Homework II

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May 7, 2025

Contents

Problem 1

1.A)

Let M be our $D > 2$ dimensional C^∞ manifold and $\phi : \mathbb{R} \times M \to M$ be a one parameter family of diffeomorphisms, that is, $\forall t \in \mathbb{R} |\ \phi_t : M \to M$ is a diffeomorphism such that,

- $\forall p \in M |\ \phi_0(p) = p$
- $\forall p \in M; \forall t, s \in \mathbb{R} |\ \phi_{t+s}(p) = (\phi_t \circ \phi_s)(p)$
- $\forall p \in M |\ \phi(p) : \mathbb{R} \to M$ is at least C^1

Then this family of diffeomorphisms define in a natural manner a vector which generate these transformations, we define at each point $p \in M$ this vector by it's action in a function $f : M \to \mathbb{R}$,

$$\xi_p(f) = \frac{\mathrm{d}}{\mathrm{d}t}((f \circ \phi_t)(p))\bigg|_{t=0}$$

And from this we define ξ as a vector field in M, this vector field has as integral curves exactly ϕ. Let's open a little bit more in some chart $x : M \to \mathbb{R}^D$,

$$\begin{aligned}
\xi_p(f) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\big(f \circ x^{-1} \circ x \circ \phi_t\big)(p)\big)\bigg|_{t=0} \\
\xi_p(f) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\big(f \circ x^{-1}\big) \circ (x \circ \phi_t)(p)\big)\bigg|_{t=0} \\
\xi_p(f) &= \partial_\mu\big(f \circ x^{-1}\big)\bigg|_{x\circ\phi_0(p)} \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(p))\bigg|_{t=0} \\
\xi_p^\mu \partial_\mu\big(f \circ x^{-1}\big)\bigg|_{x(p)} &= \partial_\mu\big(f \circ x^{-1}\big)\bigg|_{x(p)} \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(p))\bigg|_{t=0}
\end{aligned}$$

Where of curse ∂_μ is to be interpreted as the derivative of the μ-th component in the chart x. Here we have a clear definition of the values of the ξ vector field in a chart x,

$$\xi_p^\mu = \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(p))\bigg|_{t=0}$$

The term inside the derivative is just the pullback of the chart x — the chart can be seen as a $\mathbb{R}^D$-valued function —, which in it's own can be seen as a new chart x'_t defined by the transformations of the diffeomorphism family ϕ, that is,

$$x'_t = \phi_t^* x = x \circ \phi_t : M \to \mathbb{R}^D$$

All of this is consistent with our interpretation of the diffeomorphisms being a '*coordinate change*', in principle, with enough derivability of ϕ we can actually write,

$$\begin{aligned}
x'_t &= x'_0 + t\frac{\mathrm{d}}{\mathrm{d}t}(x'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big) \\
{x'_1}^\mu =: x'^\mu &= x^\mu + \xi^\mu + \cdots
\end{aligned}$$

We just restored the index to not confuse the components of the vector field ξ in the basis x with the vector field itself. That is, we showed that the transformation done by ϕ_1 is equivalent to a '*infinitesimal coordinate change*' by ξ^μ. Actually, all this we did is the special case of a more general type of derivative, the Lie Derivative, given a vector field ξ and it's family of integral curves ϕ, it's defined in terms of the pushforward of the object under analysis,

$$\pounds_\xi T = \frac{\mathrm{d}}{\mathrm{d}t}(\phi_{-t*}T)\bigg|_{t=0}$$

For a $(0,2)$ tensor, that is, for the metric,

$$\pounds_\xi g_{\mu\nu} = \nabla_\xi g_{\mu\nu} + g_{\mu\alpha}\nabla_\nu\xi^\alpha + g_{\alpha\nu}\nabla_\mu\xi^\alpha$$

And as the connection is metric compatible,

$$\pounds_\xi g_{\mu\nu} = \nabla_\nu\xi_\mu + \nabla_\mu\xi_\nu = 2\nabla_{(\mu}\xi_{\nu)}$$

This amounts for the first term in an expansion in t of the diffeomorphism transformed metric $\phi_{-t*}g = g'_{t\mu\nu}$, that is,

$$\begin{aligned}
\phi_{-t*}g_{\mu\nu} =: g'_{t\mu\nu} &= g_{\mu\nu} + t\pounds_\xi g_{\mu\nu} + \mathcal{O}\big(t^2\big) \\
g'_{t\mu\nu} &= g_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big)
\end{aligned}$$

Which also can be interpreted as the infinitesimal transformation of the metric. Imposing that the initial and transformed metric are conformally flat,

$$\begin{aligned}
\exp{(2\omega'_t)}\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big) \\
\exp\left(2\omega'_0 + 2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big)\right)\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big) \\
\exp{(2\omega)}\exp\left(2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big)\right)\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big) \\
\exp{(2\omega)}\left(1 + 2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big)\right)\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big) \\
\exp{(2\omega)}2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0}\eta_{\mu\nu} &= 2t\nabla_{(\mu}\xi_{\nu)} \\
\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} g_{\mu\nu} &= \nabla_{(\mu}\xi_{\nu)}
\end{aligned}$$

