ADTGR SEMINAR

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1. Introduction

General Relativity has been known for being a highly complex theory, to which few solutions are known, apart from highly symmetric ones, one of the main reasons of this is the non-linear character of the Action and the Equations of Motion,

$$S_{\text{EH}} = \frac{1}{2\kappa}\int\limits_M \mathrm{d}^D x\,\sqrt{|g|}g^{ab}R_{cb}{}^{c}{}_{a}$$

Apart from the non-polinomial term $\sqrt{|g|}$, and the inverse field g^{ab} showing up in the Action, inside $R_{cb}{}^{c}{}_{a}$ lies more contributions of inverse fields and derivatives of them. Thus, in the study of this theory, and couplings of it to matter, prospective of obtaining analytical solutions in non-highly symmetric scenarios is faded to doom, but, we can try to understand more of this if we look at simpler toy models, a common choice is lowering the dimension of the problem, in this case, from $3+1$ to $2+1$ dimensions, why should this be a interesting theory to study? First, the dynamical degrees of freedom of GR are tied to the dimension of the space-time, for a metric compatible, torsionless connection — which is naturally totally determined by the metric — there are $\frac{1}{2}D(D+1)$ free components of the metric, which are subject to D redundancies from diffeomorphisms, and also D Bianchi identities,this gives as a total of $\frac{1}{2}D(D-3)$ free components, in $D = 3+1$ this is our well known two polarizations of the metric, but for $D = 2+1$ this is zero, which can be interpreted as the metric having no dynamical degrees of freedom, that is, the equation of motion is merely an algebraic condition, this is compatible with our knowledge of independent components of the Riemann tensor, $\frac{1}{12}D^2(D^2-1)$, exactly 20 in $D = 3+1$, but 6 in $D = 2+1$, notice that $6 = \frac{1}{2}3(3+1)$, the same number of degrees of freedom of a symmetric $(0,2)$ tensor, in other words, in $D = 2+1$ the Riemann tensor is totally determined by the knowledge of the Ricci tensor, which is totally determined algebraically by the equations of motion. This is compatible with the metric doesn't having degrees of freedom, due to being known that dynamical propagation of gravity is linked to the Weyl tensor, and, if the Riemann tensor is totally determined by the Ricci tensor, there is no degree of freedom in the Weyl tensor, thus, no dynamics. This is our hope to solve this theory, as it's non-dynamical, we expect it to be "trivial", or at least "exact" — we have to define what we mean by this —, similarly to what was done to lower dimensional electrodynamics by Schwinger.

2. The Einstein-Hilbert Action in the Form Language

We'll begin with a quick recap of the vielbein/spin connection formalism, for now we'll keep the discussion general in D dimensions, and only later on we'll go to the special case of $D = 2+1$. The whole point of introducing the Vielbein is switch the degrees of freedom from the metric to the inertial frame basis, that is, the vielbein are a collection of D vector fields, such that at each point these collection of vectors constitute a basis of the tangent vector space, we'll use the Wald conventions, that is, the "*true*" indices which indexes which vector field we're talking about μ, and the coordinate basis index a,

$$\begin{aligned}
\operatorname{diag}\begin{pmatrix}-1 & 1 & \cdots & 1\end{pmatrix} = \eta_{\mu\nu} &= \mathbf{g}(\mathbf{e}_\mu,\mathbf{e}_\nu) = g_{ab}\,\mathrm{d}x^a\,(\mathbf{e}_\mu)\otimes\mathrm{d}x^b\,(\mathbf{e}_\nu)\\
\eta_{\mu\nu} &= g_{ab}e_\mu{}^c e_\nu{}^d\,\mathrm{d}x^a\,(\boldsymbol{\partial}_c)\otimes\mathrm{d}x^b\,(\boldsymbol{\partial}_d)\\
\eta_{\mu\nu} &= g_{ab}e_\mu{}^a e_\nu{}^b
\end{aligned}$$

From the condition that $\mathbf{g}$ must be non-degenerate we get that the matrix of components $e_\mu{}^a$ must be invertible, this ensures the existence of $e^\mu{}_a$ from which we can construct the dual vector field $\tilde{\mathbf{e}}^\mu = e^\mu{}_a\,\mathrm{d}x^a$ this ensures we have a basis of the whole tensor space, so it's possible to decompose any tensor in it,

$$\eta_{\mu\nu}\tilde{\mathbf{e}}^\mu\otimes\tilde{\mathbf{e}}^\nu = \mathbf{g}(\mathbf{e}_\mu,\mathbf{e}_\nu)\tilde{\mathbf{e}}^\mu\otimes\tilde{\mathbf{e}}^\nu = \mathbf{g}$$

