ADTGR SEMINAR

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1. Introduction

General Relativity has been known for being a highly complex theory, to which, apart from highly symmetric ones, few solutions are known. One of the main reasons of this is the non-linear character of the Action and the Equations of Motion,

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int\limits_M \mathrm{d}^D x\,\sqrt{|g|}g^{ab}R_{cb}{}^{c}{}_{a}, \quad R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$$

from the non-polinomial term $\sqrt{|g|}$, the inverse field g^{ab}, and also $R_{cb}{}^{c}{}_{a}$ where lies more contributions of inverse fields and derivatives of them. Thus, in the study of this theory, and couplings of it to matter, prospective of obtaining analytical solutions in non-highly symmetric scenarios is faded to doom, also, these peculiarities prevents the theory from being interpreted as a gauge theory in the usual sense[1], these combined motifs have proven gravity in $D = 3+1$ dimensions to be stubborn to the usual quantization methods, and even to classical non-vacuum solutions. Nevertheless, there is hope of grasping a better understanding of it — either qualitative or quantitative — by looking to simpler toy models, which, have already proven it's usefulness as in String Theory, with $D = 1+1$ gravity, so, we would like to pursue a similar line of thought and ask ourselves, what can $D = 2+1$ gravity teach us? We'll try to focus more on wether or not it can be stated as a usual gauge theory.

But first, why should $D = 2+1$ be any easier of dealing with than $D = 3+1$? The answer lies in the number of dynamical degrees of freedom of the theory, which in gravity, are deeply tied to the space-time dimensions, in the most common realization of it, the dynamical fundamental field is considered to be the metric, $\mathbf{g}$, which is a symmetric, non-degenerate $(0,2)$–tensor, to which is associated a metric-compatible, torsionless connection $\nabla_{(\,\cdot\,)}(\,\cdot\,) : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$, which under these conditions is totally determined by the metric, hence, all of the degrees of freedom of the theory are metric ones, which are $\frac{1}{2}D(D+1)$, but, as we have redundancies/gauge transformations of D diffeomorphisms, $\phi_*\mathbf{g}\sim\mathbf{g}$, we have to discount them, also, there are D additional constrains coming from the Bianchi identity, $\nabla_a\big(R^{ab} - \frac{1}{2}Rg^{ab}\big) = 0$, so the total number of degree of freedom in this theory is, $\frac{1}{2}D(D+1) - 2D = \frac{1}{2}D(D-3)$, in $D = 3+1$ this is our well known two polarizations of the metric, but, for $D = 2+1$ this is zero, which can be interpreted as the metric having no dynamical degrees of freedom, that is, the equation of motion is merely an algebraic condition, this is compatible with our knowledge of independent components of the Riemann tensor, $\frac{1}{12}D^2(D^2-1)$, exactly 20 in $D = 3+1$, but 6 in $D = 2+1$, notice that $6 = \frac{1}{2}3(3+1)$, the same number of degrees of freedom of a symmetric $(0,2)$ tensor, in other words, in $D = 2+1$ the Riemann tensor is totally determined by the knowledge of the Ricci tensor, which is totally determined algebraically by the equations of motion. This is consistent with the metric doesn't having degrees of freedom, due to being known that dynamical propagation of gravity is linked to the Weyl tensor, and, if the Riemann tensor is totally determined by the Ricci tensor, there is no degree of freedom in the Weyl tensor, thus, no dynamics. This is our hope to "solve" this theory, as it's non-dynamical, we expect it to be "trivial", or at least "exact" — we have to define what we mean by this —, similarly to what was done to lower dimensional electrodynamics by Schwinger.

2. The Einstein-Hilbert Action in the Form Language

We'll begin with a quick recap of the vielbein/spin connection formalism, for now we'll keep the discussion general in D dimensions, and only later on we'll go to the special case of $D = 2+1$. The vielbein[2] $\tilde{\mathbf{e}}_\mu$ are a basis of the vector field space $\mathfrak{X}(M)$, notice, the index μ is only indexing which vector from the D present in the basis are we talking about, it isn't a coordinate index — one that is related to a specific component decomposition in a specific chart —, as this is not a coordinate basis — i.e. $\boldsymbol{\partial}_a$ —, we can in fact do miracles with it, as diagonalize the metric,

$$\begin{aligned}
\mathrm{diag}\begin{pmatrix}-1 & 1 & \cdots & 1\end{pmatrix} = \eta_{\mu\nu} &= \mathbf{g}(\tilde{\mathbf{e}}_\mu,\tilde{\mathbf{e}}_\nu) = g_{ab}\mathbf{d}x^a(\tilde{\mathbf{e}}_\mu)\otimes\mathbf{d}x^b(\tilde{\mathbf{e}}_\nu)\\
\eta_{\mu\nu} &= g_{ab}e_\mu{}^c e_\nu{}^d\mathbf{d}x^a(\boldsymbol{\partial}_c)\otimes\mathbf{d}x^b(\boldsymbol{\partial}_d)\\
\eta_{\mu\nu} &= g_{ab}e_\mu{}^a e_\nu{}^b
\end{aligned}$$

The whole point of introducing the vielbein is switch the degrees of freedom from the metric to the inertial frame basis, which can be seen as a downgrade, due to this process enlarging the number of degrees of freedom from $\frac{1}{2}D(D+1)$ to D^2, but, of course, this is only the naive counting, without considering the redundancies, as the number of physical degrees of freedom must be the same. For this to be true, is only possible if we enlarge also the redundancies, to kill the extra degrees of freedom we introduced, as mentioned, this could be seen as not desirable, but, for us this turn out to be essencial. What are these new redundancies? They're the choice of

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[1] With *gauge theory in the usual sense* we mean a theory in which the fundamental degree of freedom is a field $\mathbf{A}$ — usually an adjoint $\mathfrak{g}$–valued 1–form — which has as redundancy a local realization of a Lie Group G which acts on the field as $\mathbf{A}\to g\mathbf{A}g^{-1} + g\mathbf{d}g^{-1}$.

