Homework I

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Problem 1

1.A)

The Nambu-Goto Action is given by:

$$S_{\mathrm{NG}} = -\frac{T}{\mathrm{c}} \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_{0}^{\pi} \mathrm{d}\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)} \tag{1.1}$$

Where we made the abbreviations,

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$$

The choice of the static gauge, together with considering the string being stretched only along the X^1 direction can be written as,

$$X^\mu(\tau, \sigma) = (\mathrm{c}\tau, f(\sigma), 0, \cdots, 0), \quad f(0) = 0 \ \& \ f(\pi) = a$$

Let's first compute what are the equations of motion,

$$\begin{aligned}
\delta S_{\mathrm{NG}} &= -\frac{T}{\mathrm{c}} \int \mathrm{d}^2\sigma \, \frac{2\left(\dot{X} \cdot X'\right)\left(\dot{X}^\alpha \delta X'_\alpha + X'^\alpha \delta \dot{X}_\alpha\right) - 2\dot{X}^2 X'^\alpha \delta X'_\alpha - 2X'^2 \dot{X}^\alpha \delta \dot{X}_\alpha}{2\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}} \\
&= -\frac{T}{\mathrm{c}} \int \mathrm{d}^2\sigma \, \frac{\delta \dot{X}_\alpha \left[\left(\dot{X} \cdot X'\right) X'^\alpha - X'^2 \dot{X}^\alpha\right] + \delta X'_\alpha \left[\left(\dot{X} \cdot X'\right)\dot{X}^\alpha - \dot{X}^2 X'^\alpha\right]}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}}
\end{aligned}$$

We define the conjugate momenta as to simplify our expression,

$$\mathcal{P}^{\tau\alpha} = -\frac{T}{\mathrm{c}} \frac{\left(\dot{X} \cdot X'\right) X'^\alpha - X'^2 \dot{X}^\alpha}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}} \tag{1.2}$$

$$\mathcal{P}^{\sigma\alpha} = -\frac{T}{\mathrm{c}} \frac{\left(\dot{X} \cdot X'\right) \dot{X}^\alpha - \dot{X}^2 X'^\alpha}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}} \tag{1.3}$$

So that our variation of the Action is,

$$\delta S_{\mathrm{NG}} = \int \mathrm{d}^2\sigma \left\{ \delta \dot{X}_\alpha \mathcal{P}^{\tau\alpha} + \delta X'_\alpha \mathcal{P}^{\sigma\alpha} \right\}$$

$$\delta S_{\mathrm{NG}} = \int \mathrm{d}^2\sigma \left\{ \frac{\partial}{\partial \tau}[\delta X_\alpha \mathcal{P}^{\tau\alpha}] - \delta X_\alpha \frac{\partial}{\partial \tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial \sigma}[\delta X_\alpha \mathcal{P}^{\sigma\alpha}] - \delta X_\alpha \frac{\partial}{\partial \sigma}\mathcal{P}^{\sigma\alpha} \right\}$$

$$\delta S_{\mathrm{NG}} = -\int \mathrm{d}^2\sigma\, \delta X_\alpha \left\{ \frac{\partial}{\partial \tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial \sigma}\mathcal{P}^{\sigma\alpha} \right\} + \int_0^\pi \mathrm{d}\sigma\, [\delta X_\alpha \mathcal{P}^{\tau\alpha}]\Big|_{\tau=-\infty}^{\tau=+\infty} + \int_{-\infty}^{+\infty} \mathrm{d}\tau\, [\delta X_\alpha \mathcal{P}^{\sigma\alpha}]\Big|_{\sigma=0}^{\sigma=\pi}$$

From imposing the Stationary Action Principle, we can easily read out both the Equations of Motion,

$$\frac{\partial}{\partial \tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial \sigma}\mathcal{P}^{\sigma\alpha} = 0 \tag{1.4}$$

And the Boundary Conditions

$$\delta X_\alpha \mathcal{P}^{\tau\alpha}\Big|_{\tau=-\infty}^{\tau=+\infty} = 0 = \delta X_\alpha \mathcal{P}^{\sigma\alpha}\Big|_{\sigma=0}^{\sigma=\pi} \tag{1.5}$$

With this three equations in hand, we just have to compute if the stretched string in the Static Gauge is a solution of them, first we calculate the derivatives,

$$\dot{X} = (\mathrm{c}, 0, \cdots, 0), \quad X' = (0, f'(\sigma), 0, \cdots, 0) \tag{1.6}$$

So now it's trivial that,

$$\dot{X} \cdot X' = 0, \quad \dot{X} \cdot \dot{X} = -\mathrm{c}^2, \quad X' \cdot X' = f'^2 \tag{1.7}$$

Plugging in those in 1.2,1.3:

$$\mathcal{P}^{\tau\alpha} = -\frac{T}{\mathrm{c}} \frac{-f'^2 \dot{X}^\alpha}{\sqrt{\mathrm{c}^2 f'^2}} = \frac{T}{\mathrm{c}} f'(1, 0, \cdots, 0) \tag{1.8}$$

$$\mathcal{P}^{\sigma\alpha} = -\frac{T}{\mathrm{c}} \frac{\mathrm{c}^2 X'^\alpha}{\sqrt{\mathrm{c}^2 f'^2}} = -T(0, 1, 0, \cdots, 0) \tag{1.9}$$

From where follows,

$$\frac{\partial}{\partial \tau}\mathcal{P}^{\tau\alpha} = \frac{T}{\mathrm{c}} \frac{\partial f'}{\partial \tau}(1, 0, \cdots, 0) = 0$$
$$\frac{\partial}{\partial \sigma}\mathcal{P}^{\sigma\alpha} = 0$$

Hence,

$$\frac{\partial}{\partial \tau}\mathcal{P}^{\tau\alpha} + \frac{\partial}{\partial \sigma}\mathcal{P}^{\sigma\alpha} = 0 + 0 = 0$$

That is, the Equations of Motion, 1.4, are satisfied for this string configuration! Now for the Boundary Conditions — 1.5 —, the first one, is trivially satisfied, that is due to the variations of the target space position, X, to which the Action is variated by, because, the initial and final time configuration of X are fixed given conditions, to change them would mean to solve another problem of initial conditions, so the variation δX must be zero at the initial and final times,

$$\delta X_\alpha \Big|_{\tau=-\infty}^{\tau=+\infty} = 0 \Rightarrow \delta X_\alpha \mathcal{P}^{\tau\alpha} \Big|_{\tau=-\infty}^{\tau=+\infty} = 0$$

What confirms the first Boundary Condition is true. For the second one, let's write the non null contributions to the Boundary Condition,

$$\delta X_\alpha \mathcal{P}^{\sigma\alpha} \Big|_{\sigma=0}^{\sigma=\pi} = \delta X_1 \mathcal{P}^{\sigma 1} \Big|_{\sigma=0}^{\sigma=\pi}$$

This is the case due to all the $\mathcal{P}^{\sigma\alpha}$ components being zero, except for $\alpha = 1$. But we have completely fixed X_1 at the endpoints, as know as the Dirichlet Boundary Conditions

$$X_1(\tau, 0) = 0, \quad X_1(\tau, \pi) = a \Rightarrow \delta X_1 \Big|_{\sigma=0}^{\sigma=\pi} = 0$$

Hence,

$$\delta X_\alpha \mathcal{P}^{\sigma\alpha} \Big|_{\sigma=0}^{\sigma=\pi} = \delta X_1 \mathcal{P}^{\sigma 1} \Big|_{\sigma=0}^{\sigma=\pi} = 0$$

Showing that our string configuration do satisfy the Boundary Conditions. There are two more constrains we have to verify, which follow from 1.2,

$$\begin{aligned} \mathcal{P}^{\tau\alpha} {X'}_\alpha &= 0 \\ \mathcal{P}^{\tau\alpha} \mathcal{P}^{\tau}_{\alpha} + \frac{T^2}{\mathrm{c}^2} X'^2 &= 0 \end{aligned}$$

First let us show that these are the right constrains,

$$\mathcal{P}^{\tau\alpha} {X'}_\alpha = -\frac{T}{\mathrm{c}} \frac{\left(\dot{X} \cdot X'\right) X' \cdot X' - X'^2 \dot{X} \cdot X'}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}} = 0$$

And,

$$\mathcal{P}^{\tau\alpha} \mathcal{P}^{\tau}_{\alpha} = \frac{T^2}{\mathrm{c}^2} \frac{\left(\dot{X} \cdot X'\right)^2 X'^2 + X'^4 \dot{X}^2 - 2\left(\dot{X} \cdot X'\right)^2 X'^2}{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}$$

$$\mathcal{P}^{\tau\alpha} \mathcal{P}^{\tau}_{\alpha} = \frac{T^2}{\mathrm{c}^2} X'^2 \frac{-\left(\dot{X} \cdot X'\right)^2 + X'^2 \dot{X}^2}{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X} \cdot \dot{X}\right)\left(X' \cdot X'\right)}$$

$$\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} = -\frac{T^2}{\mathrm{c}^2}{X'}^2 \Rightarrow \mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} + \frac{T^2}{\mathrm{c}^2}{X'}^2 = 0$$

Now we'll prove that these two constrains are true for our string configuration, this is easy, as we already have computed all the needed vectors, 1.6,1.8,

$$\mathcal{P}^{\tau\alpha}{X'}_{\alpha} = \frac{T}{\mathrm{c}}f'(1,0,\cdots,0)\cdot(0,f',0,\cdots,0)^{\mathrm{T}} = 0$$

And,

$$\begin{aligned}
\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} &= \frac{T^2}{\mathrm{c}^2}f'^2(1,0,\cdots,0)\cdot(-1,0,\cdots,0)^{\mathrm{T}} = -\frac{T^2}{\mathrm{c}^2}f'^2 \\
\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} &= -\frac{T^2}{\mathrm{c}^2}(0,f',\cdots,0)\cdot(0,f',\cdots,0)^{\mathrm{T}} = -\frac{T^2}{\mathrm{c}^2}X'^2 \\
\mathcal{P}^{\tau\alpha}\mathcal{P}^{\tau}_{\alpha} + \frac{T^2}{\mathrm{c}^2}X'^2 &= 0
\end{aligned}$$

This finishes our confirmation that indeed this string configuration is a proper solution.

1.B)

To evaluate the Nambu-Goto Action in this solution, we just have to make use of 1.7 in 1.1,

$$\begin{aligned}
S_{\text{NG-static}} &= -\frac{T}{\mathrm{c}}\int_{-\infty}^{+\infty}\mathrm{d}\tau\int_{0}^{\pi}\mathrm{d}\sigma\sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\cdot\dot{X}\right)\left(X'\cdot X'\right)} \\
S_{\text{NG-static}} &= -\frac{T}{\mathrm{c}}\int_{-\infty}^{+\infty}\mathrm{d}\tau\int_{0}^{\pi}\mathrm{d}\sigma\sqrt{\mathrm{c}^2 f'^2} = -T\int_{-\infty}^{+\infty}\mathrm{d}\tau\int_{0}^{\pi}\mathrm{d}\sigma\, f' \\
S_{\text{NG-static}} &= -T\int_{-\infty}^{+\infty}\mathrm{d}\tau\,(f(\pi)-f(0)) = -T\int_{-\infty}^{+\infty}\mathrm{d}\tau\, a
\end{aligned}$$

If we argue that the Action is of the form,

$$S = \int\mathrm{d}t\,[K - V]$$

Where K is the kinetic energy and V is the potential energy. As in our configuration everything is static, we shouldn't expect any kinetic energy present in the Action/Lagrangian, in other words, all the contribution of the action is solely from the potential energy, thus, making this identification,

$$\begin{aligned}
S_{\text{NG-static}} &= -\int_{-\infty}^{+\infty}\mathrm{d}\tau\,Ta = -\int_{-\infty}^{+\infty}\mathrm{d}\tau\,V \\
V &= Ta
\end{aligned}$$

This is a hint that T may be interpreted as energy per length, or, the tension of the string.

Problem 2

2.A)

The Polyakov Action is given by,

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{2.1}$$

With h^{ab} being the world-sheet metric, $h = \|\mathrm{Det}[h_{ab}]\|$, and $g_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$ the target space metric. A Poincare transformation of the fields X is,

$$\begin{aligned} X^\mu(\tau,\sigma) &\to \tilde{X}^\mu(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \\ \partial_a X^\mu &\to \partial_a \tilde{X}^\mu = \Lambda^\mu{}_\nu \partial_a X^\nu \end{aligned}$$

With of course Λ satisfying the defining property of a Lorentz transformation,

$$g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

The transformed Action is,

$$\begin{aligned} \tilde{S}_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}g_{\mu\nu}\partial_a \tilde{X}^\mu \partial_b \tilde{X}^\nu \\ \tilde{S}_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \partial_a X^\alpha \partial_b X^\beta \\ \tilde{S}_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}g_{\alpha\beta}\partial_a X^\alpha \partial_b X^\beta = S_{\mathrm{P}} \end{aligned}$$

Hence the Poincare group is indeed a global symmetry of the Action. To obtain the conserved currents we have to first know what are the equations of motion,

$$\begin{aligned} \delta S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left\{\sqrt{h}h^{ab}g_{\mu\nu}2\partial_a X^\mu \delta\partial_b X^\nu + \sqrt{h}\delta h^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\sqrt{h}h_{ab}\delta h^{ab}h^{cd}g_{\mu\nu}\partial_c X^\mu \partial_d X^\nu\right\} \\ \delta S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left\{\sqrt{h}h^{ab}2\partial_a X^\mu \delta\partial_b X_\mu + \sqrt{h}\delta h^{ab}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\sqrt{h}h_{ab}\delta h^{ab}\partial_c X^\mu \partial^c X_\mu\right\} \\ \delta S_{\mathrm{P}} &= -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_b\Big[\sqrt{h}\partial^b X^\mu \delta X_\mu\Big] + \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_b\Big[\sqrt{h}\partial^b X^\mu\Big]\delta X_\mu \\ &\quad - \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\delta h^{ab}\Big[\sqrt{h}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\sqrt{h}h_{ab}\partial_c X^\mu \partial^c X_\mu\Big] \end{aligned}$$

Each of the three terms has to vanish independently, the first of them is just a Boundary Condition,

$$\int \mathrm{d}^2\sigma\,\partial_b\Big[\sqrt{h}\partial^b X^\mu \delta X_\mu\Big] = 0 \tag{2.2}$$

The second gives the equations for X,

$$\partial_a\left[\sqrt{h}\partial^a X^\mu\right]=0 \tag{2.3}$$

And the last one give the equations for h,

$$\sqrt{h}\partial_a X^\mu\partial_b X_\mu-\frac{1}{2}\sqrt{h}h_{ab}\partial_c X^\mu\partial^c X_\mu=0 \tag{2.4}$$

Armed with these, we can consider now just a variation on X, which is a symmetry of the Action, in our case this will be a Poincare transformation,

$$\begin{aligned}
\delta S_{\mathrm{P}}&=-\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{h}\partial^b X^\mu\partial_b\delta X_\mu\\
\delta S_{\mathrm{P}}&=-\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\left\{\partial_b\left[\sqrt{h}\partial^b X^\mu\delta X_\mu\right]-\partial_b\left[\sqrt{h}\partial^b X^\mu\right]\delta X_\mu\right\}
\end{aligned}$$

Imposing the fields to obey the equations of motion, 2.3, the second term vanishes identically. And also using our already derived result that $\tilde{S}_{\mathrm{P}}=S_{\mathrm{P}}\Rightarrow\delta S_{\mathrm{P}}=0$, we get the simple expression, for δX being the variation under a Poincare transformation,

$$\delta S_{\mathrm{P}}=-\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu\delta X_\mu\right]=0$$

In the case of a pure translation, $\delta X=\tilde{X}-X=a$,

$$-\frac{1}{2\pi\alpha'}a_\mu\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu\right]=0$$

From where we can read the conserved current associated with translations,

$$\mathcal{P}^{b\mu}=-\frac{\sqrt{h}}{2\pi\alpha'}\partial^b X^\mu,\quad \partial_b\mathcal{P}^{b\mu}=0 \tag{2.5}$$

We can do the same for a Lorentz transformation, $\delta X^\mu=\tilde{X}^\mu-X^\mu=\omega^\mu{}_\nu X_\nu$, with of course $\omega_{\mu\nu}=-\omega_{\nu\mu}$, being the infinitesimal part of the Lorentz transformation, $\Lambda=\mathbb{1}+\omega$,

$$\begin{aligned}
-\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu\omega_{\mu\nu}X^\nu\right]&=0\\
-\frac{1}{2\pi\alpha'}\omega_{\mu\nu}\int \mathrm{d}^2\sigma\,\partial_b\left[\sqrt{h}\partial^b X^\mu X^\nu-\sqrt{h}\partial^b X^\nu X^\mu\right]&=0
\end{aligned}$$

So that the conserved current associated with Lorentz transformations is,

$$\mathcal{M}^{b\mu\nu}=-\frac{\sqrt{h}}{2\pi\alpha'}\left[X^\mu\partial^b X^\nu-X^\nu\partial^b X^\mu\right]=X^\mu\mathcal{P}^{b\nu}-X^\nu\mathcal{P}^{b\mu},\quad \partial_b\mathcal{M}^{b\mu\nu}=0$$

Lastly, the conserved charges that follow from the conserved currents are,

$$P^{\mu} = \int \mathrm{d}\sigma\, \mathcal{P}^{\tau\mu} = -\frac{1}{2\pi\alpha'} \int \mathrm{d}\sigma\, \sqrt{h}\partial^{\tau} X^{\mu} \tag{2.6}$$

$$M^{\mu\nu} = \int \mathrm{d}\sigma\, \mathcal{M}^{\tau\mu\nu} = -\frac{1}{2\pi\alpha'} \int \mathrm{d}\sigma\, \sqrt{h}[X^{\mu}\partial^{\tau} X^{\nu} - X^{\nu}\partial^{\tau} X^{\mu}] \tag{2.7}$$

2.B)

We now turn to the matter of verifying that the conserved charges derived here do obey the Poincare algebra, for this we'll need the Poisson Brackets, which are defined with respect to $X^{\mu}(t,\sigma)$, and it's conjugate momentum $\frac{\partial\mathcal{L}}{\partial\partial_{\tau}X^{\mu}} = \mathcal{P}^{\tau}{}_{\mu} \equiv \Pi_{\mu}(\tau,\sigma)$, which fortunately we already computed. The metric h does not enter in the Poisson Brackets because it's not dynamical, it has three degrees of freedom, but we also have three gauge redundancies, just enough to make it non-dynamical. The fundamental Poisson Bracket relations are,

$$\{X^{\mu}(\tau,\sigma), X^{\nu}(\tau,\sigma')\} = 0, \quad \{\Pi^{\mu}(\tau,\sigma), \Pi^{\nu}(\tau,\sigma')\} = 0, \quad \{X^{\mu}(\tau,\sigma), \Pi^{\nu}(\tau,\sigma')\} = \delta(\sigma-\sigma')g^{\mu\nu}$$

Just for completeness, we'll rewrite 2.6,2.7 in function of the canonical variables,

$$P^{\mu} = \int \mathrm{d}\sigma\, \Pi^{\mu}$$
$$M^{\mu\nu} = \int \mathrm{d}\sigma\, [X^{\mu}\Pi^{\nu} - X^{\nu}\Pi^{\mu}]$$

We'll start by the $P-P$ — we'll not keep track of the τ dependence in the conserved charges, because, they are conserved. But nevertheless, everything is assumed to be evaluated at equal τ —,

$$\{P^{\mu}, P^{\nu}\} = \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \{\Pi^{\mu}(\tau,\sigma), \Pi^{\nu}(\tau,\sigma')\} = 0$$

Next the $P-M$,

$$\begin{aligned}
\left\{P^{\mu}, M^{\alpha\beta}\right\} &= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left\{\Pi^{\mu}(\tau,\sigma), X^{\alpha}(\tau,\sigma')\Pi^{\beta}(\tau,\sigma') - X^{\beta}(\tau,\sigma')\Pi^{\alpha}(\tau,\sigma')\right\} \\
&= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left[X^{\alpha}(\tau,\sigma')\left\{\Pi^{\mu}(\tau,\sigma), \Pi^{\beta}(\tau,\sigma')\right\} + \{\Pi^{\mu}(\tau,\sigma), X^{\alpha}(\tau,\sigma')\}\Pi^{\beta}(\tau,\sigma') - (\alpha \leftrightarrow \beta)\right] \\
&= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left[-\Pi^{\beta}(\tau,\sigma')g^{\mu\alpha}\delta(\sigma-\sigma') - (\alpha \leftrightarrow \beta)\right] \\
&= \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left[-\Pi^{\beta}(\tau,\sigma')g^{\mu\alpha}\delta(\sigma-\sigma') + \Pi^{\alpha}(\tau,\sigma')g^{\mu\beta}\delta(\sigma-\sigma')\right] \\
&= \int \mathrm{d}\sigma\, \left[\Pi^{\alpha}(\tau,\sigma)g^{\mu\beta} - \Pi^{\beta}(\tau,\sigma)g^{\mu\alpha}\right] \\
\left\{P^{\mu}, M^{\alpha\beta}\right\} &= g^{\mu\beta}P^{\alpha} - g^{\mu\alpha}P^{\beta}
\end{aligned}$$

And lastly, the $M-M$,

$$\left\{M^{\mu\nu}, M^{\alpha\beta}\right\} = \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \left\{X^{\mu}(\tau,\sigma)\Pi^{\nu}(\tau,\sigma) - X^{\nu}(\tau,\sigma)\Pi^{\mu}(\tau,\sigma), X^{\alpha}(\tau,\sigma')\Pi^{\beta}(\tau,\sigma') - X^{\beta}(\tau,\sigma')\Pi^{\alpha}(\tau,\sigma')\right\}$$

$$
\begin{aligned}
&= \int \mathrm{d}\sigma\,\mathrm{d}\sigma' \left\{X^\mu(\tau,\sigma)\Pi^\nu(\tau,\sigma) - X^\nu(\tau,\sigma)\Pi^\mu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\right\} - (\alpha \leftrightarrow \beta) \\
&= \left[\int \mathrm{d}\sigma\,\mathrm{d}\sigma' \left\{X^\mu(\tau,\sigma)\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\right\} - (\alpha \leftrightarrow \beta)\right] - (\mu \leftrightarrow \nu)
\end{aligned}
$$

Notice that,

$$
\begin{aligned}
&\left\{X^\mu(\tau,\sigma)\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\right\} \\
&= \left\{X^\mu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\right\}\Pi^\nu(\tau,\sigma) + X^\mu(\tau,\sigma)\left\{\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\Pi^\beta(\tau,\sigma')\right\} \\
&= X^\alpha(\tau,\sigma')\left\{X^\mu(\tau,\sigma), \Pi^\beta(\tau,\sigma')\right\}\Pi^\nu(\tau,\sigma) + X^\mu(\tau,\sigma)\{\Pi^\nu(\tau,\sigma), X^\alpha(\tau,\sigma')\}\Pi^\beta(\tau,\sigma') \\
&= X^\alpha(\tau,\sigma')g^{\mu\beta}\delta(\sigma-\sigma')\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\alpha}\delta(\sigma-\sigma')\Pi^\beta(\tau,\sigma')
\end{aligned}
$$

Using this back in our expression,

$$
\begin{aligned}
\left\{M^{\mu\nu}, M^{\alpha\beta}\right\} &= \left[\int \mathrm{d}\sigma\, X^\alpha(\tau,\sigma)g^{\mu\beta}\Pi^\nu(\tau,\sigma) - \Pi^\mu(\tau,\sigma)g^{\nu\alpha}\Pi^\beta(\tau,\sigma) - (\alpha \leftrightarrow \beta)\right] - (\mu \leftrightarrow \nu) \\
&= \int \mathrm{d}\sigma \left[X^\alpha(\tau,\sigma)g^{\mu\beta}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\alpha}\Pi^\beta(\tau,\sigma)\right] \\
&\qquad - \int \mathrm{d}\sigma \left[X^\beta(\tau,\sigma)g^{\mu\alpha}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\beta}\Pi^\alpha(\tau,\sigma)\right] - (\mu \leftrightarrow \nu) \\
&= \int \mathrm{d}\sigma \left[X^\alpha(\tau,\sigma)g^{\mu\beta}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\alpha}\Pi^\beta(\tau,\sigma)\right] \\
&\qquad - \int \mathrm{d}\sigma \left[X^\beta(\tau,\sigma)g^{\mu\alpha}\Pi^\nu(\tau,\sigma) - X^\mu(\tau,\sigma)g^{\nu\beta}\Pi^\alpha(\tau,\sigma)\right] \\
&\qquad - \int \mathrm{d}\sigma \left[X^\alpha(\tau,\sigma)g^{\nu\beta}\Pi^\mu(\tau,\sigma) - X^\nu(\tau,\sigma)g^{\mu\alpha}\Pi^\beta(\tau,\sigma)\right] \\
&\qquad + \int \mathrm{d}\sigma \left[X^\beta(\tau,\sigma)g^{\nu\alpha}\Pi^\mu(\tau,\sigma) - X^\nu(\tau,\sigma)g^{\mu\beta}\Pi^\alpha(\tau,\sigma)\right]
\end{aligned}
$$

Collecting the terms with same metric index,

$$
\begin{aligned}
\left\{M^{\mu\nu}, M^{\alpha\beta}\right\} &= g^{\mu\beta}\int \mathrm{d}\sigma\,[X^\alpha(\tau,\sigma)\Pi^\nu(\tau,\sigma) - X^\nu(\tau,\sigma)\Pi^\alpha(\tau,\sigma)] \\
&\qquad + g^{\nu\beta}\int \mathrm{d}\sigma\,[X^\mu(\tau,\sigma)\Pi^\alpha(\tau,\sigma) - X^\alpha(\tau,\sigma)\Pi^\mu(\tau,\sigma)] \\
&\qquad + g^{\mu\alpha}\int \mathrm{d}\sigma\left[X^\nu(\tau,\sigma)\Pi^\beta(\tau,\sigma) - X^\beta(\tau,\sigma)\Pi^\nu(\tau,\sigma)\right] \\
&\qquad + g^{\nu\alpha}\int \mathrm{d}\sigma\left[X^\beta(\tau,\sigma)\Pi^\mu(\tau,\sigma) - X^\mu(\tau,\sigma)\Pi^\beta(\tau,\sigma)\right] \\
\left\{M^{\mu\nu}, M^{\alpha\beta}\right\} &= g^{\mu\beta}M^{\alpha\nu} + g^{\nu\beta}M^{\mu\alpha} + g^{\mu\alpha}M^{\nu\beta} + g^{\nu\alpha}M^{\beta\mu} \\
\left\{M^{\mu\nu}, M^{\alpha\beta}\right\} &= g^{\mu\alpha}M^{\nu\beta} - g^{\mu\beta}M^{\nu\alpha} + g^{\nu\beta}M^{\mu\alpha} - g^{\nu\alpha}M^{\mu\beta}
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
\{P^\mu, P^\nu\} &= 0 \\
\left\{P^\mu, M^{\alpha\beta}\right\} &= g^{\mu\beta}P^\alpha - g^{\mu\alpha}P^\beta \\
\left\{M^{\mu\nu}, M^{\alpha\beta}\right\} &= g^{\mu\alpha}M^{\nu\beta} - g^{\mu\beta}M^{\nu\alpha} + g^{\nu\beta}M^{\mu\alpha} - g^{\nu\alpha}M^{\mu\beta} \qquad (2.8)
\end{aligned}
$$

Which is exactly the algebra of the Poincare Group!

