Homework III

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Problem 1

1.A)

For an operator $\mathcal{O}$ to be BRST closed, it means $[Q_{\text{BRST}}, \mathcal{O}] = 0$, just remembering the usual BRST transformations[1],

$$[Q_{\text{BRST}}, X^\mu] = \big(c\partial + \tilde{c}\bar{\partial}\big)X^\mu$$

$$[Q_{\text{BRST}}, b] = T^X + T^g$$

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[1] We're using a graded commutator notation, that is, it's to be interpreted as either a commutator or an anti-commutator depending on the statistic of what is inside.

$$[Q_{\text{BRST}}, c] = c\partial c$$

with,

$$T^X = \frac{1}{\alpha'} : \partial X^\mu \partial X_\mu :, \quad T^g =: c\partial b : -2 : b\partial c :$$

so,

$$[Q_{\text{BRST}}, V^a] = \lambda^a \epsilon_\mu [Q_{\text{BRST}}, c\partial X^\mu \exp(\mathrm{i}k \cdot X)]$$
$$[Q_{\text{BRST}}, V^a] = \lambda^a \epsilon_\mu [Q_{\text{BRST}}, c]\partial X^\mu \exp(\mathrm{i}k \cdot X) - \lambda^a \epsilon_\mu c[Q_{\text{BRST}}, \partial X^\mu \exp(\mathrm{i}k \cdot X)]$$
$$[Q_{\text{BRST}}, V^a] = \lambda^a \epsilon_\mu c\partial c \partial X^\mu \exp(\mathrm{i}k \cdot X)$$
$$- \lambda^a \epsilon_\mu c\partial X^\mu [Q_{\text{BRST}}, \exp(\mathrm{i}k \cdot X)] - \lambda^a \epsilon_\mu c[Q_{\text{BRST}}, \partial X^\mu]\exp(\mathrm{i}k \cdot X)$$
$$[Q_{\text{BRST}}, V^a] = \lambda^a \epsilon_\mu c\partial c \partial X^\mu \exp(\mathrm{i}k \cdot X)$$
$$- \lambda^a \epsilon_\mu c\partial X^\mu [Q_{\text{BRST}}, \exp(\mathrm{i}k \cdot X)] - \lambda^a \epsilon_\mu c\partial[Q_{\text{BRST}}, X^\mu]\exp(\mathrm{i}k \cdot X)$$
$$[Q_{\text{BRST}}, V^a] = \lambda^a \epsilon_\mu c\partial c \partial X^\mu \exp(\mathrm{i}k \cdot X)$$
$$- \lambda^a \epsilon_\mu c\partial X^\mu [Q_{\text{BRST}}, \exp(\mathrm{i}k \cdot X)] - \lambda^a \epsilon_\mu c\partial(c\partial + \tilde{c}\bar{\partial})X^\mu \exp(\mathrm{i}k \cdot X)$$
$$[Q_{\text{BRST}}, V^a] = \lambda^a \epsilon_\mu c\partial c \partial X^\mu \exp(\mathrm{i}k \cdot X)$$
$$- \lambda^a \epsilon_\mu c\partial X^\mu [Q_{\text{BRST}}, \exp(\mathrm{i}k \cdot X)] - \lambda^a \epsilon_\mu c\partial(c\partial + \tilde{c}\bar{\partial})X^\mu \exp(\mathrm{i}k \cdot X)$$
$$- \lambda^a \epsilon_\mu cc\partial - \lambda^a \epsilon_\mu c\partial(\tilde{c}\bar{\partial})X^\mu \exp(\mathrm{i}k \cdot X)$$

1.B)

1.C)

1.D)

Problem 2

2.A)

2.B)

2.C)

2.D)

Problem 3

3.A)

3.B)

3.C)

3.D)

3.E)

3.F)

A   Faddeev-Popov Gauge Fixing

Our Action functional is,

$$S_X + \lambda\chi = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int_M d^2\sigma\,\sqrt{h}R + \frac{\lambda}{2\pi}\int_{\partial M} ds\,K \qquad (A.1)$$

we would like to define the quantum theory by means of the path integral, that is, we expect that,

$$Z \overset{?}{=} \int \mathcal{D}X\mathcal{D}h \exp\left(-S_X[X,h] - \lambda\chi\right) \qquad (A.2)$$

should give a well defined theory, but, already from A.2 there're several problems that arise, one of them is: *What should be interpreted from the path integral itself? We haven't defined any manifold to our metric h and scalar fields X to live in, also, even if we had defined such, the path integral relies on explicit coordinate points, $\mathcal{D}h = \prod_\sigma dh_{ab}(\sigma)$, which are highly dependent on charts.*

    This is a valid claim, our way to avoid it is to *define $\mathcal{D}h$* to mean: *Sum over all **allowed** two dimensional Riemannian manifolds, and all possible metric structures in these.* Here, **allowed** requires a prescription, which manifolds are or aren't allowed impacts the obtained string theory. Happily, every two dimensional manifold has a definite value for the Euler Characteristic χ, hence, we can sort them out by it,

$$Z \overset{?}{=} \sum_{\{M\}}\int_{\mathrm{Met}(M)} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h] - \lambda\chi\right)$$

$$Z \overset{?}{=} \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}}\int_{\mathrm{Met}(M_\chi)} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h]\right) \qquad (A.3)$$

Where M is to be understood as a two dimensional Riemannian manifold and M_χ is one with Euler Characteristic χ, $\mathrm{Met}(M_\chi)$ is the space of all metrics which can be assigned to M_χ, we have written $\sum_{\{M_\chi\}}$ in the special case of there being more than one manifold with same Euler Characteristic[2], also, the functional integral over X should be read as integrating over all maps from M_χ to $\mathbb{R}^{1,D-1}$. While this is better defined than before, i.e. not coordinate dependent, we still have a few problems, first, it's know that A.1 has a Gauge Group of $\mathrm{Diff}(M)\times\mathrm{Weyl}(M)$, but, in our second try of a definition of the path integral, we're integrating the metrics over $\mathrm{Met}(M_\chi)$, it's clear that may happen of two elements of $\mathrm{Met}(M_\chi)$ be equivalent under a $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ transformation, to put in more clear terms, we're worried if exists $h', h \in \mathrm{Met}(M_\chi)$ such,

$$h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

the existence of those kinds of elements is troublesome, as $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ is a infinite dimensional group of redundancies, this means we're over-counting physical configurations by a infinite amount. The solution is to look for an equivalence class of metrics under this Gauge Group action,

$$\mathcal{M}_\chi = \mathrm{Met}(M_\chi)/\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$$

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[2]As we're interested only in Differentiable Manifolds, more than manifold should read: More than one equivalence class of Differentiable Manifolds.

the equivalence class is to be understood as[3],

$$h' \sim h \Leftrightarrow h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

that is, two metrics are the same representative of an element of $\mathscr{M}_\chi$ iff they differ by a composition of a Diffeomorphism and Weyl transformation. We'll denote a given composition of a Diffeomorphism followed by a Weyl transformation by ζ,

$$h' = \zeta \circ h$$

Notice that the set of equivalence class of metrics, or, the set of inequivalent $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ metrics $\mathscr{M}_\chi$ is highly dependent on the topology of M_χ, for example, for $M_\chi \cong \mathbb{R}^2 \cong \mathbb{C}$, it's trivial, there is just one point in the set $\mathscr{M}_\chi$, in other words, every metric is equivalent, which isn't true for more complex topologies.