[2] We'll denote the *vector* — (1,0) tensor — vielbein with a tilde, only to be distinguishable from the associated *covetor* — (0,1) tensor — vielbein, which we'll denote without the tilde due to being way more important to us.

labeling μ in $\tilde{\mathbf{e}}_\mu$, as long as the new relabel also respect the defining property of the vielbein it's a redundant transformation, notice that these transformations are exactly local Lorentz ones, that is, given a set of functions $\Lambda^\mu{}_\nu : M \to \mathbb{R}$, a new set of vector fields $\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu$ is a vielbein iff,

$$\begin{aligned} \mathbf{g}\big(\Lambda^\alpha{}_\mu \tilde{\mathbf{e}}_\alpha, \Lambda^\beta{}_\nu \tilde{\mathbf{e}}_\beta\big) &= \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\beta) \\ \eta_{\mu\nu} &= \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} \end{aligned}$$

in other words, $\Lambda^\mu{}_\nu$ must be a local $SO(1, D-1)$ element[3], with this new redundancy taken in account the degrees of freedom match, D^2 from the vielbein, minus D diffeomorphisms $\phi^*\tilde{\mathbf{e}}_\mu \sim \tilde{\mathbf{e}}_\mu$, D Bianchi identities and $\frac{1}{2}D(D-1)$ local $SO(1, D-1)$ transformations, giving $D^2 - 2D - \frac{1}{2}D(D-1) = \frac{1}{2}D(D-3)$ exactly the same counting using only the metric! Notice that now the redundancies are diffeomorphisms and local $SO(1, D-1)$ transformations, this already seems like we're gauging the whole Poincaré group, which will come up later.

From the condition that $\mathbf{g}$ must be non-degenerate we get that the matrix of components $e_\mu{}^a$ must be invertible, this ensures the existence of $e^\mu{}_a$,from which we can construct the dual vector field $\mathbf{e}^\mu = e^\mu{}_a \mathbf{d}x^a$ this ensures we have a basis of the whole tensor space, so it's possible to decompose any tensor in it,

$$\eta_{\mu\nu}\mathbf{e}^\mu \otimes \mathbf{e}^\nu = \mathbf{g}(\tilde{\mathbf{e}}_\mu, \tilde{\mathbf{e}}_\nu)\mathbf{e}^\mu \otimes \mathbf{e}^\nu = \mathbf{g}$$

Up to now we have been considering a metric compatible torsionless affine connection, but, this turn out to not be the optimal choice, as for this kind of connection there is an differential/algebraical restraint between the connection and the metric, at least that what we should expect in a coordinate basis. What we would like is to have a connection linearly independent of the metric/vielbein, but without sacrificing the metricity condition, this is completely hopeless in a coordinate basis, but, in a non-coordinate basis this is achievable! We just have to remind that an affine connection can be defined in any basis of $\mathfrak{X}(M)$, what is usually done is, $\nabla_{\mathbf{X}}\boldsymbol{\partial}_b = X^a \Gamma_a{}^c{}_b \boldsymbol{\partial}_c$, but it's much more interesting to define it with respect to the vielbein basis,

$$\nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu) = \boldsymbol{\omega}(\mathbf{X})^\mu{}_\nu \tilde{\mathbf{e}}_\mu = X^a \omega_a{}^\mu{}_\nu \tilde{\mathbf{e}}_\mu$$

Where $\boldsymbol{\omega}^\mu{}_\nu = \omega_a{}^\mu{}_\nu \mathbf{d}x^a$ is named the spin connection, it can be seen as a $\mathfrak{gl}(1, D-1)$–valued $(0, 1)$ tensor, or, as we'll adopt here, a $\mathfrak{gl}(1, D-1)$–valued 1–form. The ease of working with the vielbein is that the Lorentz index μ does not change upon coordinate/chart/diffeomorphism transformations, it acts as if was an internal symmetry, thus, $\boldsymbol{\omega}^\mu{}_\nu$ do transform exactly as a tensor should, this is already a enormous dichotomy with the standard formulation where the Christoffel connection doesn't transform in a good manner. Now we'll impose the metric compatibility of the connection, this is another scenario where the vielbein formalism como to hand, as this condition imposes no additional differential/algebraical constrains among the vielbein and spin connection.

$$\begin{aligned} \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu)) &= X^a \omega_a{}^\mu{}_\nu \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\mu), \quad \text{symmetrize } \alpha \leftrightarrow \nu \\ \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu)) + \mathbf{g}(\tilde{\mathbf{e}}_\nu, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\alpha)) &= X^a \omega_a{}^\mu{}_\nu \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\mu) + X^a \omega_a{}^\mu{}_\alpha \mathbf{g}(\tilde{\mathbf{e}}_\nu, \tilde{\mathbf{e}}_\mu), \quad \text{metric symmetry} \\ \mathbf{g}(\tilde{\mathbf{e}}_\alpha, \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\nu)) + \mathbf{g}(\nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\alpha), \tilde{\mathbf{e}}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha}, \quad \text{Leibnitz rule} \\ \nabla_{\mathbf{X}}(\mathbf{g}(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\nu)) - \nabla_{\mathbf{X}}(\mathbf{g})(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha}, \quad \nabla_{\mathbf{X}}(\eta_{\alpha\nu}) = 0 \\ -\nabla_{\mathbf{X}}(\mathbf{g})(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha}, \quad \text{metricity} \\ -\omega_{a\nu\alpha} &= \omega_{a\alpha\nu} \end{aligned}$$

That is, the metric compatible spin connection is anti-symmetric in the non-coordinate indices, exactly the property satisfied by the generators of the $SO(1, D-1)$ group, thus, the spin connection can be seen as a $\mathfrak{so}(1, D-1)$-valued 1-form, it has everything in it's favor to be interpreted as a gauge field of the $SO(1, D-1)$ group, to confirm this notice how it changes under a gauge transformation of the vielbein basis $\tilde{\mathbf{e}}_\nu \to \Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu$,

$$\begin{aligned} \nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu)\tilde{\mathbf{e}}_\mu + \Lambda^\mu{}_\nu \nabla_{\mathbf{X}}(\tilde{\mathbf{e}}_\mu), \quad \text{connection definition} \\ \nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \mathbf{X}(\Lambda^\mu{}_\nu)\tilde{\mathbf{e}}_\mu + \Lambda^\mu{}_\nu \boldsymbol{\omega}(\mathbf{X})^\alpha{}_\mu \tilde{\mathbf{e}}_\alpha, \quad \Lambda^{-1\beta}{}_\alpha \Lambda^\sigma{}_\beta = \delta^\sigma{}_\alpha \\ \nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \mathbf{d}\Lambda^\mu{}_\nu(\mathbf{X})\Lambda^{-1\beta}{}_\mu \Lambda^\sigma{}_\beta \tilde{\mathbf{e}}_\sigma + \Lambda^\mu{}_\nu \boldsymbol{\omega}(\mathbf{X})^\alpha{}_\mu \Lambda^{-1\beta}{}_\alpha \Lambda^\sigma{}_\beta \tilde{\mathbf{e}}_\sigma \\ \nabla_{\mathbf{X}}(\Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu) &= \Big(\Lambda^{-1\beta}{}_\mu \mathbf{d}\Lambda^\mu{}_\nu + \Lambda^{-1\beta}{}_\alpha \boldsymbol{\omega}^\alpha{}_\mu \Lambda^\mu{}_\nu\Big)(\mathbf{X})\Lambda^\sigma{}_\beta \tilde{\mathbf{e}}_\sigma \end{aligned}$$

