

Homework III

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Problem 1

1.A)

For an operator \mathcal{O} to be BRST closed, it means $[Q_{\text{BRST}}, \mathcal{O}] = 0$, and by now we're very familiar with a commutator being written as a contour integral, that is,

$$[Q_{\text{BRST}}, \mathcal{O}(w, \bar{w})] = \frac{1}{2\pi i} \oint_{C_w} (dz j_{\text{BRST}}(z) - d\bar{z} \tilde{j}_{\text{BRST}}(\bar{z})) \mathcal{O}(w, \bar{w}) \quad (1.1)$$

Where the BRST current is given by,

$$j_{\text{BRST}}(z) = -\frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) + : bc \partial c : (z) + \frac{3}{2} : \partial^2 c : (z) \quad (1.2)$$

And also C_w is any closed contour encircling counterclockwise the point w . Actually, this definition is only good for closed strings — in which operators can be inserted at any point —, but, for the open string — our case here —, operators have to be inserted at the boundary $\text{Im}(w) = 0$. Hence, as we have to our disposal only half of the complex plane, it's impossible to have a closed curve C_w that encloses a point at $\text{Im}(w) = 0$ — unless the point itself belongs to the curve, under which the Cauchy residue theorem stops holding —. Thus, the definition of the commutator for the open string is different,

$$[Q_{\text{BRST}}, \mathcal{O}(w, \bar{w})] = \frac{1}{2\pi i} \int_{C'_w} (dz j_{\text{BRST}}(z) - d\bar{z} \tilde{j}_{\text{BRST}}(\bar{z})) \mathcal{O}(w, \bar{w}) \quad (1.3)$$

Where C'_w is any counterclockwise oriented open curve starting and ending at $\text{Im}(z) = 0$, such that the point w , $\text{Im}(w) = 0$, lies in the interior of C'_w . To get a simpler expression, we can do the so called *doubling trick*. The open string boundary conditions forces us to have at $\text{Im}(z) = 0$ the following,

$$\bar{\partial} X^\mu = \partial X^\mu, \quad \tilde{b} = b, \quad \tilde{c} = c, \quad \text{Im}(z) = 0$$

Among other things, this also imply that at $\text{Im}(z) = 0$ we have $\tilde{j}_{\text{BRST}} = j_{\text{BRST}}$. The doubling trick consists of we assigning j_{BRST} as being a operator over the whole \mathbb{C} . In the upper half it's already well defined, but in the lower half we define it as,

$$j_{\text{BRST}}(z) = \tilde{j}_{\text{BRST}}(z^*), \quad \text{Im}(z) < 0$$

Due to the open string boundary conditions this definition is continuous at $\text{Im}(z) = 0$. We can use this to simplify eq. (1.3),

$$\begin{aligned} [Q_{\text{BRST}}, \mathcal{O}(w, \bar{w})] &= \frac{1}{2\pi i} \int_{C'_w} dz j_{\text{BRST}}(z) \mathcal{O}(w, \bar{w}) - \frac{1}{2\pi i} \int_{C'_w} d\bar{z} \tilde{j}_{\text{BRST}}(\bar{z}) \mathcal{O}(w, \bar{w}) \\ [Q_{\text{BRST}}, \mathcal{O}(w, \bar{w})] &= \frac{1}{2\pi i} \int_{C'_w} dz j_{\text{BRST}}(z) \mathcal{O}(w, \bar{w}) - \frac{1}{2\pi i} \int_{\bar{C}'_w} dz j_{\text{BRST}}(z) \mathcal{O}(w, \bar{w}) \\ [Q_{\text{BRST}}, \mathcal{O}(w, \bar{w})] &= \frac{1}{2\pi i} \int_{C'_w} dz j_{\text{BRST}}(z) \mathcal{O}(w, \bar{w}) + \frac{1}{2\pi i} \int_{\bar{C}'_w^{-1}} dz j_{\text{BRST}}(z) \mathcal{O}(w, \bar{w}) \\ [Q_{\text{BRST}}, \mathcal{O}(w, \bar{w})] &= \frac{1}{2\pi i} \int_{C'_w \cup \bar{C}'_w^{-1}} dz j_{\text{BRST}}(z) \mathcal{O}(w, \bar{w}) \\ [Q_{\text{BRST}}, \mathcal{O}(w, \bar{w})] &= \frac{1}{2\pi i} \oint_{C_w} dz j_{\text{BRST}}(z) \mathcal{O}(w, \bar{w}) \end{aligned} \quad (1.4)$$

Here \bar{C}'_w is the complex conjugated curve to C'_w , and \bar{C}'_w^{-1} is the reverse orientation version of the curve \bar{C}'_w . Lastly, $C'_w \cup \bar{C}'_w^{-1}$ is the closed curve got by gluing the end of C'_w with the start of \bar{C}'_w^{-1} , which is not hard to see that is a counterclockwise oriented closed curve around w . The last expression, eq. (1.4), ease the calculation of the commutator in comparison to eq. (1.3), because now we can use the Cauchy residue theorem.

Our main interest here is when \mathcal{O} is a open string vertex operator, specially,

$$V^a(w, \bar{w}; \epsilon, k) = \lambda^a \epsilon_\mu : c \partial X^\mu \exp(ik \cdot X) : (w, \bar{w}) \quad (1.5)$$

It's clear that V^a is not just a holomorphic, or just an anti-holomorphic, operator. What spoils it is the $\exp(ik \cdot X)$ part. Hence, the dependence on \bar{w} is mandatory, also, as this is a open string vertex operator, it must be inserted at $\text{Im}(w) = 0 \Rightarrow w = \bar{w}$. Thus, V^a actually is not dependent on \bar{w} , and we will omit such dependence here¹.

By eq. (1.4), we only need the knowledge of the first order pole of the OPE between the BRST current and the vertex operator in order to know the commutator. Hence, we'll start computing the OPE between those, it is in the same manner we did in the second homework set. First compute the normal ordering of the expression you want to know the OPE, this will involve among the OPE, other normal ordered terms, after gathering the divergent contributions we can set up the normal ordered terms as non-singular and obtain the OPE. As the full expression of the BRST current is quite big, and has a lot of terms, we'll compute each three contributions separately,

$$\begin{aligned}
- : \frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) : &= - \frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) \\
&\quad - 2 \frac{\lambda^a \epsilon_\mu}{\alpha'} : (c \overline{\partial X^\alpha \partial X_\alpha})(z) (c \partial X^\mu \exp(ik \cdot X))(w) : \\
&\quad - 2 \frac{\lambda^a \epsilon_\mu}{\alpha'} : (c \overline{\partial X^\alpha \partial X_\alpha})(z) (c \partial X^\mu \exp(ik \cdot X))(w) : \\
&\quad - 2 \frac{\lambda^a \epsilon_\mu}{\alpha'} : (c \overline{\partial X^\alpha \partial X_\alpha})(z) (c \partial X^\mu \exp(ik \cdot X))(w) : \\
&\quad - \frac{\lambda^a \epsilon_\mu}{\alpha'} : (c \overline{\partial X^\alpha \partial X_\alpha})(z) (c \partial X^\mu \exp(ik \cdot X))(w) : \\
- : \frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) : &= - \frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) \\
&\quad - 2 \frac{\lambda^a \epsilon_\mu}{2} \eta^{\alpha\mu} \partial_z \partial_w \ln |z - w|^2 : (c \partial X_\alpha)(z) (c \exp(ik \cdot X))(w) : \\
&\quad + 2 \frac{\alpha' \lambda^a \epsilon_\mu}{4} \eta^{\alpha\mu} i k_\alpha \partial_z \partial_w \ln |z - w|^2 \partial_z \ln |z - w|^2 : c(z) (c \exp(ik \cdot X))(w) : \\
&\quad - 2 \frac{\lambda^a \epsilon_\mu}{2} i k_\alpha \partial_z \ln |z - w|^2 : (c \partial X^\alpha)(z) (c \partial X^\mu \exp(ik \cdot X))(w) : \\
&\quad + \frac{\alpha' \lambda^a \epsilon_\mu}{4} i k_\alpha i k^\alpha (\partial_z \ln |z - w|^2)^2 : c(z) (c \partial X^\mu \exp(ik \cdot X))(w) : \\
- : \frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) : &= - \frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) \\
&\quad - \frac{\lambda^a \epsilon_\mu}{(z - w)^2} : (c \partial X^\mu)(z) (c \exp(ik \cdot X))(w) : \\
&\quad + \frac{i \alpha' \lambda^a k \cdot \epsilon}{2(z - w)^3} : c(z) (c \exp(ik \cdot X))(w) : \\
&\quad - \frac{i k_\alpha \lambda^a \epsilon_\mu}{z - w} : (c \partial X^\alpha)(z) (c \partial X^\mu \exp(ik \cdot X))(w) : \\
&\quad - \frac{k^2 \alpha' \lambda^a \epsilon_\mu}{4(z - w)^2} : c(z) (c \partial X^\mu \exp(ik \cdot X))(w) :
\end{aligned}$$

now we expand each function of z in Taylor series around $z = w$, keeping only the terms which have a single simple pole,

$$- \frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) = \frac{\lambda^a \epsilon_\mu}{z - w} : \partial c c \partial X^\mu \exp(ik \cdot X) : (w)$$

¹If we where to be completely rigorous, we would need to change our normal ordering prescription.

