Homework III

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A BRST

A.1 Faddeev-Popov Gauge Fixing

We'll start with a discussion of the Faddeev-Popov procedure of gauge fixing, first, our action is,

$$S_X = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{|\mathrm{Det}[h_{ab}]|}h^{ab}\partial_a X^\mu \partial_b X_\mu$$

we would like to define the quantum theory by means of the path integral, that is, we expect that,

$$Z \stackrel{?}{=} \int \mathcal{D}X\mathcal{D}h \exp\left(-S_X[X,h]\right)$$

should give a well defined theory, but, the integral should be only over physical and inequivalent configurations of X, h, and as we know, we have Diff×Weyl gauge redundancies in this theory, this means in the integral measure we're over-counting physical configurations, this means instead of the integral $\int \mathcal{D}h$ being over the whole space of all possible metrics, it should be in the space of equivalence classes under Diff×Weyl of all possible metrics. Let $\hat{h}$ denote a generic member of the space of metrics inequivalent up to Diff×Weyl, then, for any possible metric h, it's always possible to find $\hat{h}$ such that h is,

$$h_{ab}(\sigma) = \exp\left(2\omega(\hat{\sigma})\right)\frac{\partial\hat{\sigma}^c}{\partial\sigma^a}\frac{\partial\hat{\sigma}^d}{\partial\sigma^b}\hat{h}_{cd}(\hat{\sigma})$$

that is, a composition of a Diff and Weyl transformation. We'll denote a given composition of a Diff followed by a Weyl by just ζ, so that

$$h = \zeta \circ \hat{h}$$

in this way is possible to separate the integral over all metrics $\int \mathcal{D}h$ into an integration over all inequivalent metrics $\int \mathcal{D}\hat{h}$ and an integration over all possible Diff×Weyl transformations $\int \mathcal{D}\zeta$, so that the partition function can be rewrote as,

$$Z \stackrel{?}{=} \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \exp\left(-S_X[X,\zeta\circ h]\right)$$

this still has the same problem of before, we're over-integrating the physical configurations, that is, $\hat{h}$ are the physical configurations, but we're integrating also over the whole Diff×Weyl group in $\mathcal{D}\zeta$. One way of circumventing this problem is introducing by hand a Dirac delta to force $\zeta = 0$, what also forces we to integrate only over one copy of the physical configurations,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta) \exp\left(-S_X[X,\zeta\circ h]\right)$$

but this is not the most general way, we could set $\zeta = f(\sigma)$, for a arbitrary function, and this would still give the same theory,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta - f) \exp\left(-S_X[X,\zeta\circ h]\right)$$

we can go even further and give a function $G(\zeta)$ such that the solution to $G(\zeta) = 0$ is only $\zeta = f$, so that we can use the relations between Dirac deltas to obtain,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta - f) \exp\left(-S_X[X,\zeta\circ h]\right)$$

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G}{\delta \zeta}\right]\Big|_{\zeta=f} \right| \delta(G(\zeta)) \exp\left(-S_X\left[X, \zeta \circ \hat{h}\right]\right)$$

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G}{\delta \zeta}\right]\Big|_{\zeta=f} \right| \delta(G(\zeta)) \exp\left(-S_X\left[X, \zeta \circ \hat{h}\right]\right)$$

There are some details here, as ζ is to represent both a Weyl and a Diff, it has to represent both a function ω and a vector field ξ such that,

$$\zeta \circ h = h + 2\omega h + \pounds_\xi h + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)$$
$$[\zeta \circ h]_{\mu\nu} = h_{\mu\nu} + 2\omega h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)$$

this means both $\zeta = f$ and $G(\zeta) = 0$ are in fact a collection of various equations. In particular, we'll choose

$$G_{ab}(\zeta) = \left[\tilde{h}\right]_{ab} - \left[\zeta \circ \hat{h}\right]_{ab}$$

for a particular metric $\tilde{h}$. As $G_{ab}(\zeta)$ is in fact a function of $h = \zeta \circ \hat{h}$ alone,

$$G_{ab}(\zeta) = \left[\tilde{h}\right]_{ab} - \left[\zeta \circ \hat{h}\right]_{ab} = \left[\tilde{h}\right]_{ab} - [h]_{ab} = G_{ab}(h)$$

we can rewrite as,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G_{ab}}{\delta \zeta}\right]\Big|_{\zeta=f} \right| \delta(G_{ab}(\zeta)) \exp\left(-S_X\left[X, \zeta \circ \hat{h}\right]\right)$$
$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G_{ab}}{\delta \zeta}\right]\Big|_{G_{ab}(h)=0} \right| \delta(G_{ab}(h)) \exp\left(-S_X[X, h]\right)$$