Homework I

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Problem 1

1.A)

1.B)

Problem 2

2.A)

2.B)

Problem 3

3.A)

The Gamma Function can be represented in the complex plane domain, $\mathrm{Re}(s) > 1$, as the following integral,

$$\Gamma(s) = \int_0^\infty \mathrm{d}t \exp(-t) t^{s-1}, \quad \mathrm{Re}(s) > 1 \tag{3.1}$$

Which is also the subset of the complex plane in which this integral converges, of course this representation of the Gamma Function in a open set is sufficient for obtain an analytical continuation to the whole complex plane. Obviously, the integral is invariant under relabeling the dummy variable t, we make the following choice $t \to nt$ — Assuming $n > 0$ —,

$$\Gamma(s) = \int_0^\infty \mathrm{d}(nt) \exp(-nt)(nt)^{s-1}, \quad \mathrm{Re}(s) > 1$$

$$\Gamma(s) = n^s \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

$$n^{-s}\Gamma(s) = \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

$$\sum_{n=1}^\infty n^{-s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

The sum in the left-hand side is recognized as the representation for the Zeta Function in the domain $\mathrm{Re}(s) > 1$, which is also the domain of convergence of the sum,

$$\zeta(s) = \sum_{n=1}^\infty n^{-s}, \quad \mathrm{Re}(s) > 1$$

So that,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

About the right-hand side, to be able to exchange the integral and the sum is sufficient that,

$$\int_0^\infty \mathrm{d}t \sum_{n=1}^\infty \left\| \exp(-nt) t^{s-1} \right\| < \infty, \quad \mathrm{Re}(s) > 1$$

$$\int_0^\infty \mathrm{d}t \sum_{n=1}^\infty \exp(-nt) \left\| t^{s-1} \right\| < \infty, \quad \mathrm{Re}(s) > 1$$

$$\int_0^\infty \mathrm{d}t \sum_{n=1}^{\infty} \exp(-nt) t^{\mathrm{Re}(s)-1} < \infty, \quad \mathrm{Re}(s) > 1$$

The sum now is a simple geometric series, giving,

$$\int_0^\infty \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{\exp(t) - 1} < \infty, \quad \mathrm{Re}(s) > 1$$

The dangerous behavior that could make the integral diverges is the one at $t \to 0$, an indeed, $\mathrm{Re}(s) > 1$, is sufficient for the convergence of this integral, which can be seen at,

$$\int_0^\epsilon \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{\exp(t) - 1} \approx \int_0^\epsilon \mathrm{d}t \, \frac{t^{\mathrm{Re}(s)-1}}{t + \mathcal{O}(t^2)} \approx \int_0^\epsilon t^{\mathrm{Re}(s)-2} = \left. \frac{t^{\mathrm{Re}(s)-1}}{\mathrm{Re}(s) - 1} \right|_0^\epsilon$$

Which shows the integral is really finite at $t \to 0$ with $\mathrm{Re}(s) > 1$, hence, switching the integral and the sum is justified, so,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}t \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

$$\zeta(s)\Gamma(s) = \mathrm{d}t \int_0^\infty \sum_{n=1}^{\infty} \exp(-nt) t^{s-1}, \quad \mathrm{Re}(s) > 1$$

Where again we have the sum of a geometric series, giving,

$$\zeta(s)\Gamma(s) = \int_0^\infty \mathrm{d}t \, \frac{t^{s-1}}{\exp(t) - 1}, \quad \mathrm{Re}(s) > 1$$

3.B)

The objective here is to make an analytical continuation to $\mathrm{Re}(s) > -2$ of the expression found in the later item. First of all, the reason the later expression is only well defined in $\mathrm{Re}(s) > 1$, is due to the divergence of the integrand at $t \to 0$ for $\mathrm{Re}(s) \leq 1$, this is only because $(\exp(t) - 1)^{-1}$ has a simple pole at $t = 0$, which is also the only pole of this function, so to get the Laurent series we first find the residue of it,

$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp(t) - 1}\right) = \left. \frac{t}{\exp(t) - 1} \right|_{t=0}$$

