TESTE GQ

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1. Introdução

O Espaço-Tempo (A)dS é definido como,

$$\mp\left(x^{-1}\right)^2 - \left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = \mp L^2$$

Onde o sinal de cima é para AdS, e o sinal de baixo para dS. A álgebra de isometria no embedding 5 dimensional é:

$$\left[J^{IK}, J^{LM}\right] = -4\mathrm{i}\eta^{[I|[L}J^{M]|K]}$$
$$\left[J^{IK}, J^{LM}\right] = -\mathrm{i}\eta^{IL}J^{MK} + \mathrm{i}\eta^{IM}J^{LK} + \mathrm{i}\eta^{KL}J^{MI} - \mathrm{i}\eta^{KM}J^{LI}$$

Da qual podemos interpretar como geradores de translação $J^{-1\alpha}$,

$$\left[J^{-1\alpha}, J^{-1\beta}\right] = -\mathrm{i}\eta^{-1-1}J^{\beta\alpha} + \mathrm{i}\eta^{-1\beta}J^{-1\alpha} + \mathrm{i}\eta^{\alpha-1}J^{\beta-1} - \mathrm{i}\eta^{\alpha\beta}J^{-1-1}$$
$$\left[J^{-1\alpha}, J^{-1\beta}\right] = \mp\mathrm{i}J^{\alpha\beta}$$

As outras relações de comutação são,

$$\left[J^{\alpha\beta}, J^{-1\mu}\right] = -\mathrm{i}\eta^{\alpha-1}J^{\mu\beta} + \mathrm{i}\eta^{\alpha\mu}J^{-1\beta} + \mathrm{i}\eta^{\beta-1}J^{\mu\alpha} - \mathrm{i}\eta^{\beta\mu}J^{-1\alpha}$$
$$\left[J^{\alpha\beta}, J^{-1\mu}\right] = \mathrm{i}\eta^{\alpha\mu}J^{-1\beta} - \mathrm{i}\eta^{\beta\mu}J^{-1\alpha}$$
$$\left[J^{\alpha\beta}, J^{-1\mu}\right] = -2\mathrm{i}J^{-1[\alpha}\eta^{\beta]\mu}$$

E,

$$\left[J^{\alpha\beta}, J^{\mu\nu}\right] = -4\mathrm{i}\eta^{[\alpha|[\mu}J^{\nu]|\beta]}$$

Como os geradores de translação necessitam ter dimensão, $P^\alpha = \frac{1}{L}J^{-1\alpha}$. A álgebra completa é,

$$\left[J^{\alpha\beta}, J^{\mu\nu}\right] = -4\mathrm{i}\eta^{[\alpha|[\mu}J^{\nu]|\beta]}, \quad \left[J^{\alpha\beta}, P^{\mu}\right] = -2\mathrm{i}P^{[\alpha}\eta^{\beta]\mu}, \quad \left[P^\alpha, P^\beta\right] = \mp\frac{\mathrm{i}}{L^2}J^{\alpha\beta}$$

O bilinear mais geral para essa álgebra é,

$$\langle J_{\alpha\beta}, J_{\mu\nu}\rangle = \pm 2\lambda\eta_{\alpha[\mu}\eta_{\nu]\beta}, \quad \langle J_{\alpha\beta}, P_\mu\rangle = 0, \quad \langle P_\alpha, P_\mu\rangle = \frac{\lambda}{L^2}\eta_{\alpha\mu}$$

