Homework III

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Contents

Problem 1

1.A)

For an operator $\mathcal{O}$ to be BRST closed, it means $[Q_{\text{BRST}}, \mathcal{O}] = 0$, and by now we're very familiar with a commutator being written as a contour integral, that is,

$$[Q_{\mathrm{BRST}}, \mathcal{O}(w,\bar{w})] = \frac{1}{2\pi \mathrm{i}} \oint_{C_w} \big(\mathrm{d}z\, j_{\mathrm{BRST}}(z) - \mathrm{d}\bar{z}\, \tilde{j}_{\mathrm{BRST}}(\bar{z})\big)\mathcal{O}(w,\bar{w}) \tag{1.1}$$

Where the BRST current is given by,

$$j_{\mathrm{BRST}}(z) = -\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) + : bc\partial c : (z) + \frac{3}{2} : \partial^2 c : (z) \tag{1.2}$$

And also C_w is any closed contour encircling counterclockwise the point w. Actually, this definition is only good for closed strings — in which operators can be inserted at any point —, but, for the open string — our case here —, operators have to be inserted at the boundary $\mathrm{Im}(w) = 0$. Hence, as we have to our disposal only half of the complex plane, it's impossible to have a closed curve C_w that encloses a point at $\mathrm{Im}(w) = 0$ — unless the point itself belongs to the curve, under which the Cauchy residue theorem stops holding —. Thus, the definition of the commutator for the open string is different,

$$[Q_{\mathrm{BRST}}, \mathcal{O}(w,\bar{w})] = \frac{1}{2\pi \mathrm{i}} \int_{C'_w} \big(\mathrm{d}z\, j_{\mathrm{BRST}}(z) - \mathrm{d}\bar{z}\, \tilde{j}_{\mathrm{BRST}}(\bar{z})\big)\mathcal{O}(w,\bar{w}) \tag{1.3}$$

Where C'_w is any counterclockwise oriented open curve starting and ending at $\mathrm{Im}(z) = 0$, such that the point w, $\mathrm{Im}(w) = 0$, lies in the interior of C'_w. To get a simpler expression, we can do the so called *doubling trick*. The open string boundary conditions forces us to have at $\mathrm{Im}(z) = 0$ the following,

$$\bar{\partial} X^\mu = \partial X^\mu, \quad \tilde{b} = b, \quad \tilde{c} = c, \quad \mathrm{Im}(z) = 0$$

Among other things, this also imply that at $\mathrm{Im}(z) = 0$ we have $\tilde{j}_{\mathrm{BRST}} = j_{\mathrm{BRST}}$. The doubling trick consists of we assigning j_{BRST} as being a operator over the whole $\mathbb{C}$. In the upper half it's already well defined, but in the lower half we define it as,

$$j_{\mathrm{BRST}}(z) = \tilde{j}_{\mathrm{BRST}}(z^*), \quad \mathrm{Im}(z) < 0$$

Due to the open string boundary conditions this definition is continuous at $\mathrm{Im}(z) = 0$. We can use this to simplify eq. (1.3),

$$\begin{aligned}
[Q_{\mathrm{BRST}}, \mathcal{O}(w,\bar{w})] &= \frac{1}{2\pi \mathrm{i}} \int_{C'_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z)\mathcal{O}(w,\bar{w}) - \frac{1}{2\pi \mathrm{i}} \int_{C'_w} \mathrm{d}\bar{z}\, \tilde{j}_{\mathrm{BRST}}(\bar{z}) O(w,\bar{w}) \\
[Q_{\mathrm{BRST}}, \mathcal{O}(w,\bar{w})] &= \frac{1}{2\pi \mathrm{i}} \int_{C'_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z)\mathcal{O}(w,\bar{w}) - \frac{1}{2\pi \mathrm{i}} \int_{\bar{C}'_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z) O(w,\bar{w}) \\
[Q_{\mathrm{BRST}}, \mathcal{O}(w,\bar{w})] &= \frac{1}{2\pi \mathrm{i}} \int_{C'_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z)\mathcal{O}(w,\bar{w}) + \frac{1}{2\pi \mathrm{i}} \int_{\bar{C}'^{-1}_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z) O(w,\bar{w}) \\
[Q_{\mathrm{BRST}}, \mathcal{O}(w,\bar{w})] &= \frac{1}{2\pi \mathrm{i}} \int_{C'_w \cup \bar{C}'^{-1}_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z)\mathcal{O}(w,\bar{w}) \\
[Q_{\mathrm{BRST}}, \mathcal{O}(w,\bar{w})] &= \frac{1}{2\pi \mathrm{i}} \oint_{C_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z)\mathcal{O}(w,\bar{w})
\end{aligned} \tag{1.4}$$

Here $\bar{C}'_w$ is the complex conjugated curve to C'_w, and $\bar{C}'^{-1}_w$ is the reverse orientation version of the curve $\bar{C}'_w$. Lastly, $C'_w \cup \bar{C}'^{-1}_w$ is the closed curve got by gluing the end of C'_w with the start of

$\bar{C}'^{-1}_w$, which is not hard to see that is a counterclockwise oriented closed curve around w. The last expression, eq. (1.4), ease the calculation of the commutator in comparison to eq. (1.3), because now we can use the Cauchy residue theorem.

Our main interest here is when $\mathcal{O}$ is a open string vertex operator, specially,

$$V^a(w, \bar{w}; \epsilon, k) = \lambda^a \epsilon_\mu : c\partial X^\mu \exp\left(\mathrm{i}k \cdot X\right) : (w, \bar{w}) \tag{1.5}$$

It's clear that V^a is not just a holomorphic, or just an anti-holomorphic, operator. What spoils it is the $\exp\left(\mathrm{i}k \cdot X\right)$ part. Hence, the dependence on $\bar{w}$ is mandatory, also, as this is a open string vertex operator, it must be inserted at $\mathrm{Im}(w) = 0 \Rightarrow w = \bar{w}$. Thus, V^a actually is not dependent on $\bar{w}$, and we will omit such dependence here[1].

By eq. (1.4), we only need the knowledge of the first order pole of the OPE between the BRST current and the vertex operator in order to know the commutator. Hence, we'll start computing the OPE between those, it is in the same manner we did in the second homework set. First compute the normal ordering of the expression you want to know the OPE, this will involve among the OPE, other normal ordered terms, after gathering the divergent contributions we can set up the normal ordered terms as non-singular and obtain the OPE. As the full expression of the BRST current is quite big, and has a lot of terms, we'll compute each three contributions separately,

$$\begin{aligned}
-:\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) : = &-\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) \\
&- 2\frac{\lambda^a \epsilon_\mu}{\alpha'} : (c\partial X^\alpha \partial X_\alpha)(z)(c\partial X^\mu \exp\left(\mathrm{i}k \cdot X\right))(w) : \\
&- 2\frac{\lambda^a \epsilon_\mu}{\alpha'} : (c\partial X^\alpha \partial X_\alpha)(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- 2\frac{\lambda^a \epsilon_\mu}{\alpha'} : (c\partial X^\alpha \partial X_\alpha)(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- \frac{\lambda^a \epsilon_\mu}{\alpha'} : (c\partial X^\alpha \partial X_\alpha)(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) :
\end{aligned}$$

$$\begin{aligned}
-:\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) : = &-\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) \\
&- 2\frac{\lambda^a \epsilon_\mu}{2} \eta^{\alpha\mu} \partial_z \partial_w \ln|z-w|^2 : (c\partial X_\alpha)(z)(c\exp\left(\mathrm{i}k \cdot X\right))(w) : \\
&+ 2\frac{\alpha' \lambda^a \epsilon_\mu}{4} \eta^{\alpha\mu} \mathrm{i}k_\alpha \partial_z \partial_w \ln|z-w|^2 \partial_z \ln|z-w|^2 : c(z)(c\exp(\mathrm{i}k \cdot X))(w) : \\
&- 2\frac{\lambda^a \epsilon_\mu}{2} \mathrm{i}k_\alpha \partial_z \ln|z-w|^2 : (c\partial X^\alpha)(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&+ \frac{\alpha' \lambda^a \epsilon_\mu}{4} \mathrm{i}k_\alpha \mathrm{i}k^\alpha \left(\partial_z \ln|z-w|^2\right)^2 : c(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) :
\end{aligned}$$

$$\begin{aligned}
-:\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) : = &-\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) \\
&- \frac{\lambda^a \epsilon_\mu}{(z-w)^2} : (c\partial X^\mu)(z)(c\exp\left(\mathrm{i}k \cdot X\right))(w) : \\
&+ \frac{\mathrm{i}\alpha' \lambda^a k \cdot \epsilon}{2(z-w)^3} : c(z)(c\exp(\mathrm{i}k \cdot X))(w) : \\
&- \frac{\mathrm{i}k_\alpha \lambda^a \epsilon_\mu}{z-w} : (c\partial X^\alpha)(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- \frac{k^2 \alpha' \lambda^a \epsilon_\mu}{4(z-w)^2} : c(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) :
\end{aligned}$$

[1] If we where to be completely rigorous, we would need to change our normal ordering prescription.

now we expand each function of z in Taylor series around $z = w$, keeping only the terms which have a single simple pole,

$$\begin{aligned}
-\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w;\epsilon,k) = {} & \frac{\lambda^a \epsilon_\mu}{z-w} : \partial c c \partial X^\mu \exp\left(\mathrm{i}k\cdot X\right) : (w) \\
& - \frac{\mathrm{i}\alpha' \lambda^a k\cdot\epsilon}{4(z-w)} : \partial^2 c c \exp(\mathrm{i}k\cdot X) : (w) \\
& + \frac{k^2 \alpha' \lambda^a \epsilon_\mu}{4(z-w)} : \partial c c \partial X^\mu \exp(\mathrm{i}k\cdot X) : (w) \\
& + \text{ no single simple pole} \\
-\frac{1}{\alpha'} : c\partial X^\mu \partial X_\mu : (z) V^a(w;\epsilon,k) = {} & \frac{\lambda^a \epsilon_\mu}{z-w} \left(1 + \frac{k^2\alpha'}{4}\right) : \partial c c \partial X^\mu \exp\left(\mathrm{i}k\cdot X\right) : (w) \\
& - \frac{\mathrm{i}\alpha' \lambda^a k\cdot\epsilon}{4(z-w)} : \partial^2 c c \exp(\mathrm{i}k\cdot X) : (w) \\
& + \text{ no single simple pole}
\end{aligned} \tag{1.6}$$

The second term of the BRST current OPE gives,

$$\begin{aligned}
:: bc\partial c : (z) V^a(w;\epsilon,k) : =: {} & bc\partial c : (z) V^a(w;\epsilon,k) \\
& + \lambda^a \epsilon_\mu : (bc\partial c)(z)(c\partial X^\mu \exp\left(\mathrm{i}k\cdot X\right))(w) : \\
:: bc\partial c : (z) V^a(w;\epsilon,k) : =: {} & bc\partial c : (z) V^a(w;\epsilon,k) \\
& - \frac{\lambda^a \epsilon_\mu}{z-w} : (c\partial c)(z)(\partial X^\mu \exp\left(\mathrm{i}k\cdot X\right))(w) : \\
: bc\partial c : (z) V^a(w;\epsilon,k) = {} & \frac{\lambda^a \epsilon_\mu}{z-w} : (c\partial c)(z)(\partial X^\mu \exp\left(\mathrm{i}k\cdot X\right))(w) : + \text{ regular} \\
: bc\partial c : (z) V^a(w;\epsilon,k) = {} & -\frac{\lambda^a \epsilon_\mu}{z-w} : \partial c c \partial X^\mu \exp\left(\mathrm{i}k\cdot X\right) : (w) \\
& + \text{ no single simple pole}
\end{aligned} \tag{1.7}$$

The third term of the BRST current OPE is already non-singular,

$$\begin{aligned}
: \frac{3}{2} : \partial^2 c : (z) V^a(w;\epsilon,k) : = {} & \frac{3}{2}\partial^2 c(z) V^a(w;\epsilon,k) \\
\frac{3}{2}\partial^2 c(z) V^a(w;\epsilon,k) = {} & \text{ no single simple pole}
\end{aligned} \tag{1.8}$$

Summing the three contributions, eqs. (1.6) to (1.8),

$$\begin{aligned}
j_{\mathrm{BRST}}(z) V^a(w;\epsilon,k) = {} & \frac{\lambda^a \epsilon_\mu}{z-w} \left(1 + \frac{k^2\alpha'}{4}\right) : \partial c c \partial X^\mu \exp\left(\mathrm{i}k\cdot X\right) : (w) \\
& - \frac{\mathrm{i}\alpha' \lambda^a k\cdot\epsilon}{4(z-w)} : \partial^2 c c \exp(\mathrm{i}k\cdot X) : (w) \\
& - \frac{\lambda^a \epsilon_\mu}{z-w} : \partial c c \partial X^\mu \exp\left(\mathrm{i}k\cdot X\right) : (w) \\
& + \text{ no single simple pole} \\
j_{\mathrm{BRST}}(z) V^a(w;\epsilon,k) = {} & \frac{\lambda^a \epsilon_\mu}{z-w} \frac{k^2\alpha'}{4} : \partial c c \partial X^\mu \exp\left(\mathrm{i}k\cdot X\right) : (w)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mathrm{i}\alpha'\lambda^a k\cdot\epsilon}{4(z-w)} : \partial^2 cc \exp(\mathrm{i}k\cdot X) : (w) \\
&+ \text{ no single simple pole}
\end{aligned} \tag{1.9}
$$

Using eq. (1.4),

$$
\begin{aligned}
[Q_{\mathrm{BRST}}, V^a(w,\epsilon,k)] &= \frac{1}{2\pi\mathrm{i}} \oint_{C_w} \mathrm{d}z\, j_{\mathrm{BRST}}(z) V^a(w,\bar{w}) \\
[Q_{\mathrm{BRST}}, V^a(w,\epsilon,k)] &= \frac{1}{2\pi\mathrm{i}} \oint_{C_w} \mathrm{d}z\, \frac{\lambda^a\epsilon_\mu}{z-w}\frac{k^2\alpha'}{4} : \partial c c \partial X^\mu \exp(\mathrm{i}k\cdot X) : (w) \\
&\quad - \frac{1}{2\pi\mathrm{i}} \oint_{C_w} \mathrm{d}z\, \frac{\mathrm{i}\alpha'\lambda^a k\cdot\epsilon}{4(z-w)} : \partial^2 cc \exp(\mathrm{i}k\cdot X) : (w) \\
&\quad + \frac{1}{2\pi\mathrm{i}} \oint_{C_w} \mathrm{d}z\, \text{ no single simple pole} \\
[Q_{\mathrm{BRST}}, V^a(w,\epsilon,k)] &= \lambda^a\epsilon_\mu \frac{k^2\alpha'}{4} : \partial c c \partial X^\mu \exp(\mathrm{i}k\cdot X) : (w) \\
&\quad - \frac{\mathrm{i}\alpha'\lambda^a k\cdot\epsilon}{4} : \partial^2 cc \exp(\mathrm{i}k\cdot X) : (w)
\end{aligned}
$$

The two operators showing up here, $: \partial cc\partial X^\mu \exp(\mathrm{i}k\cdot X) : (w)$ and $: \partial^2 cc \exp(\mathrm{i}k\cdot X) : (w)$, are linear independent and also non identically zero. Hence, if we want to guarantee that $[Q_{\mathrm{BRST}}, V^a(w,\epsilon,k)] = 0$ we must set the coefficients in front of the two operator to zero. As λ^a, α' are both non zero, and we cannot guarantee that $\epsilon_\mu : \partial cc\partial X^\mu \exp(\mathrm{i}k\cdot X) : (w)$ is zero, the only option is to set to zero,

$$
k^2 = k\cdot\epsilon = 0
$$

Which is what we wanted to show.

