Homework III

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1 Beta functions of Non-Abelian Gauge-Fermion Theory

1.1 Construction of the Theory

The theory under consideration is one of a set of Gauge Bosons, '$\mathbf{A}_\mu$', with a set meaning we have an underlying group structure, to which is associated a Lie Algebra, this one is restricted due to energy positivity reasons to be a direct sum over commuting compact simple and '$\mathfrak{u}(1)$' Lie Algebras. This greatly pins down the possible Lie Algebras, leaving just the ones whose structure constants are totally anti-symmetric in the all-up indices, this also mean that exists a symmetric non-degenerate 2-form, which behaves like a metric in the algebra, and with which we raise and lower the group indices. With a pertinent choice of basis we can, still consequences of compactness and simpleness, constrain this *group metric* to be in a diagonal form with all elements from the diagonal equal to one, from now on this is the choice assumed to hold. Also, with this choice, there is no need to distinguish between upper and lower indices, hence, all index will be written upstairs. From this, we can set our group/algebra to be one as,

$$\left[\mathbf{T}^a, \mathbf{T}^b\right] = \mathrm{i} f^{ab}{}_c \mathbf{T}^c$$

The most important representation of our group/algebra will be the *Adjoint Representation*, which is defined by,

$$\left(\mathbf{T}^a_{\mathrm{A}}\right)^b{}_c = -\mathrm{i} f^{ab}{}_c$$

And which is the representation belonging to the Gauge fields, '$\mathbf{A}_\mu$'. The fermionic fields will also transform in a representation of the same group/algebra '$\mathbf{T}^a_{\mathrm{f}}$', the requirement of the action being invariant through the transformation,

$$\mathbf{\Psi}(x) \to \exp\left(-\mathrm{i} g \alpha_a(x) \mathbf{T}^a_{\mathrm{f}}\right) \mathbf{\Psi}(x)$$

Imposes the necessity of replacing the standard derivative, for a covariant derivative in the group/algebra, and this by itself — as a notion of a covariant derivative — requires an additional field, which is of course our Gauge field. Thus, our derivative is,

$$\mathbf{D}_\mu = \mathbb{1}\partial_\mu - \mathrm{i} g \mathbf{A}_\mu(x)$$

This sign choice is a convention, and reflects the choice of sign in the transformation law of '$\mathbf{A}_\mu$', which is determined by the requirement of,

$$\mathbf{D}_\mu \mathbf{\Psi}(x) \to \exp\left(-\mathrm{i}g\alpha_a(x)\mathbf{T}_\mathrm{f}^a\right)\mathbf{D}_\mu \mathbf{\Psi}(x)$$

Setting '$\mathbf{U}(x) = \exp\left(-\mathrm{i}g\alpha_a(x)\mathbf{T}_\mathrm{f}^a\right)$', we get our constrain as,

$$\begin{aligned}
\mathbf{D}_\mu &\to \mathbf{U}(x)\mathbf{D}_\mu\mathbf{U}^\dagger(x) \\
\mathbb{1}\partial_\mu - \mathrm{i}g\mathbf{A}_\mu &\to \mathbf{U}(x)\partial_\mu\mathbf{U}^\dagger(x) + \mathbf{U}(x)\mathbf{U}^\dagger(x)\partial_\mu - \mathrm{i}g\mathbf{U}(x)\mathbf{A}_\mu\mathbf{U}^\dagger(x) \\
\mathbf{A}_\mu(x) &\to \mathbf{U}(x)\mathbf{A}_\mu\mathbf{U}^\dagger(x) + \frac{\mathrm{i}}{g}\mathbf{U}(x)\partial_\mu\mathbf{U}^\dagger(x)
\end{aligned}$$

Clearly then our kinetic term for the fermion will be,

$$\mathcal{L} \supset -\bar{\mathbf{\Psi}}\not{\mathbf{D}}\mathbf{\Psi}$$

Which is of course invariant under all transformations. Remain to be adjusted the kinetic term of the Gauge Field. For this we'll take some knowledge from Differential Geometry, which tell us a curvature 2-form is related to the commutator of covariant derivatives. That is,

$$\begin{aligned}
[\mathbf{D}_\mu, \mathbf{D}_\nu]\mathbf{\Psi} &= [\mathbb{1}\partial_\mu - \mathrm{i}g\mathbf{A}_\mu, \mathbb{1}\partial_\nu - \mathrm{i}g\mathbf{A}_\nu]\mathbf{\Psi} \\
&= \big(-\mathrm{i}g[\mathbb{1}\partial_\mu, \mathbf{A}_\nu] - \mathrm{i}g[\mathbf{A}_\mu, \mathbb{1}\partial_\nu] - g^2[\mathbf{A}_\mu, \mathbf{A}_\nu]\big)\mathbf{\Psi} \\
&= -\mathrm{i}g(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - \mathrm{i}g[\mathbf{A}_\mu, \mathbf{A}_\nu])\mathbf{\Psi}
\end{aligned}$$

So this commutator behaves just as a multiplication factor. We define it to be,

$$\mathbf{F}_{\mu\nu} = \frac{\mathrm{i}}{g}[\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - \mathrm{i}g[\mathbf{A}_\mu, \mathbf{A}_\nu] \tag{1.1}$$

Which has a good transformation rule,

$$\mathbf{F}_{\mu\nu} \to \mathbf{U}(x)\mathbf{F}_{\mu\nu}\mathbf{U}^\dagger(x)$$

So that we can use as a kinetic term,

$$-\frac{1}{2}\,\mathrm{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] \tag{1.2}$$

Which is clearly manifestly invariant thought a local transformation. To get a more meaningful expression we do,

$$\begin{aligned}
\mathbf{F}_{\mu\nu} &= \partial_\mu A_{c\nu}\mathbf{T}^c - \partial_\nu A_{c\mu}\mathbf{T}^c - \mathrm{i}gA_{a\mu}A_{b\nu}\left[\mathbf{T}^a, \mathbf{T}^b\right] \\
F_{c\mu\nu}\mathbf{T}^c &= \left[\partial_\mu A_{c\nu} - \partial_\nu A_{c\mu} + gf^{ab}{}_c A_{a\mu}A_{b\nu}\right]\mathbf{T}^c \\
F_{c\mu\nu} &= \partial_\mu A_{c\nu} - \partial_\nu A_{c\mu} + gf^{ab}{}_c A_{a\mu}A_{b\nu}
\end{aligned}$$

Our kinetic term is then,

$$-\frac{1}{2}\operatorname{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] = -\frac{1}{2}F_{a\mu\nu}{F_b}^{\mu\nu}\operatorname{Tr}\left[\mathbf{T}^a\mathbf{T}^b\right]$$

As was previously discussed, in our convenient choice of basis of the algebra,

$$\operatorname{Tr}\left[\mathbf{T}^a\mathbf{T}^b\right] = \frac{1}{2}\delta^{ab}$$

Thus, the kinetic term is just,

$$-\frac{1}{4}F_{c\mu\nu}F^{c\mu\nu}$$

We can finally write down our full Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{c\mu\nu}F^{c\mu\nu} - \bar{\mathbf{\Psi}}\not{\mathbf{D}}\mathbf{\Psi} - m\bar{\mathbf{\Psi}}\mathbf{\Psi}$$

Let's open further the fermionic therm,

$$\not{\mathbf{D}} = \gamma^\mu[\mathbb{1}\partial_\mu - \mathrm{i}gA_{a\mu}\mathbf{T}^a_{\mathrm{f}}]$$

Here we do not assume any specific representation for the fermion to transform, but rather label each component as 'Ψ_i', with the spinorial indices being implicitly understood. This get us,

$$\left[\not{\mathbf{D}}\right]_i^{\ j} = \gamma^\mu\left[\delta_i^{\ j}\partial_\mu - \mathrm{i}gA_{a\mu}[\mathbf{T}^a_{\mathrm{f}}]_i^{\ j}\right]$$

Now,

$$\mathcal{L} = -\frac{1}{4}F_{c\mu\nu}F^{c\mu\nu} - \bar{\Psi}^i\left[\not{\mathbf{D}}\right]_i^{\ j}\Psi_j - m\bar{\Psi}^i\Psi_i$$
$$\mathcal{L} = -\frac{1}{4}F_{c\mu\nu}F^{c\mu\nu} - \bar{\Psi}^i\not{\partial}\Psi_i + \mathrm{i}g\bar{\Psi}^i\not{A}_a[\mathbf{T}^a_{\mathrm{f}}]_i^{\ j}\Psi_j - m\bar{\Psi}^i\Psi_i$$

To continue further we have to quantize the theory, what will require also making a gauge fixing. In order to do all that let's take a look at our generating functional,

$$\int \mathcal{D}A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}\right) \tag{1.3}$$

As a gauge transformation is in fact a redundancy, by integrating over all gauge fields we're in fact over-counting the physical states. Thus, we need to cut some of the integration domain, this is done by choosing a particular gauge, and integrating over only the field configurations that satisfy it. To do this we split the integration measure '$\mathcal{D}A$' into the gauge inequivalent part, which we'll still call '$\mathcal{D}A$', and the gauge transformation parameter '$\mathcal{D}\alpha$', this is equivalent to integrating over all field configurations. We would like to throw away the '$\mathcal{D}\alpha$', but for a non-abelian theory, just throw away is not possible, so we'll have to make something more clever, as introducing by hand a Dirac delta to get rid of it,

$$\int \mathcal{D}A\mathcal{D}\alpha\mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}\right) \to \int \mathcal{D}A\mathcal{D}\alpha\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\delta(\alpha_a(x)) \exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}\right)$$

This is as saying that a theory with just an action of '$\int \mathcal{L}$' is not well defined when we go to non-abelian groups, and in fact the correct action have to be, '$-\mathrm{i}\ln\left[\delta(\alpha_a(x))\exp\left(\mathrm{i}\int\mathcal{L}\right)\right]$'. As the whole integral is invariant through any shift '$\alpha^a(x) \to \alpha_a(x) - f_a(x) \equiv G_a(x)$',

$$\begin{aligned}
&\int \mathcal{D}A\mathcal{D}\alpha\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\delta(\alpha_a - f_a)\exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}\right)\\
&\int \mathcal{D}A\mathcal{D}\alpha\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathrm{Det}\left[\frac{\delta G_a}{\delta\alpha_b}\right]\Bigg|_{\alpha_a=0}\delta(G_c)\exp\left(\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}\right)
\end{aligned}$$

Our choice of 'G_a' will be to preserve Lorentz invariance, that is, the 'R_ξ'-Gauge,

$$G_a(x) = \partial_\mu A_a{}^\mu - \omega_a(x)$$

With '$\omega_a(x)$' arbitrary. To do the determinant is necessary to know how 'G_a' transform under gauge, and as the determinant is to be evaluated at '$\alpha_a = 0$', it's sufficient to know how 'G_a' transforms under gauge to first order, that being said,

$$\begin{aligned}
\mathbf{A}_\mu(x) &\to \mathbf{U}(x)\mathbf{A}_\mu(x)\mathbf{U}^\dagger(x) + \frac{\mathrm{i}}{g}\mathbf{U}(x)\partial_\mu\mathbf{U}^\dagger(x), \quad \mathbf{U}(x) = \exp\left(-\mathrm{i}g\alpha_a(x)\mathbf{T}^a_{\mathrm{A}}\right)\\
A_{a\mu}(x)\mathbf{T}^a &\to A_{a\mu}(x)\mathbf{U}(x)\mathbf{T}^a\mathbf{U}^\dagger(x) - \partial_\mu\alpha_a(x)\mathbf{U}(x)\mathbf{T}^a\mathbf{U}^\dagger(x)\\
A_{a\mu}\mathbf{T}^a &\to A_{a\mu}\left(\mathbf{T}^a - g\alpha_b f^{ab}{}_c\mathbf{T}^c\right) - \partial_\mu\alpha_a\mathbf{T}^a\\
A_{a\mu} &\to A_{a\mu} - gA_{b\mu}\alpha_c f_a{}^{bc} - \partial_\mu\alpha_a\\
A_{a\mu} &\to A_{a\mu} - \left(\delta_a{}^c\partial_\mu\alpha_c - \mathrm{i}gA_{b\mu}\alpha_c\left(-\mathrm{i}f^b{}_a{}^c\right)\right)\\
A_{a\mu} &\to A_{a\mu} - \left(\delta_a{}^c\partial_\mu - \mathrm{i}gA_{b\mu}\left[\mathbf{T}^b_{\mathrm{A}}\right]_a{}^c\right)\alpha_c\\
A_{a\mu} &\to A_{a\mu} - [\mathbf{D}_\mu]_a{}^c\alpha_c\\
\partial^\mu A_{a\mu} &\to \partial^\mu A_{a\mu} - \partial^\mu[\mathbf{D}_\mu]_a{}^c\alpha_c
\end{aligned}$$

So we do know the transformation law for 'G_a',

$$\begin{aligned}
G_a(x) &\to G_a(x) - \partial^\mu[\mathbf{D}_\mu]_a{}^c\alpha_c\\
\frac{\delta G_a(x)}{\delta\alpha_b(y)}\Bigg|_{\alpha_b=0} &= -\partial^\mu[\mathbf{D}_\mu]_a{}^b\delta^{(4)}(x-y)
\end{aligned}$$

And finally the determinant is written as a path integral over anti-commuting variables,

$$\begin{aligned}
\mathrm{Det}\left[\frac{\delta G_a(x)}{\delta\alpha_b(y)}\right]\Bigg|_{\alpha_b=0} &\propto \int \mathcal{D}\bar{c}\mathcal{D}c\exp\left(\mathrm{i}\int \mathrm{d}^4x\,\bar{c}^a\partial^\mu[\mathbf{D}_\mu]_a{}^b c_b\right)\\
\mathrm{Det}\left[\frac{\delta G_a(x)}{\delta\alpha_b(y)}\right]\Bigg|_{\alpha_b=0} &\propto \int \mathcal{D}\bar{c}\mathcal{D}c\exp\left(-\mathrm{i}\int \mathrm{d}^4x\,\partial^\mu\bar{c}^a[\mathbf{D}_\mu]_a{}^b c_b\right)
\end{aligned}$$

Those are the so waited ghosts,

$$\mathcal{L}_{\text{gh}} = -\partial^\mu \bar{c}^a [\mathbf{D}_\mu]_a{}^b c_b$$
$$\mathcal{L}_{\text{gh}} = -\partial^\mu \bar{c}^a \left(\delta_a{}^b \partial_\mu - \mathrm{i} g A_{c\mu} [\mathbf{T}^c_{\text{A}}]_a{}^b\right) c_b$$
$$\mathcal{L}_{\text{gh}} = -\partial_\mu \bar{c}^a \partial^\mu c_a + g A_{c\mu} f_a{}^{bc} \partial^\mu \bar{c}^a c_b$$

Including this *ghost Lagrangian* in our theory, we can don't worry about over-counting in the gauge boson integration, and just switch back to integrating over all field configurations. In order to not get messy, we also are going to name,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{c\mu\nu} F^{c\mu\nu}$$
$$\mathcal{L}_{\text{f}} = -\bar{\Psi}^i \partial\!\!\!/ \Psi_i + \mathrm{i} g \bar{\Psi}^i A\!\!\!/_a [\mathbf{T}^a_{\text{f}}]_i{}^j \Psi_j - m \bar{\Psi}^i \Psi_i$$

So that our theory up to now is described by the following generating functional,

$$\int \mathcal{D}A \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}\bar{c} \mathcal{D}c\, \delta(G_c) \exp\left(\mathrm{i} \int \mathrm{d}^4 x \left(\mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{f}} + \mathcal{L}_{\text{gh}}\right)\right)$$

We still have a strange factor of '$\delta(G_c)$', which carry an arbitrary function 'ω_c'. Our theory is completely uncaring about this function, so the best way to get rid of it is to integrate over with a choice of weight. The simplest one is a Gaussian/quadratic,

$$\int \mathcal{D}(\cdots) \delta(G_c) \exp\left(\mathrm{i}(S_{\text{YM}} + S_{\text{f}} + S_{\text{gh}})\right) \to \int \mathcal{D}(\cdots) \exp\left(\mathrm{i}(S_{\text{YM}} + S_{\text{f}} + S_{\text{gh}})\right) \times$$
$$\times \int \mathcal{D}\omega\, \delta(G_c) \exp\left(-\frac{\mathrm{i}}{2\xi} \int \mathrm{d}^4 x\, \omega^a \omega_a\right)$$
$$\int \mathcal{D}(\cdots) \delta(G_c) \exp\left(\mathrm{i}(S_{\text{YM}} + S_{\text{f}} + S_{\text{gh}})\right) \to \int \mathcal{D}(\cdots) \exp\left(\mathrm{i}(S_{\text{YM}} + S_{\text{f}} + S_{\text{gh}})\right) \times$$
$$\times \exp\left(-\frac{\mathrm{i}}{2\xi} \int \mathrm{d}^4 x\, \partial^\mu A_{a\mu} \partial^\nu A^a{}_\nu\right)$$

Where we get our final additional piece of the theory,

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} \partial^\mu A_{a\mu} \partial^\nu A^a{}_\nu$$

So the full theory is described by the following functional generator,

$$\int \mathcal{D}A \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}\bar{c} \mathcal{D}c \exp\left(\mathrm{i}S_{\text{YM}} + \mathrm{i}S_{\text{f}} + \mathrm{i}S_{\text{gh}} + \mathrm{i}S_{\text{gf}}\right) \tag{1.4}$$

1.2 Renormalization and Feynman Rules

Before working out all Feynman rules and getting to compute the diagrams, we have to say a few words on the renormalizability of the theory, due to, for renormalization be possible, our Lagrangian must contain **all** up to '4' mass dimension symmetry compatible terms. Is this the case? First we have to know all symmetries of our theory:

- Poincaré Invariance
- ‘CPT’ Invariance
- Gauge Invariance
- Ghost Number Conservation
- Anti-Ghost Translation Invariance

We’re not going to prove, but, the theory constructed here already contains all terms compatible with those symmetries, so in principle we wouldn’t need to worry about this.

Let’s open the Lagrangian of the full theory, starting by,

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}} &= -\frac{1}{4}F_{e\mu\nu}F^{e\mu\nu} \\
&= -\frac{1}{4}\big(\partial_\mu A_{e\nu} - \partial_\nu A_{e\mu} + g f^{ab}{}_{e} A_{a\mu}A_{b\nu}\big)\big(\partial^\mu A^{e\nu} - \partial^\nu A^{e\mu} + g f_{cd}{}^{e} A^{c\mu}A^{d\nu}\big) \\
&= -\frac{1}{2}\partial_\mu A_{a\nu}\partial^\mu A^{a\nu} + \frac{1}{2}\partial_\mu A_{a\nu}\partial^\nu A^{a\mu} - g f^{ab}{}_{c} A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} \\
&\qquad - \frac{g^2}{4} f^{ab}{}_{e} f_{cd}{}^{e} A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\
&= \frac{1}{2}A^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)A^{b\nu} - g f^{ab}{}_{c} A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} \\
&\qquad - \frac{g^2}{4} f^{ab}{}_{e} f_{cd}{}^{e} A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu}
\end{aligned}$$

Putting all together,

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}&A^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)A^{b\nu} + \frac{1}{2\xi}A_{a\mu}\partial^\mu\partial^\nu A^{a}{}_{\nu} \\
&- g f^{ab}{}_{c} A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2}{4} f^{ab}{}_{e} f_{cd}{}^{e} A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\
&- \bar{\Psi}^i\delta_{ij}\partial\!\!\!/\Psi^j - \bar{\Psi}^i\delta_{ij} m\Psi^j + \mathrm{i}g\bar{\Psi}^i\gamma^\mu A_{a\mu}[\mathbf{T}^a_{\mathrm{f}}]_{ij}\Psi^j \\
&- \partial_\mu\bar{c}^a\partial^\mu c_a + gA_{c\mu}f^{abc}\partial^\mu\bar{c}_a c_b
\end{aligned} \tag{1.5}$$

Actually now all fields and couplings are treated as being the bare ones, that is,

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}&{A_0}^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big){A_0}^{b\nu} + \frac{1}{2\xi_0}A_{0a\mu}\partial^\mu\partial^\nu {A_0}^{a}{}_{\nu} \\
&- g_0 f^{ab}{}_{c} A_{0a\mu}A_{0b\nu}\partial^\mu {A_0}^{c\nu} - \frac{g_0^2}{4} f^{ab}{}_{e} f_{cd}{}^{e} A_{0a\mu}A_{0b\nu}{A_0}^{c\mu}{A_0}^{d\nu} \\
&- \bar{\Psi}^i_0\delta_{ij}\partial\!\!\!/\Psi^j_0 - \bar{\Psi}^i_0\delta_{ij} m_0\Psi^j_0 + \mathrm{i}g_0\bar{\Psi}^i_0\gamma^\mu A_{0a\mu}[\mathbf{T}^a_{\mathrm{f}}]_{ij}\Psi^j_0 \\
&- \partial_\mu\bar{c}^a_0\partial^\mu c_{0a} + g_0 A_{0c\mu}f^{abc}\partial^\mu\bar{c}_{0a} c_{0b}
\end{aligned} \tag{1.6}$$

So now we can really pass to the renormalized fields and couplings adding the ‘Z’ factors,

$$\begin{aligned}
\mathcal{L} = \frac{Z_A}{2}&A^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)A^{b\nu} + \frac{Z_\xi}{2\xi}A_{a\mu}\partial^\mu\partial^\nu A^{a}{}_{\nu} \\
&- gZ_{3g} f^{ab}{}_{c} A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2 Z_{4g}}{4} f^{ab}{}_{e} f_{cd}{}^{e} A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\
&- Z_\Psi\bar{\Psi}^i\delta_{ij}\partial\!\!\!/\Psi^j - Z_m\bar{\Psi}^i\delta_{ij} m\Psi^j + \mathrm{i}gZ_{g\Psi}\bar{\Psi}^i\gamma^\mu A_{a\mu}[\mathbf{T}^a_{\mathrm{f}}]_{ij}\Psi^j \\
&- Z_c\partial_\mu\bar{c}^a\partial^\mu c_a + gZ_{gc}A_{c\mu}f^{abc}\partial^\mu\bar{c}_a c_b
\end{aligned} \tag{1.7}$$

If we rearrange the terms,

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}A^{a\mu}\delta_{ab}\left(g_{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)A^{b\nu} - \bar{\Psi}^i\delta_{ij}\big(\not{\partial} + m\big)\Psi^j - \partial_\mu\bar{c}^a\delta_{ab}\partial^\mu c^b \\ &- gZ_{3g}f^{ab}{}_c A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2 Z_{4g}}{4}f^{ab}{}_e f_{cd}{}^e A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\ &+ \mathrm{i}gZ_{g\Psi}\bar{\Psi}^i\gamma^\mu A_{a\mu}[\mathbf{T}^a_{\mathrm{f}}]_{ij}\Psi^j + gZ_{gc}A_{c\mu}f^{abc}\partial^\mu\bar{c}_a c_b \\ &+ \frac{(Z_A-1)}{2}A^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)A^{b\nu} + \frac{(Z_\xi - 1)}{2\xi}A_{a\mu}\partial^\mu\partial^\nu A^a{}_\nu - (Z_c-1)\partial_\mu\bar{c}^a\delta_{ab}\partial^\mu c^b \\ &- (Z_\Psi - 1)\bar{\Psi}^i\delta_{ij}\not{\partial}\Psi^j - (Z_m-1)\bar{\Psi}^i\delta_{ij}m\Psi^j \end{aligned} \tag{1.8}$$

This isn't yet the final version, because, as we are going to use dimensional regularization, all couplings dimensions need to be reset with a choice of particular mass scale 'Λ', in 'D'-dimensional space-time we have,

$$[A] = \frac{D-2}{2}, \quad [\Psi] = \frac{D-1}{2}, \quad [c] = \frac{D-2}{2}$$

Setting '$4 - D = 2\epsilon$',

$$\begin{aligned}[gAA\partial A] &= \frac{3}{2}(D-2) + 1 + [g] = D \\ [g] &= -\frac{1}{2}(D-4) = \epsilon\end{aligned} \tag{1.9}$$

$$\begin{aligned}\big[g^2AAAA\big] &= \frac{4}{2}(D-2) + 2[g] = D \\ [g] &= \frac{1}{2}(-D+4) = \epsilon\end{aligned} \tag{1.10}$$

$$\begin{aligned}[gA\partial\bar{c}c] &= \frac{3}{2}(D-2) + 1 + [g] = D \\ [g] &= -\frac{1}{2}(D-4) = \epsilon\end{aligned} \tag{1.11}$$

$$\begin{aligned}\big[gA\bar{\Psi}\gamma\Psi\big] &= \frac{1}{2}(D-2) + (D-1) + [g] = D \\ [g] &= -\frac{1}{2}(D-4) = \epsilon\end{aligned} \tag{1.12}$$

We see here thatApplying the changes we get,

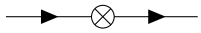
$$\begin{aligned}\mathcal{L} = &\frac{1}{2}A^{a\mu}\delta_{ab}\left(g_{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)A^{b\nu} - \bar{\Psi}^i\delta_{ij}\big(\not{\partial} + m\big)\Psi^j - \partial_\mu\bar{c}^a\delta_{ab}\partial^\mu c^b \\ &- gZ_{3g}\Lambda^\epsilon f^{ab}{}_c A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2 Z_{4g}\Lambda^{2\epsilon}}{4}f^{ab}{}_e f_{cd}{}^e A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\ &+ \mathrm{i}gZ_{g\Psi}\Lambda^\epsilon\bar{\Psi}^i\gamma^\mu A_{a\mu}[\mathbf{T}^a_{\mathrm{f}}]_{ij}\Psi^j + gZ_{gc}\Lambda^\epsilon A_{c\mu}f^{abc}\partial^\mu\bar{c}_a c_b \\ &+ \frac{(Z_A-1)}{2}A^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)A^{b\nu} + \frac{(Z_\xi - 1)}{2\xi}A_{a\mu}\partial^\mu\partial^\nu A^a{}_\nu - (Z_c-1)\partial_\mu\bar{c}^a\delta_{ab}\partial^\mu c^b \\ &- (Z_\Psi - 1)\bar{\Psi}^i\delta_{ij}\not{\partial}\Psi^j - (Z_m-1)\bar{\Psi}^i\delta_{ij}m\Psi^j \end{aligned} \tag{1.13}$$

From where the propagators Feynman rules become clear as being,

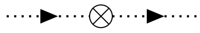
- Fermion Propagator i  j $= \frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{p}+m}{p^2+m^2}\delta_{ij}$

- Gauge Propagator $\begin{matrix} a \\ \mu \end{matrix}$ $\begin{matrix} b \\ \nu \end{matrix}$ $= \frac{1}{\mathrm{i}}\frac{\delta_{ab}}{p^2}\left(g_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}\right)$

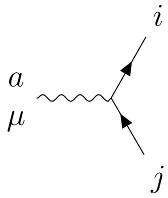
- Ghost Propagator a  b $= \frac{1}{\mathrm{i}}\frac{1}{p^2}\delta_{ab}$

- Fermion Counter-Term i  j $= (Z_\Psi - 1)\delta_{ij}\not{p} - \mathrm{i}(Z_m - 1)\delta_{ij}m$

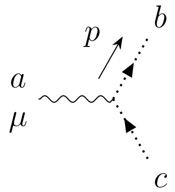
- Gauge Counter-Term $\begin{matrix} a \\ \mu \end{matrix}$  $\begin{matrix} b \\ \nu \end{matrix}$ $= -\mathrm{i}(Z_A - 1)\delta_{ab}(g_{\mu\nu}p^2 - p_\mu p_\nu) - \mathrm{i}\frac{(Z_\xi - 1)}{\xi}\delta_{ab}p_\mu p_\nu$

- Ghost Counter-Term a  b $= -\mathrm{i}(Z_c - 1)\delta_{ab}p^2$

The Fermion-Gauge and Ghost-Gauge interaction terms are also simple,

- Fermion-Gauge Interaction 

$= -gZ_{g\Psi}\Lambda^\epsilon\gamma^\mu[\mathbf{T}_\mathrm{f}^a]_{ij}$

- Ghost-Gauge Interaction 

$= gZ_{gc}\Lambda^\epsilon f^{abc}p_\mu$

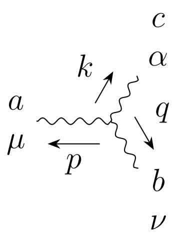
The Gauge-Gauge Interactions need to be treated a little bit more carefully, due to being present some permutation symmetries. Let's first look at the term,

$$-gZ_{3g}\Lambda^\epsilon f^{ab}{}_c A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu}$$

Certainly it must have three Gauge bosons with a momentum of one of them, but, due to this term having a 'f' factor, it must be also antisymmetric in color index,

$$-gZ_{3g}\Lambda^\epsilon f^{ab}{}_c\frac{1}{3}\left(A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} + A_{b\mu}A^{c\nu}\partial^\mu A_{a\nu} + A^{c\mu}A^a{}_\nu\partial_\mu A^{b\nu}\right)$$

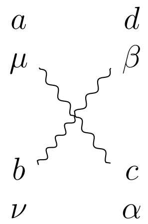
But each term can be further antisymmetrize in the last two index, what will add a factor of '2' in the denominator, this '3!' will be cancelled by the possible permutations of the leg. If we label the 'a, b, c' color index momentum as 'p, q, k', all flowing out of the vertex, the result will be,

- Gauge Cubic Interaction 

$$= gZ_{3g}\Lambda^\epsilon f^{abc}\Big((q-k)_\mu g_{\nu\alpha} + (k-p)_\nu g_{\alpha\mu} + (p-q)_\alpha g_{\mu\nu}\Big) \qquad (1.14)$$

The four Gauge vertex will also need some additional factors, but the final result is,

- Gauge Quartic Interaction



$$= -\mathrm{i}g^2 Z_{4g}\Lambda^{2\epsilon}\left[f^{ab}{}_{e} f^{cde}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) + f^{ac}{}_{e} f^{dbe}(g_{\mu\beta}g_{\alpha\nu} - g_{\mu\nu}g_{\beta\alpha}) + f^{ad}{}_{e} f^{bce}(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\beta\nu})\right] \tag{1.15}$$

This finishes all Feynman rules we'll need.

