Homework I

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Problem 1

1.A)

The Nambu-Goto Action is given by:

$$S_{\mathrm{NG}} = -\frac{T}{\mathrm{c}} \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_{0}^{\pi} \mathrm{d}\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)} \tag{1.1}$$

Where we made the abbreviations,

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$$

The choice of the static gauge, together with considering the string being stretched only along the X^1 direction can be written as,

$$X^\mu(\tau, \sigma) = (\mathrm{c}\tau, f(\sigma), 0, \cdots, 0), \quad f(0) = 0 \;\&\; f(\pi) = a$$

Let's first compute what are the equations of motion,

$$\begin{aligned}
\delta S_{\mathrm{NG}} &= -\frac{T}{\mathrm{c}} \int \mathrm{d}^2\sigma \, \frac{2\left(\dot{X} \cdot X'\right)\left(\dot{X}^\alpha \delta X'_\alpha + X'^\alpha \delta \dot{X}_\alpha\right) - 2\dot{X}^2 X'^\alpha \delta X'_\alpha - 2X'^2 \dot{X}^\alpha \delta \dot{X}_\alpha}{2\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}} \\
&= -\frac{T}{\mathrm{c}} \int \mathrm{d}^2\sigma \, \frac{\delta \dot{X}_\alpha \left[\left(\dot{X} \cdot X'\right) X'^\alpha - X'^2 \dot{X}^\alpha\right] + \delta X'_\alpha \left[\left(\dot{X} \cdot X'\right) \dot{X}^\alpha - \dot{X}^2 X'^\alpha\right]}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}}
\end{aligned}$$

We define the conjugate momenta as to simplify our expression,

$$\mathcal{P}^{\tau\alpha} = -\frac{T}{\mathrm{c}} \frac{\left(\dot{X} \cdot X'\right) X'^\alpha - X'^2 \dot{X}^\alpha}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}} \tag{1.2}$$

$$\mathcal{P}^{\sigma\alpha} = -\frac{T}{\mathrm{c}} \frac{\left(\dot{X}\cdot X'\right)\dot{X}^{\alpha} - \dot{X}^2 {X'}^{\alpha}}{\sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\cdot\dot{X}\right)\left(X'\cdot X'\right)}} \tag{1.3}$$

So that our variation of the Action is,

$$\begin{aligned}
\delta S_{\mathrm{NG}} &= \int \mathrm{d}^2\sigma \left\{\delta\dot{X}_{\alpha}\mathcal{P}^{\tau\alpha} + \delta {X'}_{\alpha}\mathcal{P}^{\sigma\alpha}\right\}\\
\delta S_{\mathrm{NG}} &= \int \mathrm{d}^2\sigma \left\{\frac{\partial}{\partial\tau}[\delta X_{\alpha}\mathcal{P}^{\tau\alpha}] - \delta X_{\alpha}\frac{\partial}{\partial\tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial\sigma}[\delta X_{\alpha}\mathcal{P}^{\sigma\alpha}] - \delta X_{\alpha}\frac{\partial}{\partial\sigma}\mathcal{P}^{\sigma\alpha}\right\}\\
\delta S_{\mathrm{NG}} &= -\int \mathrm{d}^2\sigma\, \delta X_{\alpha}\left\{\frac{\partial}{\partial\tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial\sigma}\mathcal{P}^{\sigma\alpha}\right\} + \int_0^{\pi} \mathrm{d}\sigma\,[\delta X_{\alpha}\mathcal{P}^{\tau\alpha}]\Big|_{\tau=-\infty}^{\tau=+\infty} + \int_{-\infty}^{+\infty} \mathrm{d}\tau\,[\delta X_{\alpha}\mathcal{P}^{\sigma\alpha}]\Big|_{\sigma=0}^{\sigma=\pi}
\end{aligned}$$

From imposing the Stationary Action Principle, we can easily read out both the Equations of Motion,

$$\frac{\partial}{\partial\tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial\sigma}\mathcal{P}^{\sigma\alpha} = 0 \tag{1.4}$$

And the Boundary Conditions

$$\delta X_{\alpha}\mathcal{P}^{\tau\alpha}\Big|_{\tau=-\infty}^{\tau=+\infty} = 0 = \delta X_{\alpha}\mathcal{P}^{\sigma\alpha}\Big|_{\sigma=0}^{\sigma=\pi} \tag{1.5}$$

With this three equations in hand, we just have to compute if the stretched string in the Static Gauge is a solution of them, first we calculate the derivatives,

$$\dot{X} = (\mathrm{c}, 0, \cdots, 0), \quad X' = (0, f'(\sigma), 0, \cdots, 0) \tag{1.6}$$

So now it's trivial that,

$$\dot{X}\cdot X' = 0, \quad \dot{X}\cdot\dot{X} = -\mathrm{c}^2, \quad X'\cdot X' = {f'}^2 \tag{1.7}$$

Plugging in those in 1.2,1.3:

$$\mathcal{P}^{\tau\alpha} = -\frac{T}{\mathrm{c}}\frac{-{f'}^2\dot{X}^{\alpha}}{\sqrt{\mathrm{c}^2 {f'}^2}} = \frac{T}{\mathrm{c}}f'(1, 0, \cdots, 0) \tag{1.8}$$

$$\mathcal{P}^{\sigma\alpha} = -\frac{T}{\mathrm{c}}\frac{\mathrm{c}^2 {X'}^{\alpha}}{\sqrt{\mathrm{c}^2 {f'}^2}} = -T(0, 1, 0, \cdots, 0) \tag{1.9}$$

From where follows,

$$\begin{aligned}
\frac{\partial}{\partial\tau}\mathcal{P}^{\tau\alpha} = \frac{T}{\mathrm{c}}\frac{\partial f'}{\partial\tau}(1, 0, \cdots, 0) = 0\\
\frac{\partial}{\partial\sigma}\mathcal{P}^{\sigma\alpha} = 0
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial \tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial \sigma}\mathcal{P}^{\sigma\alpha} = 0 + 0 = 0$$

That is, the Equations of Motion, 1.4, are satisfied for this string configuration! Now for the Boundary Conditions — 1.5 —, the first one, is trivially satisfied, that is due to the variations of the target space position, X, to which the Action is variated by, because, the initial and final time configuration of X are fixed given conditions, to change them would mean to solve another problem of initial conditions, so the variation δX must be zero at the initial and final times,

$$\delta X_\alpha \Big|_{\tau=-\infty}^{\tau=+\infty} = 0 \Rightarrow \delta X_\alpha \mathcal{P}^{\tau\alpha}\Big|_{\tau=-\infty}^{\tau=+\infty} = 0$$

What confirms the first Boundary Condition is true. For the second one, let's write the non null contributions to the Boundary Condition,

$$\delta X_\alpha \mathcal{P}^{\sigma\alpha}\Big|_{\sigma=0}^{\sigma=\pi} = \delta X_1 \mathcal{P}^{\sigma 1}\Big|_{\sigma=0}^{\sigma=\pi}$$

This is the case due to all the $\mathcal{P}^{\sigma\alpha}$ components being zero, except for $\alpha = 1$. But we have completely fixed X_1 at the endpoints, as know as the Dirichlet Boundary Conditions

$$X_1(\tau, 0) = 0, \quad X_1(\tau, \pi) = a \Rightarrow \delta X_1\Big|_{\sigma=0}^{\sigma=\pi} = 0$$

Hence,

$$\delta X_\alpha \mathcal{P}^{\sigma\alpha}\Big|_{\sigma=0}^{\sigma=\pi} = \delta X_1 \mathcal{P}^{\sigma 1}\Big|_{\sigma=0}^{\sigma=\pi} = 0$$

Showing that our string configuration do satisfy the Boundary Conditions. There are two more constrains we have to verify, which follow from 1.2,

$$\begin{aligned} \mathcal{P}^{\tau\alpha} {X'}_\alpha &= 0 \\ \mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} + \frac{T^2}{\mathrm{c}^2} X'^2 &= 0 \end{aligned}$$

First let us show that these are the right constrains,

$$\mathcal{P}^{\tau\alpha} {X'}_\alpha = -\frac{T}{\mathrm{c}} \frac{\left(\dot{X}\cdot X'\right) X'\cdot X' - X'^2 \dot{X}\cdot X'}{\sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\cdot\dot{X}\right)\left(X'\cdot X'\right)}} = 0$$

And,

$$\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} = \frac{T^2}{\mathrm{c}^2}\frac{\left(\dot{X}\cdot X'\right)^2 X'^2 + X'^4\dot{X}^2 - 2\left(\dot{X}\cdot X'\right)^2 X'^2}{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\cdot\dot{X}\right)\left(X'\cdot X'\right)}$$

$$\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} = \frac{T^2}{\mathrm{c}^2}X'^2\frac{-\left(\dot{X}\cdot X'\right)^2 + X'^2\dot{X}^2}{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\cdot\dot{X}\right)\left(X'\cdot X'\right)}$$

$$\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} = -\frac{T^2}{\mathrm{c}^2}X'^2 \Rightarrow \mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} + \frac{T^2}{\mathrm{c}^2}X'^2 = 0$$

Now we'll prove that these two constrains are true for our string configuration, this is easy, as we already have computed all the needed vectors, 1.6,1.8,

$$\mathcal{P}^{\tau\alpha}X'_{\alpha} = \frac{T}{\mathrm{c}}f'(1,0,\cdots,0)\cdot(0,f',0,\cdots,0)^{\mathrm{T}} = 0$$

And,

$$\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} = \frac{T^2}{\mathrm{c}^2}f'^2(1,0,\cdots,0)\cdot(-1,0,\cdots,0)^{\mathrm{T}} = -\frac{T^2}{\mathrm{c}^2}f'^2$$

$$\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} = -\frac{T^2}{\mathrm{c}^2}(0,f',\cdots,0)\cdot(0,f',\cdots,0)^{\mathrm{T}} = -\frac{T^2}{\mathrm{c}^2}X'^2$$

$$\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} + \frac{T^2}{\mathrm{c}^2}X'^2 = 0$$

This finishes our confirmation that indeed this string configuration is a proper solution.

1.B)

To evaluate the Nambu-Goto Action in this solution, we just have to make use of 1.7 in 1.1,

$$S_{\text{NG-static}} = -\frac{T}{\mathrm{c}}\int_{-\infty}^{+\infty}\mathrm{d}\tau\int_{0}^{\pi}\mathrm{d}\sigma\,\sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\cdot\dot{X}\right)\left(X'\cdot X'\right)}$$

$$S_{\text{NG-static}} = -\frac{T}{\mathrm{c}}\int_{-\infty}^{+\infty}\mathrm{d}\tau\int_{0}^{\pi}\mathrm{d}\sigma\,\sqrt{\mathrm{c}^2 f'^2} = -T\int_{-\infty}^{+\infty}\mathrm{d}\tau\int_{0}^{\pi}\mathrm{d}\sigma\, f'$$

$$S_{\text{NG-static}} = -T\int_{-\infty}^{+\infty}\mathrm{d}\tau\left(f(\pi) - f(0)\right) = -T\int_{-\infty}^{+\infty}\mathrm{d}\tau\, a$$

If we argue that the Action is of the form,

$$S = \int \mathrm{d}t\,[K - V]$$

Where K is the kinetic energy and V is the potential energy. As in our configuration everything is static, we shouldn't expect any kinetic energy present in the Action/Lagrangian, in other words, all the contribution of the action is solely from the potential energy, thus, making this identification,

$$S_{\text{NG-static}} = -\int_{-\infty}^{+\infty} \mathrm{d}\tau\, Ta = -\int_{-\infty}^{+\infty} \mathrm{d}\tau\, V$$

$$V = Ta$$

This is a hint that T may be interpreted as energy per length, or, the tension of the string.

