ADTGR SEMINAR

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1. Introduction

Einstein-Hilbert action:

$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int\limits_{M} \mathrm{d}^D x \, \sqrt{|g|} g^{ab} R_{cb}{}^{c}{}_{a}$$

The Vierbein/Tetrad formalism:

$$\begin{aligned} \eta_{\mu\nu} &= \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) \\ g_{ab} &= e^{\mu}{}_{a} e^{\nu}{}_{b} \eta_{\mu\nu} \\ \eta_{\mu\nu} &= e_{\mu}{}^{a} e_{\nu}{}^{b} g_{ab} \end{aligned}$$

A connection is defined with respect to a vector basis:

$$\begin{aligned} \nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= \boldsymbol{\omega}(\mathbf{X})^{\mu}{}_{\nu} \mathbf{e}_\mu \\ \nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= X^a \omega_{a}{}^{\mu}{}_{\nu} \mathbf{e}_\mu \\ \mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) &= X^a \omega_{a}{}^{\mu}{}_{\nu} \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\mu) \\ \mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) + \mathbf{g}(\mathbf{e}_\nu, \nabla_{\mathbf{X}}(\mathbf{e}_\alpha)) &= X^a \omega_{a}{}^{\mu}{}_{\nu} \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\mu) + X^a \omega_{a}{}^{\mu}{}_{\alpha} \mathbf{g}(\mathbf{e}_\nu, \mathbf{e}_\mu) \\ \mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) + \mathbf{g}(\nabla_{\mathbf{X}}(\mathbf{e}_\alpha), \mathbf{e}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha} \\ \nabla_{\mathbf{X}}(\mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\nu)) - \nabla_{\mathbf{X}}(\mathbf{g})(\mathbf{e}_\alpha, \mathbf{e}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha} \\ -\nabla_{\mathbf{X}}(\mathbf{g})(\mathbf{e}_\alpha, \mathbf{e}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha} \\ -\omega_{a\nu\alpha} &= \omega_{a\alpha\nu}, \quad \text{Metric compatibility} \qquad (1.1) \end{aligned}$$

Riemann curvature tensor, $\mathbf{Riem}(\mathbf{X}, \mathbf{Y}) : \mathfrak{X} \to \mathfrak{X}$:

Date: June 13, 2025.

$$
\begin{aligned}
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\big)\mathbf{e}_\mu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \Big(\nabla_{\mathbf{X}}\big(Y^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - \nabla_{\mathbf{Y}}\big(X^a\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \Big(\nabla_{\mathbf{X}}\big(Y^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - \nabla_{\mathbf{Y}}\big(X^a\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(Y^b\nabla_{\mathbf{X}}\big(\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - X^a\nabla_{\mathbf{Y}}\big(\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) \\
&\qquad +\nabla_{\mathbf{X}}\big(Y^b\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - \nabla_{\mathbf{Y}}\big(X^b\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\mathbf{e}_\nu\big) \\
&\qquad +\big(\nabla_{\mathbf{X}}\big(Y^b\big) - \nabla_{\mathbf{Y}}\big(X^b\big)\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\Big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= \big(X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\big)\mathbf{e}_\nu + X^aY^b\omega_b{}^\nu{}_\mu\omega_a{}^\alpha{}_\nu\mathbf{e}_\alpha - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\big)\mathbf{e}_\nu - X^aY^b\omega_a{}^\nu{}_\mu\omega_b{}^\alpha{}_\nu\mathbf{e}_\alpha\big) \\
&\qquad +\big(X^a\partial_aY^b + X^a\Gamma_a{}^b{}_cY^c - Y^a\partial_aX^b - Y^a\Gamma_a{}^b{}_cX^c\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu - \big(X^a\partial_aY^b - Y^a\partial_aX^b\big)\omega_b{}^\nu{}_\mu\mathbf{e}_\nu\big) \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^b\,\big(\nabla_a\big(\omega_b{}^\nu{}_\mu\big) + \omega_b{}^\alpha{}_\mu\omega_a{}^\nu{}_\alpha - \nabla_b\big(\omega_a{}^\nu{}_\mu\big) - \omega_a{}^\alpha{}_\mu\omega_b{}^\nu{}_\alpha + \Gamma_a{}^c{}_b\omega_c{}^\nu{}_\mu - \Gamma_b{}^c{}_a\omega_c{}^\nu{}_\mu\big)\,\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^b\,\big(\partial_a\omega_b{}^\nu{}_\mu - \partial_b\omega_a{}^\nu{}_\mu + \omega_a{}^\nu{}_\alpha\omega_b{}^\alpha{}_\mu - \omega_b{}^\nu{}_\alpha\omega_a{}^\alpha{}_\mu\big)\,\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^b\,(\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega})_{ab}{}^\nu{}_\mu\,\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}_\mu &= X^aY^bR_{ab}{}^\nu{}_\mu\mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\big(e_\mu{}^e\boldsymbol{\partial}_e\big) &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e_\mu{}^e\mathbf{Riem}(\mathbf{X},\mathbf{Y})\boldsymbol{\partial}_e &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e_\mu{}^eX^aY^bR_{ab}{}^c{}_e\boldsymbol{\partial}_c &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e^\mu{}_de_\mu{}^eR_{ab}{}^c{}_e &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c \\
R_{ab}{}^c{}_d &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c
\end{aligned}
$$