A ação de Einstein-Hilbert com termo de constante cosmológica é,

$$S_{\text{EH}} = \frac{1}{2\kappa}\int\limits_M \star\mathbf{R}_{\alpha\beta}\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta - \frac{\Lambda}{\kappa 4!}\int\limits_M \epsilon_{\alpha\beta\mu\nu}\mathbf{e}^\alpha\wedge\mathbf{e}^\beta\wedge\mathbf{e}^\mu\wedge\mathbf{e}^\nu$$
$$S_{\text{EH}} = \frac{1}{2\kappa}\eta_{\mu\alpha}\eta_{\beta\nu}\int\limits_M \star\mathbf{R}^{\mu\nu}\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta \pm \frac{3\cdot 2}{\kappa L^2 4!}\int\limits_M \mathbf{e}^\alpha\wedge\mathbf{e}^\beta\wedge\star(\mathbf{e}_\alpha\wedge\mathbf{e}_\beta)$$
$$S_{\text{EH}} = \frac{\pm}{4\lambda\kappa}\pm 2\lambda\eta_{\mu[\alpha}\eta_{\beta]\nu}\int\limits_M \star\mathbf{R}^{\mu\nu}\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta \pm \frac{3\cdot 2}{\lambda\kappa L^2 4!}\lambda\eta_{\mu[\alpha}\eta_{\beta]\nu}\int\limits_M \mathbf{e}^\mu\wedge\mathbf{e}^\nu\wedge\star\left(\mathbf{e}^\alpha\wedge\mathbf{e}^\beta\right)$$
$$S_{\text{EH}} = \frac{\pm}{4\lambda\kappa}\langle J_{\mu\nu}, J_{\alpha\beta}\rangle\int\limits_M \star\mathbf{R}^{\mu\nu}\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta + \frac{3}{\lambda\kappa L^2 4!}\langle J_{\mu\nu}, J_{\alpha\beta}\rangle\int\limits_M \mathbf{e}^\mu\wedge\mathbf{e}^\nu\wedge\star\left(\mathbf{e}^\alpha\wedge\mathbf{e}^\beta\right)$$
$$S_{\text{EH}} = \frac{\pm}{4\lambda\kappa}\pm\mathrm{i}L^2\langle J_{\mu\nu}, [P_\alpha, P_\beta]\rangle\int\limits_M \star\mathbf{R}^{\mu\nu}\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta + \frac{3(\pm)^2\mathrm{i}^2L^4}{\lambda\kappa L^2 4!}\langle[P_\mu, P_\nu], [P_\alpha, P_\beta]\rangle\int\limits_M \mathbf{e}^\mu\wedge\mathbf{e}^\nu\wedge\star\left(\mathbf{e}^\alpha\wedge\mathbf{e}^\beta\right)$$
$$S_{\text{EH}} = -\frac{L^2}{4\lambda\kappa}\langle\mathrm{i}J_{\mu\nu}, [\mathrm{i}P_\alpha, \mathrm{i}P_\beta]\rangle\int\limits_M \star\mathbf{R}^{\mu\nu}\wedge\mathbf{e}^\alpha\wedge\mathbf{e}^\beta - \frac{L^2}{8\lambda\kappa}\langle[\mathrm{i}P_\mu, \mathrm{i}P_\nu], [\mathrm{i}P_\alpha, \mathrm{i}P_\beta]\rangle\int\limits_M \mathbf{e}^\mu\wedge\mathbf{e}^\nu\wedge\star\left(\mathbf{e}^\alpha\wedge\mathbf{e}^\beta\right)$$
$$S_{\text{EH}} = -\frac{L^2}{2\lambda\kappa}\int\limits_M \langle\star\mathbf{R}\overset{\wedge}{,}[\mathbf{e}\overset{\wedge}{,}\mathbf{e}]\rangle - \frac{L^2}{8\lambda\kappa}\int\limits_M \langle[\mathbf{e}\overset{\wedge}{,}\mathbf{e}]\overset{\wedge}{,}\star[\mathbf{e}\overset{\wedge}{,}\mathbf{e}]\rangle$$

Com $\mathbf{R} = \frac{1}{2}\mathrm{i}J_{\mu\nu}\mathbf{R}^{\mu\nu}$ e $\mathbf{e} = \mathrm{i}P_\mu\mathbf{e}^\mu$. Note,