Problem 2

2.A)

The Polyakov Action is given by,

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\, \sqrt{h}h^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

With h^{ab} being the world-sheet metric, $h = \|\mathrm{Det}[h_{ab}]\|$, and $g_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$ the target space metric. A Poincare transformation of the fields X is,

$$\begin{aligned} X^\mu(\tau,\sigma) &\to \tilde{X}^\mu(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \\ \partial_a X^\mu &\to \partial_a \tilde{X}^\mu = \Lambda^\mu{}_\nu \partial_a X^\nu \end{aligned}$$

With of course Λ satisfying the defining property of a Lorentz transformation,

$$g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

The transformed Action is,

$$\begin{aligned} \tilde{S}_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\, \sqrt{h}h^{ab}g_{\mu\nu}\partial_a \tilde{X}^\mu \partial_b \tilde{X}^\nu \\ \tilde{S}_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\, \sqrt{h}h^{ab}g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \partial_a X^\alpha \partial_b X^\beta \\ \tilde{S}_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\, \sqrt{h}h^{ab}g_{\alpha\beta}\partial_a X^\alpha \partial_b X^\beta = S_{\mathrm{P}} \end{aligned}$$

Hence the Poincare group is indeed a global symmetry of the Action. To obtain the conserved currents we have to first know what are the equations of motion,

$$\begin{aligned} \delta S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left\{\sqrt{h}h^{ab}g_{\mu\nu}2\partial_a X^\mu \delta\partial_b X^\nu + \sqrt{h}\delta h^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\sqrt{h}h_{ab}\delta h^{ab}h^{cd}g_{\mu\nu}\partial_c X^\mu \partial_d X^\nu\right\} \\ \delta S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left\{\sqrt{h}h^{ab}2\partial_a X^\mu \delta\partial_b X_\mu + \sqrt{h}\delta h^{ab}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\sqrt{h}h_{ab}\delta h^{ab}\partial_c X^\mu \partial^c X_\mu\right\} \\ \delta S_{\mathrm{P}} &= -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\, \partial_b\Big[\sqrt{h}\partial^b X^\mu \delta X_\mu\Big] + \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\, \partial_b\Big[\sqrt{h}\partial^b X^\mu\Big]\delta X_\mu \\ &\quad - \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\, \delta h^{ab}\Big[\sqrt{h}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\sqrt{h}h_{ab}\partial_c X^\mu \partial^c X_\mu\Big] \end{aligned}$$

Each of the three terms has to vanish independently, the first of them is just a Boundary Condition,

$$\int \mathrm{d}^2\sigma\, \partial_b\Big[\sqrt{h}\partial^b X^\mu \delta X_\mu\Big] = 0 \tag{2.1}$$

The second gives the equations for X,

$$\partial_a\left[\sqrt{h}\partial^a X^\mu\right] = 0 \tag{2.2}$$

And the last one give the equations for h,

$$\sqrt{h}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\sqrt{h}h_{ab}\partial_c X^\mu \partial^c X_\mu = 0 \tag{2.3}$$

Armed with these, we can consider now just a variation on X, which is a symmetry of the Action, in our case this will be a Poincare transformation,

$$\delta S_{\mathrm{P}} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{h}\partial^b X^\mu \partial_b \delta X_\mu$$
$$\delta S_{\mathrm{P}} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\left\{\partial_b\left[\sqrt{h}\partial^b X^\mu \delta X_\mu\right] - \partial_b\left[\sqrt{h}\partial^b X^\mu\right]\delta X_\mu\right\}$$

Imposing the fields to obey the equations of motion, 2.2, the second term vanishes identically. And also using our already derived result that $\tilde{S}_{\mathrm{P}} = S_{\mathrm{P}} \Rightarrow \delta S_{\mathrm{P}} = 0$, we get the simple expression, for δX being the variation under a Poincare transformation,

$$\delta S_{\mathrm{P}} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu \delta X_\mu\right] = 0$$

In the case of a pure translation, $\delta X = \tilde{X} - X = a$,

$$-\frac{1}{2\pi\alpha'}a_\mu\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu\right] = 0$$

From where we can read the conserved current associated with translations,

$$\mathcal{P}^{b\mu} = -\frac{\sqrt{h}}{2\pi\alpha'}\partial^b X^\mu, \quad \partial_b\mathcal{P}^{b\mu} = 0 \tag{2.4}$$

We can do the same for a Lorentz transformation, $\delta X^\mu = \tilde{X}^\mu - X^\mu = \omega^\mu{}_\nu X_\nu$, with of course $\omega_{\mu\nu} = -\omega_{\nu\mu}$, being the infinitesimal part of the Lorentz transformation, $\Lambda = \mathbb{1} + \omega$,

$$-\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu \omega_{\mu\nu} X^\nu\right] = 0$$
$$-\frac{1}{2\pi\alpha'}\omega_{\mu\nu}\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu X^\nu - \sqrt{h}\partial^b X^\nu X^\mu\right] = 0$$

So that the conserved current associated with Lorentz transformations is,

$$\mathcal{M}^{b\mu\nu} = -\frac{\sqrt{h}}{2\pi\alpha'}\left[X^\mu\partial^b X^\nu - X^\nu\partial^b X^\mu\right] = X^\mu\mathcal{P}^{b\nu} - X^\nu\mathcal{P}^{b\mu}, \quad \partial_b\mathcal{M}^{b\mu\nu} = 0$$

Lastly, the conserved charges that follow from the conserved currents are,

$$P^\mu = \int \mathrm{d}\sigma\,\mathcal{P}^{\tau\mu} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma\,\sqrt{h}\partial^\tau X^\mu \tag{2.5}$$
$$M^{\mu\nu} = \int \mathrm{d}\sigma\,\mathcal{M}^{\tau\mu\nu} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma\,\sqrt{h}[X^\mu\partial^\tau X^\nu - X^\nu\partial^\tau X^\mu] \tag{2.6}$$

2.B)

We now turn to the matter of verifying that the conserved charges derived here do obey the Poincare algebra, for this we'll need the Poisson Brackets, which are defined with respect to $X^\mu(t,\sigma)$, and it's conjugate momentum $\frac{\partial \mathcal{L}}{\partial \partial_\tau X^\mu} = \mathcal{P}^\tau{}_\mu \equiv \Pi_\mu(\tau,\sigma)$, which fortunately we already computed. The metric h does not enter in the Poisson Brackets because it's not dynamical, it has three degrees of freedom, but we also have three gauge redundancies, just enough to make it non-dynamical. The fundamental Poisson Bracket relations are,

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\} = 0, \quad \{\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = 0, \quad \{X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = \delta(\sigma-\sigma')g^{\mu\nu}$$

Just for completeness, we'll rewrite 2.5,2.6 in function of the canonical variables,

$$\begin{aligned} P^\mu &= \int \mathrm{d}\sigma\, \Pi^\mu \\ M^{\mu\nu} &= \int \mathrm{d}\sigma\, [X^\mu \Pi^\nu - X^\nu \Pi^\mu] \end{aligned}$$

We'll start by the $P-P$ — we'll not keep track of the τ dependence in the conserved charges, because, they are conserved. But nevertheless, everything is assumed to be evaluated at equal τ —,

$$\{P^\mu, P^\nu\} = \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \{\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = 0$$

Next the $P-M$,

$$\begin{aligned} \left\{P^\mu, M^{\alpha\beta}\right\} &= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left\{\Pi^\mu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma') - X^\beta(\tau,\sigma')\Pi^\alpha(\tau,\sigma')\right\} \\ &= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left[X^\alpha(\tau,\sigma')\left\{\Pi^\mu(\tau,\sigma), \Pi^\beta(\tau,\sigma')\right\} + \{\Pi^\mu(\tau,\sigma), X^\alpha(\tau,\sigma')\}\Pi^\beta(\tau,\sigma') - (\alpha \leftrightarrow \beta)\right] \\ &= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left[-\Pi^\beta(\tau,\sigma')g^{\mu\alpha}\delta(\sigma-\sigma') - (\alpha \leftrightarrow \beta)\right] \\ &= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left[-\Pi^\beta(\tau,\sigma')g^{\mu\alpha}\delta(\sigma-\sigma') + \Pi^\alpha(\tau,\sigma')g^{\mu\beta}\delta(\sigma-\sigma')\right] \\ &= \int \mathrm{d}\sigma\, \left[\Pi^\alpha(\tau,\sigma)g^{\mu\beta} - \Pi^\beta(\tau,\sigma)g^{\mu\alpha}\right] \\ \left\{P^\mu, M^{\alpha\beta}\right\} &= g^{\mu\beta}P^\alpha - g^{\mu\alpha}P^\beta \end{aligned}$$

And lastly, the $M-M$,

$$\begin{aligned} \left\{M^{\mu\nu}, M^{\alpha\beta}\right\} &= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left\{X^\mu(\tau,\sigma)\Pi^\nu(\tau,\sigma) - X^\nu(\tau,\sigma)\Pi^\mu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma') - X^\beta(\tau,\sigma')\Pi^\alpha(\tau,\sigma')\right\} \\ &= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left\{X^\mu(\tau,\sigma)\Pi^\nu(\tau,\sigma) - X^\nu(\tau,\sigma)\Pi^\mu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\right\} - (\alpha \leftrightarrow \beta) \\ &= \left[\int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left\{X^\mu(\tau,\sigma)\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\right\} - (\alpha \leftrightarrow \beta)\right] - (\mu \leftrightarrow \nu) \end{aligned}$$

Notice that,

$$
\begin{aligned}
&\big\{X^\mu(\tau,\sigma)\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\big\} \\
&= \big\{X^\mu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\big\}\Pi^\nu(\tau,\sigma) + X^\mu(\tau,\sigma)\big\{\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\big\} \\
&= X^\alpha(\tau,\sigma')\big\{X^\mu(\tau,\sigma), \Pi^\beta(\tau,\sigma')\big\}\Pi^\nu(\tau,\sigma) + X^\mu(\tau,\sigma)\{\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\}\Pi^\beta(\tau,\sigma') \\
&= X^\alpha(\tau,\sigma')g^{\mu\beta}\delta(\sigma-\sigma')\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\alpha}\delta(\sigma-\sigma')\Pi^\beta(\tau,\sigma')
\end{aligned}
$$