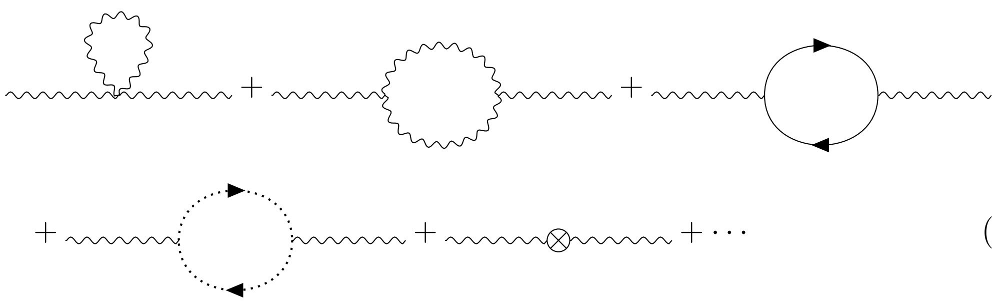
1.3 Gauge Boson Self Energy

Now we proceed to compute the renormalization of the Gauge Boson self energy to one loop, just as a remainder,

$$\begin{aligned}
\frac{1}{\mathrm{i}}\boldsymbol{\Delta}^{ab}_{\mu\nu}(p^2) &= \frac{1}{\mathrm{i}}\Delta^{ab}_{\mu\nu}(p^2) + \frac{1}{\mathrm{i}}\Delta^{ac}_{\mu\alpha}(p^2)\mathrm{i}\Pi^{cd}_{\alpha\beta}(p^2)\frac{1}{\mathrm{i}}\Delta^{db}_{\beta\nu}(p^2) + \cdots \\
\frac{1}{\mathrm{i}}\boldsymbol{\Delta}^{ab}_{\mu\nu}(p^2) &= \frac{1}{\mathrm{i}}\Delta^{ab}_{\mu\nu}(p^2) + \frac{1}{\mathrm{i}}\Delta^{ac}_{\mu\alpha}(p^2)\mathrm{i}\Pi^{cd}_{\alpha\beta}(p^2)\left\{\frac{1}{\mathrm{i}}\Delta^{db}_{\beta\nu} + \frac{1}{\mathrm{i}}\Delta^{de}_{\beta\rho}\mathrm{i}\Pi^{ef}_{\rho\sigma}\frac{1}{\mathrm{i}}\Delta^{fb}_{\sigma\nu} + \cdots\right\} \\
\frac{1}{\mathrm{i}}\boldsymbol{\Delta}^{ab}_{\mu\nu}(p^2) &= \frac{1}{\mathrm{i}}\Delta^{ab}_{\mu\nu}(p^2) + \frac{1}{\mathrm{i}}\Delta^{ac}_{\mu\alpha}(p^2)\mathrm{i}\Pi^{cd}_{\alpha\beta}(p^2)\frac{1}{\mathrm{i}}\boldsymbol{\Delta}^{db}_{\beta\nu}(p^2) \\
\frac{1}{\mathrm{i}}\Delta^{ab}_{\mu\nu}(p^2) &= \left[\delta^{ad}_{\mu\beta} - \frac{1}{\mathrm{i}}\Delta^{ac}_{\mu\alpha}(p^2)\mathrm{i}\Pi^{cd}_{\alpha\beta}(p^2)\right]\frac{1}{\mathrm{i}}\boldsymbol{\Delta}^{db}_{\beta\nu}(p^2) \\
\boldsymbol{\Delta}(p^2) &= \left[1 - \Delta(p^2)\Pi(p^2)\right]^{-1}\Delta(p^2) \\
\boldsymbol{\Delta}^{-1}{}^{ab}_{\mu\nu}(p^2) &= \Delta^{-1}{}^{ab}_{\mu\nu}(p^2) - \Pi^{ab}_{\mu\nu}(p^2) = \delta^{ab}\left(p^2 g_{\mu\nu} + \left(\frac{1}{\xi} - 1\right)p_\mu p_\nu\right) - \Pi^{ab}_{\mu\nu}(p^2)
\end{aligned} \tag{1.16}$$

Where '$\mathrm{i}\Pi^{ab}_{\mu\nu}(p^2)$' is the 1PI contributions,

$$\mathrm{i}\Pi^{ab}_{\mu\nu}(p^2) =$$



(1.17)

The first contribution is,

$$
\begin{aligned}
\mathrm{i}\Pi^{(1)ab}_{\mu\nu} &= -\mathrm{i}g^2 Z_{4g}\Lambda^{2\epsilon}\left[f^{ab}{}_{e}f^{cde}(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha})+f^{ac}{}_{e}f^{dbe}(g_{\mu\beta}g_{\nu\alpha}-g_{\mu\nu}g_{\beta\alpha})\right.\\
&\qquad\left.+f^{ad}{}_{e}f^{bce}(g_{\mu\nu}g_{\alpha\beta}-g_{\mu\alpha}g_{\beta\nu})\right]\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\delta_{cd}}{\mathrm{i}k^2}\left(g^{\alpha\beta}+(\xi-1)\frac{k^\alpha k^\beta}{k^2}\right)\\
&= -g^2 Z_{4g}\Lambda^{2\epsilon}\left[f^{a}{}_{cd}f^{cbd}(g_{\mu\beta}g_{\nu\alpha}-g_{\mu\nu}g_{\beta\alpha})\right.\\
&\qquad\left.+f^{a}{}_{cd}f^{bcd}(g_{\mu\nu}g_{\alpha\beta}-g_{\mu\alpha}g_{\beta\nu})\right]\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\left(g^{\alpha\beta}+(\xi-1)\frac{k^\alpha k^\beta}{k^2}\right)\\
&= -g^2 Z_{4g}\Lambda^{2\epsilon}f^{a}{}_{cd}f^{bcd}\left[2g_{\mu\nu}g_{\beta\alpha}-g_{\mu\alpha}g_{\beta\nu}-g_{\mu\beta}g_{\nu\alpha}\right]\times\\
&\qquad\times\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\left(g^{\alpha\beta}+(\xi-1)\frac{k^\alpha k^\beta}{k^2}\right)\\
&= -g^2 Z_{4g}\Lambda^{2\epsilon}f^{a}{}_{cd}f^{bcd}\left[2g_{\mu\nu}g_{\beta\alpha}-g_{\mu\alpha}g_{\beta\nu}-g_{\mu\beta}g_{\nu\alpha}\right]\times\\
&\qquad\times\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\left(g^{\alpha\beta}+(\xi-1)\frac{k^\alpha k^\beta}{k^2}\right)\\
&= 2^2 g Z_{4g}\Lambda^{2\epsilon}f^{a}{}_{cd}f^{bcd}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\left[g_{\mu\nu}(1-D)+(\xi-1)k_\mu k_\nu-g_{\mu\nu}(\xi-1)k^2\right]\\
&= 2g^2 Z_{4g}\Lambda^{2\epsilon}f^{a}{}_{cd}f^{bcd}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\frac{1}{D}(\xi-1)k^2 g_{\mu\nu}\\
\mathrm{i}\Pi^{(1)ab}_{\mu\nu} &= 0
\end{aligned}
$$

This impressive result is consequence of dimensional regularization, which guarantees that,

$$\int \mathrm{d}^Dk\,\left(k^2\right)^a \equiv 0$$

Besides this result, we also made use of,

$$\int \mathrm{d}^Dk\,k_\mu k_\nu f\left(k^2\right) = \frac{1}{D}g_{\mu\nu}\int \mathrm{d}^Dk\,k^2 f\left(k^2\right)$$

For the next one, we have to choose a gauge. The most easy one to do computations is the '$\xi=1$',

$$
\begin{aligned}
\mathrm{i}\Pi^{(2)ab}_{\mu\nu} &= \frac{1}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D} g Z_{3g}\Lambda^{\epsilon} f^{adc}\Big[(p-2k)_\mu g_{\alpha\beta} + (k+p)_\beta g_{\alpha\mu} + (k-2p)_\alpha g_{\mu\beta}\Big]\times \\
&\quad \times \frac{1}{\mathrm{i}}\frac{\delta_{ce}}{k^2} g^{\alpha\rho}\frac{1}{\mathrm{i}}\frac{\delta_{df}}{(k-p)^2} g^{\beta\sigma}\times \\
&\quad \times g Z_{3g}\Lambda^{\epsilon} f^{bef}\Big[(p-2k)_\nu g_{\rho\sigma} + (k-2p)_\rho g_{\sigma\nu} + (p+k)_\sigma g_{\rho\nu}\Big] \\
\mathrm{i}\Pi^{(2)ab}_{\mu\nu} &= -\frac{g^2}{2} Z_{3g}^2 \Lambda^{2\epsilon} f^{a}{}_{cd} f^{bdc}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\Big[(p-2k)_\mu g_{\alpha\beta} + (k+p)_\beta g_{\alpha\mu} + (k-2p)_\alpha g_{\mu\beta}\Big]\times \\
&\quad \times \frac{g^{\alpha\rho}g^{\beta\sigma}}{k^2(k-p)^2}\Big[(p-2k)_\nu g_{\rho\sigma} + (k-2p)_\rho g_{\sigma\nu} + (p+k)_\sigma g_{\rho\nu}\Big] \\
\mathrm{i}\Pi^{(2)ab}_{\mu\nu} &= \frac{g^2}{2} Z_{3g}^2 \Lambda^{2\epsilon} f^{a}{}_{cd} f^{bcd}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{N_{\mu\nu}}{k^2(k-p)^2} \\
\mathrm{i}\Pi^{(2)ab}_{\mu\nu} &= \frac{g^2}{2} Z_{3g}^2 \Lambda^{2\epsilon} f^{a}{}_{cd} f^{bcd}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{N_{\mu\nu}}{\left[xk^2 + (1-x)(k-p)^2\right]^2} \\
\mathrm{i}\Pi^{(2)ab}_{\mu\nu} &= \frac{g^2}{2} Z_{3g}^2 \Lambda^{2\epsilon} f^{a}{}_{cd} f^{bcd}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{N_{\mu\nu}}{\left[(k-(1-x)p)^2 + p^2 x(1-x)\right]^2}
\end{aligned}
$$

With,

$$
N_{\mu\nu} = D(p-2k)_\mu (p-2k)_\nu + g_{\mu\nu}\big(2k^2 + 5p^2 - 2k\cdot p\big) - 3[2k_\mu k_\nu + 2p_\mu p_\nu - k_\mu p_\nu - k_\nu p_\mu]
$$

We can now go ahead and to the change of integration variable, '$k \to q = k - (1-x)p$', what changes '$N_{\mu\nu}$', since the denominator is already in the form '$f(q^2)$', any odd power of 'q' in the numerator will integrate to zero, with this information in mind we can substitute in '$N_{\mu\nu}$' keeping only the non-zero terms,

$$
N_{\mu\nu} = 2g_{\mu\nu}q^2 + q_\mu q_\nu[4D-6] + g_{\mu\nu}p^2\left[2x^2 - 2x + 5\right] + p_\mu p_\nu\left[D(2x-1)^2 - 6\big(x^2 + x - 1\big)\right]
$$

Inside the integral, '$q_\mu q_\nu$' is replaceable with '$\frac{1}{D}g_{\mu\nu}q^2$', thus,

$$
N_{\mu\nu} = 6g_{\mu\nu}q^2\left[1 - \frac{1}{D}\right] + g_{\mu\nu}p^2\left[2x^2 - 2x + 5\right] + p_\mu p_\nu\left[D(2x-1)^2 - 6\big(x^2 + x - 1\big)\right]
$$

So our integral is,

$$
\mathrm{i}\Pi^{(2)ab}_{\mu\nu} = \frac{g^2}{2} Z_{3g}^2 \Lambda^{2\epsilon} f^{a}{}_{cd} f^{bcd}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{N_{\mu\nu}}{\left[q^2 + p^2 x(1-x)\right]^2}
$$

As is shown in the appendix,

$$
\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{q^2}{(q^2 + A)^2} = \frac{D}{2-D}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{A}{(q^2 + A)^2} \tag{1.18}
$$

So we can also substitute '$q^2 \to \frac{D}{2-D}p^2x(1-x)$', we got then,

$$N_{\mu\nu} = 6g_{\mu\nu}p^2x(1-x)\frac{D-1}{2-D} + g_{\mu\nu}p^2\left[2x^2-2x+5\right] + p_\mu p_\nu\left[D(2x-1)^2 - 6\left(x^2+x-1\right)\right]$$

In this way '$N_{\mu\nu}$ doesn't depend on 'q', and then can be taken outside of the integral, this integral turn out to be well known, as is calculated in the appendix,

$$\int \frac{\mathrm{d}^Dq}{(2\pi)^D}\frac{1}{\left(q^2+A\right)^2} = \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\left(p^2x(1-x)\right)^{\frac{D}{2}-2}$$

So we can put back in to get,

$$\mathrm{i}\Pi^{(2)ab}_{\mu\nu} = \frac{g^2}{2}Z_{3g}^2\Lambda^{2\epsilon}f^a{}_{cd}f^{bcd}\int_0^1 \mathrm{d}x\, N_{\mu\nu}\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\left(p^2x(1-x)\right)^{\frac{D}{2}-2}$$

As we're interested in computing the beta function, we only need to know the divergent part of each contribution, thus, as the divergence is generated by '$D \to 4$', notice that the only place where this limit has a pole is in '$\Gamma\left(2-\frac{D}{2}\right)$', so we can substitute back '$D = 4$' everywhere apart from the Gamma function, where we use '$D = 4-2\epsilon$'. We also further simplify by using the nomenclature '$T(\mathrm{A})\delta^{ab} = f^a{}_{cd}f^{bcd}$', that is, the Trace Factor of the Adjoint representation.

$$\mathrm{i}\Pi^{(2)ab}_{\mu\nu} = \mathrm{i}\delta^{ab}\frac{g^2}{32\pi^2}Z_{3g}^2\Gamma(\epsilon)T(\mathrm{A})\int_0^1 \mathrm{d}x\, N_{\mu\nu}$$

What remains to be done is the integral on 'x', which is just a polynomial integral, it's trivial and gives,

$$\mathrm{i}\Pi^{(2)ab}_{\mu\nu} = \mathrm{i}\delta^{ab}\frac{g^2}{32\pi^2}Z_{3g}^2\Gamma(\epsilon)T(\mathrm{A})\left[\frac{19}{6}g_{\mu\nu}p^2 - \frac{11}{3}p_\mu p_\nu\right]$$

Finally, using the Laurent expansion of the Gamma Function, and keeping only divergent terms,

$$\mathrm{i}\Pi^{(2)ab}_{\mu\nu} = \frac{1}{\epsilon}\mathrm{i}\delta^{ab}\frac{g^2}{32\pi^2}Z_{3g}^2T(\mathrm{A})\left[\frac{19}{6}g_{\mu\nu}p^2 - \frac{11}{3}p_\mu p_\nu\right]$$

This completes the calculus of the first diagram, the third is the fermion loop contribution,

$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = -gZ_{g\Psi}\Lambda^\epsilon[\mathbf{T}^a_\mathrm{f}]_{ij}(-1)\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\,\mathrm{Tr}\left[\gamma_\mu\frac{1}{\mathrm{i}}\frac{(-\mathrm{i}\not{k}+m)}{k^2+m}\delta^{ik}\gamma_\nu\frac{1}{\mathrm{i}}\frac{\left(-\mathrm{i}\left(\not{k}-\not{p}\right)+m\right)}{(k-p)^2+m^2}\delta^{lj}\right]\times$$
$$\times(-1)gZ_{g\Psi}\Lambda^\epsilon\left[\mathbf{T}^b_\mathrm{f}\right]_{lk}$$
$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = g^2Z_{g\Psi}^2\Lambda^{2\epsilon}\,\mathrm{Tr}\left[\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\right]\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\,\mathrm{Tr}\left[\gamma_\mu\frac{(-\mathrm{i}\not{k}+m)}{k^2+m}\gamma_\nu\frac{\left(-\mathrm{i}\left(\not{k}-\not{p}\right)+m\right)}{(k-p)^2+m^2}\right]$$

We readily identify the factor '$\mathrm{Tr}\left[\mathbf{T}_\mathrm{f}^a\mathbf{T}_\mathrm{f}^b\right] = T(\mathrm{F})\delta^{ab}$' as the Trace Factor of the Fermionic representation. The remaining trace over the gamma matrices is done remembering that only even combinations survive under the trace,

$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = \delta^{ab}g^2Z^2_{g\Psi}\Lambda^{2\epsilon}T(\mathrm{F})\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\mathrm{Tr}\left[\gamma_\mu(-\mathrm{i}\not{k}+m)\gamma_\nu\big(-\mathrm{i}(\not{k}-\not{p})+m\big)\right]}{(k^2+m^2)\big((k-p)^2+m^2\big)}$$
$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = \delta^{ab}g^2Z^2_{g\Psi}\Lambda^{2\epsilon}T(\mathrm{F})\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\left\{m^2Dg_{\mu\nu}-\mathrm{Tr}\left[\gamma_\mu\not{k}\gamma_\nu(\not{k}-\not{p})\right]\right\}}{(k^2+m^2)\big((k-p)^2+m^2\big)}$$
$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = \delta^{ab}g^2Z^2_{g\Psi}\Lambda^{2\epsilon}T(\mathrm{F})D\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{m^2g_{\mu\nu}-2k_\mu k_\nu+g_{\mu\nu}k^2+k_\mu p_\nu+k_\nu p_\mu-g_{\mu\nu}k\cdot p}{(k^2+m^2)\big((k-p)^2+m^2\big)}$$

We follow with the standard procedure of performing a Feynman reparametrization,

$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = \delta^{ab}g^2Z^2_{g\Psi}\Lambda^{2\epsilon}T(\mathrm{F})D\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{N_{\mu\nu}}{\left[x(k^2+m^2)+(1-x)\big((k-p)^2+m^2\big)\right]^2}$$
$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = \delta^{ab}g^2Z^2_{g\Psi}\Lambda^{2\epsilon}T(\mathrm{F})D\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{N_{\mu\nu}}{\left[(k-(1-x)p)^2+p^2x(1-x)+m^2\right]^2}$$

Changing integration variables to '$q = k-(1-x)p$', substituting back in '$N_{\mu\nu}$' and neglecting linear terms in 'q',

$$N_{\mu\nu} = g_{\mu\nu}q^2-2q_\mu q_\nu-g_{\mu\nu}p^2x(1-x)+2p_\mu p_\nu x(1-x)+m^2g_{\mu\nu}$$

Again, with everything inside the integral, we can do '$q_\mu q_\nu \to \frac{1}{D}g_{\mu\nu}q^2$',

$$N_{\mu\nu} = q^2g_{\mu\nu}\left(1-\frac{2}{D}\right)-g_{\mu\nu}p^2x(1-x)+2p_\mu p_\nu x(1-x)+m^2g_{\mu\nu}$$

So, up to now the contribution is,

$$\mathrm{i}\Pi^{(3)ab}_{\mu\nu} = \delta^{ab}g^2Z^2_{g\Psi}\Lambda^{2\epsilon}T(\mathrm{F})D\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{N_{\mu\nu}}{[q^2+p^2x(1-x)+m^2]^2}$$

As was stated before, inside the integral, we can replace '$q^2 \to \frac{D}{2-D}(p^2x(1-x)+m^2)$', so the '$N_{\mu\nu}$' becomes,

$$N_{\mu\nu} = 2p_\mu p_\nu x(1-x)-2p^2x(1-x)g_{\mu\nu}$$

So that it can be moved outside of the integral, and we're left with our old friendly integral,

$$\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{[q^2+p^2x(1-x)+m^2]^2} = \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\left(p^2x(1-x)+m^2\right)^{\frac{D}{2}-2}$$

So that,

$$\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = \delta^{ab} g^2 Z_{g\Psi}^2 \Lambda^{2\epsilon} T(\mathrm{F}) D \int_0^1 \mathrm{d}x\, N_{\mu\nu} \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \Gamma\left(2 - \frac{D}{2}\right) \left(p^2 x(1-x) + m^2\right)^{\frac{D}{2}-2}$$

As we're just interested in the divergent part, we can substitute back '$D = 4$' except in the terms which are divergent, that is, the Gamma function, where we do use '$D = 4 - 2\epsilon$',

$$\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = \mathrm{i}\frac{g^2}{4\pi^2} \delta^{ab} Z_{g\Psi}^2 T(\mathrm{F}) \Gamma(\epsilon) \int_0^1 \mathrm{d}x\, N_{\mu\nu}$$

The remaining integral is polynomial, trivial and gives,

$$\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = \mathrm{i}\frac{g^2}{12\pi^2} \delta^{ab} Z_{g\Psi}^2 T(\mathrm{F}) \Gamma(\epsilon) \left[p_\mu p_\nu - p^2 g_{\mu\nu}\right]$$

Also expanding the Gamma function in it's Laurent series we arrive at our final result,

$$\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = \frac{1}{\epsilon} \mathrm{i}\delta^{ab} \frac{g^2}{12\pi^2} Z_{g\Psi}^2 T(\mathrm{F}) \left[p_\mu p_\nu - p^2 g_{\mu\nu}\right]$$

Notice that this result is the one for each generation of fermions, that is, if we have more then one family, or in other words, more the one flavor of fermions, then, we need to compute this result for each flavor and sum up. Considering that all flavors transform under gauge in the same representation, we just have to multiply for the number of flavors,

$$\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = \frac{1}{\epsilon} \mathrm{i}\delta^{ab} \frac{g^2}{12\pi^2} Z_{g\Psi}^2 N_\mathrm{F} T(\mathrm{F}) \left[p_\mu p_\nu - p^2 g_{\mu\nu}\right]$$

We then proceed to our forth contribution, the ghost loop,

$$\begin{aligned}
\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} &= g Z_{gc} \Lambda^\epsilon f^{acd} (-1) \int \frac{\mathrm{d}^D k}{(2\pi)^D} k_\mu \frac{1}{\mathrm{i}} \frac{\delta_{ce}}{k^2} \frac{1}{\mathrm{i}} \frac{\delta_{fd}}{(k-p)^2} \times \\
&\quad \times g Z_{cg} \Lambda^\epsilon f^{bfe} (k-p)_\nu \\
\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} &= g^2 Z_{gc}^2 \Lambda^{2\epsilon} f^{acd} f^{b}_{\ dc} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k_\mu (k-p)_\nu}{k^2 (k-p)^2} \\
\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} &= -\delta^{ab} g^2 Z_{gc}^2 \Lambda^{2\epsilon} T(\mathrm{A}) \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k_\mu (k-p)_\nu}{\left[x k^2 + (1-x)(k-p)^2\right]^2} \\
\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} &= -\delta^{ab} g^2 Z_{gc}^2 \Lambda^{2\epsilon} T(\mathrm{A}) \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k_\mu (k-p)_\nu}{\left[(k-(1-x)p)^2 + p^2 x(1-x)\right]^2}
\end{aligned}$$

Where we already have done the Feynman parametrization, we then perform the change of integration variables, '$q = k - (1-x)p$', neglecting linear terms in 'q' in the numerator,

$$\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} = -\delta^{ab} g^2 Z_{gc}^2 \Lambda^{2\epsilon} T(\mathrm{A}) \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{q_\mu q_\nu - p_\mu p_\nu x(1-x)}{\left[q^2 + p^2 x(1-x)\right]^2}$$

Again we do the usual replacing inside the integral, '$q_\mu q_\nu \to \frac{1}{D} g_{\mu\nu} q^2 \to \frac{1}{2-D} g_{\mu\nu} p^2 x(1-x)$',

$$\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} = -\delta^{ab} g^2 Z_{gc}^2 \Lambda^{2\epsilon} T(\mathrm{A}) \left(\frac{1}{2-D} g_{\mu\nu} p^2 - p_\mu p_\nu\right) \int_0^1 \mathrm{d}x\, x(1-x) \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{\left[q^2 + p^2 x(1-x)\right]^2}$$

Again, the momentum integral gives,

$$\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} = -\delta^{ab} g^2 Z_{gc}^2 \Lambda^{2\epsilon} T(\mathrm{A}) \left(\frac{1}{2-D} g_{\mu\nu} p^2 - p_\mu p_\nu\right) \int_0^1 \mathrm{d}x\, x(1-x) \frac{\mathrm{i}(p^2 x(1-x))^{\frac{D}{2}-2}}{(4\pi)^{\frac{D}{2}}} \Gamma\left(2 - \frac{D}{2}\right)$$

Going back to '$D = 4$' except in the Gamma function,

$$\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} = \delta^{ab} \frac{\mathrm{i}}{16\pi^2} \Gamma(\epsilon) g^2 Z_{gc}^2 T(\mathrm{A}) \left(\frac{1}{2} g_{\mu\nu} p^2 + p_\mu p_\nu\right) \int_0^1 \mathrm{d}x\, x(1-x)$$
$$\mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} = \frac{1}{\epsilon} \delta^{ab} \frac{\mathrm{i}}{16\pi^2} g^2 Z_{gc}^2 T(\mathrm{A}) \left(\frac{1}{12} g_{\mu\nu} p^2 + \frac{1}{6} p_\mu p_\nu\right)$$

The final contribution is from the counter-terms, which is also the simplest one, giving,

$$\mathrm{i}\Pi^{(5)ab}_{\ \ \mu\nu} = -\mathrm{i}\delta^{ab}(Z_A - 1)\left(g_{\mu\nu} p^2 - p_\mu p_\nu\right) - \mathrm{i}\delta^{ab}(Z_\xi - 1) p_\mu p_\nu$$

So, summing all the contributions for the self-energy, we get,

$$\begin{aligned}
\mathrm{i}\Pi^{ab}_{\mu\nu} &= \mathrm{i}\Pi^{(1)ab}_{\ \ \mu\nu} + \mathrm{i}\Pi^{(2)ab}_{\ \ \mu\nu} + \mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} + \mathrm{i}\Pi^{(4)ab}_{\ \ \mu\nu} + \mathrm{i}\Pi^{(5)ab}_{\ \ \mu\nu} \\
&= \frac{\mathrm{i}\delta^{ab} g^2}{32\pi^2\epsilon} Z_{3g}^2 T(\mathrm{A}) \left[\frac{19}{6} g_{\mu\nu} p^2 - \frac{11}{3} p_\mu p_\nu\right] + \frac{\mathrm{i}\delta^{ab} g^2}{12\pi^2\epsilon} Z_{g\Psi}^2 N_\mathrm{F} T(\mathrm{F}) \left[p_\mu p_\nu - p^2 g_{\mu\nu}\right] \\
&\quad + \frac{\mathrm{i}\delta^{ab} g^2}{32\pi^2\epsilon} Z_{gc}^2 T(\mathrm{A}) \left(\frac{1}{6} g_{\mu\nu} p^2 + \frac{1}{3} p_\mu p_\nu\right) - \mathrm{i}\delta^{ab}(Z_A - 1)\left(g_{\mu\nu} p^2 - p_\mu p_\nu\right) \\
&\quad - \mathrm{i}\delta^{ab}(Z_\xi - 1) p_\mu p_\nu
\end{aligned}$$

Up to first order in 'g^2' we can neglect higher powers of 'Z', so then,

$$\begin{aligned}
\mathrm{i}\Pi^{ab}_{\mu\nu} = &\frac{\mathrm{i}\delta^{ab} g^2}{32\pi^2\epsilon} T(\mathrm{A}) \left[\frac{19}{6} g_{\mu\nu} p^2 - \frac{11}{3} p_\mu p_\nu\right] + \frac{\mathrm{i}\delta^{ab} g^2}{32\pi^2\epsilon} N_\mathrm{F} T(\mathrm{F}) \left[\frac{8}{3} p_\mu p_\nu - \frac{8}{3} p^2 g_{\mu\nu}\right] \\
&+ \frac{\mathrm{i}\delta^{ab} g^2}{32\pi^2\epsilon} T(\mathrm{A}) \left[\frac{1}{6} g_{\mu\nu} p^2 + \frac{1}{3} p_\mu p_\nu\right] - \mathrm{i}\delta^{ab}(Z_A - 1)\left(g_{\mu\nu} p^2 - p_\mu p_\nu\right) - \mathrm{i}\delta^{ab}(Z_\xi - 1) p_\mu p_\nu
\end{aligned}$$

And then we set to zero, as the renormalized self-energy should be finite. We factor the terms with 'p^2', which contribute as,

$$\mathrm{i}\delta^{ab}(Z_A-1)g_{\mu\nu}p^2 = \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}T(\mathrm{A})\frac{19}{6}g_{\mu\nu}p^2 - \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\frac{8}{3}p^2g_{\mu\nu} + \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}T(\mathrm{A})\frac{1}{6}g_{\mu\nu}p^2$$
$$Z_A = 1 + \frac{g^2}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{4}{3}N_{\mathrm{F}}T(\mathrm{F})\right]$$

And by factorizing '$p_\mu p_\nu$' we can get the 'Z_ξ' factor,

$$\begin{aligned}\mathrm{i}\delta^{ab}(Z_\xi-1)p_\mu p_\nu - \mathrm{i}\delta^{ab}(Z_A-1)p_\mu p_\nu &= -\frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}T(\mathrm{A})\frac{11}{3}p_\mu p_\nu + \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\frac{8}{3}p_\mu p_\nu \\ &\quad + \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}T(\mathrm{A})\frac{1}{3}p_\mu p_\nu \\ Z_\xi - 1 - \frac{g^2}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{4}{3}N_{\mathrm{F}}T(\mathrm{F})\right] &= -\frac{g^2}{32\pi^2\epsilon}T(\mathrm{A})\frac{10}{3} + \frac{g^2}{32\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\frac{8}{3} \\ Z_\xi &= 1\end{aligned}$$

That is, at least in this order in perturbation theory, the gauge fixing parameter isn't renormalized! In fact this is a non-perturbative result, we have '$Z_\xi = 1$' what is equivalent to the 1PI self energy being always transverse.

