2+1 GRAVITY AS A TOY MODEL FOR A GAUGE FORMULATION OF GRAVITY

VICENTE V. FIGUEIRA

1. Introduction

General Relativity is known for being a highly complex theory, to which, apart from highly symmetric ones, few solutions are known. One of the main reasons for this is the non-linear character of the Action and the Equations of Motion,

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \mathrm{d}^D x\,\sqrt{|g|}g^{ab}R_{cb}{}^{c}{}_{a}, \quad R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} \tag{1.1}$$

from the non-polinomial term $\sqrt{|g|}$, the inverse field g^{ab}, and also from $R_{cb}{}^{c}{}_{a}$, where lies more contributions of inverse fields and their derivatives. Thus, in studying this theory and its couplings to matter, the prospects of obtaining analytical solutions in non-highly symmetric scenarios is faded to doom. Also, these peculiarities prevents the theory from being interpreted as a gauge theory in the usual sense[1]. These combined motifs have proven gravity in $D = 3+1$ dimensions to be stubborn to the usual quantization methods, and even to classical non-vacuum solutions. Nevertheless, there is hope of grasping a better understanding of it — either qualitative or quantitative — by looking to simpler toy models, which, have already proven it's usefulness as in String Theory, with $D = 1+1$ gravity. We would like to pursue a similar line of thought and ask ourselves, what can $D = 2+1$ gravity teach us? We'll try to focus more on wether or not it can be stated as a usual gauge theory.

One might ask: why should $D = 2+1$ be any easier to deal with than $D = 3+1$? The answer lies in the number of dynamical degrees of freedom of the theory, which in gravity, is deeply tied to the space-time dimensions. In the most common formulation, the dynamical fundamental field is considered to be the metric, $\mathbf{g}$, a symmetric, non-degenerate $(0,2)$–tensor. Associated to it is a metric-compatible, torsionless connection $\nabla_{(\,\cdot\,)}(\,\cdot\,) : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$, which under these conditions is totally determined by the metric. Hence, all of the degrees of freedom of the theory are metric ones, which are $\frac{1}{2}D(D+1)$, but, as we have redundancies/gauge transformations of D diffeomorphisms, $\phi_*\mathbf{g}\sim\mathbf{g}$, we have to discount them. Also, there are D additional constrains coming from the Bianchi identity, $\nabla_a\big(R^{ab} - \frac{1}{2}Rg^{ab}\big) = 0$, so the total number of degree of freedom in this theory is, $\frac{1}{2}D(D+1) - 2D = \frac{1}{2}D(D-3)$. In $D = 3+1$ this is our well known two polarizations of the metric, but, for $D = 2+1$ this is zero, which can be interpreted as the metric having no dynamical degrees of freedom, that is, the equation of motion is merely an algebraic condition. This is compatible with our knowledge of independent components of the Riemann tensor, $\frac{1}{12}D^2(D^2-1)$, exactly 20 in $D = 3+1$, but 6 in $D = 2+1$. Notice that $6 = \frac{1}{2}3(3+1)$, the same number of degrees of freedom of a symmetric $(0,2)$ tensor. In other words, in $D = 2+1$ the Riemann tensor is totally determined by the knowledge of the Ricci tensor, which is totally determined algebraically by the equations of motion. This is consistent with the metric not having degrees of freedom, due to being known that dynamical propagation of gravity is linked to the Weyl tensor. If the Riemann tensor is totally determined by the Ricci tensor, there is no degree of freedom in the Weyl tensor, thus, no dynamics. This is our hope to "solve" this theory, as it's non-dynamical. We expect it to be "trivial", or at least "exact" — we have to define what we mean by this —, assertion which should be true at least classically, and hopefully, would allow for a solvable model of quantum gravity.

2. The Einstein-Hilbert Action in the Form Language

We'll begin with a quick recap of the vielbein/spin connection formalism [1, 4, 6, 7]. For now we'll keep the discussion general in D dimensions, and, only later on, we'll go to the special case of $D = 2+1$. The vielbein[2], $\tilde{\mathbf{e}}_\mu$, are a basis of the vector field space $\mathfrak{X}(M)$. Notice, the index μ is only indexing which vector from the ones present in the basis we are talking about. It isn't a coordinate index — one that is related to a specific component decomposition in a specific chart —, as this is not a coordinate basis — i.e. $\boldsymbol{\partial}_a$ —. We can in fact do miracles with it, as diagonalize the metric,

$$\begin{aligned}
\operatorname{diag}\begin{pmatrix}-1 & 1 & \cdots & 1\end{pmatrix} = \eta_{\mu\nu} &= \mathbf{g}(\tilde{\mathbf{e}}_\mu, \tilde{\mathbf{e}}_\nu) = g_{ab}\mathbf{d}x^a(\tilde{\mathbf{e}}_\mu)\otimes\mathbf{d}x^b(\tilde{\mathbf{e}}_\nu)\\
\eta_{\mu\nu} &= g_{ab}e_\mu{}^c e_\nu{}^d\mathbf{d}x^a(\boldsymbol{\partial}_c)\otimes\mathbf{d}x^b(\boldsymbol{\partial}_d)\\
\eta_{\mu\nu} &= g_{ab}e_\mu{}^a e_\nu{}^b
\end{aligned}$$

The whole point of introducing the vielbein is switch the degrees of freedom from the metric to the inertial frame basis, which can be seen as a downgrade, due to this process enlarging the number of degrees of freedom from $\frac{1}{2}D(D+1)$ to D^2. This is only the naive counting, without considering the redundancies, as the number of physical degrees of freedom must be the same. For this to be true, is only possible if we enlarge also the redundancies, to kill the extra degrees of freedom we introduced. As mentioned, this could be seen as not desirable, but, for us this turn out to be essencial. What are these new redundancies? They're the choice of labeling μ in

[1]With *gauge theory in the usual sense* we mean a theory in which the fundamental degree of freedom is a field $\mathbf{A}$ — usually an adjoint $\mathfrak{g}$–valued 1–form — which has as redundancy a local realization of a Lie Group $G\ni g$ which acts on the field as $\mathbf{A}\to g\mathbf{A}g^{-1} + g\mathbf{d}g^{-1}$.

[2]We'll denote the *vector* — (1,0) tensor — vielbein with a tilde, only to be distinguishable from the associated *covetor* — (0,1) tensor — vielbein, which we'll denote without the tilde due to being way more important to us.

$\tilde{\mathbf{e}}_\mu$. As long as the new relabel also respect the defining property of the vielbein, it's a redundant transformation. Notice that these transformations are exactly local Lorentz ones, that is, given a set of functions $\Lambda^\mu{}_\nu : M \to \mathbb{R}$, a new set of vector fields $\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu$ is a vielbein iff,

$$\mathbf{g}\big(\Lambda^\alpha{}_\mu \tilde{\mathbf{e}}_\alpha, \Lambda^\beta{}_\nu \tilde{\mathbf{e}}_\beta\big) = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\beta)$$
$$\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta}$$

In other words, $\Lambda^\mu{}_\nu$ must be a local $SO(D-1,1)$ element[3]. With this new redundancy taken into account, the degrees of freedom match, D^2 from the vielbein, minus D diffeomorphisms $\phi^* \tilde{\mathbf{e}}_\mu \sim \tilde{\mathbf{e}}_\mu$, D Bianchi identities and $\frac{1}{2}D(D-1)$ local $SO(D-1,1)$ transformations, giving $D^2 - 2D - \frac{1}{2}D(D-1) = \frac{1}{2}D(D-3)$, exactly the same counting using only the metric! Notice that now the redundancies are diffeomorphisms and local $SO(D-1,1)$ transformations. This already seems like we're gauging the whole Poincaré group, which will come up later.

From the condition that $\mathbf{g}$ must be non-degenerate, we get that the matrix of components $e_\mu{}^a$ must be invertible. This ensures the existence of $e^\mu{}_a$,from which we can construct the dual vector field $\mathbf{e}^\mu = e^\mu{}_a \mathbf{d}x^a$. This certify that we have a basis of the whole tensor space, so it's possible to decompose any tensor in it,

$$\eta_{\mu\nu} \mathbf{e}^\mu \otimes \mathbf{e}^\nu = \mathbf{g}(\tilde{\mathbf{e}}_\mu, \tilde{\mathbf{e}}_\nu) \mathbf{e}^\mu \otimes \mathbf{e}^\nu = \mathbf{g}$$

Up to now we have been considering a metric compatible torsionless affine connection, but, this turns out to not be the optimal choice, as for this kind of connection there is an differential/algebraical restraint between the connection and the metric. At least that's what we should expect in a coordinate basis. What we would like is to have a connection linearly independent of the metric/vielbein, but without sacrificing the metricity condition, this is completely hopeless in a coordinate basis. In a non-coordinate basis this is achievable! We just have to remind that, an affine connection can be defined in any basis of $\mathfrak{X}(M)$. What is usually done is $\nabla_{\mathbf{X}} \boldsymbol{\partial}_b = X^a \Gamma_a{}^c{}_b \boldsymbol{\partial}_c$, but it's much more interesting to define it with respect to the vielbein basis,

$$\nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu) = \boldsymbol{\omega}(\mathbf{X})^\mu{}_\nu \tilde{\mathbf{e}}_\mu = X^a \omega_a{}^\mu{}_\nu \tilde{\mathbf{e}}_\mu$$

