{h}(\mathbf{t})\Big]\Big) \tag{A.5}$$

This is almost in the form that we would like, notice that we're integrating over the set of representative of the inequivalent metrics, $\mathrm{d}^N\mathbf{t}$, and also over the whole group $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, $\mathcal{D}\zeta$, by construction, **every** metric in $\mathrm{Met}(M_\chi)$ can be written uniquely[5] as,

$$h=\zeta_{\mathbf{t}}\circ\hat{h}(\mathbf{t})$$

in other words, to integrate over $\mathrm{d}^N\mathbf{t}\,\mathcal{D}\zeta$ is to integrate over all metrics of the form $\zeta\circ\hat{h}(\mathbf{t})$, which is to integrate over all metrics $h=\zeta\circ\hat{h}(\mathbf{t})$ in $\mathrm{Met}(M_\chi)$! We cannot yet make this change, due to the presence of an explicit dependence in ζ at the functional delta. We'll eliminate it by means of a change of variable of the functional delta, notice that,

$$\delta\Big(\hat{h}(\mathbf{t})-\zeta\circ\hat{h}(\mathbf{t})\Big)$$

picks up just the contribution of $\zeta=0$, so it's a good candidate for a change of variables,

$$\delta\Big(\hat{h}(\mathbf{t})-\zeta\circ\hat{h}(\mathbf{t})\Big)=\delta(\zeta)\left|\mathrm{Det}\left[\frac{\delta}{\delta\zeta}\Big(\hat{h}(\mathbf{t})-\zeta\circ\hat{h}(\mathbf{t})\Big)\bigg|_{\zeta=0}\right]\right|^{-1}$$

let's compute step by step the right-hand side of this equation, as we're only interested in the solution of $\zeta=0$, what matters is just the connected component to the identity of the Gauge Group, this is parametrized by a function ω related to the Weyl transformation, and a vector field ξ related to the connected component to the identity of the Diffeomorphisms — there is an additional requirement of ξ not generating any transformation which can be undone by a Weyl transformation —, also, for ease of our manipulation, we'll write the expression inside the delta with respect to h instead of $\hat{h}(\mathbf{t})$[6], that is,

$$\delta\Big(\hat{h}(\mathbf{t})-\zeta\circ\hat{h}(\mathbf{t})\Big)=\delta\big(\zeta^{-1}\circ h-h\big)=\delta\big(\zeta^{-1}\big)\left|\mathrm{Det}\left[\frac{\delta}{\delta\zeta^{-1}}\big(\zeta^{-1}\circ h-h\big)\bigg|_{\zeta^{-1}=0}\right]\right|^{-1}$$

one might worry about the ζ^{-1} instead of the ζ, but, the integration measure $\mathcal{D}\zeta$ is formally a Haar measure in the Group, that means it's a group invariant measure, in other words, $\mathcal{D}\zeta^{-1}=\mathcal{D}\zeta$, so that we can forget about the inverse, now,

$$\begin{aligned}[\zeta\circ h]_{ab}&=[h]_{ab}+2\omega[h]_{ab}+[\pounds_\xi h]_{ab}+\mathcal{O}\big(\omega^2,\xi^2,\omega\xi\big)\\ [\zeta\circ h]_{ab}&=[h]_{ab}+2\omega[h]_{ab}+2\nabla_{(a}\xi_{b)}+\mathcal{O}\big(\omega^2,\xi^2,\omega\xi\big)\end{aligned}$$

of course ∇ here is with respect to the h metric,

$$\begin{aligned}[\zeta\circ h]_{ab}-[h]_{ab}&=2\omega[h]_{ab}+2\nabla_{(a}\xi_{b)}+\mathcal{O}\big(\omega^2,\xi^2,\omega\xi\big)\\ \frac{\delta}{\delta\zeta}([\zeta\circ h]_{ab}-[h]_{ab})&=????\end{aligned}$$

[5]With the remarks made before.

