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Super Riemann Surfaces

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Abstract: Abstract...

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1 Introduction/Motivation

The Bosonic String Theory (BST) is known to achieve several desirable properties which up to present date haven't been done in usual Quantum Field Theory, the most prominent one is it being a perturbatively renormalizable theory which contains in its spectrum a massless spin-2 particle, this perturbative computation of amplitudes in BST is almost only possible due to the heavy simplifications the anomaly free gauge group Diff(M)$\times$Weyl allows[1]. This means, as in the path integral we're integrating over metrics, the gauge redundancies permits us to forget about the metrics and to integrate over only the different kinds of topologies of two dimensional manifolds, so that in a generic string scattering situation, what would be a non-compact generic two dimensional manifold turns into a compact two dimensional manifold — a choice over the equivalence class created by the gauge group: sphere, torus, ... —, and what was the asymptotic states — the *non-compact part* of the original manifold — turns into *punctures* in the new compact two dimensional manifold. The advantages is, this process is nicely described by complex coordinates in the two dimensional (real) manifold, where the gauge transformations amounts to holomorphic change of complex coordinates, and the study of such objects, complex coordinates in two dimensional (real) manifolds, or better, one dimensional complex manifolds, has already lots of years of development in mathematics which we can borrow, these are called Riemann Surfaces[1] (RS).

Despite being a astonishing success in some points, BST still fails, at least perturbatively, to give any room to accommodate the particle zoo present at our world, principally, there are no means of introducing fermions in the target space theory, this, among other

[1] There is actually a distinction of a Riemann Surface and a two dimensional (real) manifold, every Riemann Surface is a two dimensional (real) manifold, but the converse is not true.

reasons, is the motif of pursuing other types of theories. A natural guess to overcome the fermion problem is to introduce world-sheet fermions ψ^μ[2, 3][2],

$$S \sim \int\limits_M \mathrm{d}^2 z \left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + \text{ghosts}\right) \tag{1.1}$$

which under quantization gives an analogous problem with the one present in BST[3],

$$\begin{aligned} \left[X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\right] &= \mathrm{i}\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\ \left[\psi^\mu(\tau,\sigma), \psi^\nu(\tau,\sigma')\right] = \left[\tilde{\psi}^\mu(\tau,\sigma), \tilde{\psi}^\nu(\tau,\sigma')\right] &= \pi\eta^{\mu\nu}\delta(\sigma-\sigma') \end{aligned}$$

that is, time-like fields $X^0, \psi^0, \tilde{\psi}^0$ have wrong sign commutator, which implies they will create ghost states in the theory, the resolution in BST is to use the gauge group — a.k.a. the Virasoro constrains —, to remove these non-physical states, but here, the best we could do is to use again the Virasoro constrains to get rid of the bosonic wrong sign states, and we would still had the fermionic wrong sign states. Here the only possible resolution is to find an other gauge redundancy of this theory, such that we can use it to eliminate the non-physical states. Luckily, this new action provides a possible candidate of gauge redundancy, as it has a $\mathcal{N}=1$ global supersymmetry (SUSY),

$$\delta_\epsilon X^\mu = -\epsilon\psi^\mu - \epsilon^* \tilde{\psi}^\mu \tag{1.2a}$$

$$\delta_\epsilon \psi^\mu = \epsilon \partial X^\mu, \quad \delta_\epsilon \tilde{\psi}^\mu = \epsilon^* \bar{\partial} X^\mu \tag{1.2b}$$