Here $\boldsymbol{\omega}^\mu{}_\nu = \omega_a{}^\mu{}_\nu \mathbf{d}x^a$ is named the spin connection. It can be seen as a $\mathfrak{gl}(D-1,1)$–valued $(0,1)$ tensor, or, as we'll adopt here, a $\mathfrak{gl}(D-1,1)$–valued 1–form. The advantage of working with the vielbein is that the Lorentz index μ does not change upon coordinate/chart/diffeomorphism transformations, it acts as if was an internal symmetry. Thus, $\boldsymbol{\omega}^\mu{}_\nu$ do transform exactly as a tensor should. This is already a enormous dichotomy with the standard formulation, where the Christoffel connection doesn't transform in a good manner. Now we'll impose the metric compatibility of the connection. This is another scenario where the vielbein formalism come to hand, as this condition imposes no additional differential/algebraical constrains among the vielbein and spin connection.

$$\begin{aligned}
\mathbf{g}(\tilde{\mathbf{e}}_\alpha, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu)) &= X^a \omega_a{}^\mu{}_\nu \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\mu), \quad \text{symmetrize } \alpha \leftrightarrow \nu \\
\mathbf{g}(\tilde{\mathbf{e}}_\alpha, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu)) + \mathbf{g}(\tilde{\mathbf{e}}_\nu, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\alpha)) &= X^a \omega_a{}^\mu{}_\nu \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\mu) + X^a \omega_a{}^\mu{}_\alpha \mathbf{g}(\tilde{\mathbf{e}}_\nu, \tilde{\mathbf{e}}_\mu), \quad \text{metric symmetry} \\
\mathbf{g}(\tilde{\mathbf{e}}_\alpha, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu)) + \mathbf{g}(\nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\alpha), \tilde{\mathbf{e}}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha}, \quad \text{Leibnitz rule} \\
\nabla_{\mathbf{X}}(\mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\nu)) - \nabla_{\mathbf{X}}(\mathbf{g})(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha}, \quad \nabla_{\mathbf{X}}(\eta_{\alpha\nu}) = 0 \\
-\nabla_{\mathbf{X}}(\mathbf{g})(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha}, \quad \text{metricity} \\
-\omega_{a\nu\alpha} &= \omega_{a\alpha\nu}
\end{aligned}$$

That is, the metric compatible spin connection is anti-symmetric in the non-coordinate indices, exactly the property satisfied by the generators of the $SO(D-1,1)$ group. Thus, the spin connection can be seen as a $\mathfrak{so}(D-1,1)$-valued 1-form. It has everything in its favor to be interpreted as a gauge field of the $SO(D-1,1)$ group. To confirm this, notice how it changes under a gauge transformation of the vielbein basis $\tilde{\mathbf{e}}_\nu \to \Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu$:

$$\begin{aligned}
\nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu)\tilde{\mathbf{e}}_\mu + \Lambda^\mu{}_\nu \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\mu), \quad \text{connection definition} \\
\nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \mathbf{X}(\Lambda^\mu{}_\nu)\tilde{\mathbf{e}}_\mu + \Lambda^\mu{}_\nu \boldsymbol{\omega}(\mathbf{X})^\alpha{}_\mu \tilde{\mathbf{e}}_\alpha, \quad \Lambda^{-1\beta}{}_\alpha \Lambda^\sigma{}_\beta = \delta^\sigma{}_\alpha \\
\nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \mathbf{d}\Lambda^\mu{}_\nu(\mathbf{X}) \Lambda^{-1\beta}{}_\mu \Lambda^\sigma{}_\beta \tilde{\mathbf{e}}_\sigma + \Lambda^\mu{}_\nu \boldsymbol{\omega}(\mathbf{X})^\alpha{}_\mu \Lambda^{-1\beta}{}_\alpha \Lambda^\sigma{}_\beta \tilde{\mathbf{e}}_\sigma \\
\nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \Big(\Lambda^{-1\beta}{}_\mu \mathbf{d}\Lambda^\mu{}_\nu + \Lambda^{-1\beta}{}_\alpha \boldsymbol{\omega}^\alpha{}_\mu \Lambda^\mu{}_\nu\Big)(\mathbf{X}) \Lambda^\sigma{}_\beta \tilde{\mathbf{e}}_\sigma
\end{aligned}$$

Hence, under a gauge transformation of $\tilde{\mathbf{e}}_\nu \to \Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu$, the spin connection transforms exactly as a connection of the gauge group $SO(D-1,1)$, $\boldsymbol{\omega} \to \Lambda^{-1}\boldsymbol{\omega}\Lambda + \Lambda^{-1}\mathbf{d}\Lambda$, which is what we're looking for. Before pursuing further the gauged translations, we're going to obtain a new interpretation for the Riemann tensor, using what we just learned from the spin connection. Notice, the usual interpretation of the Riemann tensor, is of it being a $(1,3)$ tensor, but, as naturally — without the need for a metric compatible torsionless connection — it's anti-symmetric in the first two entries, we can switch the point of view from a $(1,3)$ tensor to a $(1,1)$ tensor valued 2–form, or, in an even better way, an $\mathrm{End}(\mathfrak{X}(M))$–valued 2–form, which, when decomposed in the non-coordinate basis, will turn out to be a $\mathfrak{so}(D-1,1)$–valued 2–form, as we'll shown now. Starting from the definition[4],

$$\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\tilde{\mathbf{e}}_\mu = \big(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\big)\tilde{\mathbf{e}}_\mu$$

---

[3] Actually, the group is $O(D-1,1)$, but, we'll only be interested in the orientation preserving transformations.

[4] Here, care must be taken. In our conventions, the covariant derivative acts non-trivially only in vectors, and acts as a normal derivative in functions. That is, for X^a being *components* of a vector, $\nabla_{\mathbf{Y}} X^a = \mathbf{Y}(X^a) = Y^b \partial_b X^a$, in contrast to $\nabla_{\mathbf{Y}} \boldsymbol{\partial}_a = Y^b \Gamma_b{}^c{}_a \boldsymbol{\partial}_c$.

$$
\begin{aligned}
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= \nabla_{\mathbf{X}}\big(Y^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - \nabla_{\mathbf{Y}}\big(X^a\omega_a{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= Y^b\nabla_{\mathbf{X}}\big(\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - X^a\nabla_{\mathbf{Y}}\big(\omega_a{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) + \nabla_{\mathbf{X}}\big(Y^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - \nabla_{\mathbf{Y}}\big(X^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) + \big(\nabla_{\mathbf{X}}\big(Y^b\big) - \nabla_{\mathbf{Y}}\big(X^b\big)\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\big)\tilde{\mathbf{e}}_\nu + X^aY^b\omega_b{}^\nu{}_\mu\omega_a{}^\alpha{}_\nu\tilde{\mathbf{e}}_\alpha - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\big)\tilde{\mathbf{e}}_\nu - X^aY^b\omega_a{}^\nu{}_\mu\omega_b{}^\alpha{}_\nu\tilde{\mathbf{e}}_\alpha\\
&\qquad + \big(X^a\partial_aY^b - Y^a\partial_aX^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - \big(X^a\partial_aY^b - Y^a\partial_aX^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\left(\partial_a\big(\omega_b{}^\nu{}_\mu\big) + \omega_b{}^\alpha{}_\mu\omega_a{}^\nu{}_\alpha - \partial_b\big(\omega_a{}^\nu{}_\mu\big) - \omega_a{}^\alpha{}_\mu\omega_b{}^\nu{}_\alpha\right)\tilde{\mathbf{e}}_\nu\\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\left(\partial_a\omega_b{}^\nu{}_\mu - \partial_b\omega_a{}^\nu{}_\mu + \omega_a{}^\nu{}_\alpha\omega_b{}^\alpha{}_\mu - \omega_b{}^\nu{}_\alpha\omega_a{}^\alpha{}_\mu\right)\tilde{\mathbf{e}}_\nu\\
X^aY^bR_{ab}{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu &= X^aY^b\left(\mathbf{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}\right)_{ab}{}^\nu{}_\mu\,\tilde{\mathbf{e}}_\nu
\end{aligned}
$$

This settles down the interpretation of the Riemann tensor being a $\mathfrak{so}(D-1,1)$–valued 2–form, and also provides a striking resemblance to the usual gauge force field in non-abelian theories, $\mathbf{F} = \mathbf{dA} + \mathbf{A}\wedge\mathbf{A}$. It's also easily related to the usual coordinate Riemann tensor,

$$
\begin{aligned}
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^bR_{ab}{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\big(e_\mu{}^e\boldsymbol{\partial}_e\big) &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c\\
e_\mu{}^e\mathbf{Riem}(\mathbf{X},\mathbf{Y})\boldsymbol{\partial}_e &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c\\
e_\mu{}^eX^aY^bR_{ab}{}^c{}_e\boldsymbol{\partial}_c &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c\\
e^\mu{}_de_\mu{}^eR_{ab}{}^c{}_e &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c\\
R_{ab}{}^c{}_d &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c
\end{aligned}
$$

which we'll use when rewriting the Einstein-Hilbert Action. It doesn't hurt to stress that we are only assuming metricity, and not torsionless — which will come about naturally later —. Lastly, we define our Riemann Curvature 2-form,

$$
\mathbf{R}^\nu{}_\mu = \frac{1}{2}R_{ab}{}^\nu{}_\mu\mathbf{d}x^a\wedge\mathbf{d}x^b = \mathbf{d}\boldsymbol{\omega}^\nu{}_\mu + \boldsymbol{\omega}^\nu{}_\alpha\wedge\boldsymbol{\omega}^\alpha{}_\mu
$$

We'll talk more about the absence of $\mathbf{e}^\mu$ later — which signifies this is the gauge force field of only the inhomogeneous part of the Poincaré group —.