That is, under a gauge transformation of $\tilde{\mathbf{e}}_\nu \to \Lambda^\mu{}_\nu \tilde{\mathbf{e}}_\mu$, the spin connection transforms exactly as a connection of the gague group $SO(1, D-1)$, that is, $\boldsymbol{\omega} \to \Lambda^{-1}\boldsymbol{\omega}\Lambda + \Lambda^{-1}\mathbf{d}\Lambda$, which is what we're searching. Before pursuing further the gauged translations, we're going to obtain a new interpretation for the Riemann tensor, using what we just learned from the spin connection, notice, the usual interpretation of the Riemann tensor is of it being a $(1, 3)$ tensor, but, as naturally — without the need for a metric compatible torsionless connection — it's anti-symmetric in the first two entries, we can switch the point of view from a $(1, 3)$ tensor to a $(1, 1)$ tensor valued 2–form, or, in an even better way, an $\mathrm{End}(\mathfrak{X}(M))$–valued 2–form, which, when decomposed in the non-coordinate basis, will turn out to be a $\mathfrak{so}(1, D-1)$–valued 2–form, as we'll shown now. Starting from the definition[4],

$$\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\tilde{\mathbf{e}}_\mu = \big(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X}, \mathbf{Y}]}\big)\tilde{\mathbf{e}}_\mu$$

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[3] Actually the group is $O(1, D-1)$, but, we'll only be interested in the orientation preserving transformations.

[4] Here, care must be taken, in our conventions, the covariant derivative acts non-trivially only in vectors, and acts as a normal derivative in functions, that is, for X^a being *components* of a vector, $\nabla_{\mathbf{Y}} X^a = \mathbf{Y}(X^a) = Y^b \partial_b X^a$, in contrast to $\nabla_{\mathbf{Y}} \boldsymbol{\partial}_a = Y^b \Gamma_b{}^c{}_a \boldsymbol{\partial}_c$.

$$
\begin{aligned}
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= \nabla_{\mathbf{X}}\big(Y^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - \nabla_{\mathbf{Y}}\big(X^a\omega_a{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= Y^b\nabla_{\mathbf{X}}\big(\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - X^a\nabla_{\mathbf{Y}}\big(\omega_a{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) + \nabla_{\mathbf{X}}\big(Y^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - \nabla_{\mathbf{Y}}\big(X^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu\big) + \big(\nabla_{\mathbf{X}}\big(Y^b\big) - \nabla_{\mathbf{Y}}\big(X^b\big)\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - [\mathbf{X},\mathbf{Y}]^b\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\nabla_a\big(\omega_b{}^\nu{}_\mu\big)\tilde{\mathbf{e}}_\nu + X^aY^b\omega_b{}^\nu{}_\mu\omega_a{}^\alpha{}_\nu\tilde{\mathbf{e}}_\alpha - X^aY^b\nabla_b\big(\omega_a{}^\nu{}_\mu\big)\tilde{\mathbf{e}}_\nu - X^aY^b\omega_a{}^\nu{}_\mu\omega_b{}^\alpha{}_\nu\tilde{\mathbf{e}}_\alpha \\
&\quad + \big(X^a\partial_aY^b - Y^a\partial_aX^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu - \big(X^a\partial_aY^b - Y^a\partial_aX^b\big)\omega_b{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\left(\partial_a\big(\omega_b{}^\nu{}_\mu\big) + \omega_b{}^\alpha{}_\mu\omega_a{}^\nu{}_\alpha - \partial_b\big(\omega_a{}^\nu{}_\mu\big) - \omega_a{}^\alpha{}_\mu\omega_b{}^\nu{}_\alpha\right)\tilde{\mathbf{e}}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^b\left(\partial_a\omega_b{}^\nu{}_\mu - \partial_b\omega_a{}^\nu{}_\mu + \omega_a{}^\nu{}_\alpha\omega_b{}^\alpha{}_\mu - \omega_b{}^\nu{}_\alpha\omega_a{}^\alpha{}_\mu\right)\tilde{\mathbf{e}}_\nu \\
X^aY^bR_{ab}{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu &= X^aY^b\left(\mathbf{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}\right)_{ab}{}^\nu{}_\mu\,\tilde{\mathbf{e}}_\nu
\end{aligned}
$$

What settle down the interpretation of the Riemann tensor being a $\mathfrak{so}(1,D-1)$–valued 2–form, and also provides a striking resemblance to the usual gauge force field in non-abelian theories, $\mathbf{F} = \mathbf{dA} + \mathbf{A}\wedge\mathbf{A}$, it's also is easily related to the usual coordinate Riemann tensor,

$$
\begin{aligned}
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\tilde{\mathbf{e}}_\mu &= X^aY^bR_{ab}{}^\nu{}_\mu\tilde{\mathbf{e}}_\nu \\
\mathbf{Riem}(\mathbf{X},\mathbf{Y})\big(e_\mu{}^e\boldsymbol{\partial}_e\big) &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e_\mu{}^e\mathbf{Riem}(\mathbf{X},\mathbf{Y})\boldsymbol{\partial}_e &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e_\mu{}^eX^aY^bR_{ab}{}^c{}_e\boldsymbol{\partial}_c &= X^aY^bR_{ab}{}^\nu{}_\mu e_\nu{}^c\boldsymbol{\partial}_c \\
e^\mu{}_de_\mu{}^eR_{ab}{}^c{}_e &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c \\
R_{ab}{}^c{}_d &= e^\mu{}_dR_{ab}{}^\nu{}_\mu e_\nu{}^c
\end{aligned}
$$

which we'll use when rewriting the Einstein-Hilbert Action, it's of no hurt to stress that we are only assuming metricity, and not torsionless — which will come about naturally later —, lastly, we define our Riemann Curvature 2-form,

$$
\mathbf{R}^\nu{}_\mu = \frac{1}{2}R_{ab}{}^\nu{}_\mu\mathbf{d}x^a\wedge\mathbf{d}x^b = \mathbf{d}\boldsymbol{\omega}^\nu{}_\mu + \boldsymbol{\omega}^\nu{}_\alpha\wedge\boldsymbol{\omega}^\alpha{}_\mu
$$

we'll talk more about the absence of $\mathbf{e}^\mu$ later — which signifies this is the gauge force field of only the inhomogeneous part of the Poincaré group —.