$$\mathbf{R} = \frac{1}{2}\mathrm{i}\mathbf{R}_{\alpha\beta}J^{\alpha\beta} = \frac{1}{2}\mathbf{d}\boldsymbol{\omega}_{\alpha\beta}\mathrm{i}J^{\alpha\beta} + \frac{1}{2}\boldsymbol{\omega}_\alpha{}^\rho\wedge\boldsymbol{\omega}_{\rho\beta}\mathrm{i}J^{\alpha\beta} = \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}_{\mu\nu}\wedge\boldsymbol{\omega}_{\rho\sigma}\eta^{\rho\nu}\mathrm{i}J^{\mu\sigma} = \mathbf{d}\boldsymbol{\omega} + \frac{1}{8}\boldsymbol{\omega}_{\mu\nu}\wedge\boldsymbol{\omega}_{\rho\sigma}4\mathrm{i}\eta^{[\rho|[\nu}J^{\mu]|\sigma]}$$

$$\mathbf{R} = \mathbf{d}\boldsymbol{\omega} - \frac{1}{8}\boldsymbol{\omega}_{\mu\nu} \wedge \boldsymbol{\omega}_{\rho\sigma}[J^{\rho\sigma}, J^{\nu\mu}] = \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}_{\mu\nu} \wedge \boldsymbol{\omega}_{\rho\sigma}\left[\frac{\mathrm{i}}{2}J^{\mu\nu}, \frac{\mathrm{i}}{2}J^{\rho\sigma}\right] = \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]$$

Seja então,

$$\begin{aligned}
\mathbf{F} &= \mathbf{d}(\boldsymbol{\omega} + \mathbf{e}) + \frac{1}{2}[\boldsymbol{\omega} + \mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega} + \mathbf{e}] \\
\mathbf{F} &= \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}] + \frac{1}{2}[\mathbf{e} \overset{\wedge}{,} \mathbf{e}] + \mathbf{d}\mathbf{e} + [\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{e}] \\
\mathbf{F} &= \mathbf{R} + \frac{1}{2}[\mathbf{e} \overset{\wedge}{,} \mathbf{e}] + \mathbf{T}
\end{aligned}$$

Logo,

$$\begin{aligned}
\tilde{S} &= \int_M \langle \mathbf{F} \overset{\wedge}{,} \star \mathbf{F} \rangle \\
\tilde{S} &= \int_M \langle \mathbf{R} \overset{\wedge}{,} \star \mathbf{R} \rangle + \frac{1}{2}\int_M \langle \mathbf{R} \overset{\wedge}{,} \star [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \rangle + \frac{1}{2}\int_M \langle [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \overset{\wedge}{,} \star \mathbf{R} \rangle + \int_M \langle \mathbf{T} \overset{\wedge}{,} \star \mathbf{T} \rangle + \frac{1}{4}\int_M \langle [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \overset{\wedge}{,} \star [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \rangle \\
\tilde{S} &= \int_M \langle \mathbf{R} \overset{\wedge}{,} \star \mathbf{R} \rangle + \int_M \langle \mathbf{R} \overset{\wedge}{,} \star [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \rangle + \frac{1}{4}\int_M \langle [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \overset{\wedge}{,} \star [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \rangle + \int_M \langle \mathbf{T} \overset{\wedge}{,} \star \mathbf{T} \rangle
\end{aligned}$$

Isto é, a ação de Yang-Mills para o grupo de isometria de (A)dS é a ação de Einstein-Hilbert com os termos adicionais do tensor de Riemann quadrado e o tensor de torção quadrado. A equação de movimento para $\boldsymbol{\omega}$ é,