Only having a metric, or in this case, only having a vielbein, isn't sufficient to construct a theory, we also need a affine connection/covariant derivative. A useful way to define it is to define in a vector field basis, as we already have the vielbein as a vector field basis it's more convenient to define in respect to it,

$$\begin{aligned}
\nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= \boldsymbol{\omega}(\mathbf{X})^\mu{}_\nu\mathbf{e}_\mu\\
\nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= X^a\omega_a{}^\mu{}_\nu\mathbf{e}_\mu
\end{aligned}$$

Where $\boldsymbol{\omega}^\mu{}_\nu = \omega_a{}^\mu{}_\nu\,\mathrm{d}x^a$ is the spin connection, it can be seen as a $\mathfrak{gl}(1,D-1;\mathbb{R})$-valued $(0,1)$ tensor, or, as we'll adopt here, a $\mathfrak{gl}(1,D-1;\mathbb{R})$-valued 1-form. The ease of working with the vielbein is that the Lorentz index μ does not change upon coordinate/chart/diffeomorphism transformations, and it acts as if it was an internal symmetry, notice that we have a redundancy of how to

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choose the vielbein basis, due to $\Lambda^{\mu}{}_{\nu}\mathbf{e}_{\mu}$, $\Lambda^{\mu}{}_{\nu}: M \to \mathbb{R}$, being an equally good basis,

$$\begin{aligned}
\mathbf{g}\big(\Lambda^{\alpha}{}_{\mu}\mathbf{e}_{\alpha}, \Lambda^{\beta}{}_{\nu}\mathbf{e}_{\beta}\big) &= \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\mathbf{g}(\mathbf{e}_{\alpha}, \mathbf{e}_{\beta})\\
\mathbf{g}\big(\Lambda^{\alpha}{}_{\mu}\mathbf{e}_{\alpha}, \Lambda^{\beta}{}_{\nu}\mathbf{e}_{\beta}\big) &= \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta_{\alpha\beta}\\
\mathbf{g}\big(\Lambda^{\alpha}{}_{\mu}\mathbf{e}_{\alpha}, \Lambda^{\beta}{}_{\nu}\mathbf{e}_{\beta}\big) &= \eta_{\mu\nu}
\end{aligned}$$

as long as $\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta_{\alpha\beta} = \eta_{\mu\nu}$, that is, as we enlarge the number of components going from the metric to the vielbein, we also enlarge the redundancies, now, additionally to Diff(M) we have local $SO(1, D-1)$ transformations as redundancies, this is as we was gauging the whole Poincaré group, with Diff(M) being the gauging of the translations.

$$\begin{aligned}
\mathbf{g}(\mathbf{e}_{\alpha}, \nabla_{\mathbf{X}}(\mathbf{e}_{\nu})) &= X^{a}\omega_{a}{}^{\mu}{}_{\nu}\mathbf{g}(\mathbf{e}_{\alpha}, \mathbf{e}_{\mu})\\
\mathbf{g}(\mathbf{e}_{\alpha}, \nabla_{\mathbf{X}}(\mathbf{e}_{\nu})) + \mathbf{g}(\mathbf{e}_{\nu}, \nabla_{\mathbf{X}}(\mathbf{e}_{\alpha})) &= X^{a}\omega_{a}{}^{\mu}{}_{\nu}\mathbf{g}(\mathbf{e}_{\alpha}, \mathbf{e}_{\mu}) + X^{a}\omega_{a}{}^{\mu}{}_{\alpha}\mathbf{g}(\mathbf{e}_{\nu}, \mathbf{e}_{\mu})\\
\mathbf{g}(\mathbf{e}_{\alpha}, \nabla_{\mathbf{X}}(\mathbf{e}_{\nu})) + \mathbf{g}(\nabla_{\mathbf{X}}(\mathbf{e}_{\alpha}), \mathbf{e}_{\nu}) &= X^{a}\omega_{a\alpha\nu} + X^{a}\omega_{a\nu\alpha}\\
\nabla_{\mathbf{X}}(\mathbf{g}(\mathbf{e}_{\alpha}, \mathbf{e}_{\nu})) - \nabla_{\mathbf{X}}(\mathbf{g})(\mathbf{e}_{\alpha}, \mathbf{e}_{\nu}) &= X^{a}\omega_{a\alpha\nu} + X^{a}\omega_{a\nu\alpha}\\
-\nabla_{\mathbf{X}}(\mathbf{g})(\mathbf{e}_{\alpha}, \mathbf{e}_{\nu}) &= X^{a}\omega_{a\alpha\nu} + X^{a}\omega_{a\nu\alpha}\\
-\omega_{a\nu\alpha} &= \omega_{a\alpha\nu}, \quad \text{Metric compatibility}
\end{aligned} \tag{2.1}$$

