Homework III

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A BRST

A.1 Faddeev-Popov Gauge Fixing

Our Action functional is,

$$S_X + \lambda\chi = \frac{1}{4\pi\alpha'}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int\limits_M \mathrm{d}^2\sigma\,\sqrt{h}R + \frac{\lambda}{2\pi}\int\limits_{\partial M} \mathrm{d}s\,K \tag{A.1}$$

we would like to define the quantum theory by means of the path integral, that is, we expect that,

$$Z \stackrel{?}{=} \int \mathcal{D}X\mathcal{D}h \exp\left(-S_X[X,h] - \lambda\chi\right) \tag{A.2}$$

should give a well defined theory, but, already from A.2 there're several problems that arise, one of them is: *What should be interpreted from the path integral itself? We haven't defined any manifold to our metric h and scalar fields X to live in, also, even if we had defined such, the path integral relies on explicit coordinate points, $\mathcal{D}h = \prod\limits_\sigma \mathrm{d}h_{ab}(\sigma)$, which are highly dependent on charts.*

This is a valid claim, our way to avoid it is to *define* $\mathcal{D}h$ to mean: *Sum over all **allowed** two dimensional Riemannian manifolds, and all possible metric structures in these.* Here, ***allowed*** requires a prescription, which manifolds are or aren't allowed impacts the obtained string theory. Happily, every two dimensional manifold has a definite value for the Euler Characteristic χ, hence, we can sort them out by it,

$$\begin{aligned} Z &\stackrel{?}{=} \sum_{\{M\}} \int\limits_{\mathrm{Met}(M)} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h] - \lambda\chi\right) \\ Z &\stackrel{?}{=} \sum_{\{\chi\}} \exp\left(-\lambda\chi\right) \sum_{\{M_\chi\}} \int\limits_{\mathrm{Met}(M_\chi)} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h]\right) \end{aligned} \tag{A.3}$$

Where M is to be understood as a two dimensional Riemannian manifold and M_χ is one with Euler Characteristic χ, $\mathrm{Met}(M_\chi)$ is the space of all metrics which can be assigned to M_χ, we have written $\sum\limits_{\{M_\chi\}}$ in the special case of there being more than one manifold with same Euler Characteristic[1], also, the functional integral over X should be read as integrating over all maps from M_χ to $\mathbb{R}^{1,D-1}$. While this is better defined than before, i.e. not coordinate dependent, we still have a few problems, first, it's know that A.1 has a Gauge Group of $\mathrm{Diff}(M)\times\mathrm{Weyl}(M)$, but, in our second try of a definition of the path integral, we're integrating the metrics over $\mathrm{Met}(M_\chi)$, it's clear that may happen of two elements of $\mathrm{Met}(M_\chi)$ be equivalent under a $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ transformation, to put in more clear terms, we're worried if exists $h', h \in \mathrm{Met}(M_\chi)$ such,

$$h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

the existence of those kinds of elements is troublesome, as $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ is a infinite dimensional group of redundancies, this means we're over-counting physical configurations by a infinite amount. The solution is to look for an equivalence class of metrics under this Gauge Group action,

$$\mathscr{M}_\chi = \mathrm{Met}(M_\chi)/\mathrm{Diff}(M_\chi) \times \mathrm{Weyl}(M_\chi)$$

[1]As we're interested only in Differentiable Manifolds, more than manifold should read: More than one equivalence class of Differentiable Manifolds.

the equivalence class is to be understood as[2],

$$h' \sim h \Leftrightarrow h'_{ab}(\sigma'(\sigma)) = \exp\left(2\omega(\sigma)\right)\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)$$

that is, two metrics are the same representative of an element of $\mathscr{M}_\chi$ iff they differ by a composition of a Diffeomorphism and Weyl transformation. We'll denote a given composition of a Diffeomorphism followed by a Weyl transformation by ζ,

$$h' = \zeta \circ h$$

Notice that the set of equivalence class of metrics, or, the set of inequivalent $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ metrics $\mathscr{M}_\chi$ is highly dependent on the topology of M_χ, for example, for $M_\chi \cong \mathbb{R}^2 \cong \mathbb{C}$, it's trivial, there is just one point in the set $\mathscr{M}_\chi$, in other words, every metric is equivalent, which isn't true for more complex topologies.