1.4 Fermionic Self Energy

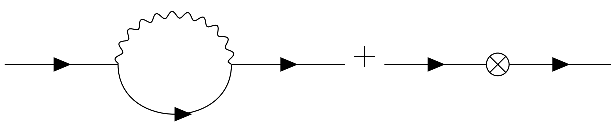
The same goes for the loop contributions of the fermion propagator,

$$\begin{aligned}\frac{1}{\mathrm{i}}\mathbf{S}(\not{p}) &= \frac{1}{\mathrm{i}}\mathrm{S}(\not{p}) + \frac{1}{\mathrm{i}}\mathrm{S}(\not{p})\mathrm{i}\Sigma(\not{p})\frac{1}{\mathrm{i}}\mathrm{S}(\not{p}) + \cdots \\ \frac{1}{\mathrm{i}}\mathbf{S}(\not{p}) &= \frac{1}{\mathrm{i}}\mathrm{S}(\not{p}) + \frac{1}{\mathrm{i}}\mathrm{S}(\not{p})\mathrm{i}\Sigma(\not{p})\left\{\frac{1}{\mathrm{i}}\mathrm{S}(\not{p}) + \frac{1}{\mathrm{i}}\mathrm{S}(\not{p})\mathrm{i}\Sigma(\not{p})\frac{1}{\mathrm{i}}\mathrm{S}(\not{p}) + \cdots\right\} \\ \frac{1}{\mathrm{i}}\mathbf{S}(\not{p}) &= \frac{1}{\mathrm{i}}\mathrm{S}(\not{p}) + \frac{1}{\mathrm{i}}\mathrm{S}(\not{p})\mathrm{i}\Sigma(\not{p})\frac{1}{\mathrm{i}}\mathbf{S}(\not{p}) \\ \mathbf{S}(\not{p}) &= \left[1 - \mathrm{S}(\not{p})\Sigma(\not{p})\right]^{-1}\mathrm{S}(\not{p}) \\ \mathbf{S}^{-1}(\not{p}) &= \mathrm{S}^{-1}(\not{p})\left[1 - \mathrm{S}(\not{p})\Sigma(\not{p})\right] = \mathrm{S}^{-1}(\not{p}) - \Sigma(\not{p}) = \mathrm{i}\not{p} + m - \Sigma(\not{p})\end{aligned}$$

Where we have suppressed, but, we also have color indices, restoring them,

$$\mathbf{S}^{-1^{ij}}(\not{p}) = (\mathrm{i}\not{p} + m)\delta^{ij} - \Sigma^{ij}(\not{p})$$

With '$\mathrm{i}\Sigma^{ij}$' being the 1PI loop contributions to the propagator, which is diagrammatically,

$\mathrm{i}\Sigma^{ij}(\not{p}) =$  $+\cdots$ (1.19)

The loop contribution is,

$$\begin{aligned}\mathrm{i}\Sigma^{ij}_{(1)} &= -gZ_{g\Psi}\Lambda^\epsilon[\mathbf{T}^a_{\mathrm{f}}]^{li}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\gamma_\mu\frac{\delta_{lk}}{\mathrm{i}}\frac{-\mathrm{i}(\not{p}-\not{k})+m}{(p-k)^2+m^2}\gamma_\nu\frac{1}{\mathrm{i}}\frac{\delta_{ab}}{k^2}g^{\mu\nu}(-1)gZ_{g\Psi}\Lambda^\epsilon\left[\mathbf{T}^b_{\mathrm{f}}\right]^{jk} \\ \mathrm{i}\Sigma^{ij}_{(1)} &= -g^2Z^2_{g\Psi}\Lambda^{2\epsilon}\delta_{ab}\left[\mathbf{T}^b_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}\right]^{ji}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\gamma_\mu\frac{-\mathrm{i}(\not{p}-\not{k})+m}{k^2\left((p-k)^2+m^2\right)}\gamma^\mu\end{aligned}$$

Using a few of Gamma matrix technology, as shown in the appendix,

$$\gamma_\mu \gamma^\mu = D$$
$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2-D)\gamma_\alpha$$

And Feynman reparametrization,

$$\mathrm{i}\Sigma^{ij}_{(1)} = -g^2 Z^2_{g\Psi} \Lambda^{2\epsilon} \delta_{ab} \big[\mathbf{T}^b_\mathrm{f} \mathbf{T}^a_\mathrm{f}\big]^{ji} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{-\mathrm{i}(2-D)\big(\not{p} - \not{k}\big) + Dm}{\big[(k-(1-x)p)^2 + (1-x)(xp^2+m^2)\big]^2}$$

Doing the change of integration variable '$q = k - (1-x)p$', and remembering that we do only keep even powers of 'q' in the numerator,

$$\mathrm{i}\Sigma^{ij}_{(1)} = -g^2 Z^2_{g\Psi} \Lambda^{2\epsilon} \delta_{ab} \big[\mathbf{T}^b_\mathrm{f} \mathbf{T}^a_\mathrm{f}\big]^{ji} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{-\mathrm{i}(2-D)\big(\not{p} - (1-x)\not{p}\big) + Dm}{\left[q^2 + (1-x)(xp^2+m^2)\right]^2}$$

Also we can identify the algebra factor '$\delta_{ab} \big[\mathbf{T}^b_\mathrm{f} \mathbf{T}^a_\mathrm{f}\big]^{ji} = C(\mathrm{F})\delta^{ij}$' as the quadratic Casimir operator, hence,

$$\mathrm{i}\Sigma^{ij}_{(1)} = -g^2 Z^2_{g\Psi} \Lambda^{2\epsilon} C(\mathrm{F})\delta^{ij} \int_0^1 \mathrm{d}x \left[-\mathrm{i}(2-D)x\not{p} + Dm\right] \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{\left[q^2 + (1-x)(xp^2+m^2)\right]^2}$$

Where we have again our beloved momentum integral giving,

$$\mathrm{i}\Sigma^{ij}_{(1)} = -g^2 Z^2_{g\Psi} \Lambda^{2\epsilon} C(\mathrm{F})\delta^{ij} \int_0^1 \mathrm{d}x \left[-\mathrm{i}(2-D)x\not{p} + Dm\right] \frac{\mathrm{i}\Gamma\big(2-\frac{D}{2}\big)}{(4\pi)^{\frac{D}{2}}} \big[(1-x)\big(xp^2+m^2\big)\big]^{\frac{D}{2}-2}$$

We go back now to '$D = 4$', keeping attention on the pole of the Gamma function,

$$\mathrm{i}\Sigma^{ij}_{(1)} = -\frac{\mathrm{i}g^2}{(4\pi)^2} Z^2_{g\Psi} C(\mathrm{F})\delta^{ij}\Gamma(\epsilon) \int_0^1 \mathrm{d}x \left[\mathrm{i}2x\not{p} + 4m\right]$$
$$\mathrm{i}\Sigma^{ij}_{(1)} = \frac{g^2}{16\pi^2} Z^2_{g\Psi} C(\mathrm{F})\delta^{ij} \frac{1}{\epsilon}\big(\not{p} - 4\mathrm{i}m\big)$$

The second diagram is the counter terms,

$$\mathrm{i}\Sigma^{ij}_{(2)} = (Z_\Psi - 1)\delta^{ij}\not{p} - \mathrm{i}(Z_m - 1)\delta^{ij} m$$

Summing the two contributions,

$$\mathrm{i}\Sigma^{ij} = \mathrm{i}\Sigma^{ij}_{(1)} + \mathrm{i}\Sigma^{ij}_{(2)}$$
$$\mathrm{i}\Sigma^{ij} = \frac{g^2\delta^{ij}}{16\pi^2\epsilon} Z^2_{g\Psi} C(\mathrm{F})\big(\not{p} - 4\mathrm{i}m\big) + (Z_\Psi - 1)\delta^{ij}\not{p} - \mathrm{i}(Z_m - 1)\delta^{ij} m$$

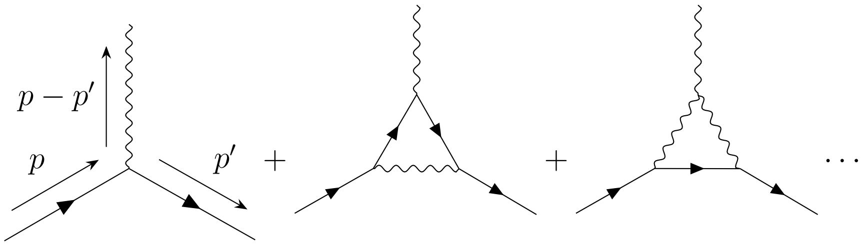
As this is only the divergent part, we must equal it to zero, what determines 'Z_Ψ' and . Also, as this is just the first order we neglect higher powers of the 'Z's.

$$
\begin{aligned}
Z_\Psi &= 1 - \frac{g^2}{16\pi^2\epsilon}C(\mathrm{F}) \\
Z_m &= 1 - 4\frac{g^2}{16\pi^2\epsilon}C(\mathrm{F})
\end{aligned}
$$

1.5 Fermion-Gauge Vertex Loop Corrections

Up to 1-loop the relevant diagrams are,

$\mathrm{i}\Gamma^{a\mu}_{ij}(p,p') =$



The first diagram gives,

$$
\mathrm{i}\Gamma^{(1)a\mu}_{\ \ ij} = -gZ_{g\Psi}\Lambda^\epsilon\gamma^\mu[\mathbf{T}^a_{\mathrm{f}}]_{ji}
$$

The second one,

$$
\begin{aligned}
\mathrm{i}\Gamma^{(2)a\mu}_{\ \ ij} &= -gZ_{g\Psi}\Lambda^\epsilon[\mathbf{T}^a_{\mathrm{f}}]_{nm}(-1)gZ_{g\Psi}\Lambda^\epsilon\big[\mathbf{T}^b_{\mathrm{f}}\big]_{li}(-1)gZ_{g\Psi}\Lambda^\epsilon[\mathbf{T}^c_{\mathrm{f}}]_{jk}\times \\
&\times\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{\mathrm{i}}\frac{\delta_{bc}g_{\alpha\beta}}{k^2}\gamma^\beta\frac{\delta^{nk}}{\mathrm{i}}\frac{-\mathrm{i}\big(\not{p}'-\not{k}\big)+m}{(p'-k)^2+m^2}\gamma^\mu\frac{\delta^{lm}}{\mathrm{i}}\frac{-\mathrm{i}\big(\not{p}-\not{k}\big)+m}{(p-k)^2+m^2}\gamma^\alpha \\
\mathrm{i}\Gamma^{(2)a\mu}_{\ \ ij} &= -\mathrm{i}g^3Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\big]_{ji}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\gamma_\alpha\frac{-\mathrm{i}\big(\not{p}'-\not{k}\big)+m}{(p'-k)^2+m^2}\gamma^\mu\frac{-\mathrm{i}\big(\not{p}-\not{k}\big)+m}{(p-k)^2+m^2}\gamma^\alpha
\end{aligned}
$$

Here we have an addendum to make. Notice that in both second and third diagrams will contain a factor '$Z^3_{g\Psi}$', which of course will later be neglected by being a higher power, that means, all the divergence generated by the second and third diagrams must be regularized by the first diagram, due to being the only one having contributing '$Z_{g\Psi}$' factor in this order in perturbation theory, and as this diagram is independent of the external momentum, we can surely state that the divergences generated at this level of perturbation theory will not have any dependence on external momentum, and hence we can make it equal to zero to facilitate the computation. In this way we get,

$$
\mathrm{i}\Gamma^{(2)a\mu}_{\ \ ij} = -\mathrm{i}g^3Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\big]_{ji}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{k^2}\gamma_\alpha\frac{\mathrm{i}\not{k}+m}{k^2+m^2}\gamma^\mu\frac{\mathrm{i}\not{k}+m}{k^2+m^2}\gamma^\alpha
$$

Recurring to some Gamma matrix relations derived in the appendix,

$$
\begin{aligned}
\gamma_\alpha\gamma^\mu\gamma^\alpha &= (2-D)\gamma^\mu \\
\gamma_\alpha\gamma^\rho\gamma^\mu\gamma^\alpha &= 2\gamma^\mu\gamma^\rho + (D-2)\gamma^\rho\gamma^\mu \\
\gamma_\alpha\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\alpha &= 2\gamma^\sigma\gamma^\rho\gamma^\mu - 2\gamma^\mu\gamma^\rho\gamma^\sigma + (2-D)\gamma^\rho\gamma^\mu\gamma^\sigma
\end{aligned}
$$

Simplifies to,

$$
\begin{aligned}
\mathrm{i}\Gamma^{(2)a\mu}_{\ ij} = -\mathrm{i}g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_\mathrm{f}\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\big]_{ji}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2(k^2+m^2)(k^2+m^2)}\times \\
\times\big(-2\not{k}\not{k}\gamma^\mu + 2\gamma^\mu\not{k}\not{k} + (D-2)\not{k}\gamma^\mu\not{k} + \mathrm{i}m[2\gamma^\mu\not{k} + (D-2)\not{k}\gamma^\mu] \\
+ \mathrm{i}m[2\not{k}\gamma^\mu + (D-2)\gamma^\mu\not{k}] + m^2(2-D)\gamma^\mu\big)
\end{aligned}
$$

Also, as already is well known to us, any linear 'k' term will integrate to zero, thus,

$$
\begin{aligned}
\mathrm{i}\Gamma^{(2)a\mu}_{\ ij} = -\mathrm{i}g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_\mathrm{f}\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\big]_{ji}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2(k^2+m^2)(k^2+m^2)}\times \\
\times\big(-2\not{k}\not{k}\gamma^\mu + 2\gamma^\mu\not{k}\not{k} + (D-2)\not{k}\gamma^\mu\not{k} + m^2(2-D)\gamma^\mu\big)
\end{aligned}
\tag{1.20}
$$

Also switching off to the Feynman parametrization,

$$
\begin{aligned}
\mathrm{i}\Gamma^{(2)a\mu}_{\ ij} &= -\mathrm{i}g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_\mathrm{f}\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\big]_{ji}\times \\
&\times 2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[xk^2 + y(k^2+m^2) + (1-x-y)(k^2+m^2)]^3}\times \\
&\times\big(k_\alpha k_\beta\big(2\gamma^\mu\gamma^\alpha\gamma^\beta - 2\gamma^\alpha\gamma^\beta\gamma^\mu + (D-2)\gamma^\alpha\gamma^\mu\gamma^\beta\big) + m^2(2-D)\gamma^\mu\big) \\
\mathrm{i}\Gamma^{(2)a\mu}_{\ ij} &= -\mathrm{i}g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_\mathrm{f}\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\big]_{ji} 2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[k^2+m^2(1-x)]^3}\times \\
&\times\big(k_\alpha k_\beta\big(2\gamma^\mu\gamma^\alpha\gamma^\beta - 2\gamma^\alpha\gamma^\beta\gamma^\mu + (D-2)\gamma^\alpha\gamma^\mu\gamma^\beta\big) + m^2(2-D)\gamma^\mu\big)
\end{aligned}
$$

The 'y' integral is trivial, and using that inside the integral we have, '$k_\alpha k_\beta \to \frac{1}{D}g_{\alpha\beta}k^2$',

$$
\begin{aligned}
\mathrm{i}\Gamma^{(2)a\mu}_{\ ij} &= -\mathrm{i}g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_\mathrm{f}\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\big]_{ji} 2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1-x}{[k^2+m^2(1-x)]^3}\times \\
&\times\left(\frac{1}{D}k^2(2\gamma^\mu\gamma_\alpha\gamma^\alpha - 2\gamma_\alpha\gamma^\alpha\gamma^\mu + (D-2)\gamma_\alpha\gamma^\mu\gamma^\alpha) + m^2(2-D)\gamma^\mu\right) \\
\mathrm{i}\Gamma^{(2)a\mu}_{\ ij} &= -\mathrm{i}g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_\mathrm{f}\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\big]_{ji} 2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1-x}{[k^2+m^2(1-x)]^3}\times \\
&\times\left(\frac{1}{D}k^2\big(2D\gamma^\mu - 2D\gamma^\mu - (2-D)^2\gamma^\mu\big) + m^2(2-D)\gamma^\mu\right) \\
\mathrm{i}\Gamma^{(2)a\mu}_{\ ij} &= -\mathrm{i}2g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\big[\mathbf{T}^c_\mathrm{f}\mathbf{T}^a_\mathrm{f}\mathbf{T}^b_\mathrm{f}\big]_{ji}(2-D)\gamma^\mu\int_0^1\mathrm{d}x\,(1-x)\times \\
&\times\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[k^2+m^2(1-x)]^3}\left(k^2\left(1-\frac{2}{D}\right) + m^2\right)
\end{aligned}
$$

For solving the momentum integral we use the results of the appendix,

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{\left[k^2+m^2(1-x)\right]^3} = \frac{1}{2}\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(3-\frac{D}{2}\right)\left[m^2(1-x)\right]^{\frac{D}{2}-3}$$
$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^2}{\left[k^2+m^2(1-x)\right]^3} = \frac{D}{4}\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\left[m^2(1-x)\right]^{\frac{D}{2}-2}$$

Factorizing the terms,

$$\mathrm{i}\Gamma^{(2)}{}^{a\mu}_{ij} = -\mathrm{i}2g^3 Z^3_{g\Psi}\Lambda^{3\epsilon}\delta_{bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}(2-D)\gamma^\mu \int_0^1 \mathrm{d}x\,(1-x)\times$$
$$\times\frac{1}{2}\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\left[m^2(1-x)\right]^{\frac{D}{2}-2}\left(\frac{D}{2}\left(1-\frac{2}{D}\right)\Gamma\left(2-\frac{D}{2}\right)+\frac{\Gamma\left(3-\frac{D}{2}\right)}{1-x}\right)$$

We can go back to '$D=4$', except in the first Gamma function,

$$\mathrm{i}\Gamma^{(2)}{}^{a\mu}_{ij} = -\frac{g^3}{8\pi^2}Z^3_{g\Psi}\delta_{bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\gamma^\mu\int_0^1 \mathrm{d}x\,((1-x)\Gamma(\epsilon)+1)$$

Neglecting the finite part,

$$\mathrm{i}\Gamma^{(2)}{}^{a\mu}_{ij} = -\frac{g^3}{16\pi^2\epsilon}Z^3_{g\Psi}\delta_{bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\gamma^\mu$$

To further simplify we have to work on the algebra factor,

$$\begin{aligned}
\delta_{bc}\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}} &= \delta_{bc}\mathbf{T}^c_{\mathrm{f}}\left\{\mathbf{T}^b_{\mathrm{f}}\mathbf{T}^a_{\mathrm{f}}+\mathrm{i}f^{ab}{}_d\mathbf{T}^d_{\mathrm{f}}\right\}\\
&= C(\mathrm{F})\mathbf{T}^a_{\mathrm{f}}+\mathrm{i}f^a{}_{bc}\mathbf{T}^b_{\mathrm{f}}\mathbf{T}^c_{\mathrm{f}}\\
&= C(\mathrm{F})\mathbf{T}^a_{\mathrm{f}}+\frac{\mathrm{i}}{2}f^a{}_{bc}\left[\mathbf{T}^b_{\mathrm{f}},\mathbf{T}^c_{\mathrm{f}}\right]\\
&= C(\mathrm{F})\mathbf{T}^a_{\mathrm{f}}+\frac{\mathrm{i}}{2}f^a{}_{bc}\mathrm{i}f^{bc}{}_d\mathbf{T}^d_{\mathrm{f}}\\
&= C(\mathrm{F})\mathbf{T}^a_{\mathrm{f}}-\frac{1}{2}f^a{}_{bc}f_d{}^{bc}\mathbf{T}^d_{\mathrm{f}}\\
&= C(\mathrm{F})\mathbf{T}^a_{\mathrm{f}}-\frac{1}{2}T(\mathrm{A})\mathbf{T}^a_{\mathrm{f}}
\end{aligned}$$

And hence,

$$\mathrm{i}\Gamma^{(2)}{}^{a\mu}_{ij} = -\frac{g^3}{16\pi^2\epsilon}Z^3_{g\Psi}\left[C(\mathrm{F})-\frac{1}{2}T(\mathrm{A})\right]\left[\mathbf{T}^a_{\mathrm{f}}\right]_{ji}\gamma^\mu$$

Let's go then to the third diagram,

$$\begin{aligned}
\mathrm{i}\Gamma^{(3)}{}^{a\mu}_{ij} = &-gZ_{g\Psi}\Lambda^\epsilon\left[\mathbf{T}^d_{\mathrm{f}}\right]_{li}(-1)gZ_{g\Psi}\Lambda^\epsilon\left[\mathbf{T}^e_{\mathrm{f}}\right]_{jk}gZ_{3g}\Lambda^\epsilon f^{abc}\times\\
&\times\int\frac{\mathrm{d}^D k}{(2\pi)^D}\gamma^\sigma\frac{\delta^{lk}}{\mathrm{i}}\frac{-\mathrm{i}\not{k}+m}{k^2+m^2}\gamma^\rho\frac{\delta_{bd}}{\mathrm{i}}\frac{g_{\rho\alpha}}{(p-k)^2}\frac{\delta_{ce}}{\mathrm{i}}\frac{g_{\beta\sigma}}{(p'-k)^2}\times\\
&\times\left[(2k-p-p')^\mu g^{\alpha\beta}+(2p'-p-k)^\alpha g^{\beta\mu}+(2p-p'-k)^\beta g^{\alpha\mu}\right]
\end{aligned}$$

As as previously discussed, we only need to consider the zero external momentum case, thus,

$$
\begin{aligned}
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} = \mathrm{i}g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\times \\
\times\int\frac{\mathrm{d}^D k}{(2\pi)^D}\gamma_\beta\frac{-\mathrm{i}\not{k}+m}{(k^2)^2(k^2+m^2)}\gamma_\alpha\left[2k^\mu g^{\alpha\beta}-k^\alpha g^{\beta\mu}-k^\beta g^{\alpha\mu}\right]
\end{aligned}
$$

Throwing away linear terms in 'k',

$$
\begin{aligned}
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\times \\
&\times\gamma_\beta\gamma_\nu\gamma_\alpha\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{2k^\mu k^\nu g^{\alpha\beta}-k^\alpha k^\nu g^{\beta\mu}-k^\beta k^\nu g^{\alpha\mu}}{(k^2)^2(k^2+m^2)} \\
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\times \\
&\times\gamma_\beta\gamma_\nu\gamma_\alpha\frac{1}{D}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{2k^2 g^{\mu\nu}g^{\alpha\beta}-k^2 g^{\alpha\nu}g^{\beta\mu}-k^2 g^{\beta\nu}g^{\alpha\mu}}{(k^2)^2(k^2+m^2)} \\
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\times \\
&\times\frac{1}{D}[2\gamma_\alpha\gamma^\mu\gamma^\alpha-2D\gamma^\mu]\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2(k^2+m^2)} \\
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\times \\
&\times\frac{2}{D}[(2-D)\gamma^\mu-D\gamma^\mu]\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2(k^2+m^2)} \\
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\frac{4}{D}(1-D)\gamma^\mu\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2(k^2+m^2)}
\end{aligned}
$$

Going now for the momentum integral,

$$
\begin{aligned}
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\frac{4}{D}(1-D)\gamma^\mu\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[xk^2+(1-x)(k^2+m^2)]^2} \\
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\frac{4}{D}(1-D)\gamma^\mu\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[k^2+m^2(1-x)]^2} \\
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} &= g^3 Z_{g\Psi}^2 Z_{3g}\Lambda^{3\epsilon} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\frac{4}{D}(1-D)\gamma^\mu\int_0^1\mathrm{d}x\,\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\left[m^2(1-x)\right]^{\frac{D}{2}-2}
\end{aligned}
$$

Switching back '$D=4$',

$$
\mathrm{i}\Gamma^{(3)a\mu}_{\ \ ij} = -3\mathrm{i}\frac{g^3}{16\pi^2}Z_{g\Psi}^2 Z_{3g} f^a_{\ bc}\left[\mathbf{T}^c_{\mathrm{f}}\mathbf{T}^b_{\mathrm{f}}\right]_{ji}\gamma^\mu\Gamma(\epsilon)
$$

The color factor can be simplified as,

$$
\begin{aligned}
f^a{}_{bc}\,\mathbf{T}_\mathrm{f}^c\mathbf{T}_\mathrm{f}^b &= \frac{1}{2} f^a{}_{bc}\left[\mathbf{T}_\mathrm{f}^c, \mathbf{T}_\mathrm{f}^b\right] \\
&= \frac{\mathrm{i}}{2} f^a{}_{bc} f^{cb}{}_d \mathbf{T}_\mathrm{f}^d \\
&= -\frac{\mathrm{i}}{2} f^a{}_{bc} f_d{}^{bc} \mathbf{T}_\mathrm{f}^d \\
&= -\frac{\mathrm{i}}{2} T(\mathrm{A}) \mathbf{T}_\mathrm{f}^a
\end{aligned}
$$

Substituting,

$$
\mathrm{i}\Gamma^{(3)}{}_{ij}^{a\mu} = -\frac{3}{2}\frac{g^3}{16\pi^2\epsilon} Z_{g\Psi}^2 Z_{3g} T(\mathrm{A}) [\mathbf{T}_\mathrm{f}^a]_{ji}\gamma^\mu
$$

Thus, putting the three contributions together,

$$
\begin{aligned}
\mathrm{i}\Gamma_{ij}^{a\mu} &= \mathrm{i}\Gamma^{(1)}{}_{ij}^{a\mu} + \mathrm{i}\Gamma^{(2)}{}_{ij}^{a\mu} + \mathrm{i}\Gamma^{(3)}{}_{ij}^{a\mu} \\
-g\gamma^\mu[\mathbf{T}_\mathrm{f}^a]_{ji} &= -gZ_{g\Psi}\gamma^\mu[\mathbf{T}_\mathrm{f}^a]_{ji} - \frac{g^3 Z_{g\Psi}^3}{16\pi^2\epsilon}\left[C(\mathrm{F}) - \frac{1}{2}T(\mathrm{A})\right][\mathbf{T}_\mathrm{f}^a]_{ji}\gamma^\mu - \frac{3g^3 Z_{g\Psi}^2}{32\pi^2\epsilon} Z_{3g} T(\mathrm{A})[\mathbf{T}_\mathrm{f}^a]_{ji}\gamma^\mu
\end{aligned}
$$

And again, we neglect higher powers of the 'Z's,

$$
\begin{aligned}
-1 &= -Z_{g\Psi} - \frac{g^2}{16\pi^2\epsilon}\left[C(\mathrm{F}) - \frac{1}{2}T(\mathrm{A})\right] - \frac{3}{2}\frac{g^2}{16\pi^2\epsilon}T(\mathrm{A}) \\
Z_{g\Psi} &= 1 - \frac{g^2}{16\pi^2\epsilon}[C(\mathrm{F}) + T(\mathrm{A})]
\end{aligned}
$$

1.6 Computation of the Beta Function

Summarizing our discoveries,

$$
\begin{cases}
Z_A &= 1 + \frac{g^2}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{4}{3}N_\mathrm{F}T(\mathrm{F})\right] \\
Z_\Psi &= 1 - \frac{g^2}{16\pi^2\epsilon}C(\mathrm{F}) \\
Z_{g\Psi} &= 1 - \frac{g^2}{16\pi^2\epsilon}[C(\mathrm{F}) + T(\mathrm{A})]
\end{cases}
$$

Looking back to the Lagrangians 1.13 and 1.6, we can do the matching between the bare and renormalized parameters,

$$
g_0 = g\frac{Z_{g\Psi}}{\sqrt{Z_A}Z_\Psi}\Lambda^\epsilon
$$

As what really shows up in the leading order is 'g^2', it's easier to work temporarily with '$\alpha = g^2$',

$$
\alpha_0 = \alpha\frac{Z_{g\Psi}^2}{Z_A Z_\Psi^2}\Lambda^{2\epsilon}
$$

Keeping only the leading pole,

$$
\begin{aligned}
\alpha_0 &= \alpha\Big(1-\frac{\alpha}{8\pi^2\epsilon}[C(\mathrm{F})+T(\mathrm{A})]\Big)\Big(1-\frac{\alpha}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A})-\frac{4}{3}N_{\mathrm{F}}T(\mathrm{F})\right]\Big)\Big(1+\frac{\alpha}{8\pi^2\epsilon}C(\mathrm{F})\Big)\Lambda^{2\epsilon}\\
\alpha_0 &= \alpha\left(1-\frac{11}{3}\frac{\alpha}{16\pi^2\epsilon}T(\mathrm{A})+\frac{4}{3}\frac{\alpha}{16\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)\Lambda^{2\epsilon}\\
0 &= \beta\left(1-\frac{11}{3}\frac{\alpha}{16\pi^2\epsilon}T(\mathrm{A})+\frac{4}{3}\frac{\alpha}{16\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)\\
&\quad-\alpha\beta\left(\frac{11}{3}\frac{1}{16\pi^2\epsilon}T(\mathrm{A})-\frac{4}{3}\frac{1}{16\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)\\
&\quad+2\epsilon\alpha\left(1-\frac{11}{3}\frac{\alpha}{16\pi^2\epsilon}T(\mathrm{A})+\frac{4}{3}\frac{\alpha}{16\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)\\
0 &= \beta\left(1-\frac{11}{3}\frac{\alpha}{8\pi^2\epsilon}T(\mathrm{A})+\frac{4}{3}\frac{\alpha}{8\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)\\
&\quad+\epsilon\alpha\left(1-\frac{11}{3}\frac{\alpha}{8\pi^2\epsilon}T(\mathrm{A})+\frac{4}{3}\frac{\alpha}{8\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)+\epsilon\alpha\\
-\epsilon\alpha &= (\beta+\epsilon\alpha)\left(1-\frac{11}{3}\frac{\alpha}{8\pi^2\epsilon}T(\mathrm{A})+\frac{4}{3}\frac{\alpha}{8\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)\\
\beta+\epsilon\alpha &= -\epsilon\alpha\left(1+\frac{11}{3}\frac{\alpha}{8\pi^2\epsilon}T(\mathrm{A})-\frac{4}{3}\frac{\alpha}{8\pi^2\epsilon}N_{\mathrm{F}}T(\mathrm{F})\right)\\
\beta &= -2\epsilon\alpha-\frac{11}{3}\frac{\alpha^2}{8\pi^2}T(\mathrm{A})+\frac{4}{3}\frac{\alpha^2}{8\pi^2}N_{\mathrm{F}}T(\mathrm{F}))
\end{aligned}
$$