Now we start the real deal of rewriting the Einstein-Hilbert Action in terms of the curvature 2–form, and the vielbein 1–form. Starting with the volume form[5],

$$
\begin{aligned}
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[g_{ab}]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^\mu{}_a\eta_{\mu\nu}e^\nu{}_b]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^\mu{}_a]\mathrm{Det}[\eta_{\mu\nu}]\mathrm{Det}[e^\nu{}_b]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{\left(\mathrm{Det}[e^\mu{}_a]\right)^2}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \mathrm{Det}[e^\mu{}_a]\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \epsilon_{\mu_1\cdots\mu_D}e^{\mu_0}{}_0\cdots e^{\mu_D}{}_{D-1}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \epsilon_{\mu_1\cdots\mu_D}e^{\mu_0}{}_0\,\mathrm{d}x^0\wedge\cdots\wedge e^{\mu_D}{}_{D-1}\,\mathrm{d}x^{D-1}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_1\cdots\mu_D}e^{\mu_1}{}_{a_1}\,\mathrm{d}x^{a_1}\wedge\cdots\wedge e^{\mu_D}{}_{a_D}\,\mathrm{d}x^{a_D}\\
\mathrm{d}^Dx\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_1\cdots\mu_D}\mathbf{e}^{\mu_1}\wedge\cdots\wedge\mathbf{e}^{\mu_D} \qquad (2.1)
\end{aligned}
$$

The other ingredient we need, is the Ricci scalar,

$$
\begin{aligned}
R &= g^{ab}R_{cb}{}^c{}_a\\
R &= e_\rho{}^ae^{\rho b}R_{cbda}e_\alpha{}^ce^{\alpha d}, \quad \text{vielbein definition}\\
R &= \eta^{\rho\sigma}\eta^{\alpha\beta}e_\rho{}^ae_\sigma{}^bR_{cbda}e_\alpha{}^ce_\beta{}^d\\
R &= \eta^{\rho\sigma}\eta^{\alpha\beta}R_{cb\beta\rho}e_\alpha{}^ce_\sigma{}^b, \quad \text{anti-symmetry in the first two index}\\
R &= \frac{1}{2}\big(\eta^{\rho\sigma}\eta^{\alpha\beta} - \eta^{\rho\alpha}\eta^{\sigma\beta}\big)R_{cb\beta\rho}e_\alpha{}^ce_\sigma{}^b\\
R &= \frac{-1}{2(D-2)!}\epsilon^{\nu_1\cdots\nu_{D-2}\beta\rho}\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^ce_\sigma{}^b \qquad (2.2)
\end{aligned}
$$

[5]While it's widely known, the Levi-Civita symbol, $\epsilon_{a_1\cdots a_D}$, in a coordinate basis isn't a tensor. For a non-coordinate basis, $\epsilon_{\mu_1\cdots\mu_D}$, it does not need additional factors of the metric determinant, one more usefulness of the vielbein basis.

Combining these two results, (2.1), (2.2), we can rewrite the EH Action in the forms language. That is, the expression we're expecting to obtain is an integral over a D–form. In the middle of the computation, we'll need to introduce also the Hodge star operator,

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \mathrm{d}^D x\,\sqrt{|g|}R$$

$$S_{\text{EH}} = \frac{-1}{4D!(D-2)!\kappa}\int_M \mathbf{e}^{\mu_1}\wedge\cdots\wedge\mathbf{e}^{\mu_D}\epsilon_{\mu_1\cdots\mu_D}\epsilon^{\nu_1\cdots\nu_{D-2}\beta\rho}\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b$$

$$S_{\text{EH}} = \frac{1}{4(D-2)!\kappa}\int_M \mathbf{e}^{\mu_1}\wedge\cdots\wedge\mathbf{e}^{\mu_D}\eta_{\mu_1}{}^{[\nu_1}\cdots\eta_{\mu_{D-2}}{}^{\nu_{D-2}}\eta_{\mu_{D-1}}{}^{\beta}\eta_{\mu_D}{}^{\rho]}\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b$$

$$S_{\text{EH}} = \frac{1}{4(D-2)!\kappa}\int_M \mathbf{e}^{\nu_1}\wedge\cdots\wedge\mathbf{e}^{\nu_{D-2}}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b$$

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \frac{1}{2(D-2)!}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b\epsilon^{\alpha\sigma}{}_{\nu_1\cdots\nu_{D-2}}\mathbf{e}^{\nu_1}\wedge\cdots\wedge\mathbf{e}^{\nu_{D-2}}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho$$

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \frac{1}{2(D-2)!}R_{cb\beta\rho}\epsilon^{\alpha\sigma}{}_{\nu_1\cdots\nu_{D-2}}e_\alpha{}^c e_\sigma{}^b e^{\nu_1}{}_{a_1}\cdots e^{\nu_{D-2}}{}_{a_{D-2}}\mathbf{d}x^{a_1}\wedge\cdots\wedge\mathbf{d}x^{a_{D-2}}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho$$

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \frac{1}{2(D-2)!}R_{cb\beta\rho}\text{Det}[e^\nu{}_a]\epsilon^{cb}{}_{a_1\cdots a_{D-2}}\mathbf{d}x^{a_1}\wedge\cdots\wedge\mathbf{d}x^{a_{D-2}}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho$$

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \frac{1}{2}R_{cb\beta\rho}\star\left(\mathbf{d}x^c\wedge\mathbf{d}x^b\right)\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho$$

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \star\mathbf{R}_{\beta\rho}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho \tag{2.3}$$

This is our final result for the EH Action. To see that it's a consistent expression, we can do the counting of the form degree. $\mathbf{R}$ is a 2–form, which makes $\star\mathbf{R}$ a $(D-2)$–form. As $\mathbf{e}$ is a 1–form, the final expression is a $D-2+2=D$–form. The next sections are dedicated to check the redundancies and equations of motion of the Action.

3. Gravity as a Gauge Theory

Let's take a step backwards to breath. This last expression was our goal to reach. It shows the Einstein-Hilbert Action[6] written exactly in terms of the vielbein and the spin connection 1–forms. In particular, the spin connection as being the connection form of the gauged Lorentz transformations shows up only inside the curvature 2–form, as it should happen to a gauge theory. This guarantees the manifestly local Lorentz invariance. To see this, we defined the local Lorentz redundancy as being $\tilde{\mathbf{e}}_\mu \to \Lambda^\nu{}_\mu\tilde{\mathbf{e}}_\nu$, with the corresponding transformation on $\boldsymbol{\omega}$ derived before. But, the vector vielbein transformation actually induces a transformation in the covector vielbein, $\mathbf{e}^\mu(\tilde{\mathbf{e}}_\nu) = \delta^\mu{}_\nu \Rightarrow \mathbf{e}^\mu \to \Lambda^{-1\mu}{}_\nu\mathbf{e}^\nu$, so that's the relevant expressions for us now are[7],

$$\begin{cases}\mathbf{e}^\mu & \to \Lambda^\mu{}_\nu\mathbf{e}^\nu \\ \boldsymbol{\omega}^\alpha{}_\beta & \to \Lambda^\alpha{}_\rho\boldsymbol{\omega}^\rho{}_\sigma\Lambda^{-1\sigma}{}_\beta + \Lambda^\alpha{}_\sigma\mathbf{d}\Lambda^{-1\sigma}{}_\beta\end{cases} \Leftrightarrow \begin{cases}\mathbf{e} & \to \Lambda\mathbf{e} \\ \boldsymbol{\omega} & \to \Lambda\boldsymbol{\omega}\Lambda^{-1} + \Lambda\mathbf{d}\Lambda^{-1}\end{cases} \tag{3.1}$$

where we also included the matrix form of these transformations. To see the invariance of the action with respect to these, we only needed to work out the transformation law of the curvature 2–form,

$$\mathbf{R} = \mathbf{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega} \to \mathbf{d}\left(\Lambda\boldsymbol{\omega}\Lambda^{-1} + \Lambda\mathbf{d}\Lambda^{-1}\right) + \left(\Lambda\boldsymbol{\omega}\Lambda^{-1} + \Lambda\mathbf{d}\Lambda^{-1}\right)\wedge\left(\Lambda\boldsymbol{\omega}\Lambda^{-1} + \Lambda\mathbf{d}\Lambda^{-1}\right)$$

$$\mathbf{R} \to \mathbf{d}\Lambda\wedge\boldsymbol{\omega}\Lambda^{-1} + \Lambda\mathbf{d}\boldsymbol{\omega}\Lambda^{-1} - \Lambda\boldsymbol{\omega}\wedge\mathbf{d}\Lambda^{-1} + \mathbf{d}\Lambda\wedge\mathbf{d}\Lambda^{-1} + \Lambda\boldsymbol{\omega}\wedge\boldsymbol{\omega}\Lambda^{-1} + \Lambda\boldsymbol{\omega}\wedge\mathbf{d}\Lambda^{-1} - \mathbf{d}\Lambda\wedge\boldsymbol{\omega}\Lambda^{-1} - \mathbf{d}\Lambda\wedge\mathbf{d}\Lambda^{-1}$$

$$\mathbf{R} \to \Lambda(\mathbf{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega})\Lambda^{-1} = \Lambda\mathbf{R}\Lambda^{-1} \tag{3.2}$$