$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp(t) - 1}\right) = \left. \frac{t}{t + \mathcal{O}(t^2)} \right|_{t=0}$$

$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp(t) - 1}\right) = \left. \frac{1}{1 + \mathcal{O}(t)} \right|_{t=0}$$

$$\mathrm{Res}_{t=0}\left(\frac{1}{\exp(t) - 1}\right) = 1$$

As this is the only pole, we get a Laurent series starting as,

$$\frac{1}{\exp(t)-1} = \frac{1}{t} + \mathcal{O}\big(t^0\big)$$

To get the following terms we just make a trivial Taylor series of the function $(\exp(t)-1)^{-1} - t^{-1}$

$$\begin{aligned}
\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\bigg|_0 &= \frac{1+t-\exp(t)}{t[\exp(t)-1]}\bigg|_0 \\
\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\bigg|_0 &= \frac{-\frac{t^2}{2}+\mathcal{O}(t^3)}{t[t+\mathcal{O}(t^2)]}\bigg|_0 \\
\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\bigg|_0 &= \frac{-\frac{t^2}{2}+\mathcal{O}(t^3)}{t^2[1+\mathcal{O}(t)]}\bigg|_0 \\
\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\bigg|_0 &= -\frac{1}{2}
\end{aligned}$$

In other words,

$$\frac{1}{\exp(t)-1} = \frac{1}{t} - \frac{1}{2} + \mathcal{O}(t)$$

The next term of the series will be,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\exp(t)-1} - \frac{1}{t}\right]\bigg|_0 &= \frac{1}{t^2} - \frac{\exp(t)}{[\exp(t)-1]^2} \\
&= \frac{\exp(t)+\exp(-t)-2-t^2}{t^2[\exp(t)+\exp(-t)-2]}\bigg|_0 \\
&= \frac{2\frac{t^4}{4!}+\mathcal{O}(t^6)}{t^2[t^2+\mathcal{O}(t^4)]}\bigg|_0 \\
&= \frac{1}{12}\frac{t^4+\mathcal{O}(t^6)}{t^4[1+\mathcal{O}(t^2)]}\bigg|_0 \\
&= \frac{1}{12}
\end{aligned}$$

So up to first order we have,

$$\frac{1}{\exp(t)-1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}\big(t^2\big) \tag{3.2}$$

Why have we done this? Because we do can soften the behavior of the integrand near $t \to 0$ if we subtract leading terms of the expansion of $(\exp(t)-1)^{-1}$, each leading term that we subtract, is equivalent to gaining a power of t in the numerator, which does soften the behavior near $t \to 0$, but also makes it worse in the region $t \to \infty$, and as our only problem is related with the small t region, we can divide the integral in two parts,

$$\zeta(s)\Gamma(s) = \int\limits_0^1 \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1} + \int\limits_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}, \quad \mathrm{Re}(s) > 1$$

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} + \frac{1}{t} - \frac{1}{2} + \frac{t}{12}\right] + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}, \quad \mathrm{Re}(s) > 1$$

Where we simply added and subtracted the leading terms of the expansion, the integral of the last three of them is trivial and can be done to give,

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] + \int_0^1 \mathrm{d}t\left[t^{s-2} - \frac{t^{s-1}}{2} + \frac{t^s}{12}\right] + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}$$

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}$$

Just what we wanted.

3.C)

Naively, this last expression should be well defined only for $\mathrm{Re}(s) > 1$, let's see this term by term, starting by the last one,

$$\int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp(t)-1}$$

This is finite for all s, as it is exponentially decaying and is bounded in the integration interval, this term is well defined for all s. The next three ones are,

$$\frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)}$$

Also these are well defined in the whole complex plane, with three poles at $s = -1, 0, 1$. Finally we have,

$$\int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right]$$

The only potential not well defined behavior that can occur is near $t = 0$, but we have already developed a series expansion for the expression in brackets, 3.2, that means, near the critical value of $t = 0$, the integrand goes like,

$$\int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp(t)-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] \approx \int_0^1 \mathrm{d}t\, t^{s-1}\mathcal{O}(t^2) \approx \int_0^1 \mathrm{d}t\, t^{s+1} = \left.\frac{t^{s+2}}{s+2}\right|_0^1$$