Thus, it's possible for us to set up a well defined version of the path integral, just replace $\mathrm{Met}(M_\chi)$ by $\mathscr{M}_\chi$[3],

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda g\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathcal{D}h \int \mathcal{D}X \exp\left(-S_X[X,h]\right) \tag{A.4}$$

where the integration is to be understood as by choosing for each equivalence class in $\mathscr{M}_\chi$ a representative element in $\mathrm{Met}(M_g)$. While this is a satisfactory definition for the path integral, it feels a little clunky, we rather have an path integral over all the possible metrics — in the sense defined before —, well, this is achievable. First, for each equivalence class of $\mathscr{M}_\chi$ elect one representative element of $\mathrm{Met}(M_\chi)$, we'll denote these elements as $\hat{h}(\mathbf{t})$ — here $\mathbf{t}$ is a parametrization of the correspondent equivalence class in $\mathscr{M}_\chi$, we haven't proved here, and won't, but $\mathscr{M}_\chi$ is a finite N dimensional manifold, hence, $\mathbf{t}$ is a N-tuple of real numbers —, by construction, these representatives are inequivalent under $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, hence,

$$\zeta_1 \circ \hat{h}(\mathbf{t}_1) = \zeta_2 \circ \hat{h}(\mathbf{t}_2) \Leftrightarrow \mathbf{t}_1 = \mathbf{t}_2 \text{ and } \zeta_1 = \zeta_2$$

so that every element in $\mathrm{Met}(M_g)$ can be written as a unique[4] composition of a given ζ into a given $\hat{h}(\mathbf{t})$. Now, we rewrite the pictorial integral over $\mathscr{M}_\chi$ is a more formal way, using the parametrization we just described,

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda g\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int \mathcal{D}X \exp\left(-S_X\left[X,\hat{h}(\mathbf{t})\right]\right)$$
$$Z = \sum_{\{\chi\}} \exp\left(-\lambda g\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)} \mathcal{D}\zeta\,\delta(\zeta) \int \mathcal{D}X \exp\left(-S_X\left[X,\hat{h}(\mathbf{t})\right]\right)$$

in the last line we introduced a one by integrating over the delta functional, as this integral picks only $\zeta = 0$, what should be understood as $\zeta = \mathrm{id}$ in the group, we can deform a little the integration to,

$$Z = \sum_{\{\chi\}} \exp\left(-\lambda g\right) \sum_{\{M_\chi\}} \int_{\mathscr{M}_\chi} \mathrm{d}^N\mathbf{t} \int_{\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)} \mathcal{D}\zeta\,\delta(\zeta) \int \mathcal{D}X \exp\left(-S_X\left[X,\zeta\circ\hat{h}(\mathbf{t})\right]\right) \tag{A.5}$$

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[2]In all charts.

[3]By making this procedure, we eliminate the redundancies of $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$ with except of a measure zero subset of transformations, known as *conformal killing group* — CKG —, so we're still over-counting the physical contributions, but, this time by a finite number, this doesn't spoil the well-definiteness of the path integral, but do spoil the normalization. There is ways of correcting this, but we'll no dwell upon.

[4]Apart from the measure zero CKG.

This is almost in the form that we would like, notice that we're integrating over the set of representative of the inequivalent metrics, $\mathrm{d}^N\mathbf{t}$, and also over the whole group $\mathrm{Diff}(M_\chi)\times\mathrm{Weyl}(M_\chi)$, $\mathcal{D}\zeta$, by construction, **every** metric in $\mathrm{Met}(M_\chi)$ can be written uniquely[5] as,

$$h = \zeta_{\mathbf{t}} \circ \hat{h}(\mathbf{t})$$

in other words, to integrate over $\mathrm{d}^N\mathbf{t}\,\mathcal{D}\zeta$ is to integrate over all metrics of the form $\zeta \circ \hat{h}(\mathbf{t})$, which is to integrate over all metrics $h = \zeta \circ \hat{h}(\mathbf{t})$ in $\mathrm{Met}(M_\chi)$! We cannot yet make this change, due to the presence of an explicit dependence in ζ at the functional delta. We'll eliminate it by means of a change of variable of the functional delta, notice that,

$$\delta\Big(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\Big)$$

picks up just the contribution of $\zeta = 0$, so it's a good candidate for a change of variables,

$$\delta\Big(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\Big) = \delta(\zeta)\left|\mathrm{Det}\left[\frac{\delta}{\delta\zeta}\Big(\hat{h}(\mathbf{t}) - \zeta \circ \hat{h}(\mathbf{t})\Big)\bigg|_{\zeta=0}\right]\right|^{-1}$$

in this way is possible to separate the integral over all metrics $\int \mathcal{D}h$ into an integration over all inequivalent metrics $\int \mathcal{D}\hat{h}$ and an integration over all possible Diff$\times$Weyl transformations $\int \mathcal{D}\zeta$, so that the partition function can be rewrote as,

$$Z \overset{?}{=} \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \exp\left(-S_X[X, \zeta \circ h]\right)$$

this still has the same problem of before, we're over-integrating the physical configurations, that is, $\hat{h}$ are the physical configurations, but we're integrating also over the whole Diff$\times$Weyl group in $\mathcal{D}\zeta$. One way of circumventing this problem is introducing by hand a Dirac delta to force $\zeta = 0$, what also forces we to integrate only over one copy of the physical configurations,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta) \exp\left(-S_X[X, \zeta \circ h]\right)$$

but this is not the most general way, we could set $\zeta = f(\sigma)$, for a arbitrary function, and this would still give the same theory,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta - f) \exp\left(-S_X[X, \zeta \circ h]\right)$$