Lastly, but not less important, we have to understand whether or not the Hodge star operator should change or not with respect to this transformation. The answer is no[8]. Despite it depending explicitly on the metric to be defined, it's dependence is only through $\sqrt{|\text{Det}[g_{ab}]|}$, which is invariant under a local Lorentz transformation, as $g_{ab} = \eta_{\mu\nu}e^\mu{}_a e^\nu{}_b$. Thus, $\star\mathbf{R} \to \Lambda(\star\mathbf{R})\Lambda^{-1}$, and the action is manifestly invariant,

$$S'_{\text{EH}} = \frac{1}{2\kappa}\int_M \Lambda_\beta{}^\tau\star\mathbf{R}_{\tau\sigma}\Lambda^{-1\sigma}{}_\rho\wedge\Lambda^\beta{}_\alpha\mathbf{e}^\alpha\wedge\Lambda^\rho{}_\mu\mathbf{e}^\mu = \frac{1}{2\kappa}\int_M \star\mathbf{R}_{\beta\rho}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho = S_{\text{EH}}$$

The natural question arises: what about the diffeomorphisms transformations? Well, it continues being a redundancy. The problem is, it does not act in the same fashion as gauge transformation should. Its action is $\mathbf{e}^\mu = e^\mu{}_a\mathbf{d}x^a \to \phi_*\mathbf{e}^\mu = e^\mu{}_b\Omega^b{}_a\mathbf{d}x^a$, with $\Omega^b{}_a : M \to \mathbb{R}$

---

[6]Actually it's the Einstein-Cartan Action, as we have not imposed the torsionless condition, but, we'll not bother distinguishing this.

[7]For convenience we'll make the change $\Lambda \leftrightarrow \Lambda^{-1}$ everywhere.

[8]As long as we work with orientation preserving transformation, otherwise it'll adquire an extra sign.

totally defined by $\phi \in \mathrm{Diff}(M)$. This differs a lot from what we would expect. Naively, $\mathbf{e}^\mu$ would be thought to be the gauge field for translations — which when made local are the diffeomorphisms —, and as such, under a diffeomorphisms gauge transformation should transform as $\mathbf{e}^\mu \to \mathbf{e}^\mu + \mathbf{d}\alpha^\mu$. This clearly isn't the case here. That poses a obstacle for us to be able to interpret gravity as being the gauge theory, in usual sense, of the group $ISO(D-1,1)$. Nevertheless, the action still is invariant, as long as we add the transformation law $\boldsymbol{\omega}^\alpha{}_\beta \to \phi_* \boldsymbol{\omega}^\alpha{}_\beta$. Remember that the Hodge star should also change, as it also depends on $\mathbf{e}^\mu$, $\star \to \phi_* \star$, and of course use the naturalness of the exterior derivative and the wedge product. Thus,

$$\mathbf{R} \to \mathbf{d}\phi_*\boldsymbol{\omega} + \phi_*\boldsymbol{\omega} \wedge \phi_*\boldsymbol{\omega} = \phi_*(\mathbf{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}) = \phi_*\mathbf{R} \Rightarrow \star\mathbf{R} \to (\phi_*\star)\phi_*\mathbf{R} = \phi_*(\star\mathbf{R})$$

This means the action is manifestly invariant,

$$S'_{\mathrm{EH}} = \frac{1}{2\kappa}\int\limits_M \phi_*(\star\mathbf{R}_{\beta\rho}) \wedge \phi_*\mathbf{e}^\beta \wedge \phi_*\mathbf{e}^\rho = \frac{1}{2\kappa}\int\limits_M \phi_*\left(\star\mathbf{R}_{\beta\rho}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho\right) = S_{\mathrm{EH}}$$

The last equality is a consequence of ϕ being a diffeomorphism[9]. Despite this, we still didn't managed to obtain a proper gauge interpretation of the diffeomorphisms. One possible way of thinking is that $\mathbf{e}^\mu$ isn't in the proper representation for a gauge field, as they must be in the adjoint representation, and by the transformation law $e^\mu{}_a \to e^\mu{}_b \Omega^b{}_a$ it seems to be in the fundamental one. We can try to change this by evoking the generators of the $\mathfrak{iso}(D-1,1)$ algebra,

$$\left[J^{\alpha\beta}, J^{\mu\nu}\right] = 4\eta^{[\alpha|[\mu}J^{\nu]|\beta]}, \quad \left[J^{\alpha\beta}, P^\mu\right] = 2P^{[\alpha}\eta^{\beta]\mu}, \quad \left[P^\mu, P^\nu\right] = 0 \tag{3.3}$$

Which naturally promote the ought to be gauge fields to any representation, $\mathbf{e}^\mu \to \mathbf{e}^\mu P_\mu, \boldsymbol{\omega}_{\alpha\beta} \to \frac{1}{2}\boldsymbol{\omega}_{\alpha\beta}J^{\alpha\beta}$, so that the only option left is the EH Action being in the following form[10],

$$S_{\mathrm{EH}} \overset{?}{=} \frac{1}{\kappa}\int\limits_M \mathrm{Tr}\left[\frac{1}{2}\star\mathbf{R}_{\alpha\beta}J^{\alpha\beta}\wedge\mathbf{e}^\mu P_\mu\wedge\mathbf{e}^\nu P_\nu\right] = \frac{1}{2\kappa}\mathrm{Tr}\left[J^{\alpha\beta}P_\mu P_\nu\right]\int\limits_M \star\mathbf{R}_{\alpha\beta}\wedge\mathbf{e}^\mu\wedge\mathbf{e}^\nu = \frac{1}{4\kappa}\mathrm{Tr}\left[J^{\alpha\beta}[P_\mu,P_\nu]\right]\int\limits_M \star\mathbf{R}_{\alpha\beta}\wedge\mathbf{e}^\mu\wedge\mathbf{e}^\nu \overset{!}{=} 0$$

With the order of the generators inside the trace[11] being not of relevance here. This is sufficient to let down the expectations of obtaining gravity as a usual sense gauge theory. However, there is a small caveat in this argument. We have overlooked the presence of the Hodge star, which naturally is dependent on the metric, and hence, the vielbein. There might be hope that, when treated correctly, it gives rise to a non-zero action. Happily, the following is true from differential geometry: given two k–forms $\boldsymbol{\alpha}, \boldsymbol{\beta}$, it holds $\boldsymbol{\alpha}\wedge\star\boldsymbol{\beta} = \boldsymbol{\beta}\wedge\star\boldsymbol{\alpha}$. Thus,

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int\limits_M \star\left(\mathbf{e}^\beta\wedge\mathbf{e}^\rho\right)\wedge\mathbf{R}_{\beta\rho} = \frac{1}{2\kappa}\int\limits_M \frac{1}{(D-2)!}\epsilon^{\beta\rho}{}_{\alpha_3\cdots\alpha_D}\mathbf{e}^{\alpha_3}\wedge\cdots\wedge\mathbf{e}^{\alpha_D}\wedge\mathbf{R}_{\beta\rho}$$

$$S_{\mathrm{EH}} = \frac{1}{2(D-2)!\kappa}\int\limits_M \epsilon_{\alpha_1\cdots\alpha_D}\mathbf{e}^{\alpha_3}\wedge\cdots\wedge\mathbf{e}^{\alpha_D}\wedge\mathbf{R}^{\alpha_1\alpha_2} \tag{3.4}$$

Again, if we want to understand $\mathbf{e}^\mu$ as the gauge field for translations, $\epsilon_{\alpha_1\cdots\alpha_D}$ has to be the result from a bilinear on the Lie Algebra $\mathfrak{iso}(D-1,1)$. Our expectation is,

$$S_{\mathrm{EH}} \overset{?}{=} \frac{1}{2(D-2)!\kappa}\mathrm{Tr}\left[J_{\alpha_1\alpha_2}P_{\alpha_3}\cdots P_{\alpha_D}\right]\int\limits_M \mathbf{e}^{\alpha_3}\wedge\cdots\wedge\mathbf{e}^{\alpha_D}\wedge\mathbf{R}^{\alpha_1\alpha_2} = \frac{1}{2\kappa}\mathrm{Tr}\left[J_{\alpha_1\alpha_2}P_{[\alpha_3}\cdots P_{\alpha_D]}\right]\int\limits_M \mathbf{e}^{\alpha_3}\wedge\cdots\wedge\mathbf{e}^{\alpha_D}\wedge\mathbf{R}^{\alpha_1\alpha_2} \overset{!}{=} 0$$

which again is hopeless, due to the anti-symmetry of the wedge product. As long as $D \geq 4$, there will be a commutator inside the bilinear that will make the whole expression zero. But here lies the magic! For $D = 2+1$, we actually have inside the bilinear just a single instance of the translation generator. Thus, it might be possible to define a proper theory in this way! Notice,

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^\mu\wedge\mathbf{R}^{\alpha\beta} \overset{?}{=} \frac{1}{2\kappa}\int\limits_M \mathrm{Tr}\left[J_{\alpha\beta}P_\mu\right]\mathbf{e}^\mu\wedge\mathbf{R}^{\alpha\beta} \tag{3.5}$$

As long as we set up the bilinear such that $\mathrm{Tr}\left[J_{\alpha\beta}P_\mu\right] = \epsilon_{\alpha\beta\mu}$, this is perfectly fine! Up to now, it seems that might be only possible to atribute the status of usual sense gauge theory to $D = 2+1$ gravity, at least as a gauging of the group $ISO(D-1,1)$. We'll see later on how might be possible to overcome this, but for now we'll follow up specializing in this lower dimensional scenario.