Using this back in our expression,

$$
\begin{aligned}
\big\{M^{\mu\nu}, M^{\alpha\beta}\big\} &= \left[\int \mathrm{d}\sigma\, X^\alpha(\tau,\sigma)g^{\mu\beta}\Pi^\nu(\tau,\sigma) - \Pi^\mu(\tau,\sigma)g^{\nu\alpha}\Pi^\beta(\tau,\sigma) - (\alpha\leftrightarrow\beta)\right] - (\mu\leftrightarrow\nu) \\
&= \int \mathrm{d}\sigma\, \big[X^\alpha(\tau,\sigma)g^{\mu\beta}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\alpha}\Pi^\beta(\tau,\sigma)\big] \\
&\qquad - \int \mathrm{d}\sigma\, \big[X^\beta(\tau,\sigma)g^{\mu\alpha}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\beta}\Pi^\alpha(\tau,\sigma)\big] - (\mu\leftrightarrow\nu) \\
&= \int \mathrm{d}\sigma\, \big[X^\alpha(\tau,\sigma)g^{\mu\beta}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\alpha}\Pi^\beta(\tau,\sigma)\big] \\
&\qquad - \int \mathrm{d}\sigma\, \big[X^\beta(\tau,\sigma)g^{\mu\alpha}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\beta}\Pi^\alpha(\tau,\sigma)\big] \\
&\qquad - \int \mathrm{d}\sigma\, \big[X^\alpha(\tau,\sigma)g^{\nu\beta}\Pi^\mu(\tau,\sigma) - X^\nu(\tau,\sigma)g^{\mu\alpha}\Pi^\beta(\tau,\sigma)\big] \\
&\qquad + \int \mathrm{d}\sigma\, \big[X^\beta(\tau,\sigma)g^{\nu\alpha}\Pi^\mu(\tau,\sigma) - X^\nu(\tau,\sigma)g^{\mu\beta}\Pi^\alpha(\tau,\sigma)\big]
\end{aligned}
$$

Collecting the terms with same metric index,

$$
\begin{aligned}
\big\{M^{\mu\nu}, M^{\alpha\beta}\big\} &= g^{\mu\beta}\int \mathrm{d}\sigma\, [X^\alpha(\tau,\sigma)\Pi^\nu(\tau,\sigma) - X^\nu(\tau,\sigma)\Pi^\alpha(\tau,\sigma)] \\
&\qquad + g^{\nu\beta}\int \mathrm{d}\sigma\, [X^\mu(\tau,\sigma)\Pi^\alpha(\tau,\sigma) - X^\alpha(\tau,\sigma)\Pi^\mu(\tau,\sigma)] \\
&\qquad + g^{\mu\alpha}\int \mathrm{d}\sigma\, \big[X^\nu(\tau,\sigma)\Pi^\beta(\tau,\sigma) - X^\beta(\tau,\sigma)\Pi^\nu(\tau,\sigma)\big] \\
&\qquad + g^{\nu\alpha}\int \mathrm{d}\sigma\, \big[X^\beta(\tau,\sigma)\Pi^\mu(\tau,\sigma) - X^\mu(\tau,\sigma)\Pi^\beta(\tau,\sigma)\big] \\
\big\{M^{\mu\nu}, M^{\alpha\beta}\big\} &= g^{\mu\beta}M^{\alpha\nu} + g^{\nu\beta}M^{\mu\alpha} + g^{\mu\alpha}M^{\nu\beta} + g^{\nu\alpha}M^{\beta\mu} \\
\big\{M^{\mu\nu}, M^{\alpha\beta}\big\} &= g^{\mu\alpha}M^{\nu\beta} - g^{\mu\beta}M^{\nu\alpha} + g^{\nu\beta}M^{\mu\alpha} - g^{\nu\alpha}M^{\mu\beta}
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
\{P^\mu, P^\nu\} &= 0 \\
\big\{P^\mu, M^{\alpha\beta}\big\} &= g^{\mu\beta}P^\alpha - g^{\mu\alpha}P^\beta \\
\big\{M^{\mu\nu}, M^{\alpha\beta}\big\} &= g^{\mu\alpha}M^{\nu\beta} - g^{\mu\beta}M^{\nu\alpha} + g^{\nu\beta}M^{\mu\alpha} - g^{\nu\alpha}M^{\mu\beta}
\end{aligned} \tag{2.7}
$$

Which is exactly the algebra of the Poincare Group!

Problem 3

3.A)

The Gamma Function can be represented in the complex plane domain, $\mathrm{Re}(s) > 1$, as the following integral,

$$\Gamma(s) = \int_0^\infty \mathrm{d}t \exp(-t) t^{s-1}, \quad \mathrm{Re}(s) > 1 \tag{3.1}$$

Which is also the subset of the complex plane in which this integral converges, of course this representation of the Gamma Function in a open set is sufficient for obtain an analytical continuation to the whole complex plane. Obviously, the integral is invariant under relabeling the dummy variable t, we make the following choice $t \to nt$ — Assuming $n > 0$ —,

$$\Gamma(s) = \int_0^\infty \mathrm{d}(nt) \exp(-nt)(nt)^{s-1}, \quad \mathrm{Re}(s) > 1$$
$$\Gamma(s) = n^s \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$
$$n^{-s}\Gamma(s) = \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$
$$\sum_{n=1}^\infty n^{-s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

The sum in the left-hand side is recognized as the representation for the Zeta Function in the domain $\mathrm{Re}(s) > 1$, which is also the domain of convergence of the sum,

$$\zeta(s) = \sum_{n=1}^\infty n^{-s}, \quad \mathrm{Re}(s) > 1$$

So that,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

About the right-hand side, to be able to exchange the integral and the sum is sufficient that,

$$\int_0^\infty \mathrm{d}t \sum_{n=1}^\infty \left\| \exp(-nt) t^{s-1} \right\| < \infty, \quad \mathrm{Re}(s) > 1$$
$$\int_0^\infty \mathrm{d}t \sum_{n=1}^\infty \exp(-nt) \left\| t^{s-1} \right\| < \infty, \quad \mathrm{Re}(s) > 1$$

$$\int_0^\infty \mathrm{d}t \sum_{n=1}^{\infty} \exp\left(-nt\right) t^{\mathrm{Re}(s)-1} < \infty, \quad \mathrm{Re}(s) > 1$$

The sum now is a simple geometric series, giving,

$$\int_0^\infty \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{\exp\left(t\right) - 1} < \infty, \quad \mathrm{Re}(s) > 1$$

The dangerous behavior that could make the integral diverges is the one at $t \to 0$, an indeed, $\mathrm{Re}(s) > 1$, is sufficient for the convergence of this integral, which can be seen at,

$$\int_0^\epsilon \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{\exp\left(t\right) - 1} \approx \int_0^\epsilon \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{t + \mathcal{O}(t^2)} \approx \int_0^\epsilon t^{\mathrm{Re}(s)-2} = \left. \frac{t^{\mathrm{Re}(s)-1}}{\mathrm{Re}(s) - 1} \right|_0^\epsilon$$

Which shows the integral is really finite at $t \to 0$ with $\mathrm{Re}(s) > 1$, hence, switching the integral and the sum is justified, so,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}t \exp\left(-nt\right) t^{s-1}, \quad \mathrm{Re}(s) > 1$$
$$\zeta(s)\Gamma(s) = \mathrm{d}t \int_0^\infty \sum_{n=1}^{\infty} \exp\left(-nt\right) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

Where again we have the sum of a geometric series, giving,

$$\zeta(s)\Gamma(s) = \int_0^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp\left(t\right) - 1}, \quad \mathrm{Re}(s) > 1$$

3.B)

The objective here is to make an analytical continuation to $\mathrm{Re}(s) > -2$ of the expression found in the later item. First of all, the reason the later expression is only well defined in $\mathrm{Re}(s) > 1$, is due to the divergence of the integrand at $t \to 0$ for $\mathrm{Re}(s) \leq 1$, this is only because $\left(\exp\left(t\right) - 1\right)^{-1}$ has a simple pole at $t = 0$, which is also the only pole of this function, so to get the Laurent series we first find the residue of it,

$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp\left(t\right) - 1}\right) = \left. \frac{t}{\exp\left(t\right) - 1} \right|_{t=0}$$
$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp\left(t\right) - 1}\right) = \left. \frac{t}{t + \mathcal{O}(t^2)} \right|_{t=0}$$
$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp\left(t\right) - 1}\right) = \left. \frac{1}{1 + \mathcal{O}(t)} \right|_{t=0}$$
$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp\left(t\right) - 1}\right) = 1$$

As this is the only pole, we get a Laurent series starting as,

$$\frac{1}{\exp{(t)}-1}=\frac{1}{t}+\mathcal{O}\big(t^0\big)$$

To get the following terms we just make a trivial Taylor series of the function $(\exp{(t)}-1)^{-1}-t^{-1}$

$$\begin{aligned}
\left[\frac{1}{\exp{(t)}-1}-\frac{1}{t}\right]\bigg|_0 &= \frac{1+t-\exp{(t)}}{t[\exp{(t)}-1]}\bigg|_0\\
\left[\frac{1}{\exp{(t)}-1}-\frac{1}{t}\right]\bigg|_0 &= \frac{-\frac{t^2}{2}+\mathcal{O}(t^3)}{t[t+\mathcal{O}(t^2)]}\bigg|_0\\
\left[\frac{1}{\exp{(t)}-1}-\frac{1}{t}\right]\bigg|_0 &= \frac{-\frac{t^2}{2}+\mathcal{O}(t^3)}{t^2[1+\mathcal{O}(t)]}\bigg|_0\\
\left[\frac{1}{\exp{(t)}-1}-\frac{1}{t}\right]\bigg|_0 &= -\frac{1}{2}
\end{aligned}$$

In other words,

$$\frac{1}{\exp{(t)}-1}=\frac{1}{t}-\frac{1}{2}+\mathcal{O}(t)$$

The next term of the series will be,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\exp{(t)}-1}-\frac{1}{t}\right]\bigg|_0 &= \frac{1}{t^2}-\frac{\exp{(t)}}{[\exp{(t)}-1]^2}\\
&= \frac{\exp{(t)}+\exp{(-t)}-2-t^2}{t^2[\exp{(t)}+\exp{(-t)}-2]}\bigg|_0\\
&= \frac{2\frac{t^4}{4!}+\mathcal{O}(t^6)}{t^2[t^2+\mathcal{O}(t^4)]}\bigg|_0\\
&= \frac{1}{12}\frac{t^4+\mathcal{O}(t^6)}{t^4[1+\mathcal{O}(t^2)]}\bigg|_0\\
&= \frac{1}{12}
\end{aligned}$$

So up to first order we have,

$$\frac{1}{\exp{(t)}-1}=\frac{1}{t}-\frac{1}{2}+\frac{t}{12}+\mathcal{O}\big(t^2\big) \tag{3.2}$$

Why have we done this? Because we do can soften the behavior of the integrand near $t\to 0$ if we subtract leading terms of the expansion of $(\exp{(t)}-1)^{-1}$, each leading term that we subtract, is equivalent to gaining a power of t in the numerator, which does soften the behavior near $t\to 0$, but also makes it worse in the region $t\to\infty$, and as our only problem is related with the small t region, we can divide the integral in two parts,

$$\zeta(s)\Gamma(s)=\int_0^1 \mathrm{d}t\,\frac{t^{s-1}}{\exp{(t)}-1}+\int_1^\infty \mathrm{d}t\,\frac{t^{s-1}}{\exp{(t)}-1},\quad \mathrm{Re}(s)>1$$

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} + \frac{1}{t} - \frac{1}{2} + \frac{t}{12}\right] + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}, \quad \mathrm{Re}(s) > 1$$

Where we simply added and subtracted the leading terms of the expansion, the integral of the last three of them is trivial and can be done to give,

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] + \int_0^1 \mathrm{d}t\left[t^{s-2} - \frac{t^{s-1}}{2} + \frac{t^s}{12}\right] + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}$$

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}$$

Just what we wanted.