We would like to write the Einstein-Hilbert action as only a function of the Vierbein and the spin connection, for this, let us explicit write the Riemann tensor as a $\mathrm{End}(\mathrm{T}M)$-valued 2-form,

$$
\begin{aligned}
\mathbf{R}^\nu{}_\mu &= \frac{1}{2}R_{ab}{}^\nu{}_\mu\,\mathrm{d}x^a\wedge\mathrm{d}x^b \\
\mathbf{R}^\nu{}_\mu &= \frac{1}{2}R_{ab}{}^\nu{}_\mu e_\alpha{}^ae^\alpha{}_ce_\beta{}^be^\beta{}_d\,\mathrm{d}x^c\wedge\mathrm{d}x^d \\
\mathbf{R}^\nu{}_\mu &= \frac{1}{2}R_{ab}{}^\nu{}_\mu e_\alpha{}^ae_\beta{}^b\tilde{\mathbf{e}}^\alpha\wedge\tilde{\mathbf{e}}^\beta
\end{aligned}
$$

Let us start by writing the volume form in terms of the Vierbein,

$$
\begin{aligned}
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[g_{ab}]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^\mu{}_a\eta_{\mu\nu}e^\nu{}_b]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^\mu{}_a]\mathrm{Det}[\eta_{\mu\nu}]\mathrm{Det}[e^\nu{}_b]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{\big(\mathrm{Det}[e^\mu{}_a]\big)^2}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \mathrm{Det}[e^\mu{}_a]\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \epsilon_{\mu_0\cdots\mu_{D-1}}e^{\mu_0}{}_0\cdots e^{\mu_{D-1}}{}_{D-1}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \epsilon_{\mu_0\cdots\mu_{D-1}}e^{\mu_0}{}_0\,\mathrm{d}x^0\wedge\cdots\wedge e^{\mu_{D-1}}{}_{D-1}\,\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_0\cdots\mu_{D-1}}e^{\mu_0}{}_{a_0}\,\mathrm{d}x^{a_0}\wedge\cdots\wedge e^{\mu_{D-1}}{}_{a_{D-1}}\,\mathrm{d}x^{a_{D-1}} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_0\cdots\mu_{D-1}}\mathbf{e}^{\mu_0}\wedge\cdots\wedge\mathbf{e}^{\mu_{D-1}}
\end{aligned}
$$