---

[9] An orientation preserving one.

[10] Here we used intuition to guess that, under a change of representation, $\boldsymbol{\omega}_{\alpha\beta} \to \frac{1}{2}\boldsymbol{\omega}_{\alpha\beta}J^{\alpha\beta}$, the right thing to do is to change accordingly $\mathbf{R}_{\alpha\beta} \to \frac{1}{2}\mathbf{R}_{\alpha\beta}J^{\alpha\beta}$. We'll show this is true in the next section.

[11] We're being sloppy here. What is usually wanted is to write non-abelian gauge theories action as an integral over a trace, which is a fitting thing to do for the adjoint representation, for semi-simple algebras — not our case —. The Killing bilinear invariant form — which can be written as a trace over the adjoint representation — is also non-degenerate, what makes possible for it to be usable in the definition of the action. When every term in the action is also in the adjoint representation of a semi-simple lie algebra, this makes it possible to extrapolates the Killing form from being a bilinear to it being a multilinear — the usual trace —. But, as we said, the $\mathfrak{iso}(D-1,1)$ algebra is not semi-simple. Hence, there is no guarantee that a non-degenerate symmetric invariant bilinear form, let down the existence of a full multilinear non-degenerate trace.

Given the found prospect of $D = 2 + 1$ gravity being a gauge theory, we have to narrow down our options. First, we'll assume we have the right bilinear over our Lie Algebra, which from inspection we deduced it should be — to stress that this is not a trace, but in fact a bilinear, we'll change the notation of it —,

$$\langle J_{\alpha\beta}, P_\mu \rangle = \epsilon_{\alpha\beta\mu}, \quad \langle J_{\alpha\beta}, J_{\mu\nu} \rangle = 0, \quad \langle P_\mu, P_\nu \rangle = 0 \tag{4.1}$$

The first one is what we have shown to be the right choice, and the last two are needed by consistency with the Lie Algebra[12]. This choice is naturally non-degenerate. As we'll write everything in terms of $\mathbf{e} = \mathbf{e}^\mu P_\mu$ and $\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^{\alpha\beta} J_{\alpha\beta}$, which themselves will be inside of the bilinear and composed with the wedge product, we'll ease notation condensing these operations using the following notation,

$$\langle \boldsymbol{\alpha} \overset{\wedge}{,} \boldsymbol{\beta} \rangle := \boldsymbol{\alpha}_I \wedge \boldsymbol{\beta}_J \left\langle T^I, T^J \right\rangle, \quad [\boldsymbol{\alpha} \overset{\wedge}{,} \boldsymbol{\beta}] := \boldsymbol{\alpha}_I \wedge \boldsymbol{\beta}_J \left[T^I, T^J\right], \quad T^I \in \mathfrak{iso}(2,1)$$

Which can be used to rewrite the curvature 2–form,

$$\mathbf{R} = \frac{1}{2}\mathbf{R}_{\alpha\beta} J^{\alpha\beta} = \frac{1}{2}\mathbf{d}\boldsymbol{\omega}_{\alpha\beta} J^{\alpha\beta} + \frac{1}{2}\boldsymbol{\omega}_\alpha{}^\rho \wedge \boldsymbol{\omega}_{\rho\beta} J^{\alpha\beta} = \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}_{\mu\nu} \wedge \boldsymbol{\omega}_{\rho\sigma}\eta^{\rho\nu} J^{\mu\sigma} = \mathbf{d}\boldsymbol{\omega} + \frac{1}{8}\boldsymbol{\omega}_{\mu\nu} \wedge \boldsymbol{\omega}_{\rho\sigma} 4\eta^{[\rho|[\nu} J^{\mu]|\sigma]}$$

$$\mathbf{R} = \mathbf{d}\boldsymbol{\omega} + \frac{1}{8}\boldsymbol{\omega}_{\mu\nu} \wedge \boldsymbol{\omega}_{\rho\sigma}[J^{\rho\sigma}, J^{\nu\mu}] = \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}_{\mu\nu} \wedge \boldsymbol{\omega}_{\rho\sigma}\left[\frac{1}{2}J^{\mu\nu}, \frac{1}{2}J^{\rho\sigma}\right] = \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}] \tag{4.2}$$

This last expression permits us to rewrite the EH Action as,

$$S_{\text{EH}} = \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int_M \mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} = \frac{1}{2\kappa}\langle P_\mu, J_{\alpha\beta}\rangle \int_M \mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} = \frac{1}{\kappa}\int_M \langle \mathbf{e} \overset{\wedge}{,} \mathbf{R}\rangle = \frac{1}{\kappa}\int_M \left\langle \mathbf{e} \overset{\wedge}{,} \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\right)\right\rangle \tag{4.3}$$

Which is not quite in the usual fashion of gauge theories, as for Yang-Mills, it can be written as,

$$S_{\text{YM}} = \frac{1}{g^2}\int_M \langle \mathbf{F} \overset{\wedge}{,} \star\mathbf{F}\rangle, \quad \mathbf{F} = \mathbf{dA} + \frac{1}{2}[\mathbf{A} \overset{\wedge}{,} \mathbf{A}]$$

The key distinction is that, the YM kinetic term is quadratic in derivatives, while EH is only linear in it. This is no obstacle, as, there is indeed a particular gauge theory possible of being formulated with only one derivative in the kinetic term. Surprising, this construction is particular to $D = 2 + 1$, as can be seen from the form degree. This theory is the Chern-Simons one,

$$S_{\text{CS}}[\mathbf{A}] = \frac{k}{4\pi}\int_M \left\langle \mathbf{A} \overset{\wedge}{,} \left(\mathbf{dA} + \frac{1}{3}[\mathbf{A} \overset{\wedge}{,} \mathbf{A}]\right)\right\rangle \tag{4.4}$$

As striking as it may be the similarities of both actions, they aren't equal. EH possesses two dynamical fields, while CS has only one. This isn't a big of a problem, because, as we're trying to interpret EH as being a gauge theory of the $ISO(2,1)$ group, what we should expect is that the connection of this new CS-like theory has values over this algebra, that is, $\mathbf{A} = \mathbf{A}_I T^I$, where $T^I \in \mathfrak{iso}(2,1)$. This resolves the question: $\mathbf{A}$ must be then a linear combination of both $\mathbf{e}, \boldsymbol{\omega}$. Hence, we'll compute what's the CS theory for a connection of the type $\mathbf{A} = \boldsymbol{\omega} + \mathbf{e}$,

$$S_{\text{CS}}[\mathbf{A}] = \frac{k}{4\pi}\int_M \left\langle (\boldsymbol{\omega} + \mathbf{e}) \overset{\wedge}{,} \left(\mathbf{d}(\boldsymbol{\omega} + \mathbf{e}) + \frac{1}{3}[\boldsymbol{\omega} + \mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega} + \mathbf{e}]\right)\right\rangle$$

$$S_{\text{CS}}[\mathbf{A}] = \frac{k}{4\pi}\int_M \left\langle \boldsymbol{\omega} \overset{\wedge}{,} \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{3}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\right)\right\rangle + \frac{k}{4\pi}\int_M \{\langle \boldsymbol{\omega} \overset{\wedge}{,} \mathbf{de}\rangle + \langle \mathbf{e} \overset{\wedge}{,} \mathbf{d}\boldsymbol{\omega}\rangle + \langle \mathbf{e} \overset{\wedge}{,} \mathbf{de}\rangle$$

$$+\frac{1}{3}\langle \boldsymbol{\omega} \overset{\wedge}{,} ([\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{e}] + [\mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega}] + [\mathbf{e} \overset{\wedge}{,} \mathbf{e}])\rangle + \frac{1}{3}\langle \mathbf{e} \overset{\wedge}{,} ([\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}] + [\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{e}] + [\mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega}] + [\mathbf{e} \overset{\wedge}{,} \mathbf{e}])\rangle\Big]$$

Now we can use the nice form of our bilinear to discard some terms. Notice that it is only non-zero when one entry is proportional to the Lorentz generators and the other to the translation generators. Thus, terms like $\langle \boldsymbol{\omega} \overset{\wedge}{,} \mathbf{d}\boldsymbol{\omega}\rangle$ will vanish, as they have Lorentz generators on the both entries. The same is true also to $\langle \boldsymbol{\omega} \overset{\wedge}{,} [\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\rangle$, $\langle \mathbf{e} \overset{\wedge}{,} \mathbf{de}\rangle$ and $\langle \mathbf{e} \overset{\wedge}{,} [\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{e}]\rangle$; the term $[\mathbf{e} \overset{\wedge}{,} \mathbf{e}]$ vanishes due to the $\mathfrak{iso}(2,1)$ algebra. Collecting the remaining terms,

$$S_{\text{CS}}[\mathbf{A}] = \frac{k}{4\pi}\int_M \left\langle \mathbf{e} \overset{\wedge}{,} \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{3}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\right) + \boldsymbol{\omega} \overset{\wedge}{,} \mathbf{de} + \frac{1}{3}\boldsymbol{\omega} \overset{\wedge}{,} ([\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{e}] + [\mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega}])\right\rangle$$

For $\boldsymbol{\alpha}, \boldsymbol{\beta}$ p, q–forms, $\langle \boldsymbol{\alpha} \overset{\wedge}{,} \boldsymbol{\beta}\rangle = \boldsymbol{\alpha}_I \wedge \boldsymbol{\beta}_J \left\langle T^I, T^J\right\rangle = (-)^{pq}\boldsymbol{\beta}_J \wedge \boldsymbol{\alpha}_I \left\langle T^J, T^I\right\rangle = (-)^{pq}\langle \boldsymbol{\beta} \overset{\wedge}{,} \boldsymbol{\alpha}\rangle$, thus,

$$S_{\text{CS}}[\mathbf{A}] = \frac{k}{4\pi}\int_M \left\langle \mathbf{e} \overset{\wedge}{,} \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{3}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\right) + \boldsymbol{\omega} \overset{\wedge}{,} \mathbf{de} + \frac{2}{3}\boldsymbol{\omega} \overset{\wedge}{,} [\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{e}]\right\rangle$$

[12] All the requirements of this bilinear are: being symmetric, invariant and non-degenerate. The requirement of invariance can be written as $[Y, \langle X, Z\rangle] = 0 \Rightarrow \langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle$, $\forall X, Y, Z \in \mathfrak{g}$. This explicit choice of bilinear we cited is not unique as the algebra is not absolutely simple.