we can go even further and give a function $G(\zeta)$ such that the solution to $G(\zeta) = 0$ is only $\zeta = f$, so that we can use the relations between Dirac deltas,

$$\delta(G(\zeta)) = \left|\mathrm{Det}\left[\frac{\delta G}{\delta\zeta}\right]\bigg|_{\zeta=f}\right|^{-1}\delta(\zeta - f)$$

to obtain,

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\delta(\zeta - f) \exp\left(-S_X[X, \zeta \circ h]\right)$$
$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta\left|\mathrm{Det}\left[\frac{\delta G}{\delta\zeta}\right]\bigg|_{\zeta=f}\right|\delta(G(\zeta)) \exp\Big(-S_X\Big[X, \zeta \circ \hat{h}\Big]\Big)$$

[5] 4.

$$Z = \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G}{\delta \zeta}\right]\Big|_{\zeta=f} \right| \delta(G(\zeta)) \exp\left(-S_X\left[X, \zeta\circ\hat{h}\right]\right)$$

There are some details here, as ζ is to represent both a Weyl and a Diff, it has to represent both a function ω and a vector field ξ such that,

$$\begin{aligned}\zeta\circ\hat{h} &= \hat{h} + 2\omega\hat{h} + \pounds_\xi\hat{h} + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)\\ \left[\zeta\circ\hat{h}\right]_{ab} &= \hat{h}_{ab} + 2\omega\hat{h}_{ab} + 2\nabla_{(a}\xi_{b)} + \mathcal{O}\big(\omega^2, \xi^2, \omega\xi\big)\end{aligned}$$

this means both $\zeta = f$ and $G(\zeta) = 0$ are in fact a collection of various equations. In particular, we'll choose

$$G_{ab}(\zeta) = \left[\tilde{h}\right]_{ab} - \left[\zeta\circ\hat{h}\right]_{ab}$$

for a particular metric $\tilde{h}$. As $G_{ab}(\zeta)$ is in fact a function of $h = \zeta\circ\hat{h}$ alone,

$$G_{ab}(\zeta) = \left[\tilde{h}\right]_{ab} - \left[\zeta\circ\hat{h}\right]_{ab} = \left[\tilde{h}\right]_{ab} - [h]_{ab} = G_{ab}(h)$$

we can rewrite as,

$$\begin{aligned}Z &= \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G_{ab}}{\delta \zeta}\right]\Big|_{\zeta=f} \right| \delta(G_{ab}(\zeta)) \exp\left(-S_X\left[X, \zeta\circ\hat{h}\right]\right)\\ Z &= \int \mathcal{D}X\mathcal{D}\hat{h}\mathcal{D}\zeta \left| \mathrm{Det}\left[\frac{\delta G_{ab}}{\delta \zeta}\right]\Big|_{G_{ab}(h)=0} \right| \delta(G_{ab}(h)) \exp\left(-S_X[X, h]\right)\end{aligned}$$

Notice that every term in the integrand depends on ζ only through $h = \zeta\circ\hat{h}$, this is what we do want, so that we can recombine the integration measure $\int \mathcal{D}\hat{h}\mathcal{D}\zeta = \int \mathcal{D}h$, the only problem in this procedure is the term,

$$\mathrm{Det}\left[\frac{\delta G_{ab}}{\delta \zeta}\right]\Big|_{G_{ab}(h)=0}$$

which is manifestly dependent on ζ, or at least looks like it is. We'll prove it only depends on ζ through $h = \zeta\circ\hat{h}$. The point is, if

$$G_{ab}(h) = \tilde{h}_{ab} - h_{ab} = \tilde{h}_{ab} - \left[\zeta\circ\hat{h}\right]_{ab} = 0$$

is needed to have a solution, then exists the transformation $\tilde{\zeta}$, such that,

$$\tilde{\zeta}\circ\hat{h} = \tilde{h}$$

as this transformation is also an element of the gauge **group**, it certainly has an inverse, $\tilde{\zeta}^{-1}\circ\tilde{\zeta}\circ\hat{h} = \hat{h}$, thus,

$$\begin{aligned}G_{ab}(h) &= \tilde{h} - \zeta\circ\hat{h}\\ G_{ab}(h) &= \tilde{h} - \zeta\circ\tilde{\zeta}^{-1}\circ\tilde{\zeta}\circ\hat{h}\\ G_{ab}(h) &= \tilde{h} - \zeta\circ\tilde{\zeta}^{-1}\circ\tilde{h}\\ G_{ab}(h) &= \tilde{h} - \zeta'\circ\tilde{h}\end{aligned}$$

where we defined the new gauge transformation $\zeta' = \zeta\circ\tilde{\zeta}^{-1}$, notice that,

$$\begin{aligned}h &= \zeta\circ\hat{h}\\ h &= \zeta\circ\tilde{\zeta}^{-1}\circ\tilde{\zeta}\circ\hat{h}\\ h &= \zeta'\circ\tilde{h}\end{aligned}$$