Riemann curvature tensor, $\mathbf{Riem}(\mathbf{X}, \mathbf{Y}) : \mathfrak{X} \to \mathfrak{X}$:

$$\begin{aligned}
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= \big(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\big)\mathbf{e}_{\mu}\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= \Big(\nabla_{\mathbf{X}}\big(Y^{b}\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big) - \nabla_{\mathbf{Y}}\big(X^{a}\omega_{a}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big) - [\mathbf{X}, \mathbf{Y}]^{b}\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\Big)\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= \Big(\nabla_{\mathbf{X}}\big(Y^{b}\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big) - \nabla_{\mathbf{Y}}\big(X^{a}\omega_{a}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big) - [\mathbf{X}, \mathbf{Y}]^{b}\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\Big)\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= \big(Y^{b}\nabla_{\mathbf{X}}\big(\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big) - X^{a}\nabla_{\mathbf{Y}}\big(\omega_{a}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big)\\
&\qquad + \nabla_{\mathbf{X}}\big(Y^{b}\big)\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu} - \nabla_{\mathbf{Y}}\big(X^{b}\big)\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu} - [\mathbf{X}, \mathbf{Y}]^{b}\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\Big)\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= \big(X^{a}Y^{b}\nabla_{a}\big(\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big) - X^{a}Y^{b}\nabla_{b}\big(\omega_{a}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big)\\
&\qquad + \big(\nabla_{\mathbf{X}}\big(Y^{b}\big) - \nabla_{\mathbf{Y}}\big(X^{b}\big)\big)\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu} - [\mathbf{X}, \mathbf{Y}]^{b}\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\Big)\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= \big(X^{a}Y^{b}\nabla_{a}\big(\omega_{b}{}^{\nu}{}_{\mu}\big)\mathbf{e}_{\nu} + X^{a}Y^{b}\omega_{b}{}^{\nu}{}_{\mu}\omega_{a}{}^{\alpha}{}_{\nu}\mathbf{e}_{\alpha} - X^{a}Y^{b}\nabla_{b}\big(\omega_{a}{}^{\nu}{}_{\mu}\big)\mathbf{e}_{\nu} - X^{a}Y^{b}\omega_{a}{}^{\nu}{}_{\mu}\omega_{b}{}^{\alpha}{}_{\nu}\mathbf{e}_{\alpha}\big)\\
&\qquad + \big(X^{a}\partial_{a}Y^{b} + X^{a}\Gamma_{a}{}^{b}{}_{c}Y^{c} - Y^{a}\partial_{a}X^{b} - Y^{a}\Gamma_{a}{}^{b}{}_{c}X^{c}\big)\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu} - \big(X^{a}\partial_{a}Y^{b} - Y^{a}\partial_{a}X^{b}\big)\omega_{b}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\big)\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= X^{a}Y^{b}\left(\nabla_{a}\big(\omega_{b}{}^{\nu}{}_{\mu}\big) + \omega_{b}{}^{\alpha}{}_{\mu}\omega_{a}{}^{\nu}{}_{\alpha} - \nabla_{b}\big(\omega_{a}{}^{\nu}{}_{\mu}\big) - \omega_{a}{}^{\alpha}{}_{\mu}\omega_{b}{}^{\nu}{}_{\alpha} + \Gamma_{a}{}^{c}{}_{b}\omega_{c}{}^{\nu}{}_{\mu} - \Gamma_{b}{}^{c}{}_{a}\omega_{c}{}^{\nu}{}_{\mu}\right)\mathbf{e}_{\nu}\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= X^{a}Y^{b}\left(\partial_{a}\omega_{b}{}^{\nu}{}_{\mu} - \partial_{b}\omega_{a}{}^{\nu}{}_{\mu} + \omega_{a}{}^{\nu}{}_{\alpha}\omega_{b}{}^{\alpha}{}_{\mu} - \omega_{b}{}^{\nu}{}_{\alpha}\omega_{a}{}^{\alpha}{}_{\mu}\right)\mathbf{e}_{\nu}\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= X^{a}Y^{b}\left(\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}\right)_{ab}{}^{\nu}{}_{\mu}\,\mathbf{e}_{\nu}\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_{\mu} &= X^{a}Y^{b}R_{ab}{}^{\nu}{}_{\mu}\mathbf{e}_{\nu}\\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\big(e_{\mu}{}^{e}\boldsymbol{\partial}_{e}\big) &= X^{a}Y^{b}R_{ab}{}^{\nu}{}_{\mu}e_{\nu}{}^{c}\boldsymbol{\partial}_{c}\\
e_{\mu}{}^{e}\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\boldsymbol{\partial}_{e} &= X^{a}Y^{b}R_{ab}{}^{\nu}{}_{\mu}e_{\nu}{}^{c}\boldsymbol{\partial}_{c}\\
e_{\mu}{}^{e}X^{a}Y^{b}R_{ab}{}^{c}{}_{e}\boldsymbol{\partial}_{c} &= X^{a}Y^{b}R_{ab}{}^{\nu}{}_{\mu}e_{\nu}{}^{c}\boldsymbol{\partial}_{c}\\
e^{\mu}{}_{d}e_{\mu}{}^{e}R_{ab}{}^{c}{}_{e} &= e^{\mu}{}_{d}R_{ab}{}^{\nu}{}_{\mu}e_{\nu}{}^{c}\\
R_{ab}{}^{c}{}_{d} &= e^{\mu}{}_{d}R_{ab}{}^{\nu}{}_{\mu}e_{\nu}{}^{c}
\end{aligned}$$

