

SCALAR PROXY

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1. INTRODUCTION

We will work most with the scalar proxy given by the lagrangian,

$$(1.1) \quad \mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2M^2}\Box\phi\Box\phi - \frac{\kappa}{2}\Box\phi\phi^2$$

The idea here is reintegrate the higher derivative term, in order to obtain a lower derivative term, but in terms of additional fields. This is easily done by,

$$(1.2) \quad \mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \Box\phi\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box\phi\phi^2$$

The new lagrangian has mixed propagator terms, to diagonalize it is also easy, we just open in terms of $\phi = h - \eta$,

$$(1.3) \quad \mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 + \eta\Box(h-\eta) + \frac{M^2}{2}\eta^2$$

$$(1.4) \quad \mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 - \partial_\mu\eta\partial^\mu(h-\eta) + \frac{M^2}{2}\eta^2$$

$$(1.5) \quad \mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2$$

The Feynman rules are easily read as,

- $h \text{-----} h = \frac{1}{i} \frac{1}{p^2}$
- $\eta \text{-----} \eta = -\frac{1}{i} \frac{1}{p^2+M^2}$
- $h_1 \text{-----} \begin{array}{c} \diagup h_2 \\ \diagdown h_3 \end{array} = i\kappa(p_1^2 + p_2^2 + p_3^2)$
- $h_1 \text{-----} \begin{array}{c} \diagup h_2 \\ \diagdown \eta_3 \end{array} = -i\kappa(p_1^2 + p_2^2 + p_3^2)$
- $h_1 \text{-----} \begin{array}{c} \diagup \eta_2 \\ \diagdown \eta_3 \end{array} = i\kappa(p_1^2 + p_2^2 + p_3^2)$
- $\eta_1 \text{-----} \begin{array}{c} \diagup \eta_2 \\ \diagdown \eta_3 \end{array} = -i\kappa(p_1^2 + p_2^2 + p_3^2)$

Which can also be seen directly from the Feynman rules of the ϕ field,

- $\phi \text{-----} \phi = \frac{1}{i} \frac{1}{p^2 + \frac{p^4}{M^2}}$

$$\bullet \quad \phi_1 \text{ --- } \begin{array}{c} \nearrow \phi_2 \\ \searrow \phi_3 \end{array} = i\kappa(p_1^2 + p_2^2 + p_3^2) = i\kappa(p_1 + p_2 + p_3)^2 - 2i\kappa(p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1) = -i\kappa(\langle 12 \rangle [12] + \langle 23 \rangle [23] + \langle 31 \rangle [31])$$

So that the four point amplitude can be computed by,

$$(1.6) \quad \begin{array}{c} \phi_2 \\ \searrow \\ \phi_1 \end{array} \xrightarrow{P} \begin{array}{c} \phi_3 \\ \swarrow \\ \phi_4 \end{array} = \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}} (\langle 12 \rangle [12] + \langle 2P \rangle [2P] + \langle P1 \rangle [P1]) (\langle 34 \rangle [34] - \langle 4P \rangle [4P] - \langle P3 \rangle [P3])$$

$$(1.7) \quad = \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}} (\langle 12 \rangle [12] - \langle P2 \rangle [2P] - \langle P1 \rangle [1P]) (\langle 34 \rangle [34] + \langle P4 \rangle [4P] + \langle P3 \rangle [3P])$$

$$(1.8) \quad = \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}} (\langle 12 \rangle [12] + \langle P | 1 + 2 | P \rangle) (\langle 34 \rangle [34] - \langle P | 3 + 4 | P \rangle)$$

$$(1.9) \quad = \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}} (\langle 12 \rangle [12] - \langle P | P | P \rangle) (\langle 34 \rangle [34] - \langle P | P | P \rangle)$$

$$(1.10) \quad = \frac{1}{i}(-i\kappa)^2 \frac{1}{P^2 + \frac{P^4}{M^2}} (\langle 12 \rangle [12] - 2P^2) (\langle 34 \rangle [34] - 2P^2)$$

$$(1.11) \quad = -i \frac{(\kappa M)^2}{s(M^2 - s)} (\langle 12 \rangle [12] + 2s) (\langle 34 \rangle [34] + 2s)$$

It's trivial to read the t and u channels from this expression,

$$(1.12) \quad \begin{array}{c} \phi_2 \quad \phi_3 \\ \searrow \quad \swarrow \\ \downarrow P \\ \phi_1 \quad \phi_4 \end{array} = -i \frac{(\kappa M)^2}{t(M^2 - t)} (\langle 23 \rangle [23] + 2t) (\langle 41 \rangle [41] + 2t)$$

$$(1.13) \quad \begin{array}{c} \phi_2 \quad \phi_4 \\ \searrow \quad \swarrow \\ \downarrow P \\ \phi_1 \quad \phi_3 \end{array} = -i \frac{(\kappa M)^2}{u(M^2 - u)} (\langle 24 \rangle [24] + 2u) (\langle 31 \rangle [31] + 2u)$$

So that the full 4-point amplitude is,

$$(1.14) \quad \begin{array}{c} \phi_2 \quad \phi_3 \\ \searrow \quad \swarrow \\ \phi_1 \quad \phi_4 \end{array} = -i \frac{(\kappa M)^2}{stu(M^2 - s)(M^2 - t)(M^2 - u)} [(\langle 12 \rangle [12] + 2s)(\langle 34 \rangle [34] + 2s)tu(M^2 - t)(M^2 - u) + (\langle 23 \rangle [23] + 2t)(\langle 41 \rangle [41] + 2t) + (\langle 24 \rangle [24] + 2u)(\langle 31 \rangle [31] + 2u)]$$

Let us specialize when 1, 2 are massless and 3, 4 are massive, then,

$$(2.2) \quad i\Pi^{(1)} = -\frac{3}{2}ig^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{i} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2}$$

$$(2.3) \quad i\Pi^{(1)} = -\frac{3}{2}g^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2}$$

$$(2.4) \quad i\Pi^{(1)} = -\frac{3}{2}g^2 \frac{i}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} (m^2)^{\frac{D}{2}-2} \frac{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)}{\Gamma(1)}$$

$$(2.5) \quad i\Pi^{(1)} = -\frac{3}{2}ig^2 \frac{(m^2)^{-\epsilon} \Gamma(\epsilon) \Gamma(1-\epsilon)}{(4\pi)^{2-\epsilon} \Gamma(2-\epsilon)}$$

$$(2.6) \quad i\Pi^{(2)} = \frac{1}{2}(ig)^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{i^2} \frac{1}{\ell^2(\ell+p)^2} \frac{(m^2 + \ell^2 + p^2 + (\ell+p)^2)^2}{\ell^2 + m^2} \frac{1}{(\ell+p)^2 + m^2}$$