Problem 3

3.A)

The Gamma Function can be represented in the complex plane domain, $\text{Re}(s) > 1$, as the following integral,

$$\Gamma(s) = \int_0^\infty \mathrm{d}t \exp(-t) t^{s-1}, \quad \text{Re}(s) > 1 \tag{3.1}$$

Which is also the subset of the complex plane in which this integral converges, of course this representation of the Gamma Function in a open set is sufficient for obtain an analytical continuation to the whole complex plane. Obviously, the integral is invariant under relabeling the dummy variable t, we make the following choice $t \to nt$ — Assuming $n > 0$ —,

$$\Gamma(s) = \int_0^\infty \mathrm{d}(nt) \exp(-nt)(nt)^{s-1}, \quad \text{Re}(s) > 1$$

$$\Gamma(s) = n^s \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \text{Re}(s) > 1$$

$$n^{-s}\Gamma(s) = \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \text{Re}(s) > 1$$

$$\sum_{n=1}^\infty n^{-s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \text{Re}(s) > 1$$

The sum in the left-hand side is recognized as the representation for the Zeta Function in the domain $\text{Re}(s) > 1$, which is also the domain of convergence of the sum,

$$\zeta(s) = \sum_{n=1}^\infty n^{-s}, \quad \text{Re}(s) > 1$$

So that,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \text{Re}(s) > 1$$

About the right-hand side, to be able to exchange the integral and the sum is sufficient that,

$$\int_0^\infty \mathrm{d}t \sum_{n=1}^\infty \left\| \exp(-nt) t^{s-1} \right\| < \infty, \quad \text{Re}(s) > 1$$

$$\int_0^\infty \mathrm{d}t \sum_{n=1}^{\infty} \exp(-nt) \left\| t^{s-1} \right\| < \infty, \quad \mathrm{Re}(s) > 1$$
$$\int_0^\infty \mathrm{d}t \sum_{n=1}^{\infty} \exp(-nt) t^{\mathrm{Re}(s)-1} < \infty, \quad \mathrm{Re}(s) > 1$$

The sum now is a simple geometric series, giving,

$$\int_0^\infty \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{\exp(t) - 1} < \infty, \quad \mathrm{Re}(s) > 1$$

The dangerous behavior that could make the integral diverges is the one at $t \to 0$, an indeed, $\mathrm{Re}(s) > 1$, is sufficient for the convergence of this integral, which can be seen at,

$$\int_0^\epsilon \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{\exp(t) - 1} \approx \int_0^\epsilon \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{t + \mathcal{O}(t^2)} \approx \int_0^\epsilon t^{\mathrm{Re}(s)-2} = \left. \frac{t^{\mathrm{Re}(s)-1}}{\mathrm{Re}(s) - 1} \right|_0^\epsilon$$

Which shows the integral is really finite at $t \to 0$ with $\mathrm{Re}(s) > 1$, hence, switching the integral and the sum is justified, so,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$
$$\zeta(s)\Gamma(s) = \mathrm{d}t \int_0^\infty \sum_{n=1}^{\infty} \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

Where again we have the sum of a geometric series, giving,

$$\zeta(s)\Gamma(s) = \int_0^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1}, \quad \mathrm{Re}(s) > 1$$

3.B)

The objective here is to make an analytical continuation to $\mathrm{Re}(s) > -2$ of the expression found in the later item. First of all, the reason the later expression is only well defined in $\mathrm{Re}(s) > 1$, is due to the divergence of the integrand at $t \to 0$ for $\mathrm{Re}(s) \leq 1$, this is only because $(\exp(t) - 1)^{-1}$ has a simple pole at $t = 0$, which is also the only pole of this function, so to get the Laurent series we first find the residue of it,

$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp(t) - 1} \right) = \left. \frac{t}{\exp(t) - 1} \right|_{t=0}$$
$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp(t) - 1} \right) = \left. \frac{t}{t + \mathcal{O}(t^2)} \right|_{t=0}$$

$$
\begin{aligned}
\operatorname{Res}_{t=0}\left(\frac{1}{\exp(t)-1}\right) &= \left.\frac{1}{1+\mathcal{O}(t)}\right|_{t=0} \\
\operatorname{Res}_{t=0}\left(\frac{1}{\exp(t)-1}\right) &= 1
\end{aligned}
$$

As this is the only pole, we get a Laurent series starting as,

$$
\frac{1}{\exp(t)-1} = \frac{1}{t} + \mathcal{O}\big(t^0\big)
$$

To get the following terms we just make a trivial Taylor series of the function $(\exp(t)-1)^{-1} - t^{-1}$

$$
\begin{aligned}
\left.\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\right|_0 &= \left.\frac{1+t-\exp(t)}{t[\exp(t)-1]}\right|_0 \\
\left.\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\right|_0 &= \left.\frac{-\frac{t^2}{2}+\mathcal{O}(t^3)}{t[t+\mathcal{O}(t^2)]}\right|_0 \\
\left.\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\right|_0 &= \left.\frac{-\frac{t^2}{2}+\mathcal{O}(t^3)}{t^2[1+\mathcal{O}(t)]}\right|_0 \\
\left.\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\right|_0 &= -\frac{1}{2}
\end{aligned}
$$

In other words,

$$
\frac{1}{\exp(t)-1} = \frac{1}{t} - \frac{1}{2} + \mathcal{O}(t)
$$

The next term of the series will be,

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\right|_0 &= \frac{1}{t^2} - \frac{\exp(t)}{[\exp(t)-1]^2} \\
&= \left.\frac{\exp(t)+\exp(-t)-2-t^2}{t^2[\exp(t)+\exp(-t)-2]}\right|_0 \\
&= \left.\frac{2\frac{t^4}{4!}+\mathcal{O}(t^6)}{t^2[t^2+\mathcal{O}(t^4)]}\right|_0 \\
&= \left.\frac{1}{12}\frac{t^4+\mathcal{O}(t^6)}{t^4[1+\mathcal{O}(t^2)]}\right|_0 \\
&= \frac{1}{12}
\end{aligned}
$$

So up to first order we have,

$$
\frac{1}{\exp(t)-1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}\big(t^2\big) \tag{3.2}
$$

Why have we done this? Because we do can soften the behavior of the integrand near $t \to 0$ if we subtract leading terms of the expansion of $(\exp(t)-1)^{-1}$, each leading term that we subtract, is equivalent to gaining a power of t in the numerator, which does soften the behavior near $t \to 0$, but also makes it worse in the region $t \to \infty$, and as our only problem is related with the small t region, we can divide the integral in two parts,

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1} + \int_1^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1}, \quad \mathrm{Re}(s) > 1$$

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t \, t^{s-1} \left[\frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} + \frac{1}{t} - \frac{1}{2} + \frac{t}{12} \right] + \int_1^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1}, \quad \mathrm{Re}(s) > 1$$

Where we simply added and subtracted the leading terms of the expansion, the integral of the last three of them is trivial and can be done to give,

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t \, t^{s-1} \left[\frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right] + \int_0^1 \mathrm{d}t \left[t^{s-2} - \frac{t^{s-1}}{2} + \frac{t^s}{12} \right] + \int_1^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1}$$

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t \, t^{s-1} \left[\frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1}$$

Just what we wanted.

3.C)

Naively, this last expression should be well defined only for $\mathrm{Re}(s) > 1$, let's see this term by term, starting by the last one,

$$\int_1^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1}$$

This is finite for all s, as it is exponentially decaying and is bounded in the integration interval, this term is well defined for all s. The next three ones are,

$$\frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)}$$

Also these are well defined in the whole complex plane, with three poles at $s = -1, 0, 1$. Finally we have,

$$\int_0^1 \mathrm{d}t \, t^{s-1} \left[\frac{1}{\exp(t) - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right]$$

The only potential not well defined behavior that can occur is near $t = 0$, but we have already developed a series expansion for the expression in brackets, 3.2, that means, near the critical value of $t = 0$, the integrand goes like,

$$\int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}\right] \approx \int_0^1 \mathrm{d}t\, t^{s-1}\mathcal{O}(t^2) \approx \int_0^1 \mathrm{d}t\, t^{s+1} = \frac{t^{s+2}}{s+2}\bigg|_0^1$$

This is well defined as long as $\mathrm{Re}(s) > -2$. Hence, the expression,

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}\right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1} \tag{3.3}$$

Is well defined as long as $\mathrm{Re}(s) > -2$. One might worry about the poles, but, these are the natural structure of $\zeta(s)\Gamma(s)$, to be well defined does not mean to don't have poles, but means the representation can be assigned a number in an unique manner. What is left to us now is to find the values $\zeta(0), \zeta(-1)$, notice that our representation has a poles in both these values of the argument, in fact these poles are structures of $\Gamma(s)$, and not $\zeta(s)$. That the Gamma Function indeed has poles in those values can be seen from,

$$\Gamma(s+1) = s\Gamma(s) \Rightarrow \begin{cases} \Gamma(0) & = \frac{\Gamma(1)}{0} \\ \Gamma(-1) & = \frac{\Gamma(0)}{-1} \end{cases}$$

And because the poles in our representation are just simple poles, they could not have been poles also in ζ, as the two functions are multiplying if there were a pole in $\zeta(0), \zeta(-1)$ they would have been apparent in our representation as double poles. Due to the absence of those, the poles at $s = 0, -1$ are indeed only due to the Gamma Function. This guarantees us that $\zeta(0), \zeta(-1)$ are both finite, and to determine those we just need to evaluate the residue of the expression. First, the residue of the Gamma Function,

$$\begin{aligned} \mathrm{Res}_{s=0}(\Gamma(s)) &= s\Gamma(s)\bigg|_{s=0} = \Gamma(s+1)\bigg|_{s=0} = \Gamma(1) = 1 \\ \mathrm{Res}_{s=-1}(\Gamma(s)) &= (s+1)\Gamma(s)\bigg|_{s=-1} = \frac{(s+1)s\Gamma(s)}{s}\bigg|_{s=-1} = \frac{\Gamma(s+2)}{s}\bigg|_{s=-1} = -1 \end{aligned}$$

As we argued that $\zeta(0), \zeta(-1)$ should be finite, what will happen is that when we multiply ζ by Γ, the residues of the poles of the Gamma Function will be multiplied by the value of the Zeta Function at that point, that is,

$$\mathrm{Res}_{s=0}(\zeta(s)\Gamma(s)) = \zeta(0)\mathrm{Res}_{s=0}(\Gamma(s))$$

But, as can be seen directly from 3.3, the only contribution for the residue at $s = 0$ will be by $-\frac{1}{2s}$, as all the other terms are finite at $s = 0$, thus,

$$\begin{aligned} \mathrm{Res}_{s=0}(\zeta(s)\Gamma(s)) &= -\frac{1}{2} = \zeta(0)\mathrm{Res}_{s=0}(\Gamma(s)) = \zeta(0) \\ \zeta(0) &= -\frac{1}{2} \end{aligned}$$

Analogously we have,

$$\text{Res}_{s=-1}(\zeta(s)\Gamma(s)) = \zeta(-1)\text{Res}_{s=-1}(\Gamma(s))$$

Again, as we discussed previously, all the terms are finite at $s = -1$, except for $\frac{1}{12(s+1)}$, hence, the residue will be,

$$\begin{aligned} \text{Res}_{s=-1}(\zeta(s)\Gamma(s)) = \frac{1}{12} &= \zeta(-1)\text{Res}_{s=-1}(\Gamma(s)) = -\zeta(-1) \\ \zeta(-1) &= -\frac{1}{12} \end{aligned} \tag{3.4}$$

As desired.

Problem 4

4.A)

First, one remark, we're going to derive everything in this problem for the **open** string with Neumann Boundary Conditions at both ends. The full trip to the classical Light-Cone gauge of the open string with NN boundary conditions is done in Appendix A, afterwards the full quantization of the same type of string in the same gauge is done in Appendix B, we're just continue here citing the needed results. These are,

$$
\begin{aligned}
X^I &= x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^I}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right), \quad \alpha_0^\mu = \sqrt{2\alpha'}p^\mu \quad A.7\\
X^- &= x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^-}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right), \quad \alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+}L_n^\perp, \quad A.8\\
X^+ &= 2\alpha' p^+\tau, \quad A.9\\
\left[x_0^-, p^+\right] &= -\mathrm{i}, \quad B.1\\
\left[\alpha_m^I, \alpha_n^J\right] &= m g^{IJ}\delta_{m+n,0}, \quad B.5\\
\left[x_0^I, \alpha_n^J\right] &= \delta_{n,0}\sqrt{2\alpha'}\mathrm{i}g^{IJ}, \quad B.6\\
\left[L_m^\perp, \alpha_n^J\right] &= -n\alpha_{n+m}^J, \quad B.9\\
\left[L_m^\perp, x_0^J\right] &= -\mathrm{i}\sqrt{2\alpha'}\alpha_m^J, \quad B.10\\
\left[L_m^\perp, L_n^\perp\right] &= (m-n)L_{m+n}^\perp + \frac{D-2}{12}\left(m^3 - m\right)\delta_{m+n,0}, \quad B.11
\end{aligned}
$$

This completes the set of all needed commutation relations. We can now discuss the Lorentz generators in the Light-cone gauge. We have already computed them in 2.7, but of course we have two remarks, neither they are quantum, nor are in the light-cone gauge, about the later, the actual quantum light-cone gauge Lorentz generators **should** satisfy the same algebra as 2.8 in the light-cone coordinates. The failure to met this requirement is related to an anomaly in this global symmetry of the Quantum Poincare Action. And about the former, we'll change the definition accordingly to ensure the quantum generators do satisfy being Hermitian. With that being said, let's evaluate the classical version of 2.7 in the light-cone gauge,

$$
\begin{aligned}
M^{\mu\nu} &\overset{?}{=} \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\left(X^\mu\dot{X}^\nu - X^\nu\dot{X}^\mu\right)\\
&\overset{?}{=} \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\left(x_0^\mu + 2\alpha' p^\mu\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)\right)\times\\
&\qquad\times\sqrt{2\alpha'}\sum_{m\in\mathbb{Z}}\alpha_m^\nu\exp\left(-\mathrm{i}m\tau\right)\cos\left(m\sigma\right) - (\mu\leftrightarrow\nu)\\
&\overset{?}{=} \frac{1}{2\pi\alpha'}\Bigg[\sqrt{2\alpha'}\sum_{m\in\mathbb{Z}}x_0^\mu\alpha_m^\nu\exp\left(-\mathrm{i}m\tau\right)\pi\delta_{m,0}\\
&\qquad + 2\alpha'\tau\sqrt{2\alpha'}\sum_{m\in\mathbb{Z}}p^\mu\alpha_m^\nu\exp\left(-\mathrm{i}m\tau\right)\pi\delta_{m,0}\\
&\qquad + \mathrm{i}2\alpha'\sum_{n\in\mathbb{Z}^*}\sum_{m\in\mathbb{Z}}\frac{\alpha_n^\mu}{n}\exp\left(-\mathrm{i}n\tau\right)\alpha_m^\nu\exp\left(-\mathrm{i}m\tau\right)\frac{\pi}{2}\left(\delta_{m,n}+\delta_{m,-n}\right)\Bigg] - (\mu\leftrightarrow\nu)
\end{aligned}
$$

$$M^{\mu\nu} \overset{?}{=} \left[\frac{1}{\sqrt{2\alpha'}}x_0^\mu\alpha_0^\nu + \tau\sqrt{2\alpha'}p^\mu\alpha_0^\nu + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\alpha_n^\mu\alpha_n^\nu\exp\left(-\mathrm{i}(n+m)\tau\right) + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\alpha_n^\mu\alpha_{-n}^\nu\right] - (\mu\leftrightarrow\nu)$$

Employing the before mentioned equality $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$,

$$\begin{aligned}
M^{\mu\nu} &\overset{?}{=} \Bigg[x_0^\mu p^\nu - x_0^\nu p^\mu + \tau\alpha_0^\mu\alpha_0^\nu - \tau\alpha_0^\nu\alpha_0^\mu + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\left(\alpha_n^\mu\alpha_{-n}^\nu - \alpha_n^\nu\alpha_{-n}^\mu\right) \\
&\qquad + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}(\alpha_n^\mu\alpha_n^\nu - \alpha_n^\nu\alpha_n^\mu)\exp\left(-\mathrm{i}2n\tau\right)\Bigg] \\
M^{\mu\nu} &\overset{?}{=} \left[x_0^\mu p^\nu - x_0^\nu p^\mu + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_n^\mu\alpha_{-n}^\nu - \alpha_n^\nu\alpha_{-n}^\mu - \alpha_{-n}^\mu\alpha_n^\nu + \alpha_{-n}^\nu\alpha_n^\mu\right)\right] \\
M^{\mu\nu} &\overset{?}{=} \left[x_0^\mu p^\nu - x_0^\nu p^\mu - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu\right)\right]
\end{aligned}$$

This is the classical version of the generators in the light-cone gauge. Does it is a good Quantum Lorentz Generator Operator? For a positive answer, it has to be both Hermitian and normal-ordered, let's consider one by one,

$$\begin{aligned}
M^{IJ} &\overset{?}{=} \left[x_0^I p^J - x_0^J p^I + \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_n^I\alpha_{-n}^J - \alpha_n^J\alpha_{-n}^I\right)\right] \\
\left(M^{IJ}\right)^\dagger - M^{IJ} &\overset{?}{=} \left[p^J x_0^I - p^I x_0^J - x_0^I p^J + x_0^J p^I - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_n^J\alpha_{-n}^I - \alpha_n^I\alpha_{-n}^J + \alpha_n^I\alpha_{-n}^J - \alpha_n^J\alpha_{-n}^I\right)\right] \\
\left(M^{IJ}\right)^\dagger - M^{IJ} &\overset{?}{=} \left[\left[p^J, x_0^I\right] - \left[p^I, x_0^J\right]\right] \\
\left(M^{IJ}\right)^\dagger - M^{IJ} &\overset{?}{=} \left[-\mathrm{i}g^{IJ} + \mathrm{i}g^{IJ}\right] = 0
\end{aligned}$$

Yes! And about normal-ordered? Yes! Then our quantum operator related to the Lorentz Generators is,

$$M^{IJ} = \left[x_0^I p^J - x_0^J p^I - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_{-n}^I\alpha_n^J - \alpha_{-n}^J\alpha_n^I\right)\right]$$

Now, already seen that it's normal ordered and looking at Hermiticity,

$$\begin{aligned}
M^{I+} &\overset{?}{=} \left[x_0^I p^+ - x_0^+ p^I - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_{-n}^I\alpha_n^+ - \alpha_{-n}^+\alpha_n^I\right)\right] \\
M^{I+} &\overset{?}{=} x_0^I p^+
\end{aligned}$$

Which is of course Hermitian, once $\left[x_0^I, p^+ = 0\right]$, hence,

$$M^{I+} = x_0^I p^+$$

By the same reasoning, that is, $x_0^+ = 0 = \alpha_n^+,\ n \neq 0$,

$$M^{-+} \overset{?}{=} x_0^- p^+$$

Which fails to be Hermitian, due the canonical commutation relations. One way to avoid this is to symmetrize it,

$$M^{-+} = \frac{1}{2}\big(x_0^- p^+ + p^+ x_0^-\big)$$

Which now is both normal ordered and hermitian. Now,

$$M^{++} \overset{?}{=} x_0^+ p^+ = 0$$

Which is expected by the anti-symmetry of the generators, we have another which is zero by the anti-symmetry,

$$M^{--} \overset{?}{=} \left[x_0^- p^- - x_0^- p^- - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(\alpha_{-n}^-\alpha_n^- - \alpha_{-n}^-\alpha_n^-\big)\right] = 0$$

At last, we have the most important generator,

$$M^{-I} \overset{?}{=} \left[x_0^- p^I - x_0^I p^- - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(\alpha_{-n}^-\alpha_n^I - \alpha_{-n}^I\alpha_n^-\big)\right]$$

Is it Hermitian? No.

$$\begin{aligned}
\big(M^{-I}\big)^\dagger - M^{-I} &\overset{?}{=} \left[p^I x_0^- - p^- x_0^I - x_0^- p^I + x_0^I p^- + \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(\alpha_{-n}^I\alpha_n^- - \alpha_{-n}^-\alpha_n^I + \alpha_{-n}^-\alpha_n^I - \alpha_{-n}^I\alpha_n^-\big)\right]\\
\big(M^{-I}\big)^\dagger - M^{-I} &\overset{?}{=} \big[\big[p^I, x_0^-\big] - \big[p^-, x_0^I\big]\big]\\
\big(M^{-I}\big)^\dagger - M^{-I} &\overset{?}{=} -\big[p^-, x_0^I\big]\\
\big(M^{-I}\big)^\dagger - M^{-I} &\overset{?}{=} -\frac{1}{\sqrt{2\alpha'}}\big[\alpha_0^-, x_0^I\big]\\
\big(M^{-I}\big)^\dagger - M^{-I} &\overset{?}{=} -\frac{1}{2\alpha' p^+}\big[L_0^\perp + a, x_0^I\big] = \frac{\mathrm{i}}{\sqrt{2\alpha'}p^+}\alpha_0^I
\end{aligned}$$

Again, if we symmetrize the $x_0^I p^-$ term this issue is resolved,

$$M^{-I} \overset{?}{=} x_0^- p^I - \frac{1}{2}\big(x_0^I p^- + p^- x_0^I\big) - \mathrm{i}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(\alpha_{-n}^-\alpha_n^I - \alpha_{-n}^I\alpha_n^-\big)$$

Is this expression normal-ordered? Yes! Due to the α_n^- being proportional to the Virasoro modes, which are already normal-ordered. Hence, the expression for this Lorentz Generator is, using Virasoro modes,

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\big(x_0^I\big(L_0^\perp + a\big) + \big(L_0^\perp + a\big)x_0^I\big) - \frac{\mathrm{i}}{\sqrt{2\alpha'}p^+}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(L_{-n}^\perp\alpha_n^I - \alpha_{-n}^I L_n^\perp\big)$$

Where we made use of our lack of knowledge about the ordering prescription by introducing $L_0^\perp + a$ whenever α_0^- is mentioned.

4.B)

This is the real deal, now is the time of truth. We'll **ensure** the Lorentz invariance of this theory, you may ask how? By making sure the Lorentz algebra is satisfied by the Light-cone Lorentz generators. We're not computing all of the commutators in the Light-cone gauge, but only $\left[M^{-I}, M^{-J}\right]$, as this is the only one with unknown constants, a. The calculus is long and cumbersome, hence, we'll split into various parts. Let's compute term by term. We define the auxiliary variables as,

$$
\begin{aligned}
A^I &= x_0^- p^I \\
B^I &= -\frac{1}{4\alpha' p^+}\left(x_0^I\left(L_0^\perp + a\right) + \left(L_0^\perp + a\right)x_0^I\right) \\
C^I &= \frac{\mathrm{i}}{\sqrt{2\alpha'}p^+}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp\right)
\end{aligned}
$$

It's clear then that the commutator is,

$$
\begin{aligned}
\left[M^{-I}, M^{-J}\right] = {} & \left[A^I, A^J\right] + \left[A^I, B^J\right] + \left[B^I, A^J\right] + \left[B^I, B^J\right] \\
& + \left[A^I, C^J\right] + \left[C^I, A^J\right] + \left[B^I, C^J\right] + \left[C^I, B^J\right] + \left[C^I, C^J\right]
\end{aligned}
$$

Which can be rewritten as,

$$
\begin{aligned}
\left[M^{-I}, M^{-J}\right] = {} & \left[A^I, A^J\right] + \left[B^I, B^J\right] + \left[C^I, C^J\right] \\
& + \left\{\left[A^I, B^J\right] + \left[A^I, C^J\right] + \left[B^I, C^J\right] - (I \leftrightarrow J)\right\}
\end{aligned}
$$

That is, for the mixed commutator, we just need to compute the anti-symmetrical part, a fact that we'll exploit,

Now, $\left[A^I, A^J\right]$,

$$
\begin{aligned}
\left[x_0^- p^I, x_0^- p^J\right] &= x_0^-\left[p^I, x_0^- p^J\right] + \left[x_0^-, x_0^- p^J\right]p^I \\
\left[x_0^- p^I, x_0^- p^J\right] &= x_0^- x_0^-\left[p^I, p^J\right] + x_0^-\left[x_0^-, p^J\right]p^I = 0
\end{aligned}
$$

That is,

$$
\left[A^I, A^J\right] = 0 \tag{4.1}
$$

Now, $\left[A^I, B^J\right]$,

$$
-\frac{1}{4\alpha'}\left[x_0^- p^I, \frac{1}{p^+}\left(x_0^J\left(L_0^\perp + a\right) + \left(L_0^\perp + a\right)x_0^J\right)\right] + \frac{1}{4\alpha'}\left[x_0^- p^J, \frac{1}{p^+}\left(x_0^I\left(L_0^\perp + a\right) + \left(L_0^\perp + a\right)x_0^I\right)\right]
$$

$$
\begin{aligned}
&= -\frac{1}{4\alpha'}x_0^-\frac{1}{p^+}\Big(\big[p^I,\big(x_0^J\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^J\big)\big]-\big[p^J,\big(x_0^I\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^I\big)\big]\Big)\\
&\qquad -\frac{1}{4\alpha'}\left[x_0^-,\frac{1}{p^+}\right]\Big(\big(x_0^J\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^J\big)p^I-\big(x_0^I\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^I\big)p^J\Big)\\
&= -\frac{\mathrm{i}}{4\alpha' p^{+2}}\Big(\big(x_0^J\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^J\big)p^I-\big(x_0^I\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^I\big)p^J\Big)\\
&= -\frac{\mathrm{i}}{4\alpha' p^{+2}}\Big(\big(2\big(L_0^\perp+a\big)x_0^J+\mathrm{i}2\alpha' p^J\big)p^I-\big(2\big(L_0^\perp+a\big)x_0^I+\mathrm{i}2\alpha' p^I\big)p^J\Big)\\
&= -\frac{\mathrm{i}}{2\alpha' p^{+2}}\big(L_0^\perp+a\big)\big(x_0^J p^I-x_0^I p^J\big)
\end{aligned}
$$

That is,

$$
\big[A^I,B^J\big]-\big[A^J,B^I\big] = -\frac{\mathrm{i}}{2\alpha' p^{+2}}\big(L_0^\perp+a\big)\big(x_0^J p^I-x_0^I p^J\big) \tag{4.2}
$$

Now, $\big[B^I,B^J\big]$,

$$
\begin{aligned}
&\frac{1}{16\alpha'^2p^{+2}}\big[x_0^I\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^I,x_0^J\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^J\big]\\
&= \frac{1}{16\alpha'^2p^{+2}}\big[2\big(L_0^\perp+a\big)x_0^I+\mathrm{i}2\alpha' p^I,2\big(L_0^\perp+a\big)x_0^J+\mathrm{i}2\alpha' p^J\big]\\
&= \frac{1}{4\alpha'^2p^{+2}}\Big\{\big[\big(L_0^\perp+a\big)x_0^I,\big(L_0^\perp+a\big)x_0^J\big]+\mathrm{i}\alpha'\Big(\big[p^I,\big(L_0^\perp+a\big)x_0^J\big]-\big[p^J,\big(L_0^\perp+a\big)x_0^I\big]\Big)\Big\}\\
&= \frac{1}{4\alpha'^2p^{+2}}\Big\{\big(L_0^\perp+a\big)\big[x_0^I,L_0^\perp\big]x_0^J+\big(L_0^\perp+a\big)\big[L_0^\perp,x_0^J\big]x_0^I+\mathrm{i}\alpha'\Big(\big[p^I,L_0^\perp\big]x_0^J-\big[p^J,L_0^\perp\big]x_0^I\Big)\Big\}\\
&= \frac{1}{4\alpha'^2p^{+2}}\Big\{\big(L_0^\perp+a\big)\mathrm{i}2\alpha' p^I x_0^J+\big(L_0^\perp+a\big)\big(-\mathrm{i}2\alpha' p_0^J\big)x_0^I\Big\}\\
&= \frac{\mathrm{i}}{2\alpha' p^{+2}}\big(L_0^\perp+a\big)\big(p^I x_0^J-p_0^J x_0^I\big)\\
&= \frac{\mathrm{i}}{2\alpha' p^{+2}}\big(L_0^\perp+a\big)\Big(x_0^J p^I+\big[p^I,x_0^J\big]-x_0^I p_0^J-\big[p_0^J,x_0^I\big]\Big)\\
&= \frac{\mathrm{i}}{2\alpha' p^{+2}}\big(L_0^\perp+a\big)\big(x_0^J p^I-x_0^I p_0^J\big)
\end{aligned}
$$

That is,

$$
\big[B^I,B^J\big] = \frac{\mathrm{i}}{2\alpha' p^{+2}}\big(L_0^\perp+a\big)\big(x_0^J p^I-x_0^I p^J\big) \tag{4.3}
$$

We neglected the lack of normal order in these expressions because we already have a nice relation, $\big[A^I,A^J\big]+\big[B^I,B^J\big]+\big(\big[A^I,B^J\big]-\big[B^I,A^J\big]\big)=0$.

