Homework III

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Contents

Problem 1

1.A)

1.B)

1.C)

1.D)

Problem 2

2.A)

2.B)

2.C)

2.D)

Problem 3

3.A)

3.B)

3.C)

3.D)

3.E)

3.F)

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A.1 Faddeev-Popov Gauge Fixing

We'll start with a discussion of the Faddeev-Popov procedure of gauge fixing, first, our action is,

$$S_X + \lambda\chi = \frac{1}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}R + \frac{\lambda}{2\pi}\int\limits_{\partial M}\mathrm{d}s\,K$$

we would like to define the quantum theory by means of the path integral, that is, we expect that,

$$Z \stackrel{?}{=} \int \mathcal{D}X\mathcal{D}h \exp\left(-S_X[X,h] - \lambda\chi\right)$$

should give a well defined theory, but, the integral should be only over physical and inequivalent configurations of X, h, and as we know, we have Diff×Weyl gauge redundancies in this theory, this means in the integral measure we're over-counting physical configurations, that is, instead of the integral $\int \mathcal{D}h$ being over the whole space of all possible metrics, it should be in the space of equivalence classes under Diff×Weyl of all possible metrics. Before correcting this over-counting, we can brake down the sum over all metrics by the value of the Euler Characteristic χ,

$$\begin{aligned}
Z &\stackrel{?}{=} \int \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h] - \lambda\chi\right) \\
Z &\stackrel{?}{=} \sum_{M_g} \int\limits_{\mathrm{Met}(M_g)} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h] - \lambda\chi\right) \\
Z &\stackrel{?}{=} \sum_{M_g} \exp\left(-\lambda g\right) \int\limits_{\mathrm{Met}(M_g)} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h]\right)
\end{aligned}$$

Where M_g is to be understood as a compact, not necessarily connected, two dimensional Riemannian manifold/surface with Euler Characteristic $\chi = g$, and $\mathrm{Met}(M_g)$ is the space of all metrics which can be assigned to M_g. While for $M_g \cong \mathbb{R}^2 \cong \mathbb{C}$ it's true that all possible metrics are Diff×Weyl equivalent to each other, for non-trivial topologies this is not true, what do happens is the *moduli space*, or, the set of equivalence classes,

$$\mathscr{M}_g = \mathrm{Met}(M_g)/\mathrm{Diff}\times\mathrm{Weyl}$$

possesses more than one element. The equivalence class is to be understood as,

$$h'_{ab} \sim h_{ab} \Leftrightarrow h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

that is, two metrics are the same representative of an element of $\mathscr{M}_g$ iff they differ by a composition of a Diff and Weyl transformation. We'll denote a given composition of a Diff followed by a Weyl by just ζ, so that

$$h' = \zeta \circ h$$

Thus, it's possible for us to set up a well defined version of the path integral, just replace $\mathrm{Met}(M_g)$ by $\mathscr{M}_g$,

$$Z = \sum_{M_g} \exp\left(-\lambda g\right) \int\limits_{\mathscr{M}_g} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h]\right)$$

where the integration is to be understood as by choosing for each equivalence class in $\mathscr{M}_g$ a representative element in $\mathrm{Met}(M_g)$. While this is a satisfactory definition for the path integral, it feels a little clunky, we rather have an path integral over all the possible metrics, well, this is achievable. First, for each equivalence class of $\mathscr{M}_g$ elect one representative element of $\mathrm{Met}(M_g)$, we'll denote these elements as $\hat{h}(t^I)$ — here t^I is a parametrization of the elements in $\mathscr{M}_g$ —, by construction, these representatives are inequivalent under Diff$\times$Weyl, hence,

$$\zeta_1 \circ \hat{h}(t^I) = \zeta_2 \circ \hat{h}(t^K) \Leftrightarrow t^I = t^K \text{ and } \zeta_1 = \zeta_2$$

so that every element in $\mathrm{Met}(M_g)$ can be written as a unique[1] composition of a given ζ into a given $\hat{h}(t^I)$

in this way is possible to separate the integral over all metrics $\int \mathcal{D}h$ into an integration over all inequivalent metrics $\int \mathcal{D}\hat{h}$ and an integration over all possible Diff$\times$Weyl transformations $\int \mathcal{D}\zeta$, so that the partition function can be rewrote as,

$$Z \stackrel{?}{=} \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \exp\left(-S_X[X, \zeta \circ h]\right)$$

this still has the same problem of before, we're over-integrating the physical configurations, that is, $\hat{h}$ are the physical configurations, but we're integrating also over the whole Diff$\times$Weyl group in $\mathcal{D}\zeta$. One way of circumventing this problem is introducing by hand a Dirac delta to force $\zeta = 0$, what also forces we to integrate only over one copy of the physical configurations,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta) \exp\left(-S_X[X, \zeta \circ h]\right)$$