For the mass renormalization we can take $p = 0$,

$$(2.7) \quad i\Pi^{(2)} = \frac{1}{2}g^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{(m^2 + 2\ell^2)^2}{\ell^4(\ell^2 + m^2)^2}$$

Let's compute the four point amplitude for this theory,

$$(2.8) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} P \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = (ig)^2 \frac{(p_1^2 + p_2^2 + (p_1 + p_2)^2 + m^2)(p_3^2 + p_4^2 + (p_3 + p_4)^2 + m^2)}{i(p_1 + p_2)^2((p_1 + p_2)^2 + m^2)}$$

First let's consider all legs massless,

$$(2.9) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} P \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = ig^2 \frac{(-s + m^2)(-s + m^2)}{(-s)(-s + m^2)} = -ig^2 \frac{(-s + m^2)}{s}$$

So,

$$(2.10) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = -ig^2 \frac{(-s + m^2)}{s} - ig^2 \frac{(-t + m^2)}{t} - ig^2 \frac{(-u + m^2)}{u} - 3ig^2$$

$$(2.11) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = -ig^2 \frac{(-s + m^2)}{s} - ig^2 \frac{(-t + m^2)}{t} - ig^2 \frac{(-u + m^2)}{u} - ig^2 \frac{s}{s} - ig^2 \frac{t}{t} - ig^2 \frac{u}{u}$$

$$(2.12) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = -ig^2 m^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right) = ig^2 m^2 \left(\frac{1}{\langle 12 \rangle [12]} + \frac{1}{\langle 14 \rangle [14]} + \frac{1}{\langle 13 \rangle [13]} \right)$$

Para uma perna massiva, ϕ_4 ,

$$(2.13) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} P \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = (ig)^2 \frac{(-s+m^2)(-s)}{i(-s)(-s+m^2)} = ig^2$$

So,

$$(2.14) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = ig^2 + ig^2 + ig^2 - 3ig^2 = 0$$

Para duas pernas massivas, $\phi_{3,4}$,

$$(2.15) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} P \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = (ig)^2 \frac{(-s+m^2)(-s-m^2)}{i(-s)(-s+m^2)} = ig^2 \frac{s+m^2}{s}$$

$$(2.16) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \\ \quad \downarrow P \\ \phi_1 \quad \quad \phi_4 \end{array} = (ig)^2 \frac{(-t)(-t)}{i(-t)(-t+m^2)} = -ig^2 \frac{t}{-t+m^2}$$

$$(2.17) \quad \begin{array}{c} \phi_2 \quad \quad \phi_4 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \\ \quad \downarrow P \\ \phi_1 \quad \quad \phi_3 \end{array} = -ig^2 \frac{u}{-u+m^2}$$

So,

$$(2.18) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = ig^2 \frac{s+m^2}{s} - ig^2 \frac{t}{-t+m^2} - ig^2 \frac{u}{-u+m^2} - 3ig^2$$

$$(2.19) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = ig^2 \frac{s+m^2}{s} - ig^2 \frac{t}{-t+m^2} - ig^2 \frac{u}{-u+m^2} - ig^2 \frac{s}{s} - ig^2 \frac{-t+m^2}{-t+m^2} - ig^2 \frac{-u+m^2}{-u+m^2}$$

$$(2.20) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = -ig^2 m^2 \left(-\frac{1}{s} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -ig^2 m^2 \left(\frac{1}{\langle 12 \rangle [12]} + \frac{1}{\langle 14 \rangle [14]} + \frac{1}{\langle 13 \rangle [13]} \right)$$

Para uma perna sem massa ϕ_1 ,

$$(2.21) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} P \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = (ig)^2 \frac{(-s)(-s-m^2)}{i(-s)(-s+m^2)} = -ig^2 \frac{s+m^2}{-s+m^2}$$

$$(2.22) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \quad \downarrow P \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = (ig)^2 \frac{(-t)(-t-m^2)}{i(-t)(-t+m^2)} = -ig^2 \frac{t+m^2}{-t+m^2}$$

$$(2.23) \quad \begin{array}{c} \phi_2 \quad \quad \phi_4 \\ \quad \searrow \quad \nearrow \\ \quad \quad \downarrow P \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_3 \end{array} = (ig)^2 \frac{(-u)(-u-m^2)}{i(-u)(-u+m^2)} = -ig^2 \frac{u+m^2}{-u+m^2}$$

$$(2.24)$$

So,

$$(2.25) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = -ig^2 \frac{s+m^2}{-s+m^2} - ig^2 \frac{t+m^2}{-t+m^2} - ig^2 \frac{u+m^2}{-u+m^2} - 3ig^2$$

$$(2.26) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = -ig^2 \frac{s+m^2}{-s+m^2} - ig^2 \frac{t+m^2}{-t+m^2} - ig^2 \frac{u+m^2}{-u+m^2} - ig^2 \frac{-s+m^2}{-s+m^2} - ig^2 \frac{-t+m^2}{-t+m^2} - ig^2 \frac{-u+m^2}{-u+m^2}$$

$$(2.27) \quad \begin{array}{c} \phi_2 \quad \quad \phi_3 \\ \quad \searrow \quad \nearrow \\ \quad \text{---} \text{---} \\ \quad \nearrow \quad \searrow \\ \phi_1 \quad \quad \phi_4 \end{array} = -ig^2 m^2 \left(\frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -ig^2 m^2 \left(\frac{1}{\langle 12 \rangle [12]} + \frac{1}{\langle 14 \rangle [14]} + \frac{1}{\langle 13 \rangle [13]} \right)$$