1.B)

We need to show that $V^a(w;\epsilon+ak,k)$, $a\in\mathbb{R}$, is just $V^a(w;\epsilon,k)$ plus an BRST exact operator. What is a BRST exact operator? By the same means of the definition of a BRST closed operator, a BRST exact operator is one such $\mathcal{O}(w,\bar{w}) = [Q_{\mathrm{BRST}}, \mathcal{O}'(w,\bar{w})]$. Let's show this is the case,

$$
\begin{aligned}
V^a(w;\epsilon+ak,k) &= \lambda^a\epsilon_\mu : c\partial X^\mu \exp(\mathrm{i}k\cdot X) : (w) + a\lambda^a k_\mu : c\partial X^\mu \exp(\mathrm{i}k\cdot X) : (w) \\
V^a(w;\epsilon+ak,k) &= V^a(w;\epsilon,k) + a\lambda^a k_\mu : c\partial X^\mu \exp(\mathrm{i}k\cdot X) : (w)
\end{aligned} \tag{1.10}
$$

The last term in the second line is our interest, notice,

$$
\lambda^a k_\mu : c\partial X^\mu \exp(\mathrm{i}k\cdot X) : (w) = \lambda^a \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left(\frac{k_\mu : c\partial X^\mu \exp(\mathrm{i}k\cdot X) : (w)}{z-w} + \text{ no single simple pole} \right) \tag{1.11}
$$

As this term has to be BRST exact, the integrand has to be an OPE of some operator with the BRST current, the choice here of operator is obvious, as we have already an $c\partial X^\mu$, what seems a remainder of the first term of the BRST current, we'll try,

$$
: j_{\mathrm{BRST}}(z) : \exp(\mathrm{i}k\cdot X) : (w) := j_{\mathrm{BRST}}(z) : \exp(\mathrm{i}k\cdot X) : (w)
$$

$$
\begin{aligned}
&- \frac{2}{\alpha'} : (c\partial X^\mu \partial X_\mu)(z) \exp(\mathrm{i}k \cdot X)(w) : \\
&- \frac{1}{\alpha'} : (c\partial X^\mu \partial X_\mu)(z) \exp(\mathrm{i}k \cdot X)(w) : \\
&+ \text{ no single simple pole}
\end{aligned}
$$

Both second and third terms of the BRST current doesn't have any wick contractions with this operator, hence, do not contribute with meaningful poles,

$$
\begin{aligned}
: j_{\mathrm{BRST}}(z) : \exp(\mathrm{i}k \cdot X) : (w) : = j_{\mathrm{BRST}}(z) : \exp(\mathrm{i}k \cdot X) : (w) \\
&- \frac{2}{\alpha'} \mathrm{i}k^\mu \frac{\alpha'}{2} \partial_z \ln |z-w|^2 : (c\partial X_\mu)(z) \exp(\mathrm{i}k \cdot X)(w) : \\
&+ \frac{1}{\alpha'} \mathrm{i}^2 k^2 \frac{\alpha'^2}{4} \left(\partial_z \ln |z-w|^2\right)^2 : c(z) \exp(\mathrm{i}k \cdot X)(w) : \\
&+ \text{ no single simple pole}
\end{aligned}
$$

Expanding on Taylor, and also using $k^2 = 0$,

$$
\begin{aligned}
j_{\mathrm{BRST}}(z) : \exp(\mathrm{i}k \cdot X) : (w) = \frac{\mathrm{i}k^\mu}{z-w} : c\partial X_\mu \exp(\mathrm{i}k \cdot X) : (w) \\
+ \text{ no single simple pole}
\end{aligned}
$$

Using this result back in eq. (1.11),

$$
\begin{aligned}
\lambda^a k_\mu : c\partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) &= \lambda^a \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left(\frac{k_\mu : c\partial X^\mu \exp(\mathrm{i}k \cdot X) : (w)}{z-w} + \text{ no single simple pole} \right) \\
\lambda^a k_\mu : c\partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) &= -\mathrm{i}\lambda^a \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} j_{\mathrm{BRST}}(z) : \exp(\mathrm{i}k \cdot X) : (w) \\
\lambda^a k_\mu : c\partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) &= -\mathrm{i}\lambda^a [Q_{\mathrm{BRST}}, : \exp(\mathrm{i}k \cdot X) : (w)]
\end{aligned}
$$

At last, substituting in eq. (1.10),

$$
V^a(w; \epsilon + ak, k) = V^a(w; \epsilon, k) - a\mathrm{i}\lambda^a [Q_{\mathrm{BRST}}, : \exp(\mathrm{i}k \cdot X) : (w)]
$$

Exactly what we wanted to show.

1.C)

This problem will be very heavy on CFT, let's state a few know results of correlation functions in CFTs,

$$
\langle \phi_i(z_i) \rangle = \delta_{h_i,0} C_i \tag{1.12}
$$

$$
\langle \phi_i(z_i) \phi_j(z_j) \rangle = \delta_{h_i,h_j} C_{ij} |z_{ij}|^{h_i+h_j} \tag{1.13}
$$

$$
\langle \phi_i(z_i) \phi_j(z_j) \phi_k(z_k) \rangle = C_{ijk} |z_{ij}|^{h_k-h_i-h_j} |z_{jk}|^{h_i-h_k-h_j} |z_{ki}|^{h_j-h_i-h_k} \tag{1.14}
$$

We'll abuse these equations. The reason they're so useful is the two point one, which is zero unless the two fields have the same conformal weight. For this we'll need to know the conformal weight of V^a — for the open string, which is inserted at $\bar{w} = w$, behaves as they were holomorphic fields —. To know the conformal weights of it is to compute the OPE with $T = -\frac{1}{\alpha'} : \partial X^\alpha \partial X_\alpha : - : \partial bc : -2 : b\partial c :$,

$$
: T(z) V^a(w) : = T(z) V^a(w) - \lambda^a \epsilon_\mu \frac{2}{\alpha'} : (\partial X^\alpha \partial X_\alpha)(z)(c\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) :
$$

$$
\begin{aligned}
&- \lambda^a \epsilon_\mu \frac{2}{\alpha'} : (\partial X^\alpha \partial X_\alpha)(z)(c \partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- \lambda^a \epsilon_\mu \frac{2}{\alpha'} : (\partial X^\alpha \partial X_\alpha)(z)(c \partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- \lambda^a \epsilon_\mu \frac{1}{\alpha'} : (\partial X^\alpha \partial X_\alpha)(z)(c \partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- \lambda^a \epsilon_\mu : (\partial b c)(z)(c \partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- 2\lambda^a \epsilon_\mu : (b \partial c)(z)(c \partial X^\mu \exp(\mathrm{i}k \cdot X))(w) :
\end{aligned}
$$

$$
\begin{aligned}
: T(z) V^a(w) : = T(z) V^a(w) &- \lambda^a \epsilon_\mu \eta^{\mu\alpha} \partial_z \partial_w \ln|z-w|^2 : \partial X_\alpha(z)(c \exp(\mathrm{i}k \cdot X))(w) : \\
&- \lambda^a \epsilon_\mu \mathrm{i}k^\alpha \partial_z \ln|z-w|^2 : \partial X_\alpha(z)(c \partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&+ \lambda^a \epsilon_\mu \frac{\alpha'}{2} \mathrm{i}k_\alpha \eta^{\alpha\mu} \partial_z \ln|z-w|^2 \partial_z \partial_w \ln|z-w|^2 : c \exp(\mathrm{i}k \cdot X) : (w) \\
&+ \lambda^a \epsilon_\mu \frac{\alpha'}{4} \mathrm{i}^2 k^2 \left(\partial_z \ln|z-w|^2\right)^2 : c \partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&- \lambda^a \epsilon_\mu \partial_z \left(\frac{1}{z-w}\right) : c(z)(\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) : \\
&- 2\lambda^a \epsilon_\mu \frac{1}{z-w} : \partial c(z)(\partial X^\mu \exp(\mathrm{i}k \cdot X))(w) :
\end{aligned}
$$

Setting now $k^2 = k \cdot \epsilon = 0$ and expanding in Taylor,

$$
\begin{aligned}
T(z) V^a(w) = {} & \lambda^a \epsilon_\mu \frac{1}{(z-w)^2} : \partial X^\mu c \exp(\mathrm{i}k \cdot X) : (w) \\
&+ \lambda^a \epsilon_\mu \frac{1}{z-w} : \partial^2 X^\mu c \exp(\mathrm{i}k \cdot X) : (w) \\
&+ \frac{\lambda^a \epsilon_\mu \mathrm{i}k^\alpha}{z-w} : \partial X_\alpha c \partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&- \lambda^a \epsilon_\mu \frac{1}{(z-w)^2} : c \partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&- \lambda^a \epsilon_\mu \frac{1}{z-w} : \partial c \partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&+ 2\lambda^a \epsilon_\mu \frac{1}{z-w} : \partial c \partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&+ \text{ regular} \\
T(z) V^a(w) = {} & + \lambda^a \epsilon_\mu \frac{1}{z-w} : c \partial^2 X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&+ \frac{\lambda^a \epsilon_\mu \mathrm{i}k^\alpha}{z-w} : c \partial X_\alpha \partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&+ \lambda^a \epsilon_\mu \frac{1}{z-w} : \partial c \partial X^\mu \exp(\mathrm{i}k \cdot X) : (w) \\
&+ \text{ regular} \\
T(z) V^a(w) = {} & \frac{1}{z-w} \partial V^a(w) + \text{ regular}
\end{aligned}
$$

An astonishing result of $h = 0$ for V^a. We'll proceed as follow, first, we'll compute the OPE VV, which is given in terms of normal ordered terms, hence, our three point correlator will be transformed into a two point correlator. By eq. (1.13) and by the zero conformal weight of V^a, the only non zero contribution between the OPE VV and the remaining V in the correlator will be of the zero conformal weight part of the VV OPE with V. At last, by eq. (1.12), the only non zero contribution to the three point correlator will be the zero conformal weight part of the OPE between the zero conformal weight part of the VV OPE with V. Before proceeding with this OPE computation, we'll need a lot's of other results which now we develop.

1.C).1 Momentum conservation delta

In further results we'll need to use the conservation of momentum $k_1 + k_2 + k_3 = 0$, which now we prove. Remember, we have as a conserved current the target-space momentum, $j^\mu(z) = \frac{\mathrm{i}}{\alpha'}\partial X^\mu(z)$, let's compute the OPE of this current with V^a,

$$
\begin{aligned}
: j^\mu(z)V^a(w) : &= j^\mu(z)V^a(w) + \frac{\mathrm{i}\lambda^a\epsilon_\nu}{\alpha'} : \partial X^\mu(z)(c\partial X^\nu \exp(\mathrm{i}k\cdot X))(w) : \\
&\quad + \frac{\mathrm{i}\lambda^a\epsilon_\nu}{\alpha'} : \partial X^\mu(z)(c\partial X^\nu \exp(\mathrm{i}k\cdot X))(w) : \\
: j^\mu(z)V^a(w) : &= j^\mu(z)V^a(w) + \frac{\mathrm{i}\lambda^a\epsilon_\nu}{2}\eta^{\mu\nu}\partial_z\partial_w \ln|z-w|^2 : c\exp(\mathrm{i}k\cdot X) : (w) \\
&\quad + \frac{\mathrm{i}^2\lambda^a\epsilon_\nu}{2} k^\mu \partial_z \ln|z-w|^2 : c\partial X^\nu \exp(\mathrm{i}k\cdot X) : (w) \\
j^\mu(z)V^a(w) &= -\frac{\mathrm{i}\lambda^a\epsilon^\mu}{2(z-w)^2} : c\exp(\mathrm{i}k\cdot X) : (w) + \frac{k^\mu V^a(w)}{2(z-w)} + \text{ regular}
\end{aligned}
$$

The conserved charge associated with the target-space conserved momentum — for the open string — is, $P^\mu = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} j^\mu(z)$, hence,

$$
\begin{aligned}
[P^\mu, V^a(w)] &= \oint_{C_w} \frac{\mathrm{d}z}{2\pi\mathrm{i}} j^\mu(z)V^a(w) \\
[P^\mu, V^a(w)] &= \oint_{C_w} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left(-\frac{\mathrm{i}\lambda^a\epsilon^\mu}{2(z-w)^2} : c\exp(\mathrm{i}k\cdot X) : (w) + \frac{k^\mu V^a(w)}{2(z-w)} + \text{ regular} \right) \\
[P^\mu, V^a(w)] &= \frac{k^\mu}{2} V^a(w)
\end{aligned} \tag{1.15}
$$

One may wonder about the usefulness of such expression. When computing $\langle \mathcal{O}(w,\bar{w})\rangle$, we're doing an expectation value of the operator $\mathcal{O}(w,\bar{w})$ over a state, in this case the vacuum. But, the vacuum itself is an eigenstate of the operator P^μ, and as $P^{\dagger\mu} = P^\mu$, the following is valid for any operator,

$$
\left(P^{\dagger\mu}\Psi, \mathcal{O}(w,\bar{w})\Psi\right) - (\Psi, \mathcal{O}(w,\bar{w})P^\mu\Psi) = \langle P^\mu \mathcal{O}(w,\bar{w})\rangle - \langle \mathcal{O}(w,\bar{w})P^\mu\rangle = \langle [P^\mu, \mathcal{O}(w,\bar{w})]\rangle = 0
$$

In particular, this is valid to,

$$
\langle [P^\mu, V^{a_1}(w_1;\epsilon_1,k_1)V^{a_2}(w_2;\epsilon_2,k_2)V^{a_3}(w_3;\epsilon_3,k_3)]\rangle = 0 \tag{1.16}
$$

But,

$$
\begin{aligned}
[P^\mu, V^{a_1}V^{a_2}V^{a_3}] &= [P^\mu, V^{a_1}]V^{a_2}V^{a_3} + V^{a_1}[P^\mu, V^{a_2}]V^{a_3} + V^{a_1}V^{a_2}[P^\mu, V^{a_3}] \\
[P^\mu, V^{a_1}V^{a_2}V^{a_3}] &= \frac{k_1^\mu}{2}V^{a_1}V^{a_2}V^{a_3} + \frac{k_2^\mu}{2}V^{a_1}V^{a_2}V^{a_3} + \frac{k_3^\mu}{2}V^{a_1}V^{a_2}V^{a_3} \\
[P^\mu, V^{a_1}V^{a_2}V^{a_3}] &= \frac{k_1^\mu + k_2^\mu + k_3^\mu}{2} V^{a_1}V^{a_2}V^{a_3}
\end{aligned}
$$

Combining this with eq. (1.16),

$$
0 = \langle [P^\mu, V^{a_1}V^{a_2}V^{a_3}]\rangle = \frac{k_1^\mu + k_2^\mu + k_3^\mu}{2} \langle V^{a_1}V^{a_2}V^{a_3}\rangle
$$

This result is the same as saying,

$$
\langle V^{a_1}V^{a_2}V^{a_3}\rangle \propto \delta^{(26)}(k_1 + k_2 + k_3)
$$

1.C).2 Trace color factors

Each vertex operator is *dressed* with respect to it's color group by λ^a. So that the final correlator also should be dressed with respect to some number $\lambda^{a_1a_2a_3}$. We can fully determine such. V^a is fermionic, hence, from the path integral point of view,

$$
\begin{aligned}
\langle V^{a_1}V^{a_3}V^{a_2}\rangle &= \lambda^{a_1a_3a_2}\,\langle :\epsilon_1\cdot\partial X\exp(\mathrm{i}k_1\cdot X)::\epsilon_3\cdot\partial X\exp(\mathrm{i}k_3\cdot X)::\epsilon_2\cdot\partial X\exp(\mathrm{i}k_2\cdot X):\rangle\\
\langle V^{a_1}V^{a_3}V^{a_2}\rangle &= -\lambda^{a_1a_3a_2}\,\langle :\epsilon_1\cdot\partial X\exp(\mathrm{i}k_1\cdot X)::\epsilon_2\cdot\partial X\exp(\mathrm{i}k_2\cdot X)::\epsilon_3\cdot\partial X\exp(\mathrm{i}k_3\cdot X):\rangle\\
\langle V^{a_1}V^{a_3}V^{a_2}\rangle &= -\frac{\lambda^{a_1a_3a_2}}{\lambda^{a_1a_2a_3}}\,\langle V^{a_1}V^{a_2}V^{a_3}\rangle
\end{aligned}
$$

But, from the path integral point of view, $\langle V^{a_1}V^{a_3}V^{a_2}\rangle$ and $\langle V^{a_1}V^{a_2}V^{a_3}\rangle$ are equal, as long as the ordering of the points stay the same, hence, the above argument proves that $-\lambda^{a_1a_3a_2} = -\lambda^{a_1a_2a_3}$. And, similar arguments can be made to prove that $\lambda^{a_1a_2a_3}$ is in fact totally antisymmetric. There is a unique, up to multiplicative factors, three index totally antisymmetric object that can be constructed out of generators of the group λ^a, this is the structure constant $f^{a_1a_2a_3}$, which has a nice representation as $\mathrm{Tr}\,[\lambda^{a_1}[\lambda^{a_2},\lambda^{a_3}]]$. This completely determines the dependence on λs,

$$\langle V^{a_1}V^{a_2}V^{a_3}\rangle \propto \mathrm{Tr}\,[\lambda^{a_1}[\lambda^{a_2},\lambda^{a_3}]]$$