Now, $\big[A^I,C^J\big]$,

$$
\begin{aligned}
&-\frac{\mathrm{i}}{\sqrt{2\alpha'}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left[x_0^-p^I,\frac{1}{p^+}\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)\right]\\
&=-\frac{\mathrm{i}}{\sqrt{2\alpha'}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left\{x_0^-\left[p^I,\frac{1}{p^+}\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)\right]+\left[x_0^-,\frac{1}{p^+}\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)\right]p^I\right\}\\
&=-\frac{\mathrm{i}}{\sqrt{2\alpha'}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left\{x_0^-\frac{1}{p^+}\big[p^I,L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big]+\left[x_0^-,\frac{1}{p^+}\right]\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)p^I\right\}\\
&=-\frac{\mathrm{i}}{\sqrt{2\alpha'}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left\{x_0^-\frac{1}{p^+}\big(\big[p^I,L_{-n}^\perp\alpha_n^J\big]-\big[p^I,\alpha_{-n}^JL_n^\perp\big]\big)+\frac{\mathrm{i}}{p^{+2}}\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)p^I\right\}\\
&=-\frac{\mathrm{i}}{\sqrt{2\alpha'}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left\{x_0^-\frac{1}{p^+}\big(\big[p^I,L_{-n}^\perp\big]\alpha_n^J+L_{-n}^\perp\big[p^I,\alpha_n^J\big]-\alpha_{-n}^J\big[p^I,L_n^\perp\big]-\big[p^I,\alpha_{-n}^J\big]L_n^\perp\big)\right.\\
&\qquad\left.+\frac{\mathrm{i}}{p^{+2}}\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)p^I\right\}\\
&=\frac{1}{\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)p^I\\
&=\frac{1}{2\alpha'p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)\alpha_0^I
\end{aligned}
$$

That is,

$$
\big[A^I,C^J\big]-\big[A^J,C^I\big]=\frac{1}{2\alpha'p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\big(L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big)\alpha_0^I-\big(L_{-n}^\perp\alpha_n^I-\alpha_{-n}^IL_n^\perp\big)\alpha_0^J\big]\tag{4.4}
$$

Now, $\big[B^I,C^J\big]$,

$$
\begin{aligned}
&\frac{\mathrm{i}}{4\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[x_0^I\big(L_0^\perp+a\big)+\big(L_0^\perp+a\big)x_0^I,L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big]\\
&=\frac{\mathrm{i}}{4\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\big[x_0^I,\big(L_0^\perp+a\big)\big]+\big(L^\perp+a\big)x_0^I+\big(L_0^\perp+a\big)x_0^I,L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big]\\
&=\frac{\mathrm{i}}{4\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big[2\big(L_0^\perp+a\big)x_0^I+\mathrm{i}\sqrt{2\alpha'}\alpha_0^I,L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\Big]\\
&=\frac{\mathrm{i}}{2\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\big(L_0^\perp+a\big)x_0^I,L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big]\\
&=\frac{\mathrm{i}}{2\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big\{\big[L_0^\perp,L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big]x_0^I+\big(L_0^\perp+a\big)\big[x_0^I,L_{-n}^\perp\big]\alpha_n^J-\big(L_0^\perp+a\big)\alpha_{-n}^J\big[x_0^I,L_n^\perp\big]\big\}\\
&=\frac{\mathrm{i}}{2\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big\{\big[L_0^\perp,L_{-n}^\perp\alpha_n^J-\alpha_{-n}^JL_n^\perp\big]x_0^I+\mathrm{i}\sqrt{2\alpha'}\big(L_0^\perp+a\big)\big(\alpha_{-n}^I\alpha_n^J-\alpha_{-n}^J\alpha_n^I\big)\Big\}\\
&=\frac{\mathrm{i}}{2\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big\{\mathrm{i}\sqrt{2\alpha'}\big(L_0^\perp+a\big)\big(\alpha_{-n}^I\alpha_n^J-\alpha_{-n}^J\alpha_n^I\big)\\
&\qquad+\big[L_0^\perp,L_{-n}^\perp\alpha_n^J\big]x_0^I-\big[L_0^\perp,\alpha_{-n}^JL_n^\perp\big]x_0^I\Big\}\\
&=\frac{\mathrm{i}}{2\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big\{\mathrm{i}\sqrt{2\alpha'}\big(L_0^\perp+a\big)\big(\alpha_{-n}^I\alpha_n^J-\alpha_{-n}^J\alpha_n^I\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +L^\perp_{-n}\big[L^\perp_0,\alpha^J_n\big]x^I_0+\big[L^\perp_0,L^\perp_{-n}\big]\alpha^J_n x^I_0-\alpha^J_{-n}\big[L^\perp_0,L^\perp_n\big]x^I_0-\big[L^\perp_0,\alpha^J_{-n}\big]L^\perp_n x^I_0\big\}\\
&=\frac{\mathrm{i}}{2\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big\{\mathrm{i}\sqrt{2\alpha'}\big(L^\perp_0+a\big)\big(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^I_n\big)\\
&\quad -nL^\perp_{-n}\alpha^J_n x^I_0+nL^\perp_{-n}\alpha^J_n x^I_0+\alpha^J_{-n}nL^\perp_n x^I_0-n\alpha^J_{-n}L^\perp_n x^I_0\big\}\\
&=\frac{\mathrm{i}}{2\alpha'\sqrt{2\alpha'}p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big\{\mathrm{i}\sqrt{2\alpha'}\big(L^\perp_0+a\big)\big(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^I_n\big)\Big\}\\
&=-\frac{1}{2\alpha'p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(L^\perp_0+a\big)\big(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^J_n\big)\\
&=-\frac{1}{2\alpha'p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big(\big[L^\perp_0+a,\alpha^I_{-n}\big]\alpha^J_n+\alpha^I_{-n}\big(L^\perp_0+a\big)\alpha^J_n-\big[L^\perp_0+a,\alpha^J_{-n}\big]\alpha^I_n-\alpha^J_{-n}\big(L^\perp_0+a\big)\alpha^I_n\Big)\\
&=-\frac{1}{2\alpha'p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big(n\alpha^I_{-n}\alpha^J_n+\alpha^I_{-n}\big(L^\perp_0+a\big)\alpha^J_n-n\alpha^J_{-n}\alpha^I_n-\alpha^J_{-n}\big(L^\perp_0+a\big)\alpha^I_n\Big)\\
&=-\frac{1}{2\alpha'p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big(\alpha^I_{-n}\big(L^\perp_0+a+n\big)\alpha^J_n-\alpha^J_{-n}\big(L^\perp_0+a+n\big)\alpha^I_n\Big)
\end{aligned}
$$

Where at the end we made some tweaks for the whole expression stay normal-ordered. Summing the anti-symmetric part,

$$
\big[B^I,C^J\big]-\big[B^J,C^I\big]=-\frac{1}{\alpha'p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\Big(\alpha^I_{-n}\big(L^\perp_0+a+n\big)\alpha^J_n-\alpha^J_{-n}\big(L^\perp_0+a+n\big)\alpha^I_n\Big)\tag{4.5}
$$

At last, $\big[C^I,C^J\big]$, this is the hardest one. So we'll need to split again into even more parts to make the result intelligible, first, notice that the full commutator is,

$$
\big[C^I,C^J\big]=-\frac{1}{2\alpha'p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\big[L^\perp_{-n}\alpha^I_n-\alpha^I_{-n}L^\perp_n,L^\perp_{-m}\alpha^J_m-\alpha^J_{-m}L^\perp_m\big]\tag{4.6}
$$

From which we subdivide as,

$$
\begin{aligned}
C^{IJ}_{11}&=-\frac{1}{2\alpha'p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\big[L^\perp_{-n}\alpha^I_n,L^\perp_{-m}\alpha^J_m\big]\\
C^{IJ}_{21}&=\frac{1}{2\alpha'p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\big[\alpha^I_{-n}L^\perp_n,L^\perp_{-m}\alpha^J_m\big]\\
C^{IJ}_{12}&=\frac{1}{2\alpha'p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\big[L^\perp_{-n}\alpha^I_n,\alpha^J_{-m}L^\perp_m\big]\\
C^{IJ}_{22}&=-\frac{1}{2\alpha'p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\big[\alpha^I_{-n}L^\perp_n,\alpha^J_{-m}L^\perp_m\big]
\end{aligned}\tag{4.7}
$$

Starting by C^{IJ}_{11},

$$
-\frac{1}{2\alpha'p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\big[L^\perp_{-n}\alpha^I_n,L^\perp_{-m}\alpha^J_m\big]
$$

$$
\begin{aligned}
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big(L^\perp_{-n} \big[\alpha^I_n, L^\perp_{-m}\alpha^J_m \big] + \big[L^\perp_{-n}, L^\perp_{-m}\alpha^J_m \big] \alpha^I_n \big) \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(L^\perp_{-n} L^\perp_{-m} \big[\alpha^I_n, \alpha^J_m \big] + L^\perp_{-n} \big[\alpha^I_n, L^\perp_{-m} \big] \alpha^J_m + L^\perp_{-m} \big[L^\perp_{-n}, \alpha^J_m \big] \alpha^I_n \right. \\
&\qquad \left. + \big[L^\perp_{-n}, L^\perp_{-m} \big] \alpha^J_m \alpha^I_n \right) \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(L^\perp_{-n} L^\perp_{-m} n g^{IJ} \delta_{n+m,0} + L^\perp_{-n} n \alpha^I_{n-m} \alpha^J_m - L^\perp_{-m} m \alpha^J_{m-n} \alpha^I_n \right. \\
&\qquad \left. + \left((-n+m) L^\perp_{-n-m} - \frac{D-2}{12} (n^3 - n) \delta_{-n-m,0} \right) \alpha^J_m \alpha^I_n \right) \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(L^\perp_{-n} n \alpha^I_{n-m} \alpha^J_m - L^\perp_{-m} m \alpha^J_{m-n} \alpha^I_n + (-n+m) L^\perp_{-n-m} \alpha^J_m \alpha^I_n \right)
\end{aligned}
$$

Notice that the last term is already normal-ordered, but, the two first ones aren't. We'll rewrite them in normal ordered form. First, change labels $n \leftrightarrow m$ in the first term. Then,

$$
\begin{aligned}
C^{IJ}_{11} &= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n - \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{n} \left(L^\perp_{-m} \alpha^I_{m-n} \alpha^J_n - L^\perp_{-m} \alpha^J_{m-n} \alpha^I_n \right) \\
C^{IJ}_{11} &= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \left(\sum_{m>n}^{\infty} + \sum_{m>0}^{n} \right) \left(L^\perp_{-m} \alpha^I_{m-n} \alpha^J_n - L^\perp_{-m} \alpha^J_{m-n} \alpha^I_n \right) \\
C^{IJ}_{11} &= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>n}^{\infty} \left(L^\perp_{-m} \alpha^I_{m-n} \alpha^J_n - L^\perp_{-m} \alpha^J_{m-n} \alpha^I_n \right) \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(L^\perp_{-m} \alpha^I_{m-n} \alpha^J_n - L^\perp_{-m} \alpha^J_{m-n} \alpha^I_n \right)
\end{aligned}
$$

It's clear in this form that the first and second ine are normal-ordered, but the third isn't, let's fix this. Also, we relabel the sum over m in the second line, we make $m \to m+n$, this makes the sum over m to be over $m > 0$,

$$
\begin{aligned}
C^{IJ}_{11} &= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>n}^{\infty} \left(L^\perp_{-m} \alpha^I_{m-n} \alpha^J_n - L^\perp_{-m} \alpha^J_{m-n} \alpha^I_n \right) \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\left(\big[L^\perp_{-m}, \alpha^I_{m-n} \big] + \alpha^I_{m-n} L^\perp_{-m} \right) \alpha^J_n - \left(\big[L^\perp_{-m}, \alpha^J_{m-n} \big] + \alpha^J_{m-n} L^\perp_{-m} \right) \alpha^I_n \right) \\
C^{IJ}_{11} &= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{\infty} \left(L^\perp_{-m-n} \alpha^I_m \alpha^J_n - L^\perp_{-m-n} \alpha^J_m \alpha^I_n \right)
\end{aligned}
$$

$$- \frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \big(\big((n-m)\alpha^I_{-n} + \alpha^I_{m-n} L^\perp_{-m} \big) \alpha^J_n - \big((n-m)\alpha^J_{-n} + \alpha^J_{m-n} L^\perp_{-m} \big) \alpha^I_n \big)$$

$$\begin{aligned} C^{IJ}_{11} = &-\frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(m L^\perp_{-m-n} \alpha^I_m \alpha^J_n - m L^\perp_{-m-n} \alpha^J_m \alpha^I_n \right) \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \big(\big((n-m)\alpha^I_{-n} + \alpha^I_{m-n} L^\perp_{-m} \big) \alpha^J_n - \big((n-m)\alpha^J_{-n} + \alpha^J_{m-n} L^\perp_{-m} \big) \alpha^I_n \big) \end{aligned}$$

As the αs commute in the second line, due to $n, m > 0$, we exchange them in the first term and them relabel $n \leftrightarrow m$ in it,

$$\begin{aligned} C^{IJ}_{11} = &-\frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(n L^\perp_{-m-n} \alpha^J_m \alpha^I_n - m L^\perp_{-m-n} \alpha^J_m \alpha^I_n \right) \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \big(\big((n-m)\alpha^I_{-n} + \alpha^I_{m-n} L^\perp_{-m} \big) \alpha^J_n - \big((n-m)\alpha^J_{-n} + \alpha^J_{m-n} L^\perp_{-m} \big) \alpha^I_n \big) \\ C^{IJ}_{11} = &-\frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (m-n) L^\perp_{-n-m} \alpha^J_m \alpha^I_n - \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} (n-m) L^\perp_{-m-n} \alpha^J_m \alpha^I_n \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \big(\big((n-m)\alpha^I_{-n} + \alpha^I_{m-n} L^\perp_{-m} \big) \alpha^J_n - \big((n-m)\alpha^J_{-n} + \alpha^J_{m-n} L^\perp_{-m} \big) \alpha^I_n \big) \\ C^{IJ}_{11} = &-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \big(\big((n-m)\alpha^I_{-n} + \alpha^I_{m-n} L^\perp_{-m} \big) \alpha^J_n - \big((n-m)\alpha^J_{-n} + \alpha^J_{m-n} L^\perp_{-m} \big) \alpha^I_n \big) \\ C^{IJ}_{11} = &-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \big(\alpha^I_{-n} \alpha^J_n - \alpha^J_{-n} \alpha^I_n \big) \sum_{m>0}^{n} (n-m) \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \big(\alpha^I_{m-n} L^\perp_{-m} \alpha^J_n - \alpha^J_{m-n} L^\perp_{-m} \alpha^I_n \big) \end{aligned}$$

The sum over m in the first line is trivial, $\sum\limits_{m>0}^{n} n = n^2$ and, $\sum\limits_{m>0}^{n} m = \frac{1}{2} n(n+1)$. In the second line we make a change of summation variables of $m \to n - m$, this makes the sum range go to $0 \leq m \leq n-1$,

$$\begin{aligned} C^{IJ}_{11} = &-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \big(\alpha^I_{-n} \alpha^J_n - \alpha^J_{-n} \alpha^I_n \big) \left(n^2 - \frac{1}{2} n(n+1) \right) \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m\geq 0}^{n-1} \big(\alpha^I_{-m} L^\perp_{m-n} \alpha^J_n - \alpha^J_{-m} L^\perp_{m-n} \alpha^I_n \big) \\ C^{IJ}_{11} = &-\frac{1}{4\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} (n-1) \big(\alpha^I_{-n} \alpha^J_n - \alpha^J_{-n} \alpha^I_n \big) \\ &- \frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m\geq 0}^{n-1} \big(\alpha^I_{-m} L^\perp_{m-n} \alpha^J_n - \alpha^J_{-m} L^\perp_{m-n} \alpha^I_n \big) \end{aligned} \tag{4.8}$$

Now, the next one, C_{12}^{IJ}, notice that $C_{21}^{IJ} = -C_{12}^{JI}$. That is, we just need to compute C_{12}^{IJ} and sum with it's anti-symmetric part.

$$
\begin{aligned}
C_{12}^{IJ} &= \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big[L_{-n}^{\perp}\alpha_n^I, \alpha_{-m}^J L_m^{\perp}\big] \\
&= \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big(L_{-n}^{\perp}\big[\alpha_n^I, \alpha_{-m}^J L_m^{\perp}\big] + \big[L_{-n}^{\perp}, \alpha_{-m}^J L_m^{\perp}\big]\alpha_n^I\big) \\
&= \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big(L_{-n}^{\perp}\alpha_{-m}^J\big[\alpha_n^I, L_m^{\perp}\big] + L_{-n}^{\perp}\big[\alpha_n^I, \alpha_{-m}^J\big]L_m^{\perp} + \big[L_{-n}^{\perp}, \alpha_{-m}^J\big]L_m^{\perp}\alpha_n^I \\
&\qquad + \alpha_{-m}^J\big[L_{-n}^{\perp}, L_m^{\perp}\big]\alpha_n^I\big) \\
&= \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big(L_{-n}^{\perp}\alpha_{-m}^J n\alpha_{n+m}^I + L_{-n}^{\perp} n g^{IJ}\delta_{n-m,0}L_m^{\perp} + m\alpha_{-m-n}^J L_m^{\perp}\alpha_n^I \\
&\qquad + \alpha_{-m}^J\bigg((-n-m)L_{-n+m}^{\perp} - \frac{D-2}{12}(n^3-n)\delta_{-n+m,0}\bigg)\alpha_n^I\bigg)
\end{aligned}
$$

It's obvious that when we anti-symmetrize the g^{IJ} term will drop out, so we'll not even attempt to normal-order it, but we won't omit it either for sake of completeness. The only relevant non-normal-ordered term is the first one,

$$
\begin{aligned}
&= \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big(n\big(\big[L_{-n}^{\perp}, \alpha_{-m}^J\big] + \alpha_{-m}^J L_{-n}^{\perp}\big)\alpha_{n+m}^I + m\alpha_{-m-n}^J L_m^{\perp}\alpha_n^I - (n+m)\alpha_{-m}^J L_{-n+m}^{\perp}\alpha_n^I \\
&\qquad + L_{-n}^{\perp} n g^{IJ}\delta_{n-m,0}L_m^{\perp}\big) - \frac{1}{2\alpha' p^{+2}}\frac{D-2}{12}\sum_{n\in\mathbb{N}^*}\frac{n^2-1}{n}\alpha_{-n}^J\alpha_n^I \\
&= \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big(nm\alpha_{-m-n}^J\alpha_{n+m}^I + n\alpha_{-m}^J L_{-n}^{\perp}\alpha_{n+m}^I + m\alpha_{-m-n}^J L_m^{\perp}\alpha_n^I - (n+m)\alpha_{-m}^J L_{-n+m}^{\perp}\alpha_n^I \\
&\qquad + L_{-n}^{\perp} n g^{IJ}\delta_{n-m,0}L_m^{\perp}\big) - \frac{1}{2\alpha' p^{+2}}\frac{D-2}{12}\sum_{n\in\mathbb{N}^*}\frac{n^2-1}{n}\alpha_{-n}^J\alpha_n^I \\
&= \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{m}\alpha_{-m}^J L_{-n}^{\perp}\alpha_{n+m}^I \\
&\quad + \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{n}\alpha_{-m-n}^J L_m^{\perp}\alpha_n^I \\
&\quad - \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \bigg[\frac{1}{n}\alpha_{-m}^J L_{-n+m}^{\perp}\alpha_n^I + \frac{1}{m}\alpha_{-m}^J L_{-n+m}^{\perp}\alpha_n^I\bigg] \\
&\quad + \frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \big(nm\alpha_{-m-n}^J\alpha_{n+m}^I + L_{-n}^{\perp} n g^{IJ}\delta_{n-m,0}L_m^{\perp}\big) \\
&\quad - \frac{1}{2\alpha' p^{+2}}\frac{D-2}{12}\sum_{n\in\mathbb{N}^*}\frac{n^2-1}{n}\alpha_{-n}^J\alpha_n^I
\end{aligned}
$$

We'll relabel in the first line $n \to n+m$, which change the range of the sum to $n > m$, and in the second line we make $m \to m+n$, and the new range will be $m > n$,

$$
C_{12}^{IJ} = \frac{1}{2\alpha' p^{+2}} \sum_{m\in\mathbb{N}^*} \frac{1}{m} \sum_{n>m}^{\infty} \alpha_{-m}^J L_{-n+m}^{\perp}\alpha_n^I
$$

$$
\begin{aligned}
&+\frac{1}{2\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\sum_{m>n}\alpha^J_{-m}L^\perp_{m-n}\alpha^I_n\\
&-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\left[\frac{1}{n}\alpha^J_{-m}L^\perp_{-n+m}\alpha^I_n+\frac{1}{m}\alpha^J_{-m}L^\perp_{-n+m}\alpha^I_n\right]\\
&+\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\left(nm\alpha^J_{-m-n}\alpha^I_{n+m}+L^\perp_{-n}ng^{IJ}\delta_{n-m,0}L^\perp_m\right)\\
&-\frac{1}{2\alpha' p^{+2}}\frac{D-2}{12}\sum_{n\in\mathbb{N}^*}\frac{n^2-1}{n}\alpha^J_{-n}\alpha^I_n
\end{aligned}
$$

$$
\begin{aligned}
C^{IJ}_{12}=&-\frac{1}{2\alpha' p^{+2}}\left[\sum_{n\in\mathbb{N}^*}\frac{1}{n}\sum_{m>0}^{n}\alpha^J_{-m}L^\perp_{-n+m}\alpha^I_n+\sum_{m\in\mathbb{N}^*}\frac{1}{m}\sum_{n>0}^{m}\alpha^J_{-m}L^\perp_{-n+m}\alpha^I_n\right]\\
&+\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\left(nm\alpha^J_{-m-n}\alpha^I_{n+m}+L^\perp_{-n}ng^{IJ}\delta_{n-m,0}L^\perp_m\right)\\
&-\frac{1}{2\alpha' p^{+2}}\frac{D-2}{12}\sum_{n\in\mathbb{N}^*}\frac{n^2-1}{n}\alpha^J_{-n}\alpha^I_n
\end{aligned}
$$

So that,

$$
\begin{aligned}
C^{IJ}_{12}-C^{JI}_{12}=&\frac{1}{2\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\sum_{m>0}^{n}\left[\alpha^I_{-m}L^\perp_{-n+m}\alpha^J_n-\alpha^J_{-m}L^\perp_{-n+m}\alpha^I_n\right]\\
&+\frac{1}{2\alpha' p^{+2}}\sum_{m\in\mathbb{N}^*}\frac{1}{m}\sum_{n>0}^{m}\left[\alpha^I_{-m}L^\perp_{-n+m}\alpha^J_n-\alpha^J_{-m}L^\perp_{-n+m}\alpha^I_n\right]\\
&-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\left(\alpha^I_{-m-n}\alpha^J_{n+m}-\alpha^J_{-m-n}\alpha^I_{n+m}\right)\\
&+\frac{1}{2\alpha' p^{+2}}\frac{D-2}{12}\sum_{n\in\mathbb{N}^*}\frac{n^2-1}{n}\left(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^I_n\right)
\end{aligned}\tag{4.9}
$$

At last, we compute C^{IJ}_{22},

$$
\begin{aligned}
C^{IJ}_{22}&=-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\left[\alpha^I_{-n}L^\perp_n,\alpha^J_{-m}L^\perp_m\right]\\
&=-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\left(\alpha^I_{-n}\left[L^\perp_n,\alpha^J_{-m}L^\perp_m\right]+\left[\alpha^I_{-n},\alpha^J_{-m}L^\perp_m\right]L^\perp_n\right)\\
&=-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\left(\alpha^I_{-n}\left[L^\perp_n,\alpha^J_{-m}\right]L^\perp_m+\alpha^J_{-m}\left[\alpha^I_{-n},L^\perp_m\right]L^\perp_n+\left[\alpha^I_{-n},\alpha^J_{-m}\right]L^\perp_mL^\perp_n\right.\\
&\quad\left.+\alpha^I_{-n}\alpha^J_{-m}\left[L^\perp_n,L^\perp_m\right]\right)\\
&=-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\left(\alpha^I_{-n}\left[L^\perp_n,\alpha^J_{-m}\right]L^\perp_m+\alpha^J_{-m}\left[\alpha^I_{-n},L^\perp_m\right]L^\perp_n+\left[\alpha^I_{-n},\alpha^J_{-m}\right]L^\perp_mL^\perp_n\right.\\
&\quad\left.+\alpha^I_{-n}\alpha^J_{-m}\left[L^\perp_n,L^\perp_m\right]\right)\\
&=-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\frac{1}{nm}\left(\alpha^I_{-n}m\alpha^J_{n-m}L^\perp_m-\alpha^J_{-m}n\alpha^I_{-n+m}L^\perp_n-ng^{IJ}\delta_{-n-m,0}L^\perp_mL^\perp_n\right.\\
&\quad\left.+\alpha^I_{-n}\alpha^J_{-m}\left((n-m)L^\perp_{n+m}+\frac{D-2}{12}(n^3-n)\delta_{n+m,0}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(m\alpha^I_{-n}\alpha^J_{n-m}L^\perp_m - n\alpha^J_{-m}\alpha^I_{-n+m}L^\perp_n + (n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} \right) \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{n}\alpha^I_{-n}\alpha^J_{n-m}L^\perp_m \\
&\qquad + \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{m}\alpha^J_{-m}\alpha^I_{-n+m}L^\perp_n \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{n}\alpha^I_{-n}\alpha^J_{n-m}L^\perp_m \\
&\qquad + \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{n}\alpha^J_{-n}\alpha^I_{n-m}L^\perp_m \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \left[\sum_{m>0}^{n} + \sum_{m>n}^{\infty} \right] \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m} \right) L^\perp_m \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>n}^{\infty} \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m} \right) L^\perp_m \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m} \right) L^\perp_m \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{\infty} \left(\alpha^I_{-n}\alpha^J_{-m} - \alpha^J_{-n}\alpha^I_{-m} \right) L^\perp_{m+n} \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m} \right) L^\perp_m \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} - \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(m\alpha^I_{-n}\alpha^J_{-m} - m\alpha^J_{-n}\alpha^I_{-m} \right) L^\perp_{m+n} \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m} \right) L^\perp_m \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} - \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm} \left(m\alpha^I_{-n}\alpha^J_{-m} - n\alpha^J_{-m}\alpha^I_{-n} \right) L^\perp_{m+n} \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m} \right) L^\perp_m \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(n-m)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{n+m} - \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \frac{1}{nm}(m-n)\alpha^I_{-n}\alpha^J_{-m}L^\perp_{m+n} \\
&\qquad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m} \right) L^\perp_m
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}\alpha^J_{n-m} - \alpha^J_{-n}\alpha^I_{n-m}\right) L^\perp_m \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}\left[\alpha^J_{n-m}, L^\perp_m\right] + \alpha^I_{-n} L^\perp_m \alpha^J_{n-m} - \alpha^J_{-n}\left[\alpha^I_{n-m}, L^\perp_m\right] - \alpha^J_{-n} L^\perp_m \alpha^I_{n-m}\right) \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n}(n-m)\alpha^J_{n} + \alpha^I_{-n} L^\perp_m \alpha^J_{n-m} - \alpha^J_{-n}(n-m)\alpha^I_{n} - \alpha^J_{-n} L^\perp_m \alpha^I_{n-m}\right) \\
&= -\frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \left(\alpha^I_{-n}\alpha^J_{n} - \alpha^J_{-n}\alpha^I_{n}\right) \sum_{m>0}^{n} (n-m) \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n} L^\perp_m \alpha^J_{n-m} - \alpha^J_{-n} L^\perp_m \alpha^I_{n-m}\right) \\
&= -\frac{1}{4\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} (n-1)\left(\alpha^I_{-n}\alpha^J_{n} - \alpha^J_{-n}\alpha^I_{n}\right) \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left(\alpha^I_{-n} L^\perp_m \alpha^J_{n-m} - \alpha^J_{-n} L^\perp_m \alpha^I_{n-m}\right) \\
C^{IJ}_{22} &= -\frac{1}{4\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} (n-1)\left(\alpha^I_{-n}\alpha^J_{n} - \alpha^J_{-n}\alpha^I_{n}\right) \\
&\quad - \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m\geq 0}^{n-1} \left(\alpha^I_{-n} L^\perp_{-m+n} \alpha^J_{m} - \alpha^J_{-n} L^\perp_{-m+n} \alpha^I_{m}\right) \qquad (4.10)
\end{aligned}
$$

Ok. Let's sum all the three expressions, they are 4.8, 4.9 and 4.10, which by the definition 4.7, when summed should recover $\left[C^I, C^J\right]$, 4.6,

$$
\begin{aligned}
\left[C^I, C^J\right] = &-\frac{1}{4\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} (n-1)\left(\alpha^I_{-n}\alpha^J_{n} - \alpha^J_{-n}\alpha^I_{n}\right) \\
&- \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m\geq 0}^{n-1} \left(\alpha^I_{-m} L^\perp_{m-n} \alpha^J_{n} - \alpha^J_{-m} L^\perp_{m-n} \alpha^I_{n}\right) \\
&+ \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m>0}^{n} \left[\alpha^I_{-m} L^\perp_{-n+m} \alpha^J_{n} - \alpha^J_{-m} L^\perp_{-n+m} \alpha^I_{n}\right] \\
&+ \frac{1}{2\alpha' {p^+}^2} \sum_{m\in\mathbb{N}^*} \frac{1}{m} \sum_{n>0}^{m} \left[\alpha^I_{-m} L^\perp_{-n+m} \alpha^J_{n} - \alpha^J_{-m} L^\perp_{-n+m} \alpha^I_{n}\right] \\
&- \frac{1}{2\alpha' {p^+}^2} \sum_{n,m\in\mathbb{N}^*} \left(\alpha^I_{-m-n}\alpha^J_{n+m} - \alpha^J_{-m-n}\alpha^I_{n+m}\right) \\
&+ \frac{1}{2\alpha' {p^+}^2} \frac{D-2}{12} \sum_{n\in\mathbb{N}^*} \frac{n^2-1}{n} \left(\alpha^I_{-n}\alpha^J_{n} - \alpha^J_{-n}\alpha^I_{n}\right) \\
&- \frac{1}{4\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} (n-1)\left(\alpha^I_{-n}\alpha^J_{n} - \alpha^J_{-n}\alpha^I_{n}\right) \\
&- \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n} \sum_{m\geq 0}^{n-1} \left(\alpha^I_{-n} L^\perp_{-m+n} \alpha^J_{m} - \alpha^J_{-n} L^\perp_{-m+n} \alpha^I_{m}\right)
\end{aligned}
$$

The second and third line cancel each other apart from the $m = 0$ and $m = n$ contributions, the same is true for the fourth and eight lines but now leaving the $m=0$ and $n=m$ terms. We also sum the first and seventh lines,

$$
\begin{aligned}
\left[C^I, C^J\right] = &-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} (n-1)\left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right) \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left(\alpha^I_0 L^\perp_{-n}\alpha^J_n - \alpha^J_0 L^\perp_{0-n}\alpha^I_n\right) \\
&+\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left[\alpha^I_{-n} L^\perp_0 \alpha^J_n - \alpha^J_{-n} L^\perp_0 \alpha^I_n\right] \\
&+\frac{1}{2\alpha' p^{+2}} \sum_{m\in\mathbb{N}^*} \frac{1}{m}\left[\alpha^I_{-m} L^\perp_0 \alpha^J_m - \alpha^J_{-m} L^\perp_0 \alpha^I_m\right] \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \left(\alpha^I_{-m-n}\alpha^J_{n+m} - \alpha^J_{-m-n}\alpha^I_{n+m}\right) \\
&+\frac{1}{2\alpha' p^{+2}} \frac{D-2}{12} \sum_{n\in\mathbb{N}^*} \frac{n^2-1}{n}\left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right) \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left(\alpha^I_{-n} L^\perp_n \alpha^J_0 - \alpha^J_{-n} L^\perp_n \alpha^I_0\right)
\end{aligned}
$$

