

# Homework I

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## Problem 1

1.A)

1.B)

## Problem 2

2.A)

2.B)

## Problem 3

### 3.A)

The Gamma Function can be represented in the complex plane domain,  $\text{Re}(s) > 1$ , as the following integral,

$$\Gamma(s) = \int_0^{\infty} dt \exp(-t)t^{s-1}, \quad \text{Re}(s) > 1 \quad (3.1)$$

Which is also the subset of the complex plane in which this integral converges, of course this representation of the Gamma Function in a open set is sufficient for obtain an analytical continuation to the whole complex plane. Obviously, the integral is invariant under relabeling the dummy variable  $t$ , we make the following choice  $t \rightarrow nt$  — Assuming  $n > 0$  —,

$$\begin{aligned} \Gamma(s) &= \int_0^{\infty} d(nt) \exp(-nt)(nt)^{s-1}, \quad \text{Re}(s) > 1 \\ \Gamma(s) &= n^s \int_0^{\infty} dt \exp(-nt)t^{s-1}, \quad \text{Re}(s) > 1 \\ n^{-s}\Gamma(s) &= \int_0^{\infty} dt \exp(-nt)t^{s-1}, \quad \text{Re}(s) > 1 \\ \sum_{n=1}^{\infty} n^{-s}\Gamma(s) &= \sum_{n=1}^{\infty} \int_0^{\infty} dt \exp(-nt)t^{s-1}, \quad \text{Re}(s) > 1 \end{aligned}$$

The sum in the left-hand side is recognized as the representation for the Zeta Function in the domain  $\text{Re}(s) > 1$ , which is also the domain of convergence of the sum,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re}(s) > 1$$

So that,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \int_0^{\infty} dt \exp(-nt)t^{s-1}, \quad \text{Re}(s) > 1$$

About the right-hand side, to be able to exchange the integral and the sum is sufficient that,

$$\begin{aligned} \int_0^{\infty} dt \sum_{n=1}^{\infty} \|\exp(-nt)t^{s-1}\| &< \infty, \quad \text{Re}(s) > 1 \\ \int_0^{\infty} dt \sum_{n=1}^{\infty} \exp(-nt)\|t^{s-1}\| &< \infty, \quad \text{Re}(s) > 1 \end{aligned}$$

$$\int_0^{\infty} dt \sum_{n=1}^{\infty} \exp(-nt) t^{\operatorname{Re}(s)-1} < \infty, \quad \operatorname{Re}(s) > 1$$

The sum now is a simple geometric series, giving,

$$\int_0^{\infty} dt \frac{t^{\operatorname{Re}(s)-1}}{\exp(t) - 1} < \infty, \quad \operatorname{Re}(s) > 1$$

The dangerous behavior that could make the integral diverges is the one at  $t \rightarrow 0$ , an indeed,  $\operatorname{Re}(s) > 1$ , is sufficient for the convergence of this integral, which can be seen at,

$$\int_0^{\epsilon} dt \frac{t^{\operatorname{Re}(s)-1}}{\exp(t) - 1} \approx \int_0^{\epsilon} dt \frac{t^{\operatorname{Re}(s)-1}}{t + \mathcal{O}(t^2)} \approx \int_0^{\epsilon} t^{\operatorname{Re}(s)-2} = \left. \frac{t^{\operatorname{Re}(s)-1}}{\operatorname{Re}(s) - 1} \right|_0^{\epsilon}$$

Which shows the integral is really finite at  $t \rightarrow 0$  with  $\operatorname{Re}(s) > 1$ , hence, switching the integral and the sum is justified, so,

$$\begin{aligned} \zeta(s)\Gamma(s) &= \sum_{n=1}^{\infty} \int_0^{\infty} dt \exp(-nt) t^{s-1}, \quad \operatorname{Re}(s) > 1 \\ \zeta(s)\Gamma(s) &= \int_0^{\infty} dt \sum_{n=1}^{\infty} \exp(-nt) t^{s-1}, \quad \operatorname{Re}(s) > 1 \end{aligned}$$

Where again we have the sum of a geometric series, giving,

$$\zeta(s)\Gamma(s) = \int_0^{\infty} dt \frac{t^{s-1}}{\exp(t) - 1}, \quad \operatorname{Re}(s) > 1$$