Cut comparison, only massless legs

$$(2.28) \quad \begin{array}{c} \quad \quad \quad \text{---} \text{---} \\ \quad \quad \quad \nearrow \quad \searrow \\ \quad \quad \quad \text{---} \text{---} \\ \quad \quad \quad \searrow \quad \nearrow \\ \quad \quad \quad \text{---} \text{---} \\ \quad \quad \quad \nearrow \quad \searrow \\ \quad \quad \quad \text{---} \text{---} \\ \quad \quad \quad \searrow \quad \nearrow \\ \quad \quad \quad \text{---} \text{---} \end{array} = \frac{ig^2 m^2 (igm^2)^2}{(im^2)^3} \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{\langle 1l \rangle [1l]} - \frac{1}{\langle 2l \rangle [2l]} \right)$$

to solve for the cuts, $l^2 = (l+3+4)^2 = (l+4)^2 = 0$,

$$(2.29) \quad l^2 = 0 \Rightarrow l = -|l\rangle[l]$$

$$(2.30) \quad 0 = (l+4)^2 = \langle l4 \rangle [l4] = 0 \Rightarrow |l\rangle = |4\rangle$$

$$(2.31) \quad 0 = (l+3+4)^2 = \langle lP_{34} \rangle [lP_{34}] + (3+4)^2 = \langle l|3+4|l\rangle + \langle 34 \rangle [34] = \langle l|3+4|4\rangle + \langle 34 \rangle [34]$$

$$(2.32) \quad \langle 43 \rangle [34] = -\langle l3 \rangle [34] \Rightarrow |l\rangle = -|4\rangle + z|3\rangle$$

$$(2.33) \quad l = -(-|4\rangle + z|3\rangle)[4]$$

The cuts are solved by this. Hence,

$$\begin{aligned} &= g^4 \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{\langle 1l \rangle [1l]} - \frac{1}{\langle 2l \rangle [2l]} \right) \\ &= g^4 \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{(-\langle 14 \rangle + z\langle 13 \rangle)[14]} - \frac{1}{(-\langle 24 \rangle + z\langle 23 \rangle)[24]} \right) \end{aligned}$$

Now for internal massive lines,

$$(2.34) \quad \begin{array}{c} \text{Diagram: A triangle with three shaded vertices. The top vertex has an incoming dashed line from the top-left labeled $l+3+4$ and an outgoing dashed line to the top-right labeled $l+4$. The bottom-left vertex has an incoming dashed line from the bottom-left labeled l and an outgoing dashed line to the bottom-right labeled l. The bottom-right vertex has an incoming dashed line from the bottom-right labeled l and an outgoing dashed line to the top-right labeled $l+4$. The right side of the triangle is dashed, while the other two sides are solid.} \end{array} = \frac{-ig^2 m^2 (-igm^2)^2}{(-im^2)^3} \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{\langle 1l \rangle [1l]} - \frac{1}{\langle 2l \rangle [2l]} \right)$$

With the cuts being, $l^2 = (l+3+4)^2 = (l+4)^2 = -m^2$,

$$(2.35) \quad 0 = (l+4)^2 - l^2 = 2l \cdot p_4$$

$$(2.36) \quad 0 = (l+4+3)^2 - l^2 = 2l \cdot (4+3) + (4+3)^2 = 2l \cdot p_3 + (4+3)^2$$

As ansatz, $l = |4\rangle[4] + \alpha|4\rangle[3] + \beta|3\rangle[4]$ satisfy both conditions above. The remaining condition is,

$$(2.37) \quad l^2 = -m^2$$

$$(2.38) \quad -\alpha\beta[43]\langle 43 \rangle = -m^2 \Rightarrow \alpha = \frac{m^2}{\beta\langle 34 \rangle[34]}$$

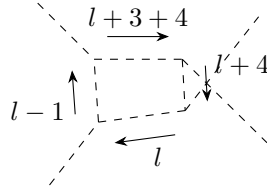
Setting now $-\beta = z$,

$$(2.39) \quad l = |4\rangle[4] - \frac{m^2}{z\langle 34 \rangle[34]} |4\rangle[3] - z|3\rangle[4]$$

The value of the diagram is,

$$\begin{aligned} &= g^4 \left(\frac{1}{\langle 12 \rangle [12]} + \frac{1}{\langle 1l \rangle [1l]} + \frac{1}{\langle 2l \rangle [2l]} \right) \\ &= g^4 \left(\frac{1}{\langle 12 \rangle [12]} + \frac{1}{-\langle 1l \rangle [1l]} + \frac{1}{-\langle 2l \rangle [2l]} \right) \\ &= g^4 \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{\langle 14 \rangle [41] - \frac{m^2}{z\langle 34 \rangle[34]} \langle 14 \rangle [31] - z\langle 13 \rangle [41]} - \frac{1}{\langle 2l \rangle [2l]} \right) \end{aligned}$$

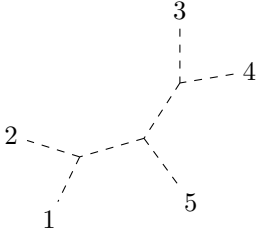
The explicit cut loop amplitude is,



Triple cut has no improvement, what about a double cut,

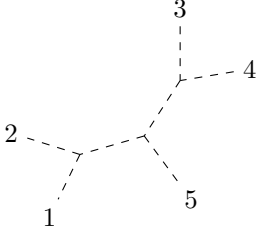
$$(2.40) \quad \begin{array}{c} \text{Diagram: A bubble diagram with two shaded vertices. The left vertex has two incoming dashed lines labeled p_1 and p_2. The right vertex has two outgoing dashed lines labeled p_3 and p_4. The top arc is labeled $\ell+3+4$ and the bottom arc is labeled ℓ.} \end{array} = \frac{(ig^2 m^2)^2}{(im^2)^2} \left(\frac{1}{\langle 12 \rangle [12]} - \frac{1}{\langle 1\ell \rangle [1\ell]} - \frac{1}{\langle 2\ell \rangle [2\ell]} \right) \left(\frac{1}{\langle 34 \rangle [34]} + \frac{1}{\langle 3\ell \rangle [3\ell]} + \frac{1}{\langle 4\ell \rangle [4\ell]} \right)$$

Five point amplitude,



$$= \frac{(ig)^3 \left(m^2 + p_1^2 + p_2^2 + (p_1 + p_2)^2 \right) \left(m^2 + p_5^2 + (p_3 + p_4)^2 + (p_1 + p_2)^2 \right) \left(m^2 + p_3^2 + p_4^2 + (p_3 + p_4)^2 \right)}{i^2 (p_1 + p_2)^2 \left((p_1 + p_2)^2 + m^2 \right) (p_3 + p_4)^2 \left((p_3 + p_4)^2 + m^2 \right)}$$

Let's consider the special case of all massless,

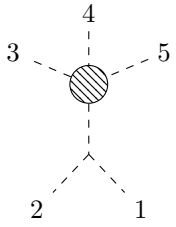


$$= \frac{ig^3}{(p_1 + p_2)^2} \frac{\left(m^2 + (p_3 + p_4)^2 + (p_1 + p_2)^2 \right) \left(m^2 + (p_3 + p_4)^2 \right)}{(p_3 + p_4)^2 \left((p_3 + p_4)^2 + m^2 \right)} = \frac{ig^3}{(p_1 + p_2)^2} \frac{\left(m^2 + (p_3 + p_4)^2 + (p_1 + p_2)^2 \right)}{(p_3 + p_4)^2}$$

