2+1 Gravity as a Gauge Theory

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We know how to quantize a class of theories with redundancies:

Yang-Mills

$$S_{\mathrm{YM}} = \frac{1}{g^2} \int_M \mathrm{Tr}\left[\mathbf{F} \wedge \star \mathbf{F}\right], \quad \mathbf{F} = \mathbf{dA} + [\mathbf{A} \overset{\wedge}{,} \mathbf{A}]$$

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It's possible to formulate GR as YM theory?

Why $D = 2 + 1$?

- No local propagating modes:

$$\text{d.o.f.} = \frac{1}{2} D(D-3)$$

Simpler but non-trivial toy model.

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- Change of variables: $\mathbf{g} \rightarrow \begin{cases} \mathbf{e}^{\mu}, & \text{Vielbein} \\ \boldsymbol{\omega}_{\alpha\beta}, & \text{Spin connection} \end{cases}$

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Can we write the integrand as a trace?

Recover of dimensionless coupling constant

- For $\mathfrak{iso}(2,1)$:

$$\mathrm{Tr}\left[J_{\alpha\beta}P_{\mu}\right] = \frac{\lambda}{\kappa}\epsilon_{\alpha\beta\mu}, \quad \mathrm{Tr}\left[P_{\nu}P_{\mu}\right] = 0, \quad \mathrm{Tr}\left[J_{\alpha\beta}J_{\mu\nu}\right] = 0$$

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- Dressing the fields with the algebra,

$$S_{\mathrm{EH}}=\frac{1}{\lambda}\int_{M}\mathrm{Tr}\left[\mathbf{R}\wedge\mathbf{e}\right],\quad \mathbf{R}=\frac{1}{2}\mathbf{R}^{\alpha\beta}J_{\alpha\beta},\ \mathbf{e}=\mathbf{e}^{\mu}P_{\mu}$$

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Not standard YM form...

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Match!

Conclusions

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- Offers a bridge between GR and usual gauge theory intuition.
- Pose some suggestions about $D = 3+1$.

Thank You!

References

[1] Edward Witten. "(2+1)-Dimensional Gravity as an Exactly Soluble System". In: *Nucl. Phys. B* 311 (1988), p. 46. DOI: 10.1016/0550-3213(88)90143-5.

[2] Edward Witten. *Three-Dimensional Gravity Revisited.* 2007. arXiv: 0706.3359 [hep-th]. URL: https://arxiv.org/abs/0706.3359.