### 3.B)

The objective here is to make an analytical continuation to  $\operatorname{Re}(s) > -2$  of the expression found in the later item. First of all, the reason the later expression is only well defined in  $\operatorname{Re}(s) > 1$ , is due to the divergence of the integrand at  $t \rightarrow 0$  for  $\operatorname{Re}(s) \leq 1$ , this is only because  $(\exp(t) - 1)^{-1}$  has a simple pole at  $t = 0$ , which is also the only pole of this function, so to get the Laurent series we first find the residue of it,

$$\begin{aligned} \operatorname{Res}_{t=0} \left( \frac{1}{\exp(t) - 1} \right) &= \left. \frac{t}{\exp(t) - 1} \right|_{t=0} \\ \operatorname{Res}_{t=0} \left( \frac{1}{\exp(t) - 1} \right) &= \left. \frac{t}{t + \mathcal{O}(t^2)} \right|_{t=0} \\ \operatorname{Res}_{t=0} \left( \frac{1}{\exp(t) - 1} \right) &= \left. \frac{1}{1 + \mathcal{O}(t)} \right|_{t=0} \\ \operatorname{Res}_{t=0} \left( \frac{1}{\exp(t) - 1} \right) &= 1 \end{aligned}$$

As this is the only pole, we get a Laurent series starting as,

$$\frac{1}{\exp(t) - 1} = \frac{1}{t} + \mathcal{O}(t^0)$$

To get the following terms we just make a trivial Taylor series of the function  $(\exp(t) - 1)^{-1} - t^{-1}$

$$\begin{aligned} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} \right] \Big|_0 &= \frac{1 + t - \exp(t)}{t[\exp(t) - 1]} \Big|_0 \\ \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} \right] \Big|_0 &= \frac{-\frac{t^2}{2} + \mathcal{O}(t^3)}{t[t + \mathcal{O}(t^2)]} \Big|_0 \\ \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} \right] \Big|_0 &= \frac{-\frac{t^2}{2} + \mathcal{O}(t^3)}{t^2[1 + \mathcal{O}(t)]} \Big|_0 \\ \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} \right] \Big|_0 &= -\frac{1}{2} \end{aligned}$$

In other words,

$$\frac{1}{\exp(t) - 1} = \frac{1}{t} - \frac{1}{2} + \mathcal{O}(t)$$

The next term of the series will be,

$$\begin{aligned} \frac{d}{dt} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} \right] \Big|_0 &= \frac{1}{t^2} - \frac{\exp(t)}{[\exp(t) - 1]^2} \\ &= \frac{\exp(t) + \exp(-t) - 2 - t^2}{t^2[\exp(t) + \exp(-t) - 2]} \Big|_0 \\ &= \frac{2\frac{t^4}{4!} + \mathcal{O}(t^6)}{t^2[t^2 + \mathcal{O}(t^4)]} \Big|_0 \\ &= \frac{1}{12} \frac{t^4 + \mathcal{O}(t^6)}{t^4[1 + \mathcal{O}(t^2)]} \Big|_0 \\ &= \frac{1}{12} \end{aligned}$$

So up to first order we have,

$$\frac{1}{\exp(t) - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2) \quad (3.2)$$

Why have we done this? Because we do can soften the behavior of the integrand near  $t \rightarrow 0$  if we subtract leading terms of the expansion of  $(\exp(t) - 1)^{-1}$ , each leading term that we subtract, is equivalent to gaining a power of  $t$  in the numerator, which does soften the behavior near  $t \rightarrow 0$ , but also makes it worse in the region  $t \rightarrow \infty$ , and as our only problem is related with the small  $t$  region, we can divide the integral in two parts,

$$\zeta(s)\Gamma(s) = \int_0^1 dt \frac{t^{s-1}}{\exp(t) - 1} + \int_1^\infty dt \frac{t^{s-1}}{\exp(t) - 1}, \quad \text{Re}(s) > 1$$

$$\zeta(s)\Gamma(s) = \int_0^1 dt t^{s-1} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} + \frac{1}{t} - \frac{1}{2} + \frac{t}{12} \right] + \int_1^\infty dt \frac{t^{s-1}}{\exp(t) - 1}, \quad \text{Re}(s) > 1$$

Where we simply added and subtracted the leading terms of the expansion, the integral of the last three of them is trivial and can be done to give,

$$\begin{aligned} \zeta(s)\Gamma(s) &= \int_0^1 dt t^{s-1} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right] + \int_0^1 dt \left[ t^{s-2} - \frac{t^{s-1}}{2} + \frac{t^s}{12} \right] + \int_1^\infty dt \frac{t^{s-1}}{\exp(t) - 1} \\ \zeta(s)\Gamma(s) &= \int_0^1 dt t^{s-1} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty dt \frac{t^{s-1}}{\exp(t) - 1} \end{aligned}$$

Just what we wanted.

