

2+1 Gravity as a Gauge Theory

Vicente V. Figueira

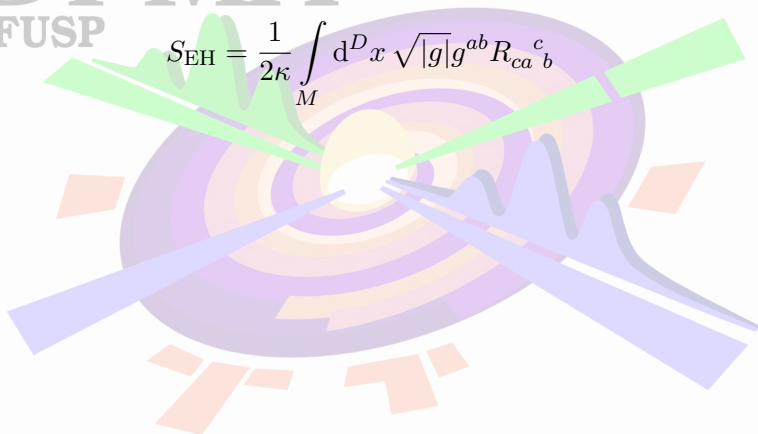
II Agorá Meeting — IFUSP

October 24, 2025

General Relativity

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$$S_{\text{EH}} = \frac{1}{2\kappa} \int_M d^D x \sqrt{|g|} g^{ab} R_{ca}{}^c{}_b$$



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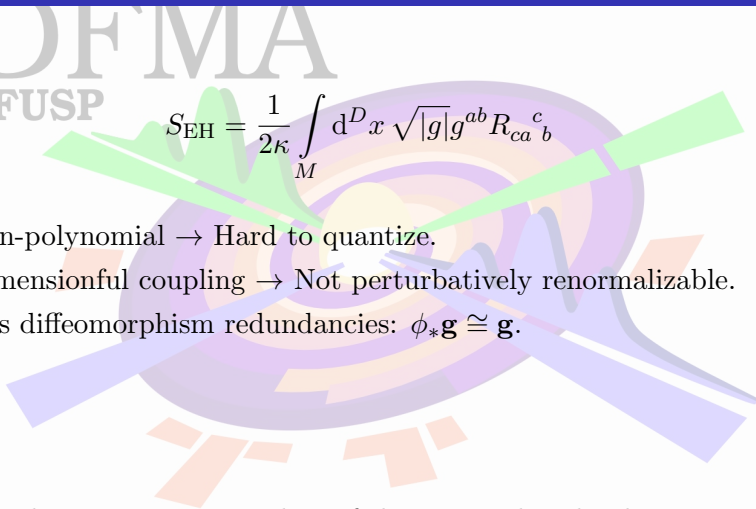
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We know how to quantize a class of theories with redundancies:

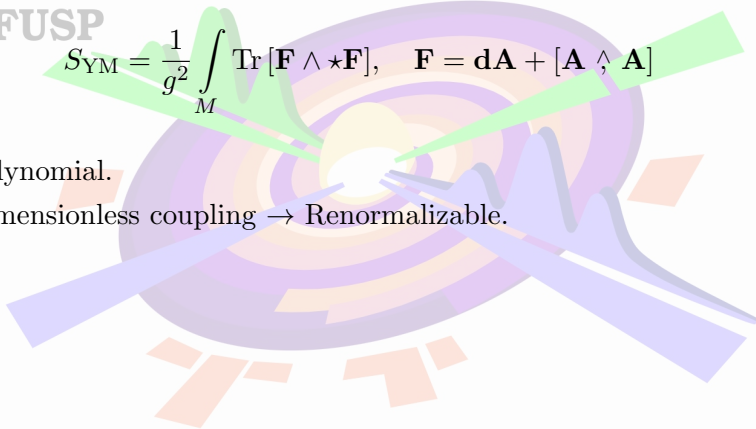
Yang-Mills

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$$S_{\text{YM}} = \frac{1}{g^2} \int_M \text{Tr} [\mathbf{F} \wedge \star \mathbf{F}], \quad \mathbf{F} = d\mathbf{A} + [\mathbf{A} \wedge \mathbf{A}]$$

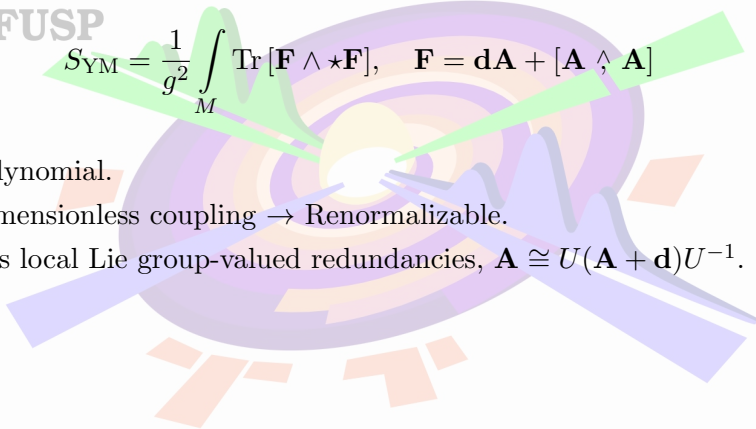
- Polynomial.