Sadly enough, this supersymmetry algebra only closes on-shell and is global instead of local, despite this, one by one these issues can be unveiled. The uplift from a global symmetry to a local redundancy can be done by means of introducing a new field in the action, the world-sheet gravitino, and the promotion of the algebra closing off-shell can also be addressed by the inclusion of an auxiliary field in the action. Both these constructions are essential to Superstring Theory (SST), and the resulting theory enjoys a superconformal gauge group, which is given by our familiar super Virasoro algebra[4],

$$\begin{cases} T(z) & \sim \partial X \partial X + \cdots \\ G(z) & \sim \psi \partial X + \cdots \end{cases} \Rightarrow \begin{cases} T(z)T(w) & \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\ T(z)G(w) & \sim \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \\ G(z)G(w) & \sim \frac{2T(w)}{(z-w)^2} \end{cases} \Rightarrow \begin{cases} [L_m, L_n] & = (m-n)L_{m+n} \\ [G_r, G_s] & = 2L_{r+s} \\ [L_m, G_r] & = \left(\frac{m}{2} - r\right)G_{m+r} \end{cases}$$

The downside here is: in going from a conformal theory — which we could benefit from developments in RS —, to a superconformal theory, there seems to be a loss of geometrical visualization — as due to $G(z)$ being fermionic is not clear how it's action on the coordinates z should be interpreted — that could affect our, before mentioned, *ease* of computing scattering amplitudes. To maintain the geometric interpretation and the off-shell supersymmetry is the role of the Super Riemann Surfaces (SRS).

[2]We'll ignore multiplicative factors and set $\alpha' = 2$ which can be restored by dimensional analysis.
[3]We're using the graded commutator notation.
[4]With the inclusion of ghosts.

2 Super Riemann Surfaces and Consequences

2.1 Definition of SRS

As we foretold, to obtain a geometrical description of the superconformal structure we need a geometrical description of the generators $T(z), G(z)$, in the bosonic case we have such a description as differential operators such $T(z) \to L_n \sim z^{n+1}\partial_z$. But, to do so for $G(z)$ — which is grassmann odd — would require to have a grassmann odd coordinate. This is the start of the history of SRS. They're a special type of **Complex supermanifolds**, so, in order to understand them, we need first to understand what is a complex supermanifold.

Definition 2.1. *A* ***complex supermanifold*** Σ *of dimension* $1|1$ *is a space locally isomorphic to* $\mathbb{C}^{1|1}$*, that is, it's locally covered by coordinate charts* $z|\theta : U \subset \Sigma \to \mathbb{C}^{1|1}$ *such that* z *is a complex grassmann even coordinate, and* θ *is a complex grassmann odd coordinate.*

Of course we have to say something about the transition functions between overlapping charts. Due to the complex nature, we really just have two kinds of choice, either we impose that all transition functions are just continuous, or that they are holomorphic. It's clear that for our context the latter is way more useful, so we'll choose it. Maybe it's needed clarification on what we mean by a transition function $(z_i|\theta_i)^{-1} \circ z_j|\theta_j : z_j|\theta_j(U_j) \subset \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ to be holomorphic.

Definition 2.2. *A function* $f(z,\theta) : U \subset \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ *is said to be holomorphic iff the expansion in powers of* θ *— which is finite due to the oddness —,*

$$f(z,\theta) = f_0(z) + \theta f_1(z)$$

has both $f_0, f_1 : U \cap \mathbb{C} \subset \mathbb{C} \to \mathbb{C}^{1|1}$ *holomorphic.*

Another definition which will be of use for us is the notion of non-zero objects in a supermanifold.

Definition 2.3. *Any* ***non-zero*** *object — function, vector, ... — is one such that is non-zero up to grassmann odd variables.*

With this we have all the ingredients to state the definition of a SRS, the point is: we want for the SRS to have a notion of superconformal transformations — which is compatible with the super Virasoro algebra from SST — as the RS had. The good thing about RS is, they don't need actually more structure than being a complex manifold, as any transformation $z \to f(z)$ is indeed a conformal transformation. Two distinct characteristic of conformal transformations in SR are: they preserve the tangent space, that is $\partial_z \to \partial_z z' \partial_{z'}$, and they preserve angles. These properties cannot be used immediately to give a prescription of super conformal transformation. If we try to set them as being all the set of transformations which preserve the tangent space of a complex supermanifold we would get actually an algebra that is way bigger than the super Virasoro algebra, and to define them as being *angle preserving* would require to introduce a metric, which is too much of additional structure.