Combining this graph with,

$$= \frac{-i3g^2 ig \left(m^2 + (p_1 + p_2)^2 \right)}{i(p_1 + p_2)^2 \left((p_1 + p_2)^2 + m^2 \right)} = \frac{-3ig^3}{(p_1 + p_2)^2}$$

We get,



$$= \frac{ig^3}{(p_1 + p_2)^2} \left[\frac{\left(m^2 + (p_3 + p_4)^2 + (p_1 + p_2)^2 \right)}{(p_3 + p_4)^2} - \frac{(p_3 + p_4)^2}{(p_3 + p_4)^2} + \frac{\left(m^2 + (p_3 + p_5)^2 + (p_1 + p_2)^2 \right)}{(p_3 + p_5)^2} - \frac{(p_3 + p_5)^2}{(p_3 + p_5)^2} + \frac{\left(m^2 + (p_5 + p_4)^2 + (p_1 + p_2)^2 \right)}{(p_5 + p_4)^2} - \frac{(p_5 + p_4)^2}{(p_5 + p_4)^2} \right]$$

$$= \frac{ig^3 \left(m^2 + (p_1 + p_2)^2 \right)}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} \right]$$

Now we have to sum the contributions of 1 being in the middle,

$$+ \text{2} \text{---} \text{5} \text{---} \text{1} \text{---} \text{3} \text{---} \text{4} + \text{perm}$$

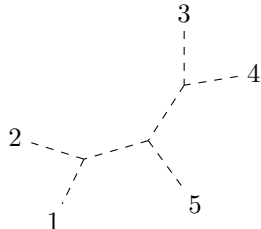
Which will be,

$$\frac{ig^3}{(p_2 + p_3)^2} \frac{m^2 + (p_2 + p_3)^2 + (p_4 + p_5)^2}{(p_4 + p_5)^2} - \frac{3ig^3}{(p_2 + p_3)^2} - \frac{3ig^3}{(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5)$$

Summing all the contributions we have,

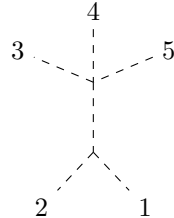
$$\begin{aligned} &= \frac{ig^3 m^2}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} \right] + (2 \leftrightarrow 3, 4, 5) \\ &\quad + ig^3 \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_2 + p_4)^2} + \frac{1}{(p_2 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_2)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_2)^2} + \frac{1}{(p_4 + p_2)^2} \right] \\ &\quad + \frac{ig^3 m^2}{(p_2 + p_3)^2 (p_4 + p_5)^2} - \frac{2ig^3}{(p_2 + p_3)^2} - \frac{2ig^3}{(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5) \\ &= \frac{ig^3 m^2}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{(p_5 + p_4)^2} \right] + (2 \leftrightarrow 3, 4, 5) \\ &\quad + \frac{ig^3 m^2}{(p_2 + p_3)^2 (p_4 + p_5)^2} + \frac{ig^3 m^2}{(p_2 + p_4)^2 (p_3 + p_5)^2} + \frac{ig^3 m^2}{(p_2 + p_5)^2 (p_4 + p_3)^2} \end{aligned}$$

By residue, any amplitude with just one massive external on-shell leg is zero. For two massive external on-shell legs, let's take as massive 1, 2,



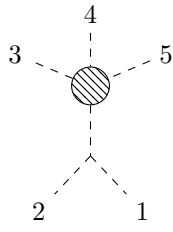
$$= ig^3 \frac{\left((p_1 + p_2)^2 - m^2 \right)}{(p_1 + p_2)^2 \left(m^2 + (p_1 + p_2)^2 \right)} \frac{\left(m^2 + (p_1 + p_2)^2 + (p_3 + p_4)^2 \right)}{(p_3 + p_4)^2}$$

Combining this graph with,



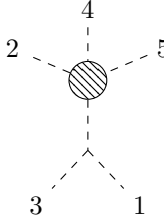
$$= -3ig^3 \frac{(p_1 + p_2)^2 - m^2}{(p_1 + p_2)^2 \left(m^2 + (p_1 + p_2)^2 \right)}$$

so,



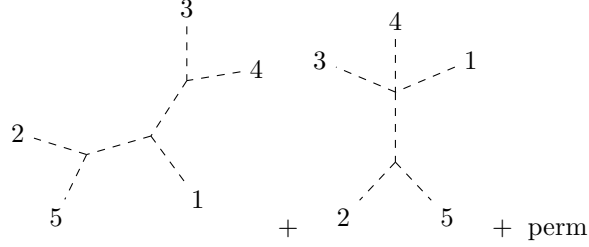
$$\begin{aligned} &= ig^3 \frac{\left((p_1 + p_2)^2 - m^2 \right)}{(p_1 + p_2)^2 \left(m^2 + (p_1 + p_2)^2 \right)} \left[\frac{\left(m^2 + (p_1 + p_2)^2 + (p_3 + p_4)^2 \right)}{(p_3 + p_4)^2} - \frac{(p_3 + p_4)^2}{(p_3 + p_4)^2} + (5 \leftrightarrow 3, 4) \right] \\ &= ig^3 \frac{\left((p_1 + p_2)^2 - m^2 \right)}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right] \end{aligned}$$

The other contributions are,



$$= \frac{ig^3(p_1 + p_3)^2}{(m^2 + (p_1 + p_3)^2)} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{(p_4 + p_5)^2} \right]$$

Now we have to sum the contributions of 1 being in the middle,



which are,

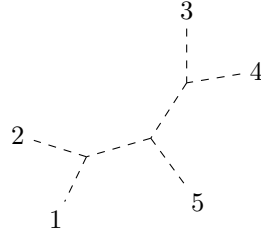
$$= ig^3 \frac{(p_2 + p_3)^2 + (p_4 + p_5)^2}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2} - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3}{(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5)$$