We would like to write the Einstein-Hilbert action as only a function of the Vierbein and the spin connection, for this, let us explicit write the Riemann tensor as a End(TM)-valued 2-form,

$$\begin{aligned}
\mathbf{R}^{\nu}{}_{\mu} &= \frac{1}{2}R_{ab}{}^{\nu}{}_{\mu}\,\mathrm{d}x^{a}\wedge\mathrm{d}x^{b}\\
\mathbf{R}^{\nu}{}_{\mu} &= \frac{1}{2}R_{ab}{}^{\nu}{}_{\mu}e_{\alpha}{}^{a}e^{\alpha}{}_{c}e_{\beta}{}^{b}e^{\beta}{}_{d}\,\mathrm{d}x^{c}\wedge\mathrm{d}x^{d}\\
\mathbf{R}^{\nu}{}_{\mu} &= \frac{1}{2}R_{ab}{}^{\nu}{}_{\mu}e_{\alpha}{}^{a}e_{\beta}{}^{b}\tilde{\mathbf{e}}^{\alpha}\wedge\tilde{\mathbf{e}}^{\beta}
\end{aligned}$$

Let us start by writing the volume form in terms of the Vierbein,

$$
\begin{aligned}
\mathrm{d}^D x\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[g_{ab}]|}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{D-1} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^{\mu}{}_a \eta_{\mu\nu} e^{\nu}{}_b]|}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{D-1} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \sqrt{|\mathrm{Det}[e^{\mu}{}_a]\mathrm{Det}[\eta_{\mu\nu}]\mathrm{Det}[e^{\nu}{}_b]|}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{D-1} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \sqrt{\left(\mathrm{Det}[e^{\mu}{}_a]\right)^2}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{D-1} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \mathrm{Det}[e^{\mu}{}_a]\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{D-1} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \epsilon_{\mu_0\cdots\mu_{D-1}} e^{\mu_0}{}_0 \cdots e^{\mu_{D-1}}{}_{D-1}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{D-1} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \epsilon_{\mu_0\cdots\mu_{D-1}} e^{\mu_0}{}_0\,\mathrm{d}x^0 \wedge \cdots \wedge e^{\mu_{D-1}}{}_{D-1}\,\mathrm{d}x^{D-1} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_0\cdots\mu_{D-1}} e^{\mu_0}{}_{a_0}\,\mathrm{d}x^{a_0} \wedge \cdots \wedge e^{\mu_{D-1}}{}_{a_{D-1}}\,\mathrm{d}x^{a_{D-1}} \\
\mathrm{d}^D x\,\sqrt{|g|} &= \frac{1}{D!}\epsilon_{\mu_0\cdots\mu_{D-1}} \mathbf{e}^{\mu_0} \wedge \cdots \wedge \mathbf{e}^{\mu_{D-1}}
\end{aligned}
$$

And now we express the Ricci scalar,

$$
\begin{aligned}
R &= g^{ab} R_{cb}{}^{c}{}_{a} \\
R &= e_{\rho}{}^{a} e^{\rho b} R_{cbda} e_{\alpha}{}^{c} e^{\alpha d} \\
R &= \eta^{\rho\sigma}\eta^{\alpha\beta} e_{\rho}{}^{a} e_{\sigma}{}^{b} R_{cbda} e_{\alpha}{}^{c} e_{\beta}{}^{d} \\
R &= \eta^{\rho\sigma}\eta^{\alpha\beta} R_{\alpha\sigma\beta\rho} \\
R &= \frac{1}{2}\left(\eta^{\rho\sigma}\eta^{\alpha\beta} - \eta^{\rho\alpha}\eta^{\sigma\beta}\right) R_{\alpha\sigma\beta\rho} \\
R &= \frac{1}{2(D-2)!}\epsilon^{\nu_0\cdots\nu_{D-3}\beta\rho}\epsilon_{\nu_0\cdots\nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho}
\end{aligned}
$$