And now we express the Ricci scalar,

$$
\begin{aligned}
R &= g^{ab} R_{cb}{}^{c}{}_{a} \\
R &= e_{\rho}{}^{a} e^{\rho b} R_{cbda} e_{\alpha}{}^{c} e^{\alpha d} \\
R &= \eta^{\rho\sigma} \eta^{\alpha\beta} e_{\rho}{}^{a} e_{\sigma}{}^{b} R_{cbda} e_{\alpha}{}^{c} e_{\beta}{}^{d} \\
R &= \eta^{\rho\sigma} \eta^{\alpha\beta} R_{\alpha\sigma\beta\rho} \\
R &= \frac{1}{2}\left(\eta^{\rho\sigma}\eta^{\alpha\beta} - \eta^{\rho\alpha}\eta^{\sigma\beta}\right) R_{\alpha\sigma\beta\rho} \\
R &= \frac{1}{2(D-2)!} \epsilon^{\nu_0 \cdots \nu_{D-3}\beta\rho} \epsilon_{\nu_0 \cdots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho}
\end{aligned}
$$

Putting everything together,

$$
\begin{aligned}
S_{\mathrm{EH}} &= \frac{1}{2\kappa} \int\limits_M \mathrm{d}^D x \sqrt{|g|} R \\
S_{\mathrm{EH}} &= \frac{1}{4D!(D-2)!\kappa} \int\limits_M \mathbf{e}^{\mu_0} \wedge \cdots \wedge \mathbf{e}^{\mu_{D-1}} \epsilon_{\mu_0 \cdots \mu_{D-1}} \epsilon^{\nu_0 \cdots \nu_{D-3}\beta\rho} \epsilon_{\nu_0 \cdots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\mathrm{EH}} &= \frac{1}{4(D-2)!\kappa} \int\limits_M \mathbf{e}^{\mu_0} \wedge \cdots \wedge \mathbf{e}^{\mu_{D-1}} \eta_{\mu_0}{}^{[\nu_0} \cdots \eta_{\mu_{D-3}}{}^{\nu_{D-3}} \eta_{\mu_{D-2}}{}^{\beta} \eta_{\mu_{D-1}}{}^{\rho]} \epsilon_{\nu_0 \cdots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\mathrm{EH}} &= \frac{1}{4(D-2)!\kappa} \int\limits_M \mathbf{e}^{\nu_0} \wedge \cdots \wedge \mathbf{e}^{\nu_{D-3}} \wedge \mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho} \epsilon_{\nu_0 \cdots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa} \int\limits_M \frac{1}{2(D-2)!} R_{\alpha\sigma\beta\rho} \epsilon^{\alpha\sigma}{}_{\nu_0 \cdots \nu_{D-3}} \mathbf{e}^{\nu_0} \wedge \cdots \wedge \mathbf{e}^{\nu_{D-3}} \wedge \mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho} \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa} \int\limits_M \star \mathbf{R}_{\beta\rho} \wedge \mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho} \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa} \int\limits_M \mathbf{R}_{\beta\rho} \wedge \star\left(\mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho}\right) \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa} \int\limits_M \frac{1}{(D-2)!} \epsilon^{\beta\rho}{}_{\alpha_0 \cdots \alpha_{D-3}} \mathbf{R}_{\beta\rho} \wedge \mathbf{e}^{\alpha_0} \wedge \cdots \wedge \mathbf{e}^{\alpha_{D-3}} \\
S_{\mathrm{EH}} &= \frac{1}{2(D-2)!\kappa} \int\limits_M \epsilon_{\alpha_0 \cdots \alpha_{D-1}} \mathbf{e}^{\alpha_0} \wedge \cdots \wedge \mathbf{e}^{\alpha_{D-3}} \wedge \mathbf{R}^{\alpha_{D-2}\alpha_{D-1}}
\end{aligned}
$$

The most interesting case here is $D = 3$,

$$
S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega}] = \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int\limits_M \mathbf{e}^{\mu} \wedge \mathbf{R}^{\alpha\beta}
$$