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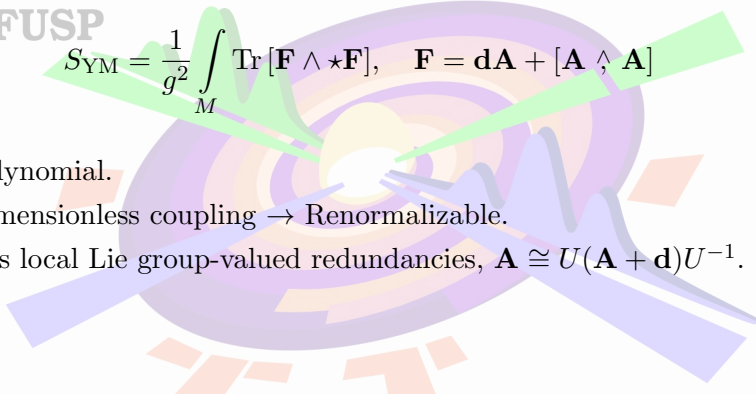
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It's possible to formulate GR as YM theory?

Why $D = 2 + 1$?

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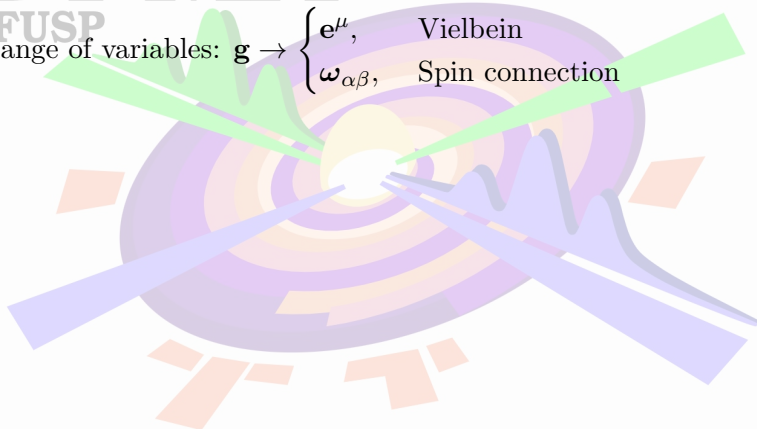
- No local propagating modes:

$$\text{d.o.f.} = \frac{1}{2}D(D-3)$$

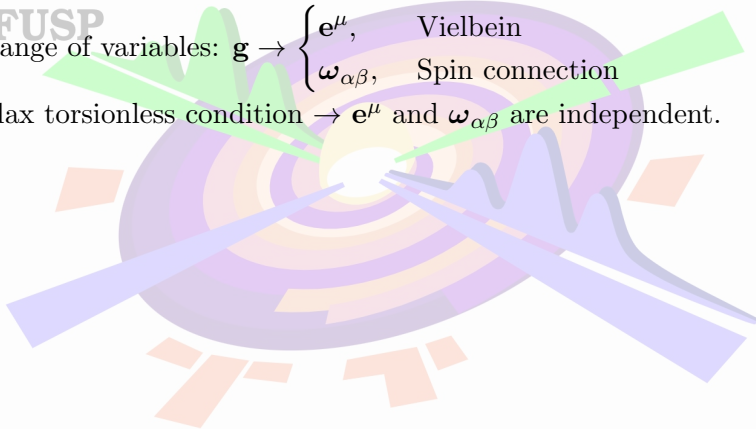
Simpler but non-trivial toy model.

Recover of polynomiality

- Change of variables: $\mathbf{g} \rightarrow \begin{cases} \mathbf{e}^\mu, & \text{Vielbein} \\ \omega_{\alpha\beta}, & \text{Spin connection} \end{cases}$



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Can we write the integrand as a trace?

Recover of dimensionless coupling constant

- For $\text{iso}(2,1)$:

$$\text{Tr}[J_{\alpha\beta}P_{\mu}] = \frac{\lambda}{\kappa}\epsilon_{\alpha\beta\mu}, \quad \text{Tr}[P_{\nu}P_{\mu}] = 0, \quad \text{Tr}[J_{\alpha\beta}J_{\mu\nu}] = 0$$

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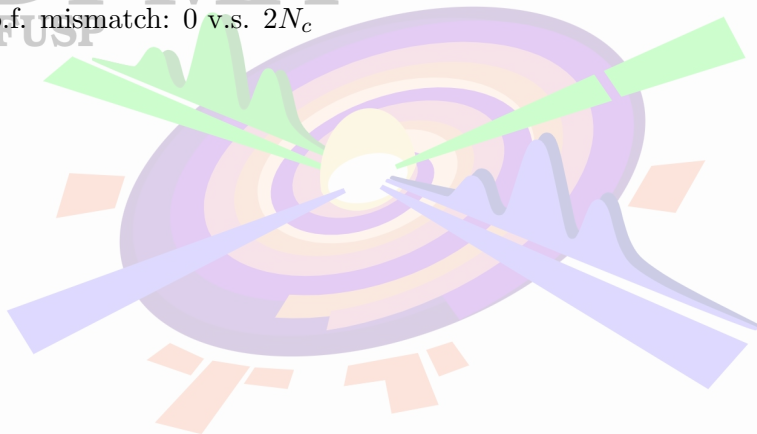
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Not standard YM form...