Putting everything together,

$$
\begin{aligned}
S_{\mathrm{EH}} &= \frac{1}{2\kappa}\int\limits_M \mathrm{d}^D x\,\sqrt{|g|} R \\
S_{\mathrm{EH}} &= \frac{1}{4D!(D-2)!\kappa}\int\limits_M \mathbf{e}^{\mu_0} \wedge \cdots \wedge \mathbf{e}^{\mu_{D-1}} \epsilon_{\mu_0\cdots\mu_{D-1}} \epsilon^{\nu_0\cdots\nu_{D-3}\beta\rho}\epsilon_{\nu_0\cdots\nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\mathrm{EH}} &= \frac{1}{4(D-2)!\kappa}\int\limits_M \mathbf{e}^{\mu_0} \wedge \cdots \wedge \mathbf{e}^{\mu_{D-1}} \eta_{\mu_0}{}^{[\nu_0} \cdots \eta_{\mu_{D-3}}{}^{\nu_{D-3}} \eta_{\mu_{D-2}}{}^{\beta} \eta_{\mu_{D-1}}{}^{\rho]} \epsilon_{\nu_0\cdots\nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\mathrm{EH}} &= \frac{1}{4(D-2)!\kappa}\int\limits_M \mathbf{e}^{\nu_0} \wedge \cdots \wedge \mathbf{e}^{\nu_{D-3}} \wedge \mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho} \epsilon_{\nu_0\cdots\nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa}\int\limits_M \frac{1}{2(D-2)!} R_{\alpha\sigma\beta\rho}\epsilon^{\alpha\sigma}{}_{\nu_0\cdots\nu_{D-3}} \mathbf{e}^{\nu_0} \wedge \cdots \wedge \mathbf{e}^{\nu_{D-3}} \wedge \mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho} \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa}\int\limits_M \star\mathbf{R}_{\beta\rho} \wedge \mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho} \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa}\int\limits_M \mathbf{R}_{\beta\rho} \wedge \star\left(\mathbf{e}^{\beta} \wedge \mathbf{e}^{\rho}\right) \\
S_{\mathrm{EH}} &= \frac{1}{2\kappa}\int\limits_M \frac{1}{(D-2)!}\epsilon^{\beta\rho}{}_{\alpha_0\cdots\alpha_{D-3}} \mathbf{R}_{\beta\rho} \wedge \mathbf{e}^{\alpha_0} \wedge \cdots \wedge \mathbf{e}^{\alpha_{D-3}} \\
S_{\mathrm{EH}} &= \frac{1}{2(D-2)!\kappa}\int\limits_M \epsilon_{\alpha_0\cdots\alpha_{D-1}} \mathbf{e}^{\alpha_0} \wedge \cdots \wedge \mathbf{e}^{\alpha_{D-3}} \wedge \mathbf{R}^{\alpha_{D-2}\alpha_{D-1}}
\end{aligned}
$$

The most interesting case here is $D = 3$,

$$
S_{\mathrm{EH}}[\mathbf{e}, \boldsymbol{\omega}] = \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^{\mu} \wedge \mathbf{R}^{\alpha\beta}
$$