The idea is to look at what is really essential for us to have, if we're hoping to get any non-trivial theory with super conformal transformations, we need some kind of derivative operator which transforms covariantly under such. This is not the end, as we need to guarantee — in a coordinate invariant way — that covariant transformations of this derivative operator — which we'll call super conformal transformations — do in fact *mix* the grassmann odd and even coordinates. While seem kind of arbitrary, this requirement already excludes lot's of possible choices as ∂_z and ∂_θ. Let's sum up what we concluded, to set up a SRS we need a notion of super conformal transformation, which we concluded is equivalent of choosing a covariant derivative — what can be seen as choosing a subspace of the tangent space $T\Sigma$ —, but no just any covariant derivative, one such that guarantees that we're in fact *coupling* both coordinates. There are just two possible choices for the dimension of the subspace we're choosing, $0|1$ or $1|0$. We argue that it's impossible to choose a $1|0$ dimensional subspace of $T\Sigma$ and guarantee that in some sense the covariance under it *couples* both coordinates, this is so because no operation done with only grassmann even variables can return a grassmann odd variable. But, the opposite is true! It's possible to compose two grassmann odd variables to give a grassmann even one. Hence, we conclude that to define a notion of super conformal transformation is needed to choose a $0|1$ dimensional subspace of $T\Sigma$, what about the condition to ensure *mixing* of coordinates? Let's save for the definition[4–6],

Definition 2.4. *A **Super Riemann Surface** Σ is a complex supermanifold of dimension $1|1$ that possesses a distinct subbundle/subspace $\mathcal{D} \subset T\Sigma$ of dimension $0|1$ that satisfy the completely non-integrability condition,*

$$\forall D \in \mathcal{D} : D \text{ is non-zero} \Rightarrow [D, D] \notin \mathcal{D}$$

$\mathcal{D}$ is said to be a complex structure of this SRS.

Here we see the beauty of this definition, as we discussed, we have the distinguished structure as a $0|1$ dimensional subspace of $T\Sigma$, and the condition that it must satisfy is the so called *completely non-integrability*, which is dependent on the graded Lie Bracket operation defined for vector fields $[\cdot, \cdot]$, what is important here is that $[D, D]$ is only non-zero if D is odd, and in this case, by statistics, it returns a even vector field. Here the hypothesis of D being non-zero is crucial, otherwise, $[D, D]$ could be zero and so, despite being even, could still belong to $\mathcal{D}$. Hence, the completely non-integrability condition provides a notion of *coupling* both coordinates without having to evoke new structure in our manifold. It's instructive to see what kind of elements belongs to this subbundle,

Lemma 2.1. *Given a SRS with distinct subbundle $\mathcal{D}$, it's always possible to construct a coordinate system $z|\theta$, such that locally, any element $D \in \mathcal{D}$ is of the form,*

$$D_\theta := D_{U_{z|\theta}} = \partial_\theta + \theta\partial_z, \quad D_\theta^2 := \frac{1}{2}[D_\theta, D_\theta] = \partial_z \tag{2.1}$$

we call this coordinate system a superconformal one.

Proof. In a given coordinate system/chart $z|\theta$ we always can decompose any element of the tangent space as,

$$D_{U_{z|\theta}} = a(z,\theta)\partial_\theta + b(z,\theta)\partial_z$$

the condition of $D_{U_{z|\theta}}$ being non-zero, as stated in definition 2.4, is equivalent to $a(z,\theta) \neq 0$, hence, it's possible to scale $\theta \to a\theta$, doing this and also expanding, due to the grassmann odd character, $b(z,\theta) = b_0(z) + \theta b_1(z)$,