$$\begin{aligned}
& -\frac{i\alpha'\lambda^ak\cdot\epsilon}{4(z-w)} : \partial^2 cc \exp(ik\cdot X) : (w) \\
& + \frac{k^2\alpha'\lambda^a\epsilon_\mu}{4(z-w)} : \partial cc \partial X^\mu \exp(ik\cdot X) : (w) \\
& + \text{no single simple pole} \\
-\frac{1}{\alpha'} : c \partial X^\mu \partial X_\mu : (z) V^a(w; \epsilon, k) &= \frac{\lambda^a\epsilon_\mu}{z-w} \left(1 + \frac{k^2\alpha'}{4} \right) : \partial cc \partial X^\mu \exp(ik\cdot X) : (w) \\
& - \frac{i\alpha'\lambda^ak\cdot\epsilon}{4(z-w)} : \partial^2 cc \exp(ik\cdot X) : (w) \\
& + \text{no single simple pole}
\end{aligned} \tag{1.6}$$

The second term of the BRST current OPE gives,

$$\begin{aligned}
& :: bc \partial c : (z) V^a(w; \epsilon, k) : =: bc \partial c : (z) V^a(w; \epsilon, k) \\
& \quad + \lambda^a \epsilon_\mu : \overline{(bc \partial c)(z)} (c \partial X^\mu \exp(ik\cdot X))(w) : \\
& :: bc \partial c : (z) V^a(w; \epsilon, k) : =: bc \partial c : (z) V^a(w; \epsilon, k) \\
& \quad - \frac{\lambda^a \epsilon_\mu}{z-w} : (c \partial c)(z) (\partial X^\mu \exp(ik\cdot X))(w) : \\
& : bc \partial c : (z) V^a(w; \epsilon, k) = \frac{\lambda^a \epsilon_\mu}{z-w} : (c \partial c)(z) (\partial X^\mu \exp(ik\cdot X))(w) : + \text{regular} \\
& : bc \partial c : (z) V^a(w; \epsilon, k) = -\frac{\lambda^a \epsilon_\mu}{z-w} : \partial cc \partial X^\mu \exp(ik\cdot X) : (w) \\
& \quad + \text{no single simple pole}
\end{aligned} \tag{1.7}$$

The third term of the BRST current OPE is already non-singular,

$$\begin{aligned}
& : \frac{3}{2} : \partial^2 c : (z) V^a(w; \epsilon, k) : = \frac{3}{2} \partial^2 c(z) V^a(w; \epsilon, k) \\
& \quad \frac{3}{2} \partial^2 c(z) V^a(w; \epsilon, k) = \text{no single simple pole}
\end{aligned} \tag{1.8}$$

Summing the three contributions, eqs. (1.6) to (1.8),

$$\begin{aligned}
j_{\text{BRST}}(z) V^a(w; \epsilon, k) &= \frac{\lambda^a \epsilon_\mu}{z-w} \left(1 + \frac{k^2\alpha'}{4} \right) : \partial cc \partial X^\mu \exp(ik\cdot X) : (w) \\
& - \frac{i\alpha'\lambda^ak\cdot\epsilon}{4(z-w)} : \partial^2 cc \exp(ik\cdot X) : (w) \\
& - \frac{\lambda^a \epsilon_\mu}{z-w} : \partial cc \partial X^\mu \exp(ik\cdot X) : (w) \\
& + \text{no single simple pole} \\
j_{\text{BRST}}(z) V^a(w; \epsilon, k) &= \frac{\lambda^a \epsilon_\mu}{z-w} \frac{k^2\alpha'}{4} : \partial cc \partial X^\mu \exp(ik\cdot X) : (w) \\
& - \frac{i\alpha'\lambda^ak\cdot\epsilon}{4(z-w)} : \partial^2 cc \exp(ik\cdot X) : (w) \\
& + \text{no single simple pole}
\end{aligned} \tag{1.9}$$

Using eq. (1.4),

$$[Q_{\text{BRST}}, V^a(w, \epsilon, k)] = \frac{1}{2\pi i} \oint_{C_w} dz j_{\text{BRST}}(z) V^a(w, \bar{w})$$

$$\begin{aligned}
[Q_{\text{BRST}}, V^a(w, \epsilon, k)] &= \frac{1}{2\pi i} \oint_{C_w} dz \frac{\lambda^a \epsilon_\mu}{z-w} \frac{k^2 \alpha'}{4} : \partial c c \partial X^\mu \exp(ik \cdot X) : (w) \\
&\quad - \frac{1}{2\pi i} \oint_{C_w} dz \frac{i \alpha' \lambda^a k \cdot \epsilon}{4(z-w)} : \partial^2 c c \exp(ik \cdot X) : (w) \\
&\quad + \frac{1}{2\pi i} \oint_{C_w} dz \text{ no single simple pole} \\
[Q_{\text{BRST}}, V^a(w, \epsilon, k)] &= \lambda^a \epsilon_\mu \frac{k^2 \alpha'}{4} : \partial c c \partial X^\mu \exp(ik \cdot X) : (w) \\
&\quad - \frac{i \alpha' \lambda^a k \cdot \epsilon}{4} : \partial^2 c c \exp(ik \cdot X) : (w)
\end{aligned}$$

The two operators showing up here, $: \partial c c \partial X^\mu \exp(ik \cdot X) : (w)$ and $: \partial^2 c c \exp(ik \cdot X) : (w)$, are linear independent and also non identically zero. Hence, if we want to guarantee that $[Q_{\text{BRST}}, V^a(w, \epsilon, k)] = 0$ we must set the coefficients in front of the two operator to zero. As λ^a, α' are both non zero, and we cannot guarantee that $\epsilon_\mu : \partial c c \partial X^\mu \exp(ik \cdot X) : (w)$ is zero, the only option is to set to zero,

$$k^2 = k \cdot \epsilon = 0$$

Which is what we wanted to show.

1.B)

We need to show that $V^a(w; \epsilon + ak, k)$, $a \in \mathbb{R}$, is just $V^a(w; \epsilon, k)$ plus an BRST exact operator. What is a BRST exact operator? By the same means of the definition of a BRST closed operator, a BRST exact operator is one such $\mathcal{O}(w, \bar{w}) = [Q_{\text{BRST}}, \mathcal{O}'(w, \bar{w})]$. Let's show this is the case,

$$\begin{aligned}
V^a(w; \epsilon + ak, k) &= \lambda^a \epsilon_\mu : c \partial X^\mu \exp(ik \cdot X) : (w) + a \lambda^a k_\mu : c \partial X^\mu \exp(ik \cdot X) : (w) \\
V^a(w; \epsilon + ak, k) &= V^a(w; \epsilon, k) + a \lambda^a k_\mu : c \partial X^\mu \exp(ik \cdot X) : (w)
\end{aligned} \tag{1.10}$$

The last term in the second line is our interest, notice,

$$\lambda^a k_\mu : c \partial X^\mu \exp(ik \cdot X) : (w) = \lambda^a \oint \frac{dz}{2\pi i} \left(\frac{k_\mu : c \partial X^\mu \exp(ik \cdot X) : (w)}{z-w} + \text{no single simple pole} \right) \tag{1.11}$$

As this term has to be BRST exact, the integrand has to be an OPE of some operator with the BRST current, the choice here of operator is obvious, as we have already an $c \partial X^\mu$, what seems a remainder of the first term of the BRST current, we'll try,

$$\begin{aligned}
: j_{\text{BRST}}(z) : \exp(ik \cdot X) : (w) &:= j_{\text{BRST}}(z) : \exp(ik \cdot X) : (w) \\
&\quad - \frac{2}{\alpha'} : (c \partial X^\mu \overline{\partial X_\mu})(z) \exp(ik \cdot X)(w) : \\
&\quad - \frac{1}{\alpha'} : (c \partial X^\mu \overline{\partial X_\mu})(z) \exp(ik \cdot X)(w) : \\
&\quad + \text{no single simple pole}
\end{aligned}$$

Both second and third terms of the BRST current doesn't have any wick contractions with this operator, hence, do not contribute with meaningful poles,

$$: j_{\text{BRST}}(z) : \exp(ik \cdot X) : (w) := j_{\text{BRST}}(z) : \exp(ik \cdot X) : (w)$$

$$\begin{aligned}
& -\frac{2}{\alpha'} i k^\mu \frac{\alpha'}{2} \partial_z \ln |z-w|^2 : (c \partial X_\mu)(z) \exp(i k \cdot X)(w) : \\
& + \frac{1}{\alpha'} i^2 k^2 \frac{\alpha'^2}{4} (\partial_z \ln |z-w|^2)^2 : c(z) \exp(i k \cdot X)(w) : \\
& + \text{no single simple pole}
\end{aligned}$$

Expanding on Taylor, and also using $k^2 = 0$,

$$\begin{aligned}
j_{\text{BRST}}(z) : \exp(i k \cdot X) : (w) &= \frac{i k^\mu}{z-w} : c \partial X_\mu \exp(i k \cdot X) : (w) \\
&+ \text{no single simple pole}
\end{aligned}$$

Using this result back in eq. (1.11),

$$\begin{aligned}
\lambda^a k_\mu : c \partial X^\mu \exp(i k \cdot X) : (w) &= \lambda^a \oint \frac{dz}{2\pi i} \left(\frac{k_\mu : c \partial X^\mu \exp(i k \cdot X) : (w)}{z-w} + \text{no single simple pole} \right) \\
\lambda^a k_\mu : c \partial X^\mu \exp(i k \cdot X) : (w) &= -i \lambda^a \oint \frac{dz}{2\pi i} j_{\text{BRST}}(z) : \exp(i k \cdot X) : (w) \\
\lambda^a k_\mu : c \partial X^\mu \exp(i k \cdot X) : (w) &= -i \lambda^a [Q_{\text{BRST}}, : \exp(i k \cdot X) : (w)]
\end{aligned}$$

At last, substituting in eq. (1.10),

$$V^a(w; \epsilon + a k, k) = V^a(w; \epsilon, k) - a i \lambda^a [Q_{\text{BRST}}, : \exp(i k \cdot X) : (w)]$$

Exactly what we wanted to show.