Now we start the real deal of rewriting the Einstein-Hilbert Action in terms of the curvature 2–form, and the vielbein 1–form, starting with the volume form[5],

$$
\begin{aligned}
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[g_{ab}]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^\mu{}_a\eta_{\mu\nu}e^\nu{}_b]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^\mu{}_a]\mathrm{Det}[\eta_{\mu\nu}]\mathrm{Det}[e^\nu{}_b]|}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \sqrt{\left(\mathrm{Det}[e^\mu{}_a]\right)^2}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \mathrm{Det}[e^\mu{}_a]\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \epsilon_{\mu_1\cdots\mu_D}e^{\mu_0}{}_0\cdots e^{\mu_D}{}_{D-1}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \epsilon_{\mu_1\cdots\mu_D}e^{\mu_0}{}_0\,\mathrm{d}x^0\wedge\cdots\wedge e^{\mu_D}{}_{D-1}\,\mathrm{d}x^{D-1} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_1\cdots\mu_D}e^{\mu_1}{}_{a_1}\,\mathrm{d}x^{a_1}\wedge\cdots\wedge e^{\mu_D}{}_{a_D}\,\mathrm{d}x^{a_D} \\
\mathrm{d}^Dx\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_1\cdots\mu_D}\mathbf{e}^{\mu_1}\wedge\cdots\wedge\mathbf{e}^{\mu_D}
\end{aligned}
$$

The other ingredient we need is the Ricci scalar,

$$
\begin{aligned}
R &= g^{ab}R_{cb}{}^c{}_a \\
R &= e_\rho{}^ae^{\rho b}R_{cbda}e_\alpha{}^ce^{\alpha d}, \quad \text{vielbein definition} \\
R &= \eta^{\rho\sigma}\eta^{\alpha\beta}e_\rho{}^ae_\sigma{}^bR_{cbda}e_\alpha{}^ce_\beta{}^d \\
R &= \eta^{\rho\sigma}\eta^{\alpha\beta}R_{cb\beta\rho}e_\alpha{}^ce_\sigma{}^b, \quad \text{anti-symmetry in the first two index} \\
R &= \frac{1}{2}\big(\eta^{\rho\sigma}\eta^{\alpha\beta} - \eta^{\rho\alpha}\eta^{\sigma\beta}\big)R_{cb\beta\rho}e_\alpha{}^ce_\sigma{}^b \\
R &= \frac{-1}{2(D-2)!}\epsilon^{\nu_1\cdots\nu_{D-2}\beta\rho}\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^ce_\sigma{}^b
\end{aligned}
$$

[5] While it's widely known the Levi-Civita symbol, $\epsilon_{a_1\cdots a_D}$, in a coordinate basis isn't a tensor, for a non-coordinate basis, $\epsilon_{\mu_1\cdots\mu_D}$, it does not need additional factors of the metric determinant, one more usefulness of the vielbein basis.

Combining these two results we can rewrite the EH Action in the forms language, that is, the expression we're expecting to obtain is an integral over a D–form, in the middle of the computation we'll need to introduce also the Hodge star operator,

$$
\begin{aligned}
S_{\text{EH}} &= \frac{1}{2\kappa}\int_M \mathrm{d}^D x\,\sqrt{|g|}R \\
S_{\text{EH}} &= \frac{-1}{4D!(D-2)!\kappa}\int_M \mathbf{e}^{\mu_1}\wedge\cdots\wedge\mathbf{e}^{\mu_D}\epsilon_{\mu_1\cdots\mu_D}\epsilon^{\nu_1\cdots\nu_{D-2}\beta\rho}\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b \\
S_{\text{EH}} &= \frac{1}{4(D-2)!\kappa}\int_M \mathbf{e}^{\mu_1}\wedge\cdots\wedge\mathbf{e}^{\mu_D}\eta_{\mu_1}{}^{[\nu_1}\cdots\eta_{\mu_{D-2}}{}^{\nu_{D-2}}\eta_{\mu_{D-1}}{}^{\beta}\eta_{\mu_D}{}^{\rho]}\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b \\
S_{\text{EH}} &= \frac{1}{4(D-2)!\kappa}\int_M \mathbf{e}^{\nu_1}\wedge\cdots\wedge\mathbf{e}^{\nu_{D-2}}\wedge\mathbf{e}^{\beta}\wedge\mathbf{e}^{\rho}\epsilon_{\nu_1\cdots\nu_{D-2}}{}^{\alpha\sigma}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b \\
S_{\text{EH}} &= \frac{1}{2\kappa}\int_M \frac{1}{2(D-2)!}R_{cb\beta\rho}e_\alpha{}^c e_\sigma{}^b\epsilon^{\alpha\sigma}{}_{\nu_1\cdots\nu_{D-2}}\mathbf{e}^{\nu_1}\wedge\cdots\wedge\mathbf{e}^{\nu_{D-2}}\wedge\mathbf{e}^{\beta}\wedge\mathbf{e}^{\rho} \\
S_{\text{EH}} &= \frac{1}{2\kappa}\int_M \frac{1}{2(D-2)!}R_{cb\beta\rho}\epsilon^{\alpha\sigma}{}_{\nu_1\cdots\nu_{D-2}}e_\alpha{}^c e_\sigma{}^b e^{\nu_1}{}_{a_1}\cdots e^{\nu_{D-2}}{}_{a_{D-2}}\mathbf{d}x^{a_1}\wedge\cdots\wedge\mathbf{d}x^{a_{D-2}}\wedge\mathbf{e}^{\beta}\wedge\mathbf{e}^{\rho} \\
S_{\text{EH}} &= \frac{1}{2\kappa}\int_M \frac{1}{2(D-2)!}R_{cb\beta\rho}\mathrm{Det}[e^\nu{}_a]\epsilon^{cb}{}_{a_1\cdots a_{D-2}}\mathbf{d}x^{a_1}\wedge\cdots\wedge\mathbf{d}x^{a_{D-2}}\wedge\mathbf{e}^{\beta}\wedge\mathbf{e}^{\rho} \\
S_{\text{EH}} &= \frac{1}{2\kappa}\int_M \frac{1}{2}R_{cb\beta\rho}\star\left(\mathbf{d}x^c\wedge\mathbf{d}x^b\right)\wedge\mathbf{e}^{\beta}\wedge\mathbf{e}^{\rho} \\
S_{\text{EH}} &= \frac{1}{2\kappa}\int_M \star\mathbf{R}_{\beta\rho}\wedge\mathbf{e}^{\beta}\wedge\mathbf{e}^{\rho} \qquad (2.1)
\end{aligned}
$$

This is our final result for the EH Action, to see that it's a consistent expression we can do the counting of the form degree, $\mathbf{R}$ is a 2–form, which makes $\star\mathbf{R}$ a $(D-2)$–form, as $\mathbf{e}$ is a 1–form, the final expression is a $D-2+2=D$–form. To check the redundancies and equations of motion of the Action is the matter of the next sections.