$$D_{U_{z|\theta}} = \partial_\theta + (b_0 + \theta b_1)\partial_z$$

which now we compute the graded Lie Bracket,

$$\begin{aligned}
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= [\partial_\theta + (b_0+\theta b_1)\partial_z, \partial_\theta + (b_0+\theta b_1)\partial_z] \\
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= 2[\partial_\theta, (b_0+\theta b_1)\partial_z] + [(b_0+\theta b_1)\partial_z, (b_0+\theta b_1)\partial_z] \\
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= 2\partial_\theta(b_0+\theta b_1)\partial_z + 2(b_0+\theta b_1)[\partial_\theta,\partial_z] + 2(b_0+\theta b_1)\big(\partial_z(b_0+\theta b_1)\partial_z + (b_0+\theta b_1)\partial_z^2\big) \\
\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] &= 2b_1\partial_z + 2\theta b_1\partial_z b_0\partial_z - 2\theta b_0\partial_z b_1\partial_z \qquad (2.2)
\end{aligned}$$

so, the only way $\left[D_{U_{z|\theta}}, D_{U_{z|\theta}}\right] \in \mathcal{D}$ could possibly be true is if the above expression is identically zero, which is only possible if $b_1 = 0$. Thus, the completely non-integrability condition gives the requirement $b_1 \neq 0$, hence, we can perform a further change of coordinates $\theta \to -b_1^{-1}b_0 + \theta$, $z \to b_1 z$,

$$D_\theta \coloneqq D_{U_{z|\theta}} = \partial_\theta + \theta\partial_z$$

And lastly, setting $b_1 = 1, b_0 = 0$ in eq. (2.2) we get, $D_\theta^2 = \partial_z$. ∎

The existence of such structure is what settles apart a SRS from a generic complex supermanifold. To grasp a better understanding of this non-integrability condition we have to take a look at the second equation in (2.1), for a usual coordinate basis of the tangent space we always have $[\partial_I, \partial_J] = 0$, which kind of induces a splitting of the tangent space as $\mathbb{C}^{1|0} \times \cdots \times \mathbb{C}^{0|1} \times \cdots$, but of course, as this is a coordinate basis, this splitting is not some kind of inner property of the manifold, it's a coordinate dependent gimmick. But, with our definition being coordinate independent, we're saying that existis a global splitting of the tangent space $T\Sigma \cong \mathbb{C}^{1|0} \times \mathbb{C}^{0|1} \cong T\Sigma/\mathcal{D} \times \mathcal{D} \cong \mathcal{D}^2 \times \mathcal{D}$.

We will now pursue why does this structure is able to reconstruct the off-shell super Virasoro algebra in SST.

2.2 Superconformality in SRS

As we foretold, having a distinct subbundle $\mathcal{D}$, is possible for us to define super conformal coordinate changes. The idea is, vector fields naturally introduce coordinate changes by integral curves, hence,

Definition 2.5. *A vector field $W \in T\Sigma$ is said to generate a super conformal coordinate transformation on a SRS Σ, if it preserves the subbundle $\mathcal{D}$, that is,*

$$W \in T\Sigma \text{ generates superconformal transformation} \Leftrightarrow \forall D \in \mathcal{D} : [W, D] \in \mathcal{D}.$$

As good as having a formal definition may be, our true interest here is to obtain these transformations in a given basis, in particular, a superconformal basis,

Lemma 2.2. *The set of all vector fields that generate superconformal transformation can be decomposed in a basis, with a superconformal coordinate system, of one even and one odd vector fields such,*

$$G_f = f(z)(\partial_\theta - \theta\partial_z), \quad L_g = g(z)\partial_z + \frac{1}{2}g'(z)\theta\partial_\theta \tag{2.3}$$

Proof. We will compute it using D_θ and a decomposition of $W = a\partial_\theta + b\partial_z$

$$[W, D_\theta] = [a\partial_\theta + b\partial_z, D_\theta] = a[\partial_\theta, D_\theta] \mp D_\theta a\partial_\theta + b[\partial_z, D_\theta] \mp D_\theta b\partial_z$$
$$[W, D_\theta] = a\partial_z \mp D_\theta a\partial_\theta \mp D_\theta b\partial_z = \mp D_\theta a\partial_\theta \mp (D_\theta b \mp a)\partial_z$$