Relabeling the forth line to $m \to n$, we can sum most of the lines,

$$
\begin{aligned}
\left[C^I, C^J\right] = &-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} (n-1)\left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right) \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left(\alpha^I_0 L^\perp_{-n}\alpha^J_n - \alpha^J_0 L^\perp_{0-n}\alpha^I_n\right) \\
&+\frac{1}{\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left[\alpha^I_{-n} L^\perp_0 \alpha^J_n - \alpha^J_{-n} L^\perp_0 \alpha^I_n\right] \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \left(\alpha^I_{-m-n}\alpha^J_{n+m} - \alpha^J_{-m-n}\alpha^I_{n+m}\right) \\
&+\frac{1}{2\alpha' p^{+2}} \frac{D-2}{12} \sum_{n\in\mathbb{N}^*} \frac{n^2-1}{n}\left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right) \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left(\alpha^I_{-n} L^\perp_n \alpha^J_0 - \alpha^J_{-n} L^\perp_n \alpha^I_0\right) \\
\left[C^I, C^J\right] = &-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left[\left(L^\perp_{-n}\alpha^J_n - \alpha_{-n} L^\perp_n\right)\alpha^I_0 - \left(L^\perp_{0-n}\alpha^I_n - \alpha^I_{-n} L^\perp_n\right)\alpha^J_0\right] \\
&+\frac{1}{\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left[\alpha^I_{-n} L^\perp_0 \alpha^J_n - \alpha^J_{-n} L^\perp_0 \alpha^I_n\right] \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n,m\in\mathbb{N}^*} \left(\alpha^I_{-m-n}\alpha^J_{n+m} - \alpha^J_{-m-n}\alpha^I_{n+m}\right) \\
&-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right)\left[-\frac{D-2}{12}\frac{n^2-1}{n} + (n-1)\right] \\
\left[C^I, C^J\right] = &-\frac{1}{2\alpha' p^{+2}} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left[\left(L^\perp_{-n}\alpha^J_n - \alpha_{-n} L^\perp_n\right)\alpha^I_0 - \left(L^\perp_{0-n}\alpha^I_n - \alpha^I_{-n} L^\perp_n\right)\alpha^J_0\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\alpha^I_{-n}L^\perp_0\alpha^J_n-\alpha^J_{-n}L^\perp_0\alpha^I_n\big]\\
&-\frac{1}{2\alpha' p^{+2}}\sum_{n,m\in\mathbb{N}^*}\big(\alpha^I_{-m-n}\alpha^J_{n+m}-\alpha^J_{-m-n}\alpha^I_{n+m}\big)\\
&-\frac{1}{\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\big(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^I_n\big)\left[n\left(\frac{1}{2}-\frac{D-2}{24}\right)+\frac{1}{n}\left(\frac{D-2}{24}\right)-\frac{1}{2}\right]
\end{aligned}
$$

The third line in this expression is a little bit off, it depends only in $n+m$, hence, we can change the two sums to just one sum over $u=n+m>1$, of course just doing this is not enough, because we would be undercounting as there are various n,m such that $n+m=u>1$, in fact, this is a not hard to solve combinatorial problem, we have $n+m=u$ 'objects', to subdivide into two, n,m. As each one has to be $n,m>0$, the number of 'objects' are in fact $n+m-2$. It's simpler to solve visually by imagining we have a line of $n+m-2$ balls and some kind of wall separating them into two sets, the possibles arrangements are $\frac{(n+m-2+1)!}{(n+m-2)!1!}=n+m-1$, hence, we have the equality,

$$
\sum_{n,m\in\mathbb{N}^*}f(n+m)=\sum_{n>1}^{\infty}(n-1)f(n)\equiv\sum_{n\in\mathbb{N}^*}(n-1)f(n)
$$

Using this fact in our expression get to us,

$$
\begin{aligned}
\big[C^I,C^J\big]=&-\frac{1}{2\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\big(L^\perp_{-n}\alpha^J_n-\alpha_{-n}L^\perp_n\big)\alpha^I_0-\big(L^\perp_{0-n}\alpha^I_n-\alpha^I_{-n}L^\perp_n\big)\alpha^J_0\big]\\
&+\frac{1}{\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\alpha^I_{-n}L^\perp_0\alpha^J_n-\alpha^J_{-n}L^\perp_0\alpha^I_n\big]\\
&-\frac{1}{2\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}(n-1)\big(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^I_n\big)\\
&-\frac{1}{\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\big(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^I_n\big)\left[n\left(\frac{1}{2}-\frac{D-2}{24}\right)+\frac{1}{n}\left(\frac{D-2}{24}\right)-\frac{1}{2}\right]\\
\big[C^I,C^J\big]=&-\frac{1}{2\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\big(L^\perp_{-n}\alpha^J_n-\alpha_{-n}L^\perp_n\big)\alpha^I_0-\big(L^\perp_{0-n}\alpha^I_n-\alpha^I_{-n}L^\perp_n\big)\alpha^J_0\big]\\
&+\frac{1}{\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\alpha^I_{-n}L^\perp_0\alpha^J_n-\alpha^J_{-n}L^\perp_0\alpha^I_n\big]\\
&-\frac{1}{\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\big(\alpha^I_{-n}\alpha^J_n-\alpha^J_{-n}\alpha^I_n\big)\left[n\left(1-\frac{D-2}{24}\right)+\frac{1}{n}\left(\frac{D-2}{24}\right)-1\right]
\end{aligned}
\tag{4.11}
$$

Now, what remains is to sum all the three terms, namely, 4.4, 4.5 and 4.11, which, as we already had $\big[A^I,A^J\big]+\big[B^I,B^J\big]+\big(\big[A^I,B^J\big]-\big[B^I,A^J\big]\big)=0$, should recover the full commutator $\big[M^{-I},M^{-J}\big]$,

$$
\begin{aligned}
\big[M^{-I},M^{-J}\big]&=\big[A^I,C^J\big]-\big[A^J,C^I\big]+\big[B^I,C^J\big]-\big[C^I,B^J\big]+\big[C^I,C^J\big]\\
\big[M^{-I},M^{-J}\big]&=\frac{1}{2\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big[\big(L^\perp_{-n}\alpha^J_n-\alpha^J_{-n}L^\perp_n\big)\alpha^I_0-\big(L^\perp_{-n}\alpha^I_n-\alpha^I_{-n}L^\perp_n\big)\alpha^J_0\big]\\
&\quad-\frac{1}{\alpha' p^{+2}}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\big(\alpha^I_{-n}\big(L^\perp_0+a+n\big)\alpha^J_n-\alpha^J_{-n}\big(L^\perp_0+a+n\big)\alpha^I_n\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left[\left(L^\perp_{-n}\alpha^J_n - \alpha_{-n}L^\perp_n\right)\alpha^I_0 - \left(L^\perp_{0-n}\alpha^I_n - \alpha^I_{-n}L^\perp_n\right)\alpha^J_0\right] \\
&+ \frac{1}{\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left[\alpha^I_{-n}L^\perp_0\alpha^J_n - \alpha^J_{-n}L^\perp_0\alpha^I_n\right] \\
&- \frac{1}{\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right)\left[n\left(1-\frac{D-2}{24}\right) + \frac{1}{n}\left(\frac{D-2}{24}\right) - 1\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[M^{-I}, M^{-J}\right] &= -\frac{1}{\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \frac{1}{n}\left(\alpha^I_{-n}(a+n)\alpha^J_n - \alpha^J_{-n}(a+n)\alpha^I_n\right) \\
&\quad - \frac{1}{\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right)\left[n\left(1-\frac{D-2}{24}\right) + \frac{1}{n}\left(\frac{D-2}{24}\right) - 1\right] \\
\left[M^{-I}, M^{-J}\right] &= -\frac{1}{\alpha' {p^+}^2} \sum_{n\in\mathbb{N}^*} \left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right)\left[n\left(1-\frac{D-2}{24}\right) + \frac{1}{n}\left(\frac{D-2}{24} + a\right)\right]
\end{aligned}
$$

To impose $\left[M^{-I}, M^{-J}\right] = 0$ is to impose the charges in the light-cone gauge satisfy the Lorentz algebra — Although we didn't computed all the commutators, granted that this one is zero, all the others will be —, that is, the quantum theory possesses a global Lorentz invariance. Let's see what this constrain implies, first, all the αs are independent operators, this means the expression has to vanish term by term, also, the combination of αs in each term cannot be zero for all states of the Hilbert space, hence, we're forced to conclude that what should be zero is the prefactor,

$$
\begin{aligned}
0 &= n\left(1-\frac{D-2}{24}\right) + \frac{1}{n}\left(\frac{D-2}{24} + a\right) \\
0 &= n^2\left(1-\frac{D-2}{24}\right) + \left(a + \frac{D-2}{24}\right)
\end{aligned}
$$

This has to vanish for every n, and both D and a cannot depend on n, thus, the coefficient of n^2 has to vanish,

$$
\begin{aligned}
1 - \frac{D-2}{24} &= 0 \\
D &= 26
\end{aligned}
$$

This implies,

$$
\begin{aligned}
0 &= a + \frac{D-2}{24} \\
a &= -1
\end{aligned}
$$

That is, imposing Lorentz invariance fixes the normal-ordering constant and the space-time dimension!

Problem 5

5.A)

We're assuming the dimension of the theory in consideration is $D = 26$, so we'll assume the compactification is done in the $X^{D-1} = X^{25}$ target space direction. The usual mode expansion is,

$$
\begin{aligned}
X^{25}(\tau,\sigma) &= x_0^{25} + \sqrt{2\alpha'}\alpha_0^{25}\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\alpha_n^{25}\mathrm{e}^{\mathrm{i}n\sigma} + \bar{\alpha}_n^{25}\mathrm{e}^{-\mathrm{i}n\sigma}\right) \\
UX^{25}(\tau,\sigma)U^{-1} &= Ux_0^{25}U^{-1} + \sqrt{2\alpha'}U\alpha_0^{25}U^{-1}\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(U\alpha_n^{25}U^{-1}\mathrm{e}^{\mathrm{i}n\sigma} + U\bar{\alpha}_n^{25}U^{-1}\mathrm{e}^{-\mathrm{i}n\sigma}\right) \\
-X^{25}(\tau,\sigma) &= Ux_0^{25}U^{-1} + \sqrt{2\alpha'}U\alpha_0^{25}U^{-1}\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(U\alpha_n^{25}U^{-1}\mathrm{e}^{\mathrm{i}n\sigma} + U\bar{\alpha}_n^{25}U^{-1}\mathrm{e}^{-\mathrm{i}n\sigma}\right)
\end{aligned} \tag{5.1}
$$

But, what is also true is,

$$
-X^{25}(\tau,\sigma) = -x_0^{25} + \sqrt{2\alpha'}\left(-\alpha_0^{25}\right)\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(-\alpha_n^{25}\mathrm{e}^{\mathrm{i}n\sigma} - \bar{\alpha}_n^{25}\mathrm{e}^{-\mathrm{i}n\sigma}\right) \tag{5.2}
$$

But as both 5.1 and 5.2 are different representations of the same operator, and, the decomposition in Fourier modes is unique, we must have an equality of the two equations term by term, that is,

$$
Ux_0^{25}U^{-1} = -x_0^{25}, \quad U\bar{\alpha}_n^{25}U^{-1} = -\bar{\alpha}_n^{25}, \quad U\alpha_n^{25}U^{-1} = -\alpha_n^{25}
$$

We can summarize the action of U in all operators as,

$$
Ux_0^{\mu}U^{-1} = (-1)^{g^{25I}}x_0^{\mu}, \quad U\bar{\alpha}_n^{\mu}U^{-1} = (-1)^{g^{25I}}\bar{\alpha}_n^{\mu}, \quad U\alpha_n^{\mu}U^{-1} = (-1)^{g^{25I}}\alpha_n^{\mu}
$$

So that we can study whether of not the light-cone gauge Hamiltonian is invariant under this transformation,

$$
\begin{aligned}
H &= \bar{L}_0^{\perp} + L_0^{\perp} - 2 = \frac{1}{2}\bar{\alpha}_0^I\bar{\alpha}_0^I + \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{n\in\mathbb{N}^*}\bar{\alpha}_{-n}^I\bar{\alpha}_n^I + \sum_{n\in\mathbb{N}^*}\alpha_{-n}^I\alpha_n^I - 2 \\
UHU^{-1} &= \frac{1}{2}U\bar{\alpha}_0^IU^{-1}U\bar{\alpha}_0^IU^{-1} + \frac{1}{2}U\alpha_0^IU^{-1}U\alpha_0^IU^{-1} \\
&\quad + \sum_{n\in\mathbb{N}^*}U\bar{\alpha}_{-n}^IU^{-1}U\bar{\alpha}_n^IU^{-1} + \sum_{n\in\mathbb{N}^*}U\alpha_{-n}^IU^{-1}U\alpha_n^IU^{-1} - 2 \\
UHU^{-1} &= \frac{1}{2}\left[(-1)^{g^{25I}}\bar{\alpha}_0^I\right]\left[(-1)^{g^{25I}}\bar{\alpha}_0^I\right] + \frac{1}{2}\left[(-1)^{g^{25I}}\alpha_0^I\right]\left[(-1)^{g^{25I}}\alpha_0^I\right] \\
&\quad + \sum_{n\in\mathbb{N}^*}\left[(-1)^{g^{25I}}\bar{\alpha}_{-n}^I\right]\left[(-1)^{g^{25I}}\bar{\alpha}_n^I\right] + \sum_{n\in\mathbb{N}^*}\left[(-1)^{g^{25I}}\alpha_{-n}^I\right]\left[(-1)^{g^{25I}}\alpha_n^I\right] - 2 \\
UHU^{-1} &= \frac{1}{2}(-1)^{2g^{25I}}\bar{\alpha}_0^I\bar{\alpha}_0^I + \frac{1}{2}(-1)^{2g^{25I}}\alpha_0^I\alpha_0^I
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{n\in\mathbb{N}^*} (-1)^{2g^{25I}} \bar{\alpha}^I_{-n}\bar{\alpha}^I_n + \sum_{n\in\mathbb{N}^*} (-1)^{2g^{25I}} \alpha^I_{-n}\alpha^I_n - 2 \\
UHU^{-1} &= \frac{1}{2}(-1)^{2g^{25I}} \bar{\alpha}^I_0\bar{\alpha}^I_0 + \frac{1}{2}(-1)^{2g^{25I}} \alpha^I_0\alpha^I_0 \\
&\quad + \sum_{n\in\mathbb{N}^*} (-1)^{2g^{25I}} \bar{\alpha}^I_{-n}\bar{\alpha}^I_n + \sum_{n\in\mathbb{N}^*} (-1)^{2g^{25I}} \alpha^I_{-n}\alpha^I_n - 2, \quad 2g^{25I} = \{0,2\} \to (-1)^{2g^{25I}} = 1 \\
UHU^{-1} &= \frac{1}{2}\bar{\alpha}^I_0\bar{\alpha}^I_0 + \frac{1}{2}\alpha^I_0\alpha^I_0 + \sum_{n\in\mathbb{N}^*} \bar{\alpha}^I_{-n}\bar{\alpha}^I_n + \sum_{n\in\mathbb{N}^*} \alpha^I_{-n}\alpha^I_n - 2 \\
UHU^{-1} &= H
\end{aligned}
$$

Hence, the Hamiltonian is invariant.

5.B)

The closed string vacuum is defined by 25 numbers, namely, q^+, q^I, we'll reserve the name p for the momentum operators, and use q for the eigenvalues of those. In this way the vacuum satisfy,

$$
p^+ \left|q^+, q^I\right\rangle = q^+ \left|q^+, q^I\right\rangle, \quad p^J \left|q^+, q^I\right\rangle = q^J \left|q^+, q^I\right\rangle, \quad \alpha^I_{-n} \left|q^+, q^I\right\rangle = 0, \; n \geq 1
$$

Let's look at which of those properties does the new state defined by the action of U over the vacuum fails to met,

$$
p^+U\left|q^+, q^I\right\rangle = UU^{-1}p^+U\left|q^+, q^I\right\rangle = U(-1)^{g25+}p^+\left|q^+, q^I\right\rangle = Uq^+\left|q^+, q^I\right\rangle = q^+U\left|q^+, q^I\right\rangle
$$

So it still an eigenvector of p^+ with eigenvalue q^+,

$$
\begin{aligned}
p^JU\left|q^+, q^I\right\rangle &= UU^{-1}p^JU\left|q^+, q^I\right\rangle = U(-1)^{g^{25J}}p^J\left|q^+, q^I\right\rangle \\
p^JU\left|q^+, q^I\right\rangle &= (-1)^{g^{25J}}Uq^J\left|q^+, q^I\right\rangle = (-1)^{g^{25J}}q^JU\left|q^+, q^I\right\rangle
\end{aligned}
$$

That is, the state still a eigenvector of p^J, but, the presence of $(-1)^{g^{25J}}$ makes the eigenvalue of the p^{25} momentum operator to have the opposite sign then before, before concluding that this state is also a vacuum, we have to ensure that all the annihilation operators still annihilate it,

$$
\alpha^J_{-n}U\left|q^+, q^I\right\rangle = UU^{-1}\alpha^J_{-n}U\left|q^+, q^I\right\rangle = U(-1)^{g^{25J}}\alpha^J_{-n}\left|q^+, q^I\right\rangle = 0
$$

So now we can be sure that $U\left|q^+, q^I\right\rangle$ is a closed string vacuum with the same eigenvalues as the former, with exception of the q^{25}. This can be written in a simplified manner using the convention of upper case roman letters to non-light-cone gauge target space index $I = 2, \cdots, 25$, and lower case roman letters to non-compactified non-light-cone gauge target space index $i = 2, \cdots, 24$,

$$
U\left|q^+, q^i, q^{25}\right\rangle = \left|q^+, q^i, -q^{25}\right\rangle \tag{5.3}
$$

This has to be true as we shown that this new state has all the correct eigenvalues.

We know the mass operator is,

$$
\begin{aligned}
M^2 &= -p^2 = 2p^+p^- - p^Ip^I = 2p^+\sqrt{\frac{2}{\alpha'}}\alpha_0^- - p^Ip^I \\
M^2 &= 2p^+\frac{2}{\alpha'}\frac{1}{p^+}\left(L_0^\perp + a\right) - \frac{2}{\alpha'}\alpha_0^I\alpha_0^I \\
M^2 &= \frac{2}{\alpha'}\left(\frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p>0}\alpha_{-p}^I\alpha_p^I + a + \bar{a}\right) - \frac{2}{\alpha'}\alpha_0^I\alpha_0^I \\
&\quad + \frac{2}{\alpha'}\left(\frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p>0}\bar{\alpha}_{-p}^I\bar{\alpha}_p^I\right) \\
\frac{\alpha'}{2}M^2 &= N^\perp + \bar{N}^\perp + a + \bar{a} \\
\frac{\alpha'}{2}M^2 &= N^\perp + \bar{N}^\perp - 2
\end{aligned}
$$

Where we also defined the number operators,

$$
\begin{aligned}
N^\perp &= \sum_{p>0}\alpha_{-p}^I\alpha_p^I \\
\bar{N}^\perp &= \sum_{p>0}\bar{\alpha}_{-p}^I\bar{\alpha}_p^I
\end{aligned}
$$

Hence, for a state to have zero mass, it has to have eigenvalues $N^\perp + \bar{N}^\perp = 2$, but, by the level matching, $N^\perp = \bar{N}^\perp$, this fixes the spectrum as states with eigenvalues $N^\perp = \bar{N}^\perp = 1$, that is, is necessary to have exactly one α_{-1}^I and one $\bar{\alpha}_{-1}^J$, so the most general state is given by a sum over linear combination of those,

$$
\begin{aligned}
|\xi\rangle &= \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\left(q^+, q^i, q^{25}\right)\alpha_{-1}^I\bar{\alpha}_{-1}^J \left|q^+, q^i, q^{25}\right\rangle \\
U|\xi\rangle &= \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\left(q^+, q^i, q^{25}\right)U\alpha_{-1}^I U^{-1}U\bar{\alpha}_{-1}^J U^{-1}U \left|q^+, q^i, q^{25}\right\rangle \\
U|\xi\rangle &= \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\left(q^+, q^i, q^{25}\right)(-1)^{g^{25I}+g^{25J}}\alpha_{-1}^I\bar{\alpha}_{-1}^j \left|q^+, q^i, -q^{25}\right\rangle \\
U|\xi\rangle &= \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{+\infty}^{-\infty} \mathrm{d}\left(-q^{25}\right)\, \xi_{IJ}\left(q^+, q^i, -q^{25}\right)(-1)^{g^{25I}+g^{25J}}\alpha_{-1}^I\bar{\alpha}_{-1}^J \left|q^+, q^i, q^{25}\right\rangle \\
U|\xi\rangle &= \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\left(q^+, q^i, -q^{25}\right)(-1)^{g^{25I}+g^{25J}}\alpha_{-1}^I\bar{\alpha}_{-1}^J \left|q^+, q^i, q^{25}\right\rangle
\end{aligned}
$$

This new state $U|\xi\rangle$ still have mass zero, because it still has one α_{-1}^I and one $\bar{\alpha}_{-1}^J$, to impose invariance under U is simply to impose $U|\xi\rangle = |\xi\rangle$, this condition can be red from the last expression as,

$$\xi_{IJ}\big(q^+,q^i,-q^{25}\big)(-1)^{g^{25I}+g^{25J}} = \xi_{IJ}\big(q^+,q^i,q^{25}\big)$$

Notice that with this information we can write,

$$|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle$$

$$\begin{aligned}|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \Bigg[&\int_0^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle \\ &+\int_{-\infty}^{0} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle\Bigg]\end{aligned}$$

$$\begin{aligned}|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \Bigg[&\int_0^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle \\ &-\int_{0}^{-\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle\Bigg]\end{aligned}$$

$$\begin{aligned}|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \Bigg[&\int_0^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle \\ &+\int_{0}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,-q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,-q^{25}\right\rangle\Bigg]\end{aligned}$$

$$\begin{aligned}|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \Bigg[&\int_0^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle \\ &+\int_{0}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)(-1)^{g^{25I}+g^{25J}}\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,-q^{25}\right\rangle\Bigg]\end{aligned}$$

It's clear to see that, if we manually force the integration in q^{25} to be from $-\infty$ to $+\infty$, we'll under the same procedure get the same two terms, hence, we can just integrate over the reals and divide by two after,

$$\begin{aligned}|\xi\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \Bigg[&\int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,q^{25}\right\rangle \\ &+\int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)(-1)^{g^{25I}+g^{25J}}\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+,q^i,-q^{25}\right\rangle\Bigg]\end{aligned}$$

$$|\xi\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+,q^i,q^{25}\big)\times$$

$$\times \left[\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+, q^i, q^{25}\right\rangle + (-1)^{g^{25I}+g^{25J}}\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+, q^i, -q^{25}\right\rangle\right]$$

In this form it's clear that this state $|\xi\rangle$ is U-invariant,

$$U|\xi\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+, q^i, q^{25}\big)\times$$
$$\times \left[U\alpha^I_{-1}U^{-1}U\bar{\alpha}^J_{-1}U^{-1}U\left|q^+, q^i, q^{25}\right\rangle + (-1)^{g^{25I}+g^{25J}}U\alpha^I_{-1}U^{-1}U\bar{\alpha}^J_{-1}U^{-1}U\left|q^+, q^i, -q^{25}\right\rangle\right]$$
$$U|\xi\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+, q^i, q^{25}\big)\times$$
$$\times \left[(-1)^{g^{25I}+g^{25J}}\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+, q^i, -q^{25}\right\rangle + (-1)^{2g^{25I}+2g^{25J}}\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+, q^i, q^{25}\right\rangle\right]$$
$$U|\xi\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{23}} \mathrm{d}^{23}q^i \int_{-\infty}^{+\infty} \mathrm{d}q^{25}\, \xi_{IJ}\big(q^+, q^i, q^{25}\big)\times$$
$$\times \left[(-1)^{g^{25I}+g^{25J}}\alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+, q^i, -q^{25}\right\rangle + \alpha^I_{-1}\bar{\alpha}^J_{-1}\left|q^+, q^i, q^{25}\right\rangle\right]$$
$$U|\xi\rangle = |\xi\rangle$$

With $\xi_{IJ}(q^+, q^i, q^{25})$ being arbitrary, in other words, we found a U-independent basis of the massless excitations of the string, the basis can be subdivided in 4 categories,

$$\alpha^i_{-1}\bar{\alpha}^j_{-1}\left|q^+, q^i, q^{25}\right\rangle + (-1)^{g^{25i}+g^{25j}}\alpha^i_{-1}\bar{\alpha}^j_{-1}\left|q^+, q^i, -q^{25}\right\rangle = \alpha^i_{-1}\bar{\alpha}^j_{-1}\left|q^+, q^i, q^{25}\right\rangle + \alpha^i_{-1}\bar{\alpha}^j_{-1}\left|q^+, q^i, -q^{25}\right\rangle$$
$$\alpha^{25}_{-1}\bar{\alpha}^j_{-1}\left|q^+, q^i, q^{25}\right\rangle + (-1)^{g^{2525}+g^{25j}}\alpha^{25}_{-1}\bar{\alpha}^j_{-1}\left|q^+, q^i, -q^{25}\right\rangle = \alpha^{25}_{-1}\bar{\alpha}^j_{-1}\left|q^+, q^i, q^{25}\right\rangle - \alpha^{25}_{-1}\bar{\alpha}^J_{-1}\left|q^+, q^i, -q^{25}\right\rangle$$
$$\alpha^i_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, q^{25}\right\rangle + (-1)^{g^{25i}+g^{2525}}\alpha^i_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, -q^{25}\right\rangle = \alpha^i_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, q^{25}\right\rangle - \alpha^i_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, -q^{25}\right\rangle$$
$$\alpha^{25}_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, q^{25}\right\rangle + (-1)^{g^{2525}+g^{2525}}\alpha^{25}_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, -q^{25}\right\rangle = \alpha^{25}_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, q^{25}\right\rangle + \alpha^{25}_{-1}\bar{\alpha}^{25}_{-1}\left|q^+, q^i, -q^{25}\right\rangle$$

5.C)

We proceed here solving the light-cone gauge classical equation of motion for the **open** string with the boundary condition,

$$X^{25}(\tau, \sigma + 2\pi) = -X^{25}(\tau, \sigma) \tag{5.4}$$

And closed string boundary condition on the other coordinates. Much of this follows in the same lines of what as already done in Appendix C, we start from the light-cone gauge equation of motion A.4, decomposing our solution as,

$$0 = \partial_+\partial_- X^\mu \Rightarrow X^\mu(\tau, \sigma) = X^\mu_L\big(\sigma^+\big) + X^\mu_R\big(\sigma^-\big)$$

Such that the boundary condition 5.4 implies,

$$X^{25}(\tau, \sigma + 2\pi) = -X^{25}(\tau, \sigma)$$

$$
\begin{aligned}
X_L^{25}(\sigma^+ + 2\pi) + X_R^{25}(\sigma^- - 2\pi) &= -X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-) \\
X_L^{25}(\sigma^+ + 2\pi) + X_L^{25}(\sigma^+) &= -X_R^{25}(\sigma^-) - X_R^{25}(\sigma^- - 2\pi)
\end{aligned}
$$

As the variables $\sigma^\pm$ are independent, this basically states that the derivatives of these modes are antiperiodic,

$$
\begin{aligned}
X'^{25}_L(\sigma^+ + 2\pi) + X'^{25}_L(\sigma^+) = 0 = -X'^{25}_R(\sigma^-) - X'^{25}_R(\sigma^- - 2\pi) \\
X'^{25}_L(\sigma^+ + 2\pi) = -X'^{25}_L(\sigma^+), \quad X'^{25}_R(\sigma^- + 2\pi) = -X'^{25}_R(\sigma^-)
\end{aligned}
$$

As a periodic function can be written as the Fourier sum over integers, an antiperiodic function can be written as a Fourier sum over half-integers,

$$
\begin{aligned}
X'^{25}_L(\sigma^+) &= \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{\alpha}_n^{25} \exp\left(-\mathrm{i}n\sigma^+\right) \\
X'^{25}_R(\sigma^-) &= \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \alpha_n^{25} \exp\left(-\mathrm{i}n\sigma^-\right)
\end{aligned}
$$

Integrating those we get,

$$
\begin{aligned}
X_L^{25}(\sigma^+) &= x_L + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\bar{\alpha}_n^{25} \exp\left(-\mathrm{i}n\sigma^+\right) \\
X_R^{25}(\sigma^-) &= x_R + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\alpha_n^{25} \exp\left(-\mathrm{i}n\sigma^-\right)
\end{aligned}
$$

And finally we can sum the two contributions to get,

$$
X^{25} = x_L + x_R + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^{25}\mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^{25}\mathrm{e}^{\mathrm{i}n\sigma}\right)
$$

Which from the condition 5.4, can be seen to imply, $x_L + x_R = 0$, thus, the most general solution with Closed string boundary condition in all coordinates apart from the X^{25} with a twist boundary condition is,

$$
\begin{aligned}
X^i &= x_0^i + \sqrt{2\alpha'}\alpha_0^i\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^*} \frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^i\mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^i\mathrm{e}^{\mathrm{i}n\sigma}\right) \\
X^{25} &= \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^{25}\mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^{25}\mathrm{e}^{\mathrm{i}n\sigma}\right)
\end{aligned}
$$

5.D)

Let's compute the commutation between the αs. It's not needed to compute $[\alpha_n^i, \alpha_m^j]$, as we already done this, and this new boundary condition won't change it, as is clear from D.4.