Thus, it's possible for us to set up a well defined version of the path integral, just replace $\mathrm{Met}(M_\chi)$ by $\mathscr{M}_\chi$,

$$Z = \sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathcal{D}h\int\mathcal{D}X\exp\left(-S_X[X,h]\right) \tag{A.4}$$

where the integration is to be understood as by choosing for each equivalence class in $\mathscr{M}_\chi$ a representative element in $\mathrm{Met}(M_g)$. While this is a satisfactory definition for the path integral, it feels a little clunky, we rather have an path integral over all the possible metrics — in the sense defined before —, well, this is achievable. First, for each equivalence class of $\mathscr{M}_\chi$ elect one representative element of $\mathrm{Met}(M_\chi)$, we'll denote these elements as $\hat{h}(\mathbf{t})$ — here $\mathbf{t}$ is a parametrization of the correspondent equivalence class in $\mathscr{M}_\chi$, we haven't proved here, and won't, but $\mathscr{M}_\chi$ is a finite N dimensional manifold, hence, $\mathbf{t}$ is a N-tuple of real numbers —, by construction, these representatives are inequivalent under $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, hence,

$$\zeta_1 \circ \hat{h}(\mathbf{t}_1) = \zeta_2 \circ \hat{h}(\mathbf{t}_2) \Leftrightarrow \mathbf{t}_1 = \mathbf{t}_2 \text{ and } \zeta_1 = \zeta_2$$

so that every element in $\mathrm{Met}(M_g)$ can be written as a unique[4] composition of a given ζ into a given $\hat{h}(\mathbf{t})$. Now, we rewrite the pictorial integral over $\mathscr{M}_\chi$ is a more formal way, using the parametrization we just described,

$$Z = \sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int\mathcal{D}X\exp\left(-S_X\left[X,\hat{h}(\mathbf{t})\right]\right)$$

$$Z = \sum_{\{\chi\}}\exp\left(-\lambda\chi\right)\sum_{\{M_\chi\}}\int_{\mathscr{M}_\chi}\mathrm{d}^N\mathbf{t}\int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)}\mathcal{D}\zeta\delta(\zeta)\int\mathcal{D}X\exp\left(-S_X\left[X,\hat{h}(\mathbf{t})\right]\right)$$

in the last line we introduced a one by integrating[5] over the delta functional, as this integral picks only $\zeta = 0$, what should be understood as $\zeta = \mathrm{id}$ in the group, we can deform a little the

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[3]In all charts.

[4]The uniqueness or not depends on a few factors, here we'll always, unless specified otherwise, interpret $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ as the group *generated by* all possible compositions of Diffeomorphisms and Weyl transformations, but a element of it, ζ, is not to be interpreted as a unique composition of Diffeomorphism and Weyl factors, as there might be some Diffeomorphism which are equivalent to Weyl transformations, what is indeed true is that every element ζ of the Gauge Group is a unique combination of an element of $\mathrm{Diff}(M_\chi)/\mathrm{Weyl}(M_\chi)$ and an element of $\mathrm{Weyl}(M_\chi)$.

[5]Again, following the same remarks made before, the integral over $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ should not be interpreted as integrating over the whole of $\mathrm{Diff}(M_\chi)$ and after integrating over the whole $\mathrm{Weyl}(M_\chi)$, this would for sure be an over-counting, but rather should be interpreted as integrating over the whole group *generated by* compositions of $\mathrm{Diff}(M_\chi)$ and $\mathrm{Weyl}(M_\chi)$, which is equivalent of integrating over the whole $\mathrm{Diff}(M_\chi)/\mathrm{Weyl}(M_\chi)$, and after integrating over the whole $\mathrm{Weyl}(M_\chi)$.

integration to,

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathcal{M}_\chi} \mathrm{d}^N \mathbf{t} \int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)} \mathcal{D}\zeta\,\delta(\zeta) \int \mathcal{D}X \exp\left(-S_X\left[X, \zeta \circ \hat{h}(\mathbf{t})\right]\right) \quad (A.5)$$

This is almost in the form that we would like, notice that we're integrating over the set of representative of the inequivalent metrics, $\mathrm{d}^N\mathbf{t}$, and also over the whole group $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, $\mathcal{D}\zeta$, by construction, **every** metric in $\mathrm{Met}(M_\chi)$ can be written uniquely[6] as,

$$h = \zeta_\mathbf{t} \circ \hat{h}(\mathbf{t})$$

in other words, to integrate over $\mathrm{d}^N\mathbf{t}\,\mathcal{D}\zeta$ is to integrate over all metrics of the form $\zeta \circ \hat{h}(\mathbf{t})$, which is to integrate over all metrics $h = \zeta \circ \hat{h}(\mathbf{t})$ in $\mathrm{Met}(M_\chi)$! We cannot yet make this change, due to the presence of an explicit dependence in ζ at the functional delta. We'll eliminate it by means of a change of variable of the functional delta, notice that,

$$\delta\left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\right)$$

picks up just the contribution of $\zeta = 0$, so it's a good candidate for a change of variables,

$$\delta\left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\right) = \delta(\zeta)\left|\mathrm{Det}\left[\frac{\delta}{\delta\zeta}\left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\right)\Big|_{\zeta=0}\right]\right|^{-1}$$

let's compute step by step the right-hand side of this equation, as we're only interested in the solution of $\zeta = 0$, what matters is just the connected component to the identity of the Gauge Group, this is parametrized by a function ω related to the Weyl transformation, and a vector field ξ related to the connected component to the identity of the Diffeomorphisms — there is an additional requirement of ξ not generating any transformation which can be undone by a Weyl transformation —, also, for ease of our manipulation, we'll write the expression inside the delta with respect to h instead of $\hat{h}(\mathbf{t})$[7], that is,

$$\delta\left(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\right) = \delta\left(\zeta^{-1} \circ h - h\right) = \delta\left(\zeta^{-1}\right)\left|\mathrm{Det}\left[\frac{\delta}{\delta\zeta^{-1}}\left(\zeta^{-1} \circ h - h\right)\Big|_{\zeta^{-1}=0}\right]\right|^{-1}$$

one might worry about the ζ^{-1} instead of the ζ, but, the integration measure $\mathcal{D}\zeta$ is formally a Haar measure in the Group, that means it's a group invariant measure, in other words, $\mathcal{D}\zeta^{-1} = \mathcal{D}\zeta$, so that we can forget about the inverse, now,

$$[\zeta \circ h]_{ab} = [h]_{ab} + 2\omega[h]_{ab} + [\pounds_\xi h]_{ab} + \mathcal{O}\left(\omega^2, \xi^2, \omega\xi\right)$$
$$[\zeta \circ h]_{ab} = [h]_{ab} + 2\omega[h]_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}\left(\omega^2, \xi^2, \omega\xi\right)$$

of course ∇ here is with respect to the h metric,

$$[\zeta \circ h]_{ab} - [h]_{ab} = 2\omega[h]_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}\left(\omega^2, \xi^2, \omega\xi\right)$$
$$\frac{\delta}{\delta\zeta}\left([\zeta \circ h]_{ab} - [h]_{ab}\right) = ????$$

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[6]With the remarks made before.

