SCALAR PROXY

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1. Introduction

We will work most with the scalar proxy given by the lagrangian,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2M^2}\Box\phi\Box\phi - \frac{\kappa}{2}\Box\phi\phi^2 \tag{1.1}$$

The idea here is reintegrate the higher derivative term, in order to obtain a lower derivative term, but in terms of additional fields. This is easily done by,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \Box\phi\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box\phi\phi^2 \tag{1.2}$$

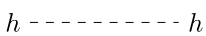
The new lagrangian has mixed propagator terms, to diagonalize it is also easy, we just open in terms of $\phi = h - \eta$,

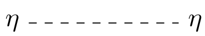
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 + \eta\Box(h-\eta) + \frac{M^2}{2}\eta^2 \tag{1.3}$$

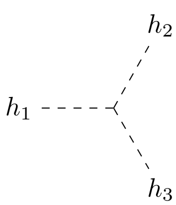
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 - \partial_\mu\eta\partial^\mu(h-\eta) + \frac{M^2}{2}\eta^2 \tag{1.4}$$

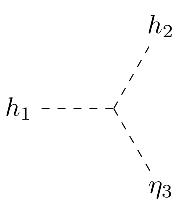
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 \tag{1.5}$$

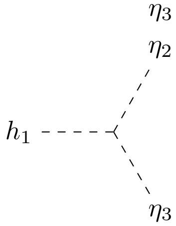
The Feynman rules are easily red as,

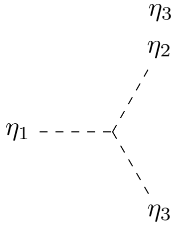
- 

$= \frac{1}{\mathrm{i}}\frac{1}{p^2}$
- 

$= -\frac{1}{\mathrm{i}}\frac{1}{p^2+M^2}$
- 

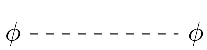
$= \mathrm{i}\kappa\big(p_1^2 + p_2^2 + p_3^2\big)$
- 

$= -\mathrm{i}\kappa\big(p_1^2 + p_2^2 + p_3^2\big)$
- 

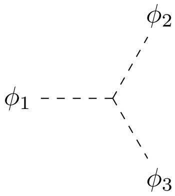
$= \mathrm{i}\kappa\big(p_1^2 + p_2^2 + p_3^2\big)$
- 

$= -\mathrm{i}\kappa\big(p_1^2 + p_2^2 + p_3^2\big)$

Which can also be seen directly from the Feynman rules of the ϕ field,

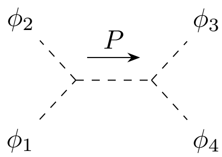
- 

$= \frac{1}{\mathrm{i}}\frac{1}{p^2+\frac{p^4}{M^2}}$

- 

$= \mathrm{i}\kappa\big(p_1^2 + p_2^2 + p_3^2\big) = \mathrm{i}\kappa(p_1 + p_2 + p_3)^2 - 2\mathrm{i}\kappa(p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1) = -\mathrm{i}\kappa(\langle 12\rangle[12] + \langle 23\rangle[23] + \langle 31\rangle[31])$

So that the four point amplitude can be computed by,



$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle 12\rangle[12] + \langle 2P\rangle[2P] + \langle P1\rangle[P1])(\langle 34\rangle[34] - \langle 4P\rangle[4P] - \langle P3\rangle[P3]) \tag{1.6}$$

$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle 12\rangle[12] - \langle P2\rangle[2P] - \langle P1\rangle[1P])(\langle 34\rangle[34] + \langle P4\rangle[4P] + \langle P3\rangle[3P]) \tag{1.7}$$

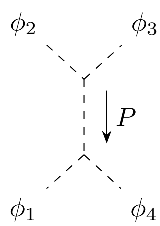
$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle 12\rangle[12] + \langle P|\, 1+2 |P]) (\langle 34\rangle[34] - \langle P|\, 3+4 |P]) \tag{1.8}$$

$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}(\langle 12\rangle[12] - \langle P|\, P |P]) (\langle 34\rangle[34] - \langle P|\, P |P]) \tag{1.9}$$

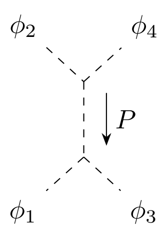
$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}}\big(\langle 12\rangle[12] - 2P^2\big)\big(\langle 34\rangle[34] - 2P^2\big) \tag{1.10}$$

$$= -\mathrm{i}\frac{(\kappa M)^2}{s(M^2 - s)}(\langle 12\rangle[12] + 2s)(\langle 34\rangle[34] + 2s) \tag{1.11}$$