1.C).3 VV OPE

Now we compute the VV OPE, for now we forget about the color factors,

$$
\begin{aligned}
:V_1V_2: = V_1V_2 &+ \epsilon_{1\mu}\epsilon_{2\nu} : (c\partial X^\mu\exp(\mathrm{i}k_1\cdot X))(w_1)(c\partial X^\nu\exp(\mathrm{i}k_2\cdot X))(w_2):\\
&+ \epsilon_{1\mu}\epsilon_{2\nu} : (c\partial X^\mu\exp(\mathrm{i}k_1\cdot X))(w_1)(c\partial X^\nu\exp(\mathrm{i}k_2\cdot X))(w_2):\\
&- \epsilon_{1\mu}\epsilon_{2\nu} : (c\partial X^\nu\exp(\mathrm{i}k_2\cdot X))(w_2)(c\partial X^\mu\exp(\mathrm{i}k_1\cdot X))(w_1):\\
&- \epsilon_{1\mu}\epsilon_{2\nu} : (c\partial X^\nu\exp(\mathrm{i}k_2\cdot X))(w_2)(c\partial X^\mu\exp(\mathrm{i}k_1\cdot X))(w_1):\\
:V_1V_2: = V_1V_2 &+ \frac{\alpha'}{2(w_1-w_2)^2}\epsilon_1\cdot\epsilon_2 : (c\exp(\mathrm{i}k_1\cdot X))(w_1)(c\exp(\mathrm{i}k_2\cdot X))(w_2):\\
&+ \frac{\alpha'\mathrm{i}}{2(w_1-w_2)}\epsilon_1\cdot k_2\epsilon_{2\nu} : (c\exp(\mathrm{i}k_1\cdot X))(w_1)(c\partial X^\nu\exp(\mathrm{i}k_2\cdot X))(w_2):\\
&+ \frac{\alpha'\mathrm{i}}{2(w_2-w_1)}\epsilon_{1\mu}\epsilon_2\cdot k_1 : (c\partial X^\mu\exp(\mathrm{i}k_1\cdot X))(w_1)(c\exp(\mathrm{i}k_2\cdot X))(w_2):\\
&- \frac{\alpha'^2\mathrm{i}^2}{4(w_2-w_1)(w_1-w_2)}\epsilon_1\cdot k_2\epsilon_2\cdot k_1 : (c\exp(\mathrm{i}k_1\cdot X))(w_1)(c\exp(\mathrm{i}k_2\cdot X))(w_2):
\end{aligned}
\tag{1.17}
$$

There are no contraction between two exponential because they are regular, as we know, $:\exp(\mathrm{i}k_1\cdot X):(z):\exp(\mathrm{i}k_2\cdot X):(w) = |z-w|^{\alpha' k_1\cdot k_2} :\exp(\mathrm{i}k_1\cdot X)(z)\exp(\mathrm{i}k_2\cdot X)(w):$, but, using the on-shell conditions and the conservation of momentum, $k_1\cdot k_2 = \frac{1}{2}(k_1+k_2)^2 = \frac{1}{2}k_3^2 = 0$, hence, the product is regular. As we mentioned before, we only need the zero conformal weight part of this OPE, to obtain this is easy, first, expand the OPE in Taylor around $w_1 = w_2$, then do the counting by using that c has -1, X has 0 and $\exp(\mathrm{i}k\cdot X)$ has $\frac{\alpha'}{4}k^2 = 0$. Other way is to look for the $w_1 - w_2$ prefactor after doing the Taylor expansion, the zero conformal weight contributions have no such prefactor. Doing this,

$$
\begin{aligned}
V_1V_2 = &-\frac{\alpha'}{4}\epsilon_1\cdot\epsilon_2 : \partial^2 c\exp(\mathrm{i}k_1\cdot X)c\exp(\mathrm{i}k_2\cdot X):(w_2)\\
&- \frac{\alpha'\mathrm{i}}{2}k_{1\mu}\epsilon_1\cdot\epsilon_2 : \partial c\partial X^\mu\exp(\mathrm{i}k_1\cdot X)c\exp(\mathrm{i}k_2\cdot X):(w_2)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\alpha' \mathrm{i}}{2}\epsilon_1 \cdot k_2 \epsilon_{2\nu} : \partial c \exp(\mathrm{i}k_1 \cdot X) c \partial X^\nu \exp(\mathrm{i}k_2 \cdot X) : (w_2) \\
&+ \frac{\alpha' \mathrm{i}}{2}\epsilon_{1\mu} \epsilon_2 \cdot k_1 : \partial c \partial X^\mu \exp(\mathrm{i}k_1 \cdot X) c \exp(\mathrm{i}k_2 \cdot X) : (w_2) \\
&+ \frac{\alpha'^2}{8}\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 : \partial^2 (c \exp(\mathrm{i}k_1 \cdot X)) c \exp(\mathrm{i}k_2 \cdot X) : (w_2) \\
&+ \ h = 0 \text{ terms}
\end{aligned}
$$

$$
\begin{aligned}
V_1 V_2 = &- \frac{\alpha'}{4}\left(\epsilon_1 \cdot \epsilon_2 - \frac{\alpha'}{2}\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1\right) : \partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) \\
&- \frac{\alpha' \mathrm{i}}{2}\left(k_{1\mu}\epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot k_2 \epsilon_{2\mu} - \epsilon_{1\mu}\epsilon_2 \cdot k_1 - \frac{\alpha'}{2} k_{1\mu}\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1\right) : \partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) \\
&+ \ h \neq 0 \text{ terms}
\end{aligned}
\tag{1.18}
$$

The $h \neq 0$ part of the OPE do not contributes to the three point function, so we can forget about it. Now, we compute the OPE of this OPE with the last V. We do it step by step, first with the first term of eq. (1.18). Again, the exponential does not need to be contracted with another exponential, also we only keep terms with zero conformal weight,

$$
\begin{aligned}
: \partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) V_3 =&: : \partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) V_3 : \\
&+ \epsilon_{3\alpha} : (c \partial X^\alpha \exp(\mathrm{i}k_3 \cdot X))(w_3)(\partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2) : \\
: \partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) V_3 =&: : \partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) V_3 : \\
&- \frac{\alpha' \mathrm{i}\epsilon_3 \cdot (k_1 + k_2)}{2(w_3 - w_2)} : (\partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2)(c \exp(\mathrm{i}k_3 \cdot X))(w_3) : \\
: \partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) V_3 =& \frac{\alpha' \mathrm{i}\epsilon_3 \cdot (k_1 + k_2)}{2} : \partial^2 c \partial c c \exp(\mathrm{i}(k_1 + k_2 + k_3) \cdot X) : (w_3) \\
&+ \ h \neq 0 \text{ terms}
\end{aligned}
\tag{1.19}
$$

$$
: \partial^2 c c \exp(\mathrm{i}(k_1 + k_2) \cdot X) : (w_2) V_3 = \ h \neq 0 \text{ terms} \tag{1.20}
$$

Where we used $k_1 + k_2 = -k_3$. Now the second term of eq. (1.18),

$$
\begin{aligned}
: \partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X) :(w_2) V_3 = &- \epsilon_{3\alpha} : (\partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2)(c \partial X^\alpha \exp(\mathrm{i}k_3 \cdot X))(w_3) : \\
&- \epsilon_{3\alpha} : (\partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2)(c \partial X^\alpha \exp(\mathrm{i}k_3 \cdot X))(w_3) : \\
&+ \epsilon_{3\alpha} : (c \partial X^\alpha \exp(\mathrm{i}k_3 \cdot X))(w_3)(\partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2) : \\
&+ \epsilon_{3\alpha} : (c \partial X^\alpha \exp(\mathrm{i}k_3 \cdot X))(w_3)(\partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2) : \\
&+ \ h \neq 0 \text{ terms} \\
: \partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X) :(w_2) V_3 = &- \frac{\alpha' \epsilon_3^\mu}{2(w_2 - w_3)^2} : (\partial c c \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2)(c \exp(\mathrm{i}k_3 \cdot X))(w_3) : \\
&- \frac{\alpha' \mathrm{i} k_3^\mu \epsilon_{3\alpha}}{2(w_2 - w_3)} : (\partial c c \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2)(c \partial X^\alpha \exp(\mathrm{i}k_3 \cdot X))(w_3) : \\
&- \frac{\alpha' \mathrm{i}(k_1 + k_2) \cdot \epsilon_3}{2(w_3 - w_2)} : (\partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2)(c \exp(\mathrm{i}k_3 \cdot X))(w_3) : \\
&+ \frac{\alpha'^2 \mathrm{i}^2 k_3^\mu (k_1 + k_2) \cdot \epsilon_3}{4(w_3 - w_2)(w_2 - w_3)} : (\partial c c \exp(\mathrm{i}(k_1 + k_2) \cdot X))(w_2)(c \exp(\mathrm{i}k_3 \cdot X))(w_3) : \\
&+ \ h \neq 0 \text{ terms} \\
: \partial c c \partial X^\mu \exp(\mathrm{i}(k_1 + k_2) \cdot X) :(w_2) V_3 = &- \frac{\alpha' \epsilon_3^\mu}{2} : \partial^2 c \partial c c \exp(\mathrm{i}(k_1 + k_2 + k_3) \cdot X) : (w_3)
\end{aligned}
$$

$$+ \ h \neq 0 \text{ terms} \tag{1.21}$$

Where we used $k_1 + k_2 = -k_3$. Putting together eqs. (1.18) to (1.21),

$$\begin{aligned} V_1 V_2 V_3 = \frac{\alpha'^2 \mathrm{i}}{4} &: \partial^2 c \partial c c \exp(\mathrm{i}(k_1 + k_2 + k_3) \cdot X) : (w_3) \times \\ &\times \left[\epsilon_3^\mu \left(k_{1\mu} \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot k_2 \epsilon_{2\mu} - \epsilon_{1\mu} \epsilon_2 \cdot k_1 - \frac{\alpha'}{2} k_{1\mu} \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \right) \right] \end{aligned}$$

Using that $\epsilon_3 \cdot k_1 = -\epsilon_3 \cdot k_2$ and similar,

$$\begin{aligned} V_1 V_2 V_3 = \frac{\alpha'^2 \mathrm{i}}{8} &: \partial^2 c \partial c c \exp(\mathrm{i}(k_1 + k_2 + k_3) \cdot X) : (w_3) \times \\ &\times \left[\epsilon_3 \cdot k_{12} \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot k_{23} \epsilon_2 \cdot \epsilon_3 + \epsilon_2 \cdot k_{31} \epsilon_3 \cdot \epsilon_1 + \frac{\alpha'}{8} \epsilon_3 \cdot k_{12} \epsilon_1 \cdot k_{23} \epsilon_2 \cdot k_{31} \right] \end{aligned}$$

In which $k_{ij} = k_i - k_j$. So the expectation value is, now dressing with color factors,

$$\begin{aligned} \langle V^{a_1} V^{a_2} V^{a_3} \rangle = \mathrm{Tr}\,[\lambda^{a_1}[\lambda^{a_2}, \lambda^{a_3}]] \frac{\alpha'^2 \mathrm{i}}{8} &\big\langle : \partial^2 c \partial c c \exp(\mathrm{i}(k_1 + k_2 + k_3) \cdot X) : (w_3) \big\rangle \times \\ &\times \left[\epsilon_3 \cdot k_{12} \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot k_{23} \epsilon_2 \cdot \epsilon_3 + \epsilon_2 \cdot k_{31} \epsilon_3 \cdot \epsilon_1 + \frac{\alpha'}{8} \epsilon_3 \cdot k_{12} \epsilon_1 \cdot k_{23} \epsilon_2 \cdot k_{31} \right] \end{aligned}$$

The expectation value on the right hand side cannot depend on the point w_3, as there is just a single operator inside it, hence, it is just a scalar factor. In fact it's from this term that we get the conservation of momentum delta, this is easier to see from the path integral formalism, where we have $\int \mathcal{D}X \exp(\mathrm{i} \sum k \cdot X) \propto \delta^{(26)}(k_1 + k_2 + k_3)$. We won't derive this is detail as we already have derived that there must be a conservation of momentum delta inside it, and as we already argued that this expectation value is a scalar — because it has conformal weight zero —, and by the on-shell conditions $k_1^2 = k_2^2 = k_3^2 = 0$ and the conservation of momentum no scalar can be build out of the momentums $k_i \cdot k_j = 0$, hence, this expectation value is just the conservation of momentum delta times a number,

$$\begin{aligned} \langle V^{a_1} V^{a_2} V^{a_3} \rangle = C\, \mathrm{Tr}\,[\lambda^{a_1}[\lambda^{a_2}, \lambda^{a_3}]] \frac{\alpha'^2 \mathrm{i}}{8} &\delta^{(26)}(k_1 + k_2 + k_3) \times \\ &\times \left[\epsilon_3 \cdot k_{12} \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot k_{23} \epsilon_2 \cdot \epsilon_3 + \epsilon_2 \cdot k_{31} \epsilon_3 \cdot \epsilon_1 + \frac{\alpha'}{8} \epsilon_3 \cdot k_{12} \epsilon_1 \cdot k_{23} \epsilon_2 \cdot k_{31} \right] \end{aligned}$$

The three first kinematic factors are the usual Yang-Mills three point interaction, the fourth one is the first stringy contribution, as can be seen from the power counting of α'.

1.D)

We just want the part of the correlator proportional to $\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4$, which can be seen from eq. (1.17), we just need to add the part corresponding to the exponential contraction,

$$V_1 V_2 \sim -\frac{\alpha' \epsilon_1 \cdot \epsilon_2}{2(w_1 - w_2)^{2 - \alpha' k_1 \cdot k_2}} : (c \exp(\mathrm{i} k_1 \cdot X))(w_1)(c \exp(\mathrm{i} k_2 \cdot X))(w_2) :$$

$$V_1V_2 \sim -\frac{\alpha'\epsilon_1\cdot\epsilon_2}{2(w_1-w_2)^{-\alpha' k_1\cdot k_2}} : \partial^2 cc\exp(\mathrm{i}(k_1+k_2)\cdot X) : (w_2)$$

We don't need the factor of $: V_1V_2 :$ because it just generates terms as $\epsilon_1\cdot\epsilon_3, \epsilon_2\cdot\epsilon_3$, and others which we aren't interested on. Thus,

$$V_1V_2V_3 \sim -\frac{\alpha'\epsilon_1\cdot\epsilon_2\epsilon_{3\mu}}{2w_{12}^{-\alpha' k_1\cdot k_2}} w_{23}^{\alpha'(k_1+k_2)\cdot k_3} : (\partial^2 cc\exp(\mathrm{i}(k_1+k_2)\cdot X))(w_2)(c\partial^\mu X\exp(\mathrm{i}k_3\cdot X))(w_3) :$$
$$V_1V_2V_3 \sim -\frac{\alpha'\epsilon_1\cdot\epsilon_2\epsilon_{3\mu}}{2w_{12}^{-\alpha' k_1\cdot k_2}} w_{23}^{1-\alpha' k_4\cdot k_3} : \partial^2 c\partial cc\partial^\mu X\exp(\mathrm{i}(k_1+k_2+k_3)\cdot X) : (w_3)$$

Where we used the conservation of momentum $k_1+k_2=-k_3-k_4$, which holds but we haven't proved, and also the on-shell conditions $k_3^2=0$. It is not possible to do any more contractions, because those would generate terms as $\epsilon_3\cdot k_1, \epsilon_3\cdot k_2$. Now we include the U_4 operator,

$$V_1V_2V_3U_4 \sim$$
$$-\int\limits_{\mathrm{Im}(w_4)} \frac{\alpha'\epsilon_1\cdot\epsilon_2\epsilon_{3\mu}\epsilon_{4\nu}}{2w_{12}^{-\alpha' k_1\cdot k_2}} w_{23}^{1-\alpha' k_4\cdot k_3} : (\partial^2 c\partial cc\partial^\mu X\exp(\mathrm{i}(k_1+k_2+k_3)\cdot X))(w_3)(\partial^\nu X\exp(\mathrm{i}k_4\cdot X))(w_4) :$$
$$\int\limits_{\mathrm{Im}(w_4)} \frac{\alpha'^2\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4}{4w_{12}^{-\alpha' k_1\cdot k_2}} \frac{w_{23}^{1-\alpha' k_4\cdot k_3}}{(w_3-w_4)^2} : (\partial^2 c\partial cc\exp(\mathrm{i}(k_1+k_2+k_3)\cdot X))(w_3)(\exp(\mathrm{i}k_4\cdot X))(w_4) :$$