This is well defined as long as $\mathrm{Re}(s) > -2$. Hence, the expression,

$$\zeta(s)\Gamma(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left[\frac{1}{\exp{(t)}-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right] + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{\exp{(t)}-1} \tag{3.3}$$

Is well defined as long as $\mathrm{Re}(s) > -2$. One might worry about the poles, but, these are the natural structure of $\zeta(s)\Gamma(s)$, to be well defined does not mean to don't have poles, but means the representation can be assigned a number in an unique manner. What is left to us now is to find the values $\zeta(0), \zeta(-1)$, notice that our representation has a poles in both these values of the argument, in fact these poles are structures of $\Gamma(s)$, and not $\zeta(s)$. That the Gamma Function indeed has poles in those values can be seen from,

$$\Gamma(s+1) = s\Gamma(s) \Rightarrow \begin{cases} \Gamma(0) & = \frac{\Gamma(1)}{0} \\ \Gamma(-1) & = \frac{\Gamma(0)}{-1} \end{cases}$$

And because the poles in our representation are just simple poles, they could not have been poles also in ζ, as the two functions are multiplying if there were a pole in $\zeta(0), \zeta(-1)$ they would have been apparent in our representation as double poles. Due to the absence of those, the poles at $s = 0, -1$ are indeed only due to the Gamma Function. This guarantees us that $\zeta(0), \zeta(-1)$ are both finite, and to determine those we just need to evaluate the residue of the expression. First, the residue of the Gamma Function,

$$\begin{aligned} \mathrm{Res}_{s=0}(\Gamma(s)) &= s\Gamma(s)\Big|_{s=0} = \Gamma(s+1)\Big|_{s=0} = \Gamma(1) = 1 \\ \mathrm{Res}_{s=-1}(\Gamma(s)) &= (s+1)\Gamma(s)\Big|_{s=-1} = \frac{(s+1)s\Gamma(s)}{s}\Big|_{s=-1} = \frac{\Gamma(s+2)}{s}\Big|_{s=-1} = -1 \end{aligned}$$

As we argued that $\zeta(0), \zeta(-1)$ should be finite, what will happen is that when we multiply ζ by Γ, the residues of the poles of the Gamma Function will be multiplied by the value of the Zeta Function at that point, that is,

$$\mathrm{Res}_{s=0}(\zeta(s)\Gamma(s)) = \zeta(0)\mathrm{Res}_{s=0}(\Gamma(s))$$

But, as can be seen directly from 3.3, the only contribution for the residue at $s = 0$ will be by $-\frac{1}{2s}$, as all the other terms are finite at $s = 0$, thus,

$$\begin{aligned} \mathrm{Res}_{s=0}(\zeta(s)\Gamma(s)) &= -\frac{1}{2} = \zeta(0)\mathrm{Res}_{s=0}(\Gamma(s)) = \zeta(0) \\ \zeta(0) &= -\frac{1}{2} \end{aligned}$$

Analogously we have,

$$\mathrm{Res}_{s=-1}(\zeta(s)\Gamma(s)) = \zeta(-1)\mathrm{Res}_{s=-1}(\Gamma(s))$$

Again, as we discussed previously, all the terms are finite at $s = -1$, except for $\frac{1}{12(s+1)}$, hence, the residue will be,

$$
\begin{aligned}
\mathrm{Res}_{s=-1}(\zeta(s)\Gamma(s)) = \frac{1}{12} &= \zeta(-1)\mathrm{Res}_{s=-1}(\Gamma(s)) = -\zeta(-1) \\
\zeta(-1) &= -\frac{1}{12}
\end{aligned}
$$

Problem 4

4.A)

4.B)

Problem 5

5.A)

5.B)

5.C)

5.D)

5.E)

Problem 6

6.A)

6.B)

6.C)

6.D)

6.E)

6.F)

6.G)