We just need to compute $[\alpha^i_n,\alpha^{25}_m],[\alpha^i_n,\bar\alpha^{25}_m],[\bar\alpha^i_n,\alpha^{25}_m],[\bar\alpha^i_n,\bar\alpha^{25}_m],[\alpha^{25}_n,\alpha^{25}_m],[\alpha^{25}_n,\bar\alpha^{25}_m],[\bar\alpha^{25}_n,\bar\alpha^{25}_m]$, the procedure is the same of D.4, in particular, as the quantization conditions are the same, D.1, we do actually get the same set of relations as D.2 and D.3.

$$
\begin{aligned}
&\left[\left(\dot X^I \pm X'^I\right)(\tau,\sigma),\left(\dot X^J \pm X'^J\right)(\tau,\sigma')\right] = \pm 4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\\
&\left[\left(\dot X^I \pm X'^I\right)(\tau,\sigma),\left(\dot X^J \mp X'^J\right)(\tau,\sigma')\right] = 0
\end{aligned}
$$

So the computation is exactly equal D.4. Now let's do it case by case, starting by $[\alpha^i_n,\bar\alpha^{25}_m]$,

$$
\begin{aligned}
0 &= \left[\left(\dot X^i - X'^i\right)(\tau,\sigma),\left(\dot X^{25} + X'^{25}\right)(\tau,\sigma')\right],\quad \sigma,\sigma'\in[0,2\pi]\\
0 &= \left[\left(\dot X^i - X'^i\right)(\tau,\sigma),\left(\dot X^{25} + X'^{25}\right)(\tau,\sigma')\right]\\
0 &= \sqrt{2\alpha'}\sum_{n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)}\sqrt{2\alpha'}\sum_{m'\in\mathbb{Z}+\frac12}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)}\left[\alpha^i_{n'},\bar\alpha^{25}_{m'}\right]\\
0 &= \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{-\mathrm{i}n\sigma}\int_0^{2\pi}\frac{\mathrm{d}\sigma'}{2\pi}\mathrm{e}^{\mathrm{i}m\sigma'}\sum_{n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)}\sum_{m'\in\mathbb{Z}+\frac12}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)}\left[\alpha^i_{n'},\bar\alpha^{25}_{m'}\right]\\
0 &= \sum_{n'\in\mathbb{Z}}\sum_{m'\in\mathbb{Z}+\frac12}\delta_{n,n'}\delta_{m,m'}\left[\alpha^i_{n'},\bar\alpha^{25}_{m'}\right]\\
0 &= \left[\alpha^i_n,\bar\alpha^{25}_m\right]
\end{aligned}
$$

Now, $[\alpha^i_n,\alpha^{25}_m]$,

$$
\begin{aligned}
-4\pi\alpha'\mathrm{i}g^{i25}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') &= \left[\left(\dot X^i - X'^i\right)(\tau,\sigma),\left(\dot X^{25} - X'^{25}\right)(\tau,\sigma')\right],\quad \sigma,\sigma'\in[0,2\pi]\\
0 &= \left[\left(\dot X^i - X'^i\right)(\tau,\sigma),\left(\dot X^{25} - X'^{25}\right)(\tau,\sigma')\right]\\
0 &= \sqrt{2\alpha'}\sum_{n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)}\sqrt{2\alpha'}\sum_{m'\in\mathbb{Z}+\frac12}\mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)}\left[\alpha^i_{n'},\alpha^{25}_{m'}\right]\\
0 &= \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{-\mathrm{i}n\sigma}\int_0^{2\pi}\frac{\mathrm{d}\sigma'}{2\pi}\mathrm{e}^{-\mathrm{i}m\sigma'}\sum_{n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)}\sum_{m'\in\mathbb{Z}+\frac12}\mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)}\left[\alpha^i_{n'},\alpha^{25}_{m'}\right]\\
0 &= \sum_{n'\in\mathbb{Z}}\sum_{m'\in\mathbb{Z}+\frac12}\delta_{n,n'}\delta_{m,m'}\left[\alpha^i_{n'},\alpha^{25}_{m'}\right]\\
0 &= \left[\alpha^i_n,\alpha^{25}_m\right]
\end{aligned}
$$

Now $[\bar\alpha^i_n,\bar\alpha^{25}_m]$,

$$
\begin{aligned}
4\pi\alpha'\mathrm{i}g^{i25}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') &= \left[\left(\dot X^i + X'^i\right)(\tau,\sigma),\left(\dot X^{25} + X'^{25}\right)(\tau,\sigma')\right],\quad \sigma,\sigma'\in[0,2\pi]\\
0 &= \left[\left(\dot X^i + X'^i\right)(\tau,\sigma),\left(\dot X^{25} + X'^{25}\right)(\tau,\sigma')\right]\\
0 &= \sqrt{2\alpha'}\sum_{n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau+\sigma)}\sqrt{2\alpha'}\sum_{m'\in\mathbb{Z}+\frac12}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)}\left[\bar\alpha^i_{n'},\bar\alpha^{25}_{m'}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 &= \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{\mathrm{i}n\sigma} \int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi} \mathrm{e}^{\mathrm{i}m\sigma'} \sum_{n'\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'(\tau+\sigma)} \sum_{m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)} \left[\bar{\alpha}^i_{n'}, \bar{\alpha}^{25}_{m'}\right] \\
0 &= \sum_{n'\in\mathbb{Z}} \sum_{m'\in\mathbb{Z}+\frac{1}{2}} \delta_{n,n'}\delta_{m,m'} \left[\alpha^i_{n'}, \bar{\alpha}^{25}_{m'}\right] \\
0 &= \left[\alpha^i_n, \bar{\alpha}^{25}_m\right]
\end{aligned}
$$

Now, $[\bar{\alpha}^i_n, \alpha^{25}_m]$,

$$
\begin{aligned}
0 &= \left[\left(\dot{X}^i + X'^i\right)(\tau,\sigma), \left(\dot{X}^{25} - X'^{25}\right)(\tau,\sigma')\right], \quad \sigma,\sigma' \in [0,2\pi] \\
0 &= \left[\left(\dot{X}^i + X'^i\right)(\tau,\sigma), \left(\dot{X}^{25} - X'^{25}\right)(\tau,\sigma')\right] \\
0 &= \sqrt{2\alpha'} \sum_{n'\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'(\tau+\sigma)} \sqrt{2\alpha'} \sum_{m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)} \left[\bar{\alpha}^i_{n'}, \alpha^{25}_{m'}\right] \\
0 &= \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{\mathrm{i}n\sigma} \int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi} \mathrm{e}^{-\mathrm{i}m\sigma'} \sum_{n'\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'(\tau+\sigma)} \sum_{m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)} \left[\bar{\alpha}^i_{n'}, \alpha^{25}_{m'}\right] \\
0 &= \sum_{n'\in\mathbb{Z}} \sum_{m'\in\mathbb{Z}+\frac{1}{2}} \delta_{n,n'}\delta_{m,m'} \left[\alpha^i_{n'}, \alpha^{25}_{m'}\right] \\
0 &= \left[\alpha^i_n, \alpha^{25}_m\right]
\end{aligned}
$$

Now $[\alpha^{25}_n, \alpha^{25}_m]$,

$$
\begin{aligned}
\left[\left(\dot{X}^{25} - X'^{25}\right)(\tau,\sigma), \left(\dot{X}^{25} - X'^{25}\right)(\tau,\sigma')\right] &= -4\pi\alpha'\mathrm{i}g^{2525}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \\
2\alpha' \sum_{n',m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)}\mathrm{e}^{-\mathrm{i}m'(\tau-\sigma')} \left[\alpha^{25}_{n'}, \alpha^{25}_{m'}\right] &= -4\pi\alpha'\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \\
\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{-\mathrm{i}n\sigma} \sum_{n',m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)}\mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)} \left[\alpha^{25}_{n'}, \alpha^{25}_{m'}\right] &= -2\pi\mathrm{i}\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{-\mathrm{i}n\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \\
\sum_{m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n\tau}\mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)} \left[\alpha^{25}_{n}, \alpha^{25}_{m'}\right] &= n\mathrm{e}^{-\mathrm{i}n\sigma'} \\
\int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi}\mathrm{e}^{-\mathrm{i}m\sigma'} \sum_{m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n\tau}\mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)} \left[\alpha^{25}_{n}, \alpha^{25}_{m'}\right] &= n\int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi}\mathrm{e}^{-\mathrm{i}m\sigma'}\mathrm{e}^{-\mathrm{i}n\sigma'} \\
\mathrm{e}^{-\mathrm{i}(n+m)\tau}\left[\alpha^{25}_n, \alpha^{25}_m\right] &= n\delta_{n+m,0} \\
\left[\alpha^{25}_n, \alpha^{25}_m\right] &= n\delta_{n+m,0}
\end{aligned}
$$

Now, $[\bar{\alpha}^{25}_n, \bar{\alpha}^{25}_m]$,

$$
\begin{aligned}
\left[\left(\dot{X}^{25} + X'^{25}\right)(\tau,\sigma), \left(\dot{X}^{25} + X'^{25}\right)(\tau,\sigma')\right] &= 4\pi\alpha'\mathrm{i}g^{2525}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \\
2\alpha' \sum_{n',m'\in\mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n'(\tau+\sigma)}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma')} \left[\bar{\alpha}^{25}_{n'}, \bar{\alpha}^{25}_{m'}\right] &= 4\pi\alpha'\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')
\end{aligned}
$$

$$\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{\mathrm{i}n\sigma} \sum_{n',m' \in \mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n'(\tau+\sigma)} \mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)} \left[\bar{\alpha}^{25}_{n'}, \bar{\alpha}^{25}_{m'}\right] = 2\pi\mathrm{i} \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{\mathrm{i}n\sigma} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma - \sigma')$$

$$\sum_{m' \in \mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n\tau} \mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)} \left[\bar{\alpha}^{25}_{n}, \bar{\alpha}^{25}_{m'}\right] = n\mathrm{e}^{\mathrm{i}n\sigma'}$$

$$\int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi} \mathrm{e}^{\mathrm{i}m\sigma'} \sum_{m' \in \mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n\tau} \mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)} \left[\bar{\alpha}^{25}_{n}, \bar{\alpha}^{25}_{m'}\right] = n \int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi} \mathrm{e}^{\mathrm{i}m\sigma'} \mathrm{e}^{\mathrm{i}n\sigma'}$$

$$\mathrm{e}^{-\mathrm{i}(n+m)\tau} \left[\bar{\alpha}^{25}_{n}, \bar{\alpha}^{25}_{m}\right] = n\delta_{n+m,0}$$

$$\left[\bar{\alpha}^{25}_{n}, \bar{\alpha}^{25}_{m}\right] = n\delta_{n+m,0}$$

And by last, $[\alpha^{25}_n, \bar{\alpha}^{25}_m]$,

$$\left[\left(\dot{X}^{25} - X'^{25}\right)(\tau,\sigma), \left(\dot{X}^{25} + X'^{25}\right)(\tau,\sigma')\right] = 0$$

$$2\alpha' \sum_{n',m' \in \mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)} \mathrm{e}^{-\mathrm{i}m'(\tau+\sigma')} \left[\alpha^{25}_{n'}, \bar{\alpha}^{25}_{m'}\right] = 0$$

$$\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{-\mathrm{i}n\sigma} \sum_{n',m' \in \mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n'(\tau-\sigma)} \mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)} \left[\alpha^{25}_{n'}, \bar{\alpha}^{25}_{m'}\right] = 0$$

$$\sum_{m' \in \mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n\tau} \mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)} \left[\alpha^{25}_{n}, \bar{\alpha}^{25}_{m'}\right] = 0$$

$$\int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi} \mathrm{e}^{\mathrm{i}m\sigma'} \sum_{m' \in \mathbb{Z}+\frac{1}{2}} \mathrm{e}^{-\mathrm{i}n\tau} \mathrm{e}^{-\mathrm{i}m'(\tau-\sigma)} \left[\alpha^{25}_{n}, \bar{\alpha}^{25}_{m'}\right] = 0$$

$$\mathrm{e}^{-\mathrm{i}(n+m)\tau} \left[\alpha^{25}_{n}, \bar{\alpha}^{25}_{m}\right] = 0$$

$$\left[\alpha^{25}_{n}, \bar{\alpha}^{25}_{m}\right] = 0$$

What finishes all the commutation relations. We have to define the new Virasoro operators for this case, those appear in the constrain, namely A.6,

$$\dot{X}^- - X'^- = \frac{1}{4\alpha' p^+} \left(\dot{X}^I - X'^I\right)^2$$

$$\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^-_n \exp\left(-\mathrm{i}n(\tau+\sigma)\right) = \frac{2\alpha'}{4\alpha' p^+} \sum_{p,q \in \mathbb{Z}} \alpha^i_p \alpha^i_q \exp\left(-\mathrm{i}(p+q)(\tau+\sigma)\right) + \frac{2\alpha'}{4\alpha' p^+} \sum_{p,q \in \mathbb{Z}+\frac{1}{2}} \alpha^{25}_p \alpha^{25}_q \exp\left(-\mathrm{i}(p+q)(\tau+\sigma)\right)$$

$$\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^-_n \exp\left(-\mathrm{i}n(\tau+\sigma)\right) = \frac{1}{2p^+} \sum_{p,n \in \mathbb{Z}} \alpha^i_p \alpha^i_{n-p} \exp\left(-\mathrm{i}n(\tau+\sigma)\right) + \frac{1}{2p^+} \sum_{p \in \mathbb{Z}+\frac{1}{2}} \sum_{n \in \mathbb{Z}} \alpha^{25}_p \alpha^{25}_{n-p} \exp\left(-\mathrm{i}n(\tau+\sigma)\right)$$

$$\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^-_n \exp\left(-\mathrm{i}n(\tau+\sigma)\right) = \frac{1}{p^+} \sum_{n \in \mathbb{Z}} \left(\frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha^i_p \alpha^i_{n-p} + \frac{1}{2} \sum_{p \in \mathbb{Z}+\frac{1}{2}} \alpha^{25}_p \alpha^{25}_{n-p}\right) \exp\left(-\mathrm{i}n(\tau+\sigma)\right)$$

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}\left(L_n^\perp + a\right)$$

From where is easy to see what should be the Virasoro operators, in particular the $n = 0$ one it's the normal ordered version of,

$$\begin{aligned}
L_0^\perp + a &= \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_p^i\alpha_{-p}^i + \frac{1}{2}\sum_{p\in\mathbb{Z}+\frac{1}{2}} \alpha_p^{25}\alpha_{-p}^{25} \\
&= \frac{1}{2}\sum_{p\geq 0} \alpha_p^i\alpha_{-p}^i + \frac{1}{2}\sum_{p<0} \alpha_p^i\alpha_{-p}^i + \frac{1}{2}\sum_{p\geq\frac{1}{2}} \alpha_p^{25}\alpha_{-p}^{25} + \frac{1}{2}\sum_{p\leq-\frac{1}{2}} \alpha_p^{25}\alpha_{-p}^{25} \\
&= \frac{1}{2}\sum_{p\geq 0}\left(\left[\alpha_p^i, \alpha_{-p}^i\right] + \alpha_{-p}^i\alpha_p^i\right) + \frac{1}{2}\sum_{p>0}\alpha_{-p}^i\alpha_p^i + \frac{1}{2}\sum_{p\geq\frac{1}{2}}\left(\left[\alpha_p^{25}, \alpha_{-p}^{25}\right] + \alpha_{-p}^{25}\alpha_p^{25}\right) + \frac{1}{2}\sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} \\
&= \frac{1}{2}\sum_{p\geq 0}\left(23p + \alpha_{-p}^i\alpha_p^i\right) + \frac{1}{2}\sum_{p>0}\alpha_{-p}^i\alpha_p^i + \frac{1}{2}\sum_{p\geq\frac{1}{2}}\left(p + \alpha_{-p}^{25}\alpha_p^{25}\right) + \frac{1}{2}\sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} \\
&= \frac{1}{2}\sum_{p\geq 0}\alpha_{-p}^i\alpha_p^i + \frac{1}{2}\sum_{p>0}\alpha_{-p}^i\alpha_p^i + \frac{1}{2}\sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} + \frac{1}{2}\sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} + \frac{23}{2}\sum_{p=1}^{\infty} p + \frac{1}{2}\sum_{p=\frac{1}{2}}^{\infty} p \\
&= \frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p>0}\alpha_{-p}^i\alpha_p^i + \sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} + \frac{23}{2}\sum_{p=1}^{\infty} p + \frac{1}{2}\sum_{p=\frac{1}{2}}^{\infty} p
\end{aligned}$$

It's clear from here that the first four normal-ordered terms correspond to $L_0^\perp$, while the two last sums are the normal ordering constants, which, are in fact divergent, and need to be regulated, we'll choose the most convenient type of regulator, the zeta function, because we already derived much of what we'll need, that is, 3.4, so that the first sum is already $-\frac{23}{24}$,

$$\begin{aligned}
a &= -\frac{23}{24} + \frac{1}{2}\sum_{p=\frac{1}{2}}^{\infty} p \\
a &= -\frac{23}{24} + \frac{1}{4}\sum_{p=\frac{1}{2}}^{\infty} 2p \\
a &= -\frac{23}{24} + \frac{1}{4}\sum_{p=1,3,5,\cdots} p \\
a &= -\frac{23}{24} + \frac{1}{4}\sum_{p=1,3,5,\cdots} p + \frac{1}{4}\sum_{p=2,4,6,\cdots} p - \frac{1}{4}\sum_{p=2,4,6,\cdots} p \\
a &= -\frac{23}{24} + \frac{1}{4}\sum_{p=1}^{\infty} p - \frac{1}{4}\sum_{p=1,2,3,\cdots} 2p \\
a &= -\frac{23}{24} + \frac{1}{4}\sum_{p=1}^{\infty} p - \frac{1}{2}\sum_{p=1}^{\infty} p \\
a &= -\frac{23}{24} - \frac{1}{4}\sum_{p=1}^{\infty} p \\
a &= -\frac{23}{24} + \frac{1}{48} = -\frac{15}{16}
\end{aligned}$$

Actually this is not the end of it, we would have to use the constrain equation with the plus sign to derive a similar expression for $\bar{L}_0^{\perp}$, from where we would have,

$$\sqrt{2\alpha'}\bar{\alpha}_0^- = \frac{1}{p^+}\left(\bar{L}_0^{\perp} + \bar{a}\right) = \frac{1}{p^+}\left(L_0^{\perp} + a\right)$$

As the level matching condition. No need to say, the computation is exactly the same to obtain $\bar{a} = a$. But, nevertheless, this is relevant for writing the Mass Operator,

$$\begin{aligned}
M^2 &= -p^2 = 2p^+p^- - p^Ip^I = 2p^+\sqrt{\frac{2}{\alpha'}}\alpha_0^- - p^ip^i - p^{25}p^{25} \\
M^2 &= 2p^+\frac{2}{\alpha'}\frac{1}{p^+}\left(L_0^{\perp} + a\right) - p^ip^i - p^{25}p^{25} \\
M^2 &= \frac{2}{\alpha'}\left(L_0^{\perp} + \bar{L}_0^{\perp} + a + \bar{a}\right) - p^ip^i - p^{25}p^{25}
\end{aligned}$$

We seen also that $p^{25} = 0$, and additionally, we substitute in this the expression we found for the Virasoro modes,

$$\begin{aligned}
M^2 &= \frac{2}{\alpha'}\left(\frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p>0}\alpha_{-p}^i\alpha_p^i + \sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} + a + \bar{a}\right) - p^ip^i \\
M^2 &= \frac{2}{\alpha'}\left(\frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p>0}\alpha_{-p}^i\alpha_p^i + \sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} + a + \bar{a}\right) - \frac{2}{\alpha'}\alpha_0^i\alpha_0^i \\
&\quad + \frac{2}{\alpha'}\left(\frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{p>0}\bar{\alpha}_{-p}^i\bar{\alpha}_p^i + \sum_{p\geq\frac{1}{2}}\bar{\alpha}_{-p}^{25}\bar{\alpha}_p^{25}\right) \\
\frac{\alpha'}{2}M^2 &= N^{\perp} + \bar{N}^{\perp} + a + \bar{a} \\
\frac{\alpha'}{2}M^2 &= N^{\perp} + \bar{N}^{\perp} - 2\frac{15}{16} \\
\frac{\alpha'}{2}M^2 &= N^{\perp} + \bar{N}^{\perp} - \frac{15}{8}
\end{aligned}$$

Where we defined the number operators,

$$\begin{aligned}
N^{\perp} &= \sum_{p>0}\alpha_{-p}^i\alpha_p^i + \sum_{p\geq\frac{1}{2}}\alpha_{-p}^{25}\alpha_p^{25} \\
\bar{N}^{\perp} &= \sum_{p>0}\bar{\alpha}_{-p}^i\bar{\alpha}_p^i + \sum_{p\geq\frac{1}{2}}\bar{\alpha}_{-p}^{25}\bar{\alpha}_p^{25}
\end{aligned}$$

5.E)

As mentioned, twisted states doesn't have a zero mode oscillator, hence, it don't possesses conserved momentum in the identified direction, we can label a twisted vacuum as,

$$\left|q^+, q^i\right\rangle \equiv \left|q^+, q^i, q^{25} = 0\right\rangle$$

It's obvious that the vacuum has $N^{\perp} = \bar{N}^{\perp} = 0$, hence, the mass is,

$$M^2 \left|q^+, q^i\right\rangle = \frac{2}{\alpha'}\left(N^{\perp} + \bar{N}^{\perp} - \frac{15}{8}\right)\left|q^+, q^i\right\rangle$$
$$M^2 \left|q^+, q^i\right\rangle = -\frac{15}{4\alpha'}\left|q^+, q^i\right\rangle$$

That is, the mass of the vacuum state is $M^2 = -\frac{15}{4\alpha'}$. Now the first excited state, we must always remember to impose the level matching condition, $N^{\perp} = \bar{N}^{\perp}$, usually, the lowest value for the number operator is one, but, here in the twisted case, we do have creation operators with half-integer value, $\alpha^{25}_{-\frac{1}{2}}, \bar{\alpha}^{25}_{-\frac{1}{2}}$, so, the lowest excited state is the state with, $N^{\perp} = \bar{N}^{\perp} = \frac{1}{2}$, which is the state,

$$\alpha^{25}_{\frac{1}{2}}\bar{\alpha}^{25}_{\frac{1}{2}}\left|q^+, q^i\right\rangle$$
$$M^2\alpha^{25}_{\frac{1}{2}}\bar{\alpha}^{25}_{\frac{1}{2}}\left|q^+, q^i\right\rangle = \frac{2}{\alpha'}\left(N^{\perp} + \bar{N}^{\perp} - \frac{15}{8}\right)\alpha^{25}_{-\frac{1}{2}}\bar{\alpha}^{25}_{-\frac{1}{2}}\left|q^+, q^i\right\rangle$$
$$M^2\alpha^{25}_{\frac{1}{2}}\bar{\alpha}^{25}_{\frac{1}{2}}\left|q^+, q^i\right\rangle = \frac{2}{\alpha'}\left(\frac{1}{2} + \frac{1}{2} - \frac{15}{8}\right)\alpha^{25}_{-\frac{1}{2}}\bar{\alpha}^{25}_{-\frac{1}{2}}\left|q^+, q^i\right\rangle$$
$$M^2\alpha^{25}_{\frac{1}{2}}\bar{\alpha}^{25}_{\frac{1}{2}}\left|q^+, q^i\right\rangle = \frac{2}{\alpha'}\left(\frac{8}{8} - \frac{15}{8}\right)\alpha^{25}_{-\frac{1}{2}}\bar{\alpha}^{25}_{-\frac{1}{2}}\left|q^+, q^i\right\rangle$$
$$M^2\alpha^{25}_{\frac{1}{2}}\bar{\alpha}^{25}_{\frac{1}{2}}\left|q^+, q^i\right\rangle = -\frac{7}{4\alpha'}\alpha^{25}_{-\frac{1}{2}}\bar{\alpha}^{25}_{-\frac{1}{2}}\left|q^+, q^i\right\rangle$$

That is, the mass of the first excited state is $M^2 = -\frac{7}{4}$.

Problem 6

6.A)

We define the Parity operator such that,

$$\Omega X^\mu(\tau,\sigma)\Omega^{-1} = X^\mu(\tau, l-\sigma)$$

Let's take a look how this operator acts on each string mode, case by case,

- Action in the Closed string

$$\begin{aligned}
X^\mu(\tau,\sigma) &= x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^\mu \mathrm{e}^{\mathrm{i}n\sigma}\right)\\
\Omega X^\mu(\tau,\sigma)\Omega^{-1} &= \Omega x_0^\mu\Omega^{-1} + \sqrt{2\alpha'}\Omega\alpha_0^\mu\Omega^{-1}\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\Omega\bar{\alpha}_n^\mu\Omega^{-1} \mathrm{e}^{-\mathrm{i}n\sigma} + \Omega\alpha_n^\mu\Omega^{-1} \mathrm{e}^{\mathrm{i}n\sigma}\right)\\
X^\mu(\tau,2\pi-\sigma) &= \Omega x_0^\mu\Omega^{-1} + \sqrt{2\alpha'}\Omega\alpha_0^\mu\Omega^{-1}\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\Omega\bar{\alpha}_n^\mu\Omega^{-1} \mathrm{e}^{-\mathrm{i}n\sigma} + \Omega\alpha_n^\mu\Omega^{-1} \mathrm{e}^{\mathrm{i}n\sigma}\right)
\end{aligned}$$

But, we also have,

$$\begin{aligned}
X^\mu(\tau,2\pi-\sigma) &= x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n(2\pi-\sigma)} + \alpha_n^\mu \mathrm{e}^{\mathrm{i}n(2\pi-\sigma)}\right)\\
X^\mu(\tau,2\pi-\sigma) &= x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^\mu \mathrm{e}^{\mathrm{i}n\sigma} + \alpha_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma}\right)
\end{aligned}$$

As the Fourier decomposition is unique, we just have to match term by term of the expansion, which gives,

$$\begin{aligned}
\Omega x_0^\mu\Omega^{-1} &= x_0^\mu\\
\Omega\alpha_n^\mu\Omega^{-1} &= \bar{\alpha}_n^\mu
\end{aligned}$$

As $\bar{\alpha}_0^\mu = \alpha_0^\mu$.

- Action in the NN Open string

$$\begin{aligned}
X^\mu(\tau,\sigma) &= x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\alpha_n^\mu\cos(n\sigma)\\
\Omega X^\mu(\tau,\sigma)\Omega^{-1} &= \Omega x_0^\mu\Omega^{-1} + \sqrt{2\alpha'}\Omega\alpha_0^\mu\Omega^{-1}\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\Omega\alpha_n^\mu\Omega^{-1}\cos(n\sigma)\\
X^\mu(\tau,\pi-\sigma) &= \Omega x_0^\mu\Omega^{-1} + \sqrt{2\alpha'}\Omega\alpha_0^\mu\Omega^{-1}\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\Omega\alpha_n^\mu\Omega^{-1}\cos(n\sigma)
\end{aligned}$$

But,

$$X^\mu(\tau,\pi-\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\alpha_n^\mu\cos\left(n(\pi-\sigma)\right)$$
$$X^\mu(\tau,\pi-\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}(-\alpha_n^\mu)\cos\left(n\sigma\right)$$

Matching term by term of the Fourier expansion,

$$\Omega x_0^\mu\Omega^{-1} = x_0, \quad \Omega\alpha_0^\mu\Omega^{-1} = \alpha_0^\mu, \quad \Omega\alpha_n^\mu\Omega^{-1} = -\alpha_n,\ n\in\mathbb{Z}^*$$

- Action in the DD Open string

We didn't derive the DD Open string solution, but we'll argue here what it should be from our knowledge of the NN Open string, the diference of the boundary conditions are instead of imposing $\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,\pi) = 0$, we impose $\partial_\tau X^\mu(\tau,0) = \partial_\tau X^\mu(\tau,\pi)$. From the same arguments preceding Equation A.4, we get that our solution should be $X^\mu = \frac{1}{2}(f(\sigma^+) + g(\sigma^-))$, and imposing the first boundary condition amounts to fixing, $g'^\mu(\tau) = -f'^\mu(\tau) \Rightarrow g^\mu(\tau) = -f^\mu(\tau) + 2x_0^\mu$, so that our solution is $X^\mu = \frac{1}{2}(f(\sigma^+) - f(\sigma^-) + 2x_0^\mu)$, imposing the second boundary condition gives, $f'^\mu(\tau+\pi) = f'^\mu(\tau-\pi)$, which implies that f'^μ is a periodic function with period 2π, and now we can just make the usual Fourier mode expansion,

$$f'^\mu\left(\sigma^+\right) = f_1^\mu + \sqrt{2\alpha'}\sum_{n\in\mathbb{N}^*}\left(\alpha_{-n}^\mu\mathrm{e}^{\mathrm{i}n\sigma^+} + \alpha_n^\mu\mathrm{e}^{-\mathrm{i}n\sigma}\right)$$
$$f^\mu\left(\sigma^+\right) = f_0^\mu + f_1^\mu\sigma^+ - \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{N}^*}\frac{1}{n}\left(\alpha_{-n}^\mu\mathrm{e}^{\mathrm{i}n\sigma^+} - \alpha_n^\mu\mathrm{e}^{-\mathrm{i}n\sigma^+}\right)$$
$$f^\mu\left(\sigma^+\right) = f_0^\mu + f_1^\mu\sigma^+ + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\alpha_n^\mu\mathrm{e}^{-\mathrm{i}n\sigma^+}$$

So that our solution is,

$$X^\mu(\tau,\sigma) = x_0^\mu + \frac{1}{2}\left(f^\mu\left(\sigma^+\right) - f^\mu\left(\sigma^-\right)\right)$$
$$X^\mu(\tau,\sigma) = x_0^\mu + f_1^\mu\sigma + \frac{1}{2}\mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\left(\mathrm{e}^{-\mathrm{i}n\sigma^+} - \mathrm{e}^{-\mathrm{i}n\sigma^-}\right)$$
$$X^\mu(\tau,\sigma) = x_0^\mu + f_1^\mu\sigma + \frac{1}{2}\mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\mathrm{e}^{-\mathrm{i}n\tau}\left(\mathrm{e}^{-\mathrm{i}n\sigma} - \mathrm{e}^{\mathrm{i}n\sigma}\right)$$
$$X^\mu(\tau,\sigma) = x_0^\mu + f_1^\mu\sigma + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\mathrm{e}^{-\mathrm{i}n\tau}\sin\left(n\sigma\right)$$
$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\sigma + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\mathrm{e}^{-\mathrm{i}n\tau}\sin\left(n\sigma\right)$$

We just called $f_1^\mu = \sqrt{2\alpha'}\alpha^\mu$. So now we can continue,

$$\Omega X^\mu(\tau,\sigma)\Omega^{-1} = \Omega x_0^\mu\Omega^{-1} + \sqrt{2\alpha'}\Omega\alpha_0^\mu\Omega^{-1}\sigma + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\Omega\alpha_n^\mu\Omega^{-1}}{n}\mathrm{e}^{-\mathrm{i}n\tau}\sin\left(n\sigma\right)$$

$$X^\mu(\tau,\pi-\sigma) = \Omega x_0^\mu\Omega^{-1} + \sqrt{2\alpha'}\Omega\alpha_0^\mu\Omega^{-1}\sigma + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\Omega\alpha_n^\mu\Omega^{-1}}{n}\mathrm{e}^{-in\tau}\sin(n\sigma)$$

But,

$$X^\mu(\tau,\pi-\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu(\pi-\sigma) + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\mathrm{e}^{-in\tau}\sin(n(\pi-\sigma))$$
$$X^\mu(\tau,\pi-\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\pi - \sqrt{2\alpha'}\alpha_0^\mu\sigma + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^\mu}{n}\mathrm{e}^{-in\tau}\sin(n\sigma)$$

Matching term by term,

$$\Omega x_0^\mu\Omega^{-1} = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\pi, \quad \Omega\alpha_0^\mu\Omega^{-1} = -\alpha_0^\mu, \quad \Omega\alpha_n^\mu\Omega^{-1} = \alpha_n^\mu,\ n\in\mathbb{Z}^*$$

6.B)

Let's recall the expression for the Hamiltonian, B.13 and D.10, respectively for the open and closed string,

$$H = \begin{cases} L_0^\perp - 1 \\ L_0^\perp + \bar{L}_0^\perp - 2 \end{cases}$$

And also let us recall the expression for the Virasoro zero mode,

$$L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p>0}\alpha_{-p}^I\alpha_p^I$$
$$\bar{L}_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p>0}\bar{\alpha}_{-p}^I\bar{\alpha}_p^I$$

- Action on the Closed string

$$\Omega L_0^\perp\Omega^{-1} = \frac{1}{2}\Omega\alpha_0^I\Omega^{-1}\Omega\alpha_0^I\Omega^{-1} + \sum_{p>0}\Omega\alpha_{-p}^I\Omega^{-1}\Omega\alpha_p^I\Omega^{-1}$$
$$\Omega L_0^\perp\Omega^{-1} = \frac{1}{2}\bar{\alpha}_0^I\bar{\alpha}_0^I + \sum_{p>0}\bar{\alpha}_{-p}^I\bar{\alpha}_p^I$$
$$\Omega L_0^\perp\Omega^{-1} = \bar{L}_0^\perp$$

And,

$$\Omega\bar{L}_0^\perp\Omega^{-1} = \frac{1}{2}\Omega\bar{\alpha}_0^I\Omega^{-1}\Omega\bar{\alpha}_0^I\Omega^{-1} + \sum_{p>0}\Omega\bar{\alpha}_{-p}^I\Omega^{-1}\Omega\bar{\alpha}_p^I\Omega^{-1}$$
$$\Omega\bar{L}_0^\perp\Omega^{-1} = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p>0}\alpha_{-p}^I\alpha_p^I$$
$$\Omega\bar{L}_0^\perp\Omega^{-1} = L_0^\perp$$

Hence,

$$
\begin{aligned}
\Omega H \Omega^{-1} &= \Omega L_0^{\perp} \Omega + \Omega \bar{L}_0^{\perp} \Omega^{-1} - 2 \\
\Omega H \Omega^{-1} &= \bar{L}_0^{\perp} + L_0^{\perp} - 2 = H
\end{aligned}
$$

That is, the closed string is indeed invariant with respect to this symmetry.