[7]We would have to carry out the $\mathbf{t}$ dependence in h also, but, soon it will disappear as matter of uniting the integrals $\mathrm{d}^N\mathbf{t}\,\mathcal{D}\zeta$ so we won't keep track of it anymore.

The ζ derivative actually has two parts, the derivative with respect to ω and the other with respect to ξ, let's do one by one,

$$\frac{\delta}{\delta\omega(\sigma')}([\zeta \circ h]_{ab} - [h]_{ab})(\sigma)\bigg|_{\zeta=0} = 2\delta^{(2)}(\sigma - \sigma')h_{ab}(\sigma)$$

and for the ξ,

$$\frac{\delta}{\delta\xi^c(\sigma')}([\zeta \circ h]_{ab} - [h]_{ab})(\sigma)\bigg|_{\zeta=0} = 2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma - \sigma')$$

Thus,

$$\frac{\delta}{\delta\zeta}([\zeta \circ h]_{ab} - [h]_{ab})\bigg|_{\zeta=0} = 2\delta^{(2)}(\sigma - \sigma')h_{ab}(\sigma) + 2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma - \sigma')$$

$$\mathrm{Det}\left[\frac{\delta}{\delta\zeta}([\zeta \circ h]_{ab} - [h]_{ab})\bigg|_{\zeta=0}\right] = \mathrm{Det}\left[2\delta^{(2)}(\sigma - \sigma')h_{ab}(\sigma) + 2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma - \sigma')\right]$$

the determinant can be computed by means of path integral of Grassmannian variables,

$$\mathrm{Det}\left[2\delta^{(2)}(\sigma - \sigma')h_{ab} + 2h_{c(b}\nabla_{a)}\delta^{(2)}(\sigma - \sigma')\right] = \int \mathcal{D}b\mathcal{D}c\mathcal{D}d \exp\left(-\frac{1}{2\pi}\int \mathrm{d}^2\sigma \sqrt{h}b^{ab}\left[h_{ab}d + h_{c(b}\nabla_{a)}c^c\right]\right)$$

$$\mathrm{Det}\left[\frac{\delta}{\delta\zeta}([\zeta \circ h]_{ab} - [h]_{ab})\bigg|_{\zeta=0}\right] = \int \mathcal{D}b\mathcal{D}c\mathcal{D}d \exp\left(-S_{\mathrm{gh}}[b, c, d, h]\right)$$

Substituting all of this back into our path integral,

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)} \mathcal{D}\zeta\delta(\zeta) \int \mathcal{D}X \exp\left(-S_X\left[X, \zeta \circ \hat{h}(\mathbf{t})\right]\right)$$

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)} \mathcal{D}\zeta\delta\left(\zeta^{-1} \circ h - h\right) \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d \exp\left(-S_X - S_{\mathrm{gh}}\right)$$

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int_{\mathrm{Met}(M_\chi)} \mathcal{D}h\delta\left(\hat{h} - h\right) \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d \exp\left(-S_X[X, h] - S_{\mathrm{gh}}[b, c, d, h]\right)$$

$$Z = \int \mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\delta\left(\hat{h} - h\right) \exp\left(-S_X[X, h] - S_{\mathrm{gh}}[b, c, d, h] - \lambda\chi\right)$$

where $\hat{h}$ is a family of choices of representatives of the equivalence classes of the Gauge equivalent metrics, of course this choice is dependent on the equivalence class h lies in, so, in a certain sense we have $\hat{h} = \hat{h}[h]$,

$$Z = \int \mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\delta\left(\hat{h}[h] - h\right) \exp\left(-S_X[X, h] - S_{\mathrm{gh}}[b, c, d, h] - \lambda\chi\right)$$

we express the delta functional in terms of a path integral,

$$Z = \int \mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\mathcal{D}B \exp\left(\frac{\mathrm{i}}{4\pi}\int \mathrm{d}^2\sigma \sqrt{h}B^{ab}\left(\hat{h}_{ab}[h] - h_{ab}\right)\right) \exp\left(-S_X - S_{\mathrm{gh}} - \lambda\chi\right)$$

$$Z = \int \mathcal{D}h\mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}d\mathcal{D}B \exp\left(-S_X[X, h] - S_{\mathrm{gh}}[b, c, d, h] - S_{\mathrm{gf}}[B, h] - \lambda\chi\right)$$

where we lastly defined the Gauge Fixing Action. This is the final expression for our path integral with the identifications,

$$S_X[X,h] + \lambda\chi = \frac{1}{4\pi\alpha'} \int_M \mathrm{d}^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi} \int_M \mathrm{d}^2\sigma \sqrt{h} R + \frac{\lambda}{2\pi} \int_{\partial M} \mathrm{d}s\, K \qquad (\mathrm{A.6a})$$

$$S_{\mathrm{gh}}[b,c,d,h] = \frac{1}{2\pi} \int_M \mathrm{d}^2\sigma \sqrt{h} b^{ab} [h_{ab} d + \nabla_a c_b] \qquad (\mathrm{A.6b})$$

$$S_{\mathrm{gf}}[B,h] = -\frac{\mathrm{i}}{4\pi} \int_M \mathrm{d}^2\sigma \sqrt{h} B^{ab} \left(\hat{h}_{ab}[h] - h_{ab}\right) \qquad (\mathrm{A.6c})$$