Finally we take the limit '$\epsilon\to 0$' and get the so awaited result,

$$\beta = -\frac{11}{3}\frac{\alpha^2}{8\pi^2}T(\mathrm{A})+\frac{4}{3}\frac{\alpha^2}{8\pi^2}N_{\mathrm{F}}T(\mathrm{F}))$$

Or, changing from 'α' to 'g',

$$\beta = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}T(\mathrm{A})-\frac{4}{3}N_{\mathrm{F}}T(\mathrm{F})\right]$$

2 Beta Functions of Non-Abelian Gauge-Scalar Theory

2.1 Construction of the Theory

We had a lot to say about Gauge theory coupled with fermions, to our luck, many of the things said there can be reused here, because, the Gauge part of the theory stays the same, there are no changes in it, the changes happen in the matter content, and in the covariant derivatives. All the motivation is the same, we start with a given group 'G', which associated algebra is a direct sum over commuting compact simple and '$\mathfrak{u}(1)$' Lie Algebras, this group is supposedly a global symmetry of the Lagrangian, that means, our complex scalar field '$\mathbf{\Phi}(x)$' transform under some representation of the group, so, the statement of the Lagrangian being invariant through the group action is summarized in,

$$\mathbf{\Phi}(x) \to \exp\left(-\mathrm{i}g\alpha_a \mathbf{T}^a_{\mathrm{s}}\right)\mathbf{\Phi}(x)$$

Leaving the Lagrangian invariant. Here we assume the complex scalar field transforms under a generic representation of the group. We would like to promote such a global transformation into a local one,

$$\mathbf{\Phi}(x) \to \exp\left(-\mathrm{i}g\alpha_a(x) \mathbf{T}^a_{\mathrm{s}}\right)\mathbf{\Phi}(x)$$

Which, as discussed before, will require a notion of covariant derivative, that require Gauge fields,

$$\mathbf{D}_\mu = \mathbb{1}\partial_\mu - \mathrm{i}g\mathbf{A}_\mu$$

And clearly our kinetic term will be,

$$\mathcal{L} \supset -[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger \mathbf{D}^\mu \mathbf{\Phi}$$

To get our full Lagrangian we need to include also all terms compatible with the required symmetries, thus, there are still missing terms apart from the kinetic terms of both the Gauge and the complex scalar fields, those missing terms are the beloved quadratic and quartic couplings of the scalar. Thus, our full Lagrangian is,

$$\mathcal{L} = -\frac{1}{2}\operatorname{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] - [\mathbf{D}_\mu \mathbf{\Phi}]^\dagger \mathbf{D}^\mu \mathbf{\Phi} - m^2 \mathbf{\Phi}^\dagger\mathbf{\Phi} - \frac{\lambda}{4}\left[\mathbf{\Phi}^\dagger\mathbf{\Phi}\right]^2$$

We now should under go the procedure of gauge fixing to properly quantize the theory, but, luckily, we have already done this, and more than that, we have already seen that this procedure does not depend upon the matter content of the theory. Thus, in the Gauge field part of the theory we must add the Gauge fixing terms and also the ghosts terms. Rewriting here these terms to remind,

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} = {} & \frac{1}{2}A^{a\mu}\delta_{ab}\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)A^{b\nu} + \frac{1}{2\xi}A_{a\mu}\partial^\mu\partial^\nu A^a{}_\nu \\ & - gf^{ab}{}_c A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2}{4}f^{ab}{}_e f_{cd}{}^e A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\ & - \partial^\mu \bar{c}^a \partial^\mu c_a + gA_{c\mu}f^{abc}\partial^\mu \bar{c}_a c_b\end{aligned}$$

So that our full theory generating functional is given by this part and the scalar one,

$$Z = \int \mathcal{D}A\mathcal{D}\Phi^\dagger\mathcal{D}\Phi \exp\left\{\mathrm{i}S_{\text{YM}} + \mathrm{i}S_{\text{gf}} + \mathrm{i}S_{\text{gh}} + \mathrm{i}S_{\text{s}}\right\}$$

2.2 Renormalization and Feynman Rules

We have already open up all terms in the Gauge+Ghost Lagrangian in the anterior problem, what remains to do now is open the scalar terms in this Lagrangian. We start by the kinetic term,

$$\begin{aligned}
-[\mathbf{D}_\mu\boldsymbol{\Phi}]^\dagger\mathbf{D}^\mu\boldsymbol{\Phi} &= -[\partial_\mu\mathbb{1}\boldsymbol{\Phi} - \mathrm{i}g\mathbf{A}_\mu\boldsymbol{\Phi}]^\dagger[\partial^\mu\mathbb{1}\boldsymbol{\Phi} - \mathrm{i}g\mathbf{A}^\mu\boldsymbol{\Phi}]\\
&= -\delta_{ij}\left[\partial_\mu\Phi^i - \mathrm{i}gA_{a\mu}[\mathbf{T}^a_{\text{s}}]^i{}_l\Phi^l\right]^\dagger\left[\partial^\mu\Phi^j - \mathrm{i}gA^\mu_b\left[\mathbf{T}^b_{\text{s}}\right]^j{}_k\Phi^k\right]\\
&= -\delta_{ij}\left[\partial_\mu\Phi^{i\dagger} + \mathrm{i}gA_{a\mu}[\mathbf{T}^{a*}_{\text{s}}]^i{}_l\Phi^{l\dagger}\right]\left[\partial^\mu\Phi^j - \mathrm{i}gA^\mu_b\left[\mathbf{T}^b_{\text{s}}\right]^j{}_k\Phi^k\right]\\
&= -\delta_{ij}\left[\partial_\mu\Phi^{i\dagger} + \mathrm{i}gA_{a\mu}[\mathbf{T}^{a}_{\text{s}}]_l{}^i\Phi^{l\dagger}\right]\left[\partial^\mu\Phi^j - \mathrm{i}gA^\mu_b\left[\mathbf{T}^b_{\text{s}}\right]^j{}_k\Phi^k\right]\\
&= -\delta_{ij}\partial_\mu\Phi^{i\dagger}\partial^\mu\Phi^j + \mathrm{i}g\delta_{ij}\partial_\mu\Phi^{i\dagger}A^\mu_b\left[\mathbf{T}^b_{\text{s}}\right]^j{}_k\Phi^k - \mathrm{i}g\delta_{ij}A_{a\mu}[\mathbf{T}^a_{\text{s}}]_l{}^i\Phi^{l\dagger}\partial^\mu\Phi^j\\
&\qquad - \delta_{ij}g^2A_{a\mu}[\mathbf{T}^a_{\text{s}}]_l{}^i\Phi^{l\dagger}A^\mu_b\left[\mathbf{T}^b_{\text{s}}\right]^j{}_k\Phi^k\\
&= -\delta_{ij}\partial_\mu\Phi^{i\dagger}\partial^\mu\Phi^j\\
&\qquad - \mathrm{i}gA_{a\mu}[\mathbf{T}^a_{\text{s}}]_{ij}\Phi^{i\dagger}\partial^\mu\Phi^j + \mathrm{i}gA_{a\mu}[\mathbf{T}^a_{\text{s}}]_{ij}\partial^\mu\Phi^{i\dagger}\Phi^j\\
&\qquad - \delta_{ij}g^2A_{a\mu}A^\mu_b[\mathbf{T}^a_{\text{s}}]_l{}^i\left[\mathbf{T}^b_{\text{s}}\right]^j{}_k\Phi^{l\dagger}\Phi^k\\
&= -\delta_{ij}\partial_\mu\Phi^{i\dagger}\partial^\mu\Phi^j\\
&\qquad - \mathrm{i}gA_{a\mu}[\mathbf{T}^a_{\text{s}}]_{ij}\left(\Phi^{i\dagger}\partial^\mu\Phi^j - \partial^\mu\Phi^{i\dagger}\Phi^j\right)\\
&\qquad - g^2A_{a\mu}A^\mu_b\left[\mathbf{T}^a_{\text{s}}\mathbf{T}^b_{\text{s}}\right]_{ij}\Phi^{i\dagger}\Phi^j
\end{aligned}$$

The quadratic term is trivial,

$$-m^2\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi} = -m^2\delta_{ij}\Phi^{i\dagger}\Phi^j$$

And finally the quartic one,

$$\begin{aligned}
-\frac{\lambda}{4}\left[\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi}\right]^2 &= -\frac{\lambda}{4}\left[\delta_{ij}\Phi^{i\dagger}\Phi^j\right]^2\\
&= -\frac{\lambda}{4}\delta_{ij}\delta_{kl}\Phi^{i\dagger}\Phi^j\Phi^{k\dagger}\Phi^l
\end{aligned}$$

Putting all together we have the full Lagrangian as,

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}A^{a\mu}\delta_{ab}&\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)A^{b\nu} + \frac{1}{2\xi}A_{a\mu}\partial^\mu\partial^\nu A^a{}_\nu\\
&- gf^{ab}{}_cA_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2}{4}f^{ab}{}_ef_{cd}{}^eA_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu}\\
&- \partial_\mu\bar{c}^a\partial^\mu c_a + gA_{c\mu}f^{abc}\partial^\mu\bar{c}_ac_b\\
&- \delta_{ij}\partial_\mu\Phi^{i\dagger}\partial^\mu\Phi^j - m^2\delta_{ij}\Phi^{i\dagger}\Phi^j - \frac{\lambda}{4}\delta_{ij}\delta_{kl}\Phi^{i\dagger}\Phi^j\Phi^{k\dagger}\Phi^l\\
&- \mathrm{i}gA_{a\mu}[\mathbf{T}^a_{\text{s}}]_{ij}\left(\Phi^{i\dagger}\partial^\mu\Phi^j - \partial^\mu\Phi^{i\dagger}\Phi^j\right) - g^2A_{a\mu}A^\mu_b\left[\mathbf{T}^a_{\text{s}}\mathbf{T}^b_{\text{s}}\right]_{ij}\Phi^{i\dagger}\Phi^j
\end{aligned}$$

This should really be the bare fields and parameters,

$$\begin{aligned}\mathcal{L} = \frac{1}{2}{A_0}^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big){A_0}^{b\nu} + \frac{1}{2\xi}A_{0a\mu}\partial^\mu\partial^\nu {A_0}^a{}_\nu \\ - gf^{ab}{}_c A_{0a\mu}A_{0b\nu}\partial^\mu {A_0}^{c\nu} - \frac{g^2}{4}f^{ab}{}_e f_{cd}{}^e A_{0a\mu}A_{0b\nu}{A_0}^{c\mu}{A_0}^{d\nu} \\ - \partial_\mu {\bar{c}_0}{}^a\partial^\mu c_{0a} + gA_{0c\mu}f^{abc}\partial^\mu \bar{c}_{0a}c_{0b} \\ - \delta_{ij}\partial_\mu {\Phi_0}^{i\dagger}\partial^\mu {\Phi_0}^j - m^2\delta_{ij}{\Phi_0}^{i\dagger}{\Phi_0}^j - \frac{\lambda}{4}\delta_{ij}\delta_{kl}{\Phi_0}^{i\dagger}{\Phi_0}^j{\Phi_0}^{k\dagger}{\Phi_0}^l \\ - \mathrm{i}gA_{0a\mu}[\mathbf{T}^a_{\mathrm{s}}]_{ij}\big({\Phi_0}^{i\dagger}\partial^\mu{\Phi_0}^j - \partial^\mu{\Phi_0}^{i\dagger}{\Phi_0}^j\big) - g^2A_{0a\mu}A_{0b}^\mu\big[\mathbf{T}^a_{\mathrm{s}}\mathbf{T}^b_{\mathrm{s}}\big]_{ij}{\Phi_0}^{i\dagger}{\Phi_0}^j\end{aligned}$$

So now we pass to the renormalized fields and parameters,

$$\begin{aligned}\mathcal{L} = \frac{1}{2}Z_A A^{a\mu}\delta_{ab}\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)A^{b\nu} + \frac{1}{2\xi}A_{a\mu}\partial^\mu\partial^\nu A^a{}_\nu \\ - gZ_{3g}f^{ab}{}_c A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2}{4}Z_{4g}f^{ab}{}_e f_{cd}{}^e A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\ - Z_c\partial_\mu\bar{c}^a\partial^\mu c_a + gZ_{gc}A_{c\mu}f^{abc}\partial^\mu\bar{c}_a c_b \\ - \delta_{ij}Z_\Phi\partial_\mu\Phi^{i\dagger}\partial^\mu\Phi^j - m^2Z_m\delta_{ij}\Phi^{i\dagger}\Phi^j - \frac{\lambda}{4}Z_\lambda\delta_{ij}\delta_{kl}\Phi^{i\dagger}\Phi^j\Phi^{k\dagger}\Phi^l \\ - \mathrm{i}gZ_{\Phi 1g}A_{a\mu}[\mathbf{T}^a_{\mathrm{s}}]_{ij}\big(\Phi^{i\dagger}\partial^\mu\Phi^j - \partial^\mu\Phi^{i\dagger}\Phi^j\big) - g^2Z_{\Phi 2g}A_{a\mu}A_b^\mu\big[\mathbf{T}^a_{\mathrm{s}}\mathbf{T}^b_{\mathrm{s}}\big]_{ij}\Phi^{i\dagger}\Phi^j\end{aligned}$$

Where we used that '$Z_\xi = 1$', that is, the Gauge fixing term do not get corrections, a result we showed before to 1-loop, but it's actually true to any loop. Just rearranging the terms and already setting '$\xi = 1$',

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\delta_{ab}g_{\mu\nu}A^{a\mu}\partial^2A^{b\nu} + \delta_{ij}\Phi^{i\dagger}\big(\partial^2 - m^2\big)\Phi^j - \partial_\mu\bar{c}^a\partial^\mu c_a \\ - gZ_{3g}f^{ab}{}_c A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2}{4}Z_{4g}f^{ab}{}_e f_{cd}{}^e A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\ + gZ_{gc}A_{c\mu}f^{abc}\partial^\mu\bar{c}_a c_b \\ - \frac{\lambda}{4}Z_\lambda\delta_{ij}\delta_{kl}\Phi^{i\dagger}\Phi^j\Phi^{k\dagger}\Phi^l \\ - \mathrm{i}gZ_{\Phi 1g}A_{a\mu}[\mathbf{T}^a_{\mathrm{s}}]_{ij}\big(\Phi^{i\dagger}\partial^\mu\Phi^j - \partial^\mu\Phi^{i\dagger}\Phi^j\big) - g^2Z_{\Phi 2g}A_{a\mu}A_b^\mu\big[\mathbf{T}^a_{\mathrm{s}}\mathbf{T}^b_{\mathrm{s}}\big]_{ij}\Phi^{i\dagger}\Phi^j \\ + \frac{1}{2}(Z_A - 1)A^{a\mu}\delta_{ab}g_{\mu\nu}\partial^2A^{b\nu} + \delta_{ij}(Z_\Phi - 1)\Phi^{i\dagger}\partial^2\Phi^j - m^2(Z_m - 1)\delta_{ij}\Phi^{i\dagger}\Phi^j \\ - (Z_c - 1)\partial_\mu\bar{c}^a\delta_{ab}\partial^\mu c^b\end{aligned}$$

This isn't yet the final version, because, as we are going to use dimensional regularization, all couplings dimensions need to be reset with a choice of particular mass scale 'Λ', in 'D'-dimensional space-time we have,

$$[A] = \frac{D-2}{2}, \quad [\Phi] = \frac{D-2}{2}, \quad [c] = \frac{D-2}{2}$$

Setting '$4 - D = 2\epsilon$',

$$
\begin{aligned}
[gAA\partial A] &= \frac{3}{2}(D-2)+1+[g]=D \\
[g] &= -\frac{1}{2}(D-4)=\epsilon
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
\left[g^2 AAAA\right] &= \frac{4}{2}(D-2)+2[g]=D \\
[g] &= \frac{1}{2}(-D+4)=\epsilon
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
[gA\partial\bar{c}c] &= \frac{3}{2}(D-2)+1+[g]=D \\
[g] &= -\frac{1}{2}(D-4)=\epsilon
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
\left[gA\Phi^\dagger\partial\Phi\right] &= \frac{1}{2}(D-2)+(D-2)+1+[g]=D \\
[g] &= -\frac{1}{2}(D-4)=\epsilon
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
\left[g^2 AA\Phi^\dagger\Phi\right] &= (D-2)+(D-2)+2[g]=D \\
[g] &= \frac{1}{2}(4-D)=\epsilon
\end{aligned} \tag{2.5}
$$

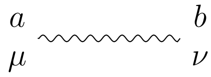
$$
\begin{aligned}
\left[\lambda\Phi^\dagger\Phi\Phi^\dagger\Phi\right] &= 2(D-2)+[\lambda]=D \\
[\lambda] &= 4-D=2\epsilon
\end{aligned} \tag{2.6}
$$

Applying the changes we get,

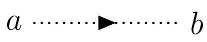
$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}&\delta_{ab}g_{\mu\nu}A^{a\mu}\partial^2 A^{b\nu} + \delta_{ij}\Phi^{i\dagger}\left(\partial^2-m^2\right)\Phi^j - \partial_\mu\bar{c}^a\partial^\mu c_a \\
&- gZ_{3g}\Lambda^\epsilon f^{ab}{}_c A_{a\mu}A_{b\nu}\partial^\mu A^{c\nu} - \frac{g^2}{4}Z_{4g}\Lambda^{2\epsilon}f^{ab}{}_e f_{cd}{}^e A_{a\mu}A_{b\nu}A^{c\mu}A^{d\nu} \\
&+ gZ_{gc}\Lambda^\epsilon A_{c\mu}f^{abc}\partial^\mu\bar{c}_a c_b \\
&- \frac{\lambda}{4}Z_\lambda\Lambda^{2\epsilon}\delta_{ij}\delta_{kl}\Phi^{i\dagger}\Phi^j\Phi^{k\dagger}\Phi^l \\
&- \mathrm{i}gZ_{\Phi 1g}\Lambda^\epsilon A_{a\mu}[\mathbf{T}^a_{\mathrm{s}}]_{ij}\left(\Phi^{i\dagger}\partial^\mu\Phi^j - \partial^\mu\Phi^{i\dagger}\Phi^j\right) - g^2 Z_{\Phi 2g}\Lambda^{2\epsilon}A_{a\mu}A^\mu_b\left[\mathbf{T}^a_{\mathrm{s}}\mathbf{T}^b_{\mathrm{s}}\right]_{ij}\Phi^{i\dagger}\Phi^j \\
&+ \frac{1}{2}(Z_A-1)A^{a\mu}\delta_{ab}g_{\mu\nu}\partial^2 A^{b\nu} + \delta_{ij}(Z_\Phi-1)\Phi^{i\dagger}\partial^2\Phi^j - m^2(Z_m-1)\delta_{ij}\Phi^{i\dagger}\Phi^j \\
&- (Z_c-1)\partial_\mu\bar{c}^a\delta_{ab}\partial^\mu c^b
\end{aligned}
$$

From where we can clearly read the propagators Feynman rules,

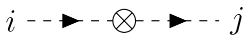
- Scalar Propagator $i \dashrightarrow j \; = \frac{1}{\mathrm{i}}\frac{1}{p^2+m^2}\delta_{ij}$

- Gauge Propagator 

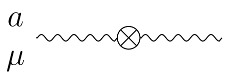
$= \frac{1}{\mathrm{i}} \frac{\delta_{ab}}{p^2} g_{\mu\nu}$

- Ghost Propagator 

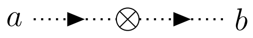
$= \frac{1}{\mathrm{i}} \frac{1}{p^2} \delta_{ab}$

- Scalar Counter-Term 

$= -\mathrm{i}(Z_\Phi - 1)\delta_{ij} p^2 - \mathrm{i}(Z_m - 1)\delta_{ij} m$

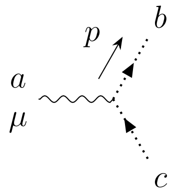
- Gauge Counter-Term 

$= -\mathrm{i}(Z_A - 1)\delta_{ab}(g_{\mu\nu} p^2 - p_\mu p_\nu)$

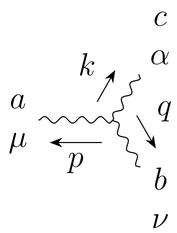
- Ghost Counter-Term 

$= -\mathrm{i}(Z_c - 1)\delta_{ab} p^2$

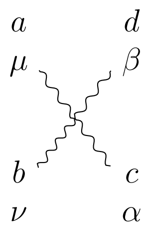
For the interactions Feynman rules we already got all the Gauge-Gauge and Gauge-Ghost rules from the last problem,

- Ghost-Gauge Interaction 

$= g Z_{gc} \Lambda^\epsilon f^{abc} p_\mu$

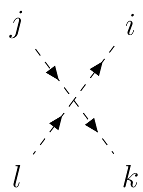
- Gauge Cubic Interaction 

$$= g Z_{3g} \Lambda^\epsilon f^{abc} \Big((q-k)_\mu g_{\nu\alpha} + (k-p)_\nu g_{\alpha\mu} + (p-q)_\alpha g_{\mu\nu} \Big) \tag{2.7}$$

- Gauge Quartic Interaction 

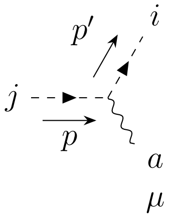
$$\begin{aligned} = -\mathrm{i} g^2 Z_{4g} \Lambda^{2\epsilon} \big[& f^{ab}{}_e f^{cde}(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) \\ & + f^{ac}{}_e f^{dbe}(g_{\mu\beta} g_{\alpha\nu} - g_{\mu\nu} g_{\beta\alpha}) \\ & + f^{ad}{}_e f^{bce}(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\beta\nu}) \big] \end{aligned} \tag{2.8}$$

What remains to be done is the Gauge-Scalar and Scalar-Scalar Feynman rules, we start with the Scalar-Scalar one, which requires a simple symmetrization,

- Scalar Quartic Interaction 

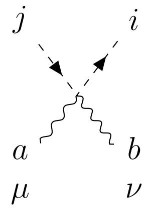
$= -\mathrm{i}\frac{\lambda}{2} Z_\lambda \Lambda^{2\epsilon} [\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}]$

Now to the Gauge-Scalar Cubic interaction, the derivative coupling gives a factor of '$\mathrm{i}p^\mu$' for 'Φ', and a factor of '$-\mathrm{i}p^\mu$' for the '$\Phi^\dagger$', so we get,

- Ghost-Scalar Cubic Interaction 

$= \mathrm{i}gZ_{\Phi 1g}\Lambda^\epsilon[\mathbf{T}^a_\mathrm{s}]_{ij}(p+p')^\mu$

The Gauge-Scalar Quartic interaction also needs a symmetrization, and gives,

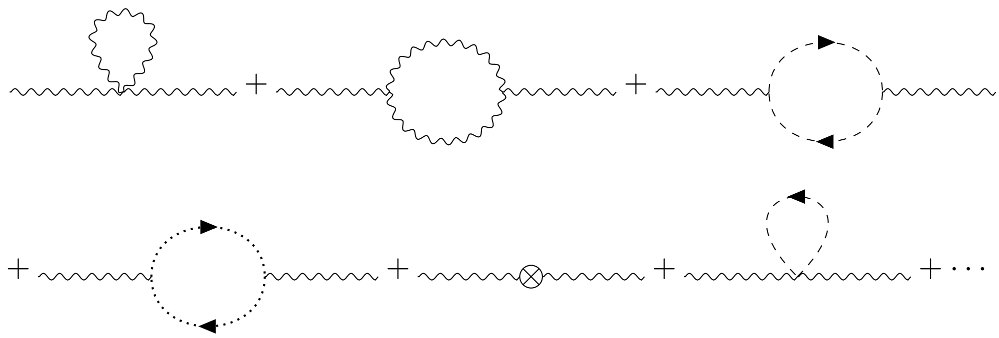
- Scalar Quartic Interaction 

$= -\mathrm{i}g^2 Z_{\Phi 2g}\Lambda^{2\epsilon}g_{\mu\nu}\Big(\big[\mathbf{T}^a_\mathrm{s}\mathbf{T}^b_\mathrm{s}\big]_{ij} + \big[\mathbf{T}^a_\mathrm{s}\mathbf{T}^b_\mathrm{s}\big]_{ij}\Big)$

This finishes all Feynman rules we'll need.