Chern-Simons

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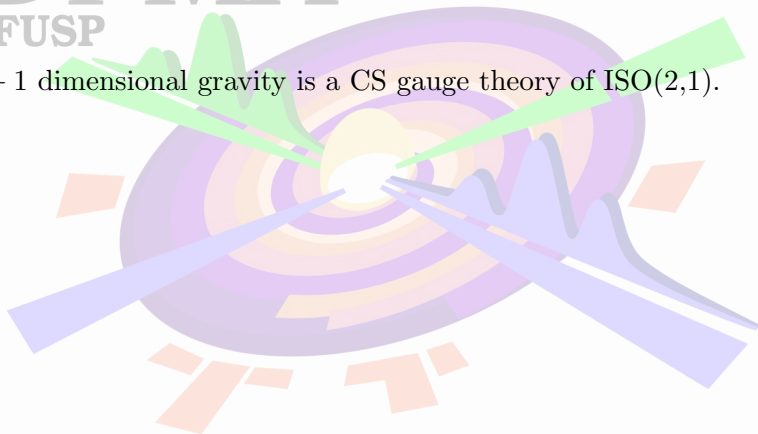
$$S_{\text{CS}} = \frac{1}{\lambda} \int_M \left\langle \mathbf{e} \frown \left(d\boldsymbol{\omega} + \frac{1}{2} [\boldsymbol{\omega} \frown \boldsymbol{\omega}] \right) \right\rangle = S_{\text{EH}}$$

Match!

Conclusions

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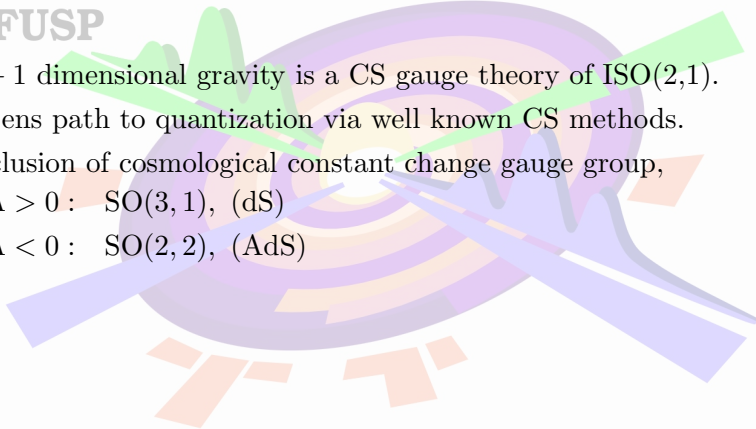


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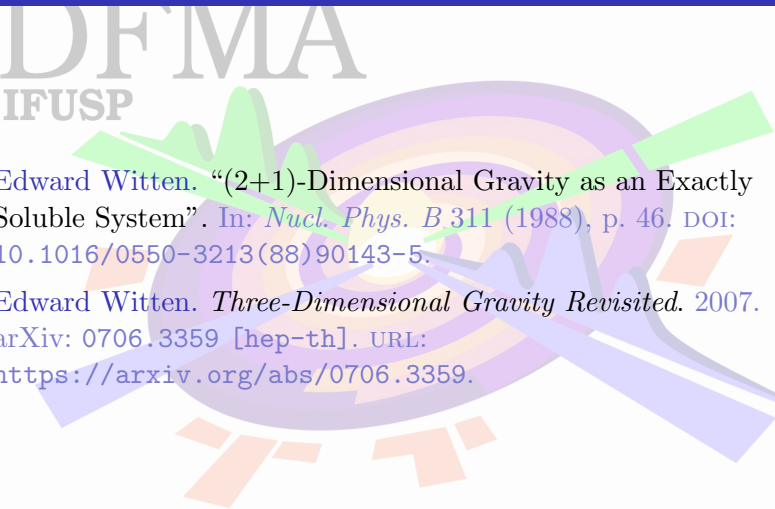
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- Pose some suggestions about $D = 3 + 1$.

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Thank You!

References

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- [1] Edward Witten. “(2+1)-Dimensional Gravity as an Exactly Soluble System”. In: *Nucl. Phys. B* 311 (1988), p. 46. DOI: 10.1016/0550-3213(88)90143-5.
- [2] Edward Witten. *Three-Dimensional Gravity Revisited*. 2007. arXiv: 0706.3359 [hep-th]. URL: <https://arxiv.org/abs/0706.3359>.