1.C)

1.D)

Problem 2

2.A)

2.B)

CANCELED

2.C)

2.D)

Problem 3

3.A)

3.B)

3.C)

3.D)

3.E)

3.F)

A Faddeev-Popov Gauge Fixing

Our Action functional is,

$$S_X + \lambda\chi = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi} \int_M d^2\sigma \sqrt{h} R + \frac{\lambda}{2\pi} \int_{\partial M} ds K \quad (\text{A.1})$$

we would like to define the quantum theory by means of the path integral, that is, we expect that,

$$Z \stackrel{?}{=} \int \mathcal{D}X \mathcal{D}h \exp(-S_X[X, h] - \lambda\chi) \quad (\text{A.2})$$

should give a well defined theory, but, already from A.2 there're several problems that arise, one of them is: *What should be interpreted from the path integral itself? We haven't defined any manifold to our metric h and scalar fields X to live in, also, even if we had defined such, the path integral relies on explicit coordinate points, $\mathcal{D}h = \prod_\sigma dh_{ab}(\sigma)$, which are highly dependent on charts.*

This is a valid claim, our way to avoid it is to *define* $\mathcal{D}h$ to mean: *Sum over all **allowed** two dimensional Riemannian manifolds, and all possible metric structures in these.* Here, **allowed** requires a prescription, which manifolds are or aren't allowed impacts the obtained string theory. Happily, every two dimensional manifold has a definite value for the Euler Characteristic χ , hence, we can sort them out by it,

$$\begin{aligned} Z &\stackrel{?}{=} \sum_{\{M\}_{\text{Met}(M)}} \int \mathcal{D}h \int \mathcal{D}X \exp(-S_X[X, h] - \lambda\chi) \\ Z &\stackrel{?}{=} \sum_{\{\chi\}} \exp(-\lambda\chi) \sum_{\{M_\chi\}_{\text{Met}(M_\chi)}} \int \mathcal{D}h \int \mathcal{D}X \exp(-S_X[X, h]) \end{aligned} \quad (\text{A.3})$$

Where M is to be understood as a two dimensional Riemannian manifold and M_χ is one with Euler Characteristic χ , $\text{Met}(M_\chi)$ is the space of all metrics which can be assigned to M_χ , we have written $\sum_{\{M_\chi\}}$ in the special case of there being more than one manifold with same Euler Characteristic², also, the functional integral over X should be read as integrating over all maps from M_χ to $\mathbb{R}^{1,D-1}$. While this is better defined than before, i.e. not coordinate dependent, we still have a few problems, first, it's know that A.1 has a Gauge Group of $\text{Diff}(M) \times \text{Weyl}(M)$, but, in our second try of a definition of the path integral, we're integrating the metrics over $\text{Met}(M_\chi)$, it's clear that may happen of two elements of $\text{Met}(M_\chi)$ be equivalent under a $\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$ transformation, to put in more clear terms, we're worried if exists $h', h \in \text{Met}(M_\chi)$ such,

$$h'_{ab}(\sigma'(\sigma)) = \exp(2\omega(\sigma)) \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} h_{cd}(\sigma)$$

the existence of those kinds of elements is troublesome, as $\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$ is a infinite dimensional group of redundancies, this means we're over-counting physical configurations by a infinite amount. The solution is to look for an equivalence class of metrics under this Gauge Group action,

$$\mathcal{M}_\chi = \text{Met}(M_\chi) / \text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$$

²As we're interested only in Differentiable Manifolds, more than manifold should read: More than one equivalence class of Differentiable Manifolds.

the equivalence class is to be understood as³,

$$h' \sim h \Leftrightarrow h'_{ab}(\sigma'(\sigma)) = \exp(2\omega(\sigma)) \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} h_{cd}(\sigma)$$

that is, two metrics are the same representative of an element of \mathcal{M}_χ iff they differ by a composition of a Diffeomorphism and Weyl transformation. We'll denote a given composition of a Diffeomorphism followed by a Weyl transformation by ζ ,

$$h' = \zeta \circ h$$

Notice that the set of equivalence class of metrics, or, the set of inequivalent $\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$ metrics \mathcal{M}_χ is highly dependent on the topology of M_χ , for example, for $M_\chi \cong \mathbb{R}^2 \cong \mathbb{C}$, it's trivial, there is just one point in the set \mathcal{M}_χ , in other words, every metric is equivalent, which isn't true for more complex topologies.

Thus, it's possible for us to set up a well defined version of the path integral, just replace $\text{Met}(M_\chi)$ by \mathcal{M}_χ ,

$$Z = \sum_{\{x\}} \exp(-\lambda\chi) \sum_{\{M_\chi\}} \int_{\mathcal{M}_\chi} \mathcal{D}h \int \mathcal{D}X \exp(-S_X[X, h]) \quad (\text{A.4})$$

where the integration is to be understood as by choosing for each equivalence class in \mathcal{M}_χ a representative element in $\text{Met}(M_g)$. While this is a satisfactory definition for the path integral, it feels a little clunky, we rather have an path integral over all the possible metrics — in the sense defined before —, well, this is achievable. First, for each equivalence class of \mathcal{M}_χ elect one representative element of $\text{Met}(M_\chi)$, we'll denote these elements as $\hat{h}(\mathbf{t})$ — here \mathbf{t} is a parametrization of the correspondent equivalence class in \mathcal{M}_χ , we haven't proved here, and won't, but \mathcal{M}_χ is a finite N dimensional manifold, hence, \mathbf{t} is a N -tuple of real numbers —, by construction, these representatives are inequivalent under $\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$, hence,

$$\zeta_1 \circ \hat{h}(\mathbf{t}_1) = \zeta_2 \circ \hat{h}(\mathbf{t}_2) \Leftrightarrow \mathbf{t}_1 = \mathbf{t}_2 \text{ and } \zeta_1 = \zeta_2$$

so that every element in $\text{Met}(M_g)$ can be written as a unique⁴ composition of a given ζ into a given $\hat{h}(\mathbf{t})$. Now, we rewrite the pictorial integral over \mathcal{M}_χ is a more formal way, using the parametrization we just described,

$$\begin{aligned} Z &= \sum_{\{x\}} \exp(-\lambda\chi) \sum_{\{M_\chi\}} \int_{\mathcal{M}_\chi} d^N \mathbf{t} \int \mathcal{D}X \exp\left(-S_X[X, \hat{h}(\mathbf{t})]\right) \\ Z &= \sum_{\{x\}} \exp(-\lambda\chi) \sum_{\{M_\chi\}} \int_{\mathcal{M}_\chi} d^N \mathbf{t} \int_{\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)} \mathcal{D}\zeta \delta(\zeta) \int \mathcal{D}X \exp\left(-S_X[X, \hat{h}(\mathbf{t})]\right) \end{aligned}$$

in the last line we introduced a one by integrating⁵ over the delta functional, as this integral picks only $\zeta = 0$, what should be understood as $\zeta = \text{id}$ in the group, we can deform a little the

³In all charts.

⁴The uniqueness or not depends on a few factors, here we'll always, unless specified otherwise, interpret $\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$ as the group *generated by* all possible compositions of Diffeomorphisms and Weyl transformations, but a element of it, ζ , is not to be interpreted as a unique composition of Diffeomorphism and Weyl factors, as there might be some Diffeomorphism which are equivalent to Weyl transformations, what is indeed true is that every element ζ of the Gauge Group is a unique combination of an element of $\text{Diff}(M_\chi)/\text{Weyl}(M_\chi)$ and an element of $\text{Weyl}(M_\chi)$.

⁵Again, following the same remarks made before, the integral over $\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$ should not be interpreted as integrating over the whole of $\text{Diff}(M_\chi)$ and after integrating over the whole $\text{Weyl}(M_\chi)$, this would for sure be an over-counting, but rather should be interpreted as integrating over the whole group *generated by* compositions of $\text{Diff}(M_\chi)$ and $\text{Weyl}(M_\chi)$, which is equivalent of integrating over the whole $\text{Diff}(M_\chi)/\text{Weyl}(M_\chi)$, and after integrating over the whole $\text{Weyl}(M_\chi)$.

integration to,

$$Z = \sum_{\{\chi\}} \exp(-\lambda\chi) \sum_{\{M_\chi\} \mathcal{M}_\chi} \int d^N \mathbf{t} \int_{\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)} \mathcal{D}\zeta \delta(\zeta) \int \mathcal{D}X \exp\left(-S_X\left[X, \zeta \circ \hat{h}(\mathbf{t})\right]\right) \quad (\text{A.5})$$