Equations of motion are,

$$
\begin{aligned}
S_{\mathrm{EH}}[\mathbf{e}+\delta\mathbf{e},\boldsymbol{\omega}]-S_{\mathrm{EH}}[\mathbf{e},\boldsymbol{\omega}] &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \delta\mathbf{e}^\mu\wedge\mathbf{R}^{\alpha\beta}=0\\
0 &= -\frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{R}^{\alpha\beta}\\
0 &= -\frac{1}{2}\epsilon_{\mu\alpha\beta}\frac{1}{2}R_{\rho\sigma}{}^{\alpha\beta}\mathbf{e}^\rho\wedge\mathbf{e}^\sigma\\
0 &= -\frac{1}{4}\epsilon_{\mu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}\star(\mathbf{e}^\rho\wedge\mathbf{e}^\sigma)\\
0 &= -\frac{1}{4}\epsilon_{\mu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}\epsilon^{\rho\sigma}{}_\kappa\mathbf{e}^\kappa\\
0 &= -\frac{1}{4}\epsilon_{\mu\alpha\beta}\epsilon^{\rho\sigma\kappa}R_{\rho\sigma}{}^{\alpha\beta}\mathbf{e}_\kappa\\
0 &= -\frac{1}{4}\eta_\mu{}^{[\rho}\eta_\alpha{}^\sigma\eta_\beta{}^{\kappa]}R_{\rho\sigma}{}^{\alpha\beta}\mathbf{e}_\kappa\\
0 &= -\frac{1}{4}R_{\rho\sigma}{}^{[\sigma\kappa}\eta_\mu{}^{\rho]}\mathbf{e}_\kappa\\
0 &= -\frac{1}{4}\big(R_{\rho\sigma}{}^{\sigma\kappa}\eta_\mu{}^\rho+R_{\rho\sigma}{}^{\kappa\rho}\eta_\mu{}^\sigma+R_{\rho\sigma}{}^{\rho\sigma}\eta_\mu{}^\kappa-R_{\rho\sigma}{}^{\rho\kappa}\eta_\mu{}^\sigma-R_{\rho\sigma}{}^{\kappa\sigma}\eta_\mu{}^\rho-R_{\rho\sigma}{}^{\sigma\rho}\eta_\mu{}^\kappa\big)\mathbf{e}_\kappa\\
0 &= -\frac{1}{4}\big(-R_\mu{}^\kappa-R_\mu{}^\kappa+R\eta_\mu{}^\kappa-R_\mu{}^\kappa-R_\mu{}^\kappa+R\eta_\mu{}^\kappa\big)\mathbf{e}_\kappa\\
0 &= -\frac{1}{4}(-4R_\mu{}^\kappa+2R\eta_\mu{}^\kappa)\mathbf{e}_\kappa\\
0 &= \left(R_{\mu\kappa}-\frac{1}{2}R\eta_{\mu\kappa}\right)\mathbf{e}^\kappa
\end{aligned}
$$