3. Gravity as a Gauge Theory

Let's take a step backwards to breath, this last expression was our goal to reach, it shows the Einstein-Hilbert Action[6] written exactly in term of the vielbein and the spin connection 1–forms, in particular, the spin connection as being the connection form of the gauged Lorentz transformations shows up only inside the curvature 2–form, as it should happen to a gauge theory, this guarantees the manifestly local Lorentz invariance, to see this, we defined the local Lorentz redundancy as being $\tilde{\mathbf{e}}_\mu \to \Lambda^\nu{}_\mu\tilde{\mathbf{e}}_\nu$, with the corresponding transformation on $\boldsymbol{\omega}$ derived before, but, the vector vielbein transformation actually induces a transformation in the covector vielbein by, $\mathbf{e}^\mu(\tilde{\mathbf{e}}_\nu)=\delta^\mu{}_\nu \Rightarrow \mathbf{e}^\mu \to \Lambda^{-1\mu}{}_\nu\mathbf{e}^\nu$, so that the relevant expressions for us now are[7],

$$
\begin{cases}\mathbf{e}^\mu & \to \Lambda^\mu{}_\nu\mathbf{e}^\nu \\ \boldsymbol{\omega}^\alpha{}_\beta & \to \Lambda^\alpha{}_\rho\boldsymbol{\omega}^\rho{}_\sigma\Lambda^{-1\sigma}{}_\beta + \Lambda^\alpha{}_\sigma\mathbf{d}\Lambda^{-1\sigma}{}_\beta\end{cases} \Leftrightarrow \begin{cases}\mathbf{e} & \to \Lambda\mathbf{e} \\ \boldsymbol{\omega} & \to \Lambda\boldsymbol{\omega}\Lambda^{-1}+\Lambda\mathbf{d}\Lambda^{-1}\end{cases}
$$

where we also included the matrix form of these transformations, to see the invariance of the action with respect to these is only needed to work out the transformation law of the curvature 2–form,

$$
\begin{aligned}
&\mathbf{R}=\mathbf{d}\boldsymbol{\omega}+\boldsymbol{\omega}\wedge\boldsymbol{\omega}\to\mathbf{d}\left(\Lambda\boldsymbol{\omega}\Lambda^{-1}+\Lambda\mathbf{d}\Lambda^{-1}\right)+\left(\Lambda\boldsymbol{\omega}\Lambda^{-1}+\Lambda\mathbf{d}\Lambda^{-1}\right)\wedge\left(\Lambda\boldsymbol{\omega}\Lambda^{-1}+\Lambda\mathbf{d}\Lambda^{-1}\right) \\
&\mathbf{R}\to\mathbf{d}\Lambda\wedge\boldsymbol{\omega}\Lambda^{-1}+\Lambda\mathbf{d}\boldsymbol{\omega}\Lambda^{-1}-\Lambda\boldsymbol{\omega}\wedge\mathbf{d}\Lambda^{-1}+\mathbf{d}\Lambda\wedge\mathbf{d}\Lambda^{-1}+\Lambda\boldsymbol{\omega}\wedge\boldsymbol{\omega}\Lambda^{-1}+\Lambda\boldsymbol{\omega}\wedge\mathbf{d}\Lambda^{-1}-\mathbf{d}\Lambda\wedge\boldsymbol{\omega}\Lambda^{-1}-\mathbf{d}\Lambda\wedge\mathbf{d}\Lambda^{-1} \\
&\mathbf{R}\to\Lambda(\mathbf{d}\boldsymbol{\omega}+\boldsymbol{\omega}\wedge\boldsymbol{\omega})\Lambda^{-1}=\Lambda\mathbf{R}\Lambda^{-1}
\end{aligned}
$$

and lastly, but not less important, we have to understand whether or not the Hodge star operator should change or not with respect to this transformation, the answer is no[8], despite it depending explicitly on the metric to be defined, it's dependence is only through $\sqrt{|\mathrm{Det}[g_{ab}]|}$, which is invariant under a local Lorentz transformation, as $g_{ab}=\eta_{\mu\nu}e^\mu{}_a e^\nu{}_b$, thus, $\star\mathbf{R}\to\Lambda(\star\mathbf{R})\Lambda^{-1}$, and the action is manifestly invariant,

$$
S'_{\text{EH}}=\frac{1}{2\kappa}\int_M \Lambda_\beta{}^\tau\star\mathbf{R}_{\tau\sigma}\Lambda^{-1\sigma}{}_\rho\wedge\Lambda^\beta{}_\alpha\mathbf{e}^\alpha\wedge\Lambda^\rho{}_\mu\mathbf{e}^\mu=\frac{1}{2\kappa}\int_M\star\mathbf{R}_{\beta\rho}\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\rho=S_{\text{EH}}
$$

[6]Actually it's the Einstein-Cartan Action, as we have not imposed the torsionless condition, but, we'll not bother distinguishing this.

[7]For convenience we'll make the change $\Lambda\leftrightarrow\Lambda^{-1}$ everywhere.

[8]As long as we work with orientation preserving transformation, otherwise it'll adquire an extra sign.

$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int_M \mathbf{R}_{\beta\rho} \wedge \star\big(\mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho}\big)$$

$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int_M \frac{1}{(D-2)!} \epsilon^{\beta\rho}{}_{\alpha_1 \cdots \alpha_{D-2}} \mathbf{R}_{\beta\rho} \wedge \mathbf{e}^{\alpha_1} \wedge \cdots \wedge \mathbf{e}^{\alpha_{D-2}}$$

$$S_{\mathrm{EH}} = \frac{1}{2(D-2)!\kappa} \int_M \epsilon_{\alpha_1 \cdots \alpha_D} \mathbf{e}^{\alpha_1} \wedge \cdots \wedge \mathbf{e}^{\alpha_{D-2}} \wedge \mathbf{R}^{\alpha_{D-1}\alpha_D}$$

For $D = 4$ this gives,

$$S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega}] = \frac{1}{4\kappa} \epsilon_{\mu\nu\alpha\beta} \int_M \mathbf{e}^{\mu} \wedge \mathbf{e}^{\nu} \wedge \mathbf{R}^{\alpha\beta}$$