The term $\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\big|_{t=0}$ is fully determined by ξ, to see this just contract both sides with the metric,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} g^{\mu\nu}g_{\mu\nu} &= g^{\mu\nu}\nabla_{(\mu}\xi_{\nu)} \\
\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} D &= \nabla_\mu\xi^\mu
\end{aligned}$$

Substituting this back in the original equation,

$$\nabla_\alpha \xi^\alpha g_{\mu\nu} = D\nabla_{(\mu}\xi_{\nu)} \tag{1.1}$$

This is the condition upon ξ that ensures the diffeomorphism maintain the conformally flatness of the metric. Now we'll solve it. First, apply ∇^ν in both sides,

$$\begin{aligned}
\nabla^\nu \nabla_\alpha \xi^\alpha g_{\mu\nu} &= D\nabla^\nu \nabla_{(\mu}\xi_{\nu)} \\
\frac{2}{D}\nabla_\mu \nabla_\alpha \xi^\alpha &= \nabla^\nu \nabla_\mu \xi_\nu + \nabla^\nu \nabla_\nu \xi_\mu \\
\frac{2}{D}\nabla_\mu \nabla_\alpha \xi^\alpha &= \nabla^\alpha \nabla_\mu \xi_\alpha + \nabla^\alpha \nabla_\alpha \xi_\mu \\
\frac{2}{D}\nabla_\mu \nabla_\alpha \xi^\alpha - \nabla_\mu \nabla^\alpha \xi_\alpha &= (\nabla^\alpha \nabla_\mu - \nabla_\mu \nabla^\alpha)\xi_\alpha + \nabla^\alpha \nabla_\alpha \xi_\mu \\
\frac{2}{D}\nabla_\mu \nabla_\alpha \xi^\alpha - \nabla_\mu \nabla_\alpha \xi^\alpha &= R^\alpha{}_{\mu\alpha}{}^\beta \xi_\beta + \nabla^\alpha \nabla_\alpha \xi_\mu \\
\left(\frac{2}{D} - 1\right)\nabla_\mu \nabla_\alpha \xi^\alpha &= R_\mu{}^\alpha \xi_\alpha + \nabla^\alpha \nabla_\alpha \xi_\mu
\end{aligned}$$

Apply ∇_ν to the both sides,

$$\begin{aligned}
\left(\frac{2}{D} - 1\right)\nabla_\nu \nabla_\mu \nabla_\alpha \xi^\alpha &= \nabla_\nu\big(R_\mu{}^\alpha \xi_\alpha\big) + \nabla_\nu \nabla^\alpha \nabla_\alpha \xi_\mu \\
\left(\frac{2}{D} - 1\right)\nabla_\nu \nabla_\mu \nabla_\alpha \xi^\alpha &= \nabla_\nu\big(R_\mu{}^\alpha \xi_\alpha\big) + (\nabla_\nu \nabla^\alpha - \nabla^\alpha \nabla_\nu)\nabla_\alpha \xi_\mu + \nabla^\alpha \nabla_\nu \nabla_\alpha \xi_\mu \\
\left(\frac{2}{D} - 1\right)\nabla_\nu \nabla_\mu \nabla_\alpha \xi^\alpha &= \nabla_\nu\big(R_\mu{}^\alpha \xi_\alpha\big) + R_\nu{}^\alpha{}_\alpha{}^\beta \nabla_\beta \xi_\mu + R_\nu{}^\alpha{}_\mu{}^\beta \nabla_\alpha \xi_\beta + \nabla^\alpha \nabla_\nu \nabla_\alpha \xi_\mu \\
\left(\frac{2}{D} - 1\right)\nabla_\nu \nabla_\mu \nabla_\alpha \xi^\alpha &= \nabla_\nu\big(R_\mu{}^\alpha \xi_\alpha\big) - R_\nu{}^\beta \nabla_\beta \xi_\mu + R_\nu{}^\alpha{}_\mu{}^\beta \nabla_\alpha \xi_\beta + \nabla^\alpha(\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu)\xi_\mu + \nabla^\alpha \nabla_\alpha \nabla_\nu \xi_\mu \\
\left(\frac{2}{D} - 1\right)\nabla_\nu \nabla_\mu \nabla_\alpha \xi^\alpha &= \nabla_\nu\big(R_\mu{}^\alpha \xi_\alpha\big) - R_\nu{}^\alpha \nabla_\alpha \xi_\mu + R_\nu{}^\alpha{}_\mu{}^\beta \nabla_\alpha \xi_\beta + \nabla^\alpha\big(R_{\nu\alpha\mu}{}^\beta \xi_\beta\big) + \nabla^\alpha \nabla_\alpha \nabla_\nu \xi_\mu
\end{aligned}$$