B BRST Quantization

Following the action principle derived from the Faddeev-Popov Gauge Fixing A.6, we can describe it's BRST symmetry by the transformations of the *matter fields* under Gauge, we know the following,

$$X^\mu(\sigma) \to X'^\mu(\sigma'(\sigma)) = X^\mu(\sigma)$$

$$h_{ab}(\sigma) \to h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial \sigma^c}{\partial \sigma'^a}\frac{\partial \sigma^d}{\partial \sigma'^b}h_{cd}(\sigma)$$

which have an *infinitesimal* form,

$$\delta X^\mu = \mathcal{L}_\xi X^\mu = \xi^a \partial_a X^\mu$$

$$\delta h_{ab} = 2\omega h_{ab} + \mathcal{L}_\xi h_{ab} = 2\omega h_{ab} + \nabla_a\xi_b + \nabla_b\xi_a = 2\omega h_{ab} + \xi^c\partial_c h_{ab} + h_{ac}\partial_b\xi^c + h_{bc}\partial_a\xi^c$$

the BRST transformation can be obtained from these by the substitution inferred from the Faddeev-Popov gauge fix, that is, $\xi^a \to i\epsilon c^a$ and $\omega \to i\epsilon d$, where ϵ is a Grassmannian parametrization of the BRST transformation,

$$\delta_{\text{BRST}}X^\mu = i\epsilon c^a \partial_a X^\mu \tag{B.1a}$$

$$\delta_{\text{BRST}}h_{ab} = 2i\epsilon\left(dh_{ab} + \nabla_{(a}c_{b)}\right) = 2i\epsilon dh_{ab} + i\epsilon(c^c\partial_c h_{ab} + h_{ac}\partial_b c^c + h_{bc}\partial_a c^c) \tag{B.1b}$$

while the BRST transformation of the non-matter fields can be obtained by the BRST procedure, which are[8],

$$\delta_{\text{BRST}}\left(\sqrt{h}B_{ab}\right) = 0 \tag{B.1c}$$

$$\delta_{\text{BRST}}\left(\sqrt{h}b_{ab}\right) = -\epsilon\sqrt{h}B_{ab} \tag{B.1d}$$

$$\delta_{\text{BRST}}d = i\epsilon c^a \partial_a d \tag{B.1e}$$

$$\delta_{\text{BRST}}c^a = i\epsilon c^b \nabla_b c^a = i\epsilon c^b \partial_b c^a \tag{B.1f}$$

the proof of nilpotency of these transformations is done in Appendix C. For the BRST quantization to be realized we need two things,

$$\delta_{\text{BRST}}(S_X + S_{\text{gh}} + S_{\text{gf}}) = 0$$
$$i\epsilon(S_{\text{gh}} + S_{\text{gf}}) = \delta_{\text{BRST}}\mathcal{O}$$

for $\mathcal{O}$ some composition of fields, we'll check both of these, and for this it'll be necessary to know the BRST transformations of a few more fields,

- h^{ab}

$$0 = \delta_{\text{BRST}}\delta_a{}^c$$
$$0 = \delta_{\text{BRST}}\left(h_{ab}h^{bc}\right)$$
$$0 = h_{ab}\delta_{\text{BRST}}h^{bc} + h^{bc}\delta_{\text{BRST}}h_{ab}$$
$$h_{ab}\delta_{\text{BRST}}h^{bc} = -2i\epsilon h^{bc}\left(dh_{ab} + \nabla_{(a}c_{b)}\right)$$
$$h^{da}h_{ab}\delta_{\text{BRST}}h^{bc} = -2i\epsilon h^{da}h^{bc}\left(dh_{ab} + \nabla_{(a}c_{b)}\right)$$
$$\delta^d{}_b\delta_{\text{BRST}}h^{bc} = -2i\epsilon\left(dh^{dc} + \nabla^{(d}c^{c)}\right)$$
$$\delta_{\text{BRST}}h^{dc} = -2i\epsilon\left(dh^{dc} + \nabla^{(d}c^{c)}\right) \tag{B.2a}$$

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[8]The first two equations might look a bit odd, but they are in fact a consequence of normalization of $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(i\int \mathrm{d}^2\sigma \sqrt{h}B(\sigma)f(\sigma)\right)$, instead of choosing $\delta(f(\sigma)) \propto \int \mathcal{D}B \exp\left(i\int \mathrm{d}^2\sigma B(\sigma)f(\sigma)\right)$.

- $\sqrt{h} = \sqrt{\mathrm{Det}[h_{ab}]}$

$$\delta_{\mathrm{BRST}}\sqrt{h} = \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\mathrm{Det}[h_{ab}])$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\exp\left(\ln\left(\mathrm{Det}[h_{ab}]\right)\right))$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = \frac{1}{2}\frac{1}{\sqrt{h}}\delta_{\mathrm{BRST}}(\exp\left(\mathrm{Tr}\left[\ln\left(h_{ab}\right)\right]\right))$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = \frac{1}{2}\frac{1}{\sqrt{h}}(\exp\left(\mathrm{Tr}\left[\ln\left(h_{ab}\right)\right]\right))\delta_{\mathrm{BRST}}(\mathrm{Tr}\left[\ln\left(h_{ab}\right)\right])$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = \frac{1}{2}\frac{1}{\sqrt{h}}h\,\mathrm{Tr}\left[\delta_{\mathrm{BRST}}(\ln\left(h_{ab}\right))\right]$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = \frac{1}{2}\sqrt{h}\,\mathrm{Tr}\left[h^{ca}\delta_{\mathrm{BRST}}h_{ab}\right]$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = \frac{1}{2}\sqrt{h}h^{ba}\delta_{\mathrm{BRST}}h_{ab}$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = i\epsilon\sqrt{h}h^{ba}\left(dh_{ab} + \nabla_{(a}c_{b)}\right)$$

$$\delta_{\mathrm{BRST}}\sqrt{h} = i\epsilon\sqrt{h}(2d + \nabla_a c^a) \tag{B.2b}$$

- $\sqrt{h}h^{ab}$

$$\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right) = \delta_{\mathrm{BRST}}\left(\sqrt{h}\right)h^{ab} + \sqrt{h}\delta_{\mathrm{BRST}}\left(h^{ab}\right)$$

$$\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right) = i\epsilon\sqrt{h}(2d + \nabla_c c^c)h^{ab} - 2i\epsilon\sqrt{h}\left(dh^{ab} + \nabla^{(a}c^{b)}\right)$$

$$\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right) = 2i\epsilon\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^{(a}c^{b)}\right) \tag{B.2c}$$

So now we can proceed to compute the two conditions, we'll start by the second one, luckily, the BRST procedure already provides a natural candidate for $\mathcal{O}$,

$$\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) = -\frac{1}{4\pi}\int d^2\sigma\,\delta_{\mathrm{BRST}}\left(\sqrt{h}b_{ab}\right)\left(\hat{h}^{ab} - h^{ab}\right)$$
$$-\frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}b_{ab}\delta_{\mathrm{BRST}}\hat{h}^{ab}$$
$$+\frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}b_{ab}\delta_{\mathrm{BRST}}\left(h^{ab}\right)$$

$$\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) = \epsilon\frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}B_{ab}\left(\hat{h}^{ab} - h^{ab}\right)$$
$$+ i\epsilon\frac{1}{2\pi}\int d^2\sigma\,\sqrt{h}b_{ab}(dh^{ab} + \nabla^a c^b)$$

$$\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) = i\epsilon S_{\mathrm{gf}} + i\epsilon S_{\mathrm{gh}}$$

$$\delta_{\mathrm{BRST}}\left(-\frac{1}{4\pi}\int d^2\sigma\,\sqrt{h}b_{ab}\left(\hat{h}^{ab} - h^{ab}\right)\right) = i\epsilon(S_{\mathrm{gf}} + S_{\mathrm{gh}})$$

so what remains to be *checked* is the first condition, actually we have to *impose* it, let's do it part by part, starting with,

$$\delta_{\mathrm{BRST}}S_X = \frac{1}{2\pi\alpha'}\int_M d^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu\partial_b\delta_{\mathrm{BRST}}X_\mu + \frac{1}{4\pi\alpha'}\int_M d^2\sigma\,\delta_{\mathrm{BRST}}\left(\sqrt{h}h^{ab}\right)\partial_a X^\mu\partial_b X_\mu$$