$$\begin{aligned}
0 = \delta_{\boldsymbol{\omega}}\tilde{S} &= 2\int_M \langle \delta_{\boldsymbol{\omega}}\mathbf{R} \overset{\wedge}{,} \star \mathbf{R} \rangle + \int_M \langle \delta_{\boldsymbol{\omega}}\mathbf{R} \overset{\wedge}{,} \star [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] \rangle + 2\int_M \langle \delta_{\boldsymbol{\omega}}\mathbf{T} \overset{\wedge}{,} \star \mathbf{T} \rangle \\
0 &= 2\int_M \left\langle \mathbf{d}\delta\boldsymbol{\omega} + [\delta\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}] \overset{\wedge}{,} \star \left(\mathbf{R} + \frac{1}{2}[\mathbf{e} \overset{\wedge}{,} \mathbf{e}]\right) \right\rangle + 2\int_M \langle [\delta\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{e}] \overset{\wedge}{,} \star \mathbf{T} \rangle \\
0 &= \int_M \left\langle \delta\boldsymbol{\omega} \overset{\wedge}{,} \mathbf{d} \star \left(\mathbf{R} + \frac{1}{2}[\mathbf{e} \overset{\wedge}{,} \mathbf{e}]\right) + \left[\boldsymbol{\omega} \overset{\wedge}{,} \star \left(\mathbf{R} + \frac{1}{2}[\mathbf{e} \overset{\wedge}{,} \mathbf{e}]\right)\right] + [\mathbf{e} \overset{\wedge}{,} \star \mathbf{T}] \right\rangle \\
0 &= \mathbf{d}_{\nabla} \star \mathbf{R} + \frac{1}{2}\mathbf{d}_{\nabla} \star [\mathbf{e} \overset{\wedge}{,} \mathbf{e}] + [\mathbf{e} \overset{\wedge}{,} \star \mathbf{d}_{\nabla}\mathbf{e}]
\end{aligned}$$

2. Transformações

A transformação de calibre mais geral é,

$$\mathbf{F} \to \mathrm{U}\mathbf{F}\mathrm{U}^{-1}, \quad \mathrm{U} = \exp\left(-\mathrm{i}a_\mu P^\mu + \frac{\mathrm{i}}{2}\theta_{\mu\nu}J^{\mu\nu}\right)$$

Que é gerada pela transformação,

$$\boldsymbol{\omega} + \mathbf{e} \to \mathrm{U}(\boldsymbol{\omega} + \mathbf{e} + \mathbf{d})\mathrm{U}^{-1}$$