Where no more contractions are possible. As we have just a single normal ordered operator inside the expectation value, by eq. (1.12), it has to have zero conformal weight for it to be non zero, we expand in Taylor the only non zero contribution,

$$\langle V_1V_2V_3U_4\rangle \sim$$
$$\int\limits_{\mathrm{Im}(w_4)} \frac{\alpha'^2\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4}{4w_{12}^{-\alpha' k_1\cdot k_2}} \frac{w_{23}^{1-\alpha' k_4\cdot k_3}}{(w_3-w_4)^2} \langle : \partial^2 c\partial cc\exp(\mathrm{i}(k_1+k_2+k_3+k_4)\cdot X) : (w_4)\rangle$$

We use the result we argued in the last item,

$$\langle V_1V_2V_3U_4\rangle \sim$$
$$C\int\limits_{\mathrm{Im}(w_4)} \frac{\alpha'^2\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4}{4w_{12}^{-\alpha' k_1\cdot k_2}} \frac{w_{23}^{1-\alpha' k_4\cdot k_3}}{(w_3-w_4)^2}\delta^{(26)}(k_1+k_2+k_3+k_4)$$

And extract the $\alpha'\to 0$ limit by taking off the overall factor of α'^2,

$$\frac{1}{\alpha'^2}\langle V_1V_2V_3U_4\rangle\Big|_{\alpha'\to 0} \to C\delta^{(26)}(k_1+k_2+k_3+k_4)\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 w_{23}\int\limits_{\mathrm{Im}(w_4)}\frac{1}{(w_3-w_4)^2}$$

Problem 2

2.A)

The computation follows almost the same route as the one done in the last problem, but now the BRST current is,

$$j_{\text{BRST}} =: cT^m + \gamma G^m + bc\partial c + \frac{3}{4}\partial c\beta\gamma + \frac{1}{4}c\partial\beta\gamma - \frac{3}{4}c\beta\partial\gamma - b\gamma^2 : \tag{2.1}$$

Where T^m are the matter energy momentum tensors, and G^m is the supersymmetric counterpart,

$$T^m = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : -\frac{1}{2} : \psi^\mu \partial \psi_\mu :$$
$$G^m = \mathrm{i}\sqrt{\frac{2}{\alpha'}} : \psi^\mu \partial X_\mu :$$

We want to compute the OPE between the j_{BRST} and the vertex operator,

$$V^a = \lambda^a \epsilon_\mu : c\delta(\gamma)\psi^\mu \exp(\mathrm{i}k\cdot X) :$$

To compute the OPE of the first term of j_{BRST} we use that ψ^μ has conformal weight $\frac{1}{2}$ and $\exp(\mathrm{i}k\cdot X)$ has conformal weight $\frac{\alpha'}{4}k^2$, and remembering we just care to simple poles in the whole OPE,

$$(cT^m)(z)V^a(w) = c(z)\left(\frac{\alpha'}{4}k^2 + \frac{1}{2}\right)\frac{V^a(w)}{(z-w)^2} + \text{ no single pole}$$
$$(cT^m)(z)V^a(w) = \left(\frac{\alpha'}{4}k^2 + \frac{1}{2}\right)\frac{: \partial c V^a : (w)}{z-w} + \text{ no single pole}$$

Now the second term,

$$(\gamma G^m)(z)V^a(w) = \mathrm{i}\lambda^a\epsilon_\mu\sqrt{\frac{2}{\alpha'}} : (\gamma\psi^\nu\partial X_\nu)(z)(c\delta(\gamma)\psi^\mu \exp(\mathrm{i}k\cdot X))(w) : + \text{ no single pole}$$
$$(\gamma G^m)(z)V^a(w) = -\mathrm{i}\lambda^a\epsilon_\mu\sqrt{\frac{2}{\alpha'}}\frac{\alpha'}{2(z-w)^2}\mathrm{i}k^\mu : \gamma(z)(c\delta(\gamma)\exp(\mathrm{i}k\cdot X))(w) : + \text{ no single pole}$$
$$(\gamma G^m)(z)V^a(w) = \lambda^a\sqrt{\frac{\alpha'}{2}}\frac{\epsilon\cdot k}{z-w} : \partial\gamma c\delta(\gamma)\exp(\mathrm{i}k\cdot X) : (w) + \text{ no single pole}$$

The third one,

$$: bc\partial c : (z)V^a(w) = \lambda^a\epsilon_\mu : (bc\partial c)(z)(c\delta(\gamma)\psi^\mu\exp(\mathrm{i}k\cdot X))(w) :$$
$$: bc\partial c : (z)V^a(w) = \lambda^a\frac{\epsilon_\mu}{z-w} : (c\partial c)(z)(\delta(\gamma)\psi^\mu\exp(\mathrm{i}k\cdot X))(w) :$$
$$: bc\partial c : (z)V^a(w) = -\frac{1}{z-w} : \partial c V^a : (w) + \text{ regular}$$

For the fourth term we need the following contraction, $: \beta(z)\delta(\gamma)(w) :$. This can be done with the bosonized version of the $\beta\gamma$ system, the OPE provides the wick contraction,

$$\beta(z)\delta(\gamma)(w) =: \exp(-\phi)\partial\xi : (z) : \exp(-\phi) : (w)$$

$$\beta(z)\delta(\gamma)(w) = |z-w|^{-1} : (\exp(-\phi)\partial)\xi(z)\exp(-\phi)(w) :$$
$$\beta(z)\delta(\gamma)(w) = |z-w|^{-1} : \exp(-\phi)\partial\xi\exp(-\phi) : (w) + \text{ regular}$$
$$\beta(z)\delta(\gamma)(w) = |z-w|^{-1} : \beta\delta(\gamma) : (w) + \text{ regular}$$

Hence, the OPE with the fourth terms is,

$$\frac{3}{4} : \partial c\beta\gamma : (z)V^a = \frac{3}{4(z-w)}\lambda^a\epsilon_\mu : (\partial c\beta\gamma)(z)(c\delta(\gamma)\psi^\mu\exp(\mathrm{i}k\cdot X))(w) := \text{regular}$$

Due to $\gamma(z)\delta(\gamma)(w) \sim \mathcal{O}(z-w)$. With the fifth term,

$$\frac{1}{4} : c\partial\beta\gamma : V^a(w) = -\lambda^a\epsilon_\mu\frac{1}{4(z-w)^2} : (c\beta\gamma)(z)(c\delta(\gamma)\psi^\mu\exp(\mathrm{i}k\cdot X))(w) :$$
$$\frac{1}{4} : c\partial\beta\gamma : V^a(w) = -\lambda^a\epsilon_\mu\frac{1}{4} : \partial c\beta\partial\gamma c\delta(\gamma)\psi^\mu\exp(\mathrm{i}k\cdot X) : (w) + \text{ regular}$$
$$\frac{1}{4} : c\partial\beta\gamma : V^a(w) = \text{ regular}$$

The sixty term,

$$-\frac{3}{4} : c\beta\partial\gamma : V^a(w) = -\lambda^a\epsilon_\mu\frac{3}{4(z-w)} : (c\beta\partial\gamma)(z)(c\delta(\gamma)\psi^\mu\exp(\mathrm{i}k\cdot X))(w) :$$
$$-\frac{3}{4} : c\beta\partial\gamma : V^a(w) = -\lambda^a\epsilon_\mu\frac{3}{4} : \partial c\beta\partial\gamma c\delta(\gamma)\psi^\mu\exp(\mathrm{i}k\cdot X) : (w) + \text{ regular}$$
$$-\frac{3}{4} : c\beta\partial\gamma : V^a(w) = \text{ regular}$$

Hence,

$$j_{\text{BRST}}(z)V^a(w) = \frac{1}{z-w}\left(\frac{\alpha'}{2}k^2 - \frac{1}{2}\right) : c\partial V^a : (w) + \lambda^a\sqrt{\frac{\alpha'}{2}}\frac{\epsilon\cdot k}{z-w} : \partial\gamma c\delta(\gamma)\exp(\mathrm{i}k\cdot X) : (w) + \text{ regular}$$
$$[Q_{\text{BRST}}, V^a(w)] = \oint_{C_w}\frac{\mathrm{d}z}{2\pi\mathrm{i}}j_{\text{BRST}}(z)V^a(w)$$

So that we can read from the imposition of $[Q_{\text{BRST}}, V^a] = 0$, the conditions $k^2 = \frac{1}{\alpha'}$ and $\epsilon\cdot k = 0$.

2.B)

CANCELED

2.C)

2.D)

Problem 3

3.A)

As the fermions have occupation number either 0 or 1,

$$H_{\mathrm{NS}} = \sum_{i=2}^{9} \sum_{r=\frac{1}{2}}^{\infty} r n_r^i - \frac{1}{6}$$

Where $n_r^i \in \{0, 1\}$. Hence, the partition function is,

$$\chi^{(--)}(\tau) = \mathrm{Tr}\left[\exp\left(2\pi \mathrm{i} \tau H_{\mathrm{NS}}\right)\right]$$

The trace is to sum over all different possible combinations of n_r^i, and setting $q = \exp(2\pi\mathrm{i}\tau)$,

$$\chi^{(--)}(\tau) = q^{-\frac{1}{6}} \sum_{\{n_r^i\} \in 2^{\{0,1\}}} q^{\sum\limits_{i=2}^{9} \sum\limits_{r=\frac{1}{2}}^{\infty} r n_r^i}$$

$$\chi^{(--)}(\tau) = q^{-\frac{1}{6}} \sum_{\{n_r^i\} \in 2^{\{0,1\}}} \prod_{i=2}^{9} \prod_{r=\frac{1}{2}}^{\infty} q^{r n_r^i}$$

$$\chi^{(--)}(\tau) = q^{-\frac{1}{6}} \prod_{i=2}^{9} \prod_{r=\frac{1}{2}}^{\infty} \left(\sum_{\{n_r\} \in \{0,1\}} q^{r n_r} \right)$$

$$\chi^{(--)}(\tau) = q^{-\frac{1}{6}} \prod_{r=\frac{1}{2}}^{\infty} (1 + q^r)^8$$

$$\chi^{(--)}(\tau) = q^{-\frac{4}{24}} \left(\prod_{n=1}^{\infty} \left(1 + q^{n-\frac{1}{2}}\right) \right)^8$$

$$\chi^{(--)}(\tau) = \left(q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 + q^{n-\frac{1}{2}}\right)\left(1 + q^{n-\frac{1}{2}}\right) \right)^4$$

$$\chi^{(--)}(\tau) = \left(\frac{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0|\tau)}{\eta(\tau)} \right)^4$$

3.B)

Done in the last item.

3.C)

The R sector is similar,

$$\chi^{(+-)}(\tau) = q^{\frac{1}{3}} \sum_{\{n_m^i\} \in 2^{\{0,1\}}} q^{\sum\limits_{i=2}^{9} \sum\limits_{m=1}^{\infty} m n_m^i}$$

$$\chi^{(+-)}(\tau) = q^{\frac{1}{3}} \sum_{\{n^i_m\}\in 2^{\{0,1\}}} \prod_{i=2}^{9}\prod_{m=1}^{\infty} q^{mn^i_m}$$

$$\chi^{(+-)}(\tau) = q^{\frac{1}{3}} \prod_{i=2}^{9}\prod_{m=1}^{\infty} \left(\sum_{\{n_m\}\in\{0,1\}} q^{mn_m} \right)$$

$$\chi^{(+-)}(\tau) = q^{\frac{1}{3}} \prod_{m=1}^{\infty} (1+q^m)^8$$

$$\chi^{(+-)}(\tau) = q^{\frac{8}{24}} \left(\frac{1}{2} \prod_{m=1}^{\infty} (1+q^m)\big(1+q^{m-1}\big) \right)^4$$

$$\chi^{(+-)}(\tau) = \left(q^{\frac{2}{24}} \frac{1}{2} \prod_{m=1}^{\infty} \left(1+q^{m+\frac{1}{2}-\frac{1}{2}}\right)\left(1+q^{m-\frac{1}{2}-\frac{1}{2}}\right) \right)^4$$

$$\chi^{(+-)}(\tau) = \left(q^{\frac{12\frac{1}{4}-1}{24}} \frac{1}{2} \prod_{m=1}^{\infty} \left(1+q^{m+\frac{1}{2}-\frac{1}{2}}\right)\left(1+q^{m-\frac{1}{2}-\frac{1}{2}}\right) \right)^4$$

$$\chi^{(+-)}(\tau) = \frac{1}{2^4} \left(q^{\frac{\frac{1}{2^2}}{2}-\frac{1}{24}} \prod_{m=1}^{\infty} \left(1+q^{m+\frac{1}{2}-\frac{1}{2}}\right)\left(1+q^{m-\frac{1}{2}-\frac{1}{2}}\right) \right)^4$$

$$\chi^{(+-)}(\tau) = \frac{1}{2^4} \left(\frac{\vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(0|\tau)}{\eta(\tau)} \right)^4$$

3.D)

With the fermion number operator the calculus is similar,

$$\chi^{(-+)}(\tau) = \mathrm{Tr}\left[\exp\left(2\pi \mathrm{i}\tau H_{\mathrm{NS}}\right)(-1)^F\right]$$

$$\chi^{(-+)}(\tau) = q^{-\frac{1}{6}} \sum_{\{n^i_r\}\in 2^{\{0,1\}}} q^{\sum\limits_{i=2}^{9}\sum\limits_{r=\frac{1}{2}}^{\infty} rn^i_r} (-1)^{\sum\limits_{i=2}^{9}\sum\limits_{r=\frac{1}{2}}^{\infty} n^i_r-1}$$

$$\chi^{(-+)}(\tau) = q^{-\frac{1}{6}} \sum_{\{n^i_r\}\in 2^{\{0,1\}}} \prod_{i=2}^{9}\prod_{r=\frac{1}{2}}^{\infty} q^{rn^i_r}(-)(-1)^{n^i_r}$$

$$\chi^{(-+)}(\tau) = q^{-\frac{1}{6}} \prod_{i=2}^{9}\prod_{r=\frac{1}{2}}^{\infty} \left(\sum_{\{n_r\}\in\{0,1\}} q^{rn_r}(-)(-1)^{n_r} \right)$$

$$\chi^{(-+)}(\tau) = q^{-\frac{1}{6}} \prod_{r=\frac{1}{2}}^{\infty} (-)^8(1-q^r)^8$$

$$\chi^{(-+)}(\tau) = q^{-\frac{4}{24}} \left(\prod_{n=1}^{\infty} \left(1-q^{n-\frac{1}{2}}\right) \right)^8$$

$$\chi^{(-+)}(\tau) = \left(q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1-q^{n-\frac{1}{2}}\right)\left(1-q^{n-\frac{1}{2}}\right) \right)^4$$

$$\chi^{(-+)}(\tau) = \left(q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 + q^{n-\frac{1}{2}} \mathrm{e}^{2\pi \mathrm{i} \frac{1}{2}}\right)\left(1 + q^{n-\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i} \frac{1}{2}}\right)\right)^4$$

$$\chi^{(-+)}(\tau) = \left(\frac{\vartheta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(0|\tau)}{\eta(\tau)} \right)^4$$