It's not clear how this should be interpreted as a gauge theory, due to we wanting to interpret $\boldsymbol{\omega}$ as the gauge 1-form of Lorentz transformations and $\tilde{\mathbf{e}}$ as the gauge 1-form of translations, the Einstein Hilbert Action should then be something of the form,

$$\int_M \mathrm{Tr}\left[\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{F}\right]$$

for a combined gauge field $\mathbf{A}$, where both $\boldsymbol{\omega}$ and $\mathbf{e}$ contribute in some way, sadly, there is no such gauge theory in this form, first, due to the trace being of three lie algebra valued forms, this vanishes, secondly, this is not invariant under a gauge transformation $\mathbf{A} \to g^{-1}\mathbf{A}g + g^{-1}\,\mathrm{d}g$. So, if it's to $3+1$ gravity to be a gauge theory in this sense, it certainly is realized in a different way, but, an interesting case happens to $D = 2+1$,

$$S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega}] = \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \mathbf{e}^{\mu} \wedge \mathbf{R}^{\alpha\beta}$$

This already seems a lot like a proper gauge theory of the Chern-Simons type, Equations of motion are,

$$\begin{aligned}
S_{\mathrm{EH}}[\mathbf{e} + \delta\mathbf{e}, \boldsymbol{\omega}] - S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega}] &= \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \delta\mathbf{e}^{\mu} \wedge \mathbf{R}^{\alpha\beta} = 0 \\
0 &= -\frac{1}{2} \epsilon_{\mu\alpha\beta} \mathbf{R}^{\alpha\beta} \\
0 &= -\frac{1}{2} \epsilon_{\mu\alpha\beta} \frac{1}{2} R_{\rho\sigma}{}^{\alpha\beta} \mathbf{e}^{\rho} \wedge \mathbf{e}^{\sigma} \\
0 &= -\frac{1}{4} \epsilon_{\mu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \star\left(\mathbf{e}^{\rho} \wedge \mathbf{e}^{\sigma}\right) \\
0 &= -\frac{1}{4} \epsilon_{\mu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \epsilon^{\rho\sigma}{}_{\kappa} \mathbf{e}^{\kappa} \\
0 &= -\frac{1}{4} \epsilon_{\mu\alpha\beta} \epsilon^{\rho\sigma\kappa} R_{\rho\sigma}{}^{\alpha\beta} \mathbf{e}_{\kappa} \\
0 &= -\frac{3!}{4} \eta_{\mu}{}^{[\rho} \eta_{\alpha}{}^{\sigma} \eta_{\beta}{}^{\kappa]} R_{\rho\sigma}{}^{\alpha\beta} \mathbf{e}_{\kappa} \\
0 &= -\frac{3!}{4} R_{\rho\sigma}{}^{[\sigma\kappa} \eta_{\mu}{}^{\rho]} \mathbf{e}_{\kappa} \\
0 &= -\frac{1}{4}\big(R_{\rho\sigma}{}^{\sigma\kappa} \eta_{\mu}{}^{\rho} + R_{\rho\sigma}{}^{\kappa\rho} \eta_{\mu}{}^{\sigma} + R_{\rho\sigma}{}^{\rho\sigma} \eta_{\mu}{}^{\kappa} - R_{\rho\sigma}{}^{\rho\kappa} \eta_{\mu}{}^{\sigma} - R_{\rho\sigma}{}^{\kappa\sigma} \eta_{\mu}{}^{\rho} - R_{\rho\sigma}{}^{\sigma\rho} \eta_{\mu}{}^{\kappa}\big) \mathbf{e}_{\kappa} \\
0 &= -\frac{1}{4}\big(-R_{\mu}{}^{\kappa} - R_{\mu}{}^{\kappa} + R\eta_{\mu}{}^{\kappa} - R_{\mu}{}^{\kappa} - R_{\mu}{}^{\kappa} + R\eta_{\mu}{}^{\kappa}\big) \mathbf{e}_{\kappa} \\
0 &= -\frac{1}{4}(-4R_{\mu}{}^{\kappa} + 2R\eta_{\mu}{}^{\kappa}) \mathbf{e}_{\kappa} \\
0 &= \left(R_{\mu\kappa} - \frac{1}{2} R\eta_{\mu\kappa}\right) \mathbf{e}^{\kappa}
\end{aligned}$$

And,

$$\begin{aligned}
S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega} + \delta\boldsymbol{\omega}] - S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega}] &= \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \mathbf{e}^{\mu} \wedge (\mathrm{d}\delta\boldsymbol{\omega} + \delta\boldsymbol{\omega} \wedge \boldsymbol{\omega} + \boldsymbol{\omega} \wedge \delta\boldsymbol{\omega})^{\alpha\beta} = 0 \\
0 &= \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \Big(-\mathrm{d}\big(\mathbf{e}^{\mu} \wedge \delta\boldsymbol{\omega}^{\alpha\beta}\big) + \mathrm{d}\mathbf{e}^{\mu} \wedge \delta\boldsymbol{\omega}^{\alpha\beta} + \mathbf{e}^{\mu} \wedge (\delta\boldsymbol{\omega} \wedge \boldsymbol{\omega})^{\alpha\beta} + \mathbf{e}^{\mu} \wedge (\boldsymbol{\omega} \wedge \delta\boldsymbol{\omega})^{\alpha\beta}\Big) \\
0 &= \frac{1}{2} \epsilon_{\mu\alpha\beta} \int_M \big(\mathrm{d}\mathbf{e}^{\mu} \wedge \delta\boldsymbol{\omega}^{\alpha\beta} + \mathbf{e}^{\mu} \wedge \delta\boldsymbol{\omega}^{\alpha\gamma} \wedge \boldsymbol{\omega}_{\gamma}{}^{\beta} + \mathbf{e}^{\mu} \wedge \boldsymbol{\omega}^{\alpha\gamma} \wedge \delta\boldsymbol{\omega}_{\gamma}{}^{\beta}\big)
\end{aligned}$$