2.3 Gauge Boson Self Energy

Well, we already discussed and computed a lot of things when we computed the Gauge Self Energy in problem 1, so we're going to reutilize most of it here, mostly because the self energy differs only in two diagrams,

$\mathrm{i}\Pi^{ab}_{\mu\nu}\big(p^2\big) =$ 

The differences are the Fermion loop which was replaced by the Scalar loop and the new Scalar bubble. We start by the Scalar loop,

$$\begin{aligned}
\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} &= \mathrm{i}gZ_{\Phi 1g}\Lambda^\epsilon[\mathbf{T}^a_\mathrm{s}]_{ij}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}(2k-p)_\mu\frac{1}{\mathrm{i}}\frac{\delta^{ik}}{k^2+m^2}(2k-p)_\nu\frac{1}{\mathrm{i}}\frac{\delta^{jl}}{(k-p)^2+m^2}\times \\
&\qquad \times\,\mathrm{i}gZ_{\Phi 1g}\Lambda^\epsilon\big[\mathbf{T}^b_\mathrm{s}\big]_{lk} \\
&= g^2Z^2_{\Phi 1g}\Lambda^{2\epsilon}\,\mathrm{Tr}\,\big[\mathbf{T}^b_\mathrm{s}\mathbf{T}^a_\mathrm{s}\big]\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{(2k-p)_\mu(2k-p)_\nu}{(k^2+m^2)\big((k-p)^2+m^2\big)} \\
&= g^2Z^2_{\Phi 1g}\Lambda^{2\epsilon}\delta^{ab}T(\mathrm{S})\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{(2k-p)_\mu(2k-p)_\nu}{(k^2+m^2)\big((k-p)^2+m^2\big)}
\end{aligned}$$

Going for the Feynman parametrization,

$$\begin{aligned}
\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} &= g^2 Z^2_{\Phi 1g}\Lambda^{2\epsilon}\delta^{ab}T(\mathrm{S})\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{(2k-p)_\mu(2k-p)_\nu}{\left[x(k^2+m^2)+(1-x)\big((k-p)^2+m^2\big)\right]^2}\\
&= g^2 Z^2_{\Phi 1g}\Lambda^{2\epsilon}\delta^{ab}T(\mathrm{S})\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{(2k-p)_\mu(2k-p)_\nu}{\left[(k-p(1-x))^2+p^2x(1-x)+m^2\right]^2}
\end{aligned}$$

Switching integration variables to '$q = k - p(1-x)$', and neglecting linear terms in 'q' in the numerator, which when integrated get us zeros,

$$\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = g^2 Z^2_{\Phi 1g}\Lambda^{2\epsilon}\delta^{ab}T(\mathrm{S})\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^D q}{(2\pi)^D}\frac{4q_\mu q_\nu+(1-2x)^2p_\mu p_\nu}{[q^2+p^2x(1-x)+m^2]^2}$$

Now we use our bag of tricks to simplify this expression, starting by the replacement, '$q_\mu q_\nu \to \frac{1}{D}g_{\mu\nu}q^2$', and the other already mentioned property, '$q^2 \to \frac{D}{2-D}[p^2x(q-x)+m^2]$' which again we reforce is deduced in the appendix,

$$\begin{aligned}
\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = g^2 Z^2_{\Phi 1g}\Lambda^{2\epsilon}\delta^{ab}T(\mathrm{S})\int_0^1 \mathrm{d}x&\left[\frac{4g_{\mu\nu}}{2-D}\big(p^2x(1-x)+m^2\big)+(1-2x)^2p_\mu p_\nu\right]\times\\
&\times\int\frac{\mathrm{d}^D q}{(2\pi)^D}\frac{1}{[q^2+p^2x(1-x)+m^2]^2}
\end{aligned}$$

The remaining integral is very familiar to us,

$$\int\frac{\mathrm{d}^D q}{(2\pi)^D}\frac{1}{[q^2+p^2x(1-x)+m^2]^2} = \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\big(p^2x(1-x)+m^2\big)^{\frac{D}{2}-2}$$

So,

$$\begin{aligned}
\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = g^2 Z^2_{\Phi 1g}\Lambda^{2\epsilon}\delta^{ab}T(\mathrm{S})\int_0^1 \mathrm{d}x&\left[\frac{4g_{\mu\nu}}{2-D}\big(p^2x(1-x)+m^2\big)+(1-2x)^2p_\mu p_\nu\right]\times\\
&\times\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(2-\frac{D}{2}\right)\big(p^2x(1-x)+m^2\big)^{\frac{D}{2}-2}
\end{aligned}$$

Now going back to '$D = 4$' except in the Gamma function,

$$\mathrm{i}\Pi^{(3)ab}_{\ \ \mu\nu} = \frac{\mathrm{i}g^2}{16\pi^2}Z^2_{\Phi 1g}\delta^{ab}T(\mathrm{S})\Gamma(\epsilon)\int_0^1 \mathrm{d}x\left[-2g_{\mu\nu}\big(p^2x(1-x)+m^2\big)+(1-2x)^2p_\mu p_\nu\right]$$

What remains to be done is the 'x' integral, which is trivial,

$$
\begin{aligned}
\mathrm{i}\Pi^{(3)ab}_{\mu\nu} &= \frac{\mathrm{i}g^2}{16\pi^2} Z^2_{\Phi 1g}\delta^{ab}T(\mathrm{S})\Gamma(\epsilon)\left[-2g_{\mu\nu}\left(\frac{1}{6}p^2+m^2\right)+\frac{1}{3}p_\mu p_\nu\right] \\
&= \frac{\mathrm{i}g^2}{16\pi^2} Z^2_{\Phi 1g}\delta^{ab}T(\mathrm{S})\Gamma(\epsilon)\left[-\frac{1}{3}\big(g_{\mu\nu}p^2-p_\mu p_\nu\big)-2g_{\mu\nu}m^2\right]
\end{aligned}
$$

Now, the Scalar bubble,

$$
\begin{aligned}
\mathrm{i}\Pi^{(6)ab}_{\mu\nu} &= -\mathrm{i}g^2 Z_{\Phi 2g}\Lambda^{2\epsilon}g_{\mu\nu}\Big(\big[\mathbf{T}^a_{\mathrm{s}}\mathbf{T}^b_{\mathrm{s}}\big]_{ij}+\big[\mathbf{T}^a_{\mathrm{s}}\mathbf{T}^b_{\mathrm{s}}\big]_{ij}\Big)\times \\
&\qquad \times\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{\mathrm{i}}\frac{\delta^{ij}}{k^2+m^2} \\
&= -2g^2 Z_{\Phi 2g}\Lambda^{2\epsilon}g_{\mu\nu}\,\mathrm{Tr}\,\big[\mathbf{T}^a_{\mathrm{s}}\mathbf{T}^b_{\mathrm{s}}\big]\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2+m^2}
\end{aligned}
$$

Substituting the Trace factor, and also integrating the momentum integral, which is carefully done in the appendix,

$$
\begin{aligned}
\mathrm{i}\Pi^{(6)ab}_{\mu\nu} &= -2g^2 Z_{\Phi 2g}\Lambda^{2\epsilon}g_{\mu\nu}\delta^{ab}T(\mathrm{S})\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\big(m^2\big)^{\frac{D}{2}-1} \\
&= -2\frac{\mathrm{i}g^2}{16\pi^2}g^2 Z_{\Phi 2g}g_{\mu\nu}\delta^{ab}T(\mathrm{S})\Gamma(\epsilon-1)m^2
\end{aligned}
$$

Now using the Laurent expansion of Gamma and keeping only the divergent terms,

$$
\mathrm{i}\Pi^{(6)ab}_{\mu\nu} = m^2\frac{\mathrm{i}g^2}{8\pi^2\epsilon}g^2 Z_{\Phi 2g}g_{\mu\nu}\delta^{ab}T(\mathrm{S})
$$

So, summing all the contributions for the self-energy, including the ones derived in the last problem,

$$
\begin{aligned}
\mathrm{i}\Pi^{ab}_{\mu\nu} &= \mathrm{i}\Pi^{(1)ab}_{\mu\nu}+\mathrm{i}\Pi^{(2)ab}_{\mu\nu}+\mathrm{i}\Pi^{(3)ab}_{\mu\nu}+\mathrm{i}\Pi^{(4)ab}_{\mu\nu}+\mathrm{i}\Pi^{(5)ab}_{\mu\nu}+\mathrm{i}\Pi^{(6)ab}_{\mu\nu} \\
&= \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}Z^2_{3g}T(\mathrm{A})\left[\frac{19}{6}g_{\mu\nu}p^2-\frac{11}{3}p_\mu p_\nu\right]+\frac{\mathrm{i}g^2}{16\pi^2\epsilon}Z^2_{\Phi 1g}\delta^{ab}T(\mathrm{S})\left[-\frac{1}{3}\big(g_{\mu\nu}p^2-p_\mu p_\nu\big)-2g_{\mu\nu}m^2\right] \\
&\qquad +\frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}Z^2_{gc}T(\mathrm{A})\left(\frac{1}{6}g_{\mu\nu}p^2+\frac{1}{3}p_\mu p_\nu\right)-\mathrm{i}\delta^{ab}(Z_A-1)\big(g_{\mu\nu}p^2-p_\mu p_\nu\big) \\
&\qquad +m^2\frac{\mathrm{i}g^2}{8\pi^2\epsilon}g^2 Z_{\Phi 2g}g_{\mu\nu}\delta^{ab}T(\mathrm{S})
\end{aligned}
$$

Keeping only the leading terms in ‘Z’, and also grouping the similar terms,

$$
\begin{aligned}
\mathrm{i}\Pi^{ab}_{\mu\nu} &= \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}T(\mathrm{A})\left[\frac{19}{6}g_{\mu\nu}p^2 - \frac{11}{3}p_\mu p_\nu\right] + \frac{\mathrm{i}g^2}{16\pi^2\epsilon}\delta^{ab}T(\mathrm{S})\left[-\frac{1}{3}\big(g_{\mu\nu}p^2 - p_\mu p_\nu\big)\right] \\
&\quad + \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}T(\mathrm{A})\left(\frac{1}{6}g_{\mu\nu}p^2 + \frac{1}{3}p_\mu p_\nu\right) - \mathrm{i}\delta^{ab}(Z_A - 1)\big(g_{\mu\nu}p^2 - p_\mu p_\nu\big) \\
&= \frac{\mathrm{i}\delta^{ab}g^2}{32\pi^2\epsilon}T(\mathrm{A})\left[\frac{20}{6}g_{\mu\nu}p^2 - \frac{10}{3}p_\mu p_\nu\right] - \frac{\mathrm{i}g^2}{16\pi^2\epsilon}\delta^{ab}\frac{1}{3}T(\mathrm{S})\big[g_{\mu\nu}p^2 - p_\mu p_\nu\big] \\
&\quad - \mathrm{i}\delta^{ab}(Z_A - 1)\big(g_{\mu\nu}p^2 - p_\mu p_\nu\big) \\
&= \frac{\mathrm{i}\delta^{ab}g^2}{16\pi^2\epsilon}\frac{5}{3}T(\mathrm{A})\big[g_{\mu\nu}p^2 - p_\mu p_\nu\big] - \frac{\mathrm{i}g^2}{16\pi^2\epsilon}\delta^{ab}\frac{1}{3}T(\mathrm{S})\big[g_{\mu\nu}p^2 - p_\mu p_\nu\big] \\
&\quad - \mathrm{i}\delta^{ab}(Z_A - 1)\big(g_{\mu\nu}p^2 - p_\mu p_\nu\big)
\end{aligned}
$$

As this is only the divergent piece, we must set it to zero,

$$
\begin{aligned}
\mathrm{i}\delta^{ab}(Z_A - 1)\big(g_{\mu\nu}p^2 - p_\mu p_\nu\big) &= \frac{\mathrm{i}\delta^{ab}g^2}{16\pi^2\epsilon}\frac{1}{3}(5T(\mathrm{A}) - T(\mathrm{A}))\big[g_{\mu\nu}p^2 - p_\mu p_\nu\big] \\
Z_A &= 1 + \frac{g^2}{16\pi^2\epsilon}\frac{1}{3}(5T(\mathrm{A}) - T(\mathrm{A}))
\end{aligned}
$$

Getting us then the Wave-function renormalization factor.

2.4 Further Counter-Terms

Now, what we should do for continuing the computation is: Computing the Wave-Function renormalization factor for the scalar, and then computing the renormalization factor of the Gauge-Scalar interaction. But, we have something to our use here, there is a combination of the 'Z' factors which is in fact independent of the matter content. For this we have to look at the relation of the bare to renormalized parameters,

$$
g_0^2 = \frac{Z_{3g}^2}{Z_A^3}g^2\Lambda^{2\epsilon} = \frac{Z_{4g}}{Z_A^4}g^2\Lambda^{2\epsilon} = \frac{Z_{\Phi 1g}^2}{Z_A Z_\Phi^2}g^2\Lambda^{2\epsilon} = \frac{Z_{\Phi 2g}}{Z_A Z_\Phi}g^2\Lambda^{2\epsilon}
$$

What is related to the Slavnov-Taylor relations. Instead of computing '$Z_{\Phi 1g}$' and 'Z_Φ' explicitly, we're just using these relations to get that as,

$$
g_0^2 = \frac{Z_{\Phi 1g}^2}{Z_A Z_\Phi^2}g^2\Lambda^{2\epsilon}
$$

is a bare parameter, it's independent of the matter content, and thus we may equal it to the same combinations of counter-terms derived with fermions in the first problem. That is,

$$
g_0^2 = \frac{Z_{\Phi 1g}^2}{Z_A Z_\Phi^2}g^2\Lambda^{2\epsilon} = g^2\frac{Z_{g\Psi}^2}{Z_A Z_\Psi^2}\Lambda^{2\epsilon}
$$

And now we just use the already computed factors of '$Z_{g\Psi}$' and 'Z_Ψ'.

2.5 Computation of the Beta Function

With the remarks made in the last sub-section, we summarize the relevant results here,

$$\begin{cases} Z_A &= 1 + \frac{g^2}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right] \\ Z_\Psi &= 1 - \frac{g^2}{16\pi^2\epsilon}C(\mathrm{F}) \\ Z_{g\Psi} &= 1 - \frac{g^2}{16\pi^2\epsilon}[C(\mathrm{F}) + T(\mathrm{A})] \end{cases}$$

This may seem odd, due to the explicit dependence of both 'Z_Ψ' and '$Z_{g\Psi}$' in the fermion representation factor '$C(\mathrm{F})$' which is clearly not present in our theory, but, as we have argued, the combination '$Z_{g\Psi}^2 Z_\Psi^{-2}$' is matter content independent, and hence, shouldn't depend on '$C(\mathrm{F})$'. Let's show this,

$$\begin{aligned} Z_{g\Psi}^2 Z_\Psi^{-2} &= \left[1 - \frac{g^2}{16\pi^2\epsilon}[C(\mathrm{F}) + T(\mathrm{A})]\right]^2 \left[1 - \frac{g^2}{16\pi^2\epsilon}C(\mathrm{F})\right]^{-2} \\ &= \left[1 - \frac{g^2}{8\pi^2\epsilon}[C(\mathrm{F}) + T(\mathrm{A})]\right]\left[1 + \frac{g^2}{8\pi^2\epsilon}C(\mathrm{F})\right] \end{aligned}$$

Where we proceed as usual keeping only the leading singularities,

$$\begin{aligned} Z_{g\Psi}^2 Z_\Psi^{-2} &= 1 - \frac{g^2}{8\pi^2\epsilon}[C(\mathrm{F}) + T(\mathrm{A})] + \frac{g^2}{8\pi^2\epsilon}C(\mathrm{F}) \\ Z_{g\Psi}^2 Z_\Psi^{-2} &= 1 - \frac{g^2}{8\pi^2\epsilon}T(\mathrm{A}) \end{aligned}$$

As was expected, it doesn't depend on any form on the fermions, and only on the Gauge part of the theory. We can without guilt state that,

$$\begin{aligned} g_0^2 &= \frac{Z_{\Phi 1g}^2}{Z_A Z_\Phi^2} g^2\Lambda^{2\epsilon} = g^2\frac{Z_{g\Psi}^2}{Z_A Z_\Psi^2} g^2\Lambda^{2\epsilon} \\ g_0^2 &= \left[1 + \frac{g^2}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right]^{-1}\left[1 - \frac{g^2}{8\pi^2\epsilon}T(\mathrm{A})\right] g^2\Lambda^{2\epsilon} \\ g_0^2 &= \left[1 - \frac{g^2}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right]\left[1 - \frac{g^2}{8\pi^2\epsilon}T(\mathrm{A})\right] g^2\Lambda^{2\epsilon} \\ g_0^2 &= g^2\Lambda^{2\epsilon}\left[1 - \frac{g^2}{16\pi^2\epsilon}\left[\frac{5}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right] - \frac{g^2}{8\pi^2\epsilon}T(\mathrm{A})\right] \\ g_0^2 &= g^2\Lambda^{2\epsilon}\left[1 - \frac{g^2}{16\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right] \end{aligned}$$

So now we just follow the usual procedure, calling '$g^2 = \alpha$' just for convenience,

$$
\begin{aligned}
\Lambda\frac{\mathrm{d}\alpha_0}{\mathrm{d}\Lambda} = 0 &= (\beta + 2\epsilon\alpha)\Lambda^{2\epsilon}\left[1 - \frac{\alpha}{16\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right] - \alpha\Lambda^{2\epsilon}\frac{\beta}{16\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right] \\
0 &= (\beta + 2\epsilon\alpha)\left[1 - \frac{\alpha}{16\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right] - \alpha\frac{\beta}{16\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right] \\
0 &= (\beta + \epsilon\alpha)\left[1 - \frac{\alpha}{8\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right] + \epsilon\alpha \\
-\epsilon\alpha &= (\beta + \epsilon\alpha)\left[1 - \frac{\alpha}{8\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right] \\
\beta + \epsilon\alpha &= -\epsilon\alpha\left[1 + \frac{\alpha}{8\pi^2\epsilon}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]\right] \\
\beta &= -2\epsilon\alpha - \frac{\alpha^2}{8\pi^2}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]
\end{aligned}
$$

Now, changing for 'g',

$$
\begin{aligned}
2g\beta &= -2\epsilon g^2 - \frac{g^4}{8\pi^2}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right] \\
\beta &= -\epsilon g - \frac{g^3}{16\pi^2}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]
\end{aligned}
$$

And passing now to the '$\epsilon \to 0$' limit,

$$
\beta = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}T(\mathrm{A}) - \frac{1}{3}T(\mathrm{S})\right]
$$

As we desired.

3 Not the Standard Model

The Lagrangian of our theory is very similar to the one already wrote down in the last problem, that is,

$$\mathcal{L} = -[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] - \frac{1}{2} \operatorname{Tr}\left[\mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu}\right] - V\left(\mathbf{\Phi}^\dagger \mathbf{\Phi}\right)$$

The kinetic term of the scalar was already open up in the last problem, being,

$$\begin{aligned} -[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] = & -\delta_{ij} \partial_\mu \Phi^{i\dagger} \partial^\mu \Phi^j \\ & - \mathrm{i} g A_{a\mu} [\mathbf{T}_\mathrm{s}^a]_{ij} \left(\Phi^{i\dagger} \partial^\mu \Phi^j - \partial^\mu \Phi^{i\dagger} \Phi^j\right) \\ & - g^2 A_{a\mu} A_b^\mu \left[\mathbf{T}_\mathrm{s}^a \mathbf{T}_\mathrm{s}^b\right]_{ij} \Phi^{i\dagger} \Phi^j \end{aligned}$$

Setting to the scalar to transform under the adjoint representation,

$$[\mathbf{T}_\mathrm{s}^a]_{bc} = -\mathrm{i} f^a{}_{bc}$$

We have the kinetic term of the scalar as being,

$$\begin{aligned} -[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] = & -\delta_{ab} \partial_\mu \Phi^{a\dagger} \partial^\mu \Phi^b \\ & - g A_{a\mu} f^a{}_{bc} \left(\Phi^{b\dagger} \partial^\mu \Phi^c - \partial^\mu \Phi^{b\dagger} \Phi^c\right) \\ & + g^2 A_{a\mu} A_b^\mu f^a{}_{ce} f^{be}{}_d \Phi^{c\dagger} \Phi^d \end{aligned}$$

Now we invoke that the algebra is the '$\mathfrak{su}(2)$' algebra, so the structure constants are just the Levi-Civita symbol,

$$\begin{aligned} -[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] = & -\delta_{ab} \partial_\mu \Phi^{a\dagger} \partial^\mu \Phi^b \\ & - g A_{a\mu} \epsilon^a{}_{bc} \left(\Phi^{b\dagger} \partial^\mu \Phi^c - \partial^\mu \Phi^{b\dagger} \Phi^c\right) \\ & + g^2 A_{a\mu} A_b^\mu \epsilon_e{}^a{}_c \epsilon^e{}_d{}^b \Phi^{c\dagger} \Phi^d \end{aligned}$$

The Levi-Civita contraction is well known, being just,

$$\epsilon_e{}^a{}_c \epsilon^e{}_d{}^b = \delta^a{}_d \delta_c{}^b - \delta^{ab} \delta_{cd}$$

Thus,

$$\begin{aligned} -[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] = & -\delta_{ab} \partial_\mu \Phi^{a\dagger} \partial^\mu \Phi^b \\ & - g A_{a\mu} \epsilon^a{}_{bc} \left(\Phi^{b\dagger} \partial^\mu \Phi^c - \partial^\mu \Phi^{b\dagger} \Phi^c\right) \\ & + g^2 A_{a\mu} A_b^\mu \left[\delta^a{}_d \delta_c{}^b - \delta^{ab} \delta_{cd}\right] \Phi^{c\dagger} \Phi^d \\ -[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] = & -\delta_{ab} \partial_\mu \Phi^{a\dagger} \partial^\mu \Phi^b \\ & - g A_{a\mu} \epsilon^a{}_{bc} \left(\Phi^{b\dagger} \partial^\mu \Phi^c - \partial^\mu \Phi^{b\dagger} \Phi^c\right) \\ & + g^2 A_{a\mu} A_b^\mu \left[\Phi^{b\dagger} \Phi^a - \delta^{ab} \mathbf{\Phi}^\dagger \mathbf{\Phi}\right] \end{aligned}$$

So now we shift the fields by a vector, '$\Phi^a \to \Phi^a + v^a$',

$$
\begin{aligned}
-[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] = -\delta_{ab}\partial_\mu \Phi^{a\dagger}\partial^\mu \Phi^b & \\
& - gA_{a\mu}\epsilon^a{}_{bc}\big((\Phi^{b\dagger} + v^{b\dagger})\partial^\mu \Phi^c - \partial^\mu \Phi^{b\dagger}(\Phi^c + v^c)\big) \\
& + g^2 A_{a\mu}A_b^\mu\big[(\Phi^{b\dagger} + v^{b\dagger})(\Phi^a + v^a) - \delta^{ab}\big(\mathbf{\Phi}^\dagger\mathbf{\Phi} + \boldsymbol{v}^\dagger\mathbf{\Phi} + \mathbf{\Phi}^\dagger\boldsymbol{v} + \boldsymbol{v}^\dagger\boldsymbol{v}\big)\big]
\end{aligned}
$$

Keeping only the terms that depend only upon '$\boldsymbol{A}_\mu$' and '$\boldsymbol{v}$',

$$
-[\mathbf{D}_\mu \mathbf{\Phi}]^\dagger [\mathbf{D}^\mu \mathbf{\Phi}] \supset -\frac{1}{2}(-2)g^2 A_{a\mu}A_b^\mu\left[v^{b\dagger}v^a - \delta^{ab}\boldsymbol{v}^\dagger\boldsymbol{v}\right]
$$

That is, the mass matrix, generated by a vacuum of '$\langle\mathbf{\Phi}\rangle = \boldsymbol{v}$', in the Gauge-Bosons, is

$$
M^2_{ab} = -2g^2\left[v_b^\dagger v_a - \delta_{ab}\boldsymbol{v}^\dagger\boldsymbol{v}\right]
$$

Of course, in this form, this matrix isn't diagonal, hence is not clear what are the eigenvalues, that is, the masses. To this matter, let's compute them,

$$
\begin{aligned}
M^2_{ab}w^b &= M^2 w_a \\
-2g^2\left[v_b^\dagger v_a - \delta_{ab}\boldsymbol{v}^\dagger\boldsymbol{v}\right]w^b &= M^2 w_a
\end{aligned}
$$

Here, and all along this problem, we're assuming '$\boldsymbol{v}^\dagger\boldsymbol{v} \neq 0$', otherwise the vacuum is trivial, that is, '$\langle\mathbf{\Phi}\rangle = 0$', and none of the problem makes sense. But, what is not certain it's true is, '$\boldsymbol{v}^\dagger\boldsymbol{w} = 0$'. Let us divide in cases,

- $\boldsymbol{v}^\dagger\boldsymbol{w} = 0$

$$
\begin{aligned}
M^2_{ab}w^b &= M^2 w_a \\
2g^2\delta_{ab}\boldsymbol{v}^\dagger\boldsymbol{v}w^b &= M^2 w_a \\
2g^2\boldsymbol{v}^\dagger\boldsymbol{v}w_a &= M^2 w_a \\
2g^2\boldsymbol{v}^\dagger\boldsymbol{v} &= M^2
\end{aligned}
$$

The other possible case is,

- $\boldsymbol{v}^\dagger\boldsymbol{w} \neq 0$

$$
\begin{aligned}
M^2_{ab}w^b &= M^2 w_a \\
-2g^2\left[v_b^\dagger v_a - \delta_{ab}\boldsymbol{v}^\dagger\boldsymbol{v}\right]w^b &= M^2 w_a \\
-2g^2 v^{a\dagger}\left[v_b^\dagger v_a - \delta_{ab}\boldsymbol{v}^\dagger\boldsymbol{v}\right]w^b &= M^2 v^{a\dagger}w_a \\
-2g^2\left[\boldsymbol{v}^\dagger\boldsymbol{w}\boldsymbol{v}^\dagger\boldsymbol{v} - \boldsymbol{v}^\dagger\boldsymbol{w}\boldsymbol{v}^\dagger\boldsymbol{v}\right] &= M^2\boldsymbol{v}^\dagger\boldsymbol{w} \\
0 &= M^2
\end{aligned}
$$

Let us summarize our results,

$$\begin{cases} M^2 = 0 & \Leftrightarrow \boldsymbol{v}^\dagger \boldsymbol{w} \neq 0 \\ M^2 = 2g^2 \boldsymbol{v}^\dagger \boldsymbol{v} & \Leftrightarrow \boldsymbol{v}^\dagger \boldsymbol{w} = 0 \end{cases}$$

Ok, but this still left us with the question, how many eigenvectors of each eigenvalue we have? To answer this question we have to notice a very important fact, the Gauge-Boson mass matrix is Hermitian!

$$\begin{aligned} M^2_{ab} &= -2g^2 \left[v^\dagger_b v_a - \delta_{ab} \boldsymbol{v}^\dagger \boldsymbol{v} \right] \\ M^2_{ba} &= -2g^2 \left[v^\dagger_a v_b - \delta_{ba} \boldsymbol{v}^\dagger \boldsymbol{v} \right] \\ {M^2_{ba}}^* &= -2g^2 \left[v_a v^\dagger_b - \delta_{ba} \boldsymbol{v}^\dagger \boldsymbol{v} \right] = {M^2}^\dagger_{ab} \\ M^2_{ab} &= {M^2}^\dagger_{ab} \end{aligned}$$

But this is not enough, as we had already said, '$\boldsymbol{v}^\dagger \boldsymbol{v} \neq 0$', otherwise, '$M^2_{ab}$' would be a null matrix, which is of course Hermitian, but, to which doesn't hold the property we're going to state now, *the eigenvectors of a Hermitian matrix belonging to different eigenvalues are orthogonal to each other, and, the eigenvectors constitute a basis of the space.* The last statement is equivalent to say that all eigenvectors are linearly independent, and of course, by definition of eigenvector, they must be non-null. Our space is 3-dimensional, hence, there are only 2 linearly independent vectors satisfying '$\boldsymbol{v}^\dagger \boldsymbol{w} = 0$' for a given '$\boldsymbol{v}$', for '$\boldsymbol{v}^\dagger \boldsymbol{w} \neq 0$' there is just 1, and as this set needs to be linear independent and complete we got the right answer,

$$\begin{cases} M^2 = 0 & \Leftrightarrow \boldsymbol{v}^\dagger \boldsymbol{w} \neq 0, & \text{1 degree of freedom} \\ M^2 = 2g^2 \boldsymbol{v}^\dagger \boldsymbol{v} & \Leftrightarrow \boldsymbol{v}^\dagger \boldsymbol{w} = 0, & \text{2 degree of freedom} \end{cases}$$

In other words, we broke two generators, and just one of them was left unbroken, so, one of the Gauge-Bosons is left still massless, while the other two becomes massive with mass '$M^2 = 2g^2 \boldsymbol{v}^\dagger \boldsymbol{v}$'. As shown, this result is independent of what '$\boldsymbol{v}$' is, in particular the direction of it. Of course the modulus is directly related to the mass, but, the direction of it is not relevant, as we can always make a rotation of the basis of the group generators to brake/unbroken a different one. To the matter of what is the group breaking pattern, we started with the group '$SU(2)$' which has three generators, as we broke two of them, we are left with just one generator. There is just one such group, continuous, over '$\mathbb{C}$' and with just one generator, the '$U(1)$' group. Hence the group breaking pattern is,

$$SU(2) \to U(1)$$

4 Anomalies

Our theory under consideration here is a single Dirac Fermion coupled to an '$U(1)$' Gauge-Boson,

$$\mathcal{L} = -\bar{\Psi}\not{D}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

With of course our covariant derivative being,

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu$$

Besides the trivial '$U(1)$' global, gauged, symmetry,

$$\begin{aligned}\Psi &\to \exp(-\mathrm{i}g\alpha)\Psi \\ \bar{\Psi} &\to \bar{\Psi}\exp(\mathrm{i}g\alpha)\end{aligned}$$

our theory has also another symmetry,

$$\begin{aligned}\Psi &\to \exp(-\mathrm{i}g\gamma_5\alpha)\Psi \\ \bar{\Psi} &\to \bar{\Psi}\exp(-\mathrm{i}g\gamma_5\alpha)\end{aligned}$$

This is true due to,

$$\{\gamma_5, \gamma_\mu\} = 0$$

What implies that,

$$\begin{aligned}-\bar{\Psi}\not{D}\Psi &\to -\bar{\Psi}\exp(-\mathrm{i}g\gamma_5\alpha)\not{D}\exp(-\mathrm{i}g\gamma_5\alpha)\Psi \\ -\bar{\Psi}\not{D}\Psi &\to -\bar{\Psi}\exp(-\mathrm{i}g\gamma_5\alpha)\exp(\mathrm{i}g\gamma_5\alpha)\not{D}\Psi \\ -\bar{\Psi}\not{D}\Psi &\to -\bar{\Psi}\not{D}\Psi\end{aligned}$$

For completeness, we're going to get the associated Noether charges,

$$\begin{aligned}S[\bar{\Psi}, \Psi, A_\mu, \alpha] &= \int \mathrm{d}^4x\, \mathcal{L}\big(\bar{\Psi}\exp(\mathrm{i}g\alpha), \exp(-\mathrm{i}g\alpha)\Psi, A_\mu\big) = \int \mathrm{d}^4x\, \mathcal{L}\big(\bar{\Psi}, \Psi, A_\mu\big) \\ \frac{\mathrm{d}S}{\mathrm{d}\alpha} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\int \mathrm{d}^4x\, \mathcal{L}\big(\bar{\Psi}\exp(\mathrm{i}g\alpha), \exp(-\mathrm{i}g\alpha)\Psi, A_\mu\big) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{L}\big(\bar{\Psi}, \Psi, A_\mu\big) = 0 \\ 0 &= \frac{\mathrm{d}\bar{\Psi}\exp(\mathrm{i}g\alpha)}{\mathrm{d}\alpha}\frac{\delta S}{\delta\bar{\Psi}(y)} - \frac{\delta S}{\delta\Psi(y)}\frac{\mathrm{d}\exp(-\mathrm{i}g\alpha)\Psi}{\mathrm{d}\alpha} \\ 0 &= \mathrm{i}g\bar{\Psi}\exp(\mathrm{i}g\alpha)\frac{\delta S}{\delta\bar{\Psi}(y)} + \frac{\delta S}{\delta\Psi(y)}\mathrm{i}g\exp(-\mathrm{i}g\alpha)\Psi\end{aligned}$$

First we get,

$$
\begin{aligned}
\frac{\delta S}{\delta\bar{\Psi}(y)} &= \frac{1}{\bar{\epsilon}}\big\{S[\bar{\Psi}(x)+\bar{\epsilon}\delta(x-y),\Psi,A_\mu]-S[\bar{\Psi},\Psi,A_\mu]\big\}\\
\frac{\delta S}{\delta\bar{\Psi}(y)} &= \frac{1}{\bar{\epsilon}}\int \mathrm{d}^4x\,\big\{-\big(\bar{\Psi}(x)+\bar{\epsilon}\delta(x-y)\big)\not{D}\Psi(x)+\bar{\Psi}(x)\not{D}\Psi(x)\big\}\\
\frac{\delta S}{\delta\bar{\Psi}(y)} &= -\frac{1}{\bar{\epsilon}}\int \mathrm{d}^4x\,\bar{\epsilon}\delta(x-y)\not{D}\Psi(x)\\
\frac{\delta S}{\delta\bar{\Psi}(y)} &= -\not{D}\Psi(y) = -\gamma^\mu\partial_\mu\Psi+\mathrm{i}gA_\mu\gamma^\mu\Psi
\end{aligned}
$$