And also, forgetting about the fermionic zero modes,

$$\chi^{(++)}(\tau) = \mathrm{Tr}\left[\exp\left(2\pi \mathrm{i}\tau H_{\mathrm{R}}\right)(-1)^F\right]$$

$$\chi^{(++)}(\tau) \propto q^{\frac{1}{3}} \sum_{\{n^i_m\}\in 2^{\{0,1\}}} q^{\sum\limits_{i=2}^{9}\sum\limits_{m=1}^{\infty} mn^i_m} (-1)^{\sum\limits_{i=2}^{9}\sum\limits_{m=1}^{\infty} n^i_m}$$

$$\chi^{(++)}(\tau) \propto q^{\frac{1}{3}} \sum_{\{n^i_m\}\in 2^{\{0,1\}}} \prod_{i=2}^{9}\prod_{m=1}^{\infty} q^{mn^i_m}(-1)^{n^i_m}$$

$$\chi^{(++)}(\tau) \propto q^{\frac{1}{3}} \prod_{i=2}^{9}\prod_{m=1}^{\infty} \left(\sum_{\{n_m\}\in\{0,1\}} q^{mn_m}(-1)^{n^i_m} \right)$$

$$\chi^{(++)}(\tau) \propto q^{\frac{1}{3}} \prod_{m=1}^{\infty} (1-q^m)^8$$

$$\chi^{(++)}(\tau) \propto q^{\frac{8}{24}} \left(\prod_{m=1}^{\infty} (1-q^m)\left(1-q^{m-1}\right) \right)^4$$

$$\chi^{(++)}(\tau) \propto \left(q^{\frac{2}{24}} \prod_{m=1}^{\infty} \left(1 + q^{m+\frac{1}{2}-\frac{1}{2}} \mathrm{e}^{2\pi \mathrm{i}\frac{1}{2}}\right)\left(1 + q^{m-\frac{1}{2}-\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i}\frac{1}{2}}\right)\right)^4$$

$$\chi^{(++)}(\tau) \propto \left(q^{\frac{12\frac{1}{4}-1}{24}} \prod_{m=1}^{\infty} \left(1 + q^{m+\frac{1}{2}-\frac{1}{2}} \mathrm{e}^{2\pi \mathrm{i}\frac{1}{2}}\right)\left(1 + q^{m-\frac{1}{2}-\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i}\frac{1}{2}}\right)\right)^4$$

$$\chi^{(++)}(\tau) \propto \left(q^{\frac{\frac{1}{2^2}}{2}-\frac{1}{24}} \mathrm{e}^{2\pi \mathrm{i}\frac{1}{2}\frac{1}{2}} \prod_{m=1}^{\infty} \left(1 + q^{m+\frac{1}{2}-\frac{1}{2}} \mathrm{e}^{2\pi \mathrm{i}\frac{1}{2}}\right)\left(1 + q^{m-\frac{1}{2}-\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i}\frac{1}{2}}\right)\right)^4$$

$$\chi^{(++)}(\tau) \propto \left(\frac{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(0|\tau)}{\eta(\tau)} \right)^4 \propto 0$$

3.E)

To find the modular invariant holomorphic combination of $\eta, \vartheta_2, \vartheta_3, \vartheta_4$ might seem a difficult task, but let's look at what we have at hands, each fermionic partition function is of the form $\chi \propto \frac{\vartheta^4}{\eta^4}$. As we seek a modular invariant combination to be partition function, we may take this form as inspiration. Sadly, both the transformations, $\tau \to \tau + 1, -\frac{1}{\tau}$ mix the functions. Hence, the best we can do is to set the invariant modular combination to be a sum of the former partition functions,

$$a_2\frac{\vartheta_2^4}{\eta^4} + a_3\frac{\vartheta_3^4}{\eta^4} + a_4\frac{\vartheta_4^4}{\eta^4}$$

Under $\tau \to \tau + 1$,

$$a_2 \frac{\vartheta_2^4}{\eta^4} + a_3 \frac{\vartheta_3^4}{\eta^4} + a_4 \frac{\vartheta_4^4}{\eta^4} \to \mathrm{e}^{\mathrm{i}\pi\frac{2}{3}} a_2 \frac{\vartheta_2^4}{\eta^4} + \mathrm{e}^{-\mathrm{i}\pi\frac{1}{3}} a_3 \frac{\vartheta_4^4}{\eta^4} + \mathrm{e}^{-\mathrm{i}\pi\frac{1}{3}} a_4 \frac{\vartheta_3^4}{\eta^4} \tag{3.1}$$

This combination is not invariant. Nevertheless, let's look at the other transformation,

$$a_2 \frac{\vartheta_2^4}{\eta^4} + a_3 \frac{\vartheta_3^4}{\eta^4} + a_4 \frac{\vartheta_4^4}{\eta^4} \to a_2 \frac{\vartheta_4^4}{\eta^4} + a_3 \frac{\vartheta_3^4}{\eta^4} + a_4 \frac{\vartheta_2^4}{\eta^4}$$

This fixes without doubt $a_2 = a_4 = a, a_3 = b$. Now we can go back to eq. (3.1). To fix it we need a way to deal with the extra factor of $\mathrm{e}^{-\mathrm{i}\pi\frac{1}{3}}$. A possible way is to insert more factors of η, lets look how adding η^{-8} change the transformation under $\tau \to \tau + 1$,

$$\frac{1}{\eta^{12}} \left(a\vartheta_2^4 + b\vartheta_3^4 + a\vartheta_4^4 \right) \to -\frac{1}{\eta^{12}} \left(-a\vartheta_2^4 + b\vartheta_4^4 + a\vartheta_3^4 \right)$$

As long as $a = -b$ the expression is invariant. But, also, this spoils the invariance with respect to the $\tau \to -\frac{1}{\tau}$ transformation. Let just rewrite our expression here, setting $a = -1$

$$\frac{1}{\eta^{12}} \left(\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 \right)$$

As we saw, this is invariant under $\tau \to \tau + 1$, but isn't under $\tau \to -\frac{1}{\tau}$. In order to transform this expression into a fully modular invariant one we need to multiply it by an expression which is invariant under $\tau \to \tau + 1$ and also transform correctly under $\tau \to -\frac{1}{\tau}$. As under $\tau \to -\frac{1}{\tau}$,

$$\frac{1}{\eta^{12}} \left(\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 \right) \to \frac{1}{(-\mathrm{i}\tau)^4 \eta^{12}} \left(\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 \right)$$

We need a $\tau \to \tau + 1$ invariant expression which transforms as $(-\mathrm{i}\tau)^4$ under $\tau \to -\frac{1}{\tau}$. The sad part here is that the only $\tau \to \tau + 1$ invariant expression which transform as powers of $(-\mathrm{i}\tau)$ that can be constructed out of $\eta, \vartheta_2, \vartheta_3, \vartheta_4$ is $\vartheta_2\vartheta_3\vartheta_4\eta^{-3}$, which is invariant under $\tau \to -\frac{1}{\tau}$. Thus, in order to match the variation we have to include factors of $\mathrm{Im}(\tau)$, which is invariant under $\tau \to \tau + 1$, but, transforms as $\mathrm{Im}(\tau) \to \frac{1}{|\tau|^2}\mathrm{Im}(\tau)$ under $\tau \to -\frac{1}{\tau}$. That is, it isn't holomorphic. Even more sadly is the fact that $\mathrm{Im}(\tau)$ cannot be split into a part which transform as $-\mathrm{i}\tau$ and another that transforms as $\mathrm{i}\bar{\tau}$. Hence, the best we can do to obtain an expression which is separately left and right modular invariant is,

$$\frac{1}{\mathrm{Im}(\tau)^4} \frac{1}{\eta^{12}} \left(\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 \right) \frac{1}{\bar{\eta}^{12}} \left(\bar{\vartheta}_2^4 - \bar{\vartheta}_3^4 + \bar{\vartheta}_4^4 \right)$$

Separately, the left and right modes are invariant under $\tau \to \tau + 1$, but, neither of them are invariant under $\tau \to -\frac{1}{\tau}$, they transform as $(-\mathrm{i}\tau)^{-4}$ and as $(\mathrm{i}\bar{\tau})^{-4}$, which together accounts for the transformation of $\mathrm{Im}(\tau)^4$ as $|\tau|^8$. If, hypothetically, would be possible for us to do a splitting $\frac{1}{\mathrm{Im}(\tau)} = w(\tau)\bar{w}(\bar{\tau})$, such that both $w, \bar{w}$ are invariant under $\tau \to \tau + 1$ and under $\tau \to -\frac{1}{\tau}$ transform as, $w, \bar{w} \to (-\mathrm{i}\tau)w, (\mathrm{i}\bar{\tau})\bar{w}$, then, the following would be holomorphic modular invariant,

$$\frac{w^4}{\eta^{12}} \left(\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 \right)$$

3.F)

The identity

$$\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 = 0$$

Implies the vanishing of the last found expression. The interpretation is simple, this is saying that all the contributions from NS bosons cancel the contribution of the R fermions. To see this more clearly we can use the results from the last itens to obtain,

$$\begin{aligned}\frac{1}{\eta^4}\left(\vartheta_3^4-\vartheta_2^4-\vartheta_2^4\right) &\propto \chi^{(--)}-\chi^{(--)}-\chi^{(+-)}\\ 0=\frac{1}{\eta^4}\left(\vartheta_3^4-\vartheta_2^4-\vartheta_2^4\right) &\propto \operatorname{Tr}\left[\exp\left(2\pi\mathrm{i}H_{\mathrm{NS}}\right)\left(1-(-1)^F\right)\right]-\operatorname{Tr}\left[\exp\left(2\pi\mathrm{i}H_{\mathrm{R}}\right)\right]\end{aligned}$$

This is showing to us that this vanishing is exactly due to the supersymmetric matching between bosonic and fermionic states.

To obtain a new modular invariant expression without the assumption of separately modular invariance we can make use of $|\eta|, |\vartheta_i|$. This trivializes the $\tau \to \tau+1$ on η, ϑ_2, and forces ϑ_3, ϑ_4 to be symmetric in our expression. To maintain invariance under $\tau \to -\frac{1}{\tau}$ forces our expression to be symmetric in exchange of ϑ_2, ϑ_4, and also imposes that the sum of the powers of $\vartheta_2, \vartheta_3, \vartheta_4$ — which are the same — is exactly minus the power of η.

We concluded that our expression have to be of one of this three options,

$$\left(\frac{|\vartheta_2||\vartheta_3||\vartheta_4|}{|\eta|^3}\right)^\alpha, \quad \left(\frac{|\vartheta_2|^\beta+|\vartheta_3|^\beta+|\vartheta_4|^\beta}{|\eta|^\beta}\right)^\alpha, \quad \left(\frac{|\vartheta_2|^\beta|\vartheta_3|^\beta+|\vartheta_2|^\beta|\vartheta_4|^\beta+|\vartheta_4|^\beta|\vartheta_3|^\beta}{|\eta|^{2\beta}}\right)^\alpha$$

These are the only options which are totally symmetric in $|\vartheta_2|, |\vartheta_3|, |\vartheta_4|$ and also have the correct powers of $|\eta|$ to cancel the $\tau \to -\frac{1}{\tau}$ transformation. Luckily, due to,

$$|\vartheta_2||\vartheta_3||\vartheta_4| = 2|\eta|^3$$

The first expression is just a constant. In order to choose between the other two we need to make some assumptions, first, we would like to no have mixing terms $|\vartheta_2||\vartheta_3|$, this excludes the third term and also set $\alpha=0$ in the second. Next, we want each term to be proportional to a left mode $\propto \vartheta_i^4\eta^{-4}$ times a right mode $\propto \bar{\vartheta}_i^4\bar{\eta}^{-4}$. This sets $\beta=8$ in the second expression,

$$\frac{|\vartheta_2|^8+|\vartheta_3|^8+|\vartheta_4|^8}{|\eta|^8}$$

A Faddeev-Popov Gauge Fixing

Our Action functional is,

$$S_X + \lambda\chi = \frac{1}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}R + \frac{\lambda}{2\pi}\int\limits_{\partial M}\mathrm{d}s\,K \tag{A.1}$$

we would like to define the quantum theory by means of the path integral, that is, we expect that,

$$Z \stackrel{?}{=} \int \mathcal{D}X\mathcal{D}h \exp\left(-S_X[X,h] - \lambda\chi\right) \tag{A.2}$$

should give a well defined theory, but, already from A.2 there're several problems that arise, one of them is: *What should be interpreted from the path integral itself? We haven't defined any manifold to our metric h and scalar fields X to live in, also, even if we had defined such, the path integral relies on explicit coordinate points, $\mathcal{D}h = \prod\limits_\sigma \mathrm{d}h_{ab}(\sigma)$, which are highly dependent on charts.*

This is a valid claim, our way to avoid it is to *define $\mathcal{D}h$* to mean: *Sum over all* ***allowed*** *two dimensional Riemannian manifolds, and all possible metric structures in these.* Here, ***allowed*** requires a prescription, which manifolds are or aren't allowed impacts the obtained string theory. Happily, every two dimensional manifold has a definite value for the Euler Characteristic χ, hence, we can sort them out by it,

$$\begin{aligned} Z &\stackrel{?}{=} \sum_{\{M\}}\int\limits_{\mathrm{Met}(M)}\mathcal{D}h\int\mathcal{D}X \exp\left(-S_X[X,h]-\lambda\chi\right) \\ Z &\stackrel{?}{=} \sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int\limits_{\mathrm{Met}(M_\chi)}\mathcal{D}h\int\mathcal{D}X\exp\left(-S_X[X,h]\right) \end{aligned} \tag{A.3}$$

Where M is to be understood as a two dimensional Riemannian manifold and M_χ is one with Euler Characteristic χ, $\mathrm{Met}(M_\chi)$ is the space of all metrics which can be assigned to M_χ, we have written $\sum\limits_{\{M_\chi\}}$ in the special case of there being more than one manifold with same Euler Characteristic[2], also, the functional integral over X should be read as integrating over all maps from M_χ to $\mathbb{R}^{1,D-1}$. While this is better defined than before, i.e. not coordinate dependent, we still have a few problems, first, it's know that A.1 has a Gauge Group of $\mathrm{Diff}(M)\times\mathrm{Weyl}(M)$, but, in our second try of a definition of the path integral, we're integrating the metrics over $\mathrm{Met}(M_\chi)$, it's clear that may happen of two elements of $\mathrm{Met}(M_\chi)$ be equivalent under a $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ transformation, to put in more clear terms, we're worried if exists $h', h \in \mathrm{Met}(M_\chi)$ such,

$$h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

the existence of those kinds of elements is troublesome, as $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ is a infinite dimensional group of redundancies, this means we're over-counting physical configurations by a infinite amount. The solution is to look for an equivalence class of metrics under this Gauge Group action,

$$\mathscr{M}_\chi = \mathrm{Met}(M_\chi)/\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$$

[2] As we're interested only in Differentiable Manifolds, more than manifold should read: More than one equivalence class of Differentiable Manifolds.

the equivalence class is to be understood as[3],

$$h' \sim h \Leftrightarrow h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

that is, two metrics are the same representative of an element of $\mathscr{M}_\chi$ iff they differ by a composition of a Diffeomorphism and Weyl transformation. We'll denote a given composition of a Diffeomorphism followed by a Weyl transformation by ζ,

$$h' = \zeta \circ h$$

Notice that the set of equivalence class of metrics, or, the set of inequivalent $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ metrics $\mathscr{M}_\chi$ is highly dependent on the topology of M_χ, for example, for $M_\chi \cong \mathbb{R}^2 \cong \mathbb{C}$, it's trivial, there is just one point in the set $\mathscr{M}_\chi$, in other words, every metric is equivalent, which isn't true for more complex topologies.

Thus, it's possible for us to set up a well defined version of the path integral, just replace $\mathrm{Met}(M_\chi)$ by $\mathscr{M}_\chi$,

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h]\right) \tag{A.4}$$

where the integration is to be understood as by choosing for each equivalence class in $\mathscr{M}_\chi$ a representative element in $\mathrm{Met}(M_g)$. While this is a satisfactory definition for the path integral, it feels a little clunky, we rather have an path integral over all the possible metrics — in the sense defined before —, well, this is achievable. First, for each equivalence class of $\mathscr{M}_\chi$ elect one representative element of $\mathrm{Met}(M_\chi)$, we'll denote these elements as $\hat{h}(\mathbf{t})$ — here $\mathbf{t}$ is a parametrization of the correspondent equivalence class in $\mathscr{M}_\chi$, we haven't proved here, and won't, but $\mathscr{M}_\chi$ is a finite N dimensional manifold, hence, $\mathbf{t}$ is a N-tuple of real numbers —, by construction, these representatives are inequivalent under $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, hence,

$$\zeta_1 \circ \hat{h}(\mathbf{t}_1) = \zeta_2 \circ \hat{h}(\mathbf{t}_2) \Leftrightarrow \mathbf{t}_1 = \mathbf{t}_2 \text{ and } \zeta_1 = \zeta_2$$

so that every element in $\mathrm{Met}(M_g)$ can be written as a unique[4] composition of a given ζ into a given $\hat{h}(\mathbf{t})$. Now, we rewrite the pictorial integral over $\mathscr{M}_\chi$ is a more formal way, using the parametrization we just described,

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int \mathcal{D}X \exp\left(-S_X\left[X, \hat{h}(\mathbf{t})\right]\right)$$

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int\limits_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)} \mathcal{D}\zeta\delta(\zeta) \int \mathcal{D}X \exp\left(-S_X\left[X, \hat{h}(\mathbf{t})\right]\right)$$

in the last line we introduced a one by integrating[5] over the delta functional, as this integral picks only $\zeta = 0$, what should be understood as $\zeta = \mathrm{id}$ in the group, we can deform a little the

[3] In all charts.