It's trivial to read the t and u channels from this expression,



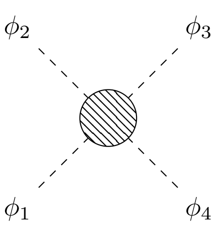
$$= -\mathrm{i}\frac{(\kappa M)^2}{t(M^2 - t)}(\langle 23\rangle[23] + 2t)(\langle 41\rangle[41] + 2t) \tag{1.12}$$



$$= -\mathrm{i}\frac{(\kappa M)^2}{u(M^2 - u)}(\langle 24\rangle[24] + 2u)(\langle 31\rangle[31] + 2u) \tag{1.13}$$

So that the full 4–point amplitude is,

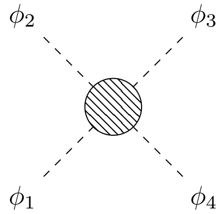
(1.14)



$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2 - s)(M^2 - t)(M^2 - u)}\big[(\langle 12\rangle[12] + 2s)(\langle 34\rangle[34] + 2s)tu\big(M^2 - t\big)\big(M^2 - u\big) + (\langle 23\rangle[23] + 2t)(\langle 41\rangle[41] + 2t$$

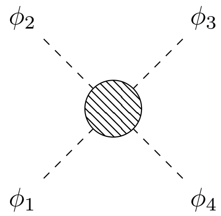
Let us specialize when $1, 2$ are massless and $3, 4$ are massive, then,

(1.15)



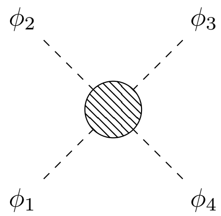
$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\Big[s\big(-s+2M^2+2s\big)tu\big(M^2-t\big)\big(M^2-u\big)+\big(-t+M^2+2t\big)\big(-t+M^2+2t\big)su\big(M^2$$

(1.16)



$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\Big[stu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+su\big(M^2+t\big)\big(M^2+t\big)\big(M^2-s\big)\big(M^2-u\big)+st$$

(1.17)



$$= -\mathrm{i}\frac{(\kappa M)^2}{tu(M^2-s)(M^2-t)(M^2-u)}\Big[tu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+u\big(M^2+t\big)^2\big(M^2-s\big)\big(M^2-u\big)+t\big(M^2+u\big)^2\big($$

2. Conformal Toy Model

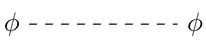
Consider the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\Box\phi\Box\phi - \frac{g}{2}\phi^2\Box\phi - \frac{g^2}{8}\phi^4 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

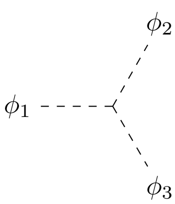
Notice the form of the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\left(\Box\phi + \frac{g}{2}\phi^2\right)^2 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

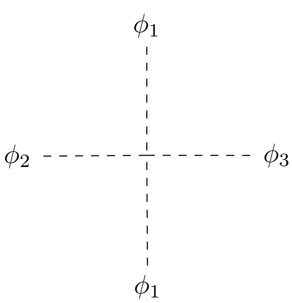
It possesses the Feynman rules,

- 

$= \frac{1}{\mathrm{i}}\frac{1}{m^2p^2+p^4}$

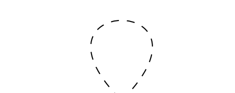
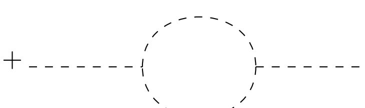
- 

$= \mathrm{i}g\big(p_1^2+p_2^2+p_3^2+m^2\big)$

- 

$= -3\mathrm{i}g^2$

Let's compute the self energy,

(2.1) $\mathrm{i}\Pi\big(p^2\big) =$  $+$  $+\cdots$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\mathrm{i}} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2} \tag{2.2}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2} \tag{2.3}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2 \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \left(m^2\right)^{\frac{D}{2}-2} \frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(1)} \tag{2.4}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2 \frac{\left(m^2\right)^{-\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-\epsilon)} \tag{2.5}$$

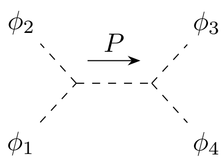
$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}(\mathrm{i}g)^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\mathrm{i}^2} \frac{1}{\ell^2(\ell+p)^2} \frac{\left(m^2 + \ell^2 + p^2 + (\ell+p)^2\right)^2}{\ell^2 + m^2} \frac{1}{(\ell+p)^2 + m^2} \tag{2.6}$$

For the mass renormalization we can take $p = 0$,

$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{\left(m^2 + 2\ell^2\right)^2}{\ell^4\left(\ell^2 + m^2\right)^2} \tag{2.7}$$