This is almost in the form that we would like, notice that we're integrating over the set of representative of the inequivalent metrics, $d^N \mathbf{t}$, and also over the whole group $\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)$, $\mathcal{D}\zeta$, by construction, **every** metric in $\text{Met}(M_\chi)$ can be written uniquely⁶ as,

$$h = \zeta_{\mathbf{t}} \circ \hat{h}(\mathbf{t})$$

in other words, to integrate over $d^N \mathbf{t} \mathcal{D}\zeta$ is to integrate over all metrics of the form $\zeta \circ \hat{h}(\mathbf{t})$, which is to integrate over all metrics $h = \zeta \circ \hat{h}(\mathbf{t})$ in $\text{Met}(M_\chi)$! We cannot yet make this change, due to the presence of an explicit dependence in ζ at the functional delta. We'll eliminate it by means of a change of variable of the functional delta, notice that,

$$\delta\left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\right)$$

picks up just the contribution of $\zeta = 0$, so it's a good candidate for a change of variables,

$$\delta\left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\right) = \delta(\zeta) \left| \text{Det} \left[\frac{\delta}{\delta\zeta} \left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t}) \right) \right]_{\zeta=0} \right|^{-1}$$

let's compute step by step the right-hand side of this equation, as we're only interested in the solution of $\zeta = 0$, what matters is just the connected component to the identity of the Gauge Group, this is parametrized by a function ω related to the Weyl transformation, and a vector field ξ related to the connected component to the identity of the Diffeomorphisms — there is an additional requirement of ξ not generating any transformation which can be undone by a Weyl transformation —, also, for ease of our manipulation, we'll write the expression inside the delta with respect to h instead of $\hat{h}(\mathbf{t})$ ⁷, that is,

$$\delta\left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\right) = \delta(\zeta^{-1} \circ h - h) = \delta(\zeta^{-1}) \left| \text{Det} \left[\frac{\delta}{\delta\zeta^{-1}} (\zeta^{-1} \circ h - h) \right]_{\zeta^{-1}=0} \right|^{-1}$$

one might worry about the ζ^{-1} instead of the ζ , but, the integration measure $\mathcal{D}\zeta$ is formally a Haar measure in the Group, that means it's a group invariant measure, in other words, $\mathcal{D}\zeta^{-1} = \mathcal{D}\zeta$, so that we can forget about the inverse, now,

$$\begin{aligned} [\zeta \circ h]_{ab} &= [h]_{ab} + 2\omega[h]_{ab} + [\mathcal{L}_\xi h]_{ab} + \mathcal{O}(\omega^2, \xi^2, \omega\xi) \\ [\zeta \circ h]_{ab} &= [h]_{ab} + 2\omega[h]_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}(\omega^2, \xi^2, \omega\xi) \end{aligned}$$

of course ∇ here is with respect to the h metric,

$$\begin{aligned} [\zeta \circ h]_{ab} - [h]_{ab} &= 2\omega[h]_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}(\omega^2, \xi^2, \omega\xi) \\ \frac{\delta}{\delta\zeta}([\zeta \circ h]_{ab} - [h]_{ab}) &= \text{????} \end{aligned}$$

⁶With the remarks made before.

⁷We would have to carry out the \mathbf{t} dependence in h also, but, soon it will disappear as matter of uniting the integrals $d^N \mathbf{t} \mathcal{D}\zeta$ so we won't keep track of it anymore.

The ζ derivative actually has two parts, the derivative with respect to ω and the other with respect to ξ , let's do one by one,

$$\left. \frac{\delta}{\delta\omega(\sigma')} ([\zeta \circ h]_{ab} - [h]_{ab})(\sigma) \right|_{\zeta=0} = 2\delta^{(2)}(\sigma - \sigma') h_{ab}(\sigma)$$

and for the ξ ,

$$\left. \frac{\delta}{\delta\xi^c(\sigma')} ([\zeta \circ h]_{ab} - [h]_{ab})(\sigma) \right|_{\zeta=0} = 2h_{c(b}\nabla_a)\delta^{(2)}(\sigma - \sigma')$$

Thus,

$$\begin{aligned} \left. \frac{\delta}{\delta\zeta} ([\zeta \circ h]_{ab} - [h]_{ab}) \right|_{\zeta=0} &= 2\delta^{(2)}(\sigma - \sigma') h_{ab}(\sigma) + 2h_{c(b}\nabla_a)\delta^{(2)}(\sigma - \sigma') \\ \text{Det} \left[\left. \frac{\delta}{\delta\zeta} ([\zeta \circ h]_{ab} - [h]_{ab}) \right|_{\zeta=0} \right] &= \text{Det} [2\delta^{(2)}(\sigma - \sigma') h_{ab}(\sigma) + 2h_{c(b}\nabla_a)\delta^{(2)}(\sigma - \sigma')] \end{aligned}$$

the determinant can be computed by means of path integral of Grassmannian variables,

$$\begin{aligned} \text{Det} [2\delta^{(2)}(\sigma - \sigma') h_{ab} + 2h_{c(b}\nabla_a)\delta^{(2)}(\sigma - \sigma')] &= \int \mathcal{D}b\mathcal{D}c\mathcal{D}d \exp \left(-\frac{1}{2\pi} \int d^2\sigma \sqrt{h} b^{ab} [h_{ab}d + h_{c(b}\nabla_a)c^c] \right) \\ \text{Det} \left[\left. \frac{\delta}{\delta\zeta} ([\zeta \circ h]_{ab} - [h]_{ab}) \right|_{\zeta=0} \right] &= \int \mathcal{D}b\mathcal{D}c\mathcal{D}d \exp (-S_{\text{gh}}[b, c, d, h]) \end{aligned}$$

Substituting all of this back into our path integral,

$$\begin{aligned} Z &= \sum_{\{\chi\}} \exp(-\lambda\chi) \sum_{\{M_\chi\} \mathcal{M}_\chi} \int d^N \mathbf{t} \int_{\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)} \mathcal{D}\zeta \delta(\zeta) \int \mathcal{D}X \exp \left(-S_X[X, \zeta \circ \hat{h}(\mathbf{t})] \right) \\ Z &= \sum_{\{\chi\}} \exp(-\lambda\chi) \sum_{\{M_\chi\} \mathcal{M}_\chi} \int d^N \mathbf{t} \int_{\text{Diff}(M_\chi) \times \text{Weyl}(M_\chi)} \mathcal{D}\zeta \delta(\zeta^{-1} \circ h - h) \int \mathcal{D}X \mathcal{D}b\mathcal{D}c\mathcal{D}d \exp(-S_X - S_{\text{gh}}) \\ Z &= \sum_{\{\chi\}} \exp(-\lambda\chi) \sum_{\{M_\chi\} \text{Met}(M_\chi)} \int \mathcal{D}h \delta(\hat{h} - h) \int \mathcal{D}X \mathcal{D}b\mathcal{D}c\mathcal{D}d \exp(-S_X[X, h] - S_{\text{gh}}[b, c, d, h]) \\ Z &= \int \mathcal{D}h \mathcal{D}X \mathcal{D}b\mathcal{D}c\mathcal{D}d \delta(\hat{h} - h) \exp(-S_X[X, h] - S_{\text{gh}}[b, c, d, h] - \lambda\chi) \end{aligned}$$

where \hat{h} is a family of choices of representatives of the equivalence classes of the Gauge equivalent metrics, of course this choice is dependent on the equivalence class h lies in, so, in a certain sense we have $\hat{h} = \hat{h}[h]$,

$$Z = \int \mathcal{D}h \mathcal{D}X \mathcal{D}b\mathcal{D}c\mathcal{D}d \delta(\hat{h}[h] - h) \exp(-S_X[X, h] - S_{\text{gh}}[b, c, d, h] - \lambda\chi)$$

we express the delta functional in terms of a path integral,

$$\begin{aligned} Z &= \int \mathcal{D}h \mathcal{D}X \mathcal{D}b\mathcal{D}c\mathcal{D}d \mathcal{D}B \exp \left(\frac{i}{4\pi} \int d^2\sigma \sqrt{h} B^{ab} (\hat{h}_{ab}[h] - h_{ab}) \right) \exp(-S_X - S_{\text{gh}} - \lambda\chi) \\ Z &= \int \mathcal{D}h \mathcal{D}X \mathcal{D}b\mathcal{D}c\mathcal{D}d \mathcal{D}B \exp(-S_X[X, h] - S_{\text{gh}}[b, c, d, h] - S_{\text{gf}}[B, h] - \lambda\chi) \end{aligned}$$

where we lastly defined the Gauge Fixing Action. This is the final expression for our path integral with the identifications,

$$S_X[X, h] + \lambda\chi = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi} \int_M d^2\sigma \sqrt{h} R + \frac{\lambda}{2\pi} \int_{\partial M} ds K \quad (\text{A.6a})$$

$$S_{\text{gh}}[b, c, d, h] = \frac{1}{2\pi} \int_M d^2\sigma \sqrt{h} b^{ab} [h_{ab} d + \nabla_a c_b] \quad (\text{A.6b})$$

$$S_{\text{gf}}[B, h] = -\frac{i}{4\pi} \int_M d^2\sigma \sqrt{h} B^{ab} (\hat{h}_{ab}[h] - h_{ab}) \quad (\text{A.6c})$$