Where the signs reefer to W being even, top sign, or odd, bottom sign. To say that $[W, D_\theta] \in \mathcal{D}$ is the same to say that $[W, D_\theta] \propto D_\theta$, hence, the condition is,

$$\mp(D_\theta b \mp a) = \mp D_\theta a\theta$$
$$D_\theta b = D_\theta a\theta \pm a$$

In this form it may seem difficult to solve it, but, we will use our virtue of foresight to propose an ansatz, $b = -a\theta$,

$$D_\theta(-a\theta) = D_\theta a\theta \pm a \Rightarrow -D_\theta a\theta \pm a = D_\theta a\theta \pm a \Rightarrow D_\theta a\theta = 0 \Rightarrow \partial_\theta a\theta = 0 \Rightarrow \begin{cases} a & = a(z) \\ b & = -a(z)\theta \end{cases}$$

With this ansatz we got exactly an solution, that is, one family of odd vector fields that generate superconformal transformations are $G_f = f(z)(\partial_\theta - \theta\partial_z)$. But, this is not the end, as we have two dimensions, one odd and one even, we know that there is one more solution to this equation,

$$D_\theta b = D_\theta a\theta \pm a$$
$$D_\theta^2 b = D_\theta^2 a\theta \pm D_\theta a \pm D_\theta a$$
$$\partial_z b = \partial_z a\theta \pm 2\partial_\theta a \pm 2\theta\partial_z a = -\partial_z a\theta \pm 2\partial_\theta a$$

it's clear that the condition $\partial_\theta a = 0$ will return our already found solution, hence, we try to solve this equation for $\partial_z a\theta = 0 \Rightarrow \partial_z b = \pm 2\partial_\theta a$, substituting back in the original equation,

$$\partial_\theta b + \theta\partial_z b = \partial_\theta a\theta \pm a$$
$$\partial_\theta b \pm 2\theta\partial_\theta a = \partial_\theta a\theta \pm a\partial_\theta\theta$$
$$\partial_\theta b + 2\partial_\theta a\theta = \partial_\theta a\theta \pm a\partial_\theta\theta$$
$$\partial_\theta b = -\partial_\theta a\theta \pm a\partial_\theta\theta = -\partial_\theta(a\theta)$$

As we already solve for $b = -a\theta$, the only other possible solution that can be obtained from here is $\partial_\theta b = \partial_\theta(a\theta) = 0 \Rightarrow a = c(z)\theta,\ b = b(z)$, and the last consistency condition gives,

$$\partial_z b = \pm 2\partial_\theta a \Rightarrow \partial_z b = 2c(z) \Rightarrow \begin{cases} a & = \frac{1}{2}\partial_z b(z)\theta \\ b & = b(z) \end{cases}$$

which give to us the second and last linear independent solution, a family of even vector fields that generate superconformal transformations $L_g = g(z)\partial_z + \frac{1}{2}g'(z)\theta\partial_\theta$. ∎

It's no coincidence the naming we utilized in eq. (2.3), as those two are the generators of the superconformal transformations, they should also be related to the SST super Virasoro generators L_n, G_r, and actually this is true, they indeed furnish a differential representation of the super Virasoro,

Lemma 2.3. *The vector field basis, eq. (2.3), of superconformal transformations for $f(z) = z^{r+\frac{1}{2}}, g(z) = -z^{n+1}$,*

$$L_n = -z^{n+1}\partial_z - \frac{n+1}{2}z^n\theta\partial_\theta, \quad G_r = z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z), \quad n \in \mathbb{Z},\ r \in \mathbb{Z} + \frac{1}{2} \tag{2.4}$$

furnishes a representation of the super Virasoro algebra for the NS sector,

$$[L_m, L_n] = (m-n)L_{m+n} \tag{2.5a}$$
$$[G_r, G_s] = 2L_{r+s} \tag{2.5b}$$
$$[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r} \tag{2.5c}$$