- Action on the Open NN string

$$
\begin{aligned}
\Omega L_0^{\perp} \Omega^{-1} &= \frac{1}{2} \Omega \alpha_0^I \Omega^{-1} \Omega \alpha_0^I \Omega^{-1} + \sum_{p>0} \Omega \alpha_{-p}^I \Omega^{-1} \Omega \alpha_p^I \Omega^{-1} \\
\Omega L_0^{\perp} \Omega^{-1} &= \frac{1}{2} \alpha_0^I \alpha_0^I + \sum_{p>0} (-1) \alpha_{-p}^I (-1) \alpha_p^I \\
\Omega L_0^{\perp} \Omega^{-1} &= L_0^{\perp}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Omega H \Omega^{-1} &= \Omega L_0^{\perp} \Omega - 1 \\
\Omega H \Omega^{-1} &= L_0^{\perp} - 1 = H
\end{aligned}
$$

That is, the open NN string is indeed invariant with respect to this symmetry.

- Action on the Open DD string

$$
\begin{aligned}
\Omega L_0^{\perp} \Omega^{-1} &= \frac{1}{2} \Omega \alpha_0^I \Omega^{-1} \Omega \alpha_0^I \Omega^{-1} + \sum_{p>0} \Omega \alpha_{-p}^I \Omega^{-1} \Omega \alpha_p^I \Omega^{-1} \\
\Omega L_0^{\perp} \Omega^{-1} &= \frac{1}{2} (-1) \alpha_0^I (-1) \alpha_0^I + \sum_{p>0} \alpha_{-p}^I \alpha_p^I \\
\Omega L_0^{\perp} \Omega^{-1} &= L_0^{\perp}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Omega H \Omega^{-1} &= \Omega L_0^{\perp} \Omega - 1 \\
\Omega H \Omega^{-1} &= L_0^{\perp} - 1 = H
\end{aligned}
$$

That is, the open DD string is indeed invariant with respect to this symmetry.

6.C)

We assume the vacuum is orientation invariant,

$$
\Omega \left| q^+, q^I \right\rangle = \left| q^+, q^I \right\rangle
$$

This is similar to the reasoning done in 5.B), the massless condition is, from 5.B), $N_0^{\perp} + \bar{N}_0^{\perp} = 2$, but, we also have the level matching condition, $N_0^{\perp} = \bar{N}_0^{\perp}$, which simply implies

$N_0^\perp = \bar{N}_0^\perp = 1$ as the unique massless state, which is a combination of the operators $\alpha_{-1}^I \bar{\alpha}_{-1}^J$, that is, the most general massless state is, 5.B),

$$|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K\, \xi_{IJ}\big(q^+, q^K\big)\alpha_{-1}^I \bar{\alpha}_{-1}^J \left|q^+, q^K\right\rangle$$

Now we impose it should be orientation invariant,

$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K\, \xi_{IJ}\big(q^+, q^K\big)\Omega\alpha_{-1}^I \Omega^{-1}\Omega\bar{\alpha}_{-1}^J \Omega^{-1}\Omega \left|q^+, q^K\right\rangle$$
$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K\, \xi_{IJ}\big(q^+, q^K\big)\bar{\alpha}_{-1}^I \alpha_{-1}^J \left|q^+, q^K\right\rangle$$
$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K\, \xi_{IJ}\big(q^+, q^K\big)\alpha_{-1}^J \bar{\alpha}_{-1}^I \left|q^+, q^K\right\rangle$$
$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K\, \xi_{JI}\big(q^+, q^K\big)\alpha_{-1}^I \bar{\alpha}_{-1}^J \left|q^+, q^K\right\rangle$$

We see here, the state $|\xi\rangle$ is only orientation invariant if $\xi_{IJ}\big(q^+, q^K\big) = \xi_{JI}\big(q^+, q^K\big)$. Thus, a possible base of massless invariant states is,

- $\frac{24\cdot 23}{2}$ Symmetric traceless states

$$\left(\alpha_{-1}^I \bar{\alpha}_{-1}^J + \alpha_{-1}^J \bar{\alpha}_{-1}^I\right) \left|q^+, q^K\right\rangle$$

- 24 Trace states

$$\alpha_{-1}^I \bar{\alpha}_{-1}^I \left|q^+, q^K\right\rangle, \quad \text{no sum}$$

6.D)

Now for the open string, the Mass operator is,

$$M^2 = -p^2 = 2p^+p^- - p^I p^I = 2p^+ \frac{1}{\sqrt{2\alpha'}}\alpha_0^- - p^I p^I$$
$$M^2 = 2p^+ \frac{1}{2\alpha'}\frac{1}{p^+}\big(L_0^\perp + a\big) - \frac{1}{2\alpha'}\alpha_0^I \alpha_0^I$$
$$M^2 = \frac{1}{\alpha'}\left(\frac{1}{2}\alpha_0^i \alpha_0^i + \sum_{p>0} \alpha_{-p}^I \alpha_p^I + a\right) - \frac{1}{2\alpha'}\alpha_0^I \alpha_0^I$$
$$\alpha' M^2 = N^\perp + a$$
$$\alpha' M^2 = N^\perp - 1$$

Where we also defined the number operator,

$$N^{\perp} = \sum_{p>0} \alpha^I_{-p}\alpha^I_p$$

Hence, for the state to be massless it has to have $N^{\perp} = 1$, which implies having just one α^I_{-1} applied to the vacuum, the most generic such state is

$$|\xi\rangle = \int_0^{\infty} \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_I\big(q^+, q^K\big)\alpha^I_{-1}\left|q^+, q^K\right\rangle$$

Now let's compute the action of the reversing orientation operator in each boundary condition case,

- NN Open String

$$\Omega\,|\xi\rangle = \int_0^{\infty} \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_I\big(q^+, q^K\big)\Omega\alpha^I_{-1}\Omega^{-1}\Omega\left|q^+, q^K\right\rangle$$
$$\Omega\,|\xi\rangle = \int_0^{\infty} \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_I\big(q^+, q^K\big)(-1)\alpha^I_{-1}\left|q^+, q^K\right\rangle$$
$$\Omega\,|\xi\rangle = -\,|\xi\rangle$$

One could argue that if we're looking to the Projective Hilbert space, equivalence classes of rays of the original Hilbert space, then $|\xi\rangle$ and $-\,|\xi\rangle$ are both equally good representatives of the same equivalence class of a single ray, thus, as long as we work with the Projective Hilbert space, they should be the same state, and hence all the massless NN open string spectrum is orientation invariant. But, as we're in fact gauging this symmetry, we're in fact restricting the original Hilbert space to a set of physical states, characterized by choosing an specific eigenvalue of the gauge transformation operator — in our case, we chose $\Omega\,|\psi\rangle = +1\,|\psi\rangle$—, that is, our Physical Hilbert space don't contain the state $|\xi\rangle$ and neither $-\,|\xi\rangle$, so that the Projective version of it won't contain none equivalence classes associated with them. Thus, none of the massless NN open string spectrum is orientation invariant.

- DD Open String

$$\Omega\,|\xi\rangle = \int_0^{\infty} \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_I\big(q^+, q^K\big)\Omega\alpha^I_{-1}\Omega^{-1}\Omega\left|q^+, q^K\right\rangle$$
$$\Omega\,|\xi\rangle = \int_0^{\infty} \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_I\big(q^+, q^K\big)\alpha^I_{-1}\left|q^+, q^K\right\rangle$$
$$\Omega\,|\xi\rangle = |\xi\rangle$$

The result is just trivial, all the massless DD open string spectrum are orientation invariant.

6.E)

Let's consider the most general massless state with Chan-Paton factors,

$$|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_{I,ij}\big(q^+, q^K\big)\alpha_{-1}^I \left|ij, q^+, q^K\right\rangle$$

$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_{I,ij}\big(q^+, q^K\big)\Omega\alpha_{-1}^I\Omega^{-1}\Omega \left|ij, q^+, q^K\right\rangle$$

$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_{I,ij}\big(q^+, q^K\big)\alpha_{-1}^I \left|ji, q^+, q^K\right\rangle$$

$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_{I,ji}\big(q^+, q^K\big)\alpha_{-1}^I \left|ij, q^+, q^K\right\rangle$$

This state is only invariant if $\xi_{I,ij}\big(q^+, q^K\big) = \xi_{I,ji}\big(q^+, q^K\big)$, this means we have a base of states as,

- $24 \cdot \frac{2N\cdot(2N-1)}{2}$ Symmetric traceless states

$$\alpha_{-1}^I\big(\left|ij, q^+, q^K\right\rangle + \left|ji, q^+, q^K\right\rangle\big)$$

- $24 \cdot 2N$ Trace states

$$\alpha_{-1}^I \left|ii, q^+, q^K\right\rangle$$

6.F)

We assume now $\Omega\left|q^+, q^I\right\rangle = -\left|q^+, q^I\right\rangle$, and follow as usual,

$$|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_{I,ij}\big(q^+, q^K\big)\alpha_{-1}^I \left|ij, q^+, q^K\right\rangle$$

$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_{I,ij}\big(q^+, q^K\big)\Omega\alpha_{-1}^I\Omega^{-1}\Omega \left|ij, q^+, q^K\right\rangle$$

$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, \xi_{I,ij}\big(q^+, q^K\big)\alpha_{-1}^I(-1) \left|ji, q^+, q^K\right\rangle$$

$$\Omega\,|\xi\rangle = \int_0^\infty \mathrm{d}q^+ \int_{\mathbb{R}^{24}} \mathrm{d}^{24}q^K \, (-1)\xi_{I,ji}\big(q^+, q^K\big)\alpha_{-1}^I \left|ij, q^+, q^K\right\rangle$$

This state is only invariant if $\xi_{I,ij}\big(q^+, q^K\big) = -\xi_{I,ji}\big(q^+, q^K\big)$, this means we have a base of states as,

- $24 \cdot \frac{2N\cdot(2N-1)}{2}$ Antisymmetric states

$$\alpha^I_{-1}\big(\big|ij, q^+, q^K\big\rangle + \big|ji, q^+, q^K\big\rangle\big)$$

6.G)

A Light-Cone Gauge for the Open String

Here, we're going to fix the gauge and solve the classical equations of motion for the open string with Neumann Boundary Conditions at both ends. To get the classical solution we just have to solve both 2.3 and 2.4. Of course, we do know the Polyakov Action has Diff×Weyl gauge freedom, hence, none of the components of the metric h are in fact dynamical, because by fixing this gauge freedom we actually fix all the components of h, what makes of 2.4 not an equation of motion for h, but merely a set of constrains regarding the choice of gauge for h. But, fixing all three components of the metric isn't the only possible choice for picking a gauge, another option is to fix τ, and two of the components of h, as we have in total 3 gauge degrees of freedom. We could in principle assign any function to τ, but what seems to be the optimal choice is to work in the Light-cone gauge. First, for any target space vector we define,

$$A^{\pm} = \frac{1}{\sqrt{2}}\left(A^0 \pm A^1\right)$$

And now we state the Light-cone gauge condition, which fixes the τ reparametrization, given by[1],

$$X^+(\tau, \sigma) = 2\alpha' p^+ \tau$$

Where p^+ is the light-cone component of the conserved charge associated with target space translations — the momentum, 2.6 —. This condition do not allow for any more reparametrizations of τ, as,

$$\begin{aligned} X'^+(\tau', \sigma') &= X^+(\tau, \sigma) \\ 2\alpha' p^+ \tau' &= 2\alpha' p^+ \tau \end{aligned}$$

Which states the only reparametrization compatible with the light-cone gauge condition is $\tau'(\tau, \sigma) = \tau$. Hence, from our initial freedom we can only make σ reparametrizations now. One way of fixing this remaining reparametrization, is to choose σ proportional to the energy carried by the string, actually, we're going to choose a reparametrization such that the energy density in the string, at fixed τ, is independent of σ. One might suspect that the energy density is given by $\mathcal{P}^{\tau 0}$, as in 2.5, but, in the light-cone coordinates, the one that has the role of energy density is $\mathcal{P}^{\tau +}$, we'll choose a sigma parametrization such that $\partial_\sigma \mathcal{P}^{\tau +} = 0$, to see how can this be done we take the transformation of this object under a reparametrization, $\tau' = \tau, \sigma' = \sigma'(\sigma, \tau)$, as,

$$\begin{aligned} \mathcal{P}^{\tau +} &= -\frac{\sqrt{h}\partial^\tau X^+}{2\pi\alpha'} = -\frac{\sqrt{h}h^{\tau\tau}}{2\pi\alpha'}2\alpha' p^+ = -\sqrt{h}h^{\tau\tau}\frac{p^+}{\pi} \\ \mathcal{P}'^{\tau +}(\tau, \sigma') &= -\frac{\partial\sigma}{\partial\sigma'}\sqrt{h}h^{\tau\tau}\frac{p^+}{\pi} = \frac{\partial\sigma}{\partial\sigma'}\mathcal{P}^{\tau +} \end{aligned}$$

Hence, given $\mathcal{P}'^{\tau +}(\tau, \sigma')$, we can choose $\sigma = \sigma(\tau, \sigma')$, so that $\partial_\sigma \mathcal{P}^{\tau +} = 0$, as long as,

$$\frac{\partial}{\partial\sigma}\left[\frac{\partial\sigma'}{\partial\sigma}\mathcal{P}'^{\tau +}\right] = 0 \Rightarrow \frac{\partial^2\sigma'}{\partial\sigma^2}\mathcal{P}'^{\tau +} + \left(\frac{\partial\sigma'}{\partial\sigma}\right)^2\frac{\partial\mathcal{P}'^{\tau +}}{\partial\sigma'} = 0$$

[1] The factor of 2 here is just convention, it get along pretty well with the domain we'll choose of $\sigma \in [0, \pi]$ as a lot of simplifications happen, α' is here just for making τ dimensionless, and p^+ is the only conserved charge with no τ, σ dependence and with the correct target space index.

This is simply a differential equation which can be solved. But this really fixes all the σ reparametrization? Let's see if it's possible to make further changes of $\sigma \to \sigma''$ preserving this condition,

$$\frac{\partial}{\partial\sigma}\left[\frac{\partial\sigma''}{\partial\sigma}\frac{\partial\sigma'}{\partial\sigma''}\right]\mathcal{P}'^{\tau+} + \left(\frac{\partial\sigma'}{\partial\sigma''}\frac{\partial\sigma''}{\partial\sigma}\right)^2\frac{\partial\mathcal{P}'^{\tau+}}{\partial\sigma'} = 0$$
$$\frac{\partial^2\sigma''}{\partial\sigma^2}\frac{\partial\sigma'}{\partial\sigma''}\mathcal{P}'^{\tau+} + \left(\frac{\partial\sigma''}{\partial\sigma}\right)^2\left[\frac{\partial^2\sigma'}{\partial\sigma''^2}\mathcal{P}'^{\tau+} + \left(\frac{\partial\sigma'}{\partial\sigma''}\right)^2\frac{\partial\mathcal{P}'^{\tau+}}{\partial\sigma'}\right] = 0$$

Imposing that σ'' also satisfy the parametrization condition,

$$\frac{\partial^2\sigma''}{\partial\sigma^2}\frac{\partial\sigma'}{\partial\sigma''}\mathcal{P}'^{\tau+} = 0 \Rightarrow \frac{\partial^2\sigma''}{\partial\sigma^2} = 0 \Rightarrow \sigma'' = a\sigma + b \tag{A.1}$$

Our condition of constancy of $\mathcal{P}^{\tau+}$ in sigma, has a residual affine reparametrization, which we can use to set $\sigma = 0$ in one of the ends of the string, and $\sigma = \pi$ in the other one. This completely fixes the σ parametrization. What remains now is to fix the Weyl redundancy, as necessarily h_{ab} has a inverse, $\mathrm{Det}[h_{ab}] \neq 0$, thus, by a Weyl transformation is always possible to make $\mathrm{Det}[h_{ab}] = -1$ — We're using Lorentzian signature —, which will be our choice of fixing the Weyl redundancy. There is no more room for any transformation now, hence, the gauge is fully fixed. Notice, as $\mathcal{P}^{\tau+}$ is constant in σ, we have the following,

$$\begin{aligned} p^+ &= \int_0^\pi \mathrm{d}\sigma\, \mathcal{P}^{\tau+} = \mathcal{P}^{\tau+}\pi = -p^+\sqrt{h}h^{\tau\tau} \\ -1 &= h^{\tau\tau} \end{aligned} \tag{A.2}$$

Furthermore, using the equation of motion 2.3 at $\mu = +$, we get,

$$\begin{aligned} 0 &= \partial_\tau\left[h^{\tau\sigma}\partial_\sigma X^+ h^{\tau\tau}\partial_\tau X^+\right] + \partial_\sigma\left[h^{\sigma\sigma}\partial_\sigma X^+ + h^{\sigma\tau}\partial_\tau X^+\right] \\ 0 &= \partial_\tau h^{\tau\tau} + \partial_\sigma h^{\sigma\tau} \\ 0 &= \partial_\sigma h^{\sigma\tau} \end{aligned}$$

But, using the Neumann Boundary conditions 2.2 at $\mu = +$,

$$\begin{aligned} 0 &= \partial^\sigma X^+ \Big|_{\sigma=0}^{\sigma=\pi} \\ 0 &= h^{\sigma\tau}\partial_\tau X^+ + h^{\sigma\sigma}\partial_\sigma X^+ \Big|_{\sigma=0}^{\sigma=\pi} \\ 0 &= h^{\tau\sigma}\Big|_{\sigma=0}^{\sigma=\pi} \Rightarrow h^{\tau\sigma}\Big|_{\sigma=0} = h^{\tau\sigma}\Big|_{\sigma=\pi} = 0 \end{aligned}$$

Together with $\partial_\sigma h^{\tau\sigma} = 0$, this simply states that $h^{\tau\sigma} = h^{\sigma\tau} \equiv 0$. This, with $h^{\tau\tau} = -1$ and the determinant condition, says that $h = \mathrm{Diag}\big(-1 \quad 1\big)$. That simplifies the equations of motion 2.3 to,

$$\partial_\tau\partial_\tau X^\mu - \partial_\sigma\partial_\sigma X^\mu = \ddot{X}^\mu - X''^\mu = 0$$

And also the consistency conditions 2.4 as,

$$
\begin{aligned}
0 =& \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab}\Big(-\dot{X}^2 + X'^2\Big)\\
0 =& \begin{cases} \partial_\tau X^\mu \partial_\tau X_\mu + \frac{1}{2}\Big(-\dot{X}^2 + X'^2\Big) & = \frac{1}{2}\Big[\dot{X}^2 + X'^2\Big]\\ \partial_\tau X^\mu \partial_\sigma X_\mu & = \dot{X}\cdot X'\\ \partial_\sigma X^\mu \partial_\sigma X_\mu - \frac{1}{2}\Big(-\dot{X}^2 + X'^2\Big) & = \frac{1}{2}\Big[\dot{X}^2 + X'^2\Big] \end{cases}
\end{aligned}
$$

These constrains can be recast as,

$$
\begin{aligned}
\dot{X}^2 + X'^2 \pm &= 0\\
\Big(\dot{X} \pm X'\Big)^2 &= 0
\end{aligned}
\tag{A.3}
$$

Lastly, but not less important, we have the Boundary Conditions, 2.2, which take the form,

$$
\partial_\sigma X^\mu \Big|_{\sigma=0} = \partial_\sigma X^\mu \Big|_{\sigma=\pi} = 0
$$

We have now to solve the equation of motion with the right Boundary conditions, for this, is easier to change coordinates to $\sigma^\pm = \tau \pm \sigma$, in which the new equations of motion read,

$$
\begin{aligned}
0 &= \frac{\partial^2}{\partial \tau^2} X^\mu - \frac{\partial^2}{\partial \sigma^2} X^\mu\\
0 &= \frac{\partial}{\partial \tau}\left[\frac{\partial \sigma^+}{\partial \tau}\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial \sigma^-}{\partial \tau}\frac{\partial}{\partial \sigma^-} X^\mu\right] - \frac{\partial}{\partial \sigma}\left[\frac{\partial \sigma^+}{\partial \sigma}\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial \sigma^-}{\partial \sigma}\frac{\partial}{\partial \sigma^-} X^\mu\right]\\
0 &= \frac{\partial}{\partial \tau}\left[\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial}{\partial \sigma^-} X^\mu\right] - \frac{\partial}{\partial \sigma}\left[\frac{\partial}{\partial \sigma^+} X^\mu - \frac{\partial}{\partial \sigma^-} X^\mu\right]\\
0 &= \left(\frac{\partial}{\partial \sigma^+} + \frac{\partial}{\partial \sigma^-}\right)\left[\frac{\partial}{\partial \sigma^+} X^\mu + \frac{\partial}{\partial \sigma^-} X^\mu\right] - \left(\frac{\partial}{\partial \sigma^+} - \frac{\partial}{\partial \sigma^-}\right)\left[\frac{\partial}{\partial \sigma^+} X^\mu - \frac{\partial}{\partial \sigma^-} X^\mu\right]\\
0 &= \partial_+ \partial_- X^\mu
\end{aligned}
\tag{A.4}
$$

This is easily solved for,

$$
X^\mu = \frac{1}{2}\big(f^\mu(\sigma^+) + g^\mu(\sigma^-)\big)
$$

Imposing the boundary condition at $\sigma = 0$,

$$
X'^\mu(\tau, \sigma = 0) = \frac{1}{2}\Big(f'^\mu(\tau) - g'^\mu(\tau)\Big) = 0
$$

This states that g^μ is equal to f^μ apart from a constant, which we'll absorb in the definition of f^μ. Now, the boundary condition on $\sigma = \pi$,

$$
X^\mu = \frac{1}{2}\big(f^\mu(\sigma^+) + f^\mu(\sigma^-)\big)
$$

$$X'^{\mu}(\tau,\pi) = \frac{1}{2}\big(f'^{\mu}(\tau+\pi) - f'^{\mu}(\tau-\pi)\big) = 0$$

That is, f'^{μ} is periodic with period 2π. The most general real function with period 2π is[2],

$$f'^{\mu}(u) = f_1^{\mu} + \sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(\alpha_n^{*\mu}\exp\left(\mathrm{i}nu\right) + \alpha_n^{\mu}\exp\left(-\mathrm{i}nu\right)\right)$$
$$f^{\mu}(u) = f_0^{\mu} + f_1^{\mu}u - \mathrm{i}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_n^{*\mu}\exp\left(\mathrm{i}nu\right) - \alpha_n^{\mu}\exp\left(-\mathrm{i}nu\right)\right)$$

Defining $\alpha_{-n}^{\mu} = \alpha_n^{*\mu}$,

$$f^{\mu}(u) = f_0^{\mu} + f_1^{\mu}u + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^{\mu}}{n}\exp\left(-\mathrm{i}nu\right)$$

Now, back in X,

$$X^{\mu} = f_0^{\mu} + \frac{1}{2}f_1^{\mu}\left(\sigma^+ + \sigma^-\right) + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^{\mu}}{n}\frac{1}{2}\left[\exp\left(-\mathrm{i}n\sigma^+\right) + \exp\left(-\mathrm{i}n\sigma^-\right)\right]$$
$$X^{\mu} = f_0^{\mu} + f_1^{\mu}\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^{\mu}}{n}\exp\left(-\mathrm{i}n\tau\right)\frac{1}{2}\left[\exp\left(-\mathrm{i}n\sigma\right) + \exp\left(\mathrm{i}n\sigma\right)\right]$$
$$X^{\mu} = f_0^{\mu} + f_1^{\mu}\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^{\mu}}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)$$

Notice that,

$$p^{\mu} = \int_0^{\pi}\mathrm{d}\sigma\,\mathcal{P}^{\tau\mu} = \int_0^{\pi}\mathrm{d}\sigma\,\frac{\partial_\tau X^{\mu}}{2\pi\alpha'}, \quad \text{Because } h = \mathrm{Diag}\begin{pmatrix}-1 & 1\end{pmatrix}$$
$$p^{\mu} = \frac{1}{2\pi\alpha'}\int_0^{\pi}\mathrm{d}\sigma\left[f_1^{\mu} + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\alpha_n^{\mu}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)\right] = \frac{f_1^{\mu}}{2\alpha'}$$

So that,

$$X^{\mu} = f_0^{\mu} + 2\alpha' p^{\mu}\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^{\mu}}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right)$$

And it's clear that,

$$\frac{1}{\pi}\int_0^{\pi}\mathrm{d}\sigma\,X^{\mu}(0,\sigma) = f_0^{\mu}$$

That is, f_0^{μ} is the mean position at $\tau = 0$, so we relabel it to be $f_0^{\mu} = x_0^{\mu}$. It's also useful to define a additional mode, $\alpha_0^{\mu} = \sqrt{2\alpha'}p^{\mu}$,

[2]The α' factors here are just to make the α_n^{μ} dimensionless.

$$
\begin{aligned}
X^\mu &= x_0^\mu + 2\alpha' p^\mu \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}^*} \frac{\alpha_n^\mu}{n} \exp\left(-\mathrm{i}n\tau\right) \cos\left(n\sigma\right) \\
X^\mu &= x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}^*} \frac{\alpha_n^\mu}{n} \exp\left(-\mathrm{i}n\tau\right) \cos\left(n\sigma\right)
\end{aligned}
$$

This is not the end of it, because we need to make sure the two constrains are satisfied, namely, A.3. With the convention of uppercase latin index referring to non-light-cone components, $I = 2, \cdots, D-1$, the constrains are,

$$
\begin{aligned}
0 &= -2\left(\dot{X}^- \pm X'^-\right)\left(\dot{X}^+ \pm X'^+\right) + \left(\dot{X}^I \pm X'^I\right)^2 \\
4\alpha' p^+\left(\dot{X}^- \pm X'^-\right) &= \left(\dot{X}^I \pm X'^I\right)^2 \\
\dot{X}^- \pm X'^- &= \frac{1}{4\alpha' p^+}\left(\dot{X}^I \pm X'^I\right)^2
\end{aligned}
$$

That is, the two constrains implies that X^- is not dynamical, and, the collection of X^I fully determine X^-, apart from a single integration constant, x_0^-. To see this is helpful to note,

$$
\begin{aligned}
\dot{X}^\mu &= \sqrt{2\alpha'}\alpha_0^\mu + \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}^*} \alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right) \cos\left(n\sigma\right) \\
\dot{X}^\mu &= \sqrt{2\alpha'}\alpha_0^\mu \exp(-\mathrm{i}0\tau) \cos\left(0\sigma\right) + \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}^*} \alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right) \cos\left(n\sigma\right) \\
\dot{X}^\mu &= \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right) \cos\left(n\sigma\right)
\end{aligned}
$$