And,

$$
\begin{aligned}
\frac{\delta S}{\delta\Psi(y)} &= \frac{1}{\epsilon}\big\{S[\bar{\Psi}(x),\Psi+\epsilon\delta(x-y),A_\mu]-S[\bar{\Psi},\Psi,A_\mu]\big\}\\
\frac{\delta S}{\delta\Psi(y)} &= \frac{1}{\epsilon}\int \mathrm{d}^4x\,\big\{-\bar{\Psi}(x)\not{D}(\Psi(x)+\epsilon\delta(x-y))+\bar{\Psi}(x)\not{D}\Psi(x)\big\}\\
\frac{\delta S}{\delta\Psi(y)} &= -\frac{1}{\epsilon}\int \mathrm{d}^4x\,\bar{\Psi}(x)\not{D}\epsilon\delta(x-y)\\
\frac{\delta S}{\delta\Psi(y)} &= \int \mathrm{d}^4x\,\bar{\Psi}(x)\gamma^\mu(\partial_\mu-\mathrm{i}gA_\mu)\delta(x-y)\\
\frac{\delta S}{\delta\Psi(y)} &= \int \mathrm{d}^4x\,\big[-\partial_\mu\bar{\Psi}(x)\gamma^\mu-\bar{\Psi}(x)\gamma^\mu\mathrm{i}gA_\mu\big]\delta(x-y)\\
\frac{\delta S}{\delta\Psi(y)} &= -\partial_\mu\bar{\Psi}(y)\gamma^\mu-\bar{\Psi}(y)\gamma^\mu\mathrm{i}gA_\mu
\end{aligned}
$$

So that the equation is,

$$
\begin{aligned}
0 &= \bar{\Psi}[-\gamma^\mu\partial_\mu\Psi+\mathrm{i}gA_\mu\gamma^\mu\Psi]-\big[\partial_\mu\bar{\Psi}(y)\gamma^\mu+\bar{\Psi}(y)\gamma^\mu\mathrm{i}gA_\mu\big]\Psi\\
0 &= -\partial_\mu\big[\bar{\Psi}\gamma^\mu\Psi\big]
\end{aligned}
$$

So we got our first current, the Vector current,

$$
J^\mu_{\mathrm{V}}(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x)
$$

We do again the same procedure for the other symmetry,

$$
\begin{aligned}
S[\bar{\Psi},\Psi,A_\mu,\alpha] &= \int \mathrm{d}^4x\,\mathcal{L}\big(\bar{\Psi}\exp(-\mathrm{i}g\gamma_5\alpha),\exp(-\mathrm{i}g\gamma_5\alpha)\Psi,A_\mu\big) = \int \mathrm{d}^4x\,\mathcal{L}\big(\bar{\Psi},\Psi,A_\mu\big)\\
\frac{\mathrm{d}S}{\mathrm{d}\alpha} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\int \mathrm{d}^4x\,\mathcal{L}\big(\bar{\Psi}\exp(-\mathrm{i}g\gamma_5\alpha),\exp(-\mathrm{i}g\gamma_5\alpha)\Psi,A_\mu\big) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{L}\big(\bar{\Psi},\Psi,A_\mu\big) = 0\\
0 &= \frac{\mathrm{d}\bar{\Psi}\exp(-\mathrm{i}g\gamma_5\alpha)}{\mathrm{d}\alpha}\frac{\delta S}{\delta\bar{\Psi}(y)} - \frac{\delta S}{\delta\Psi(y)}\frac{\mathrm{d}\exp(-\mathrm{i}g\alpha)\Psi}{\mathrm{d}\alpha}\\
0 &= -\mathrm{i}g\bar{\Psi}\gamma_5\exp(\mathrm{i}g\alpha)\frac{\delta S}{\delta\bar{\Psi}(y)}+\frac{\delta S}{\delta\Psi(y)}\mathrm{i}g\gamma_5\exp(-\mathrm{i}g\alpha)\Psi\\
0 &= -\bar{\Psi}\gamma_5[-\gamma^\mu\partial_\mu\Psi+\mathrm{i}gA_\mu\gamma^\mu\Psi]+\big[-\partial_\mu\bar{\Psi}(y)\gamma^\mu-\bar{\Psi}(y)\gamma^\mu\mathrm{i}gA_\mu\big]\gamma_5\Psi\\
0 &= \bar{\Psi}\gamma_5\gamma^\mu\partial_\mu\Psi-\partial_\mu\bar{\Psi}(y)\gamma^\mu\gamma_5\Psi\\
0 &= -\partial_\mu\big[\bar{\Psi}\gamma^\mu\gamma_5\Psi\big]
\end{aligned}
$$

This is our Axial current,

$$J^{\mu}_{\mathrm{A}}(x) = \bar{\Psi}(x)\gamma^{\mu}\gamma_5\Psi(x)$$

From a tree level analysis, or, classically, as both this symmetries are indeed symmetries of the Lagrangian, what is expected is that both of the currents should obey the zero divergence,

$$\begin{aligned}\partial_{\mu}J^{\mu}_{\mathrm{V}}(x) &= 0\\ \partial_{\mu}J^{\mu}_{\mathrm{A}}(x) &= 0\end{aligned}$$

And, as we saw, this is true. But, nothing guarantees that this statement will still hold upon quantization of the theory. To check if the quantized theory do possesses theses conservations laws, we have to look at quantities of the kind,

$$\partial_{x\mu}\left\langle 0\right|\mathrm{T}\{J^{\mu}_{\mathrm{A}}(x)\mathcal{O}(y)\}\left|0\right\rangle =?$$

By our virtue of foresight, we're going to compute,

$$\begin{aligned}\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) =&\ \left\langle\Omega\right|\mathrm{T}\Big\{J^{\mu}_{\mathrm{A}}(x)J^{\alpha}_{\mathrm{V}}(y)J^{\beta}_{\mathrm{V}}(z)\Big\}\left|\Omega\right\rangle\\ \mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) =&\ \left\langle\Omega\right|\mathrm{T}\big\{\bar{\Psi}(x)\gamma^{\mu}\gamma_5\Psi(x)\bar{\Psi}(y)\gamma^{\alpha}\Psi(y)\bar{\Psi}(z)\gamma^{\beta}\Psi(z)\big\}\left|\Omega\right\rangle\end{aligned}$$

Up to first order, there're two possible Wick contraction we can perform here — of course, only two are possible because we're ignoring non-connected contributions —,

$$\begin{aligned}\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) =&\ \left\langle 0\right|\mathrm{T}\left\{\bar{\Psi}(x)\gamma^{\mu}\gamma_5\Psi(x)\bar{\Psi}(y)\gamma^{\alpha}\Psi(y)\bar{\Psi}(z)\gamma^{\beta}\Psi(z)\right\}\left|0\right\rangle\\ &+\ \left\langle 0\right|\mathrm{T}\left\{\bar{\Psi}(x)\gamma^{\mu}\gamma_5\Psi(x)\bar{\Psi}(y)\gamma^{\alpha}\Psi(y)\bar{\Psi}(z)\gamma^{\beta}\Psi(z)\right\}\left|0\right\rangle\end{aligned}$$

Now, the contractions themselves are easily evaluated, as they're just the free propagators,

$$\begin{aligned}\Psi(x)\bar{\Psi}(y) =&\ \left\langle 0\right|\mathrm{T}\big\{\Psi(x)\bar{\Psi}(y)\big\}\left|0\right\rangle\\ \Psi(x)\bar{\Psi}(y) =&\ \frac{1}{\mathrm{i}}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\exp\left(\mathrm{i}k\cdot(x-y)\right)\frac{-\mathrm{i}\not{k}}{k^2} = \frac{1}{\mathrm{i}}S(x,y)\end{aligned}$$

As every operator is already contracted, there is no more need for the Time-ordered product, neither for the vacuum expectation value. We

$$\begin{aligned}\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) =&\ \bar{\Psi}(x)\gamma^{\mu}\gamma_5\Psi(x)\bar{\Psi}(y)\gamma^{\alpha}\Psi(y)\bar{\Psi}(z)\gamma^{\beta}\Psi(z)\\ &+\bar{\Psi}(x)\gamma^{\mu}\gamma_5\Psi(x)\bar{\Psi}(y)\gamma^{\alpha}\Psi(y)\bar{\Psi}(z)\gamma^{\beta}\Psi(z)\end{aligned}$$

We now manipulate the fields to get all in the right order, but in doing this, we can not lost the track of the multiplication order, for the first one is easy, as we can write it as a trace, and we need an additional minus signs, as we need to cross an odd number of fermionic fields. For the second one, we can switch fully one vector current with the other to get,

$$
\begin{aligned}
\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) = &- \operatorname{Tr}\left[\gamma^\mu\gamma_5\Psi(x)\bar{\Psi}(y)\gamma^\alpha\Psi(y)\bar{\Psi}(z)\gamma^\beta\Psi(z)\bar{\Psi}(x)\right] \\
&+ \bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\bar{\Psi}(z)\gamma^\beta\Psi(z)\bar{\Psi}(y)\gamma^\alpha\Psi(y) \\
\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) = &- \operatorname{Tr}\left[\gamma^\mu\gamma_5\Psi(x)\bar{\Psi}(y)\gamma^\alpha\Psi(y)\bar{\Psi}(z)\gamma^\beta\Psi(z)\bar{\Psi}(x)\right] \\
&- \operatorname{Tr}\left[\gamma^\mu\gamma_5\Psi(x)\bar{\Psi}(z)\gamma^\beta\Psi(z)\bar{\Psi}(y)\gamma^\alpha\Psi(y)\bar{\Psi}(x)\right] \\
\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) = &- \operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}S(x,y)\gamma^\alpha\frac{1}{\mathrm{i}}S(y,z)\gamma^\beta\frac{1}{\mathrm{i}}S(z,x)\right] \\
&- \operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}S(x,z)\gamma^\beta\frac{1}{\mathrm{i}}S(z,y)\gamma^\alpha\frac{1}{\mathrm{i}}S(y,x)\right] \\
\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) = &- \int\frac{\mathrm{d}^4k'\,\mathrm{d}^4p'\,\mathrm{d}^4q'}{(2\pi)^{12}}\exp\left(\mathrm{i}k'\cdot(x-y)+\mathrm{i}p'\cdot(y-z)+\mathrm{i}q'\cdot(z-x)\right)\times \\
&\times\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{k}'}{k'^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{p}'}{p'^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{q}'}{q'^2}\right] \\
&- \int\frac{\mathrm{d}^4k'\,\mathrm{d}^4p'\,\mathrm{d}^4q'}{(2\pi)^{12}}\exp\left(\mathrm{i}k'\cdot(x-y)+\mathrm{i}p'\cdot(y-z)+\mathrm{i}q'\cdot(z-x)\right)\times \\
&\times\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{q}'}{q'^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{p}'}{p'^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{k}'}{k'^2}\right]
\end{aligned}
$$

Here we wisely choose the integration momentum variables so that we can group the two terms inside just one integral,

$$
\begin{aligned}
\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) = &\int\frac{\mathrm{d}^4k'\,\mathrm{d}^4p'\,\mathrm{d}^4q'}{(2\pi)^{12}}\exp\left(\mathrm{i}k'\cdot(x-y)+\mathrm{i}p'\cdot(y-z)+\mathrm{i}q'\cdot(z-x)\right)\times \\
&\times\left\{-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{k}'}{k'^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{p}'}{p'^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{q}'}{q'^2}\right]-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{q}'}{q'^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{p}'}{p'^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{k}'}{k'^2}\right]\right\}
\end{aligned}
$$

Now it's clear the appearance of a triple Fourier transform, so we work with just the momentum space to ease our calculation. Thus,

$$
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)(2\pi)^4\delta^{(4)}(k+p+q) = \int\mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,\mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z)\exp\left(-\mathrm{i}k\cdot x-\mathrm{i}p\cdot y-\mathrm{i}q\cdot z\right)
$$

This affects primarily just the space dependent part, so let's compute first,

$$
\begin{aligned}
&\int \frac{\mathrm{d}^4k'\,\mathrm{d}^4p'\,\mathrm{d}^4q'}{(2\pi)^{12}} \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z \times \\
&\qquad \times \exp\left(-\mathrm{i}k\cdot x - \mathrm{i}p\cdot y - \mathrm{i}q\cdot z\right) \exp\left(\mathrm{i}k'\cdot(x-y) + \mathrm{i}p'\cdot(y-z) + \mathrm{i}q'\cdot(z-x)\right) \\
&\int \frac{\mathrm{d}^4k'\,\mathrm{d}^4p'\,\mathrm{d}^4q'}{(2\pi)^{12}} \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z \times \\
&\qquad \times \exp\left(\mathrm{i}x\cdot(k'-k-q') + \mathrm{i}y\cdot(p'-k'-p) + \mathrm{i}z\cdot(q'-p'-q)\right) \\
&\int \mathrm{d}^4k'\,\mathrm{d}^4p'\,\mathrm{d}^4q'\,\delta^{(4)}(k'-k-q')\delta^{(4)}(p'-k'-p)\delta^{(4)}(q'-p'-q)
\end{aligned}
$$

So,

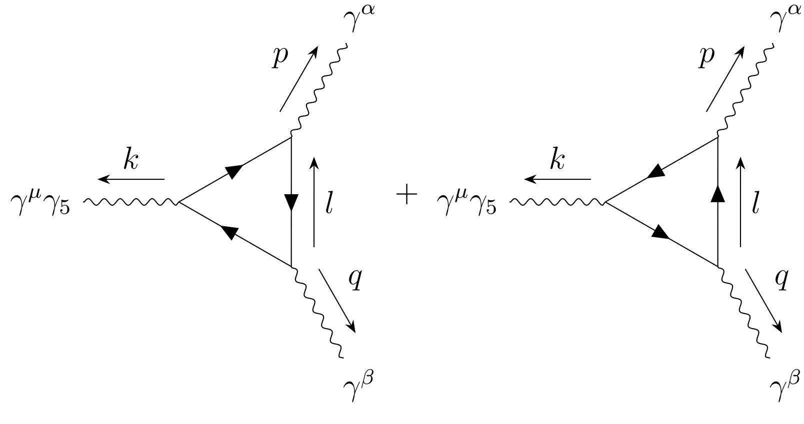
$$
\begin{aligned}
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)(2\pi)^4\delta^{(4)}(k+p+q) = &\int \mathrm{d}^4k'\,\mathrm{d}^4p'\,\mathrm{d}^4q'\,\delta^{(4)}(k'-k-q')\delta^{(4)}(p'-k'-p)\times \\
&\times \delta^{(4)}(q'-p'-q)\left(-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{k}'}{k'^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{p}'}{p'^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{q}'}{q'^2}\right]\right. \\
&\left.-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{q}'}{q'^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{p}'}{p'^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{k}'}{k'^2}\right]\right) \\
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)(2\pi)^4\delta^{(4)}(k+p+q) = &\int \mathrm{d}^4p'\,\delta^{(4)}(k+q+p)\times \\
&\times\left(-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{k}-\mathrm{i}\not{p}'-\mathrm{i}\not{q}}{(k+q+p')^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{p}'}{p'^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{p}'-\mathrm{i}\not{q}}{(q+p')^2}\right]\right. \\
&\left.-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{p}'+\mathrm{i}\not{q}}{(q+p')^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{p}'}{p'^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{k}+\mathrm{i}\not{q}+\mathrm{i}\not{p}'}{(k+q+p')^2}\right]\right)
\end{aligned}
$$

Finally we got,

$$
\begin{aligned}
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = \int\frac{\mathrm{d}^4l}{(2\pi)^4}&\left(-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{k}-\mathrm{i}\not{l}-\mathrm{i}\not{q}}{(k+q+l)^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{l}}{l^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{-\mathrm{i}\not{l}-\mathrm{i}\not{q}}{(q+l)^2}\right]\right. \\
&\left.-\operatorname{Tr}\left[\gamma^\mu\gamma_5\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{l}+\mathrm{i}\not{q}}{(q+l)^2}\gamma^\beta\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{l}}{l^2}\gamma^\alpha\frac{1}{\mathrm{i}}\frac{\mathrm{i}\not{k}+\mathrm{i}\not{q}+\mathrm{i}\not{l}}{(k+q+l)^2}\right]\right)
\end{aligned}
$$

Notice that by the form of this vacuum expectation value, we certainly can find a diagram that matches it, in this case, there are, one for each trace,

$\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) =$



But of course these diagrams are only for illustration purposes, as there is no 'γ_5' coupling to any Gauge-boson in our theory, and we're not analyzing any 3-Gauge-boson scattering. Simplifying a little,

$$
\begin{aligned}
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(q+l)^2(k+q+l)^2} \times \\
\times \left\{ \mathrm{Tr}\left[\gamma^\mu\gamma_5(\not{k}+\not{l}+\not{q})\gamma^\alpha\not{l}\gamma^\beta(\not{l}+\not{q})\right] - \mathrm{Tr}\left[\gamma^\mu\gamma_5(\not{l}+\not{q})\gamma^\beta\not{l}\gamma^\alpha(\not{k}+\not{q}+\not{l})\right]\right\}
\end{aligned}
$$

What we really are interested in computing is,

$$
\begin{aligned}
\mathrm{i}k_\mu\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(q+l)^2(k+q+l)^2} \times \\
&\times \left\{ \mathrm{Tr}\left[\not{k}\gamma_5(\not{k}+\not{l}+\not{q})\gamma^\alpha\not{l}\gamma^\beta(\not{l}+\not{q})\right] - \mathrm{Tr}\left[\not{k}\gamma_5(\not{l}+\not{q})\gamma^\beta\not{l}\gamma^\alpha(\not{k}+\not{q}+\not{l})\right]\right\} \\
\mathrm{i}k_\mu\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(q+l)^2(k+q+l)^2} \times \\
&\times \left\{ \mathrm{Tr}\left[\gamma^\alpha\not{l}\gamma^\beta\gamma_5(\not{l}+\not{q})\not{k}(\not{k}+\not{l}+\not{q})\right] - \mathrm{Tr}\left[\gamma^\beta\not{l}\gamma^\alpha\gamma_5(\not{k}+\not{q}+\not{l})\not{k}(\not{l}+\not{q})\right]\right\}
\end{aligned}
$$

But notice that,

$$
\begin{aligned}
(\not{l}+\not{q})\not{k}(\not{k}+\not{l}+\not{q}) &= (\not{l}+\not{q})\left[\not{k}+\not{l}+\not{q}-(\not{l}+\not{q})\right](\not{k}+\not{l}+\not{q}) \\
(\not{l}+\not{q})\not{k}(\not{k}+\not{l}+\not{q}) &= (\not{l}+\not{q})(k+l+q)^2-(l+q)^2(\not{k}+\not{l}+\not{q})
\end{aligned}
$$

Analogously

$$
\begin{aligned}
(\not{k}+\not{l}+\not{q})\not{k}(\not{l}+\not{q}) &= (\not{k}+\not{l}+\not{q})\left[\not{k}+\not{l}+\not{q}-(\not{l}+\not{q})\right](\not{l}+\not{q}) \\
(\not{k}+\not{l}+\not{q})\not{k}(\not{l}+\not{q}) &= (\not{l}+\not{q})(k+l+q)^2-(l+q)^2(\not{k}+\not{l}+\not{q})
\end{aligned}
$$

So that,

$$
\begin{aligned}
\mathrm{i}k_\mu\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = -\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(q+l)^2(k+q+l)^2} \times \\
\times \left[(l+k+q)^2\,\mathrm{Tr}\left[\gamma_5\gamma^\alpha\not{l}\gamma^\beta(\not{l}+\not{q})\right] - (l+q)^2\,\mathrm{Tr}\left[\gamma_5\gamma^\alpha\not{l}\gamma^\beta(\not{l}+\not{q}+\not{k})\right]\right. \\
\times \left. -(l+k+q)^2\,\mathrm{Tr}\left[\gamma_5\gamma^\beta\not{l}\gamma^\alpha(\not{l}+\not{q})\right] + (l+q)^2\,\mathrm{Tr}\left[\gamma_5\gamma^\beta\not{l}\gamma^\alpha(\not{l}+\not{q}+\not{k})\right]\right]
\end{aligned}
$$

Using now the following derived in the appendix,

$$
\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] = 4\mathrm{i}\epsilon^{\mu\nu\alpha\beta}
$$

We obtain,

$$
\begin{aligned}
\mathrm{i}k_\mu \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = -4\mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times \\
\times\left[(l+k+q)^2\epsilon^{\alpha\rho\beta\sigma}l_\rho(l+q)_\sigma - (l+q)^2\epsilon^{\alpha\rho\beta\sigma}l_\rho(l+q+k)_\sigma\right. \\
\left.\times -(l+k+q)^2\epsilon^{\beta\rho\alpha\sigma}l_\rho(l+q)_\sigma + (l+q)^2\epsilon^{\beta\rho\alpha\sigma}l_\rho(l+q+k)_\sigma\right] \\
\mathrm{i}k_\mu \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = -4\mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times \\
\times\left[(l+k+q)^2\epsilon^{\alpha\rho\beta\sigma}l_\rho q_\sigma - (l+q)^2\epsilon^{\alpha\rho\beta\sigma}l_\rho(q+k)_\sigma\right. \\
\left.\times +(l+k+q)^2\epsilon^{\alpha\rho\beta\sigma}l_\rho q_\sigma - (l+q)^2\epsilon^{\alpha\rho\beta\sigma}l_\rho(q+k)_\sigma\right] \\
\mathrm{i}k_\mu \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = 8\mathrm{i}\epsilon^{\alpha\beta\rho\sigma}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times \\
\times\left[(l+k+q)^2 l_\rho q_\sigma - (l+q)^2 l_\rho(q+k)_\sigma\right] \\
\mathrm{i}k_\mu \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = 8\mathrm{i}\epsilon^{\alpha\beta\rho\sigma}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left[\frac{l_\rho q_\sigma}{l^2(q+l)^2} - \frac{l_\rho(q+k)_\sigma}{l^2(q+l+k)^2}\right]
\end{aligned}
$$

This integral is clearly **linearly divergent**, this is a sign of trouble ahead, we proceed then carefully, first we notice that the integrand is in the form,

$$
f_\rho(l;q) = \frac{l_\rho}{l^2(q+l)^2}
$$

By Lorentz covariance, we can argue that it must be true,

$$
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) = Aq_\rho
$$

Where 'A' is some constant to be determined. Let's substitute this back in our expression,

$$
\mathrm{i}k_\mu \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = 8A\mathrm{i}\epsilon^{\alpha\beta\rho\sigma}\Big[q_\rho q_\sigma - (q+k)_\rho(q+k)_\sigma\Big] = 0
$$

Due to the anti-symmetric nature of the Levi-Civita. What a crazy result! Is this correct? As we said before, this was a linearly divergent integral, for this to be true, the integrand must behave like a constant for high momenta. As we warm up for what is coming next, suppose we have a function '$f(x)$', such that the integral,

$$
\int_{-\infty}^{+\infty} \mathrm{d}x\, f(x)
$$

Is linearly divergent. Looks like a trivial fact that a variable change such as,

$$
\int_{-\infty}^{+\infty} \mathrm{d}x\, f(x+a)
$$

Would make no diference on the integral, but this is not true. As '$f(x)$' must be a constant at infinity, all the derivatives vanish at infinity, and we can do the following,

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x+a) &= \int_{-\infty}^{+\infty}\left[f(x)+af'(x)+\frac{1}{2}a^2 f''(x)+\cdots\right]\\ &= \int_{-\infty}^{+\infty} f(x)+a(f(+\infty)-f(-\infty))\end{aligned}$$

With no more other terms as all derivatives are zero at infinity. We clearly see here that the shift we did at our integral indeed affected it, a not trivial fact. We could use this fact for changing the integral loop variable 'l' and reproduce any value for '$\mathrm{i}k_\mu\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)$', what obstructs us from just choosing a arbitrary value for these? The answer lies on the other two useful quantities '$\mathrm{i}p_\alpha\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)$' and '$\mathrm{i}q_\beta\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)$'. As we discussed before, we're interested in computing the divergence of the current beyond tree level, and, this is related to the momentum contraction of the '$\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}$'. So let's compute the other two related contractions,

$$\begin{aligned}\mathrm{i}p_\alpha\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= \int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times\\ &\times\left\{\mathrm{Tr}\left[\gamma^\mu\gamma_5(\not{k}+\not{l}+\not{q})\not{p}\not{l}\gamma^\beta(\not{l}+\not{q})\right]-\mathrm{Tr}\left[\gamma^\mu\gamma_5(\not{l}+\not{q})\gamma^\beta\not{l}\not{p}(\not{k}+\not{q}+\not{l})\right]\right\}\\ \mathrm{i}p_\alpha\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= \int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times\\ &\times\left\{-\mathrm{Tr}\left[\gamma_5\gamma^\beta(\not{l}+\not{q})\gamma^\mu(\not{k}+\not{l}+\not{q})\not{p}\not{l}\right]+\mathrm{Tr}\left[\gamma_5\gamma^\beta\not{l}\not{p}(\not{k}+\not{q}+\not{l})\gamma^\mu(\not{l}+\not{q})\right]\right\}\end{aligned}$$

Using again,

$$\begin{aligned}(\not{k}+\not{l}+\not{q})\not{p}\not{l} &= (\not{k}+\not{l}+\not{q})(-\not{k}-\not{q})\not{l}\\ &= (\not{k}+\not{l}+\not{q})(-\not{k}-\not{q}-\not{l}+\not{l})\not{l}\\ &= -(k+l+q)^2\not{l}+l^2(\not{k}+\not{l}+\not{q})\end{aligned}$$

Where we used the momentum conservation to write '$p=-q-k$'. Putting back and using the already mentioned Gamma matrices trace,

$$\begin{aligned}\mathrm{i}p_\alpha\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= \int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times\\ &\times\left((k+l+q)^2\,\mathrm{Tr}\left[\gamma_5\gamma^\beta(\not{l}+\not{q})\gamma^\mu\not{l}\right]-l^2\,\mathrm{Tr}\left[\gamma_5\gamma^\beta(\not{l}+\not{q})\gamma^\mu(\not{k}+\not{l}+\not{q})\right]\right.\\ &\left.-(k+l+q)^2\,\mathrm{Tr}\left[\gamma_5\gamma^\beta\not{l}\gamma^\mu(\not{l}+\not{q})\right]+l^2\,\mathrm{Tr}\left[\gamma_5\gamma^\beta(\not{k}+\not{q}+\not{l})\gamma^\mu(\not{l}+\not{q})\right]\right)\\ \mathrm{i}p_\alpha\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 4\mathrm{i}\epsilon^{\beta\rho\mu\sigma}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times\\ &\times\Big((k+l+q)^2(l+q)_\rho l_\sigma-l^2(l+q)_\rho(k+l+q)_\sigma\\ &-(k+l+q)^2 l_\rho(l+q)_\sigma+l^2(k+q+l)_\rho(l+q)_\sigma\Big)\end{aligned}$$

Using the antisymmetric properties of the Levi-Civita,

$$
\begin{aligned}
\mathrm{i}p_\alpha \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = 4\mathrm{i}\epsilon^{\beta\rho\mu\sigma} & \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(q+l)^2(k+q+l)^2} \times \\
& \times \Big((k+l+q)^2 q_\rho l_\sigma - l^2 (l+q)_\rho k_\sigma \\
& -(k+l+q)^2 l_\rho q_\sigma + l^2 k_\rho (l+q)_\sigma \Big) \\
\mathrm{i}p_\alpha \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = 8\mathrm{i}\epsilon^{\beta\rho\mu\sigma} & \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(q+l)^2(k+q+l)^2} \times \\
& \times \Big((k+l+q)^2 q_\rho l_\sigma - l^2 (l+q)_\rho k_\sigma \Big) \\
\mathrm{i}p_\alpha \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = 8\mathrm{i}\epsilon^{\beta\rho\mu\sigma} & \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left[\frac{q_\rho l_\sigma}{l^2 (q+l)^2} - \frac{(l+q)_\rho k_\sigma}{(q+l)^2 (k+q+l)^2} \right]
\end{aligned}
$$

Here is the catch! If shifting the momentum integral '$l + q \to l$' didn't make a difference, we could argue that this integral is also zero! But as we argued, we can't do that, so let's really compute what it is. Our starting point is the argument made with the simple single variable function before, that is,

$$
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) = \int \mathrm{d}^4 l \, (2\pi)^4 \left[f_\rho(l;q) + p^\tau \frac{\partial}{\partial l^\tau} f_\rho(l;q) + \cdots \right]
$$

But as we argued, the higher derivatives terms all go to zero, leaving us with just,

$$
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) + \int \frac{\mathrm{d}^4 l}{(2\pi)^4} p^\tau \frac{\partial}{\partial l^\tau} f_\rho(l;q)
$$

The last term is clearly a boundary term, in this case it's equivalent to integrating over the boundary of the Minkowski momentum space, something that is not really well defined, so we need to pass to the Euclidean version, what amounts to multiplying by 'i' and integrating over the 'S^3' sphere with infinite radius,

$$
\begin{aligned}
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) - \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) &= \mathrm{i} \lim_{l\to\infty} \int \frac{\mathrm{d}^3 S_\tau}{(2\pi)^4} p^\tau f_\rho(l;q) \\
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) - \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) &= \mathrm{i} \lim_{l\to\infty} \int \frac{\mathrm{d}^3 \Omega}{(2\pi)^4} p^\tau l^2 l_\tau f_\rho(l;q) \\
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) - \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) &= \mathrm{i} \lim_{l\to\infty} \int \frac{\mathrm{d}^3 \Omega}{(2\pi)^4} p^\tau l_\tau \frac{l_\rho}{(l+q)^2} \\
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) - \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) &= \mathrm{i} \lim_{l\to\infty} \int \frac{\mathrm{d}^3 \Omega}{(2\pi)^4} p^\tau l_\tau \frac{l_\rho}{l^2} \\
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) - \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) &= \mathrm{i} \lim_{l\to\infty} \int \frac{\mathrm{d}^3 \Omega}{(2\pi)^4} p^\tau \frac{1}{4} \frac{g_{\tau\rho} l^2}{l^2} \\
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) - \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) &= \mathrm{i} p_\rho \frac{1}{4} \int \frac{\mathrm{d}^3 \Omega}{(2\pi)^4}
\end{aligned}
$$