3.C)

Naively, this last expression should be well defined only for $\mathrm{Re}(s) > 1$, let's see this term by term, starting by the last one,

$$\int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}$$

This is finite for all s, as it is exponentially decaying and is bounded in the integration interval, this term is well defined for all s. The next three ones are,

$$\frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)}$$

Also these are well defined in the whole complex plane, with three poles at $s = -1, 0, 1$. Finally we have,

$$\int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right]$$

The only potential not well defined behavior that can occur is near $t = 0$, but we have already developed a series expansion for the expression in brackets, 3.2, that means, near the critical value of $t = 0$, the integrand goes like,

$$\int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] \approx \int_0^1 \mathrm{d}t\, t^{s-1}\mathcal{O}(t^2) \approx \int_0^1 \mathrm{d}t\, t^{s+1} = \left.\frac{t^{s+2}}{s+2}\right|_0^1$$

This is well defined as long as $\mathrm{Re}(s) > -2$. Hence, the expression,

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp{(t)}-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp{(t)}-1} \tag{3.3}$$

Is well defined as long as $\mathrm{Re}(s) > -2$. One might worry about the poles, but, these are the natural structure of $\zeta(s)\Gamma(s)$, to be well defined does not mean to don't have poles, but means the representation can be assigned a number in an unique manner. What is left to us now is to find the values $\zeta(0), \zeta(-1)$, notice that our representation has a poles in both these values of the argument, in fact these poles are structures of $\Gamma(s)$, and not $\zeta(s)$. That the Gamma Function indeed has poles in those values can be seen from,

$$\Gamma(s+1) = s\Gamma(s) \Rightarrow \begin{cases} \Gamma(0) & = \frac{\Gamma(1)}{0} \\ \Gamma(-1) & = \frac{\Gamma(0)}{-1} \end{cases}$$

And because the poles in our representation are just simple poles, they could not have been poles also in ζ, as the two functions are multiplying if there were a pole in $\zeta(0), \zeta(-1)$ they would have been apparent in our representation as double poles. Due to the absence of those, the poles at $s = 0, -1$ are indeed only due to the Gamma Function. This guarantees us that $\zeta(0), \zeta(-1)$ are both finite, and to determine those we just need to evaluate the residue of the expression. First, the residue of the Gamma Function,

$$\begin{aligned} \mathrm{Res}_{s=0}(\Gamma(s)) &= s\Gamma(s)\Big|_{s=0} = \Gamma(s+1)\Big|_{s=0} = \Gamma(1) = 1 \\ \mathrm{Res}_{s=-1}(\Gamma(s)) &= (s+1)\Gamma(s)\Big|_{s=-1} = \frac{(s+1)s\Gamma(s)}{s}\Big|_{s=-1} = \frac{\Gamma(s+2)}{s}\Big|_{s=-1} = -1 \end{aligned}$$

As we argued that $\zeta(0), \zeta(-1)$ should be finite, what will happen is that when we multiply ζ by Γ, the residues of the poles of the Gamma Function will be multiplied by the value of the Zeta Function at that point, that is,

$$\mathrm{Res}_{s=0}(\zeta(s)\Gamma(s)) = \zeta(0)\mathrm{Res}_{s=0}(\Gamma(s))$$

But, as can be seen directly from 3.3, the only contribution for the residue at $s = 0$ will be by $-\frac{1}{2s}$, as all the other terms are finite at $s = 0$, thus,

$$\begin{aligned} \mathrm{Res}_{s=0}(\zeta(s)\Gamma(s)) &= -\frac{1}{2} = \zeta(0)\mathrm{Res}_{s=0}(\Gamma(s)) = \zeta(0) \\ \zeta(0) &= -\frac{1}{2} \end{aligned}$$

Analogously we have,

$$\mathrm{Res}_{s=-1}(\zeta(s)\Gamma(s)) = \zeta(-1)\mathrm{Res}_{s=-1}(\Gamma(s))$$

Again, as we discussed previously, all the terms are finite at $s = -1$, except for $\frac{1}{12(s+1)}$, hence, the residue will be,

$$\begin{aligned}\mathrm{Res}_{s=-1}(\zeta(s)\Gamma(s)) = \frac{1}{12} &= \zeta(-1)\mathrm{Res}_{s=-1}(\Gamma(s)) = -\zeta(-1)\\ \zeta(-1) &= -\frac{1}{12}\end{aligned}$$

As desired.

Problem 4

4.A)

First, one remark, we're going to derive everything in this problem for the **open** string with Neumann Boundary Conditions at both ends. To get the classical solution we just have to solve both 2.2 and 2.3. Of course, we do know the Polyakov Action has Diff×Weyl gauge freedom, hence, none of the components of the metric h are in fact dynamical, because by fixing this gauge freedom we actually fix all the components of h, what makes of 2.3 not an equation of motion for h, but merely a set of constrains regarding the choice of gauge for h. But, fixing all three components of the metric isn't the only possible choice for picking a gauge, another option is to fix τ, and two of the components of h, as we have in total 3 gauge degrees of freedom. We could in principle assign any function to τ, but what seems to be the optimal choice is to work in the Light-cone gauge. First, for any target space vector we define,

$$A^{\pm} = \frac{1}{\sqrt{2}}\left(A^0 + A^1\right)$$

And now we state the Light-cone gauge condition, which is the pining τ condition, is then given by,

$$X^+(\tau, \sigma) = 2\alpha' p^+ \tau$$

Where p^+ is the light-cone component of the conserved charge associated with target space translations — the momentum, 2.5 —. This condition do not allow for any more reparametrizations of τ, as,

$$\begin{aligned} X'^+(\tau', \sigma') &= X^+(\tau, \sigma) \\ 2\alpha' p^+ \tau' &= 2\alpha' p^+ \tau \end{aligned}$$

Which states the only reparametrization compatible with the light-cone gauge condition is $\tau'(\tau, \sigma) = \tau$. Hence, from our initial freedom we can only make σ reparametrizations now. One way of fixing this remaining reparametrization, is to choose σ proportional to the energy carried by the string, actually, we're going to choose a reparametrization such that the energy density in the string, at fixed τ, is independent of σ. One might suspect that the energy density is given by $\mathcal{P}^{\tau 0}$, as in 2.4, but, in the light-cone coordinates, the one that has the role of energy density is $\mathcal{P}^{\tau +}$, we'll choose a sigma parametrization such that $\partial_\sigma \mathcal{P}^{\tau +} = 0$, to see how can this be done we take the transformation of this object under a reparametrization, $\tau' = \tau, \sigma' = \sigma'(\sigma, \tau)$, as,

$$\begin{aligned} \mathcal{P}^{\tau +} &= -\frac{\sqrt{h}\partial^\tau X^+}{2\pi\alpha'} = -\frac{\sqrt{h}h^{\tau\tau}}{2\pi\alpha'} 2\alpha' p^+ = -\sqrt{h}h^{\tau\tau}\frac{p^+}{\pi} \\ \mathcal{P}'^{\tau +}(\tau, \sigma') &= -\frac{\partial\sigma}{\partial\sigma'}\sqrt{h}h^{\tau\tau}\frac{p^+}{\pi} = \frac{\partial\sigma}{\partial\sigma'}\mathcal{P}^{\tau +} \end{aligned}$$

Hence, given $\mathcal{P}'^{\tau +}(\tau, \sigma')$, we can choose $\sigma = \sigma(\tau, \sigma')$, so that $\partial_\sigma \mathcal{P}^{\tau +} = 0$, as long as,

$$\frac{\partial}{\partial\sigma}\left[\frac{\partial\sigma'}{\partial\sigma}\mathcal{P}'^{\tau +}\right] = 0 \Rightarrow \frac{\partial^2\sigma'}{\partial\sigma^2}\mathcal{P}'^{\tau +} + \left(\frac{\partial\sigma'}{\partial\sigma}\right)^2 \frac{\partial\mathcal{P}'^{\tau +}}{\partial\sigma'} = 0$$

This is simply a differential equation which can be solved. But this really fixes all the σ reparametrization? Let's see if it's possible to make further changes of $\sigma \to \sigma''$ preserving this condition,

$$\frac{\partial}{\partial\sigma}\left[\frac{\partial\sigma''}{\partial\sigma}\frac{\partial\sigma'}{\partial\sigma''}\right]\mathcal{P}'^{\tau+} + \left(\frac{\partial\sigma'}{\partial\sigma''}\frac{\partial\sigma''}{\partial\sigma}\right)^2\frac{\partial\mathcal{P}'^{\tau+}}{\partial\sigma'} = 0$$
$$\frac{\partial^2\sigma''}{\partial\sigma^2}\frac{\partial\sigma'}{\partial\sigma''}\mathcal{P}'^{\tau+} + \left(\frac{\partial\sigma''}{\partial\sigma}\right)^2\left[\frac{\partial^2\sigma'}{\partial\sigma''^2}\mathcal{P}'^{\tau+} + \left(\frac{\partial\sigma'}{\partial\sigma''}\right)^2\frac{\partial\mathcal{P}'^{\tau+}}{\partial\sigma'}\right] = 0$$

Imposing that σ'' also satisfy the parametrization condition,

$$\frac{\partial^2\sigma''}{\partial\sigma^2}\frac{\partial\sigma'}{\partial\sigma''}\mathcal{P}'^{\tau+} = 0 \Rightarrow \frac{\partial^2\sigma''}{\partial\sigma^2} = 0 \Rightarrow \sigma'' = a\sigma + b$$

Our condition of constancy of $\mathcal{P}^{\tau+}$ in sigma, has a residual affine reparametrization, which we can use to set $\sigma = 0$ in one of the ends of the string, and $\sigma = \pi$ in the other one. This completely fixes the σ parametrization. What remains now is to fix the Weyl redundancy, as necessarily h_{ab} has a inverse, $\mathrm{Det}[h_{ab}] \neq 0$, thus, by a Weyl transformation is always possible to make $\mathrm{Det}[h_{ab}] = -1$ — We're using Lorentzian signature —, which will be our choice of fixing the Weyl redundancy. There is no more room for any transformation now, hence, the gauge is fully fixed. Notice, as $\mathcal{P}^{\tau+}$ is constant in σ, we have the following,

$$\begin{aligned} p^+ &= \int_0^\pi \mathrm{d}\sigma\, \mathcal{P}^{\tau+} = \mathcal{P}^{\tau+}\pi = -p^+\sqrt{h}h^{\tau\tau} \\ -1 &= h^{\tau\tau} \end{aligned}$$