[6]We would have to carry out the $\mathbf{t}$ dependence in h also, but, soon it will disappear as matter of uniting the integrals $\mathrm{d}^N\mathbf{t}\,\mathcal{D}\zeta$ so we won't keep track of it anymore.

The ζ derivative actually has two parts, the derivative with respect to ω and the other with respect to ξ, let's do one by one,

$$\frac{\delta}{\delta\omega(\sigma')}([\zeta\circ h]_{ab}-[h]_{ab})(\sigma)\bigg|_{\zeta=0}=2\delta^{(2)}(\sigma-\sigma')h_{ab}(\sigma)$$

and for the ξ,

$$\frac{\delta}{\delta\xi^c(\sigma')}([\zeta\circ h]_{ab}-[h]_{ab})(\sigma)\bigg|_{\zeta=0}=2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')$$

Thus,

$$\frac{\delta}{\delta\zeta}([\zeta\circ h]_{ab}-[h]_{ab})\bigg|_{\zeta=0}=2\delta^{(2)}(\sigma-\sigma')h_{ab}(\sigma)+2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')$$
$$\operatorname{Det}\left[\frac{\delta}{\delta\zeta}([\zeta\circ h]_{ab}-[h]_{ab})\bigg|_{\zeta=0}\right]=\operatorname{Det}\left[2\delta^{(2)}(\sigma-\sigma')h_{ab}(\sigma)+2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')\right]$$

the determinant can be computed by means of path integral of Grassmannian variables,

$$\operatorname{Det}\left[2\delta^{(2)}(\sigma-\sigma')h_{ab}+2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')\right]=\int\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-\frac{1}{2\pi}\int\mathrm{d}^2\sigma\sqrt{h}b^{ab}\left[h_{ab}d+h_{c(b}\nabla_{a)}c^c\right]\right)$$
$$\operatorname{Det}\left[\frac{\delta}{\delta\zeta}([\zeta\circ h]_{ab}-[h]_{ab})\bigg|_{\zeta=0}\right]=\int\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-S_{\mathrm{gh}}[b,c,d,h]\right)$$

Substituting all of this back into our path integral,

$$Z=\sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)}\mathcal{D}\zeta\delta(\zeta)\int\mathcal{D}X\exp\left(-S_X\left[X,\zeta\circ\hat{h}(\mathbf{t})\right]\right)$$
$$Z=\sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)}\mathcal{D}\zeta\delta\left(\zeta^{-1}\circ h-h\right)\int\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-S_X-S_{\mathrm{gh}}\right)$$
$$Z=\sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathrm{Met}(M_\chi)}\mathcal{D}h\delta\left(\hat{h}-h\right)\int\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]\right)$$
$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\delta\left(\hat{h}-h\right)\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]-\lambda\chi\right)$$

where $\hat{h}$ is a family of choices of representatives of the equivalence classes of the Gauge equivalent metrics, of course this choice is dependent on the equivalence class h lies in, so, in a certain sense we have $\hat{h}=\hat{h}[h]$,

$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\delta\left(\hat{h}[h]-h\right)\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]-\lambda\chi\right)$$

we express the delta functional in terms of a path integral,

$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\mathcal{D}B\exp\left(\frac{\mathrm{i}}{4\pi}\int\mathrm{d}^2\sigma\sqrt{h}B^{ab}\left(\hat{h}_{ab}[h]-h_{ab}\right)\right)\exp\left(-S_X-S_{\mathrm{gh}}-\lambda\chi\right)$$
$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\mathcal{D}B\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]-S_{\mathrm{gf}}[B,h]-\lambda\chi\right)$$

where we lastly defined the Gauge Fixing Action. This is the final expression for our path integral with the identifications,

$$S_X[X,h] + \lambda\chi = \frac{1}{4\pi\alpha'}\int_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int_M \mathrm{d}^2\sigma\,\sqrt{h}R + \frac{\lambda}{2\pi}\int_{\partial M} \mathrm{d}s\,K \tag{A.6a}$$

$$S_{\mathrm{gh}}[b,c,d,h] = \frac{1}{2\pi}\int_M \mathrm{d}^2\sigma\,\sqrt{h}b^{ab}[h_{ab}d + \nabla_a c_b] \tag{A.6b}$$

$$S_{\mathrm{gf}}[B,h] = -\frac{\mathrm{i}}{4\pi}\int_M \mathrm{d}^2\sigma\,\sqrt{h}B^{ab}\Big(\hat{h}_{ab}[h] - h_{ab}\Big) \tag{A.6c}$$