Infinitesimalmente,

$$\begin{aligned}
\mathrm{U}(\boldsymbol{\omega} + \mathbf{e} + \mathbf{d})\mathrm{U}^{-1} &= \left(\mathbb{1} - \mathrm{i}a_\mu P^\mu + \frac{\mathrm{i}}{2}\theta_{\mu\nu}J^{\mu\nu}\right)(\boldsymbol{\omega} + \mathbf{e} + \mathbf{d})\left(\mathbb{1} + \mathrm{i}a_\mu P^\mu - \frac{\mathrm{i}}{2}\theta_{\mu\nu}J^{\mu\nu}\right) \\
\boldsymbol{\omega}' + \mathbf{e}' &= \boldsymbol{\omega} + \mathbf{e} - \mathrm{i}a_\mu[P^\mu, \boldsymbol{\omega}] + \frac{\mathrm{i}}{2}\theta_{\mu\nu}[J^{\mu\nu}, \boldsymbol{\omega}] - \mathrm{i}a_\mu[P^\mu, \mathbf{e}] + \frac{\mathrm{i}}{2}\theta_{\mu\nu}[J^{\mu\nu}, \mathbf{e}] + \mathrm{i}\mathbf{d}a_\mu P^\mu - \frac{\mathrm{i}}{2}\mathbf{d}\theta_{\mu\nu}J^{\mu\nu} \\
\boldsymbol{\omega}' + \mathbf{e}' &= \boldsymbol{\omega} + \mathbf{e} - \mathrm{i}a_\mu\frac{\mathrm{i}}{2}\boldsymbol{\omega}_{\alpha\beta}\left[P^\mu, J^{\alpha\beta}\right] + \frac{\mathrm{i}}{2}\theta_{\mu\nu}\frac{\mathrm{i}}{2}\boldsymbol{\omega}_{\alpha\beta}\left[J^{\mu\nu}, J^{\alpha\beta}\right] - \mathrm{i}a_\mu\mathrm{i}\mathbf{e}_\alpha[P^\mu, P^\alpha] + \frac{\mathrm{i}}{2}\theta_{\mu\nu}\mathrm{i}\mathbf{e}_\alpha[J^{\mu\nu}, P^\alpha] + \mathrm{i}\mathbf{d}a_\mu P^\mu - \frac{\mathrm{i}}{2}\mathbf{d}\theta_{\mu\nu}J^{\mu\nu} \\
\boldsymbol{\omega}' + \mathbf{e}' &= \boldsymbol{\omega} + \mathbf{e} - \mathrm{i}a_\mu\frac{\mathrm{i}}{2}\boldsymbol{\omega}_{\alpha\beta}2\mathrm{i}P^{[\alpha}\eta^{\beta]\mu} - \frac{\mathrm{i}}{2}\theta_{\mu\nu}\frac{\mathrm{i}}{2}\boldsymbol{\omega}_{\alpha\beta}4\mathrm{i}\eta^{\mu[\alpha}J^{\beta]\nu} \pm \mathrm{i}a_\mu\mathrm{i}\mathbf{e}_\alpha\frac{\mathrm{i}}{L^2}J^{\mu\alpha} - \frac{\mathrm{i}}{2}\theta_{\mu\nu}\mathrm{i}\mathbf{e}_\alpha 2\mathrm{i}P^{[\mu}\eta^{\nu]\alpha} + \mathrm{i}\mathbf{d}a_\alpha P^\alpha - \frac{\mathrm{i}}{2}\mathbf{d}\theta_{\alpha\beta}J^{\alpha\beta} \\
\boldsymbol{\omega}' + \mathbf{e}' &= \boldsymbol{\omega} + \mathbf{e} + \mathrm{i}\boldsymbol{\omega}_{\alpha\beta}P^{[\alpha}a^{\beta]} + \mathrm{i}\theta^\mu{}_\beta\boldsymbol{\omega}_{\mu\alpha}J^{\alpha\beta} \mp \frac{\mathrm{i}}{L^2}a_\alpha\mathbf{e}_\beta J^{\alpha\beta} + \mathrm{i}\theta_{\alpha\beta}P^{[\alpha}\mathbf{e}^{\beta]} + \mathrm{i}\mathbf{d}a_\alpha P^\alpha - \frac{\mathrm{i}}{2}\mathbf{d}\theta_{\alpha\beta}J^{\alpha\beta} \\
\frac{\mathrm{i}}{2}\boldsymbol{\omega}'_{\alpha\beta}J^{\alpha\beta} + \mathrm{i}\mathbf{e}'_\alpha P^\alpha &= \boldsymbol{\omega} + \mathbf{e} + \mathrm{i}\boldsymbol{\omega}_{\alpha\beta}P^{[\alpha}a^{\beta]} + \mathrm{i}\theta^\mu{}_\beta\boldsymbol{\omega}_{\mu\alpha}J^{\alpha\beta} \mp \frac{\mathrm{i}}{L^2}a_\alpha\mathbf{e}_\beta J^{\alpha\beta} + \mathrm{i}\theta_{\alpha\beta}P^{[\alpha}\mathbf{e}^{\beta]} + \mathrm{i}\mathbf{d}a_\alpha P^\alpha - \frac{\mathrm{i}}{2}\mathbf{d}\theta_{\alpha\beta}J^{\alpha\beta}
\end{aligned}$$

Logo,

$$\begin{aligned}
\boldsymbol{\omega}'_{\alpha\beta} &= \boldsymbol{\omega}_{\alpha\beta} - 2\boldsymbol{\omega}_{[\alpha|\mu}\theta^\mu{}_{|\beta]} \mp \frac{2}{L^2}a_{[\alpha}\mathbf{e}_{\beta]} - \mathbf{d}\theta_{\alpha\beta} \\
\mathbf{e}'_\alpha &= \mathbf{e}_\alpha + \boldsymbol{\omega}_{\alpha\beta}a^\beta + \theta_{\alpha\beta}\mathbf{e}^\beta + \mathbf{d}a_\alpha
\end{aligned}$$