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$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu\partial_b[c^c\partial_c X_\mu] + \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^{(a}c^{b)}\right)\partial_a X^\mu\partial_b X_\mu$$

$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu((\nabla_b c^c)\partial_c X_\mu + c^c\nabla_b\nabla_c X_\mu)$$
$$+ \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^a c^b\right)\partial_a X^\mu\partial_b X_\mu$$

$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\partial_a X^\mu\partial_b X_\mu\nabla^a c^b + \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}c^c\partial_a X^\mu\nabla_c\nabla_b X_\mu$$
$$+ \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\left(\frac{1}{2}h^{ab}\nabla_c c^c - \nabla^a c^b\right)\partial_a X^\mu\partial_b X_\mu$$

$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{2\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}c^c\partial_a X^\mu\nabla_c\partial_b X_\mu + \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\nabla_c c^c\partial_a X^\mu\partial_b X_\mu$$

$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}c^c\nabla_c\big(h^{ab}\partial_a X^\mu\partial_b X_\mu\big) + \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\nabla_c c^c\partial_a X^\mu\partial_b X_\mu$$

$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}\nabla_c\big(c^c h^{ab}\partial_a X^\mu\partial_b X_\mu\big)$$

$$\delta_{\text{BRST}}S_X = \frac{\mathrm{i}\epsilon}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\partial_c\big(\sqrt{h}c^c h^{ab}\partial_a X^\mu\partial_b X_\mu\big) \tag{B.3}$$

now,

$$\delta_{\text{BRST}}S_{\text{gh}} = \delta_{\text{BRST}}\left(\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[h^{ab}d + \nabla^a c^b\big]\right)$$

$$\delta_{\text{BRST}}S_{\text{gh}} = \delta_{\text{BRST}}\left(\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\Big[h^{ab}d + \frac{1}{2}\big(-c^c\partial_c h^{ab} + h^{ac}\partial_c c^b + h^{bc}\partial_c c^a\big)\Big]\right)$$

$$\delta_{\text{BRST}}S_{\text{gh}} = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\delta_{\text{BRST}}\big(\sqrt{h}b_{ab}\big)\big[h^{ab}d + \nabla^a c^b\big]$$
$$+ \frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\delta_{\text{BRST}}\big(h^{ab}\big)d + h^{ab}\delta_{\text{BRST}}(d)\big]$$
$$- \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\delta_{\text{BRST}}(c^c)\partial_c h^{ab} + c^c\partial_c\delta_{\text{BRST}}\big(h^{ab}\big)\big]$$
$$+ \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\delta_{\text{BRST}}(h^{ac})\partial_c c^b + h^{ac}\partial_c\delta_{\text{BRST}}\big(c^b\big)\big]$$
$$+ \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\delta_{\text{BRST}}\big(h^{bc}\big)\partial_c c^a + h^{bc}\partial_c\delta_{\text{BRST}}(c^a)\big]$$

$$\delta_{\text{BRST}}S_{\text{gh}} = -\frac{\epsilon}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}B_{ab}\big[h^{ab}d + \nabla^a c^b\big]$$
$$+ \frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[-2\mathrm{i}\epsilon\big(dh^{ab} + \nabla^{(a}c^{b)}\big)d + h^{ab}\mathrm{i}\epsilon c^d\partial_d d\big]$$
$$- \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[\mathrm{i}\epsilon c^d\partial_d c^c\partial_c h^{ab} + 2\mathrm{i}\epsilon c^c\partial_c\big(dh^{ab} + \nabla^{(a}c^{b)}\big)\big]$$
$$+ \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{h}b_{ab}\big[-2\mathrm{i}\epsilon\big(dh^{ac} + \nabla^{(a}c^{c)}\big)\partial_c c^b + \mathrm{i}\epsilon h^{ac}\partial_c\big(c^d\partial_d c^b\big)\big]$$

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$$+ \frac{1}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[-2\mathrm{i}\epsilon \big(dh^{bc} + \nabla^{(b}c^{c)} \big) \partial_c c^a + \mathrm{i}\epsilon h^{bc} \partial_c \big(c^d \partial_d c^a \big) \big]$$

$$\delta_{\mathrm{BRST}} S_{\mathrm{gh}} = -\frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2\sigma \sqrt{h} B_{ab} (-\mathrm{i}\epsilon) \big[h^{ab} d + \nabla^a c^b \big]$$

$$+ \frac{\mathrm{i}\epsilon}{\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \nabla^{(a}c^{b)} d$$

$$+ \frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[c^d \partial_d c^c \partial_c h^{ab} + 2c^c \big(d\partial_c h^{ab} + \partial_c \nabla^{(a}c^{b)} \big) \big]$$

$$+ \frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[2 \big(dh^{ac} + \nabla^{(a}c^{c)} \big) \partial_c c^b - h^{ac} \partial_c \big(c^d \partial_d c^b \big) \big]$$

$$+ \frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[2 \big(dh^{bc} + \nabla^{(b}c^{c)} \big) \partial_c c^a - h^{bc} \partial_c \big(c^d \partial_d c^a \big) \big]$$

$$\delta_{\mathrm{BRST}} S_{\mathrm{gh}} = \frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2\sigma \sqrt{h} B_{ab} \delta_{\mathrm{BRST}} \big(\hat{h}^{ab} - h^{ab} \big)$$

$$+ \frac{\mathrm{i}\epsilon}{\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \nabla^{(a}c^{b)} d$$

$$+ \frac{\mathrm{i}\epsilon}{2\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} d \big[-c^c \partial_c h^{ab} + h^{ac} \partial_c c^b + h^{bc} \partial_c c^a \big]$$

$$+ \frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[c^d \partial_d c^c \partial_c h^{ab} - h^{bc} \partial_c \big(c^d \partial_d c^a \big) - h^{ac} \partial_c \big(c^d \partial_d c^b \big) \big]$$

$$+ \frac{\mathrm{i}\epsilon}{2\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[\nabla^{(a}c^{c)} \partial_c c^b + c^c \partial_c \nabla^{(a}c^{b)} + \nabla^{(b}c^{c)} \partial_c c^a \big]$$