[4] The uniqueness or not depends on a few factors, here we'll always, unless specified otherwise, interpret $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ as the group *generated by* all possible compositions of Diffeomorphisms and Weyl transformations, but a element of it, ζ, is not to be interpreted as a unique composition of Diffeomorphism and Weyl factors, as there might be some Diffeomorphism which are equivalent to Weyl transformations, what is indeed true is that every element ζ of the Gauge Group is a unique combination of an element of $\mathrm{Diff}(M_\chi)/\mathrm{Weyl}(M_\chi)$ and an element of $\mathrm{Weyl}(M_\chi)$.

[5] Again, following the same remarks made before, the integral over $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ should not be interpreted as integrating over the whole of $\mathrm{Diff}(M_\chi)$ and after integrating over the whole $\mathrm{Weyl}(M_\chi)$, this would for sure be an over-counting, but rather should be interpreted as integrating over the whole group *generated by* compositions of $\mathrm{Diff}(M_\chi)$ and $\mathrm{Weyl}(M_\chi)$, which is equivalent of integrating over the whole $\mathrm{Diff}(M_\chi)/\mathrm{Weyl}(M_\chi)$, and after integrating over the whole $\mathrm{Weyl}(M_\chi)$.

integration to,

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)} \mathcal{D}\zeta\,\delta(\zeta) \int \mathcal{D}X \exp\Big(-S_X\Big[X, \zeta\circ\hat{h}(\mathbf{t})\Big]\Big) \quad \text{(A.5)}$$

This is almost in the form that we would like, notice that we're integrating over the set of representative of the inequivalent metrics, $\mathrm{d}^N\mathbf{t}$, and also over the whole group $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, $\mathcal{D}\zeta$, by construction, **every** metric in $\mathrm{Met}(M_\chi)$ can be written uniquely[6] as,

$$h = \zeta_{\mathbf{t}} \circ \hat{h}(\mathbf{t})$$

in other words, to integrate over $\mathrm{d}^N\mathbf{t}\,\mathcal{D}\zeta$ is to integrate over all metrics of the form $\zeta\circ\hat{h}(\mathbf{t})$, which is to integrate over all metrics $h = \zeta\circ\hat{h}(\mathbf{t})$ in $\mathrm{Met}(M_\chi)$! We cannot yet make this change, due to the presence of an explicit dependence in ζ at the functional delta. We'll eliminate it by means of a change of variable of the functional delta, notice that,

$$\delta\Big(\hat{h}(\mathbf{t}) - \zeta\circ\hat{h}(\mathbf{t})\Big)$$

picks up just the contribution of $\zeta = 0$, so it's a good candidate for a change of variables,

$$\delta\Big(\hat{h}(\mathbf{t}) - \zeta\circ\hat{h}(\mathbf{t})\Big) = \delta(\zeta)\left|\mathrm{Det}\left[\frac{\delta}{\delta\zeta}\Big(\hat{h}(\mathbf{t}) - \zeta\circ\hat{h}(\mathbf{t})\Big)\bigg|_{\zeta=0}\right]\right|^{-1}$$

let's compute step by step the right-hand side of this equation, as we're only interested in the solution of $\zeta = 0$, what matters is just the connected component to the identity of the Gauge Group, this is parametrized by a function ω related to the Weyl transformation, and a vector field ξ related to the connected component to the identity of the Diffeomorphisms — there is an additional requirement of ξ not generating any transformation which can be undone by a Weyl transformation —, also, for ease of our manipulation, we'll write the expression inside the delta with respect to h instead of $\hat{h}(\mathbf{t})$[7], that is,

$$\delta\Big(\hat{h}(\mathbf{t}) - \zeta\circ\hat{h}(\mathbf{t})\Big) = \delta\big(\zeta^{-1}\circ h - h\big) = \delta\big(\zeta^{-1}\big)\left|\mathrm{Det}\left[\frac{\delta}{\delta\zeta^{-1}}\big(\zeta^{-1}\circ h - h\big)\bigg|_{\zeta^{-1}=0}\right]\right|^{-1}$$

one might worry about the ζ^{-1} instead of the ζ, but, the integration measure $\mathcal{D}\zeta$ is formally a Haar measure in the Group, that means it's a group invariant measure, in other words, $\mathcal{D}\zeta^{-1} = \mathcal{D}\zeta$, so that we can forget about the inverse, now,

$$\begin{aligned}
[\zeta\circ h]_{ab} &= [h]_{ab} + 2\omega[h]_{ab} + [\pounds_\xi h]_{ab} + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)\\
[\zeta\circ h]_{ab} &= [h]_{ab} + 2\omega[h]_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)
\end{aligned}$$

of course ∇ here is with respect to the h metric,

$$\begin{aligned}
[\zeta\circ h]_{ab} - [h]_{ab} &= 2\omega[h]_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)\\
\frac{\delta}{\delta\zeta}\big([\zeta\circ h]_{ab} - [h]_{ab}\big) &= ????
\end{aligned}$$

[6]With the remarks made before.

[7]We would have to carry out the $\mathbf{t}$ dependence in h also, but, soon it will disappear as matter of uniting the integrals $\mathrm{d}^N\mathbf{t}\,\mathcal{D}\zeta$ so we won't keep track of it anymore.

The ζ derivative actually has two parts, the derivative with respect to ω and the other with respect to ξ, let's do one by one,

$$\frac{\delta}{\delta\omega(\sigma')}([\zeta\circ h]_{ab}-[h]_{ab})(\sigma)\bigg|_{\zeta=0}=2\delta^{(2)}(\sigma-\sigma')h_{ab}(\sigma)$$

and for the ξ,

$$\frac{\delta}{\delta\xi^c(\sigma')}([\zeta\circ h]_{ab}-[h]_{ab})(\sigma)\bigg|_{\zeta=0}=2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')$$

Thus,

$$\frac{\delta}{\delta\zeta}([\zeta\circ h]_{ab}-[h]_{ab})\bigg|_{\zeta=0}=2\delta^{(2)}(\sigma-\sigma')h_{ab}(\sigma)+2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')$$

$$\operatorname{Det}\left[\frac{\delta}{\delta\zeta}([\zeta\circ h]_{ab}-[h]_{ab})\bigg|_{\zeta=0}\right]=\operatorname{Det}\left[2\delta^{(2)}(\sigma-\sigma')h_{ab}(\sigma)+2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')\right]$$

the determinant can be computed by means of path integral of Grassmannian variables,

$$\operatorname{Det}\left[2\delta^{(2)}(\sigma-\sigma')h_{ab}+2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma-\sigma')\right]=\int\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-\frac{1}{2\pi}\int\mathrm{d}^2\sigma\sqrt{h}b^{ab}\left[h_{ab}d+h_{c(b}\nabla_{a)}c^c\right]\right)$$

$$\operatorname{Det}\left[\frac{\delta}{\delta\zeta}([\zeta\circ h]_{ab}-[h]_{ab})\bigg|_{\zeta=0}\right]=\int\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-S_{\mathrm{gh}}[b,c,d,h]\right)$$

Substituting all of this back into our path integral,

$$Z=\sum_{\{\chi\}}\exp(-\lambda\chi)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)}\mathcal{D}\zeta\delta(\zeta)\int\mathcal{D}X\exp\left(-S_X\left[X,\zeta\circ\hat{h}(\mathbf{t})\right]\right)$$

$$Z=\sum_{\{\chi\}}\exp(-\lambda\chi)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)}\mathcal{D}\zeta\delta\left(\zeta^{-1}\circ h-h\right)\int\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-S_X-S_{\mathrm{gh}}\right)$$

$$Z=\sum_{\{\chi\}}\exp(-\lambda\chi)\sum_{\{M_\chi\}}\int_{\mathrm{Met}(M_\chi)}\mathcal{D}h\delta\left(\hat{h}-h\right)\int\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]\right)$$

$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\delta\left(\hat{h}-h\right)\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]-\lambda\chi\right)$$

where $\hat{h}$ is a family of choices of representatives of the equivalence classes of the Gauge equivalent metrics, of course this choice is dependent on the equivalence class h lies in, so, in a certain sense we have $\hat{h}=\hat{h}[h]$,

$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\delta\left(\hat{h}[h]-h\right)\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]-\lambda\chi\right)$$

we express the delta functional in terms of a path integral,

$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\mathcal{D}B\exp\left(\frac{\mathrm{i}}{4\pi}\int\mathrm{d}^2\sigma\sqrt{h}B^{ab}\left(\hat{h}_{ab}[h]-h_{ab}\right)\right)\exp\left(-S_X-S_{\mathrm{gh}}-\lambda\chi\right)$$

$$Z=\int\mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\mathcal{D}B\exp\left(-S_X[X,h]-S_{\mathrm{gh}}[b,c,d,h]-S_{\mathrm{gf}}[B,h]-\lambda\chi\right)$$

where we lastly defined the Gauge Fixing Action. This is the final expression for our path integral with the identifications,

$$S_X[X,h] + \lambda\chi = \frac{1}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}R + \frac{\lambda}{2\pi}\int\limits_{\partial M} \mathrm{d}s\,K \tag{A.6a}$$

$$S_{\mathrm{gh}}[b,c,d,h] = \frac{1}{2\pi}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}b^{ab}[h_{ab}d + \nabla_a c_b] \tag{A.6b}$$

$$S_{\mathrm{gf}}[B,h] = -\frac{\mathrm{i}}{4\pi}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}B^{ab}\Big(\hat{h}_{ab}[h] - h_{ab}\Big) \tag{A.6c}$$

B BRST Quantization

B.1 The BRST transformations

Following the action principle derived from the Faddeev-Popov Gauge Fixing A.6, we can describe it's BRST symmetry by the transformations of the *matter fields* under Gauge, we know the following,

$$
\begin{aligned}
X^\mu(\sigma) &\to X'^\mu(\sigma'(\sigma)) = X^\mu(\sigma) \\
h_{ab}(\sigma) &\to h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)
\end{aligned}
$$

which have an *infinitesimal* form,

$$
\begin{aligned}
&\delta X^\mu = \pounds_\xi X^\mu = \xi^a\partial_a X^\mu \\
&\delta h_{ab} = 2\omega h_{ab} + \pounds_\xi h_{ab} = 2\omega h_{ab} + \nabla_a\xi_b + \nabla_b\xi_a = 2\omega h_{ab} + \xi^c\partial_c h_{ab} + h_{ac}\partial_b\xi^c + h_{bc}\partial_a\xi^c
\end{aligned}
$$

the BRST transformation can be obtained from these by the substitution inferred from the Faddeev-Popov gauge fix, that is, $\xi^a \to \mathrm{i}\epsilon c^a$ and $\omega \to \mathrm{i}\epsilon d$, where ϵ is a Grassmannian parametrization of the BRST transformation,

$$\delta_{\mathrm{BRST}}X^\mu = \mathrm{i}\epsilon c^a\partial_a X^\mu \tag{B.1a}$$

$$\delta_{\mathrm{BRST}}h_{ab} = 2\mathrm{i}\epsilon\big(dh_{ab} + \nabla_{(a}c_{b)}\big) = 2\mathrm{i}\epsilon dh_{ab} + \mathrm{i}\epsilon(c^c\partial_c h_{ab} + h_{ac}\partial_b c^c + h_{bc}\partial_a c^c) \tag{B.1b}$$

while the BRST transformation of the non-matter fields can be obtained by the BRST procedure, which are[8],

$$\delta_{\mathrm{BRST}}\left(\sqrt{h}B_{ab}\right) = 0 \tag{B.1c}$$

$$\delta_{\mathrm{BRST}}\left(\sqrt{h}b_{ab}\right) = -\epsilon\sqrt{h}B_{ab} \tag{B.1d}$$

$$\delta_{\mathrm{BRST}}d = \mathrm{i}\epsilon c^a\partial_a d \tag{B.1e}$$

$$\delta_{\mathrm{BRST}}c^a = \mathrm{i}\epsilon c^b\nabla_b c^a = \mathrm{i}\epsilon c^b\partial_b c^a \tag{B.1f}$$

the proof of nilpotency of these transformations is done in Appendix **??**. For the BRST quantization to be realized we need two things,

$$
\begin{aligned}
\delta_{\mathrm{BRST}}(S_X + S_{\mathrm{gh}} + S_{\mathrm{gf}}) &= 0 \\
\mathrm{i}\epsilon(S_{\mathrm{gh}} + S_{\mathrm{gf}}) &= \delta_{\mathrm{BRST}}\mathcal{O}
\end{aligned}
$$

for $\mathcal{O}$ some composition of fields, we'll check both of these, and for this it'll be necessary to know the BRST transformations of a few more fields,

- h^{ab}

$$
\begin{aligned}
0 &= \delta_{\mathrm{BRST}}\delta_a{}^c \\
0 &= \delta_{\mathrm{BRST}}\big(h_{ab}h^{bc}\big) \\
0 &= h_{ab}\delta_{\mathrm{BRST}}h^{bc} + h^{bc}\delta_{\mathrm{BRST}}h_{ab} \\
h_{ab}\delta_{\mathrm{BRST}}h^{bc} &= -2\mathrm{i}\epsilon h^{bc}\big(dh_{ab} + \nabla_{(a}c_{b)}\big) \\
h^{da}h_{ab}\delta_{\mathrm{BRST}}h^{bc} &= -2\mathrm{i}\epsilon h^{da}h^{bc}\big(dh_{ab} + \nabla_{(a}c_{b)}\big) \\
\delta^d{}_b\delta_{\mathrm{BRST}}h^{bc} &= -2\mathrm{i}\epsilon\big(dh^{dc} + \nabla^{(d}c^{c)}\big) \\
\delta_{\mathrm{BRST}}h^{dc} &= -2\mathrm{i}\epsilon\big(dh^{dc} + \nabla^{(d}c^{c)}\big)
\end{aligned} \tag{B.2a}
$$

[8]The first two equations might look a bit odd, but they are in fact a consequence of normalization of $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(\mathrm{i}\int \mathrm{d}^2\sigma\, \sqrt{h}B(\sigma)f(\sigma)\right)$, instead of choosing $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(\mathrm{i}\int \mathrm{d}^2\sigma\, B(\sigma)f(\sigma)\right)$.