And,

$$
\begin{aligned}
X'^\mu &= -\mathrm{i}\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}^*} \alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right) \sin\left(n\sigma\right) \\
X'^\mu &= -\mathrm{i}\sqrt{2\alpha'}\alpha_0^\mu \exp\left(-\mathrm{i}0\tau\right) \sin(0\sigma) - \mathrm{i}\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}^*} \alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right) \sin\left(n\sigma\right) \\
X'^\mu &= -\mathrm{i}\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right) \sin\left(n\sigma\right)
\end{aligned}
$$

Which implies,

$$
\begin{aligned}
\dot{X}^\mu \pm X'^\mu &= \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \exp\left(-\mathrm{i}n\tau\right)(\cos\left(n\sigma\right) \mp \mathrm{i}\sin\left(n\sigma\right)) \\
\dot{X}^\mu \pm X'^\mu &= \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \exp\left(-\mathrm{i}n(\tau \pm \sigma)\right) \qquad \text{(A.5)}
\end{aligned}
$$

So that,

$$
\dot{X}^- \pm X'^- = \frac{1}{4\alpha' p^+}\left(\dot{X}^I \pm X'^I\right)^2
$$

$$
\begin{aligned}
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^-\exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{2\alpha'}{4\alpha' p^+}\sum_{p,q\in\mathbb{Z}}\alpha_p^I\alpha_q^I\exp\left(-\mathrm{i}(p+q)(\tau\pm\sigma)\right)\\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^-\exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{1}{2p^+}\sum_{n,p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I\exp\left(-\mathrm{i}n(\tau\pm\sigma)\right)\\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^-\exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{1}{p^+}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I\right)\exp\left(-\mathrm{i}n(\tau\pm\sigma)\right)\\
\sqrt{2\alpha'}\alpha_n^- &= \frac{1}{2p^+}\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I = \frac{1}{p^+}L_n^\perp
\end{aligned}
\tag{A.6}
$$

Hence, all the fourier modes of X^- are completely determined by the transverse modes ones, apart from of course the integration constant x_0^-. In the last passage we also defined the Virasoro modes $L_n^\perp$. This is it, we fully solved the equation of motion with all the constrains and boundary conditions, let's rewrite them here just for completeness,

$$X^I = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^I}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right) \tag{A.7}$$

$$X^- = x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}^*}\frac{\alpha_n^-}{n}\exp\left(-\mathrm{i}n\tau\right)\cos\left(n\sigma\right), \quad \alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+}L_n^\perp \tag{A.8}$$

$$X^+ = 2\alpha' p^+\tau \tag{A.9}$$

B Light-Cone Gauge Quantization of the Open String

To get a grasp on how the quantization should proceed we have to look at the classical Polyakov Action 2.1 in the Light-Cone gauge, using A.7,A.8 and A.9,

$$
\begin{aligned}
S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \\
&= -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\left(-g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + g_{\mu\nu}X'^\mu X'^\nu\right) \\
&= -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\left(2\dot{X}^-\dot{X}^+ - \dot{X}^I\dot{X}^I - 2X'^- X'^+ + X'^I X'^I\right) \\
&= -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\left(4\dot{X}^-\alpha' p^+ - \dot{X}^I\dot{X}^I + X'^I X'^I\right) \\
&= -\frac{1}{\pi}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\,\dot{X}^- p^+ - \frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\left(-\dot{X}^I\dot{X}^I + X'^I X'^I\right) \\
&= -\int \mathrm{d}\tau\,\dot{x}_0^- p^+ - \frac{1}{4\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\left(-\dot{X}^I\dot{X}^I + X'^I X'^I\right),
\end{aligned}
$$

From this we can read the canonical conjugated momenta,

$$
\begin{aligned}
p_- = -p^+ &= \frac{\delta S_{\mathrm{P}}}{\delta \dot{x}_0^-} = -p^+ \\
\mathcal{P}^{\tau I} &= \frac{\delta S_{\mathrm{P}}}{\delta \dot{X}^I} = \frac{1}{2\pi\alpha'}\dot{X}^I
\end{aligned}
$$

Hence, the Poisson Brackets in this theory will be,

$$
\begin{aligned}
\left\{X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right\} &= g^{IJ}\delta(\sigma-\sigma') \\
\left\{x_0^-, p_-\right\} &= g^{IJ} \Rightarrow \left\{x_0^-, p^+\right\} = -g^{IJ}
\end{aligned}
$$

From which we quantize following $[.,.] = \mathrm{i}\{.,.\}$,

$$
\left[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = \mathrm{i}g^{IJ}\delta(\sigma-\sigma'), \quad \left[x_0^-, p^+\right] = -\mathrm{i} \tag{B.1}
$$

Any other commutator between these 4 operators is zero. Of course now, X^- has to be considered as a function of the X^I and $\mathcal{P}^{\tau I}$, so, it has non trivial commutators. The first thing we need to do here is go from these canonical commutation relations, to the commutation relations of the modes α_n^I. The best way of doing this is working with the expression A.5, and computing the following commutator,

$$
\left[\dot{X}^I \pm X'^I, \dot{X}^I \pm X'^I\right]
$$

For this is useful to note,

$$
\begin{aligned}
\left[X^I(\tau,\sigma),X^J(\tau,\sigma')\right]=0 &\Rightarrow \left[X'^I(\tau,\sigma),X'^J(\tau,\sigma')\right]=0\\
\left[X^I(\tau,\sigma),\mathcal{P}^{\tau J}(\tau,\sigma')\right]=\mathrm{i}g^{IJ}\delta(\sigma-\sigma') &\Rightarrow \left[X'^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]=2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\\
\left[\mathcal{P}^{\tau I}(\tau,\sigma),\mathcal{P}^{\tau J}(\tau,\sigma')\right]=0 &\Rightarrow \left[\dot{X}^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]=0
\end{aligned}
$$

Hence, the non-vanishing contributions are,

$$
\begin{aligned}
\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\pm X'^J\right)(\tau,\sigma')\right]&=\pm\left[\dot{X}^I(\tau,\sigma),X'^J(\tau,\sigma')\right]\pm\left[X'^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]\\
\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\pm X'^J\right)(\tau,\sigma')\right]&=\mp 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma'}\delta(\sigma'-\sigma)\pm 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\\
\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\pm X'^J\right)(\tau,\sigma')\right]&=\pm 4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \qquad \text{(B.2)}
\end{aligned}
$$

And with opposite signs,

$$
\begin{aligned}
\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\mp X'^J\right)(\tau,\sigma')\right]&=\mp\left[\dot{X}^I(\tau,\sigma),X'^J(\tau,\sigma')\right]\pm\left[X'^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]\\
\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\mp X'^J\right)(\tau,\sigma')\right]&=\pm 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma'}\delta(\sigma'-\sigma)\pm 2\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\\
\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\mp X'^J\right)(\tau,\sigma')\right]&=0 \qquad \text{(B.3)}
\end{aligned}
$$

But these expressions, and of course X also, are only defined for $\sigma\in[0,\pi]$, as we want to use A.5 to isolate the Fourier modes, we'll need to integrate over 2π, the only option here is to find an extension of this expression to the whole interval $[0,2\pi]$. As everything is periodic in σ with 2π period, it's sufficient to find an extension to $\sigma\in[-\pi,\pi]$. See that,

$$
\begin{aligned}
\left(\dot{X}^I+X'^I\right)(\tau,\sigma)&=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\exp\left(-\mathrm{i}n(\tau+\sigma)\right),\quad \sigma\in[0,\pi]\\
\left(\dot{X}^I-X'^I\right)(\tau,\sigma)&=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\exp\left(-\mathrm{i}n(\tau-\sigma)\right),\quad \sigma\in[0,\pi]
\end{aligned}
$$

Performing a change of variables $\sigma\to-\sigma$ in the second expression,

$$
\begin{aligned}
\left(\dot{X}^I+X'^I\right)(\tau,\sigma)&=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\exp\left(-\mathrm{i}n(\tau+\sigma)\right),\quad \sigma\in[0,\pi]\\
\left(\dot{X}^I-X'^I\right)(\tau,-\sigma)&=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\exp\left(-\mathrm{i}n(\tau+\sigma)\right),\quad \sigma\in[-\pi,0]
\end{aligned}
$$

That's interesting, because we found a representation of the modes in the whole domain $[-\pi,\pi]$,

$$
A^I(\tau,\sigma)=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\exp\left(-\mathrm{i}n(\tau+\sigma)\right)=\begin{cases}\left(\dot{X}^I+X'^I\right)(\tau,\sigma), & \sigma\in[0,\pi]\\ \left(\dot{X}^I-X'^I\right)(\tau,-\sigma), & \sigma\in[-\pi,0]\end{cases} \qquad \text{(B.4)}
$$

And of course, as everything where, we still have $A^I(\tau,\sigma+2\pi) = A^I(\tau,\sigma)$. Also, by B.2 we do have, using the $+$ sign,

$$\left[A^I(\tau,\sigma), A^J(\tau,\sigma')\right] = 4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'), \quad \sigma,\sigma' \in [0,\pi]$$

By B.3, we get that when $\sigma \in [0,\pi]$ and $\sigma' \in [-\pi,0]$ this commutator is zero, which is consistent as the Dirac delta is zero in this domain. At last we use B.2 with both $-$ sign, and $-\sigma,-\sigma' \in [-\pi,0]$, which get us,

$$\left[A^I(\tau,\sigma), A^J(\tau,\sigma')\right] = 4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'), \quad \sigma,\sigma' \in [-\pi,0]$$

Putting all together, what we have is,

$$\left[A^I(\tau,\sigma), A^J(\tau,\sigma')\right] = 4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'), \quad \sigma,\sigma' \in [-\pi,\pi]$$

And, also as everything is periodic in 2π, this fully determines the commutation relation over $\sigma,\sigma' \in [0,2\pi]$. Inserting now the definition of $A^I(\tau,\sigma)$,

$$\begin{aligned}
4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') &= 2\alpha' \sum_{n',m'\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'(\tau+\sigma')}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)}\left[\alpha^I_{m'},\alpha^J_{n'}\right] \\
2\pi\mathrm{i}g^{IJ}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\mathrm{e}^{\mathrm{i}m\sigma} &= \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}m\sigma}\sum_{m',n'\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m')}\mathrm{e}^{-\mathrm{i}\sigma'n'-\mathrm{i}\sigma m'}\left[\alpha^I_{m'},\alpha^J_{n'}\right] \\
mg^{IJ}\mathrm{e}^{\mathrm{i}m\sigma'} &= \sum_{m',n'\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m')}\mathrm{e}^{-\mathrm{i}\sigma'n'}\delta_{m,m'}\left[\alpha^I_{m'},\alpha^J_{n'}\right] \\
\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}n\sigma'}mg^{IJ}\mathrm{e}^{\mathrm{i}m\sigma'} &= \int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}n\sigma'}\sum_{n'\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m)}\mathrm{e}^{-\mathrm{i}\sigma'n'}\left[\alpha^I_{m},\alpha^J_{n'}\right] \\
mg^{IJ}\delta_{m+n,0} &= \sum_{n'\in\mathbb{Z}}\delta_{n,n'}\mathrm{e}^{-\mathrm{i}\tau(n'+m)}\left[\alpha^I_{m},\alpha^J_{n'}\right] \\
mg^{IJ}\delta_{m+n,0}\mathrm{e}^{\mathrm{i}\tau(n+m)} &= \left[\alpha^I_{m},\alpha^J_{n}\right] \\
\left[\alpha^I_{m},\alpha^J_{n}\right] &= mg^{IJ}\delta_{m+n,0}
\end{aligned} \tag{B.5}$$

This is not the end, we still have one more commutation relation to get,

$$\begin{aligned}
2\pi\alpha'\mathrm{i}g^{IJ}\delta(\sigma-\sigma') &= \left[X^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right] \\
\int_0^\pi \mathrm{d}\sigma\, 2\pi\alpha'\mathrm{i}g^{IJ}\delta(\sigma-\sigma') &= \int_0^\pi \mathrm{d}\sigma\left[X^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right], \quad \int_0^\pi \mathrm{d}\sigma\cos(n\sigma) = 0,\ n\in\mathbb{Z}^* \\
2\pi\alpha'\mathrm{i}g^{IJ} &= \pi\left[x^I_0 + \sqrt{2\alpha'}\alpha^I_0\tau, \dot{X}^J(\tau,\sigma')\right] \\
2\alpha'\mathrm{i}g^{IJ} &= \sqrt{2\alpha'}\sum_{n'\in\mathbb{Z}}\exp(-\mathrm{i}n'\tau)\cos(n'\sigma')\left\{\left[x^I_0,\alpha^J_{n'}\right] + \sqrt{2\alpha'}\tau\left[\alpha^I_0,\alpha^J_{n'}\right]\right\} \\
\sqrt{2\alpha'}\mathrm{i}g^{IJ} &= \sum_{n'\in\mathbb{Z}}\exp(-\mathrm{i}n'\tau)\cos(n'\sigma')\left[x^I_0,\alpha^J_{n'}\right]
\end{aligned}$$

$$\begin{aligned}
\int_0^\pi \frac{\mathrm{d}\sigma'}{\pi}\cos(n\sigma')\sqrt{2\alpha'}\mathrm{i}g^{IJ} &= \int_0^\pi \frac{\mathrm{d}\sigma'}{\pi}\cos(n\sigma')\sum_{n'\in\mathbb{Z}}\exp(-\mathrm{i}n'\tau)\cos(n'\sigma')\left[x_0^I,\alpha_{n'}^J\right] \\
\delta_{n,0}\sqrt{2\alpha'}\mathrm{i}g^{IJ} &= \sum_{n'\in\mathbb{Z}}\exp(-\mathrm{i}n'\tau)\delta_{n',n}\left[x_0^I,\alpha_{n'}^J\right] \\
\left[x_0^I,\alpha_n^J\right] &= \delta_{n,0}\sqrt{2\alpha'}\mathrm{i}g^{IJ}\exp(-\mathrm{i}n\tau) = \delta_{n,0}\sqrt{2\alpha'}\mathrm{i}g^{IJ}
\end{aligned} \tag{B.6}$$

The last thing we need to discuss before going to the Lorentz generators is about the Virasoro operators, these were defined as,

$$L_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{n-p}^I\alpha_p \tag{B.7}$$

We should be aware of possible ambiguities in the ordering of these, as we now have the commutation relations B.5, two alphas fail to commute only if their mode number sum up to 0, but, notice that $n-p+p=0\to n=0$, hence, the only not well defined Virasoro mode is $L_0^\perp$. As the difference between any two ordering prescriptions is always proportional to the identity operator, we'll **define** $L_0^\perp$ to be on the **normal ordered** prescription — All $\alpha_n^I,\ n\geq 0$ need to be to the right of all $\alpha_n^I,\ n<0$ —, and wherever there is mention to this Virasoro mode we should use the normal ordered one plus an addition undetermined normal ordering constant,

$$L_0^\perp \to L_0^\perp + a$$

As an example, in A.6, with $n=0$, the quantum version should read,

$$\begin{aligned}
2\alpha' p^- = \sqrt{2\alpha'}\alpha_0^- &= \frac{1}{p^+}\left(L_0^\perp + a\right) \\
L_0^\perp &= \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p\in\mathbb{N}^*}\alpha_{-p}^I\alpha_p^I
\end{aligned}$$

We can write a manifestly normal ordered form for all n,

$$L_n^\perp = \frac{1}{2}\sum_{p\geq 0}\alpha_{n-p}^I\alpha_p^I + \frac{1}{2}\sum_{p<0}\alpha_p^I\alpha_{n-p}^I \tag{B.8}$$

Using these, every calculation we do is manifestly normal ordered, which will prevent us from making mistakes. As classically we had, $\left(\alpha_n^I\right)^* = \alpha_{-n}^I$, in the quantization we have, $\left(\alpha_n^I\right)^\dagger = \alpha_{-n}^I$. This allows us to conclude that, $\left(L_n^\perp\right)^\dagger = L_{-n}^\perp$. A few more properties we'll need are the commutation relations of the Virasoro modes with all the other objects, we start with,

$$\begin{aligned}
\left[L_m^\perp,\alpha_n^J\right] &= \frac{1}{2}\sum_{p\geq 0}\left[\alpha_{m-p}^I\alpha_p^I,\alpha_n^J\right] + \frac{1}{2}\sum_{p<0}\left[\alpha_p^I\alpha_{m-p}^I,\alpha_n^J\right] \\
&= \frac{1}{2}\sum_{p\geq 0}\left\{\alpha_{m-p}^I\left[\alpha_p^I,\alpha_n^J\right] + \left[\alpha_{m-p}^I,\alpha_n^J\right]\alpha_p^I\right\} + \frac{1}{2}\sum_{p<0}\left\{\alpha_p^I\left[\alpha_{m-p}^I,\alpha_n^J\right] + \left[\alpha_p^I,\alpha_n^J\right]\alpha_{m-p}^I\right\} \\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}}\left\{\alpha_{m-p}^I g^{IJ} p\delta_{p+n,0} + (m-p)\delta_{m-p+n,0}g^{IJ}\alpha_p^I\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big\{-\alpha^J_{m+n} n + (m-m-n)\alpha^J_{m+n}\big\} \\
\big[L^\perp_m, \alpha^J_n\big] &= -n\alpha^J_{n+m} \qquad \text{(B.9)}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\big[L^\perp_m, x^J_0\big] &= \frac{1}{2}\sum_{p\geq 0}\big[\alpha^I_{m-p}\alpha^I_p, x^J_0\big] + \frac{1}{2}\sum_{p<0}\big[\alpha^I_p\alpha^I_{m-p}, x^J_0\big] \\
&= \frac{1}{2}\sum_{p\geq 0}\big\{\alpha^I_{m-p}\big[\alpha^I_p, x^J_0\big] + \big[\alpha^I_{m-p}, x^J_0\big]\alpha^I_p\big\} + \frac{1}{2}\sum_{p<0}\big\{\alpha^I_p\big[\alpha^I_{m-p}, x^J_0\big] + \big[\alpha^I_p, x^J_0\big]\alpha^I_{m-p}\big\} \\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}}\Big\{-\mathrm{i}\sqrt{2\alpha'}g^{IJ}\delta_{p,0}\alpha^I_{m-p} - \mathrm{i}\sqrt{2\alpha'}g^{IJ}\delta_{m-p,0}\alpha^I_p\Big\} \\
&= -\mathrm{i}\sqrt{2\alpha'}\frac{1}{2}\big\{\alpha^J_m + \alpha^J_m\big\} \\
\big[L^\perp_m, x^J_0\big] &= -\mathrm{i}\sqrt{2\alpha'}\alpha^J_m \qquad \text{(B.10)}
\end{aligned}
$$

And lastly, but not less important, we have to know the commutation relation between the Virasoro modes themselves, this is more subtle, because we defined them being normal ordered, thus, every step of the calculation we have to make sure all terms are normal ordered,

$$
\begin{aligned}
\big[L^\perp_m, L^\perp_n\big] &= \frac{1}{2}\sum_{p\geq 0}\big[\alpha^I_{m-p}\alpha^I_p, L^\perp_n\big] + \frac{1}{2}\sum_{p<0}\big[\alpha^I_p\alpha^I_{m-p}, L^\perp_n\big] \\
&= \frac{1}{2}\sum_{p\geq 0}\big\{\alpha^I_{m-p}\big[\alpha^I_p, L^\perp_n\big] + \big[\alpha^I_{m-p}, L^\perp_n\big]\alpha^I_p\big\} \\
&\qquad + \frac{1}{2}\sum_{p<0}\big\{\alpha^I_p\big[\alpha^I_{m-p}, L^\perp_n\big] + \big[\alpha^I_p, L^\perp_n\big]\alpha^I_{m-p}\big\} \\
&= \frac{1}{2}\sum_{p\geq 0}\big\{p\alpha^I_{m-p}\alpha^I_{p+n} + (m-p)\alpha^I_{n+m-p}\alpha^I_p\big\} \\
&\qquad + \frac{1}{2}\sum_{p<0}\big\{(m-p)\alpha^I_p\alpha^I_{m+n-p} + p\alpha^I_{p+n}\alpha^I_{m-p}\big\} \\
&= \frac{1}{2}\sum_{p\geq 0}(m-p)\alpha^I_{n+m-p}\alpha^I_p + \frac{1}{2}\sum_{p<0}(m-p)\alpha^I_p\alpha^I_{m+n-p} \\
&\qquad + \frac{1}{2}\sum_{p\geq 0}p\alpha^I_{m-p}\alpha^I_{p+n} + \frac{1}{2}\sum_{p<0}p\alpha^I_{p+n}\alpha^I_{m-p} \\
&= \frac{1}{2}\sum_{p\geq 0}(m-p)\alpha^I_{n+m-p}\alpha^I_p + \frac{1}{2}\sum_{p<0}(m-p)\alpha^I_p\alpha^I_{m+n-p} \\
&\qquad + \frac{1}{2}\sum_{p\geq n}(p-n)\alpha^I_{m+n-p}\alpha^I_p + \frac{1}{2}\sum_{p<n}(p-n)\alpha^I_p\alpha^I_{m+n-p} \\
&= \frac{1}{2}\sum_{p\geq 0}(m-p)\alpha^I_{n+m-p}\alpha^I_p + \frac{1}{2}\sum_{p<0}(m-p)\alpha^I_p\alpha^I_{m+n-p} \\
&\qquad + \frac{1}{2}\sum_{p\geq 0}(p-n)\alpha^I_{m+n-p}\alpha^I_p + \frac{1}{2}\sum_{p<0}(p-n)\alpha^I_p\alpha^I_{m+n-p}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \frac{1}{2}\left(\frac{1}{2} - \frac{n}{2|n|}\right)\left[\sum_{p=n}^{-1}(p-n)\alpha^I_{m+n-p}\alpha^I_p - \sum_{p=n}^{-1}(p-n)\alpha^I_p\alpha^I_{m+n-p}\right] \\
&\qquad + \frac{1}{2}\left(\frac{1}{2} + \frac{n}{2|n|}\right)\left[-\sum_{p=0}^{n-1}(p-n)\alpha^I_{m+n-p}\alpha^I_p + \sum_{p=0}^{n-1}(p-n)\alpha^I_p\alpha^I_{m+n-p}\right] \\
&= \frac{1}{2}(m-n)\sum_{p\geq 0}\alpha^I_{n+m-p}\alpha^I_p + \frac{1}{2}(m-n)\sum_{p<0}\alpha^I_p\alpha^I_{m+n-p} \\
&\qquad + \frac{1}{2}\left(\frac{1}{2} - \frac{n}{2|n|}\right)\left[\sum_{p=n}^{-1}(p-n)\left\{\left[\alpha^I_{m+n-p}, \alpha^I_p\right] + \alpha^I_p\alpha^I_{m+n-p}\right\} - \sum_{p=n}^{-1}(p-n)\alpha^I_p\alpha^I_{m+n-p}\right] \\
&\qquad + \frac{1}{2}\left(\frac{1}{2} + \frac{n}{2|n|}\right)\left[-\sum_{p=0}^{n-1}(p-n)\alpha^I_{m+n-p}\alpha^I_p + \sum_{p=0}^{n-1}(p-n)\left\{\left[\alpha^I_p, \alpha^I_{m+n-p}\right] + \alpha^I_{m+n-p}\alpha^I_p\right\}\right] \\
&= (m-n)L^{\perp}_{m+n} \\
&\qquad - \frac{g^{II}}{2}\left(\frac{1}{2} - \frac{n}{2|n|}\right)\sum_{p=n}^{-1}(p-n)p\delta_{m+n,0} + \frac{g^{II}}{2}\left(\frac{1}{2} + \frac{n}{2|n|}\right)\sum_{p=0}^{n-1}(p-n)p\delta_{m+n,0} \\
&= (m-n)L^{\perp}_{m+n} \\
&\qquad + \frac{D-2}{2}\left(\frac{1}{2} - \frac{n}{2|n|}\right)\sum_{p=0}^{|n|-1}(-p-n)p\delta_{m+n,0} + \frac{D-2}{2}\left(\frac{1}{2} + \frac{n}{2|n|}\right)\sum_{p=0}^{|n|-1}(p-n)p\delta_{m+n,0} \\
&= (m-n)L^{\perp}_{m+n} \\
&\qquad + \frac{D-2}{2}\delta_{m+n,0}\sum_{p=0}^{|n|-1}p\left[\left(\frac{1}{2} - \frac{n}{2|n|}\right)(-p-n) + \left(\frac{1}{2} + \frac{n}{2|n|}\right)(p-n)\right] \\
&= (m-n)L^{\perp}_{m+n} \\
&\qquad + \frac{D-2}{2}\delta_{m+n,0}\sum_{p=0}^{|n|-1}p\left[-n + \frac{n}{2|n|}(p+n) + \frac{n}{2|n|}(p-n)\right] \\
&= (m-n)L^{\perp}_{m+n} + \frac{D-2}{2}\delta_{m+n,0}\frac{n}{|n|}\sum_{p=0}^{|n|-1}p[p-|n|]
\end{aligned}
$$

In the middle of the calculus we introduced factors of $\frac{1}{2}\left(1 \pm \frac{n}{|n|}\right)$ just to account for the two possible cases, $n > 0$ and $n < 0$ — Of course, the case $n = 0$ is trivial due to not being necessary to introduce any other factors to ensure the normal ordering of the expression—. Now, we're going to prove by induction that the value of the sum is,

$$
\sum_{p=0}^{|n|-1}p(p-|n|) = \frac{1}{6}\left(|n| - |n|^3\right), \quad |n| \geq 1
$$

It's trivial to check it's validity from $|n| = 1$, now, suppose it's valid for $|n| = k$,

$$
\begin{aligned}
\sum_{p=0}^{k}p(p-k-1) &= \sum_{p=0}^{k}p(p-k) - \sum_{p=0}^{k}p \\
\sum_{p=0}^{k}p(p-k-1) &= \sum_{p=0}^{k-1}p(p-k) - \sum_{p=0}^{k}p
\end{aligned}
$$

$$\sum_{p=0}^{k} p(p-k-1) = \frac{1}{6}\big(k-k^3\big) - \frac{1}{2}k(k+1)$$
$$\sum_{p=0}^{k} p(p-k-1) = \frac{1}{6}\big(-3k^2-2k-k^3\big)$$
$$\sum_{p=0}^{k} p(p-k-1) = \frac{1}{6}\big(k+1-1-3k-3k^2-k^3\big) = \frac{1}{6}\big(k+1-(k+1)^3\big)$$

Which finishes our proof. Hence,

$$\big[L^\perp_m, L^\perp_n\big] = (m-n)L^\perp_{m+n} + \frac{D-2}{2}\delta_{m+n,0}\frac{n}{|n|}\sum_{p=0}^{|n|-1} p[p-|n|]$$
$$\big[L^\perp_m, L^\perp_n\big] = (m-n)L^\perp_{m+n} + \frac{D-2}{12}\delta_{m+n,0}\frac{n}{|n|}\big(|n|-|n|^3\big)$$
$$\big[L^\perp_m, L^\perp_n\big] = (m-n)L^\perp_{m+n} + \frac{D-2}{12}\delta_{m+n,0}\big(n-n^3\big)$$
$$\big[L^\perp_m, L^\perp_n\big] = (m-n)L^\perp_{m+n} + \frac{D-2}{12}\big(m^3-m\big)\delta_{m+n,0} \tag{B.11}$$

As completeness, we're going to derive what should be the Hamiltonian of this theory, of course, the Hamiltonian isn't the X^0 evolution operator, but rather the τ evolution, we get simply from the Legendre transform,

$$H = -p^+\dot{x}_0^- + \int_0^\pi \mathrm{d}\sigma\, \mathcal{P}^{\tau I}\dot{X}^I - L$$
$$H = 2\pi\alpha'\int_0^\pi \mathrm{d}\sigma\, \mathcal{P}^{\tau I}\mathcal{P}^{\tau I} - \pi\alpha'\int_0^\pi \mathrm{d}\sigma\, \mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\, X'^I X'^I$$
$$H = \pi\alpha'\int_0^\pi \mathrm{d}\sigma\left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi^2\alpha'^2}X'^I X'^I\right)$$

But, notice, by the constrains,

$$\dot{X}^- + X'^- = \frac{1}{4\alpha' p^+}\left(\dot{X}^I + X'^I\right)^2$$
$$\dot{X}^- - X'^- = \frac{1}{4\alpha' p^+}\left(\dot{X}^I - X'^I\right)^2$$

Implies,

$$\dot{X}^- = \frac{1}{4\alpha' p^+}\left(\dot{X}^I\dot{X}^I + X'^I X'^I\right)$$
$$\dot{X}^- = \frac{4\pi^2\alpha'^2}{4\alpha' p^+}\left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi^2\alpha'^2}X'^I X'^I\right)$$

$$
\begin{aligned}
\frac{p^+}{\pi}\dot{X}^- &= \pi\alpha'\left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi^2\alpha'^2}X'^I X'^I\right) \\
\frac{p^+}{\pi}\int\limits_0^\pi \mathrm{d}\sigma\,\dot{X}^- &= \pi\alpha'\int\limits_0^\pi \mathrm{d}\sigma\left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi^2\alpha'^2}X'^I X'^I\right)
\end{aligned}
$$

Thus,

$$
H = \frac{2\pi\alpha' p^+}{\pi}\int\limits_0^\pi \mathrm{d}\sigma\,\frac{1}{2\pi\alpha'}\dot{X}^- = 2\alpha' p^+ p^- \tag{B.12}
$$

We can put it in a more useful form,

$$
\begin{aligned}
H &= 2\alpha' p^+ p^- = 2\alpha' p^+ \frac{1}{\sqrt{2\alpha'}}\frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\left(L_0^\perp + a\right) \\
H &= L_0^\perp + a
\end{aligned} \tag{B.13}
$$