So, summing all the contributions,

$$\begin{aligned} &= ig^3 \frac{((p_1 + p_2)^2 - m^2)}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right] \\ &\quad + \frac{ig^3(p_1 + p_3)^2}{(m^2 + (p_1 + p_3)^2)} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{(p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\ &\quad + ig^3 \frac{(p_2 + p_3)^2 + (p_4 + p_5)^2}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2} - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3}{(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5) \\ &= -ig^3 \frac{m^2}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right] \\ &\quad + ig^3 \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right] \\ &\quad + ig^3 \left[\frac{1}{2p_2 \cdot p_4} + \frac{1}{2p_2 \cdot p_5} + \frac{1}{(p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\ &\quad - ig^3 \frac{m^2}{(2p_1 \cdot p_3)} \left[\frac{1}{2p_2 \cdot p_4} + \frac{1}{2p_2 \cdot p_5} + \frac{1}{(p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\ &\quad + ig^3 \frac{-m^2 + 2p_2 \cdot p_3 + (p_4 + p_5)^2}{(2p_2 \cdot p_3)(p_4 + p_5)^2} - \frac{3ig^3}{2p_2 \cdot p_3} - \frac{3ig^3}{(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5) \\ &= -\frac{ig^3 m^2}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right] \\ &\quad + ig^3 \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right] \\ &\quad + ig^3 \left[\frac{1}{2p_2 \cdot p_4} + \frac{1}{2p_2 \cdot p_5} + \frac{1}{(p_4 + p_5)^2} + \frac{1}{2p_2 \cdot p_3} + \frac{1}{2p_2 \cdot p_5} + \frac{1}{(p_3 + p_5)^2} + \frac{1}{2p_2 \cdot p_4} + \frac{1}{2p_2 \cdot p_3} + \frac{1}{(p_4 + p_3)^2} \right] \\ &\quad - \frac{ig^3 m^2}{(2p_1 \cdot p_3)} \left[\frac{1}{2p_2 \cdot p_4} + \frac{1}{2p_2 \cdot p_5} + \frac{1}{(p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \end{aligned}$$

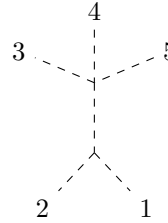
$$\begin{aligned}
& -\frac{ig^3m^2}{(2p_2 \cdot p_3)(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5) \\
& -\frac{2ig^3}{2p_2 \cdot p_3} - \frac{2ig^3}{(p_4 + p_5)^2} - \frac{2ig^3}{2p_2 \cdot p_4} - \frac{2ig^3}{(p_3 + p_5)^2} - \frac{2ig^3}{2p_2 \cdot p_5} - \frac{2ig^3}{(p_4 + p_3)^2} \\
& = -\frac{ig^3m^2}{(p_1 + p_2)^2} \left[\frac{1}{(p_3 + p_4)^2} + \frac{1}{(p_5 + p_4)^2} + \frac{1}{(p_3 + p_5)^2} \right] \\
& - \frac{ig^3m^2}{(2p_1 \cdot p_3)} \left[\frac{1}{2p_2 \cdot p_4} + \frac{1}{2p_2 \cdot p_5} + \frac{1}{(p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& - \frac{ig^3m^2}{(2p_2 \cdot p_3)(p_4 + p_5)^2} + (3 \leftrightarrow 4, 5)
\end{aligned}$$

Is almost the same of the all massless, but with a different denominator in the 1,2 channel. Now with three massive legs, being 3, 4, 5,



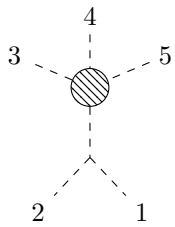
$$= ig^3 \frac{\left((p_1 + p_2)^2 + (p_3 + p_4)^2 \right) \left(-m^2 + (p_3 + p_4)^2 \right)}{(p_1 + p_2)^2 (p_3 + p_4)^2 \left(m^2 + (p_3 + p_4)^2 \right)}$$

With,



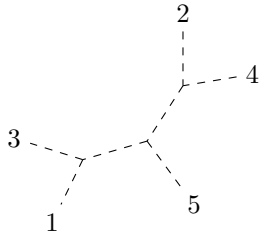
$$= \frac{-i3g^2ig \left(m^2 + (p_1 + p_2)^2 \right)}{i(p_1 + p_2)^2 \left((p_1 + p_2)^2 + m^2 \right)} = \frac{-3ig^3}{(p_1 + p_2)^2}$$

So,

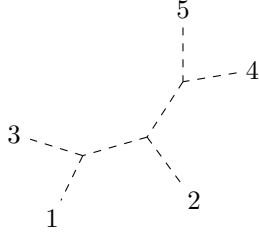


$$\begin{aligned}
& = ig^3 \frac{\left((p_1 + p_2)^2 + (p_3 + p_4)^2 \right) \left(-m^2 + (p_3 + p_4)^2 \right)}{(p_1 + p_2)^2 (p_3 + p_4)^2 \left(m^2 + (p_3 + p_4)^2 \right)} - \frac{3ig^3}{(p_1 + p_2)^2} \\
& = \frac{ig^3}{(p_1 + p_2)^2} \left[\frac{\left((p_1 + p_2)^2 + (p_3 + p_4)^2 \right) \left(-m^2 + (p_3 + p_4)^2 \right)}{(p_3 + p_4)^2 \left(m^2 + (p_3 + p_4)^2 \right)} - \frac{(p_3 + p_4)^2 \left(m^2 + (p_3 + p_4)^2 \right)}{(p_3 + p_4)^2 \left(m^2 + (p_3 + p_4)^2 \right)} \right] + (5 \leftrightarrow 3, 4) \\
& = \frac{ig^3}{(p_1 + p_2)^2} \left[\frac{(p_1 + p_2)^2 \left(-m^2 + (p_3 + p_4)^2 \right) - 2m^2(p_3 + p_4)^2}{(p_3 + p_4)^2 \left(m^2 + (p_3 + p_4)^2 \right)} \right] + (5 \leftrightarrow 3, 4)
\end{aligned}$$

The other topology is,

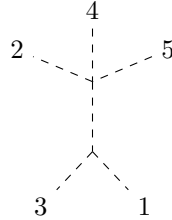


$$\begin{aligned}
&= ig^3 \frac{\left(m^2 + p_3^2 + (p_1 + p_3)^2\right) \left(m^2 + p_5^2 + (p_1 + p_3)^2 + (p_2 + p_4)^2\right) \left(m^2 + p_4^2 + (p_2 + p_4)^2\right)}{(p_1 + p_3)^2 \left(m^2 + (p_1 + p_3)^2\right) (p_2 + p_4)^2 \left(m^2 + (p_2 + p_4)^2\right)} \\
&= ig^3 \frac{\left((p_1 + p_3)^2 + (p_2 + p_4)^2\right)}{\left(m^2 + (p_1 + p_3)^2\right) \left(m^2 + (p_2 + p_4)^2\right)}
\end{aligned}$$



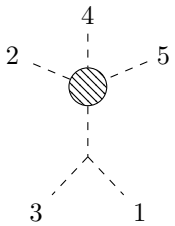
$$\begin{aligned}
&= ig^3 \frac{\left(m^2 + p_3^2 + (p_1 + p_3)^2\right) \left(m^2 + (p_1 + p_3)^2 + (p_5 + p_4)^2\right) \left(m^2 + p_5^2 + p_4^2 + (p_5 + p_4)^2\right)}{(p_1 + p_3)^2 \left(m^2 + (p_1 + p_3)^2\right) (p_5 + p_4)^2 \left(m^2 + (p_5 + p_4)^2\right)} \\
&= ig^3 \frac{\left(m^2 + (p_1 + p_3)^2 + (p_5 + p_4)^2\right) \left(-m^2 + (p_5 + p_4)^2\right)}{\left(m^2 + (p_1 + p_3)^2\right) (p_5 + p_4)^2 \left(m^2 + (p_5 + p_4)^2\right)}
\end{aligned}$$