The Lie Algebra invariance[13] of the bilinear can be expressed as $\langle [T^I, T^J], T^K \rangle = \langle T^I, [T^J, T^K] \rangle$, which fortunately implies $\langle \boldsymbol{\omega} \curlywedge [\boldsymbol{\omega} \curlywedge \mathbf{e}] \rangle = \langle [\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}] \curlywedge \mathbf{e} \rangle = \langle \mathbf{e} \curlywedge [\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}] \rangle$. Hence,

$$S_{\mathrm{CS}}[\mathbf{A}] = \frac{k}{4\pi} \int_M \left\langle \mathbf{e} \curlywedge \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{3}[\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}] \right) + \boldsymbol{\omega} \curlywedge \mathbf{de} + \frac{2}{3}\mathbf{e} \curlywedge [\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}] \right\rangle = \frac{k}{4\pi} \int_M \langle \mathbf{e} \curlywedge (\mathbf{d}\boldsymbol{\omega} + [\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}]) + \boldsymbol{\omega} \curlywedge \mathbf{de} \rangle$$

Now, we integrate by parts, $\langle \boldsymbol{\omega} \curlywedge \mathbf{de} \rangle = -\mathbf{d} \langle \boldsymbol{\omega} \curlywedge \mathbf{e} \rangle + \langle \mathbf{d}\boldsymbol{\omega} \curlywedge \mathbf{e} \rangle$,

$$S_{\mathrm{CS}}[\mathbf{A}] = \frac{k}{4\pi} \int_M \langle \mathbf{e} \curlywedge (\mathbf{d}\boldsymbol{\omega} + [\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}]) - \mathbf{d}(\boldsymbol{\omega} \curlywedge \mathbf{e}) + \mathbf{d}\boldsymbol{\omega} \curlywedge \mathbf{e} \rangle = \frac{k}{2\pi} \int_M \left\langle \mathbf{e} \curlywedge \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}] \right) \right\rangle - \frac{k}{4\pi} \int_{\partial M} \langle \boldsymbol{\omega} \curlywedge \mathbf{e} \rangle$$

$$S_{\mathrm{CS}}[\mathbf{A}] = \frac{k\kappa}{2\pi} \frac{1}{\kappa} \int_M \langle \mathbf{e} \curlywedge \mathbf{R} \rangle - \frac{k}{4\pi} \int_{\partial M} \langle \boldsymbol{\omega} \curlywedge \mathbf{e} \rangle = \frac{k\kappa}{2\pi} \left(S_{\mathrm{EH}} + \frac{1}{2\kappa} \int_{\partial M} \langle \mathbf{e} \curlywedge \boldsymbol{\omega} \rangle \right) \tag{4.5}$$

This is one of the main results we wanted to reach! We successfully described $D = 2+1$ gravity by means of a *usual sense* gauge theory of the group $ISO(2,1)$! Of course we didn't managed to describe it as an Yang-Mills type, mainly due to it being impossible, as Yang-Mills has non trivial bulk dynamics, $\mathbf{d}_\nabla \star \mathbf{F} = 0$, a property that $D = 2+1$ gravity don't possesses. As we discussed in the beginning, this is totally compatible with the Chern-Simons dynamics in the bulk, $\mathbf{F} = 0 \Rightarrow \mathbf{A} = g^{-1}\mathbf{d}g$, trivial. Despite this seeming at least disappointing, as we have switched from a trivial theory to another trivial theory, the beauty of Chern-Simons don't really lies in the bulk dynamics, but in the boundary of the manifold, as it's used in Condensed Matter theory to describe topological insulators — in which the non-trivial part of the system lies in the boundary —. This rises hopes of finding non-trivial dynamics in the boundary for $D = 2+1$ gravity, despite of already being found non trivial solutions to the bulk of $D = 2+1$ gravity, the BTZ Black Hole, which is consistent with the Chern-Simons interpretation, as it's bulk dynamics can be made less trivial with the inclusion of more complex topologies of the space-time[14]. The fact that classical $D = 2+1$ gravity has trivial bulk dynamics — with possible non-trivial boundary dynamics — does not say much about how a possible quantization of it should be done. It's clear that quantization with respect to the metric is highly intricate due to the — apparent — non-polinomial character of the EH Action, in contrast with it's cousin, the $D = 1+1$ gravity, which despite also being trivial, has an way clearer process of quantization. As the Action is a topological term — proportional to the genus —, the quantization procedure is only to sum over all possible topologies taken into account possible moduli spaces. Therefore, the relationship of $D = 2+1$ gravity with Chern-Simons open doors to a possible more straightforward way of quantizing it. As the quantization of Chern-Simons Theory is well known, this toy model derived here shows that the claim of gravity being non-polinomial might not be the end of the story, at least for $D = 2+1$.

As we mentioned before, the whole application of the CS theory is to non-trivial boundaries, or at least, to non-trivial topologies, as otherwise there is no non-trivial solutions. At least this rises an eyebrow when looking at (4.5), where we see a contribution of a boundary term, which has a funny looking structure. To see what role is it playing here, let's try to obtain the equations of motion of $\boldsymbol{\omega}$ of the EH Action,

$$0 = S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega} + \delta\boldsymbol{\omega}] - S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega}] = \frac{1}{\kappa} \int_M \left\langle \mathbf{e} \curlywedge \left(\mathbf{d}\delta\boldsymbol{\omega} + \frac{1}{2}[\delta\boldsymbol{\omega} \curlywedge \boldsymbol{\omega}] + \frac{1}{2}[\boldsymbol{\omega} \curlywedge \delta\boldsymbol{\omega}] \right) \right\rangle$$

$$0 = \frac{1}{\kappa} \int_M \langle -\mathbf{d}(\mathbf{e} \curlywedge \delta\boldsymbol{\omega}) + \mathbf{de} \curlywedge \delta\boldsymbol{\omega} + \mathbf{e} \curlywedge [\boldsymbol{\omega} \curlywedge \delta\boldsymbol{\omega}] \rangle$$

$$0 = -\frac{1}{\kappa} \int_{\partial M} \langle \mathbf{e} \curlywedge \delta\boldsymbol{\omega} \rangle + \frac{1}{\kappa} \int_M \langle \mathbf{de} \curlywedge \delta\boldsymbol{\omega} + [\mathbf{e} \curlywedge \boldsymbol{\omega}] \curlywedge \delta\boldsymbol{\omega} \rangle = -\frac{1}{\kappa} \int_{\partial M} \langle \mathbf{e} \curlywedge \delta\boldsymbol{\omega} \rangle + \frac{1}{\kappa} \int_M \langle (\mathbf{de} + [\boldsymbol{\omega} \curlywedge \mathbf{e}]) \curlywedge \delta\boldsymbol{\omega} \rangle$$

The boundary equation of motion is $\mathbf{de} + [\boldsymbol{\omega} \curlywedge \mathbf{e}] = 0$, which is our familiar torsionless condition! In addition to our nice equation of motion, we must also guarantee with the right boundary conditions that this boundary term in the variation of the Action is zero. Of course the condition $\mathbf{e}\big|_{\partial M} = 0$ is not desirable, and we can't impose $\delta\boldsymbol{\omega}\big|_{\partial M} = 0$ otherwise the system would be overconstrained — due to the hyperbolic nature of the equations of motion —. The only acceptable boundary condition is $\delta\mathbf{e}\big|_{\partial M} = 0$, which forces us to add a boundary term in the EH Action to cancel this variation and to obtain an well defined initial value problem. The form of this term is straightforward to see. From the equations of motion, we got,

$$S_{\mathrm{EH}} = \frac{1}{\kappa} \int_M \langle \mathbf{e} \curlywedge \mathbf{R} \rangle \to \frac{1}{\kappa} \int_M \langle \mathbf{e} \curlywedge \mathbf{R} \rangle + \frac{1}{\kappa} \int_{\partial M} \langle \mathbf{e} \curlywedge \boldsymbol{\omega} \rangle \tag{4.6}$$

contribution which is called the Gibbons-Hawking-York term. Notice that our Chern-Simons Action naturally incorporates this term, but, with an additional half in from of it. This undermines the purpose of it: to make the initial value problem well defined. However, this is no surprise, as it's well known that the initial value problem of the Chern-Simons Action for manifolds with boundary is not well defined; it's not even gauge invariant. Nevertheless, all of this is resolved by the introduction of Wess-Zumino-Witen model in the boundary. We'll not discuss this, and as long as we're in a boundaryless manifold, everything holds tight.