$$\delta_{\mathrm{BRST}} S_{\mathrm{gh}} = -\delta_{\mathrm{BRST}} S_{\mathrm{gf}}$$

$$+ \frac{\mathrm{i}\epsilon}{\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \nabla^{(a}c^{b)} d$$

$$+ \frac{\mathrm{i}\epsilon}{\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} d \nabla^{(a}c^{b)}$$

$$- \frac{\mathrm{i}\epsilon}{2\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \nabla^{(a|} \big(c^d \nabla_d c^{|b)} \big)$$

$$+ \frac{\mathrm{i}\epsilon}{2\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[\nabla^{(a}c^{c)} \nabla_c c^b + c^c \nabla_c \nabla^{(a}c^{b)} + \nabla^{(b}c^{c)} \nabla_c c^a \big]$$

$$\delta_{\mathrm{BRST}} (S_{\mathrm{gh}} + S_{\mathrm{gf}}) = -\frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[\nabla^a c^d \nabla_d c^b + \nabla^b c^d \nabla_d c^a + c^d \nabla^a \nabla_d c^b + c^d \nabla^b \nabla_d c^a \big]$$

$$+ \frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[\nabla^a c^c \nabla_c c^b + \nabla^c c^a \nabla_c c^b + \nabla^b c^c \nabla_c c^a + \nabla^c c^b \nabla_c c^a \big]$$

$$+ \frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[c^c \nabla_c \nabla^a c^b + c^c \nabla_c \nabla^b c^a \big]$$

$$\delta_{\mathrm{BRST}} (S_{\mathrm{gh}} + S_{\mathrm{gf}}) = -\frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[c^c \nabla^a \nabla_c c^b + c^c \nabla^b \nabla_c c^a \big]$$

$$+ \frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} \big[c^c \nabla_c \nabla^a c^b + c^c \nabla_c \nabla^b c^a \big]$$

$$\delta_{\mathrm{BRST}} (S_{\mathrm{gh}} + S_{\mathrm{gf}}) = -\frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} c^c \big[R_d{}^{ba}{}_c + R_d{}^{ab}{}_c \big] c^d$$

$$\delta_{\mathrm{BRST}} (S_{\mathrm{gh}} + S_{\mathrm{gf}}) = -\frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} c^c c^d \big[R_d{}^{ba}{}_c + R_d{}^{ab}{}_c \big]$$

$$\delta_{\mathrm{BRST}} (S_{\mathrm{gh}} + S_{\mathrm{gf}}) = -\frac{\mathrm{i}\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} c^c c^d \big[R_d{}^{ba}{}_c - R_c{}^{ba}{}_d + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d \big]$$

$$\delta_{\text{BRST}}(S_{\text{gh}} + S_{\text{gf}}) = -\frac{i\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} c^c c^d \left[R^a{}_{cd}{}^b - R^a{}_{dc}{}^b + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d \right]$$

$$\delta_{\text{BRST}}(S_{\text{gh}} + S_{\text{gf}}) = -\frac{i\epsilon}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h} b_{ab} c^c c^d \left[R_c{}^{ab}{}_d - R_d{}^{ab}{}_c + R_d{}^{ab}{}_c - R_c{}^{ab}{}_d \right]$$

$$\delta_{\text{BRST}}(S_{\text{gh}} + S_{\text{gf}}) = 0$$

so that the imposition $\delta_{\text{BRST}}(S_X + S_{\text{gh}} + S_{\text{gf}}) = 0$, is equivalent to,

C   Closure of BRST transformations

Nilpotency of BRST,

$$\delta_1 X^\mu = \mathrm{i}\epsilon_1 c^a \partial_a X^\mu$$

$$\delta_2\delta_1 X^\mu = \mathrm{i}\epsilon_1 \delta_2(c^a)\partial_a X^\mu + \mathrm{i}\epsilon_1 c^a \partial_a(\delta_2(X^\mu))$$

$$\delta_2\delta_1 X^\mu = \mathrm{i}\epsilon_1(\mathrm{i}\epsilon_2)c^c\nabla_c(c^a)\partial_a X^\mu + \mathrm{i}\epsilon_1 c^a \partial_a\big((\mathrm{i}\epsilon_2 c^b\partial_b)X^\mu\big)$$

$$\delta_2\delta_1 X^\mu = -\epsilon_1\epsilon_2 c^c\nabla_c(c^a)\partial_a X^\mu + \epsilon_1\epsilon_2 c^a \partial_a\big(c^b\partial_b X^\mu\big)$$

$$\delta_2\delta_1 X^\mu = -\epsilon_1\epsilon_2 c^c\nabla_c(c^a)\nabla_a X^\mu + \epsilon_1\epsilon_2 c^a \nabla_a\big(c^b\nabla_b X^\mu\big)$$

$$\delta_2\delta_1 X^\mu = -\epsilon_1\epsilon_2 c^c\nabla_c(c^a)\nabla_a X^\mu + \epsilon_1\epsilon_2 c^a \nabla_a\big(c^b\big)\nabla_b X^\mu + \epsilon_1\epsilon_2 c^a c^b \nabla_a\nabla_b X^\mu$$

$$\delta_2\delta_1 X^\mu = \epsilon_1\epsilon_2 c^a c^b \nabla_a\nabla_b X^\mu, \quad \text{using the ghost statistics}$$

$$\delta_2\delta_1 X^\mu = \epsilon_1\epsilon_2 c^a c^b \nabla_{[a}\nabla_{b]} X^\mu, \quad \text{as } X^\mu \text{ is a world-sheet scalar: } \nabla_a\nabla_b X^\mu = \nabla_b\nabla_a X^\mu$$

$$\delta_2\delta_1 X^\mu = 0$$

$$\delta_1 h_{ab} = 2\mathrm{i}\epsilon_1 d h_{ab} + \mathrm{i}\epsilon_1(c^c\partial_c h_{ab} + h_{ac}\partial_b c^c + h_{bc}\partial_a c^c)$$

$$\begin{aligned}
\delta_2\delta_1 h_{ab} = {}& 2\mathrm{i}\epsilon_1\delta_2(d)h_{ab} + 2\mathrm{i}\epsilon_1 d\delta_2(h_{ab})\\
&+ \mathrm{i}\epsilon_1(\delta_2(c^c)\partial_c h_{ab} + c^c\partial_c\delta_2(h_{ab}))\\
&+ \mathrm{i}\epsilon_1(\delta_2(h_{ac})\partial_b c^c + h_{ac}\partial_b\delta_2(c^c))\\
&+ \mathrm{i}\epsilon_1(\delta_2(h_{bc})\partial_a c^c + h_{bc}\partial_a\delta_2(c^c))
\end{aligned}$$