And,

$$
\begin{aligned}
S_{\mathrm{EH}}[\mathbf{e},\boldsymbol{\omega}+\delta\boldsymbol{\omega}]-S_{\mathrm{EH}}[\mathbf{e},\boldsymbol{\omega}] &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^\mu\wedge(\mathrm{d}\delta\boldsymbol{\omega}+\delta\boldsymbol{\omega}\wedge\boldsymbol{\omega}+\boldsymbol{\omega}\wedge\delta\boldsymbol{\omega})^{\alpha\beta}=0\\
0 &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \Big(-\mathrm{d}\big(\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}\big)+\mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}+\mathbf{e}^\mu\wedge(\delta\boldsymbol{\omega}\wedge\boldsymbol{\omega})^{\alpha\beta}+\mathbf{e}^\mu\wedge(\boldsymbol{\omega}\wedge\delta\boldsymbol{\omega})^{\alpha\beta}\Big)\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \big(\mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}+\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\gamma}\wedge\boldsymbol{\omega}_\gamma{}^\beta+\mathbf{e}^\mu\wedge\boldsymbol{\omega}^{\alpha\gamma}\wedge\delta\boldsymbol{\omega}_\gamma{}^\beta\big)\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \big(\mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}-\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\gamma{}^\beta\wedge\delta\boldsymbol{\omega}^{\alpha\gamma}+\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\alpha{}_\gamma\wedge\delta\boldsymbol{\omega}^{\gamma\beta}\big)\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}-\frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\gamma{}^\beta\wedge\delta\boldsymbol{\omega}^{\alpha\gamma}+\frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\alpha{}_\gamma\wedge\delta\boldsymbol{\omega}^{\gamma\beta}\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}-\frac{1}{2}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\beta{}^\gamma\wedge\delta\boldsymbol{\omega}^{\alpha\beta}+\frac{1}{2}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\gamma{}_\alpha\wedge\delta\boldsymbol{\omega}^{\alpha\beta}\\
0 &= \frac{1}{2}\int\limits_M \Big(\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu-\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\beta{}^\gamma+\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\gamma{}_\alpha\Big)\wedge\delta\boldsymbol{\omega}^{\alpha\beta}\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu-\frac{1}{2}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\beta{}^\gamma+\frac{1}{2}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\gamma{}_\alpha\\
0 &= \frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu-\frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\beta{}^\gamma+\frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\gamma{}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu-\eta_\gamma{}^{[\beta}\eta_\mu{}^{\nu]}\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\beta{}^\gamma+\eta_\mu{}^{[\nu}\eta_\gamma{}^{\alpha]}\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\gamma{}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu-\frac{1}{2}\big(\eta_\gamma{}^\beta\eta_\mu{}^\nu-\eta_\gamma{}^\nu\eta_\mu{}^\beta\big)\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\beta{}^\gamma+\frac{1}{2}\big(\eta_\mu{}^\nu\eta_\gamma{}^\alpha-\eta_\mu{}^\alpha\eta_\gamma{}^\nu\big)\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\gamma{}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu+\frac{1}{2}\eta_\gamma{}^\nu\eta_\mu{}^\beta\mathbf{e}^\mu\wedge\boldsymbol{\omega}_\beta{}^\gamma-\frac{1}{2}\eta_\mu{}^\alpha\eta_\gamma{}^\nu\mathbf{e}^\mu\wedge\boldsymbol{\omega}^\gamma{}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu+\frac{1}{2}\mathbf{e}^\beta\wedge\boldsymbol{\omega}_\beta{}^\nu-\frac{1}{2}\mathbf{e}^\alpha\wedge\boldsymbol{\omega}^\nu{}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu+\boldsymbol{\omega}^\nu{}_\alpha\wedge\mathbf{e}^\alpha
\end{aligned}
$$

It's not really feasible to give this a gauge theory approach, only if we're in 2+1, in this case there is an isomorphism, $\mathbf{e}^\mu \to \mathbf{e}_{\alpha\beta} = \epsilon_{\mu\alpha\beta}\mathbf{e}^\mu$, so that,

$$
\begin{aligned}
S_{\text{EH}} &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} \\
S_{\text{EH}} &= -\frac{1}{2\kappa}\int\limits_M \mathbf{e}_{\beta\alpha} \wedge \mathbf{R}^{\alpha\beta} \\
S_{\text{EH}} &= -\frac{1}{2\kappa}\int\limits_M \operatorname{Tr}\left[\mathbf{e} \wedge \mathbf{R}\right] \\
S_{\text{EH}} &= -\frac{1}{2\kappa}\int\limits_M \operatorname{Tr}\left[\mathbf{e} \wedge \left(\mathrm{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\right)\right]
\end{aligned}
$$

This is a lot similar to Chern-Simons theory,

$$
S_{\text{CS}}[\mathbf{A}] = k\int\limits_M \operatorname{Tr}\left[\mathbf{A} \wedge \mathrm{d}\mathbf{A} + \frac{1}{3}\mathbf{A} \wedge [\mathbf{A} \overset{\wedge}{,} \mathbf{A}]\right]
$$