- $\sqrt{h} = \sqrt{\mathrm{Det}[h_{ab}]}$

$$\begin{aligned}
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\mathrm{Det}[h_{ab}]) \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\exp(\ln(\mathrm{Det}[h_{ab}]))) \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\exp(\mathrm{Tr}[\ln(h_{ab})])) \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}(\exp(\mathrm{Tr}[\ln(h_{ab})]))\delta_{\mathrm{BRST}}(\mathrm{Tr}[\ln(h_{ab})]) \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\frac{1}{\sqrt{h}}h\,\mathrm{Tr}[\delta_{\mathrm{BRST}}(\ln(h_{ab}))] \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\sqrt{h}\,\mathrm{Tr}[h^{ca}\delta_{\mathrm{BRST}}h_{ab}] \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \frac{1}{2}\sqrt{h}h^{ba}\delta_{\mathrm{BRST}}h_{ab} \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \mathrm{i}\epsilon\sqrt{h}h^{ba}\big(dh_{ab} + \nabla_{(a}c_{b)}\big) \\
\delta_{\mathrm{BRST}}\sqrt{h} &= \mathrm{i}\epsilon\sqrt{h}(2d + \nabla_a c^a) \qquad \text{(B.2b)}
\end{aligned}$$

- $\sqrt{h}h^{ab}$

$$\begin{aligned}
\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right) &= \delta_{\mathrm{BRST}}\left(\sqrt{h}\right)h^{ab} + \sqrt{h}\delta_{\mathrm{BRST}}\big(h^{ab}\big) \\
\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right) &= \mathrm{i}\epsilon\sqrt{h}(2d + \nabla_c c^c)h^{ab} - 2\mathrm{i}\epsilon\sqrt{h}\big(dh^{ab} + \nabla^{(a}c^{b)}\big) \\
\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right) &= 2\mathrm{i}\epsilon\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^{(a}c^{b)}\right) \qquad \text{(B.2c)}
\end{aligned}$$

So now we can proceed to compute the two conditions, we'll start by the second one, luckily, the BRST procedure already provides a natural candidate for $\mathcal{O}$,

$$\begin{aligned}
\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) &= -\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\delta_{\mathrm{BRST}}\left(\sqrt{h}b_{ab}\right)\left(\hat{h}^{ab} - h^{ab}\right) \\
&\quad - \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\delta_{\mathrm{BRST}}\hat{h}^{ab} \\
&\quad + \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\delta_{\mathrm{BRST}}\big(h^{ab}\big) \\
\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) &= \epsilon\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}\left(\hat{h}^{ab} - h^{ab}\right) \\
&\quad + \mathrm{i}\epsilon\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big(dh^{ab} + \nabla^a c^b\big) \\
\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) &= \mathrm{i}\epsilon S_{\mathrm{gf}} + \mathrm{i}\epsilon S_{\mathrm{gh}} \\
\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) &= \mathrm{i}\epsilon(S_{\mathrm{gf}} + S_{\mathrm{gh}})
\end{aligned}$$

so what remains to be *checked* is the first condition, let's do it part by part, starting with,

$$\delta_{\mathrm{BRST}}S_X = \frac{1}{2\pi\alpha'}\int_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu\partial_b\delta_{\mathrm{BRST}}X_\mu + \frac{1}{4\pi\alpha'}\int_M \mathrm{d}^2\sigma\,\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right)\partial_a X^\mu\partial_b X_\mu$$

$$
\begin{aligned}
\delta_{\mathrm{BRST}} S_X &= \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} h^{ab} \partial_a X^\mu \partial_b [c^c \partial_c X_\mu] + \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^{(a} c^{b)} \right) \partial_a X^\mu \partial_b X_\mu \\
\delta_{\mathrm{BRST}} S_X &= \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} h^{ab} \partial_a X^\mu ((\nabla_b c^c) \partial_c X_\mu + c^c \nabla_b \nabla_c X_\mu) \\
&\quad + \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^a c^b \right) \partial_a X^\mu \partial_b X_\mu \\
\delta_{\mathrm{BRST}} S_X &= \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} \partial_a X^\mu \partial_b X_\mu \nabla^a c^b + \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} h^{ab} c^c \partial_a X^\mu \nabla_c \nabla_b X_\mu \\
&\quad + \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^a c^b \right) \partial_a X^\mu \partial_b X_\mu \\
\delta_{\mathrm{BRST}} S_X &= \frac{\mathrm{i}\epsilon}{2\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} h^{ab} c^c \partial_a X^\mu \nabla_c \partial_b X_\mu + \frac{\mathrm{i}\epsilon}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} h^{ab} \nabla_c c^c \partial_a X^\mu \partial_b X_\mu \\
\delta_{\mathrm{BRST}} S_X &= \frac{\mathrm{i}\epsilon}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} c^c \nabla_c \big(h^{ab} \partial_a X^\mu \partial_b X_\mu \big) + \frac{\mathrm{i}\epsilon}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} h^{ab} \nabla_c c^c \partial_a X^\mu \partial_b X_\mu \\
\delta_{\mathrm{BRST}} S_X &= \frac{\mathrm{i}\epsilon}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \sqrt{h} \nabla_c \big(c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \big) \\
\delta_{\mathrm{BRST}} S_X &= \frac{\mathrm{i}\epsilon}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma\, \partial_c \Big(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \Big)
\end{aligned} \tag{B.3}
$$

now,

$$
\begin{aligned}
\delta_{\mathrm{BRST}} S_{\mathrm{gh}} &= \delta_{\mathrm{BRST}} \left(\frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[h^{ab} d + \nabla^a c^b \big] \right) \\
\delta_{\mathrm{BRST}} S_{\mathrm{gh}} &= \delta_{\mathrm{BRST}} \left(\frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \left[h^{ab} d + \frac{1}{2} \big(-c^c \partial_c h^{ab} + h^{ac} \partial_c c^b + h^{bc} \partial_c c^a \big) \right] \right) \\
\delta_{\mathrm{BRST}} S_{\mathrm{gh}} &= \frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \delta_{\mathrm{BRST}} \Big(\sqrt{h} b_{ab} \Big) \big[h^{ab} d + \nabla^a c^b \big] \\
&\quad + \frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}} \big(h^{ab} \big) d + h^{ab} \delta_{\mathrm{BRST}}(d) \big] \\
&\quad - \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}}(c^c) \partial_c h^{ab} + c^c \partial_c \delta_{\mathrm{BRST}} \big(h^{ab} \big) \big] \\
&\quad + \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}}(h^{ac}) \partial_c c^b + h^{ac} \partial_c \delta_{\mathrm{BRST}} \big(c^b \big) \big] \\
&\quad + \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}} \big(h^{bc} \big) \partial_c c^a + h^{bc} \partial_c \delta_{\mathrm{BRST}}(c^a) \big] \\
\delta_{\mathrm{BRST}} S_{\mathrm{gh}} &= -\frac{\epsilon}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} B_{ab} \big[h^{ab} d + \nabla^a c^b \big] \\
&\quad + \frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[-2\mathrm{i}\epsilon \big(d h^{ab} + \nabla^{(a} c^{b)} \big) d + h^{ab} \mathrm{i}\epsilon c^d \partial_d d \big] \\
&\quad - \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\mathrm{i}\epsilon c^d \partial_d c^c \partial_c h^{ab} + 2\mathrm{i}\epsilon c^c \partial_c \big(d h^{ab} + \nabla^{(a} c^{b)} \big) \big] \\
&\quad + \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[-2\mathrm{i}\epsilon \big(d h^{ac} + \nabla^{(a} c^{c)} \big) \partial_c c^b + \mathrm{i}\epsilon h^{ac} \partial_c \big(c^d \partial_d c^b \big) \big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[-2\mathrm{i}\epsilon\big(dh^{bc}+\nabla^{(b}c^{c)}\big)\partial_c c^a+\mathrm{i}\epsilon h^{bc}\partial_c\big(c^d\partial_d c^a\big)\big]\\
\delta_{\mathrm{BRST}}S_{\mathrm{gh}}={}&-\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}(-\mathrm{i}\epsilon)\big[h^{ab}d+\nabla^a c^b\big]\\
&+\frac{\mathrm{i}\epsilon}{\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\nabla^{(a}c^{b)}d\\
&+\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[c^d\partial_d c^c\partial_c h^{ab}+2c^c\big(d\partial_c h^{ab}+\partial_c\nabla^{(a}c^{b)}\big)\big]\\
&+\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[2\big(dh^{ac}+\nabla^{(a}c^{c)}\big)\partial_c c^b-h^{ac}\partial_c\big(c^d\partial_d c^b\big)\big]\\
&+\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[2\big(dh^{bc}+\nabla^{(b}c^{c)}\big)\partial_c c^a-h^{bc}\partial_c\big(c^d\partial_d c^a\big)\big]\\
\delta_{\mathrm{BRST}}S_{\mathrm{gh}}={}&\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}\delta_{\mathrm{BRST}}\Big(\hat{h}^{ab}-h^{ab}\Big)\\
&+\frac{\mathrm{i}\epsilon}{\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\nabla^{(a}c^{b)}d\\
&+\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}d\big[-c^c\partial_c h^{ab}+h^{ac}\partial_c c^b+h^{bc}\partial_c c^a\big]\\
&+\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[c^d\partial_d c^c\partial_c h^{ab}-h^{bc}\partial_c\big(c^d\partial_d c^a\big)-h^{ac}\partial_c\big(c^d\partial_d c^b\big)\big]\\
&+\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\nabla^{(a}c^{c)}\partial_c c^b+c^c\partial_c\nabla^{(a}c^{b)}+\nabla^{(b}c^{c)}\partial_c c^a\big]\\
\delta_{\mathrm{BRST}}S_{\mathrm{gh}}={}&-\delta_{\mathrm{BRST}}S_{\mathrm{gf}}\\
&+\frac{\mathrm{i}\epsilon}{\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\nabla^{(a}c^{b)}d\\
&+\frac{\mathrm{i}\epsilon}{\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}d\nabla^{(a}c^{b)}\\
&-\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\nabla^{(a|}\big(c^d\nabla_d c^{|b)}\big)\\
&+\frac{\mathrm{i}\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\nabla^{(a}c^{c)}\nabla_c c^b+c^c\nabla_c\nabla^{(a}c^{b)}+\nabla^{(b}c^{c)}\nabla_c c^a\big]\\
\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}})={}&-\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\nabla^a c^d\nabla_d c^b+\nabla^b c^d\nabla_d c^a+c^d\nabla^a\nabla_d c^b+c^d\nabla^b\nabla_d c^a\big]\\
&+\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\nabla^a c^c\nabla_c c^b+\nabla^c c^a\nabla_c c^b+\nabla^b c^c\nabla_c c^a+\nabla^c c^b\nabla_c c^a\big]\\
&+\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[c^c\nabla_c\nabla^a c^b+c^c\nabla_c\nabla^b c^a\big]\\
\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}})={}&-\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[c^c\nabla^a\nabla_c c^b+c^c\nabla^b\nabla_c c^a\big]\\
&+\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[c^c\nabla_c\nabla^a c^b+c^c\nabla_c\nabla^b c^a\big]\\
\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}})={}&-\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}c^c\big[R_d{}^{ba}{}_c+R_d{}^{ab}{}_c\big]c^d\\
\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}})={}&-\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}c^c c^d\big[R_d{}^{ba}{}_c+R_d{}^{ab}{}_c\big]\\
\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}})={}&-\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}c^c c^d\big[R_d{}^{ba}{}_c-R_c{}^{ba}{}_d+R_d{}^{ab}{}_c-R_c{}^{ab}{}_d\big]
\end{aligned}
$$

$$\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}}) = -\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}c^c c^d\left[R^a{}_{cd}{}^b - R^a{}_{dc}{}^b + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d\right]$$
$$\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}}) = -\frac{\mathrm{i}\epsilon}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}c^c c^d\left[R_c{}^{ab}{}_d - R_d{}^{ab}{}_c + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d\right]$$
$$\delta_{\mathrm{BRST}}(S_{\mathrm{gh}}+S_{\mathrm{gf}}) = 0$$

so that the condition $\delta_{\mathrm{BRST}}(S_X+S_{\mathrm{gh}}+S_{\mathrm{gf}})=0$, is equivalent to,

$$\delta_{\mathrm{BRST}}(S_X+S_{\mathrm{gh}}+S_{\mathrm{gf}}) = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\partial_c\Big(\sqrt{h}c^c h^{ab}\partial_a X^\mu\partial_b X_\mu\Big)$$
$$\delta_{\mathrm{BRST}}(S_X+S_{\mathrm{gh}}+S_{\mathrm{gf}}) = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_{\partial M} \mathrm{d}s\, n_c\sqrt{h}c^c h^{ab}\partial_a X^\mu\partial_b X_\mu = 0$$

where n_c is a inwards unitary normal vector to boundary ∂M. Of course, if we're dealing with the closed string $\partial M=\varnothing$ and the right-hand side is identically zero, the problem only shows up in the open string, as $\partial M\neq\varnothing$, but, in this case we have more then just a boundary, as the boundary exists we have to specify the boundary conditions for our fields, this is telling us what might be a good boundary condition to impose[9],

$$n_c c^c\Big|_{\partial M} = 0$$

which is what we're going to choose[10], so,

$$\delta_{\mathrm{BRST}}(S_X+S_{\mathrm{gh}}+S_{\mathrm{gf}}) = 0$$

our theory is BRST invariant!

B.2 The BRST Charge

Have proven our theory is BRST invariant, we'll now comute the BRST Charge associated with this symmetry, this is an easy computation, is just necessary to follow the steps needed to derive B.3, but now with ϵ being world-sheet dependent,

$$\delta_{\mathrm{BRST}}S_X = \frac{\mathrm{i}}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu\partial_b[\epsilon c^c\partial_c X_\mu] + \frac{\mathrm{i}}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\epsilon\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^{(a}c^{b)}\right)\partial_a X^\mu\partial_b X_\mu$$

notice, in the first term, the ∂_b when acting on $c^c\partial_c X_\mu$ will sum-up with the second term to give rise just to B.3, but now with ϵ non-constant, so we have,

$$\delta_{\mathrm{BRST}}S_X = \frac{\mathrm{i}}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\partial_b(\epsilon)\sqrt{h}h^{ab}\partial_a X^\mu c^c\partial_c X_\mu + \frac{\mathrm{i}}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\epsilon\partial_c\Big(\sqrt{h}c^c h^{ab}\partial_a X^\mu\partial_b X_\mu\Big)$$
$$\delta_{\mathrm{BRST}}S_X = \frac{\mathrm{i}}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\partial_b\Big[\epsilon\sqrt{h}h^{ab}\partial_a X^\mu c^c\partial_c X_\mu\Big]$$

[9]Any other choice here is too strong, the boundary condition must in some way depend on n_a, other wise it'll be just a world-sheet scalar and will be zero everywhere, as is the case if we try to set $h^{ab}\partial_a X^\mu\partial_b X_\mu\big|_{\partial M}=0$, other possible choices are $c^c\big|_{\partial M}=0$, which again, is too restrictive to give any dynamics to our theory.

[10]Contrary to beliefs, this condition cannot be inferred solely from the equations of motion, what can be done is to infer $n^a b_{ab}\big|_{\partial M}=0$ from the EoM and then try to mimic a condition for c. But, here we're seeing a way which the boundary condition on c is heavily suggested.