$$\begin{aligned}
\delta_2\delta_1 h_{ab} = {}& 2\mathrm{i}\epsilon_1\mathrm{i}\epsilon_2 h_{ab}c^c\partial_c d + 2\mathrm{i}\epsilon_1 d 2\mathrm{i}\epsilon_2\big(dh_{ab} + \nabla_{(a}c_{b)}\big)\\
&+ \mathrm{i}\epsilon_1\big(\mathrm{i}\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} + c^c\partial_c\big(2\mathrm{i}\epsilon_2\big(dh_{ab} + \nabla_{(a}c_{b)}\big)\big)\big)\\
&+ \mathrm{i}\epsilon_1\big(2\mathrm{i}\epsilon_2\big(dh_{ac} + \nabla_{(a}c_{c)}\big)\partial_b c^c + h_{ac}\partial_b\big(\mathrm{i}\epsilon_2 c^d\partial_d c^c\big)\big)\\
&+ \mathrm{i}\epsilon_1\big(2\mathrm{i}\epsilon_2\big(dh_{bc} + \nabla_{(b}c_{c)}\big)\partial_a c^c + h_{bc}\partial_a\big(\mathrm{i}\epsilon_2 c^d\partial_d c^c\big)\big)
\end{aligned}$$

$$\begin{aligned}
\delta_2\delta_1 h_{ab} = {}& -2\epsilon_1\epsilon_2 h_{ab}c^c\partial_c d + 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)}\\
&- \epsilon_1\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} + 2\epsilon_1\epsilon_2 c^c\partial_c\big(dh_{ab} + \nabla_{(a}c_{b)}\big)\\
&- 2\epsilon_1\epsilon_2\big(dh_{ac} + \nabla_{(a}c_{c)}\big)\partial_b c^c - \epsilon_1\epsilon_2 h_{ac}\partial_b\big(c^d\partial_d c^c\big)\\
&- 2\epsilon_1\epsilon_2\big(dh_{bc} + \nabla_{(b}c_{c)}\big)\partial_a c^c - \epsilon_1\epsilon_2 h_{bc}\partial_a\big(c^d\partial_d c^c\big)
\end{aligned}$$

$$\begin{aligned}
\delta_2\delta_1 h_{ab} = {}& 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - \epsilon_1\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} + 2\epsilon_1\epsilon_2 c^c\big(d\partial_c h_{ab} + \partial_c\nabla_{(a}c_{b)}\big)\\
&- 2\epsilon_1\epsilon_2\big(dh_{ac} + \nabla_{(a}c_{c)}\big)\partial_b c^c - \epsilon_1\epsilon_2 h_{ac}\partial_b\big(c^d\partial_d c^c\big)\\
&- 2\epsilon_1\epsilon_2\big(dh_{bc} + \nabla_{(b}c_{c)}\big)\partial_a c^c - \epsilon_1\epsilon_2 h_{bc}\partial_a\big(c^d\partial_d c^c\big)
\end{aligned}$$

$$\begin{aligned}
\delta_2\delta_1 h_{ab} = {}& 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - 2\epsilon_1\epsilon_2 dc^c\partial_c h_{ab} - 2\epsilon_1\epsilon_2 dh_{ac}\partial_b c^c - 2\epsilon_1\epsilon_2 dh_{bc}\partial_a c^c\\
&- \epsilon_1\epsilon_2 c^d\partial_d c^c\partial_c h_{ab} - \epsilon_1\epsilon_2 h_{bc}\partial_a\big(c^d\partial_d c^c\big) - \epsilon_1\epsilon_2 h_{ac}\partial_b\big(c^d\partial_d c^c\big)\\
&- 2\epsilon_1\epsilon_2 \partial_c\nabla_{(a}c_{b)}c^c - 2\epsilon_1\epsilon_2 \nabla_{(a}c_{c)}\partial_b c^c - 2\epsilon_1\epsilon_2 \nabla_{(b}c_{c)}\partial_a c^c
\end{aligned}$$

$$\begin{aligned}
\delta_2\delta_1 h_{ab} = {}& 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - 2\epsilon_1\epsilon_2 d[c^c\partial_c h_{ab} + h_{ac}\partial_b c^c + h_{bc}\partial_a c^c]\\
&- \epsilon_1\epsilon_2\big[c^d\partial_d c^c\partial_c h_{ab} + h_{bc}\partial_a\big(c^d\partial_d c^c\big) + h_{ac}\partial_b\big(c^d\partial_d c^c\big)\big]\\
&- 2\epsilon_1\epsilon_2\big[\partial_c\nabla_{(a}c_{b)}c^c + \nabla_{(a}c_{c)}\partial_b c^c + \nabla_{(b}c_{c)}\partial_a c^c\big]
\end{aligned}$$

$$\begin{aligned}
\delta_2\delta_1 h_{ab} = {}& 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)} - 4\epsilon_1\epsilon_2 d\nabla_{(a}c_{b)}\\
&- 2\epsilon_1\epsilon_2 \nabla_{(a|}\big(c^c\nabla_c c_{|b)}\big)\\
&- 2\epsilon_1\epsilon_2\big[\nabla_c\nabla_{(a}c_{b)}c^c + \nabla_{(a}c_{c)}\nabla_b c^c + \nabla_{(b}c_{c)}\nabla_a c^c\big]
\end{aligned}$$

$$\delta_2\delta_1 h_{ab} = -2\epsilon_1\epsilon_2\big[\nabla_{(a|}\big(c^c\nabla_c c_{|b)}\big) + \nabla_c\nabla_{(a}c_{b)}c^c + \nabla_{(a}c_{c)}\nabla_b c^c + \nabla_{(b}c_{c)}\nabla_a c^c\big]$$

$$\delta_2\delta_1 h_{ab} = -\epsilon_1\epsilon_2 \left[\nabla_a(c^c\nabla_c c_b) + \nabla_b(c^c\nabla_c c_a) + \nabla_c\nabla_a c_b c^c + \nabla_c\nabla_b c_a c^c \right.$$
$$\left. + \nabla_a c_c \nabla_b c^c + \nabla_b c_c \nabla_a c^c + \nabla_c c_a \nabla_b c^c + \nabla_c c_b \nabla_a c^c \right]$$

$$\delta_2\delta_1 h_{ab} = -\epsilon_1\epsilon_2 \left[c^c\nabla_a\nabla_c c_b + c^c\nabla_b\nabla_c c_a + \nabla_c\nabla_a c_b c^c + \nabla_c\nabla_b c_a c^c \right.$$
$$\left. + \nabla_a c^c\nabla_c c_b + \nabla_b c^c\nabla_c c_a + \nabla_a c_c\nabla_b c^c + \nabla_b c_c\nabla_a c^c + \nabla_c c_a\nabla_b c^c + \nabla_c c_b\nabla_a c^c \right]$$