---

[13]Ibid.

[14]In $D = 2+1$ dimensions the existence of a Black Hole makes the first Homotopy Class of the spatial slices non-trivial, what can be understood as a non-trivial topology.

5. Inclusion of a Cosmological Constant

After successfully rewriting the EH Action in terms of a gauge theory, the next natural desire is to incorporate a cosmological constant, as it's of relevance for our universe. First, we must work out what is the appearance of this term in the form language, let's recall the action with the cosmological constant,

$$S_{\text{EH}} \to \frac{1}{2\kappa}\int_M \mathrm{d}^D x\,\sqrt{|g|}(R-2\Lambda) \tag{5.1}$$

All we need is to rewrite the volume form, which we have already done in (2.1), just substituting it back (2.1) and (3.5),

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M \epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge\mathbf{R}^{\alpha\beta} - \frac{\Lambda}{\kappa 3!}\int_M \epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta$$

In the same line as we have been doing, we want the action to be writable in terms of the variables $\mathbf{e}=\mathbf{e}^\mu P_\mu$. The first term we already showed how can be done. For the second one is where some subtleties starts to appear. First, the striking difference between these two terms is that the second one has three instances of a gauge field. To be able to insert inside our bilinear three instances of the gauge field is only possible through using the anti-symmetry of the wedge product to produce a commutator, that is, we expect,

$$-\frac{\Lambda}{\kappa 3!}\int_M \epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta \overset{?}{=} -\frac{\Lambda}{\kappa 3!}\int_M \langle P_\mu,[P_\alpha,P_\beta]\rangle\,\mathbf{e}^\mu\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta = -\frac{\Lambda}{\kappa 3!}\int_M \langle \mathbf{e} \curlywedge [\mathbf{e}\curlywedge\mathbf{e}]\rangle \overset{!}{=} 0$$

This is the only possible way of inserting three terms inside a bilinear. Yet, it's flawed. The reason is obvious, $[P_\alpha,P_\beta]=0$. This means that it's inconsistent to add a cosmological constant and still requires the gauge group to be $ISO(2,1)$. Is this correct? Yes! When introducing the cosmological constant, the solutions will of course be related to the (A)dS space-times, and in particular to the vacuum (A)dS space-times. The isometry group isn't $ISO(D-1,1)$, but actually is $SO(D-1,2)$ for AdS — $\Lambda<0$ — and $SO(D,1)$ for dS — $\Lambda>0$ —. Thus, alike to the $\Lambda=0$ case we worked out, we concluded that we should gauge $ISO(D-1,1)$, the isometry group of Minkowski space-time. Analogously, with $\Lambda\neq 0$ we can only suppose that we must gauge the respective isometry groups of the (A)dS space-times. Does this allows the inclusion of a cosmological constant? First, we have to look what is to be taken as translations from the groups $SO(3,1)$ and $SO(2,2)$. We'll not dwell on details here, but, all of the 6 generators from these two groups can be rearranged in a similar form of the algebra of $ISO(2,1)$ — which can be seen as a non-central extension of the Poincaré group/algebra. This is consistent with the interpretation of Poincaré group being a Inönü-Wigner contraction of (A)dS —,

$$\left[J^{\alpha\beta},J^{\mu\nu}\right]=4\eta^{[\alpha|[\mu}J^{\nu]|\beta]},\quad \left[J^{\alpha\beta},P^\mu\right]=2P^{[\alpha}\eta^{\beta]\mu},\quad [P^\mu,P^\nu]=\pm\frac{1}{L^2}J^{\mu\nu}\quad \begin{cases}+\to AdS\\ -\to dS\end{cases} \tag{5.2}$$

in which L is the radius of the (A)dS space-time, it's related to the cosmological constant by, $|\Lambda|=L^{-2}$. This perfectly matches with our guess to the form of this term,

$$-\frac{\Lambda}{\kappa 3!}\int_M \epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta = -\frac{\Lambda}{\kappa 3!}\int_M \langle P_\mu, J_{\alpha\beta}\rangle\,\mathbf{e}^\mu\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta \overset{!}{=} \frac{1}{\kappa 3!}\int_M \langle P_\mu,[P_\alpha,P_\beta]\rangle\,\mathbf{e}^\mu\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta = \frac{1}{\kappa 3!}\int_M \langle \mathbf{e}\curlywedge[\mathbf{e}\curlywedge\mathbf{e}]\rangle$$

As long as the invariant symmetric non-degenerate bilinear still exists for these groups, a fact that would have to be checked. In fact, indeed there is such a bilinear, which is sadly not as pretty as our former[15],

$$\langle J_{\alpha\beta},P_\mu\rangle=\epsilon_{\alpha\beta\mu},\quad \langle J_{\alpha\beta},J_{\mu\nu}\rangle=\epsilon^\rho{}_{\alpha\beta}\epsilon_{\rho\mu\nu},\quad \langle P_\mu,P_\nu\rangle=\pm\frac{1}{L^2}\eta_{\mu\nu} \tag{5.3}$$

The existence of this bilinear is almost a miracle happening only in $D=2+1$. For $D\neq 3$, the algebras of the isometry groups of (A)dS allows only one bilinear, the Killing form, which if used here in $D=2+1$ would not give the desired result, as we'll comment later on. Now we can rework the form of the Chern-Simons Action, but, as the bilinear isn't that simple it's not really possible to get rid of the majority of the unwanted terms. So, we'll have to play a little with definitions and set the connection $\mathbf{A}^\pm=\boldsymbol{\omega}\pm\mathbf{e}$,

$$S_{\text{CS}}\left[\mathbf{A}^\pm\right]=\frac{k}{4\pi}\int_M \left\langle(\boldsymbol{\omega}\pm\mathbf{e})\curlywedge\left(\mathbf{d}(\boldsymbol{\omega}\pm\mathbf{e})+\frac{1}{3}[\boldsymbol{\omega}\pm\mathbf{e}\curlywedge\boldsymbol{\omega}\pm\mathbf{e}]\right)\right\rangle$$

$$\begin{aligned}S_{\text{CS}}\left[\mathbf{A}^\pm\right]=&\frac{k}{4\pi}\int_M \left\langle\boldsymbol{\omega}\curlywedge\left(\mathbf{d}\boldsymbol{\omega}+\frac{1}{3}[\boldsymbol{\omega}\curlywedge\boldsymbol{\omega}]\right)\right\rangle \pm\frac{k}{4\pi}\int_M\Big\{\langle\boldsymbol{\omega}\curlywedge\mathbf{de}\rangle+\langle\mathbf{e}\curlywedge\mathbf{d}\boldsymbol{\omega}\rangle\mp\langle\mathbf{e}\curlywedge\mathbf{de}\rangle\\ &+\frac{1}{3}\left\langle\boldsymbol{\omega}\curlywedge([\boldsymbol{\omega}\curlywedge\mathbf{e}]+[\mathbf{e}\curlywedge\boldsymbol{\omega}]\mp[\mathbf{e}\curlywedge\mathbf{e}])\right\rangle+\frac{1}{3}\left\langle\mathbf{e}\curlywedge([\boldsymbol{\omega}\curlywedge\boldsymbol{\omega}]\mp[\boldsymbol{\omega}\curlywedge\mathbf{e}]\mp[\mathbf{e}\curlywedge\boldsymbol{\omega}]+[\mathbf{e}\curlywedge\mathbf{e}])\right\rangle\Big\}\end{aligned}$$

Here lies the reason for us to considering the connection with a generic sign $\pm$: if we sum $S_{\text{CS}}[\mathbf{A}^+]-S_{\text{CS}}[\mathbf{A}^-]$ all the unwanted terms will vanish due them being even powers of $\mathbf{e}$,

$$S_{\text{CS}}\left[\mathbf{A}^+\right]-S_{\text{CS}}\left[\mathbf{A}^-\right]=2\frac{k}{4\pi}\int_M\left\{\langle\boldsymbol{\omega}\curlywedge\mathbf{de}\rangle+\langle\mathbf{e}\curlywedge\mathbf{d}\boldsymbol{\omega}\rangle+\frac{1}{3}\left\langle\boldsymbol{\omega}\curlywedge([\boldsymbol{\omega}\curlywedge\mathbf{e}]+[\mathbf{e}\curlywedge\boldsymbol{\omega}])\right\rangle+\frac{1}{3}\left\langle\mathbf{e}\curlywedge([\boldsymbol{\omega}\curlywedge\boldsymbol{\omega}]+[\mathbf{e}\curlywedge\mathbf{e}])\right\rangle\right\}$$

---

[15] This bilinear is not unique.