Proof. The computation is very straightforward, we start with $L - L$,

$$[L_m, L_n] = \left[z^{m+1}\partial_z + \frac{m+1}{2}z^m\theta\partial_\theta, z^{n+1}\partial_z + \frac{n+1}{2}z^n\theta\partial_\theta\right]$$
$$[L_m, L_n] = \left[z^{m+1}\partial_z, z^{n+1}\partial_z\right] + \left[z^{m+1}\partial_z, \frac{n+1}{2}z^n\theta\partial_\theta\right] + \left[\frac{m+1}{2}z^m\theta\partial_\theta, z^{n+1}\partial_z\right]$$
$$[L_m, L_n] = z^{m+1}\partial_z z^{n+1}\partial_z - z^{n+1}\partial_z z^{m+1}\partial_z + \frac{n+1}{2}z^{m+1}\theta\partial_z z^n\partial_\theta - \frac{m+1}{2}z^{n+1}\theta\partial_z z^m\partial_\theta$$
$$[L_m, L_n] = (n+1)z^{m+n+1}\partial_z - (m+1)z^{n+m+1}\partial_z + \frac{(n+1)n}{2}z^{m+n}\theta\partial_\theta - \frac{(m+1)m}{2}z^{n+m}\theta\partial_\theta$$
$$[L_m, L_n] = -(m-n)z^{m+n+1}\partial_z + \left(n^2 + n - m^2 - m\right)\frac{z^{n+m}}{2}\theta\partial_\theta$$
$$[L_m, L_n] = -(m-n)z^{m+n+1}\partial_z - (m-n)(n+m+1)\frac{z^{n+m}}{2}\theta\partial_\theta = (m-n)L_{m+n}$$

now $G - G$,

$$[G_r, G_s] = \left[z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z), z^{s+\frac{1}{2}}(\partial_\theta - \theta\partial_z)\right]$$
$$[G_r, G_s] = -\left(z^{r+\frac{1}{2}}\theta\partial_z z^{s+\frac{1}{2}} + z^{s+\frac{1}{2}}\theta\partial_z z^{r+\frac{1}{2}}\right)(\partial_\theta - \theta\partial_z) + z^{r+s+1}[(\partial_\theta - \theta\partial_z), (\partial_\theta - \theta\partial_z)]$$
$$[G_r, G_s] = -\left(s + \frac{1}{2} + r + \frac{1}{2}\right)z^{r+s}\theta\partial_\theta - 2z^{r+s+1}\partial_z = 2\left(-z^{r+s+1}\partial_z - \frac{r+s+1}{2}z^{r+s}\theta\partial_\theta\right)$$
$$[G_r, G_s] = 2L_{r+s}$$

and lastly $L-G$,

$$
\begin{aligned}
[L_m, G_r] &= \left[-z^{m+1}\partial_z - \frac{m+1}{2}z^m\theta\partial_\theta, z^{r+\frac{1}{2}}(\partial_\theta - \theta\partial_z)\right] \\
[L_m, G_r] &= -z^{m+1}\partial_z z^{r+\frac{1}{2}}\partial_\theta + \frac{m+1}{2}z^{m+r+\frac{1}{2}}\partial_\theta + z^{m+1}\partial_z z^{r+\frac{1}{2}}\theta\partial_z - z^{r+\frac{1}{2}}\theta\partial_z z^{m+1}\partial_z \\
&\quad + \frac{m+1}{2}z^{m+r+\frac{1}{2}}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} + \frac{1}{2} - r - \frac{1}{2}\right)z^{m+r+\frac{1}{2}}\partial_\theta + \left(r + \frac{1}{2} - \frac{m}{2} - \frac{1}{2}\right)z^{m+r+\frac{1}{2}}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right)z^{m+r+\frac{1}{2}}(\partial_\theta - \theta\partial_z) = \left(\frac{m}{2} - r\right)G_{m+r}
\end{aligned}
$$