The remaining integral is the area of a unit '3'-sphere, what is also computed in the appendix as being, '$2\pi^2$',

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) - \int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l;q) = \frac{\mathrm{i}}{32\pi^2} p_\rho$$

A beautiful result! That is,

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f_\rho(l+p;q) = A q_\rho + \frac{\mathrm{i}}{32\pi^2} p_\rho$$

Going back to our original expression,

$$\begin{aligned}
\mathrm{i}p_\alpha \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 8\mathrm{i}\epsilon^{\beta\rho\mu\sigma} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} [q_\rho f_\sigma(l;q) - k_\sigma f_\rho(l+q;k)] \\
\mathrm{i}p_\alpha \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 8\mathrm{i}\epsilon^{\beta\rho\mu\sigma} \left[A q_\rho q_\sigma - k_\sigma \left(A k_\rho + \frac{\mathrm{i}}{32\pi^2} q_\rho \right) \right] \\
\mathrm{i}p_\alpha \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= -\frac{1}{4\pi^2} \epsilon^{\beta\mu\rho\sigma} q_\rho k_\sigma
\end{aligned}$$

Interesting! And the final one,

$$\begin{aligned}
\mathrm{i}q_\beta \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} & \frac{1}{l^2 (q+l)^2 (k+q+l)^2} \times \\
& \times \left\{ \mathrm{Tr}\left[\gamma^\mu \gamma_5 (\not{k} + \not{l} + \not{q}) \gamma^\alpha \not{l} \not{q} (\not{l} + \not{q})\right] - \mathrm{Tr}\left[\gamma^\mu \gamma_5 (\not{l} + \not{q}) \not{q} \not{l} \gamma^\alpha (\not{k} + \not{q} + \not{l})\right] \right\}
\end{aligned}$$

Using again,

$$\begin{aligned}
(\not{l} + \not{q}) \not{q} \not{l} &= (\not{l} + \not{q})(\not{l} + \not{q} - \not{l}) \not{l} \\
&= (l+q)^2 \not{l} - l^2 (\not{l} + \not{q})
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{i}q_\beta \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2 (q+l)^2 (k+q+l)^2} \times \\
& \quad \times \Big((q+l)^2 \,\mathrm{Tr}\left[\gamma^\mu \gamma_5 (\not{k} + \not{l} + \not{q}) \gamma^\alpha \not{l}\right] - l^2 \,\mathrm{Tr}\left[\gamma^\mu \gamma_5 (\not{k} + \not{l} + \not{q}) \gamma^\alpha (\not{l} + \not{q})\right] \\
& \quad -(q+l)^2 \,\mathrm{Tr}\left[\gamma^\mu \gamma_5 \not{l} \gamma^\alpha (\not{k} + \not{q} + \not{l})\right] + l^2 \,\mathrm{Tr}\left[\gamma^\mu \gamma_5 (\not{l} + \not{q}) \gamma^\alpha (\not{k} + \not{q} + \not{l})\right] \Big) \\
\mathrm{i}q_\beta \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 4\mathrm{i}\epsilon^{\mu\rho\alpha\sigma} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2 (q+l)^2 (k+q+l)^2} \times \\
& \quad \times \Big(-(q+l)^2 (k+q)_\rho l_\sigma + l^2 k_\rho (l+q)_\sigma \\
& \quad +(q+l)^2 l_\rho (k+q)_\sigma - l^2 (l+q)_\rho k_\sigma \Big) \\
\mathrm{i}q_\beta \tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 8\mathrm{i}\epsilon^{\mu\alpha\rho\sigma} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2 (q+l)^2 (k+q+l)^2} \times \\
& \quad \times \Big((q+l)^2 (k+q)_\rho l_\sigma - l^2 k_\rho (l+q)_\sigma \Big)
\end{aligned}$$

Now we put it in the more familiar form,

$$
\begin{aligned}
\mathrm{i}q_\beta\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 8\mathrm{i}\epsilon^{\mu\alpha\rho\sigma}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\left[\frac{(k+q)_\rho l_\sigma}{l^2(k+q+l)^2}-\frac{k_\rho(l+q)_\sigma}{(q+l)^2(k+q+l)^2}\right]\\
\mathrm{i}q_\beta\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 8\mathrm{i}\epsilon^{\mu\alpha\rho\sigma}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\left[(k+q)_\rho f_\sigma(l;k+q)-k_\rho f_\sigma(l+q;k)\right]\\
\mathrm{i}q_\beta\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= 8\mathrm{i}\epsilon^{\mu\alpha\rho\sigma}\left[(k+q)_\rho A(k+q)_\sigma-k_\rho\left(Ak_\sigma+\frac{\mathrm{i}}{32\pi^2}q_\sigma\right)\right]\\
\mathrm{i}q_\beta\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) &= \frac{1}{4\pi^2}\epsilon^{\mu\alpha\rho\sigma}k_\rho q_\sigma
\end{aligned}
$$

So two of them are non-zero while one is indeed zero. But is this the only choice? We could go back to our original expression,

$$
\begin{aligned}
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = &\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2(q+l)^2(k+q+l)^2}\times\\
&\times\left\{\mathrm{Tr}\left[\gamma^\mu\gamma_5(\not{k}+\not{l}+\not{q})\gamma^\alpha\not{l}\gamma^\beta(\not{l}+\not{q})\right]-\mathrm{Tr}\left[\gamma^\mu\gamma_5(\not{l}+\not{q})\gamma^\beta\not{l}\gamma^\alpha(\not{k}+\not{q}+\not{l})\right]\right\}
\end{aligned}
$$

And try to regularize it as,

$$
\begin{aligned}
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q) = &\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{(l+k+q)_\rho l_\nu(l+q)_\sigma}{l^2(q+l)^2(k+q+l)^2}\times\\
&\times\left\{\mathrm{Tr}\left[\gamma^\mu\gamma_5\gamma^\rho\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\sigma\right]-\mathrm{Tr}\left[\gamma^\mu\gamma_5\gamma^\sigma\gamma^\beta\gamma^\nu\gamma^\alpha\gamma^\rho\right]\right\}
\end{aligned}
$$

We have just one integral which is linearly divergent, so we can shift the integration variable to see what kind of choices we're allowed to make,

$$
I_{\rho\nu\sigma}(k,q;l+a)=\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{(l+a+k+q)_\rho(l+a)_\nu(l+a+q)_\sigma}{(l+a)^2(q+l+a)^2(k+q+l+a)^2}
$$

As we saw,

$$
\begin{aligned}
I_{\rho\nu\sigma}(k,q;l+a)-I_{\rho\nu\sigma}(k,q;l) &= \mathrm{i}\lim_{l\to\infty}\int\frac{\mathrm{d}^3S_\tau}{(2\pi)^4}a^\tau\frac{(l+k+q)_\rho l_\nu(l+q)_\sigma}{l^2(q+l)^2(k+q+l)^2}\\
I_{\rho\nu\sigma}(k,q;l+a)-I_{\rho\nu\sigma}(k,q;l) &= \mathrm{i}\lim_{l\to\infty}\int\frac{\mathrm{d}^3S_\tau}{(2\pi)^4}a^\tau\frac{l_\rho l_\nu l_\sigma}{l^2l^2l^2}\\
I_{\rho\nu\sigma}(k,q;l+a)-I_{\rho\nu\sigma}(k,q;l) &= \mathrm{i}\lim_{l\to\infty}\int\frac{\mathrm{d}^3\Omega}{(2\pi)^4}a^\tau l_\tau\frac{l_\rho l_\nu l_\sigma}{l^2l^2}\\
I_{\rho\nu\sigma}(k,q;l+a)-I_{\rho\nu\sigma}(k,q;l) &= \mathrm{i}\lim_{l\to\infty}\int\frac{\mathrm{d}^3\Omega}{(2\pi)^4}a^\tau\frac{1}{4\cdot 3!}(g_{\tau\rho}g_{\nu\sigma}+g_{\tau\nu}g_{\rho\sigma}+g_{\tau\sigma}g_{\nu\rho})\frac{l^2l^2}{l^2l^2}\\
I_{\rho\nu\sigma}(k,q;l+a)-I_{\rho\nu\sigma}(k,q;l) &= \mathrm{i}\frac{2\pi^2}{(2\pi)^4}a^\tau\frac{1}{4\cdot 3!}(g_{\tau\rho}g_{\nu\sigma}+g_{\tau\nu}g_{\rho\sigma}+g_{\tau\sigma}g_{\nu\rho})\\
I_{\rho\nu\sigma}(k,q;l+a)-I_{\rho\nu\sigma}(k,q;l) &= \frac{\mathrm{i}}{32\pi^2 6}a^\tau(g_{\tau\rho}g_{\nu\sigma}+g_{\tau\nu}g_{\rho\sigma}+g_{\tau\sigma}g_{\nu\rho})
\end{aligned}
$$

We have now to multiply this factor by the Gamma matrices trace, let's start with,

$$
\begin{aligned}
&(g_{\tau\rho}g_{\nu\sigma}+g_{\tau\nu}g_{\rho\sigma}+g_{\tau\sigma}g_{\nu\rho})\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\rho\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\sigma\right]\\
&\big(-2g_{\tau\rho}\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\rho\gamma^\alpha\gamma^\beta\right]+g_{\tau\nu}g_{\rho\sigma}\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\rho\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\sigma\right]-2g_{\tau\sigma}\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\alpha\gamma^\beta\gamma^\sigma\right]\big)\\
&8\mathrm{i}g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}+8\mathrm{i}g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}+2g_{\tau\nu}\operatorname{Tr}\left[\gamma^\mu\gamma_5\big(\gamma^\beta\gamma^\alpha\gamma^\nu-\gamma^\alpha\gamma^\nu\gamma^\beta-\gamma^\nu\gamma^\alpha\gamma^\beta\big)\right]\\
&8\mathrm{i}g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}+8\mathrm{i}g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}-8\mathrm{i}g_{\tau\nu}\big[\epsilon^{\mu\beta\alpha\nu}-\epsilon^{\mu\alpha\nu\beta}-\epsilon^{\mu\nu\alpha\beta}\big]\\
&8\mathrm{i}g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}+8\mathrm{i}g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}+8\mathrm{i}g_{\tau\nu}\epsilon^{\mu\nu\alpha\beta}\\
&8\mathrm{i}g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}+8\mathrm{i}g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}+8\mathrm{i}g_{\tau\nu}\epsilon^{\mu\nu\alpha\beta}
\end{aligned}
$$

The other contribution is,

$$
\begin{aligned}
&(g_{\tau\rho}g_{\nu\sigma}+g_{\tau\nu}g_{\rho\sigma}+g_{\tau\sigma}g_{\nu\rho})\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\sigma\gamma^\beta\gamma^\nu\gamma^\alpha\gamma^\rho\right]\\
&\big(-2g_{\tau\rho}\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\beta\gamma^\alpha\gamma^\rho\right]+g_{\tau\nu}g_{\rho\sigma}\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\sigma\gamma^\beta\gamma^\nu\gamma^\alpha\gamma^\rho\right]-2g_{\tau\sigma}\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\sigma\gamma^\beta\gamma^\alpha\right]\big)\\
&8\mathrm{i}g_{\tau\rho}\epsilon^{\mu\beta\alpha\rho}+8\mathrm{i}g_{\tau\sigma}\epsilon^{\mu\sigma\beta\alpha}+2g_{\tau\nu}\operatorname{Tr}\left[\gamma^\mu\gamma_5\big(\gamma^\alpha\gamma^\beta\gamma^\nu-\gamma^\beta\gamma^\nu\gamma^\alpha-\gamma^\nu\gamma^\beta\gamma^\alpha\big)\right]\\
&-8\mathrm{i}g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}-8\mathrm{i}g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}-8\mathrm{i}g_{\tau\nu}\big[\epsilon^{\mu\alpha\beta\nu}-\epsilon^{\mu\beta\nu\alpha}-\epsilon^{\mu\nu\beta\alpha}\big]\\
&-8\mathrm{i}g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}-8\mathrm{i}g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}-8\mathrm{i}g_{\tau\nu}\epsilon^{\mu\nu\alpha\beta}
\end{aligned}
$$

Subtracting the two we have,

$$
\begin{aligned}
&(g_{\tau\rho}g_{\nu\sigma}+g_{\tau\nu}g_{\rho\sigma}+g_{\tau\sigma}g_{\nu\rho})\big\{\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\rho\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\sigma\right]-\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\sigma\gamma^\beta\gamma^\nu\gamma^\alpha\gamma^\rho\right]\big\}\\
&=16\mathrm{i}\big[g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}+g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}+g_{\tau\nu}\epsilon^{\mu\nu\alpha\beta}\big]
\end{aligned}
$$

So that finally,

$$
\begin{aligned}
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)-\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)&=[I_{\rho\nu\sigma}(k,q;l+a)-I_{\rho\nu\sigma}(k,q;l)]\times\\
&\quad\times\big\{\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\rho\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\sigma\right]-\operatorname{Tr}\left[\gamma^\mu\gamma_5\gamma^\sigma\gamma^\beta\gamma^\nu\gamma^\alpha\gamma^\rho\right]\big\}\\
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)-\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)&=\frac{\mathrm{i}}{32\pi^2 6}a^\tau 16\mathrm{i}\big[g_{\tau\rho}\epsilon^{\mu\rho\alpha\beta}+g_{\tau\sigma}\epsilon^{\mu\alpha\beta\sigma}+g_{\tau\nu}\epsilon^{\mu\nu\alpha\beta}\big]\\
\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)-\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)&=\frac{1}{4\pi^2}a_\nu\epsilon^{\nu\mu\alpha\beta}
\end{aligned}
$$

Now we make use of what we already had computed to get,

$$
\begin{aligned}
k_\mu\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)-k_\mu\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)&=\frac{1}{4\pi^2}a_\nu k_\mu\epsilon^{\nu\mu\alpha\beta}\\
k_\mu\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)&=\frac{1}{4\pi^2}a_\nu k_\mu\epsilon^{\nu\mu\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
p_\alpha\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)-p_\alpha\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)&=\frac{1}{4\pi^2}a_\nu p_\alpha\epsilon^{\nu\mu\alpha\beta}\\
p_\alpha\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)&=-\frac{1}{4\pi^2}p_\alpha k_\nu\epsilon^{\nu\mu\alpha\beta}+\frac{1}{4\pi^2}a_\nu p_\alpha\epsilon^{\nu\mu\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
q_\beta\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)-q_\beta\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)&=\frac{1}{4\pi^2}a_\nu q_\beta\epsilon^{\nu\mu\alpha\beta}\\
q_\beta\mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)&=\frac{1}{4\pi^2}q_\beta k_\nu\epsilon^{\nu\mu\alpha\beta}+\frac{1}{4\pi^2}a_\nu q_\beta\epsilon^{\nu\mu\alpha\beta}
\end{aligned}
$$

We have total freedom in choosing the vector 'a', but, nevertheless, this isn't enough to set all these quantities to zero. If we want to set the first one to zero, the only option is to choose '$a \propto k$', with this choice the second one can be made zero by only choosing '$a = k$', and this choice already gives the third one non-zero! The best we can do is to choose two of them to be zero, and hence, the preferred choice, as the vector current is Gauged in our theory, is to choose 'a' to be such that both '$q_\beta \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)$' and '$p_\alpha \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a)$', are zero. From the third condition we see that 'a' needs to have a part like '$-k$' plus any vector parallel to 'q', and from the second condition we see that the part parallel has to be '$2q$', hence, our unique choice to maintain the vectorial current with zero divergence is,

$$a = 2q - k$$

With this choice all contractions are,

$$\begin{aligned}
k_\mu \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a) &= \frac{1}{4\pi^2}(2q-k)_\nu k_\mu \epsilon^{\nu\mu\alpha\beta} \\
k_\mu \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a) &= \frac{1}{4\pi^2}(2q+p+q)_\nu(-p-q)_\mu \epsilon^{\nu\mu\alpha\beta} \\
k_\mu \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a) &= \frac{1}{4\pi^2}[-3q_\nu p_\mu - p_\nu q_\mu]\epsilon^{\nu\mu\alpha\beta} \\
k_\mu \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a) &= -\frac{1}{2\pi^2} q_\nu p_\mu \epsilon^{\nu\mu\alpha\beta} \\
p_\alpha \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a) &= 0 \\
q_\beta \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q;a) &= 0
\end{aligned}$$

As desired. The discrepancy between the said in the problem '$\frac{1}{4\pi^2}$', and the found here '$\frac{1}{2\pi^2}$' is to requiring or not that the vector current is indeed conserved, here we approached requiring it, but if we didn't, we could had chosen for 'a' just '$a = q$', and that would have recovered the result '$\frac{1}{4\pi^2}$'. This in fact prove that the axial current isn't conserved, due to,

$$\begin{aligned}
\partial_{x\mu} \mathrm{i}\Delta^{\mu\alpha\beta}(x,y,z) &= \partial_{x\mu} \langle \Omega | J^\mu_\mathrm{A}(x) J^\alpha_\mathrm{V}(y) J^\beta_\mathrm{V}(z) | \Omega \rangle \\
&= \partial_{x\mu} \int \mathrm{d}^4k \, \mathrm{d}^4p \, \mathrm{d}^4q \, \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)(2\pi)^4 \delta^{(4)}(k+p+q) \exp\left(\mathrm{i}k\cdot x + \mathrm{i}p\cdot y + \mathrm{i}q\cdot z\right) \\
&= \int \mathrm{d}^4k \, \mathrm{d}^4p \, \mathrm{d}^4q \, \mathrm{i}k_\mu \mathrm{i}\tilde{\Delta}^{\mu\alpha\beta}(k,p,q)(2\pi)^4 \delta^{(4)}(k+p+q) \exp\left(\mathrm{i}k\cdot x + \mathrm{i}p\cdot y + \mathrm{i}q\cdot z\right) \\
&= -\int \mathrm{d}^4k \, \mathrm{d}^4p \, \mathrm{d}^4q \, \mathrm{i}\frac{1}{2\pi^2} q_\nu p_\mu \epsilon^{\nu\mu\alpha\beta}(2\pi)^4 \delta^{(4)}(k+p+q) \exp\left(\mathrm{i}k\cdot x + \mathrm{i}p\cdot y + \mathrm{i}q\cdot z\right)
\end{aligned}$$

Hence, what we proved is,

$$\partial_{x\mu} \langle \Omega | J^\mu_\mathrm{A}(x) J^\alpha_\mathrm{V}(y) J^\beta_\mathrm{V}(z) | \Omega \rangle = -8\mathrm{i}\pi^2 \epsilon^{\nu\mu\alpha\beta} \int \mathrm{d}^4p \, \mathrm{d}^4q \, q_\nu p_\mu \exp\left(\mathrm{i}p\cdot(y-x) + \mathrm{i}q\cdot(z-x)\right)$$

In other words, the Ward Identity does not hold, and thus the current don't have zero divergence.

5 Sine-Gordon Theory

Our theory under consideration is the following Lagrangian in a 1+1 space-time,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{a}{b}(1 - \cos(b\Phi))$$

5.1 Exact Solutions

We're going to verify that the two expressions,

$$\Phi^\pm(x) = \frac{4}{b}\arctan\left[\exp\left\{\pm x \cdot v\sqrt{\frac{ab}{v^2}}\right\}\right]$$

Are in fact two independent solutions of the theory, with 'v' a 2-vector, an integration constant. To do this, we first state the equations of motion,

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \\ 0 &= -a\sin(b\Phi) + \partial_\mu \partial^\mu \Phi \end{aligned}$$

So the we can just compute the derivatives of the expressions,

$$\begin{aligned}
\partial_\mu \Phi^\pm &= \frac{4}{b}\frac{(\pm)v_\mu\sqrt{\frac{ab}{v^2}}}{1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}}\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\} \\
\partial^\mu\partial_\mu \Phi^\pm &= -\frac{4}{b}\frac{(\pm)v_\mu\sqrt{\frac{ab}{v^2}}(\pm)2v^\mu\sqrt{\frac{ab}{v^2}}}{\left[1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\right]^2}\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\} \\
&\quad + \frac{4}{b}\frac{(\pm)v_\mu\sqrt{\frac{ab}{v^2}}(\pm)v^\mu\sqrt{\frac{ab}{v^2}}}{1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}}\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\} \\
\partial^\mu\partial_\mu \Phi^\pm &= -8\frac{a}{\left[1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\right]^2}\exp\left\{\pm 3x\cdot v\sqrt{\frac{ab}{v^2}}\right\} \\
&\quad + 4\frac{a}{1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}}\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\} \\
\partial^\mu\partial_\mu \Phi^\pm &= \frac{4a\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\}}{\left[1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\right]^2}\left[-2\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}+1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\right] \\
\partial^\mu\partial_\mu \Phi^\pm &= \frac{4a\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\}}{\left[1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\right]^2}\left[1-\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\right] \qquad (5.1)
\end{aligned}$$

This seems a little bit hopeless, but, now let's take a look at,

$$-a\sin\left(b\Phi^{\pm}\right) = -a\sin\left(b\frac{4}{b}\arctan\left[\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\}\right]\right)$$

It's clear that this will be cumbersome, hence, we'll compactify the notation, making,

$$\theta = \exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\}$$

So,

$$-a\sin\left(b\Phi^{\pm}\right) = -a\sin\left(4\arctan\theta\right)$$

The objective here is to use trigonometric identities to make the argument inside any trigonometric function just 'arctan θ', and then rewrite all the trigonometric functions in terms of tangents, we start with simple double angle identities,

$$\begin{aligned}
-a\sin\left(b\Phi^{\pm}\right) &= -a\sin\left(4\arctan\theta\right)\\
&= -2a\sin\left(2\arctan\theta\right)\cos\left(2\arctan\theta\right)\\
&= -4a\sin\left(\arctan\theta\right)\cos\left(\arctan\theta\right)\left[2\cos^2\left(\arctan\theta\right)-1\right]
\end{aligned}$$

The first objective is already achieved, everything has as argument 'arctan θ', now, we have to use the relations,

$$\begin{cases}\tan^2 z+1 &= \frac{1}{\cos^2 z}\\ \frac{1}{\tan^2 z}+1 &= \frac{1}{\sin^2 z}\end{cases} \Rightarrow \begin{cases}\cos z &= \frac{1}{\sqrt{\tan^2 z+1}}\\ \sin z &= \frac{\tan z}{\sqrt{\tan^2 z+1}}\end{cases}$$

To get an equation with just tangents,

$$\begin{aligned}
-a\sin\left(b\Phi^{\pm}\right) &= -4a\sin\left(\arctan\theta\right)\cos\left(\arctan\theta\right)\left[2\cos^2\left(\arctan\theta\right)-1\right]\\
&= -4a\frac{\tan\left(\arctan\theta\right)}{\sqrt{\tan^2\left(\arctan\theta\right)+1}}\frac{1}{\sqrt{\tan^2\left(\arctan\theta\right)+1}}\left[\frac{2}{\tan^2\left(\arctan\theta\right)+1}-1\right]
\end{aligned}$$

And now we just use the trivial property '$\tan\left(\arctan\theta\right)=\theta$',

$$\begin{aligned}
-a\sin\left(b\Phi^{\pm}\right) &= -4a\frac{\tan\left(\arctan\theta\right)}{\sqrt{\tan^2\left(\arctan\theta\right)+1}}\frac{1}{\sqrt{\tan^2\left(\arctan\theta\right)+1}}\left[\frac{2}{\tan^2\left(\arctan\theta\right)+1}-1\right]\\
&= -4a\frac{\theta}{\sqrt{1+\theta^2}}\frac{1}{\sqrt{1+\theta^2}}\left[\frac{2}{1+\theta^2}-1\right]\\
&= -4a\frac{\theta}{\left[1+\theta^2\right]^2}\left[2-\theta^2-1\right]\\
&= -4a\frac{\theta}{\left[1+\theta^2\right]^2}\left[1-\theta^2\right]
\end{aligned}$$

If we switch back to, '$\theta = \exp\left\{\pm x \cdot v\sqrt{\frac{ab}{v^2}}\right\}$', we get,

$$
\begin{aligned}
-a\sin\left(b\Phi^{\pm}\right) &= -4a\frac{\theta}{\left[1+\theta^2\right]^2}\left[1-\theta^2\right] \\
-a\sin\left(b\Phi^{\pm}\right) &= -4a\frac{\exp\left\{\pm x \cdot v\sqrt{\frac{ab}{v^2}}\right\}}{\left[1+\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\}\right]^2}\left[1-\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\}\right] \qquad (5.2)
\end{aligned}
$$

If we take a look at the two expressions 5.1 and 5.2, we can notice that the two are identical, apart from a minus sign, this implies that,

$$
-a\sin\left(b\Phi^{\pm}\right) + \partial_\mu\partial^\mu\Phi^{\pm} = 0
$$

Is true for any 2-vector 'v', confirming the expression is indeed a exact solution of the theory. Actually, we have some subtleties, as if those are to be solutions, they better be real, what is equivalent to say,

$$
\frac{ab}{v^2} \geq 0
$$

Both 'a, b' are Real parameters, but, for the vacuum at '$\Phi = 0$' to be *stable*, it's needed '$\frac{a}{b} > 0$' — the case with '$\frac{a}{b} = 0$' is trivial —, what implies '$ab > 0$', and that we do need '$v^2 \geq 0$'. We're going to trow away under the carpet the singular case '$v^2 = 0$', and for now on just consider the '$v^2 > 0$' case. This allows us to do some simplifications, as, with '$v^2 > 0$' we know for sure that isn't possible to have '$v^1 = 0$', and, by the way 'v' appears in the solution, any rescaling of 'v' is irrelevant, thus we can without any loss of generality choose both '$v^2 = 1$', '$v^1 = 1$' and also '$v^0 = u$'. One could argue about the choice '$v^1 = -1$', and until now there is nothing we can say about to reject it, but, we'll say in advance that the sign of 'v^1' is the same sign of the energies of those modes, hence, to modes of negative energies not be present, we force '$v^1 = 1$' which, as we're going to show later, implies the energy is positive.