A transformação associada com θ são apenas boosts e rotações, o que realmente nos interessa são as transformações parametrizadas por a. Essas têm de ser relacionadas aos difeomorfismos. Para isso olhemos,

$$\pounds_\xi\mathbf{e}_\alpha = \mathbf{d}(\mathbf{e}_\alpha(\xi)) + (\mathbf{d}\mathbf{e}_\alpha)(\xi)$$

$$\pounds_\xi \mathbf{e}_\alpha = \mathbf{d}\xi_\alpha + \big(\mathbf{d}\mathbf{e}_\alpha + \boldsymbol{\omega}_\alpha{}^\beta \wedge \mathbf{e}_\beta\big)(\xi) - \big(\boldsymbol{\omega}_\alpha{}^\beta \wedge \mathbf{e}_\beta\big)(\xi)$$
$$\pounds_\xi \mathbf{e}_\alpha = \mathbf{d}\xi_\alpha + \mathbf{T}_\alpha(\xi) - \big(\boldsymbol{\omega}_\alpha{}^\beta \wedge \mathbf{e}_\beta\big)(\xi)$$
$$\pounds_\xi \mathbf{e}_\alpha = \mathbf{d}\xi_\alpha + \mathbf{T}_\alpha(\xi) - \boldsymbol{\omega}_\alpha{}^\beta(\xi)\mathbf{e}_\beta + \boldsymbol{\omega}_\alpha{}^\beta \xi_\beta$$

E também,

$$\pounds_\xi \boldsymbol{\omega}_{\alpha\beta} = \mathbf{d}(\boldsymbol{\omega}_{\alpha\beta}(\xi)) + (\mathbf{d}\boldsymbol{\omega}_{\alpha\beta})(\xi)$$
$$\pounds_\xi \boldsymbol{\omega}_{\alpha\beta} = -\mathbf{d}(-\boldsymbol{\omega}_{\alpha\beta}(\xi)) + \big(\mathbf{d}\boldsymbol{\omega}_{\alpha\beta} + \boldsymbol{\omega}_\alpha{}^\mu \wedge \boldsymbol{\omega}_{\mu\beta}\big)(\xi) - \big(\boldsymbol{\omega}_\alpha{}^\mu \wedge \boldsymbol{\omega}_{\mu\beta}\big)(\xi)$$
$$\pounds_\xi \boldsymbol{\omega}_{\alpha\beta} = -\mathbf{d}(-\boldsymbol{\omega}_{\alpha\beta}(\xi)) + \mathbf{R}_{\alpha\beta}(\xi) - \boldsymbol{\omega}_\alpha{}^\mu(\xi)\boldsymbol{\omega}_{\mu\beta} + \boldsymbol{\omega}_\alpha{}^\mu \boldsymbol{\omega}_{\mu\beta}(\xi)$$
$$\pounds_\xi \boldsymbol{\omega}_{\alpha\beta} = -\mathbf{d}(-\boldsymbol{\omega}_{\alpha\beta}(\xi)) + \mathbf{R}_{\alpha\beta}(\xi) - 2\boldsymbol{\omega}_{[\alpha|}{}^\mu(-)\boldsymbol{\omega}_{\mu|\beta]}(\xi)$$

Isso impõe duas condições,

$$\mathbf{T}_\alpha = 0, \quad \mathbf{R}_{\alpha\beta}(\xi) = \mp\frac{2}{L^2}\xi_{[\alpha}\mathbf{e}_{\beta]}$$

$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right] = -a_\mu\left[J^{\alpha\beta}, P^\mu\right]$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right] = 2\mathrm{i}a_\mu P^{[\alpha}\eta^{\beta]\mu} = 2\mathrm{i}P^{[\alpha}a^{\beta]}$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_2 = \left[2\mathrm{i}P^{[\alpha}a^{\beta]}, \mathrm{i}a_\mu P^\mu\right]$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_2 = -2a_\mu a^{[\beta}\left[P^{\alpha]}, P^\mu\right]$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_2 = \pm 2\frac{\mathrm{i}}{L^2}a_\mu a^{[\beta}J^{\alpha]\mu}$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_3 = \left[\pm 2\frac{\mathrm{i}}{L^2}a_\nu a^{[\beta}J^{\alpha]\nu}, \mathrm{i}a_\mu P^\mu\right]$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_3 = \pm\frac{\mathrm{i}}{L^2}a_\mu a_\nu\big(a^\beta P^\alpha \eta^{\nu\mu} - a^\beta P^\nu \eta^{\alpha\mu} - a^\alpha P^\beta \eta^{\nu\mu} + a^\alpha P^\nu \eta^{\beta\mu}\big)$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_3 = \pm\frac{\mathrm{i}}{L^2}\big(a^\beta P^\alpha a^2 - a^\beta a_\nu P^\nu a^\alpha - a^\alpha P^\beta a^2 + a^\alpha a_\nu P^\nu a^\beta\big)$$
$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_3 = \pm\frac{a^2}{L^2}2\mathrm{i}P^{[\alpha}a^{\beta]}$$