■

As we have mentioned before, this last result is one of the main reasons why SRS are useful, they make the superconformal algebra on-shell, it's instructive to see explicitly from eq. (2.5c) the realization of SUSY, $\left[G_{-\frac{1}{2}}, G_{-\frac{1}{2}}\right] = 2L_{-1} = -2\partial_z$. Nevertheless, there are still many points left to explanation, as stated in lemma 2.3, the resulting super Virasoro algebra from SRS are in the NS sector, that is, they only work for states with NS boundary conditions. As a reminder, NS boundary conditions are imposed in the world-sheet fermions (1.1) as $\psi^\mu(w+2\pi\mathrm{i}) = -\psi^\mu(w)$, with $w = \ln z$ the cylinder coordinate. We will not prove here, but, it's impossible to reconstruct the super Virasoro algebra in the R sector — $\psi^\mu(w+2\pi\mathrm{i}) = \psi^\mu(w) \Rightarrow G_r,\ r \in \mathbb{Z}$ — with (2.3). While this may seem a big problem, there is an interesting interpretation of why this is the case, and in fact this is no obstruction to the introduction of R sector fermions, for now we just cite what is the form of the super Virasoro generators for the R sector, and later on there will be a explanation on why this should be true,

Lemma 2.4. *The following vector fields,*

$$
L_n = -z^{n+1}\partial_z - \frac{n}{2}z^n\theta\partial_\theta, \quad G_r = z^r(\partial_\theta - \theta z\partial_z), \quad n, r \in \mathbb{Z} \tag{2.6}
$$

furnishes a differential representation of the super Virasoro algebra for the R sector,

$$
[L_m, L_n] = (m-n)L_{m+n} \tag{2.7a}
$$

$$
[G_r, G_s] = 2L_{r+s} \tag{2.7b}
$$

$$
[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r} \tag{2.7c}
$$

Proof. Again, the calculation is very straightforward, we start with $L - L$,

$$
\begin{aligned}
[L_m, L_n] &= \left[z^{m+1}\partial_z + \frac{m}{2}z^m\theta\partial_\theta, z^{n+1}\partial_z + \frac{n}{2}z^n\theta\partial_\theta\right] \\
[L_m, L_n] &= \left[z^{m+1}\partial_z, z^{n+1}\partial_z\right] + \left[z^{m+1}\partial_z, \frac{n}{2}z^n\theta\partial_\theta\right] + \left[\frac{m}{2}z^m\theta\partial_\theta, z^{n+1}\partial_z\right] \\
[L_m, L_n] &= z^{m+1}\partial_z z^{n+1}\partial_z - z^{n+1}\partial_z z^{m+1}\partial_z + \frac{n}{2}z^{m+1}\theta\partial_z z^n\partial_\theta - \frac{m}{2}z^{n+1}\theta\partial_z z^m\partial_\theta \\
[L_m, L_n] &= (n+1)z^{m+n+1}\partial_z - (m+1)z^{n+m+1}\partial_z + \frac{n^2}{2}z^{m+n}\theta\partial_\theta - \frac{m^2}{2}z^{n+m}\theta\partial_\theta \\
[L_m, L_n] &= -(m-n)z^{m+n+1}\partial_z + \left(n^2 - m^2\right)\frac{z^{n+m}}{2}\theta\partial_\theta \\
[L_m, L_n] &= -(m-n)z^{m+n+1}\partial_z - (m-n)(n+m)\frac{z^{n+m}}{2}\theta\partial_\theta = (m-n)L_{m+n}
\end{aligned}
$$