Also,



$$\begin{aligned}
&= -3ig^2 g \frac{\left(m^2 + p_3^2 + (p_3 + p_1)^2\right)}{(p_3 + p_1)^2 \left(m^2 + (p_3 + p_1)^2\right)} \\
&= -3ig^3 \frac{1}{\left(m^2 + (p_3 + p_1)^2\right)}
\end{aligned}$$

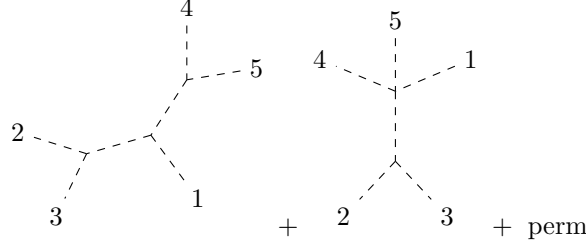
So,



$$\begin{aligned}
&= \frac{ig^3}{m^2 + (p_1 + p_3)^2} \left[\frac{\left((p_1 + p_3)^2 + (p_2 + p_4)^2\right) - m^2 - (p_2 + p_4)^2}{\left(m^2 + (p_2 + p_4)^2\right)} + \frac{\left((p_1 + p_3)^2 + (p_2 + p_5)^2\right) - m^2 - (p_2 + p_5)^2}{\left(m^2 + (p_2 + p_5)^2\right)} \right] \\
&\quad + \frac{ig^3}{m^2 + (p_1 + p_3)^2} \left[\frac{\left(m^2 + (p_1 + p_3)^2 + (p_4 + p_5)^2\right) \left(-m^2 + (p_4 + p_5)^2\right) - (p_4 + p_5)^2 \left(m^2 + (p_4 + p_5)^2\right)}{(p_4 + p_5)^2 \left(m^2 + (p_4 + p_5)^2\right)} \right] \\
&= \frac{ig^3 \left((p_1 + p_3)^2 - m^2\right)}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{ig^3}{m^2 + (p_1 + p_3)^2} \left[\frac{-m^4 - m^2(p_1 + p_3)^2 + (p_1 + p_3)^2(p_4 + p_5)^2 - m^2(p_4 + p_5)^2}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} \right] \\
& = \frac{ig^3((p_1 + p_3)^2 - m^2)}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& \quad - \frac{ig^3 m^2}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5)
\end{aligned}$$

Now, with 1 in the middle,



Which is,

$$\begin{aligned}
& = \frac{ig^3(m^2 + p_3^2 + (p_2 + p_3)^2)(m^2 + p_1^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2)(m^2 + p_5^2 + p_4^2 + (p_5 + p_4)^2)}{(p_2 + p_3)^2(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3(-m^2 + (p_4 + p_5)^2)}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} \\
& = \frac{ig^3(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2)(-m^2 + (p_5 + p_4)^2)}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3(-m^2 + (p_4 + p_5)^2)}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5)
\end{aligned}$$

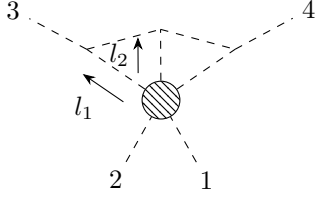
Summing all the contributions we have,

$$\begin{aligned}
& ig^3 \frac{(-m^2 + (p_5 + p_4)^2)}{(p_5 + p_4)^2(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3 m^2}{(p_1 + p_2)^2(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& + \frac{ig^3((p_1 + p_3)^2 - m^2)}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& - \frac{ig^3 m^2}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& + \frac{ig^3(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2)(-m^2 + (p_5 + p_4)^2)}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} - \frac{3ig^3(-m^2 + (p_4 + p_5)^2)}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3 m^2}{(p_1 + p_2)^2(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& + \frac{ig^3((p_1 + p_3)^2 - m^2)}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& - \frac{ig^3 m^2}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5)
\end{aligned}$$

$$\begin{aligned}
& + \frac{ig^3(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2)(-m^2 + (p_5 + p_4)^2)}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} + \frac{2ig^3m^2}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} - \frac{2ig^3}{(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{(p_1 + p_2)^2} \frac{1}{(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& + ig^3 \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] \\
& + ig^3 \left[\frac{1}{m^2 + (p_2 + p_3)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_3 + p_5)^2} \right] \\
& + ig^3 \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_3)^2} + \frac{1}{m^2 + (p_4 + p_3)^2} \right] \\
& + \frac{ig^3(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2)(-m^2 + (p_5 + p_4)^2)}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{3ig^3}{m^2 + (p_2 + p_3)^2} + \frac{ig^3m^2}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} - \frac{2ig^3}{(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{(p_1 + p_2)^2} \frac{1}{(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& + \frac{ig^3(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2)(-m^2 + (p_5 + p_4)^2)}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{ig^3}{m^2 + (p_2 + p_3)^2} + \frac{ig^3m^2}{(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} - \frac{ig^3}{(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{(p_1 + p_2)^2} \frac{1}{(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& + \frac{ig^3(m^2 + (p_2 + p_3)^2 + (p_5 + p_4)^2)(-m^2 + (p_5 + p_4)^2)}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& + ig^3 \frac{-(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2) + m^2(m^2 + (p_2 + p_3)^2) - (p_4 + p_5)^2(m^2 + (p_2 + p_3)^2)}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{(p_1 + p_2)^2} \frac{1}{(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3m^2}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& + ig^3 \frac{-m^4 + m^2(p_4 + p_5)^2 - m^2(p_2 + p_3)^2 + (p_2 + p_3)^2(p_4 + p_5)^2 - m^2(p_5 + p_4)^2 + (p_4 + p_5)^4}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5)
\end{aligned}$$

$$\begin{aligned}
& + ig^3 \frac{-(p_4 + p_5)^2 m^2 - (p_4 + p_5)^4 + m^4 + m^2(p_2 + p_3)^2 - (p_4 + p_5)^2 m^2 - (p_2 + p_3)^2 (p_4 + p_5)^2}{(m^2 + (p_2 + p_3)^2)(p_4 + p_5)^2(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3 m^2}{(p_1 + p_2)^2} \frac{1}{(m^2 + (p_5 + p_4)^2)} + (3 \leftrightarrow 4, 5) \\
& - \frac{2ig^3 m^2}{m^2 + (p_1 + p_3)^2} \left[\frac{1}{m^2 + (p_2 + p_4)^2} + \frac{1}{m^2 + (p_2 + p_5)^2} + \frac{1}{m^2 + (p_4 + p_5)^2} \right] + (3 \leftrightarrow 4, 5) \\
& - 2ig^3 m^2 \frac{1}{(m^2 + (p_2 + p_3)^2)(m^2 + (p_4 + p_5)^2)} + (3 \leftrightarrow 4, 5)
\end{aligned}$$