A relação de recorrência é clara,

$$\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_{2n+1} = \left(\pm\frac{a^2}{L^2}\right)^n 2\mathrm{i}P^{[\alpha}a^{\beta]}, \quad \left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_{2n} = \left(\pm\frac{a^2}{L^2}\right)^n 2\mathrm{i}\frac{a_\mu}{a^2}a^{[\beta}J^{\alpha]\mu}$$

Logo,

$$\mathrm{U}\mathrm{i}J^{\alpha\beta}\mathrm{U}^{-1} = \mathrm{i}J^{\alpha\beta} + \sum_{n=1}^{\infty}\frac{\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_n}{n!}$$
$$\mathrm{U}\mathrm{i}J^{\alpha\beta}\mathrm{U}^{-1} = \mathrm{i}J^{\alpha\beta} + \sum_{n=1}^{\infty}\frac{\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{\left[\mathrm{i}J^{\alpha\beta}, \mathrm{i}a_\mu P^\mu\right]_{2n+1}}{(2n+1)!}$$
$$\mathrm{U}\mathrm{i}J^{\alpha\beta}\mathrm{U}^{-1} = \mathrm{i}J^{\alpha\beta} + 2\mathrm{i}P^{[\alpha}a^{\beta]}\sqrt{\frac{L^2}{a^2}}\sum_{n=0}^{\infty}(\pm)^n\left(\sqrt{\frac{a^2}{L^2}}\right)^{2n+1}\frac{1}{(2n+1)!} + 2\mathrm{i}\frac{a_\mu}{a^2}a^{[\beta}J^{\alpha]\mu}\sum_{n=1}^{\infty}(\pm)^n\left(\sqrt{\frac{a^2}{L^2}}\right)^{2n}\frac{1}{(2n)!}$$
$$\mathrm{U}\mathrm{i}J^{\alpha\beta}\mathrm{U}^{-1} = \mathrm{i}J^{\alpha\beta} + 2\mathrm{i}P^{[\alpha}a^{\beta]}\sqrt{\frac{L^2}{a^2}}\sin(\mathrm{h})\left(\sqrt{\frac{a^2}{L^2}}\right) + 2\mathrm{i}\frac{a_\mu}{a^2}a^{[\beta}J^{\alpha]\mu}\cos(\mathrm{h})\left(\sqrt{\frac{a^2}{L^2}}\right) - 2\mathrm{i}\frac{a_\mu}{a^2}a^{[\beta}J^{\alpha]\mu}$$

E não menos importante,

$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu] = \pm\frac{\mathrm{i}a_\mu}{L^2}J^{\alpha\mu}$$
$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_2 = \left[\pm\frac{\mathrm{i}a_\mu}{L^2}J^{\alpha\mu}, \mathrm{i}a_\nu P^\nu\right]$$
$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_2 = \mp\frac{a_\mu a_\nu}{L^2}[J^{\alpha\mu}, P^\nu]$$
$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_2 = \pm 2\mathrm{i}\frac{a_\mu a_\nu}{L^2}P^{[\alpha}\eta^{\mu]\nu}$$
$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_2 = \pm\mathrm{i}\frac{1}{L^2}\big(P^\alpha a^2 + a^\alpha a_\beta P^\beta\big)$$
$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_3 = \left[\pm\mathrm{i}\frac{1}{L^2}\big(P^\alpha a^2 + a^\alpha a_\beta P^\beta\big), \mathrm{i}a_\mu P^\mu\right]$$

$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_3 = \mp\frac{1}{L^2}a_\mu\big[\big(P^\alpha a^2 + a^\alpha a_\beta P^\beta\big), P^\mu\big]$$

