SCALAR PROXY

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1. Cut solutions

Of course in each amplitude we have different cut solutions. Now let us solve them,

1.1. **all massless.** The cut condition is,

$$k_1^2 = k_2^2 = (3-k_1)^2 = (3-k_1-k_2)^2 = (3+4-k_1-k_2)^2 = 0$$

The first and third condition enforces $k_1 = -|k_1]\langle 3|$. But the fourth and fifth conditions enforces $3-k_1-k_2 = n$, with $n \cdot 4 = 0$ & $n^2 = 0$. Lastly, the second condition imposes $(3-k_1-n)^2 = -23 \cdot n + 2k_1 \cdot n = 0$, that is,

$$[3n]\langle n3\rangle = [k_1 n]\langle n3\rangle$$

which has two solutions, $|n] = |k_1] - |3]$ & $|n\rangle = z|4\rangle$ or $|n\rangle = |3\rangle$ & $|n] = z|4]$. When working with scalar particles it's better to choose the first solution, as this avoids singularities in denominators such as $(k_1 \cdot k_2)^{-1}$. Hence, the solution we're going to choose is,

$$\begin{cases} k_1 &= -|k_1]\langle 3| \\ k_2 &= -|3]\langle 3| + |k_1]\langle 3| + z(|k_1] - |3])\langle 4| \end{cases}$$

1.2. **massive legs first topology.** Our approach to massive legs is to shift the solution with massless, in order to obtain a well behaved solution in the $m^2 \to 0$ limit. For this topology the cut constrains are,

$$l_1^2 = l_2^2 = (3-l_1)^2 = -m^2 \ \&\ (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = 0$$

The ideia here is to define, $l_i = k_i + \alpha_i q_i$ (no sum), with $q_i^2 = 0$ and $\alpha_i = -m^2(2k_i \cdot q_i)^{-1}$, then, q_i, k_i are not allowed to have any dependence on m^2. The first and second constrains are already satisfied. The third one gives,

$$-23 \cdot l_1 = 0 \to 3 \cdot (k_1 + \alpha_1 q_1) = 0 \to 3 \cdot q_1 = 0$$

As $|q_1\rangle = |3\rangle$ is forbidden, $|q_1] = |3]$. The fourth and fifth constrains imposes,

$$\begin{cases} -n \cdot (\alpha_1 q_1 + \alpha_2 q_2) + \alpha_1 \alpha_2 q_1 \cdot q_2 &= 0 \\ 4 \cdot (\alpha_1 q_1 + \alpha_2 q_2) &= 0 \end{cases}$$

This imposes actually $q_1 \cdot q_2 = 0$, for this to be true we have to options, either $|q_2] = |3]$, or $|q_2\rangle = |q_1\rangle$. If we choose the first, we can shift k_1 by 3 such to make $|q_1\rangle = |4\rangle$, this imposes further $|q_2\rangle = |4\rangle$. Hence, a possible solution is,

$$q_1 = q_2 = -|3]\langle 4|$$

1.3. **massive legs second topology.** The constrains now are slightly different,

$$l_1^2 = (3-l_1)^2 = 0 \ \&\ l_2^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

Which has as solution $q_2 = -|4]\langle 3|$

1.4. **massive legs third topology.** Now the constrain is difficult to solve,

$$l_2^2 = 0 \ \&\ l_1^2 = (3-l_1)^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

The first, second and third constrains give, $l_2 = -|l_2]\langle l_2|$, $l_1 = -|l_1]\langle 3| - \alpha|3]\langle l_1|$, which is get by shifting $|l_1]$. Now, the fourth constrain gives,

$$l_2 \cdot (3-l_1) = 0$$

Expanding $|l_2] = x|l_1] + y|3]$,

$$\begin{aligned} x[l_1 3]\langle 3 l_2\rangle - y[3 l_1]\langle 3 l_2\rangle - x\alpha[l_1 3]\langle l_1 l_2\rangle &= 0 \\ (x+y)\langle 3 l_2\rangle &= x\alpha\langle l_1 l_2\rangle \end{aligned}$$

The fifth constrain gives,

$$\begin{aligned} 4 \cdot (3-l_1-l_2) &= 0 \\ [4|(-|3]\langle 3| + |l_1]\langle 3| + \alpha|3]\langle l_1| + x|l_1]\langle l_2| + y|3]\langle l_2|)|4\rangle &= 0 \\ -[43]\langle 34\rangle + [4l_1]\langle 34\rangle + \alpha[43]\langle l_1 4\rangle + x[4l_1]\langle l_2 4\rangle + y[43]\langle l_2 4\rangle &= 0 \\ [4l_1](\langle 34\rangle + x\langle l_2 4\rangle) + [43](y\langle l_2 4\rangle - \langle 34\rangle + \alpha\langle l_1 4\rangle) &= 0 \end{aligned}$$

This fixes, $x\langle l_2 4\rangle = -\langle 34\rangle = -y\langle l_2 4\rangle - \alpha\langle l_1 4\rangle$, hence, $|l_2\rangle = -\frac{1}{x}|3\rangle + \mu|4\rangle$, & $|l_1\rangle = \frac{1}{\alpha}\left(1+\frac{y}{x}\right)|3\rangle + \nu|4\rangle$ Plugging this back on the other constrain gives,

$$
\begin{aligned}
(x+y)\mu\langle 34\rangle &= x\alpha\Big(\frac{\mu}{\alpha}\Big(1+\frac{y}{x}\Big)\langle 34\rangle - \frac{\nu}{x}\langle 43\rangle\Big)\\
(x+y)\mu\langle 34\rangle &= \mu(x+y)\langle 34\rangle - \alpha\nu\langle 43\rangle
\end{aligned}
$$

This forces $\nu = 0$, which is not a possible solution. The only way to get around this is to impose $|l_1\rangle = |l_2\rangle$. This imposes $x+y=0$ and $|l_2\rangle = z|4\rangle$. But this itself is only a solution if we shift $|l_1] \to |l_1] + |3]$. That is,

$$
\begin{cases}
l_1 &= -|3]\langle 3| - |l_1]\langle 3| - \frac{m^2}{[l_1 3]\langle 43\rangle}|3]\langle 4|\\
l_2 &= -z(|3] - |l_1])\langle 4|
\end{cases}
$$