Furthermore, using the equation of motion 2.2 at $\mu = +$, we get,

$$\begin{aligned} 0 &= \partial_\tau\left[h^{\tau\sigma}\partial_\sigma X^+ h^{\tau\tau}\partial_\tau X^+\right] + \partial_\sigma\left[h^{\sigma\sigma}\partial_\sigma X^+ + h^{\sigma\tau}\partial_\tau X^+\right] \\ 0 &= \partial_\tau h^{\tau\tau} + \partial_\sigma h^{\sigma\tau} \\ 0 &= \partial_\sigma h^{\sigma\tau} \end{aligned}$$

But, using the Neumann Boundary conditions 2.1 at $\mu = +$,

$$\begin{aligned} 0 &= \partial^\sigma X^+\Big|_{\sigma=0}^{\sigma=\pi} \\ 0 &= h^{\sigma\tau}\partial_\tau X^+ + h^{\sigma\sigma}\partial_\sigma X^+\Big|_{\sigma=0}^{\sigma=\pi} \\ 0 &= h^{\tau\sigma}\Big|_{\sigma=0}^{\sigma=\pi} \Rightarrow h^{\tau\sigma}\Big|_{\sigma=0} = h^{\tau\sigma}\Big|_{\sigma=\pi} = 0 \end{aligned}$$

Together with $\partial_\sigma h^{\tau\sigma} = 0$, this simply states that $h^{\tau\sigma} = h^{\sigma\tau} \equiv 0$. This, with $h^{\tau\tau} = -1$, and with the determinant condition, says that $h = \mathrm{Diag}\begin{pmatrix}-1 & 1\end{pmatrix}$. That simplifies the equations of motion 2.2 to,

$$\partial_\tau\partial_\tau X^\mu - \partial_\sigma\partial_\sigma X^\mu = \ddot{X}^\mu - X''^\mu = 0$$

And also the consistency conditions 2.3 as,

$$
\begin{aligned}
0 =&\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab}\Big(-\dot{X}^2 + X'^2\Big) \\
0 =& \begin{cases} \partial_\tau X^\mu \partial_\tau X_\mu + \frac{1}{2}\Big(-\dot{X}^2 + X'^2\Big) & = \frac{1}{2}\Big[\dot{X}^2 + X'^2\Big] \\ \partial_\tau X^\mu \partial_\sigma X_\mu & = \dot{X} \cdot X' \\ \partial_\sigma X^\mu \partial_\sigma X_\mu - \frac{1}{2}\Big(-\dot{X}^2 + X'^2\Big) & = \frac{1}{2}\Big[\dot{X}^2 + X'^2\Big] \end{cases}
\end{aligned}
$$

These constrains can be recast as,

$$
\Big(\dot{X} \pm X'\Big)^2 = 0 \tag{4.1}
$$

Lastly, but not less important, we have the Boundary Conditions, 2.1, which take the form,

$$
\partial_\sigma X^\mu \Big|_{\sigma=0} = \partial_\sigma X^\mu \Big|_{\sigma=\pi} = 0
$$

We have now to solve the equation of motion with the right Boundary conditions, for this, is easier to change coordinates to $\sigma^\pm = \tau \pm \sigma$,

$$
\begin{aligned}
0 &= \frac{\partial^2}{\partial \tau^2} X^\mu - \frac{\partial^2}{\partial \sigma^2} X^\mu \\
0 &= \frac{\partial}{\partial \tau}\left[\frac{\partial \sigma^+}{\partial \tau}\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial \sigma^-}{\partial \tau}\frac{\partial}{\partial \sigma^-} X^\mu\right] - \frac{\partial}{\partial \sigma}\left[\frac{\partial \sigma^+}{\partial \sigma}\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial \sigma^-}{\partial \sigma}\frac{\partial}{\partial \sigma^-} X^\mu\right] \\
0 &= \frac{\partial}{\partial \tau}\left[\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial}{\partial \sigma^-} X^\mu\right] - \frac{\partial}{\partial \sigma}\left[\frac{\partial}{\partial \sigma^+} X^\mu - \frac{\partial}{\partial \sigma^-} X^\mu\right] \\
0 &= \left(\frac{\partial}{\partial \sigma^+} + \frac{\partial}{\partial \sigma^-}\right)\left[\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial}{\partial \sigma^-} X^\mu\right] - \left(\frac{\partial}{\partial \sigma^+} - \frac{\partial}{\partial \sigma^-}\right)\left[\frac{\partial}{\partial \sigma^+} X^\mu - \frac{\partial}{\partial \sigma^-} X^\mu\right] \\
0 &= \partial_+ \partial_- X^\mu
\end{aligned}
$$

This is easily solved for,

$$
X^\mu = \frac{1}{2}\big(f^\mu(\sigma^+) + g^\mu(\sigma^-)\big)
$$

Imposing the boundary condition at $\sigma = 0$,

$$
X'^\mu(\tau, \sigma = 0) = \frac{1}{2}\big(f'^\mu(\tau) - g'^\mu(\tau)\big) = 0
$$

This states that g^μ is equal to f^μ apart from a constant, which we'll absorb in the definition of f^μ. Now, the boundary condition on $\sigma = \pi$,

$$
\begin{aligned}
X^\mu &= \frac{1}{2}\big(f^\mu(\sigma^+) + f^\mu(\sigma^-)\big) \\
X'^\mu(\tau, \pi) &= \frac{1}{2}\big(f'^\mu(\tau + \pi) - f'^\mu(\tau - \pi)\big) = 0
\end{aligned}
$$

That is, f'^μ is periodic with period 2π. The most general real function with period 2π is,

$$
\begin{aligned}
f'^\mu(u) &= f_1^\mu + \sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(\alpha_n^{*\mu}\exp\left(\mathrm{i}nu\right) + \alpha_n^\mu \exp\left(-\mathrm{i}nu\right)\right) \\
f^\mu(u) &= f_0^\mu + f_1^\mu u - \mathrm{i}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_n^{*\mu}\exp\left(\mathrm{i}nu\right) - \alpha_n^\mu \exp\left(-\mathrm{i}nu\right)\right)
\end{aligned}
$$

Defining $\alpha_{-n}^\mu = \alpha_n^{*\mu}$,

$$
f^\mu(u) = f_0^\mu + f_1^\mu u + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\exp\left(-\mathrm{i}nu\right)
$$

Now, back in X,

$$
\begin{aligned}
X^\mu &= f_0^\mu + \frac{1}{2}f_1^\mu\left(\sigma^+ + \sigma^-\right) + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\frac{1}{2}\left[\exp\left(-\mathrm{i}n\sigma^+\right) + \exp\left(-\mathrm{i}n\sigma^-\right)\right] \\
X^\mu &= f_0^\mu + f_1^\mu \tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\exp\left(-\mathrm{i}n\tau\right)\frac{1}{2}\left[\exp\left(-\mathrm{i}n\sigma\right) + \exp\left(\mathrm{i}n\sigma\right)\right] \\
X^\mu &= f_0^\mu + f_1^\mu \tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)
\end{aligned}
$$

Setting $f_0^\mu = x_0^\mu$, and also noting that,

$$
\begin{aligned}
p^\mu &= \int_0^\pi \mathrm{d}\sigma\, \mathcal{P}^{\tau\mu} = \int_0^\pi \mathrm{d}\sigma\, \frac{\partial_\tau X^\mu}{2\pi\alpha'} \\
p^\mu &= \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\left[f_1^\mu + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\alpha_n^\mu\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)\right] = \frac{f_1^\mu}{2\alpha'}
\end{aligned}
$$

So that,

$$
X^\mu = x_0^\mu + 2\alpha' p^\mu \tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)
$$

Also is useful to define a additional mode, $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$.

This is not the end of it, because we need to make sure the two constrains are satisfied, namely, 4.1. With the convention of uppercase latin index referring to non-light-cone components, $I = 2, \cdots, D-1$, the constrains are,

$$
\begin{aligned}
0 &= -2\left(\dot{X}^- \pm X'^-\right)\left(\dot{X}^+ \pm X'^+\right) + \left(\dot{X}^I \pm X'^I\right)^2 \\
4\alpha' p^+\left(\dot{X}^- \pm X'^-\right) &= \left(\dot{X}^I \pm X'^I\right)^2 \\
\dot{X}^- \pm X'^- &= \frac{1}{4\alpha' p^+}\left(\dot{X}^I \pm X'^I\right)^2
\end{aligned}
$$

That is, the two constrains implies that X^- is not dynamical, and, the collection of X^I fully determine X^-, apart from a single integration constant, x_0^-. To see this is helpful to note,

$$\begin{aligned}
\dot{X}^\mu &= \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)\\
X' &= -\mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right)\sin\left(n\sigma\right)
\end{aligned}$$

Which implies,

$$\dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) \tag{4.2}$$

So that,

$$\begin{aligned}
\dot{X}^- \pm X'^- &= \frac{1}{4\alpha'\pi}\left(\dot{X}^I \pm X'^I\right)^2\\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{2\alpha'}{4\alpha' p^+}\sum_{p,q\in\mathbb{Z}}\alpha_p^I\alpha_q^I \exp\left(-\mathrm{i}(p+q)(\tau\pm\sigma)\right)\\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{1}{2p^+}\sum_{n,p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right)\\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{1}{p^+}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I\right)\exp\left(-\mathrm{i}n(\tau\pm\sigma)\right)\\
\sqrt{2\alpha'}\alpha_n^- &= \frac{1}{2p^+}\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I = \frac{1}{p^+}L_n^\perp
\end{aligned} \tag{4.3}$$

Hence, all the fourier modes of X^- are completely determined by the transverse modes ones, apart from of course the integration constant x_0^-. In the last passage we also defined the Virasoro modes $L_n^\perp$. This is it, we fully solved the equation of motion with all the constrains and boundary conditions, the degrees of freedom we found are:

$$X^I(\tau,\sigma), p^+, x_0^-$$

From this list of classical degrees of freedom we can start the quantization of our theory! This is done by imposing the canonical commutation relations dictated by the Poisson bracket between the degrees of freedom and their correspondent conjugated momentum, those are given by,

$$\left[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = \mathrm{i}g^{IJ}\delta(\sigma-\sigma'), \quad \left[x_0^-, p^+\right] = -\mathrm{i}$$