B BRST Quantization

Following the action principle derived from the Faddeev-Popov Gauge Fixing A.6, we can describe it's BRST symmetry by the transformations of the *matter fields* under Gauge, we know the following,

$$
\begin{aligned}
X^\mu(\sigma) &\to X'^\mu(\sigma'(\sigma)) = X^\mu(\sigma)\\
h_{ab}(\sigma) &\to h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)
\end{aligned}
$$

which have an *infinitesimal* form,

$$
\begin{aligned}
\delta X^\mu &= \xi^a\partial_a X^\mu\\
\delta h_{ab} &= 2\omega h_{ab} + \nabla_a\xi_b + \nabla_b\xi_a
\end{aligned}
$$

the BRST transformation can be obtained from these by the substitution inferred from the Faddeev-Popov gauge fix, that is, $\xi_a \to \mathrm{i}\epsilon c_a$ and $\omega \to \mathrm{i}\epsilon d$, where ϵ is a Grassmannian parametrization of the BRST transformation,

$$
\begin{aligned}
\delta_{\mathrm{BRST}}X^\mu &= \mathrm{i}\epsilon c^a\partial_a X^\mu\\
\delta_{\mathrm{BRST}}h_{ab} &= 2\mathrm{i}\epsilon d h_{ab} + 2\mathrm{i}\epsilon\nabla_{(a}c_{b)}
\end{aligned}
$$

This can be checked to be the right transformation by looking at how S_X transforms under it,

$$
\delta_{\mathrm{BRST}}S_X = \frac{1}{2\pi\alpha'}\int_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu\partial_b\delta_{\mathrm{BRST}}X_\mu + \frac{1}{4\pi\alpha'}\int_M \mathrm{d}^2\sigma\,\delta_{\mathrm{BRST}}\Big(\sqrt{h}h^{ab}\Big)\partial_a X^\mu\partial_b X_\mu
$$

first, let's understood each variation part by part,

$$
\begin{aligned}
0 &= \delta_{\mathrm{BRST}}\delta_a{}^c\\
0 &= \delta_{\mathrm{BRST}}\big(h_{ab}h^{bc}\big)\\
0 &= h_{ab}\delta_{\mathrm{BRST}}h^{bc} + h^{bc}\delta_{\mathrm{BRST}}h_{ab}\\
h_{ab}\delta_{\mathrm{BRST}}h^{bc} &= -2\mathrm{i}\epsilon h^{bc}\big(dh_{ab} + \nabla_{(a}c_{b)}\big)\\
h^{da}h_{ab}\delta_{\mathrm{BRST}}h^{bc} &= -2\mathrm{i}\epsilon h^{da}h^{bc}\big(dh_{ab} + \nabla_{(a}c_{b)}\big)\\
\delta^d{}_b\delta_{\mathrm{BRST}}h^{bc} &= -2\mathrm{i}\epsilon\big(dh^{dc} + \nabla^{(d}c^{c)}\big)\\
\delta_{\mathrm{BRST}}h^{dc} &= -2\mathrm{i}\epsilon\big(dh^{dc} + \nabla^{(d}c^{c)}\big)
\end{aligned}
$$