Now, let's do the cuts, consider a two loop four point amplitude with five cuts,



$$= -\frac{g^3}{m^4} [\mathcal{A}_5(1, 2, l_{1h}, l_{2h}, l_{3h}) - \mathcal{A}_5(1, 2, l_{1h}, l_{2\eta}, l_{3\eta}) - \mathcal{A}_5(1, 2, l_{1\eta}, l_{2h}, l_{3\eta}) - \mathcal{A}_5(1, 2, l_{1\eta}, l_{2\eta}, l_{3h}) + 2\mathcal{A}_5(1, 2, l_{1\eta}, l_{2\eta}, l_{3\eta})]$$

3. CUT SOLUTIONS

Of course in each amplitude we have different cut solutions. Now let us solve them,

3.1. all massless. The cut condition is,

$$k_1^2 = k_2^2 = (3 - k_1)^2 = (3 - k_1 - k_2)^2 = (3 + 4 - k_1 - k_2)^2 = 0$$

The first and third condition enforces $k_1 = -|k_1]\langle 3|$. But the fourth and fifth conditions enforces $3 - k_1 - k_2 = n$, with $n \cdot 4 = 0$ & $n^2 = 0$. Lastly, the second condition imposes $(3 - k_1 - n)^2 = -23 \cdot n + 2k_1 \cdot n = 0$, that is,

$$[3n]\langle n3 \rangle = [k_1 n]\langle n3 \rangle$$

which has two solutions, $|n] = |k_1] - |3]$ & $|n] = z|4]$ or $|n\rangle = |3\rangle$ & $|n\rangle = z|4\rangle$. When working with scalar particles it's better to choose the first solution, as this avoids singularities in denominators such as $(k_1 \cdot k_2)^{-1}$. Hence, the solution we're going to choose is,

$$\begin{cases} k_1 &= -|k_1]\langle 3| \\ k_2 &= -|3]\langle 3| + |k_1]\langle 3| + z(|k_1] - |3])\langle 4| \end{cases}$$

3.2. massive legs first topology. Our approach to massive legs is to shift the solution with massless, in order to obtain a well behaved solution in the $m^2 \rightarrow 0$ limit. For this topology the cut constrains are,

$$l_1^2 = l_2^2 = (3 - l_1)^2 = -m^2 \text{ \& } (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = 0$$

The idea here is to define, $l_i = k_i + \alpha_i q_i$ (no sum), with $q_i^2 = 0$ and $\alpha_i = -m^2(2k_i \cdot q_i)^{-1}$, then, q_i, k_i are not allowed to have any dependence on m^2 . The first and second constrains are already satisfied. The third one gives,

$$-23 \cdot l_1 = 0 \rightarrow 3 \cdot (k_1 + \alpha_1 q_1) = 0 \rightarrow 3 \cdot q_1 = 0$$

As $|q_1\rangle = |3\rangle$ is forbidden, $|q_1] = |3]$. The fourth and fifth constrains imposes,

$$\begin{cases} -n \cdot (\alpha_1 q_1 + \alpha_2 q_2) + \alpha_1 \alpha_2 q_1 \cdot q_2 &= 0 \\ 4 \cdot (\alpha_1 q_1 + \alpha_2 q_2) &= 0 \end{cases}$$

This imposes actually $q_1 \cdot q_2 = 0$, for this to be true we have two options, either $|q_2] = |3]$, or $|q_2\rangle = |q_1\rangle$. If we choose the first, we can shift k_1 by 3 such to make $|q_1\rangle = |4\rangle$, this imposes further $|q_2\rangle = |4\rangle$. Hence, a possible solution is,

$$q_1 = q_2 = -|3]\langle 4|$$

3.3. massive legs second topology. The constrains now are slightly different,

$$l_1^2 = (3 - l_1)^2 = 0 \text{ \& } l_2^2 = (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = -m^2$$

Which has as solution $q_2 = -|4]\langle 3|$

3.4. massive legs third topology. Now the constrain is difficult to solve,

$$l_2^2 = 0 \text{ \& } l_1^2 = (3 - l_1)^2 = (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = -m^2$$

The second and third constrains give, $l_1 = -|k_1]\langle 3| - \alpha|3]\langle l_1|$. Now, the fourth and fifth constrains gives,

$$3 - l_1 - l_2 = -z(|k_1] - |3])\langle 4| + \beta|4]\langle n|$$

With of course $\beta = -\frac{m^2}{z\langle 4n\rangle[4]([k_1] - |3])}$. At last the second constrain gives,

$$\begin{aligned} l_2 &= -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n| \\ l_2^2 = 0 &= -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4]([k_1] - |3]) \\ 0 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\ -2m^2 &= \beta\langle 3n\rangle[4]([3] - |k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\ -2m^2 &= -\frac{m^2}{z\langle 4n\rangle[4]([k_1] - |3])}\langle 3n\rangle[4]([3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]}z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \end{aligned}$$

For the quadratic term in m^2 to vanish is necessary $\langle l_1 n\rangle = 0 \rightarrow |l_1\rangle \propto |n\rangle$, thus,

$$\begin{aligned} -2m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle}z\langle 4l_1\rangle \\ -2 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle}z\langle 4n\rangle \rightarrow \langle 3n\rangle = -z\langle 4n\rangle \end{aligned}$$

The best parametrization is $|n\rangle = |l_1\rangle = |4\rangle - \frac{1}{z}|3\rangle$.