Now we follow the usual means, grouping the remaining terms, integrating by parts, and using the invariance of the bilinear form,

$$S_{\mathrm{CS}}[\mathbf{A}^+] - S_{\mathrm{CS}}[\mathbf{A}^-] = \frac{k}{2\pi}\int\limits_M \left\{ -\mathbf{d}\left\langle \boldsymbol{\omega} \mathbin{\overset{\wedge}{,}} \mathbf{e}\right\rangle + \left\langle \mathbf{d}\boldsymbol{\omega} \mathbin{\overset{\wedge}{,}} \mathbf{e}\right\rangle + \left\langle \mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{d}\boldsymbol{\omega}\right\rangle + \frac{2}{3}\left\langle [\boldsymbol{\omega} \mathbin{\overset{\wedge}{,}} \boldsymbol{\omega}] \mathbin{\overset{\wedge}{,}} \mathbf{e}\right\rangle + \frac{1}{3}\left\langle \mathbf{e} \mathbin{\overset{\wedge}{,}} ([\boldsymbol{\omega} \mathbin{\overset{\wedge}{,}} \boldsymbol{\omega}] + [\mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{e}])\right\rangle \right\}$$

$$S_{\mathrm{CS}}[\mathbf{A}^+] - S_{\mathrm{CS}}[\mathbf{A}^-] = \frac{k}{2\pi}\int\limits_M \left\{ -\mathbf{d}\left\langle \boldsymbol{\omega} \mathbin{\overset{\wedge}{,}} \mathbf{e}\right\rangle + 2\left\langle \mathbf{e} \mathbin{\overset{\wedge}{,}} \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \mathbin{\overset{\wedge}{,}} \boldsymbol{\omega}]\right)\right\rangle + \frac{1}{3}\left\langle \mathbf{e} \mathbin{\overset{\wedge}{,}} [\mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{e}]\right\rangle \right\}$$

$$\frac{\pi}{k\kappa}\left(S_{\mathrm{CS}}[\mathbf{A}^+] - S_{\mathrm{CS}}[\mathbf{A}^-]\right) = \frac{1}{2\kappa}\int\limits_{\partial M} \left\langle \mathbf{e} \mathbin{\overset{\wedge}{,}} \boldsymbol{\omega}\right\rangle + \frac{1}{\kappa}\int\limits_M \left\langle \mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{R}\right\rangle + \frac{1}{\kappa 3!}\int\limits_M \left\langle \mathbf{e} \mathbin{\overset{\wedge}{,}} [\mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{e}]\right\rangle \tag{5.4}$$

exactly the result we're expecting to obtain, a relation between the EH Action with a cosmological constant term, and a — actually two — Chern-Simons theory! As previously discussed this process additionally gives half of the GHY boundary term.

6. About D=3+1

With our astonishing success achieved in $D = 2+1$, a natural follow up is to ask: what can be said about $D = 3+1$? In the derivation of (3.5), we discarded $D \neq 3$ reasoning that no non-zero action could be build, from $\mathbf{e}, \boldsymbol{\omega}$ with a multilinear trace in the algebra $\mathfrak{iso}(D-1,1)$, due to the explicit appearance of $[P_\mu, P_\nu]$ inside the trace, which is naturally zero for $\mathfrak{iso}(D-1,1)$. Nevertheless, in our pursue to include the cosmological constant, we concluded that for non-zero cosmological constant space-times, we should not be looking towards $\mathfrak{iso}(D-1,1)$, but actually to $\mathfrak{so}(D,1)$ and $\mathfrak{so}(D-1,2)$, which have a non-zero translation commutator, (5.2). This means we can indeed build non-zero actions using a bilinear form in $D \neq 3$, as long as we stick to the non-zero cosmological constant case. To see how this can be done, we pick up from (3.4), with the cosmological constant term,

$$S_{\mathrm{EH}} = \frac{1}{4\kappa}\int\limits_M \epsilon_{\alpha\beta\mu\nu}\mathbf{e}^\alpha \wedge \mathbf{e}^\beta \wedge \mathbf{R}^{\mu\nu} - \frac{\Lambda}{\kappa 4!}\int\limits_M \epsilon_{\alpha\beta\mu\nu}\mathbf{e}^\alpha \wedge \mathbf{e}^\beta \wedge \mathbf{e}^\mu \wedge \mathbf{e}^\nu$$

which as always, we'll try to write as a bilinear. Notice that each two vielbein terms will have to couple to a commutator of the translations, due to the wedge anti-symmetry, so our guess has to be,

$$S_{\mathrm{EH}} \overset{?}{=} -\frac{1}{4\Lambda\kappa}\left\langle [P_\alpha, P_\beta], J_{\mu\nu}\right\rangle \int\limits_M \mathbf{e}^\alpha \wedge \mathbf{e}^\beta \wedge \mathbf{R}^{\mu\nu} - \frac{1}{\Lambda\kappa 4!}\left\langle [P_\alpha, P_\beta], [P_\mu, P_\nu]\right\rangle \int\limits_M \mathbf{e}^\alpha \wedge \mathbf{e}^\beta \wedge \mathbf{e}^\mu \wedge \mathbf{e}^\nu$$

$$S_{\mathrm{EH}} \overset{?}{=} -\frac{1}{2\Lambda\kappa}\int\limits_M \left\langle [\mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{e}] \mathbin{\overset{\wedge}{,}} \mathbf{R}\right\rangle - \frac{1}{\Lambda\kappa 4!}\int\limits_M \left\langle [\mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{e}] \mathbin{\overset{\wedge}{,}} [\mathbf{e} \mathbin{\overset{\wedge}{,}} \mathbf{e}]\right\rangle \tag{6.1}$$

this is indeed consistent, as in there exists an invariant symmetric non-degenerate bilinear form such as,

$$\langle J_{\alpha\beta}, J_{\mu\nu}\rangle = \epsilon_{\alpha\beta\mu\nu}, \quad \langle J_{\alpha\beta}, P_\mu\rangle = 0, \quad \langle P_\mu, P_\nu\rangle = \pm\frac{1}{L^2}\eta_{\mu\nu}$$

The only problem with (6.1) is the linearity in derivatives, as there is no gauge theory in $D = 3+1$ with only linear derivatives terms, so it's not clear of what can be made of it. Of course, if we're interested in non-minimal Actions, it's possible to add further terms as $\langle \mathbf{R} \mathbin{\overset{\wedge}{,}} \star\mathbf{R}\rangle$ or $\langle \mathbf{R} \mathbin{\overset{\wedge}{,}} \mathbf{R}\rangle$, which are much more similar to Yang-Mills theories, but this would go more in the interpretation of decoupling the translations from the Poincaré group and interpreting gravity as a gauge theory of $SO(D-1,1)$ coupled to a inertial frame field, which does not seem to be self consistent as for a kinetic term $\langle \mathbf{R} \mathbin{\overset{\wedge}{,}} \star\mathbf{R}\rangle$ is already coupled to the vielbein through the Hodge star, and a term like $\langle \mathbf{R} \mathbin{\overset{\wedge}{,}} \mathbf{R}\rangle$ does not provide dynamics as is topological.

7. Concluding Remarks

The two main results we have achieved here are summarized in (4.5), (5.4), that is, the equivalence of, classical, $2+1$–dimensional gravity as a gauge theory of the groups $ISO(2,1)$, $SO(3,1)$ and $SO(2,2)$, depending on the sign of the cosmological constant. What does this provides to us? Besides the realization of common sense knowledge of gravity being a gauge theory of the Poincaré Group, which we argued is not trivial to $D > 3$, it also provides an well defined quantization procedure to this lower dimensional toy model, the one inherited of the Chern-Simons. More than that, the resulting theory is even renormalizable and possesses a zero beta function[8]. This is due to the cosmological constant being a structure constant of the Lie Algebra and the κ being related to the Chern-Simons level k, which is allowed only a discrete number of values. Nevertheless, quantization of $2+1$ gravity has also been argued only to be equivalent to Chern-Simons perturbatively[9], i.e., expanded around a classical solutions, which in gravity have the constrain of the vielbein to be invertible, so it's not clear what sense should made of a non-classical solution as $\mathbf{A} = 0 \Rightarrow \mathbf{e} = \boldsymbol{\omega} = 0$, and whether or not it should be included in the path integral. According to our cousin String Theory, we should allow only for invertible metrics. This might seem to be the case here, but the matter is not settled yet.

References

[1] Steven Carlip. *Lectures in (2+1)-Dimensional Gravity*. 1995. arXiv: gr-qc/9503024 [gr-qc]. URL: https://arxiv.org/abs/gr-qc/9503024.

[2] S Deser and R Jackiw. "Three-dimensional cosmological gravity: Dynamics of constant curvature". In: *Annals of Physics* 153.2 (1984), pp. 405–416. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(84)90025-3. URL: https://www.sciencedirect.com/science/article/pii/0003491684900253.

[3] Stanley Deser, R. Jackiw, and Gerard 't Hooft. "Three-Dimensional Einstein Gravity: Dynamics of Flat Space". In: *Annals Phys.* 152 (1984), p. 220. DOI: 10.1016/0003-4916(84)90085-X.

[4] Laura Donnay. *Asymptotic dynamics of three-dimensional gravity*. 2016. arXiv: 1602.09021 [hep-th]. URL: https://arxiv.org/abs/1602.09021.

[5] Emil Martinec. "Soluble systems in quantum gravity". In: *Phys. Rev. D* 30 (6 Sept. 1984), pp. 1198–1204. DOI: 10.1103/PhysRevD.30.1198. URL: https://link.aps.org/doi/10.1103/PhysRevD.30.1198.

[6] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. Gravitation pt. 3. W. H. Freeman, 1973. ISBN: 9780716703440. URL: https://books.google.com.br/books?id=w4Gigq3tY1kC.

[7] R.M. Wald. *General Relativity*. University of Chicago Press, 1984. ISBN: 9780226870328. URL: https://books.google.com.br/books?id=ibSdQgAACAAJ.

[8] Edward Witten. "(2+1)-Dimensional Gravity as an Exactly Soluble System". In: *Nucl. Phys. B* 311 (1988), p. 46. DOI: 10.1016/0550-3213(88)90143-5.

[9] Edward Witten. *Three-Dimensional Gravity Revisited*. 2007. arXiv: 0706.3359 [hep-th]. URL: https://arxiv.org/abs/0706.3359.