B BRST Quantization

B.1 The BRST transformations

Following the action principle derived from the Faddeev-Popov Gauge Fixing A.6, we can describe it's BRST symmetry by the transformations of the *matter fields* under Gauge, we know the following,

$$\begin{aligned} X^\mu(\sigma) &\rightarrow X'^\mu(\sigma'(\sigma)) = X^\mu(\sigma) \\ h_{ab}(\sigma) &\rightarrow h'_{ab}(\sigma'(\sigma)) = \exp(2\omega(\sigma)) \frac{\partial\sigma^c}{\partial\sigma'^a} \frac{\partial\sigma^d}{\partial\sigma'^b} h_{cd}(\sigma) \end{aligned}$$

which have an *infinitesimal* form,

$$\begin{aligned} \delta X^\mu &= \mathcal{L}_\xi X^\mu = \xi^a \partial_a X^\mu \\ \delta h_{ab} &= 2\omega h_{ab} + \mathcal{L}_\xi h_{ab} = 2\omega h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a = 2\omega h_{ab} + \xi^c \partial_c h_{ab} + h_{ac} \partial_b \xi^c + h_{bc} \partial_a \xi^c \end{aligned}$$

the BRST transformation can be obtained from these by the substitution inferred from the Faddeev-Popov gauge fix, that is, $\xi^a \rightarrow i\epsilon c^a$ and $\omega \rightarrow i\epsilon d$, where ϵ is a Grassmannian parametrization of the BRST transformation,

$$\delta_{\text{BRST}} X^\mu = i\epsilon c^a \partial_a X^\mu \quad (\text{B.1a})$$

$$\delta_{\text{BRST}} h_{ab} = 2i\epsilon (dh_{ab} + \nabla_{(a} c_{b)}) = 2i\epsilon dh_{ab} + i\epsilon (c^c \partial_c h_{ab} + h_{ac} \partial_b c^c + h_{bc} \partial_a c^c) \quad (\text{B.1b})$$

while the BRST transformation of the non-matter fields can be obtained by the BRST procedure, which are⁸,

$$\delta_{\text{BRST}} (\sqrt{h} B_{ab}) = 0 \quad (\text{B.1c})$$

$$\delta_{\text{BRST}} (\sqrt{h} b_{ab}) = -\epsilon \sqrt{h} B_{ab} \quad (\text{B.1d})$$

$$\delta_{\text{BRST}} d = i\epsilon c^a \partial_a d \quad (\text{B.1e})$$

$$\delta_{\text{BRST}} c^a = i\epsilon c^b \nabla_b c^a = i\epsilon c^b \partial_b c^a \quad (\text{B.1f})$$

the proof of nilpotency of these transformations is done in Appendix ???. For the BRST quantization to be realized we need two things,

$$\begin{aligned} \delta_{\text{BRST}} (S_X + S_{\text{gh}} + S_{\text{gf}}) &= 0 \\ i\epsilon (S_{\text{gh}} + S_{\text{gf}}) &= \delta_{\text{BRST}} \mathcal{O} \end{aligned}$$

for \mathcal{O} some composition of fields, we'll check both of these, and for this it'll be necessary to know the BRST transformations of a few more fields,

- h^{ab}

$$\begin{aligned} 0 &= \delta_{\text{BRST}} \delta_a^c \\ 0 &= \delta_{\text{BRST}} (h_{ab} h^{bc}) \\ 0 &= h_{ab} \delta_{\text{BRST}} h^{bc} + h^{bc} \delta_{\text{BRST}} h_{ab} \\ h_{ab} \delta_{\text{BRST}} h^{bc} &= -2i\epsilon h^{bc} (dh_{ab} + \nabla_{(a} c_{b)}) \\ h^{da} h_{ab} \delta_{\text{BRST}} h^{bc} &= -2i\epsilon h^{da} h^{bc} (dh_{ab} + \nabla_{(a} c_{b)}) \\ \delta_b^d \delta_{\text{BRST}} h^{bc} &= -2i\epsilon (dh^{dc} + \nabla^{(d} c^{c)}) \\ \delta_{\text{BRST}} h^{dc} &= -2i\epsilon (dh^{dc} + \nabla^{(d} c^{c)}) \end{aligned} \quad (\text{B.2a})$$

⁸The first two equations might look a bit odd, but they are in fact a consequence of normalization of $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(i \int d^2\sigma \sqrt{h} B(\sigma) f(\sigma)\right)$, instead of choosing $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(i \int d^2\sigma B(\sigma) f(\sigma)\right)$.

- $\sqrt{h} = \sqrt{\text{Det}[h_{ab}]}$

$$\begin{aligned}
\delta_{\text{BRST}}\sqrt{h} &= \frac{1}{2} \frac{1}{\sqrt{h}} \delta_{\text{BRST}}(\text{Det}[h_{ab}]) \\
\delta_{\text{BRST}}\sqrt{h} &= \frac{1}{2} \frac{1}{\sqrt{h}} \delta_{\text{BRST}}(\exp(\ln(\text{Det}[h_{ab}]))) \\
\delta_{\text{BRST}}\sqrt{h} &= \frac{1}{2} \frac{1}{\sqrt{h}} \delta_{\text{BRST}}(\exp(\text{Tr}[\ln(h_{ab})])) \\
\delta_{\text{BRST}}\sqrt{h} &= \frac{1}{2} \frac{1}{\sqrt{h}} (\exp(\text{Tr}[\ln(h_{ab})])) \delta_{\text{BRST}}(\text{Tr}[\ln(h_{ab})]) \\
\delta_{\text{BRST}}\sqrt{h} &= \frac{1}{2} \frac{1}{\sqrt{h}} h \text{Tr}[\delta_{\text{BRST}}(\ln(h_{ab}))] \\
\delta_{\text{BRST}}\sqrt{h} &= \frac{1}{2} \sqrt{h} \text{Tr}[h^{ca} \delta_{\text{BRST}} h_{ab}] \\
\delta_{\text{BRST}}\sqrt{h} &= \frac{1}{2} \sqrt{h} h^{ba} \delta_{\text{BRST}} h_{ab} \\
\delta_{\text{BRST}}\sqrt{h} &= i\epsilon \sqrt{h} h^{ba} (dh_{ab} + \nabla_{(a} c_{b)}) \\
\delta_{\text{BRST}}\sqrt{h} &= i\epsilon \sqrt{h} (2d + \nabla_a c^a)
\end{aligned} \tag{B.2b}$$

- $\sqrt{h} h^{ab}$

$$\begin{aligned}
\delta_{\text{BRST}}(\sqrt{h} h^{ab}) &= \delta_{\text{BRST}}(\sqrt{h}) h^{ab} + \sqrt{h} \delta_{\text{BRST}}(h^{ab}) \\
\delta_{\text{BRST}}(\sqrt{h} h^{ab}) &= i\epsilon \sqrt{h} (2d + \nabla_c c^c) h^{ab} - 2i\epsilon \sqrt{h} (dh^{ab} + \nabla^{(a} c^{b)}) \\
\delta_{\text{BRST}}(\sqrt{h} h^{ab}) &= 2i\epsilon \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^{(a} c^{b)} \right)
\end{aligned} \tag{B.2c}$$

So now we can proceed to compute the two conditions, we'll start by the second one, luckily, the BRST procedure already provides a natural candidate for \mathcal{O} ,

$$\begin{aligned}
\delta_{\text{BRST}} \left(-\frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} (\hat{h}^{ab} - h^{ab}) \right) &= -\frac{1}{4\pi} \int d^2\sigma \delta_{\text{BRST}}(\sqrt{h} b_{ab}) (\hat{h}^{ab} - h^{ab}) \\
&\quad - \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} \delta_{\text{BRST}} \hat{h}^{ab} \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} \delta_{\text{BRST}}(h^{ab}) \\
\delta_{\text{BRST}} \left(-\frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} (\hat{h}^{ab} - h^{ab}) \right) &= \epsilon \frac{1}{4\pi} \int d^2\sigma \sqrt{h} B_{ab} (\hat{h}^{ab} - h^{ab}) \\
&\quad + i\epsilon \frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} (dh^{ab} + \nabla^a c^b) \\
\delta_{\text{BRST}} \left(-\frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} (\hat{h}^{ab} - h^{ab}) \right) &= i\epsilon S_{\text{gf}} + i\epsilon S_{\text{gh}} \\
\delta_{\text{BRST}} \left(-\frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} (\hat{h}^{ab} - h^{ab}) \right) &= i\epsilon (S_{\text{gf}} + S_{\text{gh}})
\end{aligned}$$

so what remains to be *checked* is the first condition, let's do it part by part, starting with,

$$\delta_{\text{BRST}} S_X = \frac{1}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b \delta_{\text{BRST}} X_\mu + \frac{1}{4\pi\alpha'} \int_M d^2\sigma \delta_{\text{BRST}}(\sqrt{h} h^{ab}) \partial_a X^\mu \partial_b X_\mu$$

$$\begin{aligned}
\delta_{\text{BRST}} S_X &= \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b [c^c \partial_c X_\mu] + \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^a c^b \right) \partial_a X^\mu \partial_b X_\mu \\
\delta_{\text{BRST}} S_X &= \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu ((\nabla_b c^c) \partial_c X_\mu + c^c \nabla_b \nabla_c X_\mu) \\
&\quad + \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^a c^b \right) \partial_a X^\mu \partial_b X_\mu \\
\delta_{\text{BRST}} S_X &= \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} \partial_a X^\mu \partial_b X_\mu \nabla^a c^b + \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} c^c \partial_a X^\mu \nabla_c \nabla_b X_\mu \\
&\quad + \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^a c^b \right) \partial_a X^\mu \partial_b X_\mu \\
\delta_{\text{BRST}} S_X &= \frac{i\epsilon}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} c^c \partial_a X^\mu \nabla_c \partial_b X_\mu + \frac{i\epsilon}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \nabla_c c^c \partial_a X^\mu \partial_b X_\mu \\
\delta_{\text{BRST}} S_X &= \frac{i\epsilon}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} c^c \nabla_c (h^{ab} \partial_a X^\mu \partial_b X_\mu) + \frac{i\epsilon}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \nabla_c c^c \partial_a X^\mu \partial_b X_\mu \\
\delta_{\text{BRST}} S_X &= \frac{i\epsilon}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} \nabla_c (c^c h^{ab} \partial_a X^\mu \partial_b X_\mu) \\
\delta_{\text{BRST}} S_X &= \frac{i\epsilon}{4\pi\alpha'} \int_M d^2\sigma \partial_c \left(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right) \tag{B.3}
\end{aligned}$$