$$
\begin{aligned}
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \left(\mathrm{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} - \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\gamma{}^\beta \wedge \delta\boldsymbol{\omega}^{\alpha\gamma} + \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\alpha{}_\gamma \wedge \delta\boldsymbol{\omega}^{\gamma\beta}\right) \\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \mathrm{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} - \frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\gamma{}^\beta \wedge \delta\boldsymbol{\omega}^{\alpha\gamma} + \frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\alpha{}_\gamma \wedge \delta\boldsymbol{\omega}^{\gamma\beta} \\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \mathrm{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} - \frac{1}{2}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta{}^\gamma \wedge \delta\boldsymbol{\omega}^{\alpha\beta} + \frac{1}{2}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma{}_\alpha \wedge \delta\boldsymbol{\omega}^{\alpha\beta} \\
0 &= \frac{1}{2}\int\limits_M \left(\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu - \epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta{}^\gamma + \epsilon_{\mu\gamma\beta}\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma{}_\alpha\right) \wedge \delta\boldsymbol{\omega}^{\alpha\beta} \\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu - \frac{1}{2}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta{}^\gamma + \frac{1}{2}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma{}_\alpha \\
0 &= \frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu - \frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta{}^\gamma + \frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma{}_\alpha \\
0 &= \mathrm{d}\mathbf{e}^\nu - \eta_\gamma{}^{[\beta}\eta_\mu{}^{\nu]}\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta{}^\gamma + \eta_\mu{}^{[\nu}\eta_\gamma{}^{\alpha]}\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma{}_\alpha \\
0 &= \mathrm{d}\mathbf{e}^\nu - \frac{1}{2}\left(\eta_\gamma{}^\beta\eta_\mu{}^\nu - \eta_\gamma{}^\nu\eta_\mu{}^\beta\right)\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta{}^\gamma + \frac{1}{2}\left(\eta_\mu{}^\nu\eta_\gamma{}^\alpha - \eta_\mu{}^\alpha\eta_\gamma{}^\nu\right)\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma{}_\alpha \\
0 &= \mathrm{d}\mathbf{e}^\nu + \frac{1}{2}\eta_\gamma{}^\nu\eta_\mu{}^\beta\mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta{}^\gamma - \frac{1}{2}\eta_\mu{}^\alpha\eta_\gamma{}^\nu\mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma{}_\alpha \\
0 &= \mathrm{d}\mathbf{e}^\nu + \frac{1}{2}\mathbf{e}^\beta \wedge \boldsymbol{\omega}_\beta{}^\nu - \frac{1}{2}\mathbf{e}^\alpha \wedge \boldsymbol{\omega}^\nu{}_\alpha \\
0 &= \mathrm{d}\mathbf{e}^\nu + \boldsymbol{\omega}^\nu{}_\alpha \wedge \mathbf{e}^\alpha
\end{aligned}
$$

It's not really feasible to give this a gauge theory approach, only if we're in 2+1, in this case there is an isomorphism, $\mathbf{e}^\mu \to \mathbf{e}_{\alpha\beta} = \epsilon_{\mu\alpha\beta}\mathbf{e}^\mu$, so that,

$$
\begin{aligned}
S_{\mathrm{EH}} &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} \\
S_{\mathrm{EH}} &= -\frac{1}{2\kappa}\int\limits_M \mathbf{e}_{\beta\alpha} \wedge \mathbf{R}^{\alpha\beta} \\
S_{\mathrm{EH}} &= -\frac{1}{2\kappa}\int\limits_M \mathrm{Tr}\left[\mathbf{e} \wedge \mathbf{R}\right] \\
S_{\mathrm{EH}} &= -\frac{1}{2\kappa}\int\limits_M \mathrm{Tr}\left[\mathbf{e} \wedge \left(\mathrm{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\right)\right]
\end{aligned}
$$

This is a lot similar to Chern-Simons theory,

$$
S_{\mathrm{CS}}[\mathbf{A}] = k\int\limits_M \mathrm{Tr}\left[\mathbf{A} \wedge \mathrm{d}\mathbf{A} + \frac{1}{3}\mathbf{A} \wedge [\mathbf{A} \overset{\wedge}{,} \mathbf{A}]\right]
$$

Let's try, $\mathbf{A}^x = \boldsymbol{\omega} + x\mathbf{e}$,

$$
\begin{aligned}
S_{\mathrm{CS}}[\mathbf{A}^x] &= k\int\limits_M \mathrm{Tr}\left[(\boldsymbol{\omega} + x\mathbf{e}) \wedge \mathrm{d}(\boldsymbol{\omega} + x\mathbf{e}) + \frac{1}{3}(\boldsymbol{\omega} + x\mathbf{e}) \wedge [\boldsymbol{\omega} + x\mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega} + x\mathbf{e}]\right] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= k\int\limits_M \mathrm{Tr}\Big[\boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\omega} + x\boldsymbol{\omega} \wedge \mathrm{d}\mathbf{e} + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} + x^2\mathbf{e} \wedge \mathrm{d}\mathbf{e} \\
&\qquad + \frac{2}{3}(\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e})\Big] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= k\int\limits_M \mathrm{Tr}\Big[\boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\omega} + x\,\mathrm{d}\mathbf{e} \wedge \boldsymbol{\omega} + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} - \frac{1}{2}x^2\,\mathrm{d}(\mathbf{e} \wedge \mathbf{e}) \\
&\qquad + \frac{2}{3}\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\omega} + 2x\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + 2x^2\mathbf{e} \wedge \mathbf{e} \wedge \boldsymbol{\omega} + \frac{2}{3}x^3\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\Big] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= k\int\limits_M \mathrm{Tr}\left[\boldsymbol{\omega} \wedge \left(\mathrm{d}\boldsymbol{\omega} + \frac{2}{3}\boldsymbol{\omega} \wedge \boldsymbol{\omega}\right) + x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} + x\mathbf{e} \wedge (\mathrm{d}\boldsymbol{\omega} + 2\boldsymbol{\omega} \wedge \boldsymbol{\omega}) + 2x^2\mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e}\right.
\end{aligned}
$$

$$+\frac{2}{3}x^3\mathbf{e}\wedge\mathbf{e}\wedge\mathbf{e}\Big]$$

$$S_{\text{CS}}[\mathbf{A}^x]=S_{\text{CS}}[\boldsymbol{\omega}]+k\int\limits_M \text{Tr}\left[x\,\mathrm{d}(\mathbf{e}\wedge\boldsymbol{\omega})+2x\mathbf{e}\wedge(\mathrm{d}\boldsymbol{\omega}+\boldsymbol{\omega}\wedge\boldsymbol{\omega})+2x^2\mathbf{e}\wedge\boldsymbol{\omega}\wedge\mathbf{e}+\frac{2}{3}x^3\mathbf{e}\wedge\mathbf{e}\wedge\mathbf{e}\right]$$

$$S_{\text{CS}}[\mathbf{A}^x]=S_{\text{CS}}[\boldsymbol{\omega}]+k\int\limits_M \text{Tr}\left[x\,\mathrm{d}(\mathbf{e}\wedge\boldsymbol{\omega})+2x\mathbf{e}\wedge\mathbf{R}+2x^2\mathbf{e}\wedge\boldsymbol{\omega}\wedge\mathbf{e}+\frac{2}{3}x^3\mathbf{e}\wedge\mathbf{e}\wedge\mathbf{e}\right]$$

$$S_{\text{CS}}[\mathbf{A}^x]=S_{\text{CS}}[\boldsymbol{\omega}]-4xk\kappa S_{\text{EH}}+k\int\limits_M \text{Tr}\left[x\,\mathrm{d}(\mathbf{e}\wedge\boldsymbol{\omega})+2x^2\mathbf{e}\wedge\boldsymbol{\omega}\wedge\mathbf{e}+\frac{2}{3}x^3\mathbf{e}\wedge\mathbf{e}\wedge\mathbf{e}\right]$$