Let's compute the four point amplitude for this theory,

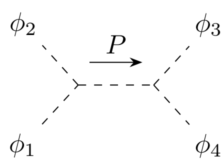
(2.8)



$$= (\mathrm{i}g)^2 \frac{\left(p_1^2 + p_2^2 + (p_1+p_2)^2 + m^2\right)\left(p_3^2 + p_4^2 + (p_3+p_4)^2 + m^2\right)}{\mathrm{i}(p_1+p_2)^2\left((p_1+p_2)^2 + m^2\right)}$$

First let's consider all legs massless,

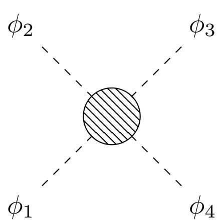
(2.9)



$$= \mathrm{i}g^2 \frac{\left(-s+m^2\right)\left(-s+m^2\right)}{(-s)(-s+m^2)} = -\mathrm{i}g^2 \frac{\left(-s+m^2\right)}{s}$$

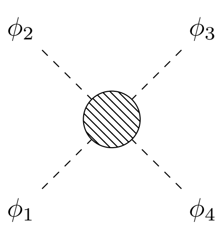
So,

(2.10)



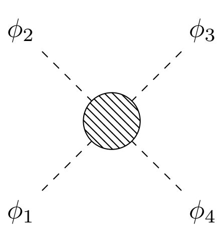
$$= -\mathrm{i}g^2 \frac{\left(-s+m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t+m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u+m^2\right)}{u} - 3\mathrm{i}g^2$$

(2.11)



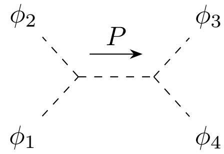
$$= -\mathrm{i}g^2 \frac{\left(-s+m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t+m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u+m^2\right)}{u} - \mathrm{i}g^2\frac{s}{s} - \mathrm{i}g^2\frac{t}{t} - \mathrm{i}g^2\frac{u}{u}$$

(2.12)



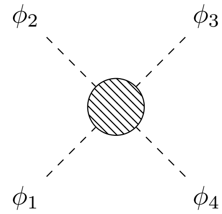
$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$$

Para uma perna massiva, ϕ_4,

(2.13) 

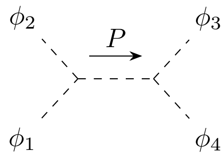
$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)(-s)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2 \tag{2.13}$$

So,

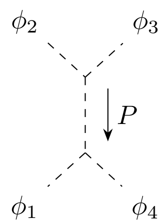
(2.14) 

$$= \mathrm{i}g^2 + \mathrm{i}g^2 + \mathrm{i}g^2 - 3\mathrm{i}g^2 = 0 \tag{2.14}$$

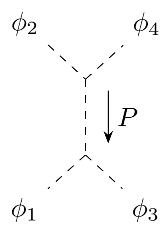
Para duas pernas massivas, $\phi_{3,4}$,

(2.15) 

$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2 \frac{s+m^2}{s} \tag{2.15}$$

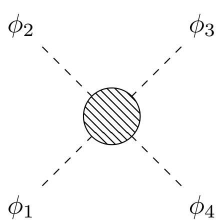
(2.16) 

$$= (\mathrm{i}g)^2 \frac{(-t)(-t)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t}{-t+m^2} \tag{2.16}$$

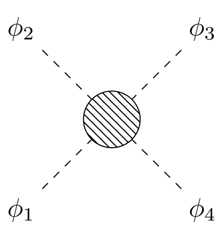
(2.17) 

$$= -\mathrm{i}g^2 \frac{u}{-u+m^2} \tag{2.17}$$

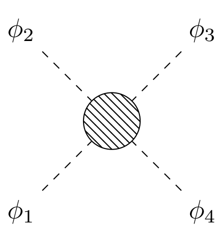
So,

(2.18) 

$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - 3\mathrm{i}g^2 \tag{2.18}$$

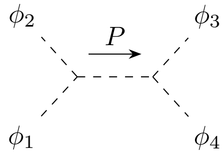
(2.19) 

$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - \mathrm{i}g^2 \frac{s}{s} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2} \tag{2.19}$$

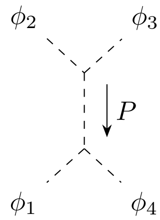
(2.20) 

$$= -\mathrm{i}g^2 m^2 \left(-\frac{1}{s} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) \tag{2.20}$$

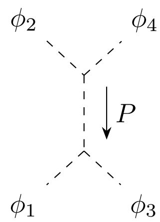
Para uma perna sem massa ϕ_1,

(2.21)


$$= (\mathrm{i}g)^2 \frac{(-s)\big(-s-m^2\big)}{\mathrm{i}(-s)(-s+m^2)} = -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2}$$