but this is not the most general way, we could set $\zeta = f(\sigma)$, for a arbitrary function, and this would still give the same theory,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta - f) \exp\left(-S_X[X, \zeta \circ h]\right)$$

we can go even further and give a function $G(\zeta)$ such that the solution to $G(\zeta) = 0$ is only $\zeta = f$, so that we can use the relations between Dirac deltas,

$$\delta(G(\zeta)) = \left|\mathrm{Det}\left[\frac{\delta G}{\delta \zeta}\right]\Big|_{\zeta=f}\right|^{-1} \delta(\zeta - f)$$

to obtain,

$$\begin{aligned}
Z &= \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta - f) \exp\left(-S_X[X, \zeta \circ h]\right) \\
Z &= \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\left|\mathrm{Det}\left[\frac{\delta G}{\delta \zeta}\right]\Big|_{\zeta=f}\right|\delta(G(\zeta)) \exp\left(-S_X\left[X, \zeta \circ \hat{h}\right]\right) \\
Z &= \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\left|\mathrm{Det}\left[\frac{\delta G}{\delta \zeta}\right]\Big|_{\zeta=f}\right|\delta(G(\zeta)) \exp\left(-S_X\left[X, \zeta \circ \hat{h}\right]\right)
\end{aligned}$$

There are some details here, as ζ is to represent both a Weyl and a Diff, it has to represent both a function ω and a vector field ξ such that,

$$\zeta \circ \hat{h} = \hat{h} + 2\omega\hat{h} + \mathscr{L}_\xi \hat{h} + \mathcal{O}(\omega^2, \xi^2, \omega\xi)$$

[1] The

$$\left[\zeta \circ \hat{h}\right]_{ab} = \hat{h}_{ab} + 2\omega\hat{h}_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)$$

this means both $\zeta = f$ and $G(\zeta) = 0$ are in fact a collection of various equations. In particular, we'll choose

$$G_{ab}(\zeta) = \left[\tilde{h}\right]_{ab} - \left[\zeta \circ \hat{h}\right]_{ab}$$

for a particular metric $\tilde{h}$. As $G_{ab}(\zeta)$ is in fact a function of $h = \zeta \circ \hat{h}$ alone,

$$G_{ab}(\zeta) = \left[\tilde{h}\right]_{ab} - \left[\zeta \circ \hat{h}\right]_{ab} = \left[\tilde{h}\right]_{ab} - [h]_{ab} = G_{ab}(h)$$

we can rewrite as,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G_{ab}}{\delta\zeta}\right]\Big|_{\zeta=f} \right| \delta(G_{ab}(\zeta)) \exp\left(-S_X\left[X, \zeta \circ \hat{h}\right]\right)$$
$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G_{ab}}{\delta\zeta}\right]\Big|_{G_{ab}(h)=0} \right| \delta(G_{ab}(h)) \exp\left(-S_X[X, h]\right)$$

Notice that every term in the integrand depends on ζ only through $h = \zeta \circ \hat{h}$, this is what we do want, so that we can recombine the integration measure $\int \mathcal{D}\hat{h}\mathcal{D}\zeta = \int \mathcal{D}h$, the only problem in this procedure is the term,

$$\mathrm{Det}\left[\frac{\delta G_{ab}}{\delta\zeta}\right]\Big|_{G_{ab}(h)=0}$$

which is manifestly dependent on ζ, or at least looks like it is. We'll prove it only depends on ζ through $h = \zeta \circ \hat{h}$. The point is, if

$$G_{ab}(h) = \tilde{h}_{ab} - h_{ab} = \tilde{h}_{ab} - \left[\zeta \circ \hat{h}\right]_{ab} = 0$$

is needed to have a solution, then exists the transformation $\tilde{\zeta}$, such that,

$$\tilde{\zeta} \circ \hat{h} = \tilde{h}$$

as this transformation is also an element of the gauge **group**, it certainly has an inverse, $\tilde{\zeta}^{-1} \circ \tilde{\zeta} \circ \hat{h} = \hat{h}$, thus,

$$\begin{aligned}
G_{ab}(h) &= \tilde{h} - \zeta \circ \hat{h} \\
G_{ab}(h) &= \tilde{h} - \zeta \circ \tilde{\zeta}^{-1} \circ \tilde{\zeta} \circ \hat{h} \\
G_{ab}(h) &= \tilde{h} - \zeta \circ \tilde{\zeta}^{-1} \circ \tilde{h} \\
G_{ab}(h) &= \tilde{h} - \zeta' \circ \tilde{h}
\end{aligned}$$

where we defined the new gauge transformation $\zeta' = \zeta \circ \tilde{\zeta}^{-1}$, notice that,

$$\begin{aligned}
h &= \zeta \circ \hat{h} \\
h &= \zeta \circ \tilde{\zeta}^{-1} \circ \tilde{\zeta} \circ \hat{h} \\
h &= \zeta' \circ \tilde{h}
\end{aligned}$$