Any other commutator between these 4 operators is zero. Of course now, X^- has to be considered as a function of the X^I and $\mathcal{P}^{\tau I}$, so, it has non trivial commutators. The first thing we need to do here is go from these canonical commutation relations, to the commutation relations of the modes α_n^I. The best way of doing this is working with the expression 4.2, and computing the following commutator,

$$\left[\dot{X}^I \pm X'^I, \dot{X}^I \pm X'^I\right]$$

For this is useful to note,

$$\begin{gathered}
\left[X^I(\tau,\sigma), X^J(\tau,\sigma')\right] = 0 \Rightarrow \left[X'^I(\tau,\sigma), X'^J(\tau,\sigma')\right] = 0 \\
\left[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = \mathrm{i}g^{IJ}\delta(\sigma-\sigma') \Rightarrow \left[X'^I(\tau,\sigma), \dot{X}^J(\tau,\sigma')\right] = 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \\
\left[\mathcal{P}^{\tau I}(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = 0 \Rightarrow \left[\dot{X}^I(\tau,\sigma), \dot{X}^J(\tau,\sigma')\right] = 0
\end{gathered}$$

Hence, the non-vanishing contributions are,

$$\begin{aligned}
\left[\left(\dot{X}^I \pm X'^I\right)(\tau,\sigma), \left(\dot{X}^I \pm X'^I\right)(\tau,\sigma')\right] &= \pm\left[\dot{X}^I(\tau,\sigma), X'^I(\tau,\sigma')\right] \pm \left[X'^I(\tau,\sigma), \dot{X}^I(\tau,\sigma')\right] \\
\left[\left(\dot{X}^I \pm X'^I\right)(\tau,\sigma), \left(\dot{X}^I \pm X'^I\right)(\tau,\sigma')\right] &= \mp 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma'}\delta(\sigma'-\sigma) \pm 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \\
\left[\left(\dot{X}^I \pm X'^I\right)(\tau,\sigma), \left(\dot{X}^I \pm X'^I\right)(\tau,\sigma')\right] &= \pm 4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \qquad (4.4)
\end{aligned}$$

And with opposite signs,

$$\begin{aligned}
\left[\left(\dot{X}^I \pm X'^I\right)(\tau,\sigma), \left(\dot{X}^I \mp X'^I\right)(\tau,\sigma')\right] &= \mp\left[\dot{X}^I(\tau,\sigma), X'^I(\tau,\sigma')\right] \pm \left[X'^I(\tau,\sigma), \dot{X}^I(\tau,\sigma')\right] \\
\left[\left(\dot{X}^I \pm X'^I\right)(\tau,\sigma), \left(\dot{X}^I \mp X'^I\right)(\tau,\sigma')\right] &= \pm 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma'}\delta(\sigma'-\sigma) \pm 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \\
\left[\left(\dot{X}^I \pm X'^I\right)(\tau,\sigma), \left(\dot{X}^I \mp X'^I\right)(\tau,\sigma')\right] &= 0 \qquad (4.5)
\end{aligned}$$

But these expressions, and of course X also, are only defined for $\sigma \in [0,\pi]$, as we want to use 4.2 to isolate the fourier modes, we'll need to integrate over 2π, the only option here is to find an extension of this expression to the whole interval $[0, 2\pi]$. As everything is periodic in σ with 2π period, it's sufficient to find an extension to $\sigma \in [-\pi,\pi]$. See that,

$$\begin{aligned}
\left(\dot{X}^I + X'^I\right)(\tau,\sigma) &= \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I \exp\left(-\mathrm{i}n(\tau+\sigma)\right), \quad \sigma\in[0,\pi] \\
\left(\dot{X}^I - X'^I\right)(\tau,\sigma) &= \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I \exp\left(-\mathrm{i}n(\tau-\sigma)\right), \quad \sigma\in[0,\pi]
\end{aligned}$$

Performing a change of variables $\sigma \to -\sigma$ in the second expression,

$$\begin{aligned}
\left(\dot{X}^I + X'^I\right)(\tau,\sigma) &= \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I \exp\left(-\mathrm{i}n(\tau+\sigma)\right), \quad \sigma\in[0,\pi] \\
\left(\dot{X}^I - X'^I\right)(\tau,-\sigma) &= \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I \exp\left(-\mathrm{i}n(\tau+\sigma)\right), \quad \sigma\in[-\pi,0]
\end{aligned}$$

That's interesting, because we found a representation of the modes in the whole domain $[-\pi,\pi]$,

$$A^I(\tau,\sigma)=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\exp\left(-\mathrm{i}n(\tau+\sigma)\right)=\begin{cases}\left(\dot{X}^I+X'^I\right)(\tau,\sigma), & \sigma\in[0,\pi]\\ \left(\dot{X}^I-X'^I\right)(\tau,-\sigma), & \sigma\in[-\pi,0]\end{cases}$$

And of course, as everything where, we still have $A(\tau,\sigma+2\pi)=A(\tau,\sigma)$. Also, by 4.4 we do have, using the $+$ sign,

$$\left[A^I(\tau,\sigma),A^J(\tau,\sigma')\right]=4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'),\quad \sigma,\sigma'\in[0,\pi]$$

By 4.5, we get that when $\sigma\in[0,\pi]$ and $\sigma'\in[-\pi,0]$ this commutator is zero, which is consistent as the Dirac delta is zero in this domain. At last we use 4.4 with both $-$ sign, and $-\sigma,-\sigma'\in[-\pi,0]$, which get us,

$$\left[A^I(\tau,\sigma),A^J(\tau,\sigma')\right]=4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'),\quad \sigma,\sigma'\in[-\pi,0]$$

Putting all together, what we have is,

$$\left[A^I(\tau,\sigma),A^J(\tau,\sigma')\right]=4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'),\quad \sigma,\sigma'\in[-\pi,\pi]$$

And, also as everything is periodic in 2π, this fully determines the commutation relation over $\sigma,\sigma'\in[0,2\pi]$. Inserting now the definition of $A(\tau,\sigma)$,

$$\begin{aligned}
4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') &= 2\alpha'\sum_{n',m'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau+\sigma')}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)}\left[\alpha^I_{m'},\alpha^J_{n'}\right]\\
2\pi\mathrm{i}g^{IJ}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\mathrm{e}^{\mathrm{i}m\sigma} &= \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}m\sigma}\sum_{m',n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\tau(n'+m')}\mathrm{e}^{-\mathrm{i}\sigma'n'-\mathrm{i}\sigma m'}\left[\alpha^I_{m'},\alpha^J_{n'}\right]\\
mg^{IJ}\mathrm{e}^{\mathrm{i}m\sigma'} &= \sum_{m',n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\tau(n'+m')}\mathrm{e}^{-\mathrm{i}\sigma'n'}\delta_{m,m'}\left[\alpha^I_{m'},\alpha^J_{n'}\right]\\
\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}n\sigma'}mg^{IJ}\mathrm{e}^{\mathrm{i}m\sigma'} &= \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}n\sigma'}\sum_{n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\tau(n'+m)}\mathrm{e}^{-\mathrm{i}\sigma'n'}\left[\alpha^I_{m},\alpha^J_{n'}\right]\\
mg^{IJ}\delta_{m+n,0} &= \sum_{n'\in\mathbb{Z}}\delta_{n,n'}\mathrm{e}^{-\mathrm{i}\tau(n'+m)}\left[\alpha^I_{m},\alpha^J_{n'}\right]\\
mg^{IJ}\delta_{m+n,0}\mathrm{e}^{\mathrm{i}\tau(n+m)} &= \left[\alpha^I_m,\alpha^J_n\right]\\
\left[\alpha^I_m,\alpha^J_n\right] &= mg^{IJ}\delta_{m+n,0}
\end{aligned}\tag{4.6}$$

This is not the end, we still have one more commutation relation to get,

$$\begin{aligned}
2\pi\alpha'\mathrm{i}g^{IJ}\delta(\sigma-\sigma') &= \left[X^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]\\
\int_0^\pi\mathrm{d}\sigma\,2\pi\alpha'\mathrm{i}g^{IJ}\delta(\sigma-\sigma') &= \int_0^\pi\mathrm{d}\sigma\left[X^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right],\quad \int_0^\pi\mathrm{d}\sigma\cos(n\sigma)=0,\ n\in\mathbb{Z}\\
2\pi\alpha'\mathrm{i}g^{IJ} &= \pi\left[x_0^I+\sqrt{2\alpha'}\alpha_0^I\tau,\dot{X}^J(\tau,\sigma')\right]
\end{aligned}$$

$$
\begin{aligned}
2\alpha' \mathrm{i} g^{IJ} &= \sqrt{2\alpha'} \sum_{n'\in\mathbb{Z}} \exp\left(-\mathrm{i} n' \tau\right) \cos\left(n'\sigma'\right) \Big\{ \left[x_0^I, \alpha_{n'}^J\right] + \sqrt{2\alpha'}\tau \left[\alpha_0^I, \alpha_{n'}^J\right] \Big\} \\
\sqrt{2\alpha'} \mathrm{i} g^{IJ} &= \sum_{n'\in\mathbb{Z}} \exp\left(-\mathrm{i} n' \tau\right) \cos\left(n'\sigma'\right) \left[x_0^I, \alpha_{n'}^J\right] \\
\int_0^\pi \frac{\mathrm{d}\sigma'}{\pi} \cos\left(n\sigma'\right) \sqrt{2\alpha'} \mathrm{i} g^{IJ} &= \int_0^\pi \frac{\mathrm{d}\sigma'}{\pi} \cos\left(n\sigma'\right) \sum_{n'\in\mathbb{Z}} \exp\left(-\mathrm{i} n' \tau\right) \cos\left(n'\sigma'\right) \left[x_0^I, \alpha_{n'}^J\right] \\
\delta_{n,0} \sqrt{2\alpha'} \mathrm{i} g^{IJ} &= \sum_{n'\in\mathbb{Z}} \exp\left(-\mathrm{i} n' \tau\right) \delta_{n',n} \left[x_0^I, \alpha_{n'}^J\right] \\
\left[x_0^I, \alpha_n^J\right] &= \delta_{n,0} \sqrt{2\alpha'} \mathrm{i} g^{IJ} \exp\left(-\mathrm{i} n\tau\right) = \delta_{n,0} \sqrt{2\alpha'} \mathrm{i} g^{IJ}
\end{aligned}
$$

The last thing we need to discuss before going to the Lorentz generators is about the Virasoro operators, these were defined as,

$$
L_n^\perp = \frac{1}{2} \sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p
$$

We should be aware of possible ambiguities in the ordering of these, as we now the commutation relations 4.6, two alphas fail to commute only if their mode number sum up to 0, but, notice that $n - p + p = 0 \to n = 0$, hence, the only not well defined Virasoro mode is $L_0^\perp$. As the difference between any two ordering prescriptions is always proportional to the identity operator, we'll **define** $L_0^\perp$ to be on the **normal ordered** prescription — All $\alpha_n^I,\ n \geq 0$ need to be to the right of all $\alpha_n^I,\ n < 0$ —, and wherever there is mention to this Virasoro mode we should use the normal ordered one plus an addition undetermined normal ordering constant,

$$
L_0^\perp \to L_0^\perp + a
$$

As an example, in 4.3, with $n = 0$, the quantum version should read,

$$
\begin{aligned}
2\alpha' p^- = \sqrt{2\alpha'} \alpha_0^- &= \frac{1}{p^+} \left(L_0^\perp + a\right) \\
L_0^\perp &= \frac{1}{2} \alpha_0^I \alpha_0^I + \sum_{p\in\mathbb{N}^*} \alpha_{-p}^I \alpha_p^I
\end{aligned}
$$