$$- \frac{\mathrm{i}}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_b \Big[\sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu\Big] + \frac{\mathrm{i}}{4\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_c \Big(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu\Big)$$

$$\delta_{\mathrm{BRST}} S_X = \frac{\mathrm{i}}{2\pi\alpha'} \int\limits_{\partial M} \mathrm{d}s\, \epsilon \sqrt{h} n^a \partial_a X^\mu c^c \partial_c X_\mu$$
$$- \frac{\mathrm{i}}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_c \Big[\sqrt{h} h^{ca} \partial_a X^\mu c^b \partial_b X_\mu\Big] + \frac{\mathrm{i}}{4\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_c \Big(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu\Big)$$

the first integral is zero, either in the closed string, $\partial M = \varnothing$, or in the open string, $n^a \partial_a X \big|_{\partial M} = 0$,

$$\delta_{\mathrm{BRST}} S_X = \frac{\mathrm{i}}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \partial_b(\epsilon) \sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu + \frac{\mathrm{i}}{4\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_c \Big(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu\Big)$$
$$\delta_{\mathrm{BRST}} S_X = \frac{\mathrm{i}}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \partial_b \Big[\epsilon \sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu\Big]$$
$$- \frac{\mathrm{i}}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_b \Big[\sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu\Big] + \frac{\mathrm{i}}{4\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_c \Big(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu\Big)$$
$$\delta_{\mathrm{BRST}} S_X = -\frac{\mathrm{i}}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \epsilon \partial_c \Big[\sqrt{h} h^{ca} \partial_a X^\mu c^b \partial_b X_\mu - \frac{1}{2} \sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu\Big]$$
$$\delta_{\mathrm{BRST}} S_X = -\frac{\mathrm{i}}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \sqrt{h} \epsilon \nabla_c \Big[h^{ca} \partial_a X^\mu c^b \partial_b X_\mu - \frac{1}{2} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu\Big]$$

as we go back now to ϵ being a constant, we know the left-hand side is zero, this forces,

$$-\frac{1}{\alpha'} \nabla_c \Big[h^{ca} \partial_a X^\mu c^b \partial_b X_\mu - \frac{1}{2} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu\Big] = 0$$
$$\nabla_c \big[c_b T^{cb}\big] = 0 \tag{B.4}$$

as we're not starters, we wrote the last line in a suggestive manner, this is just one piece of the puzzle, remains to vary the other two terms of the total action, this is not as difficult as it seems, as the total standard BRST variation is identically zero, any term that does not have a derivative of ϵ will eventually sum-up to zero,

$$\delta_{\mathrm{BRST}} S_{\mathrm{gh}} = \delta_{\mathrm{BRST}} \Bigg(\frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[h^{ab} d + \nabla^a c^b\big] \Bigg)$$
$$\delta_{\mathrm{BRST}} S_{\mathrm{gh}} = \delta_{\mathrm{BRST}} \Bigg(\frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \Big[h^{ab} d + \frac{1}{2}\big(-c^c \partial_c h^{ab} + h^{ac} \partial_c c^b + h^{bc} \partial_c c^a\big)\Big] \Bigg)$$
$$\delta_{\mathrm{BRST}} S_{\mathrm{gh}} = \frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \delta_{\mathrm{BRST}} \Big(\sqrt{h} b_{ab}\Big) \big[h^{ab} d + \nabla^a c^b\big]$$
$$+ \frac{1}{2\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}}\big(h^{ab}\big) d + h^{ab} \delta_{\mathrm{BRST}}(d)\big]$$
$$- \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}}(c^c) \partial_c h^{ab} + c^c \partial_c \delta_{\mathrm{BRST}}\big(h^{ab}\big)\big]$$
$$+ \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}}(h^{ac}) \partial_c c^b + h^{ac} \partial_c \delta_{\mathrm{BRST}}\big(c^b\big)\big]$$
$$+ \frac{1}{4\pi} \int \mathrm{d}^2\sigma\, \sqrt{h} b_{ab} \big[\delta_{\mathrm{BRST}}\big(h^{bc}\big) \partial_c c^a + h^{bc} \partial_c \delta_{\mathrm{BRST}}(c^a)\big]$$

$$\begin{aligned}
\delta_{\text{BRST}} S_{\text{gh}} = &-\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\epsilon\sqrt{h}B_{ab}\big[h^{ab}d + \nabla^a c^b\big] \\
&+\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[-2\mathrm{i}\epsilon\big(dh^{ab}+\nabla^{(a}c^{b)}\big)d + h^{ab}\mathrm{i}\epsilon c^d\partial_d d\big] \\
&-\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\mathrm{i}\epsilon c^d\partial_d c^c\partial_c h^{ab} - 2\mathrm{i}c^c\partial_c\big(\epsilon dh^{ab}+\epsilon\nabla^{(a}c^{b)}\big)\big] \\
&+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[-2\mathrm{i}\epsilon\big(dh^{ac}+\nabla^{(a}c^{c)}\big)\partial_c c^b + \mathrm{i}h^{ac}\partial_c\big(\epsilon c^d\partial_d c^b\big)\big] \\
&+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[-2\mathrm{i}\epsilon\big(dh^{bc}+\nabla^{(b}c^{c)}\big)\partial_c c^a + \mathrm{i}h^{bc}\partial_c\big(\epsilon c^d\partial_d c^a\big)\big] \\
\delta_{\text{BRST}} S_{\text{gh}} = &-\delta_{\text{BRST}} S_{\text{gf}} \\
&+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}2\mathrm{i}c^c\partial_c(\epsilon)\big[dh^{ab}+\nabla^{(a}c^{b)}\big] \\
&+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\mathrm{i}h^{ac}\partial_c(\epsilon)c^d\partial_d c^b \\
&+\frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\mathrm{i}h^{bc}\partial_c(\epsilon)c^d\partial_d c^a \\
\delta_{\text{BRST}}(S_{\text{gh}}+S_{\text{gf}}) = &\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2\sigma\,\partial_c(\epsilon)\sqrt{h}b_{ab}c^c\big[dh^{ab}+\nabla^{(a}c^{b)}\big] \\
&-\frac{\mathrm{i}}{4\pi}\int \mathrm{d}^2\sigma\,\partial_c(\epsilon)\sqrt{h}b_{ab}\big(h^{ac}c^d\partial_d c^b + h^{bc}c^d\partial_d c^a\big)
\end{aligned}$$

using the boundary conditions $n^a b_{ab}\big|_{\partial M} = n_a c^a\big|_{\partial M} = 0$,

$$\begin{aligned}
\delta_{\text{BRST}}(S_{\text{gh}}+S_{\text{gf}}) = &\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2\sigma\,\epsilon\sqrt{h}\nabla_c\big[b_{ab}c^c\big(dh^{ab}+\nabla^{(a}c^{b)}\big)\big] \\
&-\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2\sigma\,\epsilon\sqrt{h}\nabla_c\big[b_{ab}h^{ac}c^d\partial_d c^b\big] \\
\delta_{\text{BRST}}(S_{\text{gh}}+S_{\text{gf}}) = &\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2\sigma\,\epsilon\sqrt{h}\nabla_c\big[b_{ab}\big(c^c dh^{ab} + c^c\nabla^{(a}c^{b)} - h^{ac}c^d\partial_d c^b\big)\big]
\end{aligned}$$

B.3 The BRST Nilpotency

Nilpotency of BRST,

$$\begin{aligned}
\delta_1 X^\mu &= \mathrm{i}\epsilon_1 c^a\partial_a X^\mu \\
\delta_2\delta_1 X^\mu &= \mathrm{i}\epsilon_1\delta_2(c^a)\partial_a X^\mu + \mathrm{i}\epsilon_1 c^a\partial_a(\delta_2(X^\mu)) \\
\delta_2\delta_1 X^\mu &= \mathrm{i}\epsilon_1(\mathrm{i}\epsilon_2)c^c\nabla_c(c^a)\partial_a X^\mu + \mathrm{i}\epsilon_1 c^a\partial_a\Big(\big(\mathrm{i}\epsilon_2 c^b\partial_b\big)X^\mu\Big) \\
\delta_2\delta_1 X^\mu &= -\epsilon_1\epsilon_2 c^c\nabla_c(c^a)\partial_a X^\mu + \epsilon_1\epsilon_2 c^a\partial_a\big(c^b\partial_b X^\mu\big) \\
\delta_2\delta_1 X^\mu &= -\epsilon_1\epsilon_2 c^c\nabla_c(c^a)\nabla_a X^\mu + \epsilon_1\epsilon_2 c^a\nabla_a\big(c^b\nabla_b X^\mu\big) \\
\delta_2\delta_1 X^\mu &= -\epsilon_1\epsilon_2 c^c\nabla_c(c^a)\nabla_a X^\mu + \epsilon_1\epsilon_2 c^a\nabla_a\big(c^b\big)\nabla_b X^\mu + \epsilon_1\epsilon_2 c^a c^b\nabla_a\nabla_b X^\mu \\
\delta_2\delta_1 X^\mu &= \epsilon_1\epsilon_2 c^a c^b\nabla_a\nabla_b X^\mu, \quad \text{using the ghost statistics} \\
\delta_2\delta_1 X^\mu &= \epsilon_1\epsilon_2 c^a c^b\nabla_{[a}\nabla_{b]} X^\mu, \quad \text{as } X^\mu \text{ is a world-sheet scalar: } \nabla_a\nabla_b X^\mu = \nabla_b\nabla_a X^\mu \\
\delta_2\delta_1 X^\mu &= 0
\end{aligned}$$

$$\delta_1 h_{ab} = 2\mathrm{i}\epsilon_1 dh_{ab} + \mathrm{i}\epsilon_1(c^c\partial_c h_{ab} + h_{ac}\partial_b c^c + h_{bc}\partial_a c^c)$$

$$
\begin{aligned}
\delta_2\delta_1 h_{ab} = {} & 2\mathrm{i}\epsilon_1\delta_2(d)h_{ab} + 2\mathrm{i}\epsilon_1 d\delta_2(h_{ab}) \\
& + \mathrm{i}\epsilon_1(\delta_2(c^c)\partial_c h_{ab} + c^c\partial_c\delta_2(h_{ab})) \\
& + \mathrm{i}\epsilon_1(\delta_2(h_{ac})\partial_b c^c + h_{ac}\partial_b\delta_2(c^c)) \\
& + \mathrm{i}\epsilon_1(\delta_2(h_{bc})\partial_a c^c + h_{bc}\partial_a\delta_2(c^c)) \\
\delta_2\delta_1 h_{ab} = {} & 2\mathrm{i}\epsilon_1\mathrm{i}\epsilon_2 h_{ab}c^c\partial_c d + 2\mathrm{i}\epsilon_1 d 2\mathrm{i}\epsilon_2\big(dh_{ab} + \nabla_{(a}c_{b)}\big) \\
& + \mathrm{i}\epsilon_1\big(\mathrm{i}\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} + c^c\partial_c\big(2\mathrm{i}\epsilon_2\big(dh_{ab} + \nabla_{(a}c_{b)}\big)\big)\big) \\
& + \mathrm{i}\epsilon_1\big(2\mathrm{i}\epsilon_2\big(dh_{ac} + \nabla_{(a}c_{c)}\big)\partial_b c^c + h_{ac}\partial_b\big(\mathrm{i}\epsilon_2 c^d\partial_d c^c\big)\big) \\
& + \mathrm{i}\epsilon_1\big(2\mathrm{i}\epsilon_2\big(dh_{bc} + \nabla_{(b}c_{c)}\big)\partial_a c^c + h_{bc}\partial_a\big(\mathrm{i}\epsilon_2 c^d\partial_d c^c\big)\big) \\
\delta_2\delta_1 h_{ab} = {} & -2\epsilon_1\epsilon_2 h_{ab}c^c\partial_c d + 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} \\
& - \epsilon_1\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} + 2\epsilon_1\epsilon_2 c^c\partial_c\big(dh_{ab} + \nabla_{(a}c_{b)}\big) \\
& - 2\epsilon_1\epsilon_2\big(dh_{ac} + \nabla_{(a}c_{c)}\big)\partial_b c^c - \epsilon_1\epsilon_2 h_{ac}\partial_b\big(c^d\partial_d c^c\big) \\
& - 2\epsilon_1\epsilon_2\big(dh_{bc} + \nabla_{(b}c_{c)}\big)\partial_a c^c - \epsilon_1\epsilon_2 h_{bc}\partial_a\big(c^d\partial_d c^c\big) \\
\delta_2\delta_1 h_{ab} = {} & 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - \epsilon_1\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} + 2\epsilon_1\epsilon_2 c^c\big(d\partial_c h_{ab} + \partial_c\nabla_{(a}c_{b)}\big) \\
& - 2\epsilon_1\epsilon_2\big(dh_{ac} + \nabla_{(a}c_{c)}\big)\partial_b c^c - \epsilon_1\epsilon_2 h_{ac}\partial_b\big(c^d\partial_d c^c\big) \\
& - 2\epsilon_1\epsilon_2\big(dh_{bc} + \nabla_{(b}c_{c)}\big)\partial_a c^c - \epsilon_1\epsilon_2 h_{bc}\partial_a\big(c^d\partial_d c^c\big) \\
\delta_2\delta_1 h_{ab} = {} & 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - 2\epsilon_1\epsilon_2 dc^c\partial_c h_{ab} - 2\epsilon_1\epsilon_2 dh_{ac}\partial_b c^c - 2\epsilon_1\epsilon_2 dh_{bc}\partial_a c^c \\
& - \epsilon_1\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} - \epsilon_1\epsilon_2 h_{bc}\partial_a\big(c^d\partial_d c^c\big) - \epsilon_1\epsilon_2 h_{ac}\partial_b\big(c^d\partial_d c^c\big) \\
& - 2\epsilon_1\epsilon_2\partial_c\nabla_{(a}c_{b)}c^c - 2\epsilon_1\epsilon_2\nabla_{(a}c_{c)}\partial_b c^c - 2\epsilon_1\epsilon_2\nabla_{(b}c_{c)}\partial_a c^c \\
\delta_2\delta_1 h_{ab} = {} & 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - 2\epsilon_1\epsilon_2 d[c^c\partial_c h_{ab} + h_{ac}\partial_b c^c + h_{bc}\partial_a c^c] \\
& - \epsilon_1\epsilon_2\big[c^d\partial_d c^c\partial_c h_{ab} + h_{bc}\partial_a\big(c^d\partial_d c^c\big) + h_{ac}\partial_b\big(c^d\partial_d c^c\big)\big] \\
& - 2\epsilon_1\epsilon_2\big[\partial_c\nabla_{(a}c_{b)}c^c + \nabla_{(a}c_{c)}\partial_b c^c + \nabla_{(b}c_{c)}\partial_a c^c\big] \\
\delta_2\delta_1 h_{ab} = {} & 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} \\
& - 2\epsilon_1\epsilon_2\nabla_{(a|}\big(c^c\nabla_c c_{|b)}\big) \\
& - 2\epsilon_1\epsilon_2\big[\nabla_c\nabla_{(a}c_{b)}c^c + \nabla_{(a}c_{c)}\nabla_b c^c + \nabla_{(b}c_{c)}\nabla_a c^c\big] \\
\delta_2\delta_1 h_{ab} = {} & -2\epsilon_1\epsilon_2\big[\nabla_{(a|}\big(c^c\nabla_c c_{|b)}\big) + \nabla_c\nabla_{(a}c_{b)}c^c + \nabla_{(a}c_{c)}\nabla_b c^c + \nabla_{(b}c_{c)}\nabla_a c^c\big] \\
\delta_2\delta_1 h_{ab} = {} & -\epsilon_1\epsilon_2\left[\nabla_a(c^c\nabla_c c_b) + \nabla_b(c^c\nabla_c c_a) + \nabla_c\nabla_a c_b c^c + \nabla_c\nabla_b c_a c^c\right. \\
& \left. +\nabla_a c_c\nabla_b c^c + \nabla_b c_c\nabla_a c^c + \nabla_c c_a\nabla_b c^c + \nabla_c c_b\nabla_a c^c\right] \\
\delta_2\delta_1 h_{ab} = {} & -\epsilon_1\epsilon_2\left[c^c\nabla_a\nabla_c c_b + c^c\nabla_b\nabla_c c_a + \nabla_c\nabla_a c_b c^c + \nabla_c\nabla_b c_a c^c\right. \\
& \left. +\nabla_a c^c\nabla_c c_b + \nabla_b c^c\nabla_c c_a + \nabla_a c_c\nabla_b c^c + \nabla_b c_c\nabla_a c^c + \nabla_c c_a\nabla_b c^c + \nabla_c c_b\nabla_a c^c\right] \\
\delta_2\delta_1 h_{ab} = {} & -\epsilon_1\epsilon_2\left[c^c\nabla_a\nabla_c c_b + c^c\nabla_b\nabla_c c_a - c^c\nabla_c\nabla_a c_b - c^c\nabla_c\nabla_b c_a\right. \\
& \left. +\nabla_a c^c\nabla_c c_b + \nabla_b c^c\nabla_c c_a - \nabla_b c^c\nabla_c c_a - \nabla_a c^c\nabla_c c_b\right] \\
\delta_2\delta_1 h_{ab} = {} & -\epsilon_1\epsilon_2\left[c^c\nabla_a\nabla_c c_b + c^c\nabla_b\nabla_c c_a - c^c\nabla_c\nabla_a c_b - c^c\nabla_c\nabla_b c_a\right. \\
& \left. +\nabla_a c^c\nabla_c c_b + \nabla_b c^c\nabla_c c_a - \nabla_b c^c\nabla_c c_a - \nabla_a c^c\nabla_c c_b\right] \\
\delta_2\delta_1 h_{ab} = {} & -\epsilon_1\epsilon_2 c^c\left[R^d{}_{bac}c_d + R^d{}_{abc}c_d\right] \\
\delta_2\delta_1 h_{ab} = {} & -\epsilon_1\epsilon_2 c^c c^d\left[R_{dbac} + R_{dabc}\right] \\
\delta_2\delta_1 h_{ab} = {} & -\frac{1}{2}\epsilon_1\epsilon_2 c^c c^d\left[R_{dbac} - R_{cbad} + R_{dabc} - R_{cabd}\right] \\
\delta_2\delta_1 h_{ab} = {} & -\frac{1}{2}\epsilon_1\epsilon_2 c^c c^d\left[R_{dbac} - R_{cbad} + R_{bcda} - R_{bdca}\right] \\
\delta_2\delta_1 h_{ab} = {} & -\frac{1}{2}\epsilon_1\epsilon_2 c^c c^d\left[R_{dbac} - R_{cbad} + R_{cbad} - R_{dbac}\right]
\end{aligned}
$$

$\delta_2 \delta_1 h_{ab} = 0$