Equations of motion are,

$$
\begin{aligned}
S_{\mathrm{EH}}[\mathbf{e}+\delta\mathbf{e},\boldsymbol{\omega}] - S_{\mathrm{EH}}[\mathbf{e},\boldsymbol{\omega}] &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \delta\mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} = 0 \\
0 &= -\frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{R}^{\alpha\beta} \\
0 &= -\frac{1}{2}\epsilon_{\mu\alpha\beta}\frac{1}{2}R_{\rho\sigma}{}^{\alpha\beta}\mathbf{e}^\rho \wedge \mathbf{e}^\sigma \\
0 &= -\frac{1}{4}\epsilon_{\mu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta} \star (\mathbf{e}^\rho \wedge \mathbf{e}^\sigma) \\
0 &= -\frac{1}{4}\epsilon_{\mu\alpha\beta}R_{\rho\sigma}{}^{\alpha\beta}\epsilon^{\rho\sigma}{}_\kappa \mathbf{e}^\kappa \\
0 &= -\frac{1}{4}\epsilon_{\mu\alpha\beta}\epsilon^{\rho\sigma\kappa}R_{\rho\sigma}{}^{\alpha\beta}\mathbf{e}_\kappa \\
0 &= -\frac{1}{4}\eta_\mu{}^{[\rho}\eta_\alpha{}^\sigma\eta_\beta{}^{\kappa]}R_{\rho\sigma}{}^{\alpha\beta}\mathbf{e}_\kappa \\
0 &= -\frac{1}{4}R_{\rho\sigma}{}^{[\sigma\kappa}\eta_\mu{}^{\rho]}\mathbf{e}_\kappa \\
0 &= -\frac{1}{4}\big(R_{\rho\sigma}{}^{\sigma\kappa}\eta_\mu{}^\rho + R_{\rho\sigma}{}^{\kappa\rho}\eta_\mu{}^\sigma + R_{\rho\sigma}{}^{\rho\sigma}\eta_\mu{}^\kappa - R_{\rho\sigma}{}^{\rho\kappa}\eta_\mu{}^\sigma - R_{\rho\sigma}{}^{\kappa\sigma}\eta_\mu{}^\rho - R_{\rho\sigma}{}^{\sigma\rho}\eta_\mu{}^\kappa\big)\mathbf{e}_\kappa \\
0 &= -\frac{1}{4}\big(-R_\mu{}^\kappa - R_\mu{}^\kappa + R\eta_\mu{}^\kappa - R_\mu{}^\kappa - R_\mu{}^\kappa + R\eta_\mu{}^\kappa\big)\mathbf{e}_\kappa \\
0 &= -\frac{1}{4}(-4R_\mu{}^\kappa + 2R\eta_\mu{}^\kappa)\mathbf{e}_\kappa \\
0 &= \left(R_{\mu\kappa} - \frac{1}{2}R\eta_{\mu\kappa}\right)\mathbf{e}^\kappa
\end{aligned}
$$

And,

$$
\begin{aligned}
S_{\mathrm{EH}}[\mathbf{e},\boldsymbol{\omega}+\delta\boldsymbol{\omega}]-S_{\mathrm{EH}}[\mathbf{e},\boldsymbol{\omega}] &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^\mu\wedge\left(\mathrm{d}\delta\boldsymbol{\omega}+\delta\boldsymbol{\omega}\wedge\boldsymbol{\omega}+\boldsymbol{\omega}\wedge\delta\boldsymbol{\omega}\right)^{\alpha\beta}=0\\
0 &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \Big(-\mathrm{d}\big(\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}\big)+\mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}+\mathbf{e}^\mu\wedge(\delta\boldsymbol{\omega}\wedge\boldsymbol{\omega})^{\alpha\beta}+\mathbf{e}^\mu\wedge(\boldsymbol{\omega}\wedge\delta\boldsymbol{\omega})^{\alpha\beta}\Big)\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \big(\mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}+\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\gamma}\wedge{\boldsymbol{\omega}_\gamma}^\beta+\mathbf{e}^\mu\wedge\boldsymbol{\omega}^{\alpha\gamma}\wedge\delta{\boldsymbol{\omega}_\gamma}^\beta\big)\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \big(\mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}-\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\gamma}^\beta\wedge\delta\boldsymbol{\omega}^{\alpha\gamma}+\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\alpha}_\gamma\wedge\delta\boldsymbol{\omega}^{\gamma\beta}\big)\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}-\frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\gamma}^\beta\wedge\delta\boldsymbol{\omega}^{\alpha\gamma}+\frac{1}{2}\epsilon_{\mu\alpha\beta}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\alpha}_\gamma\wedge\delta\boldsymbol{\omega}^{\gamma\beta}\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\int\limits_M \mathrm{d}\mathbf{e}^\mu\wedge\delta\boldsymbol{\omega}^{\alpha\beta}-\frac{1}{2}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\beta}^\gamma\wedge\delta\boldsymbol{\omega}^{\alpha\beta}+\frac{1}{2}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\gamma}_\alpha\wedge\delta\boldsymbol{\omega}^{\alpha\beta}\\
0 &= \frac{1}{2}\int\limits_M \Big(\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu-\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\beta}^\gamma+\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\gamma}_\alpha\Big)\wedge\delta\boldsymbol{\omega}^{\alpha\beta}\\
0 &= \frac{1}{2}\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu-\frac{1}{2}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\beta}^\gamma+\frac{1}{2}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\gamma}_\alpha\\
0 &= \frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\alpha\beta}\,\mathrm{d}\mathbf{e}^\mu-\frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\alpha\gamma}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\beta}^\gamma+\frac{1}{2}\epsilon^{\alpha\beta\nu}\epsilon_{\mu\gamma\beta}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\gamma}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu-{\eta_\gamma}^{[\beta}{\eta_\mu}^{\nu]}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\beta}^\gamma+{\eta_\mu}^{[\nu}{\eta_\gamma}^{\alpha]}\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\gamma}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu-\frac{1}{2}\big({\eta_\gamma}^\beta{\eta_\mu}^\nu-{\eta_\gamma}^\nu{\eta_\mu}^\beta\big)\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\beta}^\gamma+\frac{1}{2}\big({\eta_\mu}^\nu{\eta_\gamma}^\alpha-{\eta_\mu}^\alpha{\eta_\gamma}^\nu\big)\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\gamma}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu+\frac{1}{2}{\eta_\gamma}^\nu{\eta_\mu}^\beta\mathbf{e}^\mu\wedge{\boldsymbol{\omega}_\beta}^\gamma-\frac{1}{2}{\eta_\mu}^\alpha{\eta_\gamma}^\nu\mathbf{e}^\mu\wedge{\boldsymbol{\omega}^\gamma}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu+\frac{1}{2}\mathbf{e}^\beta\wedge{\boldsymbol{\omega}_\beta}^\nu-\frac{1}{2}\mathbf{e}^\alpha\wedge{\boldsymbol{\omega}^\nu}_\alpha\\
0 &= \mathrm{d}\mathbf{e}^\nu+{\boldsymbol{\omega}^\nu}_\alpha\wedge\mathbf{e}^\alpha
\end{aligned}
$$