### 3.C)

Naively, this last expression should be well defined only for  $\text{Re}(s) > 1$ , let's see this term by term, starting by the last one,

$$\int_1^\infty dt \frac{t^{s-1}}{\exp(t) - 1}$$

This is finite for all  $s$ , as it is exponentially decaying and is bounded in the integration interval, this term is well defined for all  $s$ . The next three ones are,

$$\frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)}$$

Also these are well defined in the whole complex plane, with three poles at  $s = -1, 0, 1$ . Finally we have,

$$\int_0^1 dt t^{s-1} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right]$$

The only potential not well defined behavior that can occur is near  $t = 0$ , but we have already developed a series expansion for the expression in brackets, 3.2, that means, near the critical value of  $t = 0$ , the integrand goes like,

$$\int_0^1 dt t^{s-1} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right] \approx \int_0^1 dt t^{s-1} \mathcal{O}(t^2) \approx \int_0^1 dt t^{s+1} = \frac{t^{s+2}}{s+2} \Big|_0^1$$

This is well defined as long as  $\text{Re}(s) > -2$ . Hence, the expression,

$$\zeta(s)\Gamma(s) = \int_0^1 dt t^{s-1} \left[ \frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty dt \frac{t^{s-1}}{\exp(t) - 1} \quad (3.3)$$

Is well defined as long as  $\text{Re}(s) > -2$ . One might worry about the poles, but, these are the natural structure of  $\zeta(s)\Gamma(s)$ , to be well defined does not mean to don't have poles, but means the representation can be assigned a number in an unique manner. What is left to us now is to find the values  $\zeta(0), \zeta(-1)$ , notice that our representation has a poles in both these values of the argument, in fact these poles are structures of  $\Gamma(s)$ , and not  $\zeta(s)$ . That the Gamma Function indeed has poles in those values can be seen from,

$$\Gamma(s+1) = s\Gamma(s) \Rightarrow \begin{cases} \Gamma(0) & = \frac{\Gamma(1)}{0} \\ \Gamma(-1) & = \frac{\Gamma(0)}{-1} \end{cases}$$

And because the poles in our representation are just simple poles, they could not have been poles also in  $\zeta$ , as the two functions are multiplying if there were a pole in  $\zeta(0), \zeta(-1)$  they would have been apparent in our representation as double poles. Due to the absence of those, the poles at  $s = 0, -1$  are indeed only due to the Gamma Function. This guarantees us that  $\zeta(0), \zeta(-1)$  are both finite, and to determine those we just need to evaluate the residue of the expression. First, the residue of the Gamma Function,

$$\begin{aligned} \text{Res}_{s=0}(\Gamma(s)) &= s\Gamma(s) \Big|_{s=0} = \Gamma(s+1) \Big|_{s=0} = \Gamma(1) = 1 \\ \text{Res}_{s=-1}(\Gamma(s)) &= (s+1)\Gamma(s) \Big|_{s=-1} = \frac{(s+1)s\Gamma(s)}{s} \Big|_{s=-1} = \frac{\Gamma(s+2)}{s} \Big|_{s=-1} = -1 \end{aligned}$$

As we argued that  $\zeta(0), \zeta(-1)$  should be finite, what will happen is that when we multiply  $\zeta$  by  $\Gamma$ , the residues of the poles of the Gamma Function will be multiplied by the value of the Zeta Function at that point, that is,

$$\text{Res}_{s=0}(\zeta(s)\Gamma(s)) = \zeta(0)\text{Res}_{s=0}(\Gamma(s))$$

But, as can be seen directly from 3.3, the only contribution for the residue at  $s = 0$  will be by  $-\frac{1}{2s}$ , as all the other terms are finite at  $s = 0$ , thus,

$$\begin{aligned} \text{Res}_{s=0}(\zeta(s)\Gamma(s)) &= -\frac{1}{2} = \zeta(0)\text{Res}_{s=0}(\Gamma(s)) = \zeta(0) \\ \zeta(0) &= -\frac{1}{2} \end{aligned}$$

Analogously we have,

$$\text{Res}_{s=-1}(\zeta(s)\Gamma(s)) = \zeta(-1)\text{Res}_{s=-1}(\Gamma(s))$$

Again, as we discussed previously, all the terms are finite at  $s = -1$ , except for  $\frac{1}{12(s+1)}$ , hence, the residue will be,

$$\begin{aligned}\operatorname{Res}_{s=-1}(\zeta(s)\Gamma(s)) &= \frac{1}{12} = \zeta(-1)\operatorname{Res}_{s=-1}(\Gamma(s)) = -\zeta(-1) \\ \zeta(-1) &= -\frac{1}{12}\end{aligned}$$

As desired.



## Problem 4

4.A)

4.B)

## Problem 5

5.A)

5.B)

5.C)

5.D)

5.E)

## Problem 6

6.A)

6.B)

6.C)

6.D)

6.E)

6.F)

6.G)