now,

$$\begin{aligned}
\delta_{\text{BRST}} S_{\text{gh}} &= \delta_{\text{BRST}} \left(\frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [h^{ab} d + \nabla^a c^b] \right) \\
\delta_{\text{BRST}} S_{\text{gh}} &= \delta_{\text{BRST}} \left(\frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} \left[h^{ab} d + \frac{1}{2} (-c^c \partial_c h^{ab} + h^{ac} \partial_c c^b + h^{bc} \partial_c c^a) \right] \right) \\
\delta_{\text{BRST}} S_{\text{gh}} &= \frac{1}{2\pi} \int d^2\sigma \delta_{\text{BRST}} \left(\sqrt{h} b_{ab} \right) [h^{ab} d + \nabla^a c^b] \\
&\quad + \frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}}(h^{ab}) d + h^{ab} \delta_{\text{BRST}}(d)] \\
&\quad - \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}}(c^c) \partial_c h^{ab} + c^c \partial_c \delta_{\text{BRST}}(h^{ab})] \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}}(h^{ac}) \partial_c c^b + h^{ac} \partial_c \delta_{\text{BRST}}(c^b)] \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}}(h^{bc}) \partial_c c^a + h^{bc} \partial_c \delta_{\text{BRST}}(c^a)] \\
\delta_{\text{BRST}} S_{\text{gh}} &= -\frac{\epsilon}{2\pi} \int d^2\sigma \sqrt{h} B_{ab} [h^{ab} d + \nabla^a c^b] \\
&\quad + \frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [-2i\epsilon (dh^{ab} + \nabla^a c^b) d + h^{ab} i\epsilon c^d \partial_d d] \\
&\quad - \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [i\epsilon c^d \partial_d c^c \partial_c h^{ab} + 2i\epsilon c^c \partial_c (dh^{ab} + \nabla^a c^b)] \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [-2i\epsilon (dh^{ac} + \nabla^a c^c) \partial_c c^b + i\epsilon h^{ac} \partial_c (c^d \partial_d c^b)]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [-2i\epsilon (dh^{bc} + \nabla^{(b} c^{c)}) \partial_c c^a + i\epsilon h^{bc} \partial_c (c^d \partial_d c^a)] \\
\delta_{\text{BRST}} S_{\text{gh}} = & -\frac{i}{2\pi} \int d^2\sigma \sqrt{h} B_{ab} (-i\epsilon) [h^{ab} d + \nabla^a c^b] \\
& + \frac{i\epsilon}{\pi} \int d^2\sigma \sqrt{h} b_{ab} \nabla^{(a} c^{b)} d \\
& + \frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [c^d \partial_d c^c \partial_c h^{ab} + 2c^c (d\partial_c h^{ab} + \partial_c \nabla^{(a} c^{b)})] \\
& + \frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [2(dh^{ac} + \nabla^{(a} c^{c)}) \partial_c c^b - h^{ac} \partial_c (c^d \partial_d c^b)] \\
& + \frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [2(dh^{bc} + \nabla^{(b} c^{c)}) \partial_c c^a - h^{bc} \partial_c (c^d \partial_d c^a)] \\
\delta_{\text{BRST}} S_{\text{gh}} = & \frac{i}{2\pi} \int d^2\sigma \sqrt{h} B_{ab} \delta_{\text{BRST}} (\hat{h}^{ab} - h^{ab}) \\
& + \frac{i\epsilon}{\pi} \int d^2\sigma \sqrt{h} b_{ab} \nabla^{(a} c^{b)} d \\
& + \frac{i\epsilon}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} d [-c^c \partial_c h^{ab} + h^{ac} \partial_c c^b + h^{bc} \partial_c c^a] \\
& + \frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [c^d \partial_d c^c \partial_c h^{ab} - h^{bc} \partial_c (c^d \partial_d c^a) - h^{ac} \partial_c (c^d \partial_d c^b)] \\
& + \frac{i\epsilon}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [\nabla^{(a} c^{c)} \partial_c c^b + c^c \partial_c \nabla^{(a} c^{b)} + \nabla^{(b} c^{c)} \partial_c c^a] \\
\delta_{\text{BRST}} S_{\text{gh}} = & -\delta_{\text{BRST}} S_{\text{gf}} \\
& + \frac{i\epsilon}{\pi} \int d^2\sigma \sqrt{h} b_{ab} \nabla^{(a} c^{b)} d \\
& + \frac{i\epsilon}{\pi} \int d^2\sigma \sqrt{h} b_{ab} d \nabla^{(a} c^{b)} \\
& - \frac{i\epsilon}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} \nabla^{[a} (c^d \nabla_d c^{b]}) \\
& + \frac{i\epsilon}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [\nabla^{(a} c^{c)} \nabla_c c^b + c^c \nabla_c \nabla^{(a} c^{b)} + \nabla^{(b} c^{c)} \nabla_c c^a] \\
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) = & -\frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\nabla^a c^d \nabla_d c^b + \nabla^b c^d \nabla_d c^a + c^d \nabla^a \nabla_d c^b + c^d \nabla^b \nabla_d c^a] \\
& + \frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\nabla^a c^c \nabla_c c^b + \nabla^c c^a \nabla_c c^b + \nabla^b c^c \nabla_c c^a + \nabla^c c^b \nabla_c c^a] \\
& + \frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [c^c \nabla_c \nabla^a c^b + c^c \nabla_c \nabla^b c^a] \\
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) = & -\frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [c^c \nabla^a \nabla_c c^b + c^c \nabla^b \nabla_c c^a] \\
& + \frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [c^c \nabla_c \nabla^a c^b + c^c \nabla_c \nabla^b c^a] \\
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) = & -\frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} c^c [R_d{}^{ba}{}_c + R_d{}^{ab}{}_c] c^d \\
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) = & -\frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} c^c c^d [R_d{}^{ba}{}_c + R_d{}^{ab}{}_c] \\
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) = & -\frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} c^c c^d [R_d{}^{ba}{}_c - R_c{}^{ba}{}_d + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d]
\end{aligned}$$

$$\delta_{\text{BRST}}(S_{\text{gh}} + S_{\text{gf}}) = -\frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} c^c c^d [R_{cd}^a{}^b - R_{dc}^a{}^b + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d]$$

$$\delta_{\text{BRST}}(S_{\text{gh}} + S_{\text{gf}}) = -\frac{i\epsilon}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} c^c c^d [R_c{}^{ab}{}_d - R_d{}^{ab}{}_c + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d]$$

$$\delta_{\text{BRST}}(S_{\text{gh}} + S_{\text{gf}}) = 0$$

so that the condition $\delta_{\text{BRST}}(S_X + S_{\text{gh}} + S_{\text{gf}}) = 0$, is equivalent to,

$$\delta_{\text{BRST}}(S_X + S_{\text{gh}} + S_{\text{gf}}) = \frac{i\epsilon}{4\pi\alpha'} \int_M d^2\sigma \partial_c \left(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right)$$

$$\delta_{\text{BRST}}(S_X + S_{\text{gh}} + S_{\text{gf}}) = \frac{i\epsilon}{4\pi\alpha'} \int_{\partial M} ds n_c \sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu = 0$$

where n_c is a inwards unitary normal vector to boundary ∂M . Of course, if we're dealing with the closed string $\partial M = \emptyset$ and the right-hand side is identically zero, the problem only shows up in the open string, as $\partial M \neq \emptyset$, but, in this case we have more than just a boundary, as the boundary exists we have to specify the boundary conditions for our fields, this is telling us what might be a good boundary condition to impose⁹,

$$n_c c^c \Big|_{\partial M} = 0$$

which is what we're going to choose¹⁰, so,

$$\delta_{\text{BRST}}(S_X + S_{\text{gh}} + S_{\text{gf}}) = 0$$

our theory is BRST invariant!

B.2 The BRST Charge

Have proven our theory is BRST invariant, we'll now compute the BRST Charge associated with this symmetry, this is an easy computation, is just necessary to follow the steps needed to derive B.3, but now with ϵ being world-sheet dependent,

$$\delta_{\text{BRST}} S_X = \frac{i}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b [\epsilon c^c \partial_c X_\mu] + \frac{i}{2\pi\alpha'} \int_M d^2\sigma \epsilon \sqrt{h} \left(\frac{1}{2} h^{ab} \nabla_c c^c - \nabla^{(a} c^{b)} \right) \partial_a X^\mu \partial_b X_\mu$$

notice, in the first term, the ∂_b when acting on $c^c \partial_c X_\mu$ will sum-up with the second term to give rise just to B.3, but now with ϵ non-constant, so we have,

$$\delta_{\text{BRST}} S_X = \frac{i}{2\pi\alpha'} \int_M d^2\sigma \partial_b (\epsilon) \sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu + \frac{i}{4\pi\alpha'} \int_M d^2\sigma \epsilon \partial_c \left(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right)$$

$$\delta_{\text{BRST}} S_X = \frac{i}{2\pi\alpha'} \int_M d^2\sigma \partial_b \left[\epsilon \sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu \right]$$

⁹Any other choice here is too strong, the boundary condition must in some way depend on n_a , other wise it'll be just a world-sheet scalar and will be zero everywhere, as is the case if we try to set $h^{ab} \partial_a X^\mu \partial_b X_\mu|_{\partial M} = 0$, other possible choices are $c^c|_{\partial M} = 0$, which again, is too restrictive to give any dynamics to our theory.