It's not really feasible to give this a gauge theory approach, only if we're in 2+1, in this case there is an isomorphism, $\mathbf{e}^\mu\to\mathbf{e}_{\alpha\beta}=\epsilon_{\mu\alpha\beta}\mathbf{e}^\mu$, so that,

$$
\begin{aligned}
S_{\mathrm{EH}} &= \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^\mu\wedge\mathbf{R}^{\alpha\beta}\\
S_{\mathrm{EH}} &= -\frac{1}{2\kappa}\int\limits_M \mathbf{e}_{\beta\alpha}\wedge\mathbf{R}^{\alpha\beta}\\
S_{\mathrm{EH}} &= -\frac{1}{2\kappa}\int\limits_M \mathrm{Tr}\left[\mathbf{e}\wedge\mathbf{R}\right]\\
S_{\mathrm{EH}} &= -\frac{1}{2\kappa}\int\limits_M \mathrm{Tr}\left[\mathbf{e}\wedge\left(\mathrm{d}\boldsymbol{\omega}+\frac{1}{2}[\boldsymbol{\omega}\overset{\wedge}{,}\boldsymbol{\omega}]\right)\right]
\end{aligned}
$$

This is a lot similar to Chern-Simons theory,

$$
S_{\mathrm{CS}}[\mathbf{A}]=k\int\limits_M \mathrm{Tr}\left[\mathbf{A}\wedge\mathrm{d}\mathbf{A}+\frac{1}{3}\mathbf{A}\wedge[\mathbf{A}\overset{\wedge}{,}\mathbf{A}]\right]
$$

Let's try, $\mathbf{A}^x=\boldsymbol{\omega}+x\mathbf{e}$,

$$
\begin{aligned}
S_{\mathrm{CS}}[\mathbf{A}^x] &= k \int\limits_M \mathrm{Tr}\left[(\boldsymbol{\omega} + x\mathbf{e}) \wedge \mathrm{d}(\boldsymbol{\omega} + x\mathbf{e}) + \frac{1}{3}(\boldsymbol{\omega} + x\mathbf{e}) \wedge [\boldsymbol{\omega} + x\mathbf{e} \overset{\wedge}{,} \boldsymbol{\omega} + x\mathbf{e}]\right] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= k \int\limits_M \mathrm{Tr}\Big[\boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\omega} + x\boldsymbol{\omega} \wedge \mathrm{d}\mathbf{e} + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} + x^2 \mathbf{e} \wedge \mathrm{d}\mathbf{e} \\
&\qquad + \frac{2}{3}(\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e})\Big] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= k \int\limits_M \mathrm{Tr}\Big[\boldsymbol{\omega} \wedge \mathrm{d}\boldsymbol{\omega} + x\,\mathrm{d}\mathbf{e} \wedge \boldsymbol{\omega} + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} - \frac{1}{2}x^2\,\mathrm{d}(\mathbf{e} \wedge \mathbf{e}) \\
&\qquad + \frac{2}{3}\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\omega} + 2x\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + 2x^2 \mathbf{e} \wedge \mathbf{e} \wedge \boldsymbol{\omega} + \frac{2}{3}x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\Big] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= k \int\limits_M \mathrm{Tr}\Big[\boldsymbol{\omega} \wedge \left(\mathrm{d}\boldsymbol{\omega} + \frac{2}{3}\boldsymbol{\omega} \wedge \boldsymbol{\omega}\right) + x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + x\mathbf{e} \wedge \mathrm{d}\boldsymbol{\omega} + x\mathbf{e} \wedge (\mathrm{d}\boldsymbol{\omega} + 2\boldsymbol{\omega} \wedge \boldsymbol{\omega}) + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} \\
&\qquad + \frac{2}{3}x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\Big] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= S_{\mathrm{CS}}[\boldsymbol{\omega}] + k \int\limits_M \mathrm{Tr}\left[x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x\mathbf{e} \wedge (\mathrm{d}\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}) + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3}x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= S_{\mathrm{CS}}[\boldsymbol{\omega}] + k \int\limits_M \mathrm{Tr}\left[x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x\mathbf{e} \wedge \mathbf{R} + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3}x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right] \\
S_{\mathrm{CS}}[\mathbf{A}^x] &= S_{\mathrm{CS}}[\boldsymbol{\omega}] - 4xk\kappa S_{\mathrm{EH}} + k \int\limits_M \mathrm{Tr}\left[x\,\mathrm{d}(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3}x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right]
\end{aligned}
$$

