2+1 Dimensional Gravity as a Gauge Theory

An Exploration of Simplicity

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Introduction & Motivation

General relativity (GR), at its core, is a theory of spacetime geometry — but there are strong motivations to reformulate it as a gauge theory. Gauge theories describe interactions as consequences of local redundancies, and this perspective has proven remarkably powerful in the Standard Model of particle physics. Writing gravity as a gauge theory places it on the same conceptual footing as the other fundamental forces: the metric or vielbein becomes the gauge field of local translations, and the spin connection gauges local Lorentz transformations. This approach clarifies the role of redundancies, makes the counting of degrees of freedom more transparent, and provides a natural framework for coupling gravity to matter with spin. It also opens the door to techniques from Yang–Mills (YM) theory — such as BRST quantization, loop variables, and color-kinematics duality — which can be crucial in the search for a quantum theory of gravity.

Modern Language of Gauge Theories

We have first to define what we mean by an *usual sense gauge theory*. In the modern language, an elegant way of defining such is: Given a finite dimensional Lie algebra $\mathfrak{g}$, a *usual sense gauge theory* is any theory in which the fundamental field is a adjoint $\mathfrak{g}$ valued 1-form,

$$\mathbf{A} = A_{aI}\mathrm{T}^I\mathbf{d}x^a, \quad \mathrm{T}^I \in \mathfrak{g}$$

which enjoys redundancy by G valued 0-forms,

$$\mathbf{A} \sim U(\mathbf{A} + \mathbf{d})U^{-1}, \quad U = \exp\Big(\xi_I \mathrm{T}^I\Big).$$

For the usual YM theory, the action is written in terms of a trace in the Lie algebra — which could be any symmetric, non-degenerated, bilinear form —,

$$S_{\text{YM}} = \frac{1}{g^2}\int\limits_M \mathrm{Tr}\,[\mathbf{F} \wedge \star\mathbf{F}], \quad \mathbf{F} = \mathbf{dA} + [\mathbf{A} \overset{\wedge}{,} \mathbf{A}]$$

Vielbein & Spin Connection

The metric, g_{ab}, isn't the best variable to make a parallel between gravity and usual gauge theories. Among motives is the presence of non-polynomial terms in the Einstein-Hilbert (EH) action, such as g^{ab} and $\sqrt{|g|}$,

$$S_{\text{EH}} = \frac{1}{2\kappa}\int\limits_M \mathrm{d}^D x\,\sqrt{|g|}g^{ab}R_{cb}{}^{c}{}_{a},$$

other is that the redundancy here are the diffeomorphisms, $\mathbf{g} \sim \phi_*\mathbf{g}$, which is not in the standard form of usual gauge theories.

Hence, to connect gravity to gauge theory, we shift from the metric g_{ab} to the **vielbein** $e^{\mu}{}_{a}$ and **spin connection** $\omega^{\alpha}{}_{\beta a}$.

- **Vielbein:** Local trivialization of the metric, defines a non-coordinate frame,

$$\mathbf{e}^\mu = e^\mu{}_a\mathbf{d}x^a, \; \eta_{\mu\nu}\mathbf{e}^\mu \otimes \mathbf{e}^\nu = \mathbf{g},$$

- **Spin Connection:** Local connection in the non-coordinate frame,

$$\boldsymbol{\omega}_\nu{}^\mu = \omega_\nu{}^\mu{}_a\mathbf{d}x^a, \; \nabla_{\mathbf{X}}\mathbf{e}^\mu = \boldsymbol{\omega}(\mathbf{X})_\nu{}^\mu\mathbf{e}^\nu$$

As is usual to general relativity, we do impose metricity in the connection, but, it's suboptimal to impose torsionless, as this requires a differential/algebraical restraint between the vielbein and spin-connection.