¹⁰Contrary to beliefs, this condition cannot be inferred solely from the equations of motion, what can be done is to infer $n^a b_{ab}|_{\partial M} = 0$ from the EoM and then try to mimic a condition for c . But, here we're seeing a way which the boundary condition on c is heavily suggested.

$$\begin{aligned}
& -\frac{i}{2\pi\alpha'} \int_M d^2\sigma \epsilon \partial_b \left[\sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu \right] + \frac{i}{4\pi\alpha'} \int_M d^2\sigma \epsilon \partial_c \left(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right) \\
\delta_{\text{BRST}} S_X &= \frac{i}{2\pi\alpha'} \int_{\partial M} ds \epsilon \sqrt{h} n^a \partial_a X^\mu c^c \partial_c X_\mu \\
& -\frac{i}{2\pi\alpha'} \int_M d^2\sigma \epsilon \partial_c \left[\sqrt{h} h^{ca} \partial_a X^\mu c^b \partial_b X_\mu \right] + \frac{i}{4\pi\alpha'} \int_M d^2\sigma \epsilon \partial_c \left(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right)
\end{aligned}$$

the first integral is zero, either in the closed string, $\partial M = \emptyset$, or in the open string, $n^a \partial_a X|_{\partial M} = 0$,

$$\begin{aligned}
\delta_{\text{BRST}} S_X &= \frac{i}{2\pi\alpha'} \int_M d^2\sigma \partial_b(\epsilon) \sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu + \frac{i}{4\pi\alpha'} \int_M d^2\sigma \epsilon \partial_c \left(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right) \\
\delta_{\text{BRST}} S_X &= \frac{i}{2\pi\alpha'} \int_M d^2\sigma \partial_b \left[\epsilon \sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu \right] \\
& -\frac{i}{2\pi\alpha'} \int_M d^2\sigma \epsilon \partial_b \left[\sqrt{h} h^{ab} \partial_a X^\mu c^c \partial_c X_\mu \right] + \frac{i}{4\pi\alpha'} \int_M d^2\sigma \epsilon \partial_c \left(\sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right) \\
\delta_{\text{BRST}} S_X &= -\frac{i}{2\pi\alpha'} \int_M d^2\sigma \epsilon \partial_c \left[\sqrt{h} h^{ca} \partial_a X^\mu c^b \partial_b X_\mu - \frac{1}{2} \sqrt{h} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right] \\
\delta_{\text{BRST}} S_X &= -\frac{i}{2\pi\alpha'} \int_M d^2\sigma \sqrt{h} \epsilon \nabla_c \left[h^{ca} \partial_a X^\mu c^b \partial_b X_\mu - \frac{1}{2} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right]
\end{aligned}$$

as we go back now to ϵ being a constant, we know the left-hand side is zero, this forces,

$$\begin{aligned}
-\frac{1}{\alpha'} \nabla_c \left[h^{ca} \partial_a X^\mu c^b \partial_b X_\mu - \frac{1}{2} c^c h^{ab} \partial_a X^\mu \partial_b X_\mu \right] &= 0 \\
\nabla_c [c_b T^{cb}] &= 0
\end{aligned} \tag{B.4}$$

as we're not starters, we wrote the last line in a suggestive manner, this is just one piece of the puzzle, remains to vary the other two terms of the total action, this is not as difficult as it seems, as the total standard BRST variation is identically zero, any term that does not have a derivative of ϵ will eventually sum-up to zero,

$$\begin{aligned}
\delta_{\text{BRST}} S_{\text{gh}} &= \delta_{\text{BRST}} \left(\frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [h^{ab} d + \nabla^a c^b] \right) \\
\delta_{\text{BRST}} S_{\text{gh}} &= \delta_{\text{BRST}} \left(\frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} \left[h^{ab} d + \frac{1}{2} (-c^c \partial_c h^{ab} + h^{ac} \partial_c c^b + h^{bc} \partial_c c^a) \right] \right) \\
\delta_{\text{BRST}} S_{\text{gh}} &= \frac{1}{2\pi} \int d^2\sigma \delta_{\text{BRST}} (\sqrt{h} b_{ab}) [h^{ab} d + \nabla^a c^b] \\
& + \frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}} (h^{ab}) d + h^{ab} \delta_{\text{BRST}} (d)] \\
& - \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}} (c^c) \partial_c h^{ab} + c^c \partial_c \delta_{\text{BRST}} (h^{ab})] \\
& + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}} (h^{ac}) \partial_c c^b + h^{ac} \partial_c \delta_{\text{BRST}} (c^b)] \\
& + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [\delta_{\text{BRST}} (h^{bc}) \partial_c c^a + h^{bc} \partial_c \delta_{\text{BRST}} (c^a)]
\end{aligned}$$

$$\begin{aligned}
\delta_{\text{BRST}} S_{\text{gh}} &= -\frac{1}{2\pi} \int d^2\sigma \epsilon \sqrt{h} B_{ab} [h^{ab} d + \nabla^a c^b] \\
&\quad + \frac{1}{2\pi} \int d^2\sigma \sqrt{h} b_{ab} [-2i\epsilon (dh^{ab} + \nabla^{(a} c^{b)}) d + h^{ab} i\epsilon c^d \partial_d d] \\
&\quad - \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [i\epsilon c^d \partial_d c^c \partial_c h^{ab} - 2i\epsilon c^c \partial_c (\epsilon dh^{ab} + \epsilon \nabla^{(a} c^{b)})] \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [-2i\epsilon (dh^{ac} + \nabla^{(a} c^{c)}) \partial_c c^b + i h^{ac} \partial_c (\epsilon c^d \partial_d c^b)] \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} [-2i\epsilon (dh^{bc} + \nabla^{(b} c^{c)}) \partial_c c^a + i h^{bc} \partial_c (\epsilon c^d \partial_d c^a)] \\
\delta_{\text{BRST}} S_{\text{gh}} &= -\delta_{\text{BRST}} S_{\text{gf}} \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} 2i\epsilon c^c \partial_c (\epsilon) [dh^{ab} + \nabla^{(a} c^{b)}] \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} i h^{ac} \partial_c (\epsilon) c^d \partial_d c^b \\
&\quad + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} b_{ab} i h^{bc} \partial_c (\epsilon) c^d \partial_d c^a \\
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) &= \frac{i}{2\pi} \int d^2\sigma \partial_c (\epsilon) \sqrt{h} b_{ab} c^c [dh^{ab} + \nabla^{(a} c^{b)}] \\
&\quad - \frac{i}{4\pi} \int d^2\sigma \partial_c (\epsilon) \sqrt{h} b_{ab} (h^{ac} c^d \partial_d c^b + h^{bc} c^d \partial_d c^a)
\end{aligned}$$

using the boundary conditions $n^a b_{ab}|_{\partial M} = n_a c^a|_{\partial M} = 0$,

$$\begin{aligned}
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) &= \frac{i}{2\pi} \int d^2\sigma \epsilon \sqrt{h} \nabla_c [b_{ab} c^c (dh^{ab} + \nabla^{(a} c^{b)})] \\
&\quad - \frac{i}{2\pi} \int d^2\sigma \epsilon \sqrt{h} \nabla_c [b_{ab} h^{ac} c^d \partial_d c^b] \\
\delta_{\text{BRST}} (S_{\text{gh}} + S_{\text{gf}}) &= \frac{i}{2\pi} \int d^2\sigma \epsilon \sqrt{h} \nabla_c [b_{ab} (c^c dh^{ab} + c^c \nabla^{(a} c^{b)} - h^{ac} c^d \partial_d c^b)]
\end{aligned}$$

B.3 The BRST Nilpotency

Nilpotency of BRST,

$$\begin{aligned}
\delta_1 X^\mu &= i\epsilon_1 c^a \partial_a X^\mu \\
\delta_2 \delta_1 X^\mu &= i\epsilon_1 \delta_2 (c^a) \partial_a X^\mu + i\epsilon_1 c^a \partial_a (\delta_2 (X^\mu)) \\
\delta_2 \delta_1 X^\mu &= i\epsilon_1 (i\epsilon_2) c^c \nabla_c (c^a) \partial_a X^\mu + i\epsilon_1 c^a \partial_a ((i\epsilon_2 c^b \partial_b) X^\mu) \\
\delta_2 \delta_1 X^\mu &= -\epsilon_1 \epsilon_2 c^c \nabla_c (c^a) \partial_a X^\mu + \epsilon_1 \epsilon_2 c^a \partial_a (c^b \partial_b X^\mu) \\
\delta_2 \delta_1 X^\mu &= -\epsilon_1 \epsilon_2 c^c \nabla_c (c^a) \nabla_a X^\mu + \epsilon_1 \epsilon_2 c^a \nabla_a (c^b \nabla_b X^\mu) \\
\delta_2 \delta_1 X^\mu &= -\epsilon_1 \epsilon_2 c^c \nabla_c (c^a) \nabla_a X^\mu + \epsilon_1 \epsilon_2 c^a \nabla_a (c^b) \nabla_b X^\mu + \epsilon_1 \epsilon_2 c^a c^b \nabla_a \nabla_b X^\mu \\
\delta_2 \delta_1 X^\mu &= \epsilon_1 \epsilon_2 c^a c^b \nabla_a \nabla_b X^\mu, \quad \text{using the ghost statistics} \\
\delta_2 \delta_1 X^\mu &= \epsilon_1 \epsilon_2 c^a c^b \nabla_{[a} \nabla_{b]} X^\mu, \quad \text{as } X^\mu \text{ is a world-sheet scalar: } \nabla_a \nabla_b X^\mu = \nabla_b \nabla_a X^\mu \\
\delta_2 \delta_1 X^\mu &= 0
\end{aligned}$$