now $G - G$,

$$
\begin{aligned}
[G_r, G_s] &= [z^r(\partial_\theta - \theta z\partial_z), z^s(\partial_\theta - \theta z\partial_z)] \\
[G_r, G_s] &= -\left(z^{r+1}\theta\partial_z z^s + z^{s+1}\theta\partial_z z^r\right)(\partial_\theta - \theta z\partial_z) + z^{r+s}[(\partial_\theta - \theta z\partial_z), (\partial_\theta - \theta z\partial_z)] \\
[G_r, G_s] &= -(s+r)z^{r+s}\theta\partial_\theta - 2z^{r+s}z\partial_z = 2\left(-z^{r+s+1}\partial_z - \frac{r+s}{2}z^{r+s}\theta\partial_\theta\right) \\
[G_r, G_s] &= 2L_{r+s}
\end{aligned}
$$

and lastly $L - G$,

$$
\begin{aligned}
[L_m, G_r] &= \left[-z^{m+1}\partial_z - \frac{m}{2}z^m\theta\partial_\theta, z^r(\partial_\theta - \theta z\partial_z)\right] \\
[L_m, G_r] &= -z^{m+1}\partial_z z^r\partial_\theta + \frac{m}{2}z^{m+r}\partial_\theta + z^{m+1}\partial_z z^{r+1}\theta\partial_z - z^{r+1}\theta\partial_z z^{m+1}\partial_z \\
&\qquad + \frac{m}{2}z^{m+r+1}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right)z^{m+r}\partial_\theta + \left(r + 1 - \frac{m}{2} - 1\right)z^{m+r+1}\theta\partial_z \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right)z^{m+r}(\partial_\theta - \theta z\partial_z) = \left(\frac{m}{2} - r\right)G_{m+r}
\end{aligned}
$$

■

While might seem taken out of a hat, this shows two main points: (i) that is possible to somewhat *embed* the R sector in SRS, but, (ii) the fact that (2.6) cannot be written as (2.3), together with lemma 2.2, shows that the super Virasoro R sector cannot possibly preserve our distinct subspace $\mathcal{D}$. There is a small flaw in this argument that may shed some hope in the interpretation of R fermions, which we are going to postpone. For now we follow working with just the NS sector.

2.3 Super fields and weights

Having now constructed SRS, and also shown that they do satisfy an on-shell super Virasoro algebra, the next step is to do a matching of how a theory formulated in SRS can give the action in eq. (1.1). The main ideia here is to follow analogous to BST, first to describe objects that transform covariantly under superconformal transformations, and later use them to construct a superconformal invariant action. It's instructive to first understand how θ behaves under dilations, which for now is the closest we have to a conformal weight. As we know, dilations are the integral curves of, eq. (2.4), $L_0 = -z\partial_z - \frac{1}{2}\theta\partial_\theta$, which are trivially $z|\theta \to \lambda z|\lambda^{\frac{1}{2}}\theta$. This kind of is a reminiscent of z having conformal weight -1 and θ having conformal weight $-\frac{1}{2}$, this is not really true, as we used global transformations, to get a real definition of what may be superconformal weights we have to look at general transformations. By now we are fully aware that superconformal transformations preserve $\mathcal{D}$, other way to pose it is, superconformal transformations of a superconformal coordinate system $z|\theta$ to another superconformal coordinate system $\hat{z}|\hat{\theta}$ changes D_θ to $F(z,\theta)D_{\hat{\theta}}$, hence, it should be true that,

$$\begin{aligned} D_\theta = F(z,\theta)D_{\hat{\theta}} \Rightarrow D_\theta\hat{\theta} &= F(z,\theta)D_{\hat{\theta}}\hat{\theta} \\ D_\theta\hat{\theta} &= F(z,\theta) \Rightarrow D_\theta = D_\theta\hat{\theta}D_{\hat{\theta}} \end{aligned} \tag{2.8}$$

which is the definition we will be taking of a superconformal transformation. Plenty other consistency conditions can be obtained from eq. (2.8) by applying it to $z, \hat{z}, \theta$. Those are specially useful if one want to show what is the most general form of a superconformal transformation explicitly, but, there isn't so much of insight coming from this, and thus we won't attempt it here.

3 Punctures in SRS

A Mathematical Toolkit

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