(2.22)


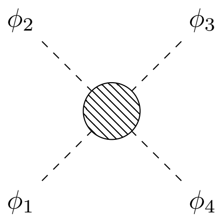
$$= (\mathrm{i}g)^2 \frac{(-t)\big(-t-m^2\big)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t+m^2}{-t+m^2}$$

(2.23)


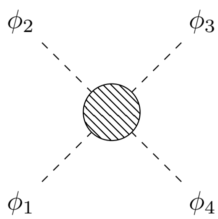
$$= (\mathrm{i}g)^2 \frac{(-u)\big(-u-m^2\big)}{\mathrm{i}(-u)(-u+m^2)} = -\mathrm{i}g^2 \frac{u+m^2}{-u+m^2}$$

(2.24)

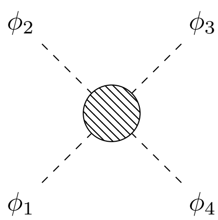
So,

(2.25)


$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - 3\mathrm{i}g^2$$

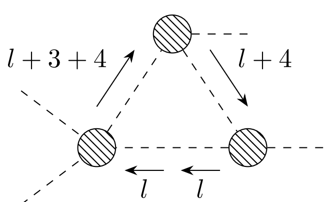
(2.26)


$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - \mathrm{i}g^2 \frac{-s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2}$$

(2.27)


$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right)$$

Cut comparison, only massless legs

(2.28)


$$= \frac{\mathrm{i}g^2 m^2 \big(\mathrm{i}gm^2\big)^2}{\left(\mathrm{i}m^2\right)^3} \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{\langle 1l \rangle [1l]} - \frac{1}{\langle 2l \rangle [2l]} \right)$$

to solve for the cuts, $l^2 = (l+3+4)^2 = (l+4)^2 = 0$,

(2.29) $$l^2 = 0 \Rightarrow l = -|l\rangle [l|$$

(2.30) $$0 = (l+4)^2 = \langle l4 \rangle [l4] = 0 \Rightarrow |l] = |4]$$

(2.31) $$0 = (l+3+4)^2 = \langle l P_{34} \rangle [l P_{34}] + (3+4)^2 = \langle l|3+4|l] + \langle 34 \rangle [34] = \langle l|3+4|4] + \langle 34 \rangle [34]$$

(2.32) $$\langle 43 \rangle [34] = -\langle l3 \rangle [34] \Rightarrow |l\rangle = -|4\rangle + z|3\rangle$$

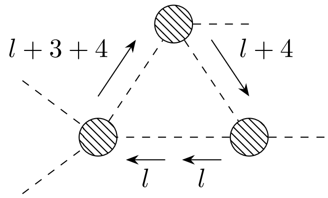
(2.33) $$l = -(-|4\rangle + z|3\rangle)[4|$$

The cuts are solved by this. Hence,

$$= g^4\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]}\right)$$
$$= g^4\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{(-\langle 14\rangle + z\langle 13\rangle)[14]} - \frac{1}{(-\langle 24\rangle + z\langle 23\rangle)[24]}\right)$$

Now for internal massive lines,

(2.34)



$$= \frac{-\mathrm{i}g^2 m^2\left(-\mathrm{i}gm^2\right)^2}{\left(-\mathrm{i}m^2\right)^3}\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]}\right)$$

With the cuts being, $l^2 = \left(l+3+4\right)^2 = \left(l+4\right)^2 = -m^2$,

(2.35) $$0 = (l+4)^2 - l^2 = 2l \cdot p_4$$

(2.36) $$0 = (l+4+3)^2 - l^2 = 2l \cdot (4+3) + (4+3)^2 = 2l \cdot p_3 + (4+3)^2$$

As ansatz, $l = |4\rangle[4| + \alpha|4\rangle[3| + \beta|3\rangle[4|$ satisfy both conditions above. The remaining condition is,

(2.37) $$l^2 = -m^2$$

(2.38) $$-\alpha\beta[43]\langle 43\rangle = -m^2 \Rightarrow \alpha = \frac{m^2}{\beta\langle 34\rangle[34]}$$

Setting now $-\beta = z$,

(2.39) $$l = |4\rangle[4| - \frac{m^2}{z\langle 34\rangle[34]}|4\rangle[3| - z|3\rangle[4|$$

The value of the diagram is,

$$= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1l\rangle[l1]} + \frac{1}{\langle 2l\rangle[l2]}\right)$$
$$= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{-\langle 1|l|1]} + \frac{1}{-\langle 2|l|2]}\right)$$