3.5. massive legs fourth topology. Now the constrain is the hardest to solve,

$$l_1^2 = l_2^2 = (3 - l_1)^2 = (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = -m^2$$

Happily, most of the work was done already in the last solution, we just have to change:

$$\begin{aligned} l_2 &= -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n| \\ l_2^2 = -m^2 \rightarrow m^2 &= -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4]([k_1] - |3]) \\ m^2 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\ -m^2 &= \beta\langle 3n\rangle[4]([3] - |k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\ -m^2 &= -\frac{m^2}{z\langle 4n\rangle[4]([k_1] - |3])}\langle 3n\rangle[4]([3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]}z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \end{aligned}$$

Again we fix $|l_1\rangle \propto |n\rangle$. The solution then is given by,

$$\begin{aligned} -m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle}z\langle 4l_1\rangle \\ -1 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle}z\langle 4n\rangle \rightarrow \langle 3n\rangle = -z\left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)\langle 4n\rangle = -ze^{\frac{1}{3}\pi i}\langle 4n\rangle \end{aligned}$$

4. AMPLITUDES EVALUATED IN THE CUTS

4.1. all massless. The expression for the amplitude is,

$$\begin{aligned} &= \frac{ig^3m^2}{2p_1 \cdot p_2} \left[\frac{1}{2k_1 \cdot k_2} + \frac{1}{2k_1 \cdot k_3} + \frac{1}{2k_2 \cdot k_3} \right] \\ &\quad + \frac{ig^3m^2}{2p_1 \cdot k_1} \left[\frac{1}{2p_2 \cdot k_2} + \frac{1}{2p_2 \cdot k_3} + \frac{1}{2k_3 \cdot k_2} \right] \\ &\quad + \frac{ig^3m^2}{2p_1 \cdot k_2} \left[\frac{1}{2k_1 \cdot p_2} + \frac{1}{2k_1 \cdot k_3} + \frac{1}{2k_3 \cdot p_2} \right] \\ &\quad + \frac{ig^3m^2}{2p_1 \cdot k_3} \left[\frac{1}{2k_1 \cdot k_2} + \frac{1}{2k_1 \cdot p_2} + \frac{1}{2p_2 \cdot k_2} \right] \\ &\quad + \frac{ig^3m^2}{2p_2 \cdot k_1 2k_2 \cdot k_3} + \frac{ig^3m^2}{2p_2 \cdot k_2 2k_1 \cdot k_3} + \frac{ig^3m^2}{2p_2 \cdot k_3 2k_2 \cdot k_1} \end{aligned}$$

4.2. two massive.

4.2.1. l_1 and l_2 massive.

$$\begin{aligned}
&= -\frac{ig^3m^2}{-2m^2+2l_1 \cdot l_2} \left[\frac{1}{2k_3 \cdot p_1} + \frac{1}{2p_2 \cdot p_1} + \frac{1}{2k_3 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2l_1 \cdot p_1)} \left[\frac{1}{2l_2 \cdot k_3} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2k_3 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2l_1 \cdot p_2)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot k_3} + \frac{1}{2p_1 \cdot k_3} \right] \\
&\quad - \frac{ig^3m^2}{(2l_1 \cdot k_3)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2p_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{2l_2 \cdot k_3 2p_1 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_1 2k_3 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_2 2p_1 \cdot k_3} \\
&= -\frac{ig^3m^2}{2k_1 \cdot k_2} \left[\frac{1}{2k_3 \cdot p_1} + \frac{1}{2p_2 \cdot p_1} + \frac{1}{2k_3 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2l_1 \cdot p_1)} \left[\frac{1}{2k_2 \cdot k_3} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2k_3 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2l_1 \cdot p_2)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2k_2 \cdot k_3} + \frac{1}{2p_1 \cdot k_3} \right] \\
&\quad - \frac{ig^3m^2}{(2k_1 \cdot k_3)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2p_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{2k_2 \cdot k_3 2p_1 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_1 2k_3 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_2 2p_1 \cdot k_3}
\end{aligned}$$

We used that $l_3 = k_3, l_2 \cdot k_3 = k_2 \cdot k_3, l_1 \cdot k_3 = k_1 \cdot k_3, 2l_1 \cdot l_2 = 2k_1 \cdot k_2 + 2m^2$.

4.2.2. l_2 and l_3 massive.

$$\begin{aligned}
&= -\frac{ig^3m^2}{-2m^2+2l_3 \cdot l_2} \left[\frac{1}{2k_1 \cdot p_1} + \frac{1}{2p_2 \cdot p_1} + \frac{1}{2k_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2l_3 \cdot p_1)} \left[\frac{1}{2l_2 \cdot k_1} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2k_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2l_3 \cdot p_2)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot k_1} + \frac{1}{2p_1 \cdot k_1} \right] \\
&\quad - \frac{ig^3m^2}{(2l_3 \cdot k_1)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2p_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{2l_2 \cdot k_1 2p_1 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_1 2k_1 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_2 2p_1 \cdot k_1} \\
&= -\frac{ig^3m^2}{2k_3 \cdot k_2} \left[\frac{1}{2k_1 \cdot p_1} + \frac{1}{2p_2 \cdot p_1} + \frac{1}{2k_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2k_3 \cdot p_1)} \left[\frac{1}{2k_2 \cdot k_1} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2k_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2k_3 \cdot p_2)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2k_2 \cdot k_1} + \frac{1}{2p_1 \cdot k_1} \right] \\
&\quad - \frac{ig^3m^2}{(2k_3 \cdot k_1)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot p_2} + \frac{1}{2p_1 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{2k_2 \cdot k_1 2p_1 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_1 2k_1 \cdot p_2} - \frac{ig^3m^2}{2l_2 \cdot p_2 2p_1 \cdot k_1}
\end{aligned}$$

We used that $l_1 = k_1, l_2 \cdot k_1 = k_2 \cdot k_1, l_3 \cdot k_1 = k_3 \cdot k_1, 2l_2 \cdot l_3 = 2k_2 \cdot k_3 + 2m^2$.

4.2.3. l_1 and l_3 massive.

$$\begin{aligned}
&= -\frac{ig^3m^2}{-2m^2+2l_3 \cdot l_1} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2p_2 \cdot p_1} + \frac{1}{2l_2 \cdot p_2} \right] \\
&\quad - \frac{ig^3m^2}{(2l_3 \cdot p_1)} \left[\frac{1}{2l_2 \cdot l_1} + \frac{1}{2l_1 \cdot p_2} + \frac{1}{2l_1 \cdot p_2} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{ig^3m^2}{(2l_3 \cdot p_2)} \left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot l_1} + \frac{1}{2p_1 \cdot l_1} \right] \\
& -\frac{ig^3m^2}{(2l_3 \cdot l_2)} \left[\frac{1}{2l_1 \cdot p_1} + \frac{1}{2l_1 \cdot p_2} + \frac{1}{2p_1 \cdot p_2} \right] \\
& -\frac{ig^3m^2}{2l_1 \cdot l_2 2p_1 \cdot p_2} - \frac{ig^3m^2}{2l_1 \cdot p_1 2l_2 \cdot p_2} - \frac{ig^3m^2}{2l_1 \cdot p_2 2p_1 \cdot l_1}
\end{aligned}$$

4.2.4. l_1, l_2 and l_3 massive.