$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_3 = \mathrm{i}\frac{1}{L^4}a_\mu\big(J^{\alpha\mu}a^2 + a^\alpha a_\beta J^{\beta\mu}\big) = \pm\frac{a^2}{L^2}(\pm)\frac{\mathrm{i}a_\mu}{L^2}J^{\alpha\mu}$$

A relação de recorrência é clara,

$$[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_{2n+1} = \left(\pm\frac{a^2}{L^2}\right)^n(\pm)\frac{\mathrm{i}a_\mu}{L^2}J^{\alpha\mu}, \quad [\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_{2n} = \left(\pm\frac{a^2}{L^2}\right)^n\mathrm{i}\left(P^\alpha + \frac{a^\alpha a_\beta}{a^2}P^\beta\right)$$

Logo,

$$\mathrm{U}\mathrm{i}P^\alpha\mathrm{U}^{-1} = \mathrm{i}P^\alpha + \sum_{n=1}^{\infty}\frac{[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_n}{n!}$$

$$\mathrm{U}\mathrm{i}P^\alpha\mathrm{U}^{-1} = \mathrm{i}P^\alpha + \sum_{n=1}^{\infty}\frac{[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{[\mathrm{i}P^\alpha, \mathrm{i}a_\mu P^\mu]_{2n+1}}{(2n+1)!}$$

$$\mathrm{U}\mathrm{i}P^\alpha\mathrm{U}^{-1} = \mathrm{i}P^\alpha \pm \frac{\mathrm{i}a_\mu}{L^2}J^{\alpha\mu}\sqrt{\frac{L^2}{a^2}}\sum_{n=0}^{\infty}(\pm)^n\left(\sqrt{\frac{a^2}{L^2}}\right)^{2n+1}\frac{1}{(2n+1)!} + \mathrm{i}\left(P^\alpha + \frac{a^\alpha a_\beta}{a^2}P^\beta\right)\sum_{n=1}^{\infty}(\pm)^n\left(\sqrt{\frac{a^2}{L^2}}\right)^{2n}\frac{1}{(2n)!}$$

$$\mathrm{U}\mathrm{i}P^\alpha\mathrm{U}^{-1} = \mathrm{i}P^\alpha \pm \frac{\mathrm{i}a_\mu}{L^2}J^{\alpha\mu}\sqrt{\frac{L^2}{a^2}}\sin(\mathrm{h})\left(\sqrt{\frac{a^2}{L^2}}\right) + \mathrm{i}\left(P^\alpha + \frac{a^\alpha a_\beta}{a^2}P^\beta\right)\cos(\mathrm{h})\left(\sqrt{\frac{a^2}{L^2}}\right) - \mathrm{i}\left(P^\alpha + \frac{a^\alpha a_\beta}{a^2}P^\beta\right)$$

Por último,

$$\mathrm{U}\mathbf{d}\mathrm{U}^{-1} = \mathrm{U}\mathrm{i}\mathbf{d}a_\mu P^\mu\mathrm{U}^{-1}$$

3. Path Integral

Let's try to formulate a path integral version of the Einstein-Cartan action, first without the cosmological constant,

$$S_{\mathrm{EC}} = \frac{1}{4\kappa}\int\epsilon_{\alpha\beta\mu\nu}\mathbf{R}^{\alpha\beta}\wedge\mathbf{e}^\mu\wedge\mathbf{e}^\nu$$

So that the path integral is,

$$Z = \int\mathcal{D}\mathbf{e}\mathcal{D}\boldsymbol{\omega}\mathrm{e}^{\mathrm{i}S_{\mathrm{EC}}}$$

What is computable are gauge invariant objects, such as,

$$\langle R\rangle = \int\mathcal{D}\mathbf{e}\mathcal{D}\boldsymbol{\omega}\epsilon_{\alpha\beta\mu\nu}\mathbf{R}^{\alpha\beta}\wedge\mathbf{e}^\mu\wedge\mathbf{e}^\nu\mathrm{e}^{\mathrm{i}S_{\mathrm{EC}}}$$