and,

$$
\begin{aligned}
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\mathrm{Det}[h_{ab}])\\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\exp\left(\ln\left(\mathrm{Det}[h_{ab}]\right)\right))\\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\exp\left(\mathrm{Tr}\left[\ln\left(h_{ab}\right)\right]\right))\\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}(\exp\left(\mathrm{Tr}\left[\ln\left(h_{ab}\right)\right]\right))\delta_{\mathrm{BRST}}(\mathrm{Tr}\left[\ln\left(h_{ab}\right)\right])\\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}h\,\mathrm{Tr}\left[\delta_{\mathrm{BRST}}(\ln\left(h_{ab}\right))\right]
\end{aligned}
$$

$$\delta_{\text{BRST}}\sqrt{h} = \frac{1}{2}\sqrt{h}\,\text{Tr}\left[h^{ca}\delta_{\text{BRST}}h_{ab}\right]$$
$$\delta_{\text{BRST}}\sqrt{h} = \frac{1}{2}\sqrt{h}h^{ba}\delta_{\text{BRST}}h_{ab}$$
$$\delta_{\text{BRST}}\sqrt{h} = \mathrm{i}\epsilon\sqrt{h}h^{ba}\big(dh_{ab} + \nabla_{(a}c_{b)}\big)$$
$$\delta_{\text{BRST}}\sqrt{h} = \mathrm{i}\epsilon\sqrt{h}(2d + \nabla_a c^a)$$

so that,

$$\delta_{\text{BRST}}\left(\sqrt{h}h^{ab}\right) = \delta_{\text{BRST}}\left(\sqrt{h}\right)h^{ab} + \sqrt{h}\delta_{\text{BRST}}\big(h^{ab}\big)$$
$$\delta_{\text{BRST}}\left(\sqrt{h}h^{ab}\right) = \mathrm{i}\epsilon\sqrt{h}(2d + \nabla_c c^c)h^{ab} - 2\mathrm{i}\epsilon\sqrt{h}\big(dh^{ab} + \nabla^{(a}c^{b)}\big)$$
$$\delta_{\text{BRST}}\left(\sqrt{h}h^{ab}\right) = 2\mathrm{i}\epsilon\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^{(a}c^{b)}\right)$$

Putting everything together now,

$$\delta_{\text{BRST}}S_X = \frac{1}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b \delta_{\text{BRST}}X_\mu + \frac{1}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\delta_{\text{BRST}}\left(\sqrt{h}h^{ab}\right)\partial_a X^\mu \partial_b X_\mu$$
$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b[c^c\partial_c X_\mu] + \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^{(a}c^{b)}\right)\partial_a X^\mu \partial_b X_\mu$$
$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu((\nabla_b c^c)\partial_c X_\mu + c^c\nabla_b\nabla_c X_\mu)$$
$$+ \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^a c^b\right)\partial_a X^\mu \partial_b X_\mu$$
$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\partial_a X^\mu \partial_b X_\mu \nabla^a c^b + \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}c^c\partial_a X^\mu \nabla_c\nabla_b X_\mu$$
$$+ \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^a c^b\right)\partial_a X^\mu \partial_b X_\mu$$
$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}c^c\partial_a X^\mu \nabla_c\partial_b X_\mu + \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\nabla_c c^c \partial_a X^\mu \partial_b X_\mu$$
$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}c^c\nabla_c\big(h^{ab}\partial_a X^\mu \partial_b X_\mu\big) + \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\nabla_c c^c \partial_a X^\mu \partial_b X_\mu$$
$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\nabla_c\big(c^c h^{ab}\partial_a X^\mu \partial_b X_\mu\big)$$
$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\partial_c\left(\sqrt{h}c^c h^{ab}\partial_a X^\mu \partial_b X_\mu\right)$$

which is a total derivative that should be zero for the theory to be BRST invariant. What should also hold is $S_{\text{gh}} + S_{\text{gf}}$ to be BRST exact, for ensuring this we need to know the BRST transformations of the ghosts and auxiliary fields[7],

$$\delta_{\text{BRST}}\left(\sqrt{h}B_{ab}\right) = 0$$

[7]The first two equations might look a bit odd, but they are in fact a consequence of normalization of $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(\mathrm{i}\int \mathrm{d}^2\sigma\,\sqrt{h}B(\sigma)f(\sigma)\right)$, instead of choosing $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(\mathrm{i}\int \mathrm{d}^2\sigma\,B(\sigma)f(\sigma)\right)$.