We symmetrize the $\mu\nu$ indices, make use of 1.1, and after contract with $g^{\mu\nu}$,

$$\begin{aligned}
\left(\frac{2}{D} - 1\right)\nabla_{(\nu}\nabla_{\mu)}\nabla_\alpha \xi^\alpha &= \nabla_{(\nu}\big(R_{\mu)}{}^\alpha \xi_\alpha\big) - R_{(\nu|}{}^\alpha \nabla_\alpha \xi_{|\mu)} + R_{(\nu|}{}^\alpha{}_{|\mu)}{}^\beta \nabla_\alpha \xi_\beta \\
&\quad + \nabla^\alpha\Big(R_{(\nu|\alpha|\mu)}{}^\beta \xi_\beta\Big) + \nabla^\alpha \nabla_\alpha \nabla_{(\nu}\xi_{\mu)} \\
\left(\frac{2}{D} - 1\right)\nabla_{(\nu}\nabla_{\mu)}\nabla_\alpha \xi^\alpha &= \nabla_{(\nu}\big(R_{\mu)}{}^\alpha \xi_\alpha\big) - R_{(\nu|}{}^\alpha \nabla_\alpha \xi_{|\mu)} + R_{(\nu|}{}^\alpha{}_{|\mu)}{}^\beta \nabla_\alpha \xi_\beta \\
&\quad + \nabla^\alpha\Big(R_{(\nu|\alpha|\mu)}{}^\beta \xi_\beta\Big) + \frac{1}{D} g_{\mu\nu} \nabla^\alpha \nabla_\alpha \nabla_\beta \xi^\beta \\
\left(\frac{2}{D} - 1\right)\nabla_\mu \nabla^\mu \nabla_\alpha \xi^\alpha &= \nabla_\mu(R^{\mu\alpha}\xi_\alpha) - R^{\mu\alpha}\nabla_\alpha \xi_\mu + R_\mu{}^{\alpha\mu\beta}\nabla_\alpha \xi_\beta \\
&\quad + \nabla^\alpha\big(R_{\mu\alpha}{}^{\mu\beta}\xi_\beta\big) + \nabla^\alpha \nabla_\alpha \nabla_\beta \xi^\beta \\
\left(\frac{2}{D} - 1\right)\nabla_\mu \nabla^\mu \nabla_\alpha \xi^\alpha &= 2\nabla_\mu(R^{\mu\alpha}\xi_\alpha) + \nabla^\alpha \nabla_\alpha \nabla_\beta \xi^\beta
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad + \nabla^{\alpha}\big(R_{\mu\alpha}{}^{\mu\beta}\xi_{\beta}\big) + \nabla^{\alpha}\nabla_{\alpha}\nabla_{\beta}\xi^{\beta} \\
&\left(\frac{2}{D} - 2\right)\nabla_{\mu}\nabla^{\mu}\nabla_{\alpha}\xi^{\alpha} = 2\nabla_{\mu}(R^{\mu\alpha}\xi_{\alpha}) \\
&(1 - D)\nabla_{\mu}\nabla^{\mu}\nabla_{\alpha}\xi^{\alpha} = D\nabla_{\mu}(R^{\mu\alpha}\xi_{\alpha})
\end{aligned} \tag{1.2}
$$

Let's focus in the left-hand side,

$$
\begin{aligned}
\nabla_{\mu}\nabla^{\mu}\nabla_{\alpha}\xi^{\alpha} &= \nabla^{\mu}\nabla_{\mu}\nabla_{\alpha}\xi^{\alpha} \\
\nabla_{\mu}\nabla^{\mu}\nabla_{\alpha}\xi^{\alpha} &= \nabla^{\mu}\partial_{\mu}\left(\frac{1}{\sqrt{-g}}\partial_{\alpha}\big(\sqrt{-g}\xi^{\alpha}\big)\right) \\
\nabla_{\mu}\nabla^{\mu}\nabla_{\alpha}\xi^{\alpha} &= \frac{1}{\sqrt{-g}}\partial^{\mu}\left(\sqrt{-g}\partial_{\mu}\left(\frac{1}{\sqrt{-g}}\partial_{\alpha}\big(\sqrt{-g}\xi^{\alpha}\big)\right)\right)
\end{aligned}
$$

1.B)

Problem 2

2.A)

2.B)

Problem 3

3.A)

3.B)

3.C)

3.D)

3.E)

3.F)

Problem 4

4.A)

4.B)

4.C)

4.D)

4.E)

Problem 5

5.A)

5.B)

Problem 6

6.A)

6.B)

6.C)

6.D)

A Appendix