We can write a manifestly normal ordered form for all n,

$$
L_n^\perp = \frac{1}{2} \sum_{p\geq 0} \alpha_{n-p}^I \alpha_p^I + \frac{1}{2} \sum_{p<0} \alpha_p^I \alpha_{m-p}^I
$$

Using these, every calculation we do is manifestly normal ordered, which will prevent us from making mistakes. As classically we had, $\left(\alpha_n^I\right)^* = \alpha_{-n}^I$, in the quantization we have, $\left(\alpha_n^I\right)^\dagger = \alpha_{-n}^I$. This allows us to conclude that, $\left(L_n^\perp\right)^\dagger = L_{-n}^\perp$. A few more properties we'll need are the commutation relations of the Virasoro modes with all the other objects, we start with,

$$
\left[L_m^\perp, \alpha_n^J\right] = \frac{1}{2} \sum_{p\geq 0} \left[\alpha_{m-p}^I \alpha_p^I, \alpha_n^J\right] + \frac{1}{2} \sum_{p<0} \left[\alpha_p^I \alpha_{m-p}^I, \alpha_n^J\right]
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{p\geq 0}\left\{\alpha^I_{m-p}\big[\alpha^I_p,\alpha^J_n\big]+\big[\alpha^I_{m-p},\alpha^J_n\big]\alpha^I_p\right\}+\frac{1}{2}\sum_{p<0}\left\{\alpha^I_p\big[\alpha^I_{m-p},\alpha^J_n\big]+\big[\alpha^I_p,\alpha^J_n\big]\alpha^I_{m-p}\right\}\\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}}\left\{\alpha^I_{m-p}g^{IJ}p\delta_{p+n,0}+(m-p)\delta_{m-p+n,0}g^{IJ}\alpha^I_p\right\}\\
&= \frac{1}{2}\big\{-\alpha^J_{m+n}n+(m-m-n)\alpha^J_{m+n}\big\}\\
\big[L^\perp_m,\alpha^J_n\big] &= -n\alpha^J_{n+m} \qquad (4.7)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\big[L^\perp_m,x^J_0\big] &= \frac{1}{2}\sum_{p\geq 0}\left[\alpha^I_{m-p}\alpha^I_p,x^J_0\right]+\frac{1}{2}\sum_{p<0}\left[\alpha^I_p\alpha^I_{m-p},x^J_0\right]\\
&= \frac{1}{2}\sum_{p\geq 0}\left\{\alpha^I_{m-p}\big[\alpha^I_p,x^J_0\big]+\big[\alpha^I_{m-p},x^J_0\big]\alpha^I_p\right\}+\frac{1}{2}\sum_{p<0}\left\{\alpha^I_p\big[\alpha^I_{m-p},x^J_0\big]+\big[\alpha^I_p,x^J_0\big]\alpha^I_{m-p}\right\}\\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}}\left\{-\mathrm{i}\sqrt{2\alpha'}g^{IJ}\delta_{p,0}\alpha^I_{m-p}-\mathrm{i}\sqrt{2\alpha'}g^{IJ}\delta_{m-p,0}\alpha^I_p\right\}\\
&= -\mathrm{i}\sqrt{2\alpha'}\frac{1}{2}\big\{\alpha^J_m+\alpha^J_m\big\}\\
\big[L^\perp_m,x^J_0\big] &= -\mathrm{i}\sqrt{2\alpha'}\alpha^J_m
\end{aligned}
$$

And lastly, but not less important, we have to know the commutation relation between the Virasoro modes themselves, this is more subtle, because we defined them being normal ordered, thus, every step of the calculation we have to make sure all terms are normal ordered,

$$
\begin{aligned}
\big[L^\perp_m,L^\perp_n\big] &= \frac{1}{2}\sum_{p\geq 0}\left[\alpha^I_{m-p}\alpha^I_p,L^\perp_n\right]+\frac{1}{2}\sum_{p<0}\left[\alpha^I_p\alpha^I_{m-p},L^\perp_n\right]\\
&= \frac{1}{2}\sum_{p\geq 0}\left\{\alpha^I_{m-p}\big[\alpha^I_p,L^\perp_n\big]+\big[\alpha^I_{m-p},L^\perp_n\big]\alpha^I_p\right\}\\
&\qquad+\frac{1}{2}\sum_{p<0}\left\{\alpha^I_p\big[\alpha^I_{m-p},L^\perp_n\big]+\big[\alpha^I_p,L^\perp_n\big]\alpha^I_{m-p}\right\}\\
&= \frac{1}{2}\sum_{p\geq 0}\left\{p\alpha^I_{m-p}\alpha^I_{p+n}+(m-p)\alpha^I_{n+m-p}\alpha^I_p\right\}\\
&\qquad+\frac{1}{2}\sum_{p<0}\left\{(m-p)\alpha^I_p\alpha^I_{m+n-p}+p\alpha^I_{p+n}\alpha^I_{m-p}\right\}\\
&= \frac{1}{2}\sum_{p\geq 0}(m-p)\alpha^I_{n+m-p}\alpha^I_p+\frac{1}{2}\sum_{p<0}(m-p)\alpha^I_p\alpha^I_{m+n-p}\\
&\qquad+\frac{1}{2}\sum_{p\geq 0}p\alpha^I_{m-p}\alpha^I_{p+n}+\frac{1}{2}\sum_{p<0}p\alpha^I_{p+n}\alpha^I_{m-p}\\
&= \frac{1}{2}\sum_{p\geq 0}(m-p)\alpha^I_{n+m-p}\alpha^I_p+\frac{1}{2}\sum_{p<0}(m-p)\alpha^I_p\alpha^I_{m+n-p}\\
&\qquad+\frac{1}{2}\sum_{p\geq n}(p-n)\alpha^I_{m+n-p}\alpha^I_p+\frac{1}{2}\sum_{p<n}(p-n)\alpha^I_p\alpha^I_{m+n-p}\\
&= \frac{1}{2}\sum_{p\geq 0}(m-p)\alpha^I_{n+m-p}\alpha^I_p+\frac{1}{2}\sum_{p<0}(m-p)\alpha^I_p\alpha^I_{m+n-p}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{p\geq 0}(p-n)\alpha^I_{m+n-p}\alpha^I_p+\frac{1}{2}\sum_{p<0}(p-n)\alpha^I_p\alpha^I_{m+n-p}\\
&+\frac{1}{2}\left(\frac{1}{2}-\frac{n}{2|n|}\right)\left[\sum_{p=n}^{-1}(p-n)\alpha^I_{m+n-p}\alpha^I_p-\sum_{p=n}^{-1}(p-n)\alpha^I_p\alpha^I_{m+n-p}\right]\\
&+\frac{1}{2}\left(\frac{1}{2}+\frac{n}{2|n|}\right)\left[-\sum_{p=0}^{n-1}(p-n)\alpha^I_{m+n-p}\alpha^I_p+\sum_{p=0}^{n-1}(p-n)\alpha^I_p\alpha^I_{m+n-p}\right]\\
=\;&\frac{1}{2}(m-n)\sum_{p\geq 0}\alpha^I_{n+m-p}\alpha^I_p+\frac{1}{2}(m-n)\sum_{p<0}\alpha^I_p\alpha^I_{m+n-p}\\
&+\frac{1}{2}\left(\frac{1}{2}-\frac{n}{2|n|}\right)\left[\sum_{p=n}^{-1}(p-n)\left\{\left[\alpha^I_{m+n-p},\alpha^I_p\right]+\alpha^I_p\alpha^I_{m+n-p}\right\}-\sum_{p=n}^{-1}(p-n)\alpha^I_p\alpha^I_{m+n-p}\right]\\
&+\frac{1}{2}\left(\frac{1}{2}+\frac{n}{2|n|}\right)\left[-\sum_{p=0}^{n-1}(p-n)\alpha^I_{m+n-p}\alpha^I_p+\sum_{p=0}^{n-1}(p-n)\left\{\left[\alpha^I_p,\alpha^I_{m+n-p}\right]+\alpha^I_{m+n-p}\alpha^I_p\right\}\right]\\
=\;&(m-n)L^{\perp}_{m+n}\\
&-\frac{g^{II}}{2}\left(\frac{1}{2}-\frac{n}{2|n|}\right)\sum_{p=n}^{-1}(p-n)p\delta_{m+n,0}+\frac{g^{II}}{2}\left(\frac{1}{2}+\frac{n}{2|n|}\right)\sum_{p=0}^{n-1}(p-n)p\delta_{m+n,0}\\
=\;&(m-n)L^{\perp}_{m+n}\\
&+\frac{D-2}{2}\left(\frac{1}{2}-\frac{n}{2|n|}\right)\sum_{p=0}^{|n|-1}(-p-n)p\delta_{m+n,0}+\frac{D-2}{2}\left(\frac{1}{2}+\frac{n}{2|n|}\right)\sum_{p=0}^{|n|-1}(p-n)p\delta_{m+n,0}\\
=\;&(m-n)L^{\perp}_{m+n}\\
&+\frac{D-2}{2}\delta_{m+n,0}\sum_{p=0}^{|n|-1}p\left[\left(\frac{1}{2}-\frac{n}{2|n|}\right)(-p-n)+\left(\frac{1}{2}+\frac{n}{2|n|}\right)(p-n)\right]\\
=\;&(m-n)L^{\perp}_{m+n}\\
&+\frac{D-2}{2}\delta_{m+n,0}\sum_{p=0}^{|n|-1}p\left[-n+\frac{n}{2|n|}(p+n)+\frac{n}{2|n|}(p-n)\right]\\
=\;&(m-n)L^{\perp}_{m+n}+\frac{D-2}{2}\delta_{m+n,0}\frac{n}{|n|}\sum_{p=0}^{|n|-1}p[p-|n|]
\end{aligned}
$$

In the middle of the calculus we introduced factors of $\frac{1}{2}\left(1\pm\frac{n}{|n|}\right)$ just to account for the two possible cases, $n>0$ and $n<0$ — Of course, the case $n=0$ is trivial due to not being necessary to introduce any other factors to ensure the normal ordering of the expression—. Now, we're going to prove by induction that the value of the sum is,

$$\sum_{p=0}^{|n|-1}p(p-|n|)=\frac{1}{6}\left(|n|-|n|^3\right),\quad |n|\geq 1$$

It's trivial to check it's validity from $|n|=1$, now, suppose it's valid for $|n|=k$,

$$\sum_{p=0}^{k}p(p-k-1)=\sum_{p=0}^{k}p(p-k)-\sum_{p=0}^{k}p$$

$$\sum_{p=0}^{k} p(p-k-1) = \sum_{p=0}^{k-1} p(p-k) - \sum_{p=0}^{k} p$$
$$\sum_{p=0}^{k} p(p-k-1) = \frac{1}{6}\big(k-k^3\big) - \frac{1}{2}k(k+1)$$
$$\sum_{p=0}^{k} p(p-k-1) = \frac{1}{6}\big(-3k^2-2k-k^3\big)$$
$$\sum_{p=0}^{k} p(p-k-1) = \frac{1}{6}\big(k+1-1-3k-3k^2-k^3\big) = \frac{1}{6}\big(k+1-(k+1)^3\big)$$