5.2 Conserved Current and Charge

These exact solutions are curious due to being topological, that is, they do not satisfy '$\Phi(x^0, x^1 = +\infty) = 0 = \Phi(x^0, x^1 = -\infty)$', neither the weaker condition, '$\Phi(x^0, x^1 = +\infty) = \Phi(x^0, x^1 = -\infty)$'. In fact we have,

$$
\begin{aligned}
\Phi^{\pm}\left(x^0, x^1\right) &= \frac{4}{b}\arctan\left[\exp\left\{\pm\left(-x^0u + x^1\right)\sqrt{\frac{ab}{v^2}}\right\}\right] \\
\Phi^{\pm}\left(x^0, x^1 = +\infty\right) &= \frac{4}{b}\arctan\left[\exp\left\{\pm\left(-x^0u + \infty\right)\sqrt{\frac{ab}{v^2}}\right\}\right] \\
\Phi^{\pm}\left(x^0, x^1 = +\infty\right) &= \frac{4}{b}\arctan\left[\exp\left\{\pm\infty\right\}\right] \\
\Phi^{\pm}\left(x^0, x^1 = +\infty\right) &= \frac{\pi}{b}(1 \pm 1)
\end{aligned}
$$

And also,

$$
\begin{aligned}
\Phi^{\pm}\big(x^0, x^1\big) &= \frac{4}{b}\arctan\left[\exp\left\{\pm\big(-x^0u + x^1\big)\sqrt{\frac{ab}{v^2}}\right\}\right] \\
\Phi^{\pm}\big(x^0, x^1 = -\infty\big) &= \frac{4}{b}\arctan\left[\exp\left\{\pm\big(-x^0u - \infty\big)\sqrt{\frac{ab}{v^2}}\right\}\right] \\
\Phi^{\pm}\big(x^0, x^1 = -\infty\big) &= \frac{4}{b}\arctan\left[\exp\left\{\mp\infty\right\}\right] \\
\Phi^{\pm}\big(x^0, x^1 = -\infty\big) &= \frac{\pi}{b}(1 \mp 1)
\end{aligned}
$$

Where clearly we have some non-zero values for,

$$
\begin{aligned}
\Phi^{\pm}\big(x^0, x^1 = +\infty\big) - \Phi^{\pm}\big(x^0, x^1 = -\infty\big) &= \frac{\pi}{b}(1 \pm 1) - \frac{\pi}{b}(1 \mp 1) \\
\Phi^{\pm}\big(x^0, x^1 = +\infty\big) - \Phi^{\pm}\big(x^0, x^1 = -\infty\big) &= \pm\frac{2\pi}{b}
\end{aligned}
$$

This result is highly suspicious, clearly these two solutions labeled by '$+, -$' are some kind of *eigenvalues* of some certain charge operator. We're going to be very straightforward and readily define an **winding number charge operator**, by,

$$
Q = \frac{b}{2\pi}\left[\Phi\big(x^0, x^1 = +\infty\big) - \Phi\big(x^0, x^1 = -\infty\big)\right]
$$

We would like to write this in a fully covariant manner, we know that this generator should come from a current, so we try to make something as a 2-vector apear,

$$
Q = \frac{b}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x^1\, \partial_1\Phi
$$

Well, '$\partial_1\Phi$' seems like a piece from a 2-vector, but, it is the spatial part, rather then the temporal part, fear not, because as we're in 1+1 dimensions we have a natural way of transforming spatial components into temporal ones, this is the Levi-Civita, which in 1+1 dimensions has 2 index, furthermore, the components are,

$$
\begin{aligned}
\epsilon_{00} &= \epsilon_{11} = 0 \\
\epsilon_{01} &= -\epsilon_{10} = 1
\end{aligned}
$$

There is! Our salvation!

$$
\begin{aligned}
Q &= \frac{b}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x^1\, \epsilon^{10}\partial_1\Phi \\
Q &= -\frac{b}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x^1\, \epsilon^{01}\partial_1\Phi \\
Q &= -\frac{b}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x^1\, \epsilon^{0\nu}\partial_\nu\Phi
\end{aligned}
$$

This is clearly a quantity that transforms as a temporal component of a 2-vector, which we'll define as our current,

$$J^\mu = -\frac{b}{2\pi}\epsilon^{\mu\nu}\partial_\nu\Phi$$

So that,

$$Q = \int_{-\infty}^{+\infty} \mathrm{d}x^1\, J^0$$

Now we can finally prove a claim we tried to omit, the time independence of this charge, it depends on the conservation of this current, which is trivially true,

$$\partial_\mu J^\mu = -\frac{b}{2\pi}\epsilon^{\mu\nu}\partial_\mu\partial_\nu\Phi = 0$$

Simply due to the anti-symmetry of the Levi-Civita. Hence,

$$\frac{\mathrm{d}Q}{\mathrm{d}x^0} = \int_{-\infty}^{+\infty} \mathrm{d}x^1\, \partial_0 J^0$$
$$\frac{\mathrm{d}Q}{\mathrm{d}x^0} = -\int_{-\infty}^{+\infty} \mathrm{d}x^1\, \partial_1 J^1$$
$$\frac{\mathrm{d}Q}{\mathrm{d}x^0} = -J^1\Big|_{x^1=-\infty}^{x^1=+\infty}$$
$$\frac{\mathrm{d}Q}{\mathrm{d}x^0} = \frac{b}{2\pi}\partial_1\Phi\Big|_{x^1=-\infty}^{x^1=+\infty}$$

This quantity **has** to be zero, because, if the derivative of the field isn't zero at infinity, then, the field blows up at infinity, what isn't an acceptable physical solution. Thus,

$$\frac{\mathrm{d}Q}{\mathrm{d}x^0} = 0$$

5.3 Energy of the Exact Solutions

To compute the energy, we need first to compute the Hamiltonian in terms of the fields, that's a simple Legendre transform, first, we define the conjugate momentum,

$$\Pi(x) = \frac{\partial\mathcal{L}(x)}{\partial\partial_0\Phi(x)}$$

And then define the Legendre transform,

$$
\begin{aligned}
\mathcal{H}(x) &= \Pi(x)\partial_0\Phi(x) - \mathcal{L}\\
\mathcal{H}(x) &= \Pi(x)\partial_0\Phi(x) + \frac{1}{2}\partial_\mu\Phi(x)\partial^\mu\Phi(x) + \frac{a}{b}(1-\cos{(b\Phi(x))})\\
\mathcal{H}(x) &= \Pi^2(x) - \frac{1}{2}[\partial_0\Phi(x)]^2 + \frac{1}{2}[\partial_1\Phi(x)]^2 + \frac{a}{b}(1-\cos{(b\Phi(x))})\\
\mathcal{H}(x) &= \frac{1}{2}[\Pi(x)]^2 + \frac{1}{2}[\partial_1\Phi(x)]^2 + \frac{a}{b}(1-\cos{(b\Phi(x))})
\end{aligned}
$$

This is the Hamiltonian density, to get the Hamiltonian we just integrate over the spatial coordinate,

$$
H = \int\limits_{-\infty}^{+\infty} \mathrm{d}x^1\,\mathcal{H}(x) = \int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \left[\frac{1}{2}\Pi^2(x) + \frac{1}{2}[\partial_1\Phi(x)]^2 + \frac{a}{b}(1-\cos{(b\Phi(x))})\right]
$$

We now use some of what we had already done,

$$
\partial_\mu\Phi^\pm = \frac{4}{b}\frac{(\pm)v_\mu\sqrt{\frac{ab}{v^2}}}{1+\exp\left\{\pm 2x\cdot v\sqrt{\frac{ab}{v^2}}\right\}}\exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\}
$$

From where we can compute already '$\partial_1\Phi$' and 'Π'. What we'll also need is an expression for the cosine, this we'll have to derive from scratch, following the same procedure we did for the sine. So, using the same 'θ' notation that we used before,

$$
\begin{aligned}
\cos\left(b\Phi^\pm\right) &= \cos{(4\arctan{(\theta)})}\\
\cos\left(b\Phi^\pm\right) &= 2\cos^2{(2\arctan{(\theta)})} - 1\\
\cos\left(b\Phi^\pm\right) &= 2\big[2\cos^2{(\arctan{(\theta)})} - 1\big]^2 - 1\\
\cos\left(b\Phi^\pm\right) &= 2\left[2\frac{1}{\tan^2{(\arctan{(\theta)})}+1} - 1\right]^2 - 1\\
\cos\left(b\Phi^\pm\right) &= 2\left[2\frac{1}{\theta^2+1} - 1\right]^2 - 1
\end{aligned}
$$

Just for remembering,

$$
\theta = \exp\left\{\pm x\cdot v\sqrt{\frac{ab}{v^2}}\right\}
$$

And we write also the derivatives in terms of 'θ',

$$
\partial_\mu\Phi^\pm = \frac{4}{b}\frac{(\pm)v_\mu\sqrt{\frac{ab}{v^2}}}{\theta^2+1}\theta
$$

So that,

$$\left[\partial_\mu \Phi^\pm\right]^2 = \frac{16a}{bv^2}\frac{v_\mu^2}{\left(\theta^2+1\right)^2}\theta^2 \tag{5.3}$$

Going to our phase space variables,

$$\left[\Pi^\pm\right]^2 = \frac{16a}{bv^2}\frac{v_0^2}{\left(\theta^2+1\right)^2}\theta^2, \qquad \left[\partial_1 \Phi^\pm\right]^2 = \frac{16a}{bv^2}\frac{v_1^2}{\left(\theta^2+1\right)^2}\theta^2$$

Now we just put together everything we already got,

$$\begin{aligned}
H^\pm &= \int_{-\infty}^{+\infty} \mathrm{d}x^1 \left[\frac{1}{2}\left[\Pi^\pm(x)\right]^2 + \frac{1}{2}\left[\partial_1 \Phi^\pm(x)\right]^2 + \frac{a}{b}\left(1-\cos\left(b\Phi^\pm(x)\right)\right)\right] \\
H^\pm &= \int_{-\infty}^{+\infty} \mathrm{d}x^1 \left[\frac{8a}{bv^2}\frac{v_0^2}{\left(\theta^2+1\right)^2}\theta^2 + \frac{8a}{bv^2}\frac{v_1^2}{\left(\theta^2+1\right)^2}\theta^2 + \frac{a}{b}\left(1-2\left[2\frac{1}{\theta^2+1}-1\right]^2+1\right)\right] \\
H^\pm &= \int_{-\infty}^{+\infty} \mathrm{d}x^1 \left[\frac{8a}{bv^2}\frac{v_0^2}{\left(\theta^2+1\right)^2}\theta^2 + \frac{8a}{bv^2}\frac{v_1^2}{\left(\theta^2+1\right)^2}\theta^2 + \frac{a}{b}\left(-\frac{8}{\left(\theta^2+1\right)^2}+\frac{8}{\theta^2+1}\right)\right] \\
H^\pm &= 8\frac{a}{b}\int_{-\infty}^{+\infty} \mathrm{d}x^1 \left[\frac{1}{v^2}\frac{v_0^2}{\left(\theta^2+1\right)^2}\theta^2 + \frac{1}{v^2}\frac{v_1^2}{\left(\theta^2+1\right)^2}\theta^2 + \frac{\theta^2}{\left(\theta^2+1\right)^2}\right] \\
H^\pm &= 8\frac{a}{b}\left[\frac{v_0^2+v_1^2}{v^2}+1\right]\int_{-\infty}^{+\infty} \mathrm{d}x^1\,\frac{\theta^2}{\left(\theta^2+1\right)^2} \\
H^\pm &= 8\frac{a}{b}\frac{v_0^2+v_1^2-v_0^2+v_1^2}{v^2}\int_{-\infty}^{+\infty} \mathrm{d}x^1\,\frac{\theta^2}{\left(\theta^2+1\right)^2} \\
H^\pm &= 16\frac{a}{b}\frac{v_1^2}{v^2}\int_{-\infty}^{+\infty} \mathrm{d}x^1\,\frac{\theta^2}{\left(\theta^2+1\right)^2}
\end{aligned}$$

So now what remains to be done is the integral,

$$H^{\pm} = 16\frac{a}{b}\frac{v_1^2}{v^2}\int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \, \frac{\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\}}{\left(\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\} + 1\right)^2}$$

$$H^{\pm} = 16\frac{a}{b}\frac{v_1^2}{v^2}(\pm)\frac{1}{2v^1}\sqrt{\frac{v^2}{ab}}\int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \, \frac{\pm 2v^1\sqrt{\frac{ab}{v^2}}\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\}}{\left(\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\} + 1\right)^2}$$

$$H^{\pm} = 16\frac{a}{b}\frac{v_1^2}{v^2}(\pm)\frac{1}{2v^1}\sqrt{\frac{v^2}{ab}}\int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \, \frac{1}{\left(\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\} + 1\right)^2}\partial_1 \exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\}$$

$$H^{\pm} = \mp 8\sqrt{\frac{a}{b^3}}\frac{v_1}{\sqrt{v^2}}\int\limits_{-\infty}^{+\infty} \mathrm{d}x^1 \, \partial_1 \frac{1}{\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\} + 1}$$

$$H^{\pm} = \mp 8\sqrt{\frac{a}{b^3}}\frac{v_1}{\sqrt{v^2}}\frac{1}{\exp\left\{\pm 2x \cdot v\sqrt{\frac{ab}{v^2}}\right\} + 1}\Bigg|_{-\infty}^{+\infty}$$

$$H^{\pm} = \mp 8\sqrt{\frac{a}{b^3}}\frac{v_1}{\sqrt{v^2}}\left[\frac{1}{\exp\left\{\pm 2(+\infty v^1 - x^0 v^0)\sqrt{\frac{ab}{v^2}}\right\} + 1} - \frac{1}{\exp\left\{\pm 2(-\infty v^1 - x^0 v^0) \cdot v\sqrt{\frac{ab}{v^2}}\right\} + 1}\right]$$

$$H^{\pm} = \mp 8\sqrt{\frac{a}{b^3}}\frac{v_1}{\sqrt{v^2}}\left[\frac{1}{\exp\{\pm\infty\} + 1} - \frac{1}{\exp\{\mp\infty\} + 1}\right]$$

$$H^{\pm} = \mp 8\sqrt{\frac{a}{b^3}}\frac{v_1}{\sqrt{v^2}}(\mp)$$

$$H^{\pm} = 8\sqrt{\frac{a}{b^3}}\frac{v_1}{\sqrt{v^2}}$$

Now we invoke the aforementioned rescaling property of 'v', which does simplify to,

$$H^{\pm} = 8\sqrt{\frac{a}{b^3}}$$

Both modes having the same energy, and also being independent of the time variable.

5.4 Expanding the Lagrangian

We're going to use the following expansion,

$$\cos\left(b\Phi(x)\right) = \sum_{n=0}^{\infty} (-1)^n \frac{b^{2n}\Phi^{2n}}{(2n)!}$$

Thus the expansion of the Lagrangian is,

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{a}{b}(1-\cos{(b\Phi)})\\
\mathcal{L} &= -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{a}{b}\left(1-1+\frac{b^2}{2}\Phi^2 - \frac{b^4}{4!}\Phi^4 + \mathcal{O}\big(b^6\big)\right)\\
\mathcal{L} &= -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{ab}{2}\Phi^2 + \frac{ab^3}{4!}\Phi^4 + \mathcal{O}\big(b^6\big)
\end{aligned}
$$

From where we can easily read the mass and quartic coupling from the quadratic and quartic factors,

$$
\begin{aligned}
m^2 &= ab\\
\lambda &= ab^3
\end{aligned}
$$

If we would like, we can invert these expressions to give,

$$
\begin{aligned}
a &= \frac{m^3}{\sqrt{\lambda}}\\
b &= \frac{\sqrt{\lambda}}{m}
\end{aligned}
$$

This is interesting, because we can write the energy of the exact solutions in terms of these parameters,

$$
\begin{aligned}
H &= 8\sqrt{\frac{m^3m^3}{\sqrt{\lambda}\sqrt{\lambda^3}}}\\
H &= 8\frac{m^3}{\lambda}
\end{aligned}
$$

This seems very wrong, at least from a dimensional analysis point of view, but isn't, this is because we usually take 'λ' to be adimensional, but, in this case it isn't. To see that we have to do the analysis of the dimensions,

$$
\begin{aligned}
\big[\mathrm{d}^2x\,\partial_\mu\Phi\partial^\mu\Phi\big] &= 0\\
-2+2+2[\Phi] &= 0 \Rightarrow [\Phi] = 0
\end{aligned}
$$

Wow! This is rather strange, to have a dimensionless field, but with this,

$$
\begin{aligned}
\big[\Phi^2 ab\big] &= 2 \Rightarrow [ab] = 0\\
\big[\Phi^4 ab^3\big] &= 2 \Rightarrow \big[ab^3\big] = 0
\end{aligned}
$$

The only possible solution to this system of equations is,

$$
[a] = 2, \quad [b] = 0
$$

So, we actually have,

$$
\big[m^2\big] = [\lambda]
$$

And in fact our Energy found has indeed the right dimensions!

A Conventions

On all this homework we used the mostly plus metric, '$g_{\mu\nu} = \mathrm{diag}\begin{pmatrix}-1 & 1 & 1 & 1\end{pmatrix}$' together with the Gamma matrices defined by,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$$

This implies that the kinetic term for a Dirac fermion is,

$$-\bar{\Psi}\not{\partial}\Psi$$

Also our definition for the fifth Gamma matrix is,

$$\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$$

Also, our definition for the Levi-Civita is,

$$\epsilon_{0123} = -\epsilon^{0123} = +1$$

B Gamma Matrix Technology

We start with the trace properties, we're going to assume here that the space-time is 'D' dimensional, except when there is a 'γ_5'. There is a small catch where, as what is the dimension of the matrices for an arbitrary space-time dimensionality, the answer is '$2^{\lfloor \frac{D}{2} \rfloor}$', but as we're always interested in the dimensional regularization, this only amounts for an additional finite term in the counter terms, so, we're just assuming the matrices are '4' dimensional,

- $\operatorname{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu}$

  This is simple,

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}\mathbb{1} \\
\operatorname{Tr}[\{\gamma^\mu, \gamma^\nu\}] &= 2g^{\mu\nu} \operatorname{Tr}[\mathbb{1}] \\
2\operatorname{Tr}[\gamma^\mu \gamma^\nu] &= 8g^{\mu\nu} \\
\operatorname{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu}
\end{aligned}$$

- $\operatorname{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta\right] = 4g^{\mu\beta}g^{\nu\alpha} - 4g^{\nu\beta}g^{\alpha\mu} + 4g^{\mu\nu}g^{\alpha\beta}$

$$\begin{aligned}
\operatorname{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta\right] &= \operatorname{Tr}\left[\gamma^\mu \gamma^\nu \left(-\gamma^\beta \gamma^\alpha + 2g^{\alpha\beta}\right)\right] \\
&= -\operatorname{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\beta \gamma^\alpha\right] + 8g^{\mu\nu}g^{\alpha\beta} \\
&= -\operatorname{Tr}\left[\gamma^\mu \left(-\gamma^\beta \gamma^\nu + 2g^{\nu\beta}\right)\gamma^\alpha\right] + 8g^{\mu\nu}g^{\alpha\beta} \\
&= \operatorname{Tr}\left[\gamma^\mu \gamma^\beta \gamma^\nu \gamma^\alpha\right] - 8g^{\nu\beta}g^{\alpha\mu} + 8g^{\mu\nu}g^{\alpha\beta} \\
&= \operatorname{Tr}\left[\left(-\gamma^\beta \gamma^\mu + 2g^{\mu\beta}\right)\gamma^\nu \gamma^\alpha\right] - 8g^{\nu\beta}g^{\alpha\mu} + 8g^{\mu\nu}g^{\alpha\beta} \\
&= -\operatorname{Tr}\left[\gamma^\beta \gamma^\mu \gamma^\nu \gamma^\alpha\right] + 8g^{\mu\beta}g^{\nu\alpha} - 8g^{\nu\beta}g^{\alpha\mu} + 8g^{\mu\nu}g^{\alpha\beta} \\
2\operatorname{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta\right] &= 8g^{\mu\beta}g^{\nu\alpha} - 8g^{\nu\beta}g^{\alpha\mu} + 8g^{\mu\nu}g^{\alpha\beta} \\
\operatorname{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta\right] &= 4g^{\mu\beta}g^{\nu\alpha} - 4g^{\nu\beta}g^{\alpha\mu} + 4g^{\mu\nu}g^{\alpha\beta}
\end{aligned}$$

- $\operatorname{Tr}[\gamma_5] = 0$

$$\begin{aligned}
\operatorname{Tr}[\gamma_5] &= \mathrm{i}\operatorname{Tr}\left[\gamma^0 \gamma^1 \gamma^2 \gamma^3\right] \\
&= \mathrm{i}4\left(g^{03}g^{12} - g^{13}g^{20} + g^{01}g^{23}\right) = 0
\end{aligned}$$

  Now we'll need another result,

- $\{\gamma_5, \gamma^\mu\} = 0$

  This has to be proved case by case,

$$\begin{aligned}
\left\{\gamma_5, \gamma^0\right\} &= \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + \mathrm{i}\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 - \gamma^1\gamma^2\gamma^3 \\
&= \mathrm{i}\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3 \\
&= -\mathrm{i}\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3 \\
&= \mathrm{i}\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3 \\
&= 0
\end{aligned}$$

  The proof for the other matrix are analogous,

- $\gamma_5\gamma_5 = \mathbb{1}$

$$\begin{aligned}
\gamma_5\gamma_5 &= \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= \gamma^0\gamma^1\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 \\
&= \gamma^0\gamma^1\gamma^0\gamma^1\gamma^2\gamma^2 \\
&= -\gamma^0\gamma^0\gamma^1\gamma^1 = \mathbb{1}
\end{aligned}$$

- $\mathrm{Tr}\left[\text{odd number of } \gamma\right] = 0$

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\right] &= \mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma_5\gamma_5\right] \\
&= (-1)^{2n+1}\,\mathrm{Tr}\left[\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma_5\right] \\
&= -\,\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\right] \\
\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\right] &= 0
\end{aligned}$$

- $\mathrm{Tr}\left[\gamma_5 \text{ odd number of } \gamma\right] = 0$

$$\begin{aligned}
\mathrm{Tr}\left[\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\right] &= \mathrm{Tr}\left[\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma_5\gamma_5\right] \\
&= (-1)^{2n+1}\,\mathrm{Tr}\left[\gamma_5\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma_5\right] \\
&= -\,\mathrm{Tr}\left[\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\right] \\
\mathrm{Tr}\left[\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\right] &= 0
\end{aligned}$$

- $\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\right] = 0$

Take '$\alpha \neq \mu$' and '$\alpha \neq \nu$', and the following is not assuming the summation convention,

$$\begin{aligned}
\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\right] &= \mathrm{Tr}\left[\gamma_5 g_{\alpha\alpha}\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\right] \\
&= -\,\mathrm{Tr}\left[\gamma_5 g_{\alpha\alpha}\gamma^\alpha\gamma^\mu\gamma^\alpha\gamma^\nu\right] \\
&= \mathrm{Tr}\left[\gamma_5 g_{\alpha\alpha}\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\alpha\right] \\
&= -\,\mathrm{Tr}\left[\gamma_5\gamma^\alpha g_{\alpha\alpha}\gamma^\alpha\gamma^\mu\gamma^\nu\right] \\
\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\right] &= 0
\end{aligned}$$

- $\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] = 4\mathrm{i}\epsilon^{\mu\nu\alpha\beta}$

$$\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] \tag{B.1}$$

This is a little bit more complicated, but we can use some tricks, we see that if we have two or more indices equal it gives zero, due to the last property,

$$\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\mu\gamma^\alpha\gamma^\beta\right] = g^{\mu\mu}\,\mathrm{Tr}\left[\gamma_5\gamma^\alpha\gamma^\beta\right] = 0$$

Hence it can only be non-zero when all indices are different from each other. in this case, as the Gamma matrices anticommute with each other, that is, this quantity is totally antisymmetric in all indices, thus, it must be proportional to '$\epsilon^{\mu\nu\alpha\beta}$' as this is the only fully antisymmetric quantity in '4' dimensions,

$$\begin{aligned}
\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\mu\gamma^\alpha\gamma^\beta\right] &\propto \epsilon^{\mu\nu\alpha\beta} \\
\mathrm{Tr}\left[\gamma_5\gamma^0\gamma^1\gamma^2\gamma^3\right] &\propto \epsilon^{0123} \\
\mathrm{i}\,\mathrm{Tr}\left[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\right] &\propto -1 \\
-\mathrm{i}\,\mathrm{Tr}\left[\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3\right] &\propto -1 \\
\mathrm{i}\,\mathrm{Tr}\left[\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3\right] &\propto -1 \\
-\mathrm{i}\,\mathrm{Tr}\left[\gamma^2\gamma^2\gamma^3\gamma^3\right] &\propto -1 \\
-4\mathrm{i} &\propto -1
\end{aligned} \tag{B.2}$$

Hence,

$$\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\mu\gamma^\alpha\gamma^\beta\right] = 4\mathrm{i}\epsilon^{\mu\nu\alpha\beta}$$

Now we got for some other equalities,

- $(\gamma^\mu)^2 = g^{\mu\mu}$

$$\begin{aligned}
\{\gamma^\mu, \gamma^\mu\} &= 2g^{\mu\mu} \\
2\gamma^\mu = 2g^{\mu\mu} & \\
(\gamma^\mu)^2 &= g^{\mu\mu}
\end{aligned}$$

- $\gamma_\mu\gamma^\mu = D$

$$\begin{aligned}
g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} &= 2g_{\mu\nu}g^{\mu\nu} \\
2\gamma_\mu\gamma^\mu &= 2D \\
\gamma_\mu\gamma^\mu &= D
\end{aligned}$$

- $\gamma_\mu\gamma^\alpha\gamma^\mu = (2-D)\gamma^\alpha$

$$\begin{aligned}
\gamma_\mu\gamma^\alpha\gamma^\mu &= \gamma_\mu(-\gamma^\mu\gamma^\alpha + 2g^{\mu\alpha}) \\
&= -\gamma_\mu\gamma^\mu\gamma^\alpha + 2\gamma^\alpha \\
&= (2-D)\gamma^\alpha
\end{aligned} \tag{B.3}$$

- $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = (D-2)\gamma^\alpha\gamma^\beta + 2\gamma^\beta\gamma^\alpha$

$$
\begin{aligned}
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= \gamma_\mu\gamma^\alpha\left(-\gamma^\mu\gamma^\beta + 2g^{\beta\mu}\right) \\
&= -\gamma_\mu\gamma^\alpha\alpha\gamma^\mu\gamma^\beta + \gamma^\beta\gamma^\alpha \\
&= (D-2)\gamma^\alpha\gamma^\beta + 2\gamma^\beta\gamma^\alpha
\end{aligned}
$$

- $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\mu = +2\gamma_\delta\gamma^\alpha\gamma^\beta - 2\gamma^\beta\gamma^\alpha\gamma^\delta + (2-D)\gamma^\alpha\gamma^\beta\gamma^\delta$

$$
\begin{aligned}
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\mu &= \gamma_\mu\gamma^\alpha\gamma^\beta\left(-\gamma^\mu\gamma^\delta + 2g^{\mu\delta}\right) \\
&= -\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\delta + 2\gamma_\delta\gamma^\alpha\gamma^\beta \\
&= -(D-2)\gamma^\alpha\gamma^\beta\gamma^\delta - 2\gamma^\beta\gamma^\alpha\gamma^\delta + 2\gamma_\delta\gamma^\alpha\gamma^\beta
\end{aligned}
$$

C Useful Integrals

The mostly useful integrals we'll need to know how to solve are,

- $\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{(k^2)^a}{(k^2+A)^b} = \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} A^{-b+\frac{D}{2}+a} \frac{\Gamma\left(b-a-\frac{D}{2}\right)\Gamma\left(a+\frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)}$

  Let's prove this,
  We first start with a Wick rotation in 'k^0',

$$\begin{aligned}\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{(k^2)^a}{(k^2+A)^b} &= \mathrm{i} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{(k^2)^a}{(k^2+A)^b} \\ &= \frac{\mathrm{i}}{(2\pi)^D} \int \mathrm{d}^{D-1}\Omega \int_0^\infty \mathrm{d}k \, k^{D-1} \frac{(k^2)^a}{(k^2+A)^b}\end{aligned}$$

  For getting what is the angular integral,

$$\begin{aligned}\int \mathrm{d}^D k \exp\left(-tk^2\right) &= \left[\int_{-\infty}^{+\infty} \mathrm{d}k \exp\left(-tk^2\right)\right]^D \\ \int \mathrm{d}^{D-1}\Omega \int_0^\infty \mathrm{d}k \, k^{D-1} \exp\left(-tk^2\right) &= \left(\frac{\pi}{t}\right)^{-\frac{D}{2}} \\ \int \mathrm{d}^{D-1}\Omega \, t^{-\frac{D}{2}} \int_0^\infty \mathrm{d}x \, x^{D-1} \exp\left(-x^2\right) &= \left(\frac{\pi}{t}\right)^{-\frac{D}{2}} \\ \int \mathrm{d}^{D-1}\Omega \, t^{-\frac{D}{2}} \frac{1}{2}\Gamma\left(\frac{D}{2}\right) &= \left(\frac{\pi}{t}\right)^{-\frac{D}{2}} \\ \int \mathrm{d}^{D-1}\Omega &= \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}\end{aligned}$$

  So we continue as,

$$
\begin{aligned}
\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{(k^2)^a}{(k^2+A)^b} &= \frac{\mathrm{i}}{(2\pi)^D} \int \mathrm{d}^{D-1}\Omega \int\limits_0^\infty \mathrm{d}k\, k^{D-1} \frac{(k^2)^a}{(k^2+A)^b} \\
&= \frac{\mathrm{i}}{(2\pi)^D} \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \frac{1}{2} \int\limits_0^\infty \mathrm{d}k^2\, k^{D-2} \frac{(k^2)^a}{(k^2+A)^b} \\
&= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} \int\limits_0^\infty \mathrm{d}k^2 \left(k^2\right)^{\frac{D-2}{2}} \frac{(k^2)^a}{(k^2+A)^b} \\
&= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} A^{1-b+\frac{D}{2}-1+a} \int\limits_0^\infty \mathrm{d}x\, (x)^{\frac{D-2}{2}} \frac{(x)^a}{(x+1)^b} \\
&= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} A^{-b+\frac{D}{2}+a} \int\limits_0^\infty \mathrm{d}x\, (x)^{\frac{D-2}{2}+a} (x+1)^{-b} \\
&= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} A^{-b+\frac{D}{2}+a} \int\limits_1^0 \mathrm{d}\left(\frac{1}{z}-1\right) \left(\frac{1}{z}-1\right)^{\frac{D-2}{2}+a} \left(\frac{1}{z}-1+1\right)^{-b} \\
&= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} A^{-b+\frac{D}{2}+a} \int\limits_0^1 \mathrm{d}z\, (1-z)^{\frac{D}{2}+a-1} z^{b-a-\frac{D}{2}-1}
\end{aligned}
$$

This can be written as the Beta function,

$$
\begin{aligned}
\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{(k^2)^a}{(k^2+A)^b} &= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} A^{-b+\frac{D}{2}+a} \int\limits_0^1 \mathrm{d}z\, (1-z)^{\frac{D}{2}+a-1} z^{b-a-\frac{D}{2}-1} \\
&= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} A^{-b+\frac{D}{2}+a} \frac{\Gamma\left(b-a-\frac{D}{2}\right)\Gamma\left(a+\frac{D}{2}\right)}{\Gamma(b)}
\end{aligned}
$$

Some other useful properties are,

- $\int \mathrm{d}^D k\, k_\alpha f(k^2) = 0$

This is only due to the integral is an odd function integrated over an even domain contain zero.

- $\int \mathrm{d}^D k\, k_\alpha k_\beta f(k^2) = \frac{1}{D} g_{\alpha\beta} \int \mathrm{d}^D k\, k^2 f(k^2)$

This a matter of Lorentz covariance, if '$\alpha \neq \beta$' the integral is zero by the same reasoning of the last property. If '$\alpha = \beta$', we can just make use of rotational/boost covariance to align the momentum in the direction of 'α', and thus this integral is equal to the one with 'k^2', but as latter has 'D' components we need to divide by 'D'.

- $\int \mathrm{d}^D k\, k_\alpha k_\beta k_\mu f(k^2) = 0$

This is the same reasoning of the integral before.

- $\int \mathrm{d}^D k\, k_\alpha k_\beta k_\mu k_\nu f(k^2) = \frac{1}{D}\frac{1}{3!}(g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\mu\beta}) \int \mathrm{d}^D k\, (k^2)^2 f(k^2)$

This is the same reasoning as before, we just have an additional '3!' due to the possible contractions.

Now we come to a trick used in the problems,

- $\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^2}{(k^2+A)^2} = \frac{D}{2-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{A}{(k^2+A)^2}$

$$
\begin{aligned}
\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^2}{(k^2+A)^2} &= \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} A^{-2+\frac{D}{2}+1} \frac{\Gamma\left(2-1-\frac{D}{2}\right)\Gamma\left(1+\frac{D}{2}\right)}{\Gamma(2)\Gamma\left(\frac{D}{2}\right)} \\
&= \frac{A}{1-\frac{D}{2}} \frac{D}{2} \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}} A^{-2+\frac{D}{2}} \frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma(2)\Gamma\left(\frac{D}{2}\right)} \\
&= \frac{AD}{2-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k^2+A)^2} \\
\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^2}{(k^2+A)^2} &= \frac{D}{2-D} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{A}{(k^2+A)^2}
\end{aligned}
$$