$$\delta_2\delta_1 h_{ab} = -\epsilon_1\epsilon_2 \left[c^c\nabla_a\nabla_c c_b + c^c\nabla_b\nabla_c c_a - c^c\nabla_c\nabla_a c_b - c^c\nabla_c\nabla_b c_a \right.$$
$$\left. + \nabla_a c^c\nabla_c c_b + \nabla_b c^c\nabla_c c_a - \nabla_b c^c\nabla_c c_a - \nabla_a c^c\nabla_c c_b \right]$$

$$\delta_2\delta_1 h_{ab} = -\epsilon_1\epsilon_2 \left[c^c\nabla_a\nabla_c c_b + c^c\nabla_b\nabla_c c_a - c^c\nabla_c\nabla_a c_b - c^c\nabla_c\nabla_b c_a \right.$$
$$\left. + \nabla_a c^c\nabla_c c_b + \nabla_b c^c\nabla_c c_a - \nabla_b c^c\nabla_c c_a - \nabla_a c^c\nabla_c c_b \right]$$

$$\delta_2\delta_1 h_{ab} = -\epsilon_1\epsilon_2 c^c \left[R^d{}_{bac}c_d + R^d{}_{abc}c_d \right]$$

$$\delta_2\delta_1 h_{ab} = -\epsilon_1\epsilon_2 c^c c^d \left[R_{dbac} + R_{dabc} \right]$$

$$\delta_2\delta_1 h_{ab} = -\frac{1}{2}\epsilon_1\epsilon_2 c^c c^d \left[R_{dbac} - R_{cbad} + R_{dabc} - R_{cabd} \right]$$

$$\delta_2\delta_1 h_{ab} = -\frac{1}{2}\epsilon_1\epsilon_2 c^c c^d \left[R_{dbac} - R_{cbad} + R_{bcda} - R_{bdca} \right]$$

$$\delta_2\delta_1 h_{ab} = -\frac{1}{2}\epsilon_1\epsilon_2 c^c c^d \left[R_{dbac} - R_{cbad} + R_{cbad} - R_{dbac} \right]$$

$$\delta_2\delta_1 h_{ab} = 0$$

$$\delta_2\Gamma^d_{be} = \delta_2\left\{ \frac{1}{2}h^{df}(\partial_b h_{ef} + \partial_e h_{bf} - \partial_f h_{be}) \right\}$$

$$\delta_2\Gamma^d_{be} = -i\epsilon_2\left(dh^{df} + \nabla^{(d}c^{f)}\right)(\partial_b h_{ef} + \partial_e h_{bf} - \partial_f h_{be})$$
$$+ i\epsilon_2 h^{df}\left(\partial_b\left[dh_{ef} + \nabla_{(e}c_{f)}\right] + \partial_e\left[dh_{bf} + \nabla_{(b}c_{f)}\right] - \partial_f\left[dh_{be} + \nabla_{(b}c_{e)}\right] \right)$$

$$\delta_2\Gamma^d_{be} = -i\epsilon_2\nabla^{(d}c^{f)}(\partial_b h_{ef} + \partial_e h_{bf} - \partial_f h_{be})$$
$$+ i\epsilon_2 h^{df}\left(h_{ef}\partial_b d + \partial_b\nabla_{(e}c_{f)} + h_{bf}\partial_e d + \partial_e\nabla_{(b}c_{f)} - h_{be}\partial_f d - \partial_f\nabla_{(b}c_{e)} \right)$$

$$h_{da}\delta_2\Gamma^d_{be} = -i\epsilon_2 h_{da}\nabla^{(d}c^{f)}(\partial_b h_{ef} + \partial_e h_{bf} - \partial_f h_{be})$$
$$+ i\epsilon_2 h_{da}h^{df}\left(h_{ef}\partial_b d + \partial_b\nabla_{(e}c_{f)} + h_{bf}\partial_e d + \partial_e\nabla_{(b}c_{f)} - h_{be}\partial_f d - \partial_f\nabla_{(b}c_{e)} \right)$$

$$h_{da}\delta_2\Gamma^d_{be} = -i\epsilon_2 h_{da}\nabla^{(d}c^{f)}(\partial_b h_{ef} + \partial_e h_{bf} - \partial_f h_{be})$$
$$+ i\epsilon_2\left(h_{ea}\partial_b d + \partial_b\nabla_{(a}c_{e)} + h_{ba}\partial_e d + \partial_e\nabla_{(b}c_{a)} - h_{eb}\partial_a d - \partial_a\nabla_{(b}c_{e)} \right)$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = -i\epsilon_2 h_{d(a}\nabla^{(d}c^{f)}\left(\partial_{b)}h_{ef} + \partial_e h_{b)f} - \partial_f h_{b)e}\right)$$
$$+ i\epsilon_2\left(h_{ba}\partial_e d + \partial_e\nabla_{(b}c_{a)} \right)$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = -i\epsilon_2\nabla_{(c}c_{(a)}h^{fc}\left(\partial_{b)}h_{ef} + \partial_e h_{b)f} - \partial_f h_{b)e}\right)$$
$$+ i\epsilon_2\left(h_{ba}\partial_e d + \partial_e\nabla_{(b}c_{a)} \right)$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = -2i\epsilon_2\nabla_{(c}c_{(a)}\Gamma^c_{b)e} + i\epsilon_2\left(h_{ba}\partial_e d + \partial_e\nabla_{(b}c_{a)} \right)$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = -i\epsilon_2\left[\nabla_c c_{(a} + \nabla_{(a}c_c \right]\Gamma^c_{b)e} + i\epsilon_2\left(h_{ba}\partial_e d + \partial_e\nabla_{(b}c_{a)} \right)$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = -\frac{1}{2}i\epsilon_2[\nabla_c c_a + \nabla_a c_c]\Gamma^c_{be} - \frac{1}{2}i\epsilon_2[\nabla_c c_b + \nabla_b c_c]\Gamma^c_{ae} + \frac{1}{2}i\epsilon_2\partial_e\nabla_b c_a + \frac{1}{2}i\epsilon_2\partial_e\nabla_a c_b$$
$$+ i\epsilon_2 h_{ba}\partial_e d$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = \frac{1}{2}i\epsilon_2[\partial_e\nabla_a c_b - \Gamma^c_{be}\nabla_a c_c - \Gamma^c_{ae}\nabla_c c_b] + \frac{1}{2}i\epsilon_2[\partial_e\nabla_b c_a - \Gamma^c_{be}\nabla_c c_a - \Gamma^c_{ae}\nabla_b c_c]$$
$$+ i\epsilon_2 h_{ba}\partial_e d$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = \frac{1}{2}i\epsilon_2\nabla_e\nabla_a c_b + \frac{1}{2}i\epsilon_2\nabla_e\nabla_b c_a + i\epsilon_2 h_{ba}\partial_e d$$

$$h_{d(a}\delta_2\Gamma^d_{b)e} = \mathrm{i}\epsilon_2\nabla_e\nabla_{(a}c_{b)} + \mathrm{i}\epsilon_2 h_{ba}\partial_e d$$