Which finishes our proof. Hence,

$$\big[L^{\perp}_m, L^{\perp}_n\big] = (m-n)L^{\perp}_{m+n} + \frac{D-2}{2}\delta_{m+n,0}\frac{n}{|n|}\sum_{p=0}^{|n|-1} p[p-|n|]$$
$$\big[L^{\perp}_m, L^{\perp}_n\big] = (m-n)L^{\perp}_{m+n} + \frac{D-2}{12}\delta_{m+n,0}\frac{n}{|n|}\big(|n|-|n|^3\big)$$
$$\big[L^{\perp}_m, L^{\perp}_n\big] = (m-n)L^{\perp}_{m+n} + \frac{D-2}{12}\delta_{m+n,0}\big(n-n^3\big)$$
$$\big[L^{\perp}_m, L^{\perp}_n\big] = (m-n)L^{\perp}_{m+n} + \frac{D-2}{12}\big(m^3-m\big)\delta_{m+n,0}$$

This completes the set of all needed commutation relations. We can now discuss the Lorentz generators in the Light-cone gauge. We have already computed them in 2.6, but of course we have two remarks, neither they are quantum, nor are in the light-cone gauge, about the later, the actual quantum light-cone gauge Lorentz generators **should** satisfy the same algebra as 2.7 in the light-cone coordinates. The failure to met this requirement is related to an anomaly in this global symmetry of the Quantum Poincare Action. And about the former, we'll change the definition accordingly to ensure the quantum generators do satisfy being Hermitian. With that being said, let's evaluate the classical version of 2.6 in the light-cone gauge,

$$\begin{aligned}
M^{\mu\nu} &\overset{?}{=} \frac{1}{2\pi\alpha'}\int_0^{\pi} \mathrm{d}\sigma\,\Big(X^{\mu}\dot{X}^{\nu} - X^{\nu}\dot{X}^{\mu}\Big) \\
&\overset{?}{=} \frac{1}{2\pi\alpha'}\int_0^{\pi} \mathrm{d}\sigma\left(x_0^{\mu} + 2\alpha' p^{\mu}\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^{\mu}}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)\right)\times \\
&\qquad \times\sqrt{2\alpha'}\sum_{m\in\mathbb{Z}}\alpha_m^{\nu}\exp\left(-\mathrm{i}m\tau\right)\cos\left(m\sigma\right) - (\mu\leftrightarrow\nu) \\
&\overset{?}{=} \frac{1}{2\pi\alpha'}\Bigg[\sqrt{2\alpha'}\sum_{m\in\mathbb{Z}} x_0^{\mu}\alpha_m^{\nu}\exp\left(-\mathrm{i}m\tau\right)\pi\delta_{m,0} \\
&\qquad + 2\alpha'\tau\sqrt{2\alpha'}\sum_{m\in\mathbb{Z}} p^{\mu}\alpha_m^{\nu}\exp\left(-\mathrm{i}m\tau\right)\pi\delta_{m,0} \\
&\qquad + \mathrm{i}2\alpha'\sum_{n\in\mathbb{Z}^*}\sum_{m\in\mathbb{Z}}\frac{\alpha_n^{\mu}}{n}\exp\left(-\mathrm{i}n\tau\right)\alpha_m^{\nu}\exp\left(-\mathrm{i}m\tau\right)\frac{\pi}{2}\left(\delta_{m,n}+\delta_{m,-n}\right)\Bigg] - (\mu\leftrightarrow\nu)
\end{aligned}$$

$$M^{\mu\nu} \overset{?}{=} \left[\frac{1}{\sqrt{2\alpha'}} x_0^\mu \alpha_0^\nu + \tau\sqrt{2\alpha'} p^\mu \alpha_0^\nu + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\alpha_n^\mu \alpha_n^\nu \exp\left(-\mathrm{i}(n+m)\tau\right) + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\alpha_n^\mu \alpha_{-n}^\nu\right] - (\mu \leftrightarrow \nu)$$

Employing the before mentioned equality $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$,

$$\begin{aligned}
M^{\mu\nu} &\overset{?}{=} \Bigg[x_0^\mu p^\nu - x_0^\nu p^\mu + \tau\alpha_0^\mu\alpha_0^\nu - \tau\alpha_0^\nu\alpha_0^\mu + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\left(\alpha_n^\mu\alpha_{-n}^\nu - \alpha_n^\nu\alpha_{-n}^\mu\right) \\
&\qquad + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}(\alpha_n^\mu\alpha_n^\nu - \alpha_n^\nu\alpha_n^\mu)\exp\left(-\mathrm{i}2n\tau\right)\Bigg] \\
M^{\mu\nu} &\overset{?}{=} \left[x_0^\mu p^\nu - x_0^\nu p^\mu + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_n^\mu\alpha_{-n}^\nu - \alpha_n^\nu\alpha_{-n}^\mu - \alpha_{-n}^\mu\alpha_n^\nu + \alpha_{-n}^\nu\alpha_n^\mu\right)\right] \\
M^{\mu\nu} &\overset{?}{=} \left[x_0^\mu p^\nu - x_0^\nu p^\mu + \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_n^\mu\alpha_{-n}^\nu - \alpha_n^\nu\alpha_{-n}^\mu\right)\right]
\end{aligned}$$

This is the classical version of the generators in the light-cone gauge. Does it is a good Quantum Lorentz Generator Operator? For a positive answer, it has to be both Hermitian and normal-ordered, let's consider one by one,

$$\begin{aligned}
M^{IJ} &\overset{?}{=} \left[x_0^I p^J - x_0^J p^I + \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_n^I\alpha_{-n}^J - \alpha_n^J\alpha_{-n}^I\right)\right] \\
\left(M^{IJ}\right)^\dagger - M^{IJ} &\overset{?}{=} \left[p^J x_0^I - p^I x_0^J - x_0^I p^J + x_0^J p^I - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_n^J\alpha_{-n}^I - \alpha_n^I\alpha_{-n}^J + \alpha_n^I\alpha_{-n}^J - \alpha_n^J\alpha_{-n}^I\right)\right] \\
\left(M^{IJ}\right)^\dagger - M^{IJ} &\overset{?}{=} \left[\left[p^J, x_0^I\right] - \left[p^I, x_0^J\right]\right] \\
\left(M^{IJ}\right)^\dagger - M^{IJ} &\overset{?}{=} \left[-\mathrm{i}g^{IJ} + \mathrm{i}g^{IJ}\right] = 0
\end{aligned}$$

Yes! And about normal-ordered? No. Both terms inside the sum are reversed normal-ordered, just reversing those, our quantum operator related to the Lorentz Generators is,

$$M^{IJ} = \left[x_0^I p^J - x_0^J p^I - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_{-n}^I\alpha_n^J - \alpha_{-n}^J\alpha_n^I\right)\right]$$

Now, already changing to the normal-ordered one, and looking at Hermiticity,

$$\begin{aligned}
M^{I+} &\overset{?}{=} \left[x_0^I p^+ - x_0^+ p^I - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_{-n}^I\alpha_n^+ - \alpha_{-n}^+\alpha_n^I\right)\right] \\
M^{I+} &\overset{?}{=} x_0^I p^+
\end{aligned}$$

Which is of course Hermitian, hence,

$$M^{I+} = x_0^I p^+$$

By the same reasoning, that is, $x_0^+ = 0 = \alpha_n^+,\ n \neq 0$,

$$M^{-+} \stackrel{?}{=} x_0^- p^+$$

Which fails to be Hermitian, due the canonical commutation relations. One way to avoid this is to symmetrize it,

$$M^{-+} = \frac{1}{2}\left(x_0^- p^+ + p^+ x_0^-\right)$$

Which now is both normal ordered and hermitian. Now,

$$M^{++} \stackrel{?}{=} x_0^+ p^+ = 0$$

Which is expected by the anti-symmetry of the generators, we have another which is zero by the anti-symmetry,

$$M^{--} \stackrel{?}{=} \left[x_0^- p^- - x_0^- p^- - \mathrm{i} \sum_{n \in \mathbb{N}^*} \frac{1}{n}\left(\alpha_{-n}^- \alpha_n^- - \alpha_{-n}^- \alpha_n^-\right)\right] = 0$$

At last, we have the most important generator,

$$M^{-I} \stackrel{?}{=} \left[x_0^- p^I - x_0^I p^- - \mathrm{i} \sum_{n \in \mathbb{N}^*} \frac{1}{n}\left(\alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^-\right)\right]$$

Is it Hermitian? No.

$$\left(M^{-I}\right)^\dagger - M^{-I} \stackrel{?}{=} \left[p^I x_0^- - p^- x_0^I - x_0^- p^I + x_0^I p^- + \mathrm{i} \sum_{n \in \mathbb{N}^*} \frac{1}{n}\left(\alpha_{-n}^I \alpha_n^- - \alpha_{-n}^- \alpha_n^I + \alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^-\right)\right]$$
$$\left(M^{-I}\right)^\dagger - M^{-I} \stackrel{?}{=} \left[\left[p^I, x_0^-\right] - \left[p^-, x_0^I\right]\right]$$
$$\left(M^{-I}\right)^\dagger - M^{-I} \stackrel{?}{=} -\left[p^-, x_0^I\right]$$
$$\left(M^{-I}\right)^\dagger - M^{-I} \stackrel{?}{=} -\frac{1}{\sqrt{2\alpha'}}\left[\alpha_0^-, x_0^I\right]$$
$$\left(M^{-I}\right)^\dagger - M^{-I} \stackrel{?}{=} -\frac{1}{2\alpha' p^+}\left[L_0^\perp + a, x_0^I\right] = \frac{\mathrm{i}}{\sqrt{2\alpha' p^+}}\alpha_0^I$$

Again, if we symmetrize the $x_0^I p^-$ term this issue is resolved,

$$M^{-I} \stackrel{?}{=} x_0^- p^I - \frac{1}{2}\left(x_0^I p^- + p^- x_0^I\right) - \mathrm{i} \sum_{n \in \mathbb{N}^*} \frac{1}{n}\left(\alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^-\right)$$

Is this expression normal-ordered? Yes! Due to the α_n^- being proportional to the Virasoro modes, which are already normal-ordered. Hence, the expression for this Lorentz Generator is, using Virasoro modes,

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I\left(L_0^\perp + a\right) + \left(L_0^\perp + a\right)x_0^I\right) - \frac{\mathrm{i}}{\sqrt{2\alpha' p^+}} \sum_{n \in \mathbb{N}^*} \frac{1}{n}\left(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp\right)$$

4.B)

Problem 5

5.A)

5.B)

5.C)

5.D)

5.E)

Problem 6

6.A)

6.B)

6.C)

6.D)

6.E)

6.F)

6.G)