We can simplify if we sum two contributions,

$$
\begin{aligned}
S_{\mathrm{CS}}[\mathbf{A}^x] - S_{\mathrm{CS}}\left[\mathbf{A}^{-x}\right] &= -8xk\kappa S_{\mathrm{EH}} + 2kx \int\limits_{\partial M} \mathrm{Tr}\left[\mathbf{e} \wedge \boldsymbol{\omega}\right] + \frac{4}{3}x^3 k \int\limits_M \mathrm{Tr}\left[\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right] \\
\frac{1}{8xk\kappa}\left(S_{\mathrm{CS}}\left[\mathbf{A}^{-x}\right] - S_{\mathrm{CS}}[\mathbf{A}^x]\right) &= S_{\mathrm{EH}} - \frac{1}{4\kappa} \int\limits_{\partial M} \mathrm{Tr}\left[\mathbf{e} \wedge \boldsymbol{\omega}\right] - \frac{x^2}{3!\kappa} \int\limits_M \mathrm{Tr}\left[\mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e}\right]
\end{aligned}
$$

And also the not so usual action,

$$
S_{\mathrm{CS}}[\mathbf{A}^x] + S_{\mathrm{CS}}\left[\mathbf{A}^{-x}\right] = 2S_{\mathrm{CS}}[\boldsymbol{\omega}] + 4x^2 k \int\limits_M \mathrm{Tr}\left[\mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e}\right]
$$

Chern-Simons Equation of Motion,

$$
\begin{aligned}
S_{\mathrm{CS}}[\mathbf{A} + \delta\mathbf{A}] - S_{\mathrm{CS}}[\mathbf{A}] &= k \int\limits_M \mathrm{Tr}\left[\delta\mathbf{A} \wedge \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathrm{d}\delta\mathbf{A} + \frac{2}{3}\delta\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \delta\mathbf{A} \wedge \mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \delta\mathbf{A}\right] \\
S_{\mathrm{CS}}[\mathbf{A} + \delta\mathbf{A}] - S_{\mathrm{CS}}[\mathbf{A}] &= k \int\limits_M \mathrm{Tr}\left[\mathrm{d}\mathbf{A} \wedge \delta\mathbf{A} - \mathrm{d}(\mathbf{A} \wedge \delta\mathbf{A}) + \mathrm{d}\mathbf{A} \wedge \delta\mathbf{A} + 2\mathbf{A} \wedge \mathbf{A} \wedge \delta\mathbf{A}\right] \\
S_{\mathrm{CS}}[\mathbf{A} + \delta\mathbf{A}] - S_{\mathrm{CS}}[\mathbf{A}] &= 2k \int\limits_M \mathrm{Tr}\left[(\mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A}) \wedge \delta\mathbf{A}\right] - k \int\limits_M \mathrm{Tr}\left[\mathrm{d}(\mathbf{A} \wedge \delta\mathbf{A})\right] \\
S_{\mathrm{CS}}[\mathbf{A} + \delta\mathbf{A}] - S_{\mathrm{CS}}[\mathbf{A}] &= 2k \int\limits_M \mathrm{Tr}\left[\mathbf{F} \wedge \delta\mathbf{A}\right] - k \int\limits_{\partial M} \mathrm{Tr}\left[\mathbf{A} \wedge \delta\mathbf{A}\right]
\end{aligned}
$$