We can simplify if we sum two contributions,

$$S_{\text{CS}}[\mathbf{A}^x]-S_{\text{CS}}\left[\mathbf{A}^{-x}\right]=-8xk\kappa S_{\text{EH}}+2kx\int\limits_{\partial M}\text{Tr}\left[\mathbf{e}\wedge\boldsymbol{\omega}\right]+\frac{4}{3}x^3k\int\limits_M\text{Tr}\left[\mathbf{e}\wedge\mathbf{e}\wedge\mathbf{e}\right]$$

$$\frac{1}{8xk\kappa}\left(S_{\text{CS}}\left[\mathbf{A}^{-x}\right]-S_{\text{CS}}[\mathbf{A}^x]\right)=S_{\text{EH}}-\frac{1}{4\kappa}\int\limits_{\partial M}\text{Tr}\left[\mathbf{e}\wedge\boldsymbol{\omega}\right]-\frac{x^2}{3!\kappa}\int\limits_M\text{Tr}\left[\mathbf{e}\wedge\mathbf{e}\wedge\mathbf{e}\right]$$

And also the not so usual action,

$$S_{\text{CS}}[\mathbf{A}^x]+S_{\text{CS}}\left[\mathbf{A}^{-x}\right]=2S_{\text{CS}}[\boldsymbol{\omega}]+4x^2k\int\limits_M\text{Tr}\left[\mathbf{e}\wedge\boldsymbol{\omega}\wedge\mathbf{e}\right]$$

Chern-Simons Equation of Motion,

$$S_{\text{CS}}[\mathbf{A}+\delta\mathbf{A}]-S_{\text{CS}}[\mathbf{A}]=k\int\limits_M\text{Tr}\left[\delta\mathbf{A}\wedge\mathrm{d}\mathbf{A}+\mathbf{A}\wedge\mathrm{d}\delta\mathbf{A}+\frac{2}{3}\delta\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}+\frac{2}{3}\mathbf{A}\wedge\delta\mathbf{A}\wedge\mathbf{A}+\frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\delta\mathbf{A}\right]$$

$$S_{\text{CS}}[\mathbf{A}+\delta\mathbf{A}]-S_{\text{CS}}[\mathbf{A}]=k\int\limits_M\text{Tr}\left[\mathrm{d}\mathbf{A}\wedge\delta\mathbf{A}-\mathrm{d}(\mathbf{A}\wedge\delta\mathbf{A})+\mathrm{d}\mathbf{A}\wedge\delta\mathbf{A}+2\mathbf{A}\wedge\mathbf{A}\wedge\delta\mathbf{A}\right]$$

$$S_{\text{CS}}[\mathbf{A}+\delta\mathbf{A}]-S_{\text{CS}}[\mathbf{A}]=2k\int\limits_M\text{Tr}\left[(\mathrm{d}\mathbf{A}+\mathbf{A}\wedge\mathbf{A})\wedge\delta\mathbf{A}\right]-k\int\limits_M\text{Tr}\left[\mathrm{d}(\mathbf{A}\wedge\delta\mathbf{A})\right]$$

$$S_{\text{CS}}[\mathbf{A}+\delta\mathbf{A}]-S_{\text{CS}}[\mathbf{A}]=2k\int\limits_M\text{Tr}\left[\mathbf{F}\wedge\delta\mathbf{A}\right]-k\int\limits_{\partial M}\text{Tr}\left[\mathbf{A}\wedge\delta\mathbf{A}\right]$$

Chern-Simons Gauge Invariance,

$$S_{\text{CS}}\left[g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right]=k\int\limits_M\text{Tr}\left[\left(g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right)\wedge g\mathbf{F}g^{-1}-\frac{1}{3}\left(g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right)\wedge\left(g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right)\wedge\left(g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right)\right]$$

$$S_{\text{CS}}\left[g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right]=k\int\limits_M\text{Tr}\Big[\mathbf{A}\wedge\mathbf{F}+\mathrm{d}g^{-1}\,g\wedge\mathbf{F}-\frac{1}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}-\frac{1}{3}g\,\mathrm{d}g^{-1}\wedge g\,\mathrm{d}g^{-1}\wedge g\,\mathrm{d}g^{-1}$$
$$-g\mathbf{A}g^{-1}\wedge g\mathbf{A}g^{-1}\wedge g\,\mathrm{d}g^{-1}-g\mathbf{A}g^{-1}\wedge g\,\mathrm{d}g^{-1}\wedge g\,\mathrm{d}g^{-1}\Big]$$

$$S_{\text{CS}}\left[g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right]=S_{\text{CS}}[\mathbf{A}]+k\int\limits_M\text{Tr}\Big[\mathrm{d}g^{-1}\,g\wedge\mathbf{F}-\frac{1}{3}\,\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g$$
$$-\,\mathrm{d}g^{-1}\,g\wedge\mathbf{A}\wedge\mathbf{A}-\mathbf{A}\wedge\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g\Big]$$

$$S_{\text{CS}}\left[g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right]=S_{\text{CS}}[\mathbf{A}]+k\int\limits_M\text{Tr}\left[\mathrm{d}g^{-1}\,g\wedge\mathrm{d}\mathbf{A}-\frac{1}{3}\,\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g+\mathrm{d}g^{-1}\,g\wedge g^{-1}\,\mathrm{d}g\wedge\mathbf{A}\right]$$

$$S_{\text{CS}}\left[g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right]=S_{\text{CS}}[\mathbf{A}]+k\int\limits_M\text{Tr}\left[\mathrm{d}g^{-1}\,g\wedge\mathrm{d}\mathbf{A}-\frac{1}{3}\,\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g-\mathrm{d}\left(\mathrm{d}g^{-1}\,g\wedge\mathbf{A}\right)-\mathrm{d}g^{-1}\,g\wedge\mathrm{d}\mathbf{A}\right]$$

$$S_{\text{CS}}\left[g\mathbf{A}g^{-1}+g\,\mathrm{d}g^{-1}\right]=S_{\text{CS}}[\mathbf{A}]-k\int\limits_M\text{Tr}\left[\frac{1}{3}\,\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g\wedge\mathrm{d}g^{-1}\,g+\mathrm{d}\left(\mathrm{d}g^{-1}\,g\wedge\mathbf{A}\right)\right]$$

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