Let's try, $\mathbf{A}^x = \boldsymbol{\omega} + x\mathbf{e}$,

$$
\begin{aligned}
S_{\text{CS}}[\mathbf{A}^x] &= k\int\limits_M \operatorname{Tr}\left[(\boldsymbol{\omega} + x\mathbf{e}) \wedge \mathrm{d}(\boldsymbol{\omega} + x\mathbf{e}) + \frac{1}{3}(\boldsymbol{\omega} + x\mathbf{e}) \wedge [\boldsymbol{\omega} + x\mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega} + x\mathbf{e}]\right] \\
S_{\text{CS}}[\mathbf{A}^x] &= k\int\limits_M \operatorname{Tr}\Big[\boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\omega} + x\boldsymbol{\omega} \wedge \mathrm{d}\mathbf{e} + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} + x^2\mathbf{e} \wedge \mathrm{d}\mathbf{e} \\
&\qquad + \frac{2}{3}(\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e})\Big] \\
S_{\text{CS}}[\mathbf{A}^x] &= k\int\limits_M \operatorname{Tr}\Big[\boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\omega} + x\,\mathrm{d}\mathbf{e} \wedge \boldsymbol{\omega} + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} - \frac{1}{2}x^2\,\mathrm{d}(\mathbf{e} \wedge \mathbf{e}) \\
&\qquad + \frac{2}{3}\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\omega} + 2x\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + 2x^2\mathbf{e} \wedge \mathbf{e} \wedge \boldsymbol{\omega} + \frac{2}{3}x^3\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\Big] \\
S_{\text{CS}}[\mathbf{A}^x] &= k\int\limits_M \operatorname{Tr}\Big[\boldsymbol{\omega} \wedge \left(\mathrm{d}\boldsymbol{\omega} + \frac{2}{3}\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\omega}\right) + x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} + x\mathbf{e} \wedge (\mathrm{d}\boldsymbol{\omega} + 2\boldsymbol{\omega} \wedge \boldsymbol{\omega}) + 2x^2\mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} \\
&\qquad + \frac{2}{3}x^3\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\Big] \\
S_{\text{CS}}[\mathbf{A}^x] &= S_{\text{CS}}[\boldsymbol{\omega}] + k\int\limits_M \operatorname{Tr}\left[x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x\mathbf{e} \wedge (\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}) + 2x^2\mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3}x^3\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right] \\
S_{\text{CS}}[\mathbf{A}^x] &= S_{\text{CS}}[\boldsymbol{\omega}] + k\int\limits_M \operatorname{Tr}\left[x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x\mathbf{e} \wedge \mathbf{R} + 2x^2\mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3}x^3\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right] \\
S_{\text{CS}}[\mathbf{A}^x] &= S_{\text{CS}}[\boldsymbol{\omega}] - 4xk\kappa S_{\text{EH}} + k\int\limits_M \operatorname{Tr}\left[x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x^2\mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3}x^3\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right]
\end{aligned}
$$

We can simplify if we sum two contributions,

$$
\begin{aligned}
S_{\text{CS}}[\mathbf{A}^x] - S_{\text{CS}}\left[\mathbf{A}^{-x}\right] &= -8xk\kappa S_{\text{EH}} + 2kx\int\limits_{\partial M} \operatorname{Tr}\left[\mathbf{e} \wedge \boldsymbol{\omega}\right] + \frac{4}{3}x^3k\int\limits_M \operatorname{Tr}\left[\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right] \\
\frac{1}{8xk\kappa}\left(S_{\text{CS}}\left[\mathbf{A}^{-x}\right] - S_{\text{CS}}[\mathbf{A}^x]\right) &= S_{\text{EH}} - \frac{1}{4\kappa}\int\limits_{\partial M} \operatorname{Tr}\left[\mathbf{e} \wedge \boldsymbol{\omega}\right] - \frac{x^2}{3!\kappa}\int\limits_M \operatorname{Tr}\left[\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right]
\end{aligned}
$$

And also the not so usual action,

$$
S_{\text{CS}}[\mathbf{A}^x] + S_{\text{CS}}\left[\mathbf{A}^{-x}\right] = 2S_{\text{CS}}[\boldsymbol{\omega}] + 4x^2k\int\limits_M \operatorname{Tr}\left[\mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e}\right]
$$