$$\delta_{\text{BRST}}\left(\sqrt{h}b_{ab}\right) = \epsilon\sqrt{h}B_{ab}$$
$$\delta_{\text{BRST}}d = -\mathrm{i}\epsilon c^a\partial_a d$$
$$\delta_{\text{BRST}}c^a = -\mathrm{i}\epsilon c^b\partial_b c^a$$

These might look a little bit *ad hoc*, and in fact are! They come from general procedures of BRST quantization. To prove BRST exactness of $S_{\text{gh}}+S_{\text{gf}}$ we have to prove $S_{\text{gh}}+S_{\text{gf}} = \delta_{\text{BRST}}\mathcal{O}$ for some combination of fields $\mathcal{O}$, luckily, the BRST procedure already has a candidate for this,

$$\begin{aligned}
\delta_{\text{BRST}}\left(\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab}-h^{ab}\right)\right) &= \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\delta_{\text{BRST}}\left(\sqrt{h}b_{ab}\right)\left(\hat{h}^{ab}-h^{ab}\right)\\
&\quad-\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\delta_{\text{BRST}}\hat{h}^{ab}\\
&\quad+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\delta_{\text{BRST}}\left(h^{ab}\right)\\
\delta_{\text{BRST}}\left(\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab}-h^{ab}\right)\right) &= \epsilon\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}\left(\hat{h}^{ab}-h^{ab}\right)\\
&\quad+\mathrm{i}\epsilon\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(dh^{ab}+\nabla^a c^b\right)\\
\delta_{\text{BRST}}\left(\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab}-h^{ab}\right)\right) &= \mathrm{i}\epsilon S_{\text{gf}}+\mathrm{i}\epsilon S_{\text{gh}}\\
\delta_{\text{BRST}}\left(\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab}-h^{ab}\right)\right) &= \mathrm{i}\epsilon(S_{\text{gf}}+S_{\text{gh}})
\end{aligned}$$

and lastly, the verification of the closure of $S_{\text{gh}}+S_{\text{gf}}$, let's do it step by step,

$$\begin{aligned}
\delta_{\text{BRST}}S_{\text{gh}} &= \frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\delta_{\text{BRST}}\left(\sqrt{h}b_{ab}\right)\left[h^{ab}d+\nabla^a c^b\right]\\
&\quad-\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left[\delta_{\text{BRST}}\left(h^{ab}\right)d+h^{ab}\delta_{\text{BRST}}d+\nabla^a\delta_{\text{BRST}}c^b\right]\\
\delta_{\text{BRST}}S_{\text{gh}} &= \frac{\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}\left[h^{ab}d+\nabla^a c^b\right]\\
&\quad-\frac{\mathrm{i}\epsilon}{\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(dh^{ab}+\nabla^a c^b\right)d\\
&\quad-\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}h^{ab}c^c\partial_c d\\
&\quad-\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\nabla^a\left(c^c\nabla_c c^b\right)\\
\delta_{\text{BRST}}S_{\text{gh}} &= \frac{\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}\left[h^{ab}d+\nabla^a c^b\right]\\
&\quad-\frac{\mathrm{i}\epsilon}{\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\nabla^a c^b d\\
&\quad-\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}h^{ab}c^c\partial_c d\\
&\quad-\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\nabla^a\left(c^c\nabla_c c^b\right)
\end{aligned}$$

and,

$$\delta_{\text{BRST}}S_{\text{gf}} = \frac{\mathrm{i}}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}\delta_{\text{BRST}}h^{ab}$$

$$\delta_{\mathrm{BRST}} S_{\mathrm{gf}} = \frac{\epsilon}{2\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} B_{ab} \big(h^{ab} d + \nabla^a c^b \big)$$