$$\delta_1 h_{ab} = 2i\epsilon_1 dh_{ab} + i\epsilon_1 (c^c \partial_c h_{ab} + h_{ac} \partial_b c^c + h_{bc} \partial_a c^c)$$

$$\begin{aligned}
\delta_2 \delta_1 h_{ab} &= 2i\epsilon_1 \delta_2(d) h_{ab} + 2i\epsilon_1 d \delta_2(h_{ab}) \\
&\quad + i\epsilon_1 (\delta_2(c^c) \partial_c h_{ab} + c^c \partial_c \delta_2(h_{ab})) \\
&\quad + i\epsilon_1 (\delta_2(h_{ac}) \partial_b c^c + h_{ac} \partial_b \delta_2(c^c)) \\
&\quad + i\epsilon_1 (\delta_2(h_{bc}) \partial_a c^c + h_{bc} \partial_a \delta_2(c^c)) \\
\delta_2 \delta_1 h_{ab} &= 2i\epsilon_1 i\epsilon_2 h_{ab} c^c \partial_c d + 2i\epsilon_1 d 2i\epsilon_2 (dh_{ab} + \nabla_{(a} c_{b)}) \\
&\quad + i\epsilon_1 (i\epsilon_2 c^d \partial_d c^c \partial_c h_{ab} + c^c \partial_c (2i\epsilon_2 (dh_{ab} + \nabla_{(a} c_{b)}))) \\
&\quad + i\epsilon_1 (2i\epsilon_2 (dh_{ac} + \nabla_{(a} c_{c)}) \partial_b c^c + h_{ac} \partial_b (i\epsilon_2 c^d \partial_d c^c)) \\
&\quad + i\epsilon_1 (2i\epsilon_2 (dh_{bc} + \nabla_{(b} c_{c)}) \partial_a c^c + h_{bc} \partial_a (i\epsilon_2 c^d \partial_d c^c)) \\
\delta_2 \delta_1 h_{ab} &= -2\epsilon_1 \epsilon_2 h_{ab} c^c \partial_c d + 4\epsilon_1 \epsilon_2 d \nabla_{(a} c_{b)} \\
&\quad - \epsilon_1 \epsilon_2 c^d \partial_d c^c \partial_c h_{ab} + 2\epsilon_1 \epsilon_2 c^c \partial_c (dh_{ab} + \nabla_{(a} c_{b)}) \\
&\quad - 2\epsilon_1 \epsilon_2 (dh_{ac} + \nabla_{(a} c_{c)}) \partial_b c^c - \epsilon_1 \epsilon_2 h_{ac} \partial_b (c^d \partial_d c^c) \\
&\quad - 2\epsilon_1 \epsilon_2 (dh_{bc} + \nabla_{(b} c_{c)}) \partial_a c^c - \epsilon_1 \epsilon_2 h_{bc} \partial_a (c^d \partial_d c^c) \\
\delta_2 \delta_1 h_{ab} &= 4\epsilon_1 \epsilon_2 d \nabla_{(a} c_{b)} - \epsilon_1 \epsilon_2 c^d \partial_d c^c \partial_c h_{ab} + 2\epsilon_1 \epsilon_2 c^c (d \partial_c h_{ab} + \partial_c \nabla_{(a} c_{b)}) \\
&\quad - 2\epsilon_1 \epsilon_2 (dh_{ac} + \nabla_{(a} c_{c)}) \partial_b c^c - \epsilon_1 \epsilon_2 h_{ac} \partial_b (c^d \partial_d c^c) \\
&\quad - 2\epsilon_1 \epsilon_2 (dh_{bc} + \nabla_{(b} c_{c)}) \partial_a c^c - \epsilon_1 \epsilon_2 h_{bc} \partial_a (c^d \partial_d c^c) \\
\delta_2 \delta_1 h_{ab} &= 4\epsilon_1 \epsilon_2 d \nabla_{(a} c_{b)} - 2\epsilon_1 \epsilon_2 d c^c \partial_c h_{ab} - 2\epsilon_1 \epsilon_2 dh_{ac} \partial_b c^c - 2\epsilon_1 \epsilon_2 dh_{bc} \partial_a c^c \\
&\quad - \epsilon_1 \epsilon_2 c^d \partial_d c^c \partial_c h_{ab} - \epsilon_1 \epsilon_2 h_{bc} \partial_a (c^d \partial_d c^c) - \epsilon_1 \epsilon_2 h_{ac} \partial_b (c^d \partial_d c^c) \\
&\quad - 2\epsilon_1 \epsilon_2 \partial_c \nabla_{(a} c_{b)} c^c - 2\epsilon_1 \epsilon_2 \nabla_{(a} c_{c)} \partial_b c^c - 2\epsilon_1 \epsilon_2 \nabla_{(b} c_{c)} \partial_a c^c \\
\delta_2 \delta_1 h_{ab} &= 4\epsilon_1 \epsilon_2 d \nabla_{(a} c_{b)} - 2\epsilon_1 \epsilon_2 d [c^c \partial_c h_{ab} + h_{ac} \partial_b c^c + h_{bc} \partial_a c^c] \\
&\quad - \epsilon_1 \epsilon_2 [c^d \partial_d c^c \partial_c h_{ab} + h_{bc} \partial_a (c^d \partial_d c^c) + h_{ac} \partial_b (c^d \partial_d c^c)] \\
&\quad - 2\epsilon_1 \epsilon_2 [\partial_c \nabla_{(a} c_{b)} c^c + \nabla_{(a} c_{c)} \partial_b c^c + \nabla_{(b} c_{c)} \partial_a c^c] \\
\delta_2 \delta_1 h_{ab} &= 4\epsilon_1 \epsilon_2 d \nabla_{(a} c_{b)} - 4\epsilon_1 \epsilon_2 d \nabla_{(a} c_{b)} \\
&\quad - 2\epsilon_1 \epsilon_2 \nabla_{[a} (c^c \nabla_{c} c_{b]}) \\
&\quad - 2\epsilon_1 \epsilon_2 [\nabla_c \nabla_{(a} c_{b)} c^c + \nabla_{(a} c_{c)} \nabla_b c^c + \nabla_{(b} c_{c)} \nabla_a c^c] \\
\delta_2 \delta_1 h_{ab} &= -2\epsilon_1 \epsilon_2 [\nabla_{[a} (c^c \nabla_{c} c_{b]}) + \nabla_c \nabla_{(a} c_{b)} c^c + \nabla_{(a} c_{c)} \nabla_b c^c + \nabla_{(b} c_{c)} \nabla_a c^c] \\
\delta_2 \delta_1 h_{ab} &= -\epsilon_1 \epsilon_2 [\nabla_a (c^c \nabla_{c} c_b) + \nabla_b (c^c \nabla_{c} c_a) + \nabla_c \nabla_a c_b c^c + \nabla_c \nabla_b c_a c^c \\
&\quad + \nabla_a c_c \nabla_b c^c + \nabla_b c_c \nabla_a c^c + \nabla_c c_a \nabla_b c^c + \nabla_c c_b \nabla_a c^c] \\
\delta_2 \delta_1 h_{ab} &= -\epsilon_1 \epsilon_2 [c^c \nabla_a \nabla_c c_b + c^c \nabla_b \nabla_c c_a + \nabla_c \nabla_a c_b c^c + \nabla_c \nabla_b c_a c^c \\
&\quad + \nabla_a c^c \nabla_c c_b + \nabla_b c^c \nabla_c c_a + \nabla_a c_c \nabla_b c^c + \nabla_b c_c \nabla_a c^c + \nabla_c c_a \nabla_b c^c + \nabla_c c_b \nabla_a c^c] \\
\delta_2 \delta_1 h_{ab} &= -\epsilon_1 \epsilon_2 [c^c \nabla_a \nabla_c c_b + c^c \nabla_b \nabla_c c_a - c^c \nabla_c \nabla_a c_b - c^c \nabla_c \nabla_b c_a \\
&\quad + \nabla_a c^c \nabla_c c_b + \nabla_b c^c \nabla_c c_a - \nabla_b c^c \nabla_c c_a - \nabla_a c^c \nabla_c c_b] \\
\delta_2 \delta_1 h_{ab} &= -\epsilon_1 \epsilon_2 [c^c \nabla_a \nabla_c c_b + c^c \nabla_b \nabla_c c_a - c^c \nabla_c \nabla_a c_b - c^c \nabla_c \nabla_b c_a \\
&\quad + \nabla_a c^c \nabla_c c_b + \nabla_b c^c \nabla_c c_a - \nabla_b c^c \nabla_c c_a - \nabla_a c^c \nabla_c c_b] \\
\delta_2 \delta_1 h_{ab} &= -\epsilon_1 \epsilon_2 c^c [R^d_{bac} c_d + R^d_{abc} c_d] \\
\delta_2 \delta_1 h_{ab} &= -\epsilon_1 \epsilon_2 c^c c^d [R_{dbac} + R_{dabc}] \\
\delta_2 \delta_1 h_{ab} &= -\frac{1}{2} \epsilon_1 \epsilon_2 c^c c^d [R_{dbac} - R_{cbad} + R_{dabc} - R_{cabd}] \\
\delta_2 \delta_1 h_{ab} &= -\frac{1}{2} \epsilon_1 \epsilon_2 c^c c^d [R_{dbac} - R_{cbad} + R_{bcd a} - R_{bdca}] \\
\delta_2 \delta_1 h_{ab} &= -\frac{1}{2} \epsilon_1 \epsilon_2 c^c c^d [R_{dbac} - R_{cbad} + R_{cbad} - R_{dbac}]
\end{aligned}$$

$$\delta_2 \delta_1 h_{ab} = 0$$