C Light-Cone Gauge for the Closed String

Here, we're going to fix the gauge and solve the classical equations of motion for the closed string. The procedure here is pretty much the same as we already done in the Appendix A, the gauge fixing conditions are very similar to the ones used there, that is,

$$\begin{aligned} X^+(\tau,\sigma) &= \alpha' p^+ \tau \\ \partial_\sigma \mathcal{P}^{\tau +} &= 0 \\ \mathrm{Det}[h_{ab}] &= -1 \end{aligned}$$

This three conditions almost fully fix the gauge, but, as discussed already in Appendix A, the second condition in this list allows for a residual gauge A.1,

$$\sigma' = a\sigma + b \tag{C.1}$$

But now we have to impose we're dealing with a closed string, that is, the σ parameter lives in a topology of a circle, in other words, exists $l \in \mathbb{R}$, such that,

$$X(\tau, l+\sigma) = X(\tau,\sigma)$$

Or in an even better description, $\sigma \sim \sigma + l \bmod l$. Notice that we can use C.1 partially to set l to whatever value we like, we'll choose $l = 2\pi$, which facilitates some computations, but, the translational part of C.1, $\sigma' = \sigma + b(\tau)$ is impossible to get fully rid off, we have no privileged point to call $\sigma = 0$, but, we can remove the τ dependence in it by choosing a specific family of $\sigma = \text{cte}$ curves, this choice will be of the $\sigma = \text{cte}$ lines to be orthogonal to the $\tau = \text{cte}$ lines, this is of course another condition on the metric,

$$h^{\sigma\tau}(\tau,\sigma_0) = 0$$

Where σ_0 is just any fixed value. This leaves again a residual reparametrization, a solid translation of the σ coordinate, $\sigma' = \sigma + a$, this cannot be resolved, and we'll have to live with. With also the same reasoning of A.2 we can obtain,

$$\begin{aligned} p^+ &= \int_{\sigma_0}^{\sigma_0+2\pi} \mathrm{d}\sigma\, \mathcal{P}^{\tau+} = \mathcal{P}^{\tau+}2\pi = -\frac{2\pi\alpha' p^+\sqrt{h}h^{\tau\tau}}{2\pi\alpha'} \\ -1 &= h^{\tau\tau} \end{aligned} \tag{C.2}$$

Furthermore, using the equation of motion 2.3 at $\mu = +$, we get,

$$\begin{aligned} 0 &= \partial_\tau\left[h^{\tau\sigma}\partial_\sigma X^+ h^{\tau\tau}\partial_\tau X^+\right] + \partial_\sigma\left[h^{\sigma\sigma}\partial_\sigma X^+ + h^{\sigma\tau}\partial_\tau X^+\right] \\ 0 &= \partial_\tau h^{\tau\tau} + \partial_\sigma h^{\sigma\tau} \\ 0 &= \partial_\sigma h^{\sigma\tau} \end{aligned}$$

Together with $h^{\tau\sigma}(\tau,\sigma_0) = 0$, this simply states that $h^{\tau\sigma} = h^{\sigma\tau} \equiv 0$. This, with $h^{\tau\tau} = -1$ and the determinant condition, says that $h = \mathrm{Diag}\begin{pmatrix}-1 & 1\end{pmatrix}$. That simplifies the equations of motion 2.3 to,

$$\partial_\tau\partial_\tau X^\mu - \partial_\sigma\partial_\sigma X^\mu = \ddot{X}^\mu - X''^\mu = 0$$

And also the consistency conditions 2.4 as,

$$0 = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}h_{ab}\Big(-\dot{X}^2 + X'^2\Big)$$
$$0 = \begin{cases} \partial_\tau X^\mu \partial_\tau X_\mu + \frac{1}{2}\Big(-\dot{X}^2 + X'^2\Big) & = \frac{1}{2}\Big[\dot{X}^2 + X'^2\Big] \\ \partial_\tau X^\mu \partial_\sigma X_\mu & = \dot{X}\cdot X' \\ \partial_\sigma X^\mu \partial_\sigma X_\mu - \frac{1}{2}\Big(-\dot{X}^2 + X'^2\Big) & = \frac{1}{2}\Big[\dot{X}^2 + X'^2\Big] \end{cases}$$

These constrains can be recast as,

$$\begin{aligned} \dot{X}^2 + X'^2 \pm &= 0 \\ \Big(\dot{X} \pm X'\Big)^2 &= 0 \end{aligned} \tag{C.3}$$

And luckily, we don't need to worry about the boundary condition 2.2, as the σ lives in a circle, there is no boundary on which we could impose this boundary equations, instead we have another consistency equation, which can be seen as an additional boundary condition, as we're trying to solve with σ in $\mathbb{R}$ instead of in a circle,

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$$

We have now to solve the equation of motion with the consistency conditions, we already computed the equations of motion in the variables $\sigma^\pm = \tau \pm \sigma$, A.4,

$$\partial_+\partial_- X^\mu = 0$$

This is easily solved for,

$$X^\mu = X_L^\mu\big(\sigma^+\big) + X_R^\mu\big(\sigma^-\big)$$

Imposing the consistency condition,

$$\begin{aligned} X^\mu(\tau, \sigma + 2\pi) &= X^\mu(\tau, \sigma) \\ X_L^\mu\big(\sigma^+ + 2\pi\big) + X_R^\mu\big(\sigma^- - 2\pi\big) &= X_L^\mu\big(\sigma^+\big) + X_R^\mu\big(\sigma^-\big) \\ X_L^\mu\big(\sigma^+ + 2\pi\big) - X_L^\mu\big(\sigma^+\big) &= X_R^\mu\big(\sigma^-\big) - X_R^\mu\big(\sigma^- - 2\pi\big) \end{aligned}$$

As $\sigma^\pm$ are independent variables,

$$X_L'^{\,\mu}\big(\sigma^+ + 2\pi\big) - X_L'^{\,\mu}\big(\sigma^+\big) = 0 = X_R'^{\,\mu}\big(\sigma^-\big) - X_R'^{\,\mu}\big(\sigma^- - 2\pi\big)$$

This states that the two functions $X'_{L,R}$ are both periodic in 2π, hence, we can expand then in Fourier modes, we write it as,

$$X_L'^{\mu}(\sigma^+) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \bar{\alpha}_n^{\mu} \exp\left(-\mathrm{i}n\sigma^+\right)$$
$$X_R'^{\mu}(\sigma^-) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^{\mu} \exp\left(-\mathrm{i}n\sigma^-\right)$$

Integrating this equation we get,

$$X_L^{\mu}(\sigma^+) = \frac{1}{2}x_{0,L}^{\mu} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^{\mu}\sigma^+ + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^*} \frac{\bar{\alpha}_n^{\mu}}{n} \exp\left(-\mathrm{i}n\sigma^+\right)$$
$$X_R^{\mu}(\sigma^-) = \frac{1}{2}x_{0,R}^{\mu} + \sqrt{\frac{\alpha'}{2}}\alpha_0^{\mu}\sigma^- + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^*} \frac{\alpha_n^{\mu}}{n} \exp\left(-\mathrm{i}n\sigma^-\right)$$

Let's see what does the consistency condition of the closed string imply also,

$$\begin{aligned} X_L^{\mu}(\sigma^+ + 2\pi) - X_L^{\mu}(\sigma^+) &= X_R^{\mu}(\sigma^-) - X_R^{\mu}(\sigma^- - 2\pi) \\ \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^{\mu}2\pi &= \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^{\mu}2\pi \\ \bar{\alpha}_0^{\mu} &= \alpha_0^{\mu} \end{aligned} \tag{C.4}$$

This allows us to sum both the parts to obtain the full solution,

$$X^{\mu}(\tau,\sigma) = \frac{1}{2}\left(x_{0,L}^{\mu} + x_{0,R}^{\mu}\right) + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^{\mu}\left(\sigma^+ + \sigma^-\right) + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^*} \frac{1}{n}\left(\bar{\alpha}_n^{\mu}\exp\left(-\mathrm{i}n\sigma^+\right) + \alpha_n^{\mu}\exp\left(-\mathrm{i}n\sigma^-\right)\right)$$
$$X^{\mu}(\tau,\sigma) = x_0^{\mu} + \sqrt{2\alpha'}\bar{\alpha}_0^{\mu}\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^*} \frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^{\mu}\mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^{\mu}\mathrm{e}^{\mathrm{i}n\sigma}\right)$$

Notice that,

$$p^{\mu} = \int_{\sigma_0}^{\sigma_0+2\pi} \mathrm{d}\sigma\, \mathcal{P}^{\tau\mu} = \int_{\sigma_0}^{\sigma_0+2\pi} \mathrm{d}\sigma\, \frac{\partial_\tau X^{\mu}}{2\pi\alpha'}, \quad \text{Because } h = \mathrm{Diag}\begin{pmatrix}-1 & 1\end{pmatrix}$$
$$p^{\mu} = \frac{1}{2\pi\alpha'}\int_{\sigma_0}^{2\pi} \mathrm{d}\sigma \left[\sqrt{2\alpha'}\bar{\alpha}_0^{\mu} + \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^*} \mathrm{e}^{-\mathrm{i}n\tau}\left(\bar{\alpha}_n^{\mu}\mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^{\mu}\mathrm{e}^{\mathrm{i}n\sigma}\right)\right] = \sqrt{\frac{2}{\alpha'}}\bar{\alpha}_0^{\mu}$$
$$\alpha_0^{\mu} = \bar{\alpha}_0^{\mu} = \sqrt{\frac{\alpha'}{2}}p^{\mu}$$

Rewriting again,

$$X^{\mu}(\tau,\sigma) = x_0^{\mu} + \sqrt{2\alpha'}\bar{\alpha}_0^{\mu}\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}^*} \frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^{\mu}\mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^{\mu}\mathrm{e}^{\mathrm{i}n\sigma}\right)$$

This is not the end of it, because we need to make sure the two constrains are satisfied, namely, C.3. With the convention of uppercase Latin index referring to non-light-cone components, $I = 2, \cdots, D-1$, the constrains are,

$$
\begin{aligned}
0 &= -2\left(\dot{X}^- \pm X'^-\right)\left(\dot{X}^+ \pm X'^+\right) + \left(\dot{X}^I \pm X'^I\right)^2 \\
2\alpha' p^+\left(\dot{X}^- \pm X'^-\right) &= \left(\dot{X}^I \pm X'^I\right)^2 \\
\dot{X}^- \pm X'^- &= \frac{1}{2\alpha' p^+}\left(\dot{X}^I \pm X'^I\right)^2
\end{aligned}
$$

That is, the two constrains implies that X^- is not dynamical, and, the collection of X^I fully determine X^-, apart from a single integration constant, x_0^-. To see this is helpful to note,

$$
\begin{aligned}
\dot{X}^\mu &= \sqrt{2\alpha'}\bar{\alpha}_0^\mu + \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*} \mathrm{e}^{-\mathrm{i}n\tau}\left(\bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^\mu \mathrm{e}^{\mathrm{i}n\sigma}\right) \\
\dot{X}^\mu &= \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \bar{\alpha}_0^\mu) + \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*} \mathrm{e}^{-\mathrm{i}n\tau}\left(\bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^\mu \mathrm{e}^{\mathrm{i}n\sigma}\right) \\
\dot{X}^\mu &= \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n\tau}\left(\bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^\mu \mathrm{e}^{\mathrm{i}n\sigma}\right)
\end{aligned}
$$

And,

$$
\begin{aligned}
X'^\mu &= \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*} \mathrm{e}^{-\mathrm{i}n\tau}\left(\bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma} - \alpha_n^\mu \mathrm{e}^{\mathrm{i}n\sigma}\right) \\
X'^\mu &= \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n\tau}\left(\bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n\sigma} - \alpha_n^\mu \mathrm{e}^{\mathrm{i}n\sigma}\right)
\end{aligned}
$$

Which implies,

$$
\dot{X}^\mu + X'^\mu = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu \exp\left(-\mathrm{i}n\sigma^+\right) \tag{C.5}
$$

$$
\dot{X}^\mu - X'^\mu = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^\mu \exp\left(-\mathrm{i}n\sigma^-\right) \tag{C.6}
$$

So that,

$$
\begin{aligned}
\dot{X}^- + X'^- &= \frac{1}{2\alpha' p^+}\left(\dot{X}^I + X'^I\right)^2 \\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \bar{\alpha}_n^- \exp\left(-\mathrm{i}n\sigma^+\right) &= \frac{2\alpha'}{2\alpha' p^+}\sum_{p,q\in\mathbb{Z}} \bar{\alpha}_p^I\bar{\alpha}_q^I \exp\left(-\mathrm{i}(p+q)\sigma^+\right) \\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \bar{\alpha}_n^- \exp\left(-\mathrm{i}n\sigma^+\right) &= \frac{1}{p^+}\sum_{n,p\in\mathbb{Z}} \bar{\alpha}_p^I\bar{\alpha}_{n-p}^I \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) \\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \bar{\alpha}_n^- \exp\left(-\mathrm{i}n\sigma^+\right) &= \frac{2}{p^+}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\sum_{p\in\mathbb{Z}} \bar{\alpha}_p^I\bar{\alpha}_{n-p}^I\right) \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right)
\end{aligned}
$$

$$\sqrt{2\alpha'}\bar{\alpha}_n^- = \frac{1}{p^+}\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I = \frac{2}{p^+}\bar{L}_n^\perp \tag{C.7}$$

And,

$$\begin{aligned}
\dot{X}^- \pm X'^- &= \frac{1}{2\alpha' p^+}\left(\dot{X}^I \pm X'^I\right)^2 \\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{2\alpha'}{2\alpha' p^+}\sum_{p,q\in\mathbb{Z}}\alpha_p^I\alpha_q^I \exp\left(-\mathrm{i}(p+q)(\tau\pm\sigma)\right) \\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{1}{p^+}\sum_{n,p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) \\
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- \exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) &= \frac{2}{p^+}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I\right)\exp\left(-\mathrm{i}n(\tau\pm\sigma)\right) \\
\sqrt{2\alpha'}\alpha_n^- &= \frac{1}{p^+}\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I = \frac{2}{p^+}L_n^\perp
\end{aligned} \tag{C.8}$$

Hence, all the Fourier modes of X^- are completely determined by the transverse modes ones, apart from of course the integration constant x_0^-. In the last passage we also defined the Virasoro modes $L_n^\perp$. This is it, we fully solved the equation of motion with all the constrains and boundary conditions, but there's a catch, imposing σ lives in a circle gives C.4, which for $\mu = -$ implies a non-trivial constrain,

$$\bar{L}_0^\perp = L_0^\perp \tag{C.9}$$

As a consequence we get the full set $\alpha_n^I, \bar{\alpha}_n^I$ cannot be chosen freely. Let's rewrite here the full solution for completeness,

$$\begin{aligned}
X^I &= x_0^I + \sqrt{2\alpha'}\bar{\alpha}_0^I\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{\alpha}_n^I\mathrm{e}^{-\mathrm{i}n\sigma} + \alpha_n^I\mathrm{e}^{\mathrm{i}n\sigma}\right) \\
X^- &= x_0^- + \frac{2}{p^+}L_0^\perp\tau + \frac{\mathrm{i}}{p^+}\sum_{n\in\mathbb{Z}^*}\frac{\mathrm{e}^{-\mathrm{i}n\tau}}{n}\left(\bar{L}_n^\perp\mathrm{e}^{-\mathrm{i}n\sigma} + L_n^\perp\mathrm{e}^{\mathrm{i}n\sigma}\right) \\
X^+ &= \alpha' p^+\tau \\
L_0^\perp &= \bar{L}_0^\perp
\end{aligned}$$

D Light-Cone Gauge Quantization of the Closed String

We already did a lot in B, in particular we should use the same quantization conditions,

$$\left[X^I(\tau,\sigma),\mathcal{P}^{\tau J}(\tau,\sigma')\right]=\mathrm{i}g^{IJ}\delta(\sigma-\sigma'),\quad \left[x_0^-,p^+\right]=-\mathrm{i} \tag{D.1}$$

Any other commutator between these 4 operators is zero. Of course now, X^- has to be considered as a function of the X^I and $\mathcal{P}^{\tau I}$, so, it has non trivial commutators. The first thing we need to do here is go from these canonical commutation relations, to the commutation relations of the modes $\alpha_n^I,\bar{\alpha}_n^I$. The best way of doing this is working with the expressions C.5 and C.6, as was done in the open string, and computing the following commutator,

$$\left[\dot{X}^I\pm X'^I,\dot{X}^J\pm X'^J\right]$$

For now everything is equal to the open string, hence, we can use some of the results derived there, as B.2 and B.3,

$$\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\pm X'^J\right)(\tau,\sigma')\right]=\pm4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma') \tag{D.2}$$

$$\left[\left(\dot{X}^I\pm X'^I\right)(\tau,\sigma),\left(\dot{X}^J\mp X'^J\right)(\tau,\sigma')\right]=0 \tag{D.3}$$

For our luck, unlike the open string, everything here is defined for the interval $\sigma\in[0,2\pi]$, hence, using C.5 and C.6 we can compute,

$$\begin{aligned}
4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')&=\left[\left(\dot{X}^I+X'^I\right)(\tau,\sigma),\left(\dot{X}^J+X'^J\right)(\tau,\sigma')\right]\\
4\pi\alpha'\mathrm{i}g^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')&=2\alpha'\sum_{n',m'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau+\sigma')}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)}\left[\bar{\alpha}_{m'}^I,\bar{\alpha}_{n'}^J\right]\\
2\pi\mathrm{i}g^{IJ}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\mathrm{e}^{\mathrm{i}m\sigma}&=\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}m\sigma}\sum_{m',n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\tau(n'+m')}\mathrm{e}^{-\mathrm{i}\sigma'n'-\mathrm{i}\sigma m'}\left[\bar{\alpha}_{m'}^I,\bar{\alpha}_{n'}^J\right]\\
mg^{IJ}\mathrm{e}^{\mathrm{i}m\sigma'}&=\sum_{m',n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\tau(n'+m')}\mathrm{e}^{-\mathrm{i}\sigma'n'}\delta_{m,m'}\left[\bar{\alpha}_{m'}^I,\bar{\alpha}_{n'}^J\right]\\
\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}n\sigma'}mg^{IJ}\mathrm{e}^{\mathrm{i}m\sigma'}&=\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\mathrm{e}^{\mathrm{i}n\sigma'}\sum_{n'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}\tau(n'+m)}\mathrm{e}^{-\mathrm{i}\sigma'n'}\left[\bar{\alpha}_m^I,\bar{\alpha}_{n'}^J\right]\\
mg^{IJ}\delta_{m+n,0}&=\sum_{n'\in\mathbb{Z}}\delta_{n,n'}\mathrm{e}^{-\mathrm{i}\tau(n'+m)}\left[\bar{\alpha}_m^I,\bar{\alpha}_{n'}^J\right]\\
mg^{IJ}\delta_{m+n,0}\mathrm{e}^{\mathrm{i}\tau(n+m)}&=\left[\bar{\alpha}_m^I,\bar{\alpha}_n^J\right]\\
\left[\bar{\alpha}_m^I,\bar{\alpha}_n^J\right]&=mg^{IJ}\delta_{m+n,0}
\end{aligned} \tag{D.4}$$

And,

$$\begin{aligned}
0&=\left[\left(\dot{X}^I+X'^I\right)(\tau,\sigma),\left(\dot{X}^J-X'^J\right)(\tau,\sigma')\right]\\
0&=2\alpha'\sum_{n',m'\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}n'(\tau-\sigma')}\mathrm{e}^{-\mathrm{i}m'(\tau+\sigma)}\left[\bar{\alpha}_{m'}^I,\alpha_{n'}^J\right]
\end{aligned}$$

$$
\begin{aligned}
0 &= \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{\mathrm{i}m\sigma} \sum_{m',n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m')} \mathrm{e}^{\mathrm{i}\sigma' n' - \mathrm{i}\sigma m'} \left[\bar{\alpha}^I_{m'}, \alpha^J_{n'}\right] \\
0 &= \sum_{m',n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m')} \mathrm{e}^{\mathrm{i}\sigma' n'} \delta_{m,m'} \left[\bar{\alpha}^I_{m'}, \alpha^J_{n'}\right] \\
0 &= \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{-\mathrm{i}n\sigma'} \sum_{n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m)} \mathrm{e}^{\mathrm{i}\sigma' n'} \left[\bar{\alpha}^I_{m}, \alpha^J_{n'}\right] \\
0 &= \sum_{n' \in \mathbb{Z}} \delta_{n,n'} \mathrm{e}^{-\mathrm{i}\tau(n'+m)} \left[\bar{\alpha}^I_{m}, \alpha^J_{n'}\right] \\
\left[\bar{\alpha}^I_{m}, \alpha^J_{n}\right] &= 0
\end{aligned} \tag{D.5}
$$

And lastly,

$$
\begin{aligned}
-4\pi\alpha' \mathrm{i} g^{IJ} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma - \sigma') &= \left[\left(\dot{X}^I - X'^I\right)(\tau, \sigma), \left(\dot{X}^J - X'^J\right)(\tau, \sigma')\right] \\
-4\pi\alpha' \mathrm{i} g^{IJ} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma - \sigma') &= 2\alpha' \sum_{n',m' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'(\tau - \sigma')} \mathrm{e}^{-\mathrm{i}m'(\tau - \sigma)} \left[\alpha^I_{m'}, \alpha^J_{n'}\right] \\
-2\pi \mathrm{i} g^{IJ} \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma - \sigma') \mathrm{e}^{-\mathrm{i}m\sigma} &= \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{-\mathrm{i}m\sigma} \sum_{m',n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m')} \mathrm{e}^{\mathrm{i}\sigma' n' + \mathrm{i}\sigma m'} \left[\alpha^I_{m'}, \alpha^J_{n'}\right] \\
m g^{IJ} \mathrm{e}^{-\mathrm{i}m\sigma'} &= \sum_{m',n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m')} \mathrm{e}^{\mathrm{i}\sigma' n'} \delta_{m,m'} \left[\alpha^I_{m'}, \alpha^J_{n'}\right] \\
\int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{-\mathrm{i}n\sigma'} m g^{IJ} \mathrm{e}^{-\mathrm{i}m\sigma'} &= \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{-\mathrm{i}n\sigma'} \sum_{n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tau(n'+m)} \mathrm{e}^{\mathrm{i}\sigma' n'} \left[\alpha^I_{m}, \alpha^J_{n'}\right] \\
m g^{IJ} \delta_{m+n,0} &= \sum_{n' \in \mathbb{Z}} \delta_{n,n'} \mathrm{e}^{-\mathrm{i}\tau(n'+m)} \left[\alpha^I_{m}, \alpha^J_{n'}\right] \\
m g^{IJ} \delta_{m+n,0} \mathrm{e}^{\mathrm{i}\tau(n+m)} &= \left[\alpha^I_{m}, \alpha^J_{n}\right] \\
\left[\alpha^I_{m}, \alpha^J_{n}\right] &= m g^{IJ} \delta_{m+n,0}
\end{aligned} \tag{D.6}
$$

This is not the end, we still have two more commutation relation to get,

$$
\begin{aligned}
2\pi\alpha' \mathrm{i} g^{IJ} \delta(\sigma - \sigma') &= \left[X^I(\tau, \sigma), \dot{X}^J(\tau, \sigma')\right] \\
\int_0^{2\pi} \mathrm{d}\sigma \, 2\pi\alpha' \mathrm{i} g^{IJ} \delta(\sigma - \sigma') &= \int_0^{2\pi} \mathrm{d}\sigma \left[X^I(\tau, \sigma), \dot{X}^J(\tau, \sigma')\right], \quad \int_0^{2\pi} \mathrm{d}\sigma \, \mathrm{e}^{-\mathrm{i}n\sigma} = 0, \ n \in \mathbb{Z}^* \\
2\pi\alpha' \mathrm{i} g^{IJ} &= 2\pi \left[x^I_0 + \sqrt{2\alpha'} \alpha^I_0 \tau, \dot{X}^J(\tau, \sigma')\right] \\
\alpha' \mathrm{i} g^{IJ} &= \sqrt{\frac{\alpha'}{2}} \sum_{n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'\tau} \left[x^I_0 + \sqrt{2\alpha'} \tau \alpha^I_0, \bar{\alpha}^J_{n'} \mathrm{e}^{-\mathrm{i}n'\sigma} + \alpha^J_{n'} \mathrm{e}^{\mathrm{i}n'\sigma}\right] \\
\sqrt{2\alpha'} \mathrm{i} g^{IJ} &= \sum_{n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'\tau} \left[x^I_0, \bar{\alpha}^J_{n'} \mathrm{e}^{-\mathrm{i}n'\sigma} + \alpha^J_{n'} \mathrm{e}^{\mathrm{i}n'\sigma}\right] \\
\int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi} \mathrm{e}^{\mathrm{i}n\sigma'} \sqrt{2\alpha'} \mathrm{i} g^{IJ} &= \int_0^{2\pi} \frac{\mathrm{d}\sigma'}{2\pi} \mathrm{e}^{\mathrm{i}n\sigma'} \sum_{n' \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}n'\tau} \left[x^I_0, \bar{\alpha}^J_{n'} \mathrm{e}^{-\mathrm{i}n'\sigma} + \alpha^J_{n'} \mathrm{e}^{\mathrm{i}n'\sigma}\right] \\
\sqrt{2\alpha'} \mathrm{i} g^{IJ} \delta_{n,0} &= \mathrm{e}^{-\mathrm{i}n\tau} \left[x^I_0, \bar{\alpha}^J_{n}\right] + \mathrm{e}^{\mathrm{i}n\tau} \left[x^J_0, \alpha^J_{-n}\right]
\end{aligned}
$$

Notice that the right-handed side cannot have τ dependence, as the left side doesn't have any, and, as the exponential are both linearly independent, both commutators have to vanish, well, this is the case if $n \neq 0$, but, if $n = 0$ there is no τ dependence anywhere, and as $\bar{\alpha}_0^\mu = \alpha_0^\mu$, we simply get the following result,

$$\left[x_0^I, \bar{\alpha}_n^J\right] = \left[x_0^I, \alpha_n^J\right] = \sqrt{\frac{\alpha'}{2}} \mathrm{i} g^{IJ} \delta_{n,0} \tag{D.7}$$

The last thing we need to discuss before going to the Lorentz generators is about the Virasoro operators, these were defined as,

$$L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p, \quad \bar{L}_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \bar{\alpha}_{n-p}^I \bar{\alpha}_p \tag{D.8}$$

Again, we have the same normal ordering problem discussed in Appendix B, what we should do is to define the Virasoro modes as normal ordered and add a normal ordering constant to the ambiguous operators, in this case these are just

$$\begin{aligned} L_0^\perp &\to L_0^\perp + a \\ \bar{L}_0^\perp &\to \bar{L}_0^\perp + \bar{a} \end{aligned}$$

For now these are not know, but, it's true that $a = \bar{a} = -1$ and $D = 26$ for the Global Poincare symmetry to hold, we'll not prove this. From here on, as we shown that the two $\bar{\alpha}$ and α decouple, it's as we had two copies of an open string, Hence, all the remaining commutation relations can be read directly from the open string, with mixed terms of barred and non-barred commuting among themselves.

As completeness, we're going to derive what should be the Hamiltonian of this theory, of course, the Hamiltonian isn't the X^0 evolution operator, but rather the τ evolution, we get simply from the Legendre transform,

$$\begin{aligned} H &= -p^+ \dot{x}_0^- + \int_0^{2\pi} \mathrm{d}\sigma\, \mathcal{P}^{\tau I} \dot{X}^I - L \\ H &= 2\pi\alpha' \int_0^{2\pi} \mathrm{d}\sigma\, \mathcal{P}^{\tau I} \mathcal{P}^{\tau I} - \pi\alpha' \int_0^{2\pi} \mathrm{d}\sigma\, \mathcal{P}^{\tau I} \mathcal{P}^{\tau I} + \frac{1}{4\pi\alpha'} \int_0^{2\pi} \mathrm{d}\sigma\, X'^I X'^I \\ H &= \pi\alpha' \int_0^{2\pi} \mathrm{d}\sigma \left(\mathcal{P}^{\tau I} \mathcal{P}^{\tau I} + \frac{1}{4\pi^2 \alpha'^2} X'^I X'^I \right) \end{aligned}$$

But, notice, by the constrains,

$$\begin{aligned} \dot{X}^- + X'^- &= \frac{1}{2\alpha' p^+} \left(\dot{X}^I + X'^I \right)^2 \\ \dot{X}^- - X'^- &= \frac{1}{2\alpha' p^+} \left(\dot{X}^I - X'^I \right)^2 \end{aligned}$$

Implies,

$$
\begin{aligned}
\dot{X}^- &= \frac{1}{2\alpha' p^+}\left(\dot{X}^I \dot{X}^I + X'^I X'^I\right) \\
\dot{X}^- &= \frac{4\pi^2\alpha'^2}{2\alpha' p^+}\left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi^2\alpha'^2} X'^I X'^I\right) \\
\frac{p^+}{2\pi}\dot{X}^- &= \pi\alpha'\left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi^2\alpha'^2} X'^I X'^I\right) \\
\frac{p^+}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\, \dot{X}^- &= \pi\alpha' \int_0^{2\pi} \mathrm{d}\sigma \left(\mathcal{P}^{\tau I}\mathcal{P}^{\tau I} + \frac{1}{4\pi^2\alpha'^2} X'^I X'^I\right)
\end{aligned}
$$

Thus,

$$
H = \frac{2\pi\alpha' p^+}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\, \frac{1}{2\pi\alpha'}\dot{X}^- = \alpha' p^+ p^- \tag{D.9}
$$

We can put it in a more useful form, using the level matching,

$$
\begin{aligned}
H &= 2\alpha' p^+ p^- = \alpha' p^+ \sqrt{\frac{2}{\alpha'}}\sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\left(L_0^\perp + a\right) \\
H &= L_0^\perp + \bar{L}_0^\perp + a + \bar{a}
\end{aligned} \tag{D.10}
$$