- **Metricity:** The connection is metric compatible, $\nabla_{\mathbf{X}}\mathbf{g} = 0 \Rightarrow \boldsymbol{\omega}_{\alpha\beta} = -\boldsymbol{\omega}_{\beta\alpha}$
- **Torsionfull (In principle):** Torsion not necessarily is absent, $\mathbf{T}^\mu = \mathbf{d}\mathbf{e}^\mu + \boldsymbol{\omega}^\mu{}_\nu \wedge \mathbf{e}^\nu \neq 0$

In this way the vielbein and the spin-connection are independent, what softens the non-polynomial character of EH action. The last needed piece is how the Riemann tensor looks in this formalism, which can be derived from the definition of the covariant derivative,

$$\mathbf{Riem}(\mathbf{X},\mathbf{Y})\mathbf{e}^\mu = \Big([\nabla_{\mathbf{X}}, \nabla_{\mathbf{Y}}] - \nabla_{[\mathbf{X},\mathbf{Y}]}\Big)\mathbf{e}^\nu$$

Which is compactly written as a 2-form,

$$\mathbf{R}^{\alpha\beta} = \mathbf{d}\boldsymbol{\omega}^{\alpha\beta} + \boldsymbol{\omega}^{\alpha\mu} \wedge \boldsymbol{\omega}_\mu{}^\beta, \quad \mathbf{R}^{\alpha\beta} = \frac{1}{2}R_{ab}{}^{\alpha\beta}\mathbf{d}x^a \wedge \mathbf{d}x^b$$

EH Action in Form Language

Using these introduced variables, we rewrite the EH action — which now is the Einstein-Cartan (EC) theory due to torsion being permissible —. This is a straightforward manipulation of tensor indexes,

$$S_{\text{EH}} = \frac{1}{2\kappa}\int\limits_M \star\mathbf{R}^{\alpha\beta} \wedge \mathbf{e}_\alpha \wedge \mathbf{e}_\beta.$$

That is valid for $D \geq 2$.

Redundancies of Gravity

We were able to rewrite the EH action in terms of two kinds of 1-forms, the vielbein and the spin-connection. What is missing to fulfill the analogy with usual gauge theories is: Dynamical 1-forms being adjoint $\mathfrak{g}$ valued, correct redundancy transformation and the existence of a bilinear, symmetric, non-degenerated form in the Lie algebra.

For standard GR, the redundancy is only diffeomorphisms, $\mathbf{g} \sim \phi_*\mathbf{g}$, but, now, as we have more dynamical fields, we gain also more redundancies. From the definition of vielbein, we see that $\mathbf{e}^\mu \to \Lambda^\mu{}_\nu\mathbf{e}^\nu$ is a redundancy whenever $\Lambda^\mu{}_\nu$ is a local $SO(D-1,1)$ element, due to $\mathbf{g}$ being preserved by this transformation,

$$\mathbf{g} = \eta_{\mu\nu}\mathbf{e}^\mu \otimes \mathbf{e}^\nu \to \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = \mathbf{g}$$

It's interesting to observe what is the action of this redundancy transformation on the spin-connection, which can be obtained from it's definition,

$$\boldsymbol{\omega}^\mu{}_\nu \to \Lambda^\mu{}_\alpha\Big(\boldsymbol{\omega}^\alpha{}_\beta + \delta^\alpha{}_\beta\mathbf{d}\Big)\Lambda^{-1\beta}{}_\nu$$

The action is invariant under those, and, additionally, this settle down the interpretation of $\boldsymbol{\omega}$ being a usual gauge field for the group $SO(D-1,1)$, is just necessary to dress it with the generators of the algebra $J^{\mu\nu}$.

The problem arises with the diffeomorphisms, these are D d.o.f. of redundancies, and being seen that the spin-connection correspond to $SO(D-1,1)$, it's natural to expect that the vielbein would correspond to gauged translations, this is also compatible with the index structure for the vielbein, which could be dressed with the translation generators P^μ. A quick comparison between the gauged translations and diffeomorphisms transformations,

$$\begin{cases}\delta\boldsymbol{\omega}_{\alpha\beta} &= 0\\ \delta\mathbf{e}^\mu &= \boldsymbol{\omega}^\mu{}_\nu\xi^\nu + \mathbf{d}\xi^\mu\end{cases} \qquad \text{vs.} \qquad \begin{cases}\pounds_\xi\boldsymbol{\omega}_{\alpha\beta} &= \mathbf{R}_{\alpha\beta}(\xi) + \mathbf{d}\big(\boldsymbol{\omega}_{\alpha\beta}(\xi)\big) + 2\boldsymbol{\omega}_{[\alpha}{}^\mu\boldsymbol{\omega}^\mu{}_{\beta]}(\xi)\\ \pounds_\xi\mathbf{e}^\mu &= \boldsymbol{\omega}^\mu{}_\nu\xi^\nu + \mathbf{d}\xi^\mu + \mathbf{T}^\mu(\xi) - \boldsymbol{\omega}^\mu{}_\nu(\xi)\mathbf{e}^\nu\end{cases}$$

Which are the same for vacuum solutions. Thus, it may be possible to express our action as a $ISO(D-1,1)$ gauge theory.

Gravity as a Gauge Theory

We were guided to interpret the vielbein and the spin-connection as gauge fields of $\mathfrak{iso}(D-1,1)$, the last piece we need to construct such theory is a bilinear — if possible multilinear —, symmetric, non-degenerated form on the Lie algebra, which sadly does not exists for $\mathfrak{iso}(D-1,1)$, apart from $D=3$. In this case it reads,

$$\big\langle J_{\alpha\beta}, P_\mu\big\rangle = \frac{1}{\kappa}\epsilon_{\alpha\beta\mu}, \quad \big\langle J_{\alpha\beta}, J_{\mu\nu}\big\rangle = 0, \quad \big\langle P_\mu, P_\nu\big\rangle = 0.$$

It's not possible to write a Yang-Mills alike action for $D=3$ due to a mismatch in the d.o.f.. But, a Chern-Simons (CS) theory has the correct number of d.o.f.,

$$S_{\text{CS}}[\mathbf{A}] = \frac{1}{2}\int\limits_M \left\langle \mathbf{A} \overset{\wedge}{,} \left(\mathbf{dA} + \frac{1}{3}[\mathbf{A} \overset{\wedge}{,} \mathbf{A}]\right)\right\rangle,$$

which, if evaluated for a gauge field of $\mathfrak{iso}(2,1)$,

$$\mathbf{A} = \frac{1}{2}\boldsymbol{\omega}_{\alpha\beta}J^{\alpha\beta} + \mathbf{e}_\mu P^\mu = \boldsymbol{\omega} + \mathbf{e},$$

returns exactly the EH action,

$$S_{\text{CS}} = \int\limits_M \left\langle \mathbf{e} \overset{\wedge}{,} \left(\mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega} \overset{\wedge}{,} \boldsymbol{\omega}]\right)\right\rangle = \frac{1}{2\kappa}\epsilon_{\mu\alpha\beta}\int\limits_M \mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} = \frac{1}{2\kappa}\int\limits_M \star\mathbf{R}^{\alpha\beta} \wedge \mathbf{e}_\alpha \wedge \mathbf{e}_\beta = S_{\text{EH}}.$$

Conclusions

- Gravity in $D = 2+1$ is expressible in terms of an usual gauge theory, a Chern-Simons one. That is, the EH (EC) action — specifically in $D=3$ — possesses beyond the usual diffeomorphism redundancy, an additional local Poincaré redundancy. These are equivalent for vacuum solutions.
- The equivalence shown for this lower dimensional toy model shed some light on a possible quantization proceedure for $D=3$ gravity, as it can inherit the well estabilished quantization proceedure of Chern-Simons theories.
- Despite not being a description of gravity in our world, the expectation is that this toy model can provide some insight into $D=4$ gravity.

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References

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