SCALAR PROXY

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1. Conformal Toy Model

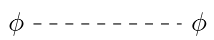
Consider the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\Box\phi\Box\phi - \frac{g}{2}\phi^2\Box\phi - \frac{g^2}{8}\phi^4 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

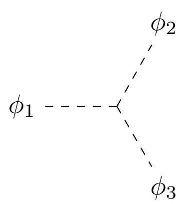
Notice the form of the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\left(\Box\phi + \frac{g}{2}\phi^2\right)^2 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

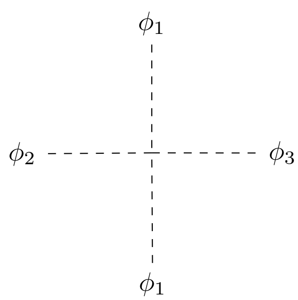
It possesses the Feynman rules,

- 

$= \frac{1}{\mathrm{i}}\frac{1}{m^2p^2+p^4}$

- 

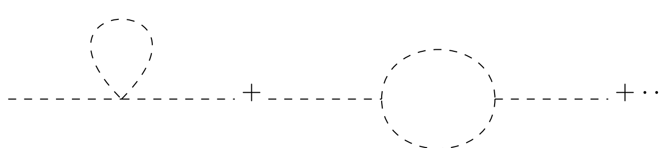
$= \mathrm{i}g\left(p_1^2 + p_2^2 + p_3^2 + m^2\right)$

- 

$= -3\mathrm{i}g^2$

Let's compute the self energy,

$$\mathrm{i}\Pi\left(p^2\right) = \tag{1.1}$$



$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\mathrm{i}}\frac{1}{\ell^2}\frac{1}{\ell^2+m^2} \tag{1.2}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\ell^2}\frac{1}{\ell^2+m^2} \tag{1.3}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}\left(m^2\right)^{\frac{D}{2}-2}\frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(1)} \tag{1.4}$$

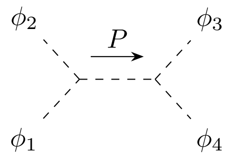
$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2\frac{\left(m^2\right)^{-\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-\epsilon)} \tag{1.5}$$

$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}(\mathrm{i}g)^2\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\mathrm{i}^2}\frac{1}{\ell^2(\ell+p)^2}\frac{\left(m^2+\ell^2+p^2+(\ell+p)^2\right)^2}{\ell^2+m^2}\frac{1}{(\ell+p)^2+m^2} \tag{1.6}$$

For the mass renormalization we can take $p = 0$,

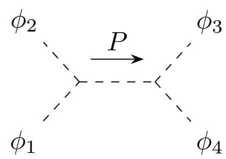
$$\mathrm{i}\Pi^{(2)} = \frac{1}{2} g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{\left(m^2 + 2\ell^2\right)^2}{\ell^4 \left(\ell^2 + m^2\right)^2} \tag{1.7}$$

Let's compute the four point amplitude for this theory,



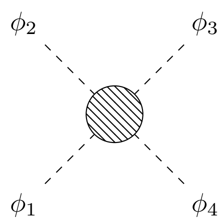
$$= (\mathrm{i}g)^2 \frac{\left(p_1^2 + p_2^2 + (p_1 + p_2)^2 + m^2\right)\left(p_3^2 + p_4^2 + (p_3 + p_4)^2 + m^2\right)}{\mathrm{i}(p_1 + p_2)^2 \left((p_1 + p_2)^2 + m^2\right)} \tag{1.8}$$

First let's consider all legs massless,

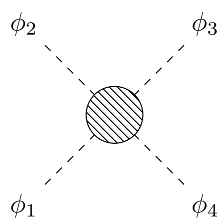


$$= \mathrm{i}g^2 \frac{\left(-s + m^2\right)\left(-s + m^2\right)}{(-s)(-s + m^2)} = -\mathrm{i}g^2 \frac{\left(-s + m^2\right)}{s} \tag{1.9}$$

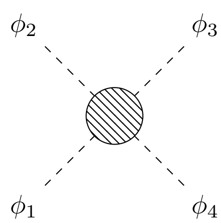
So,



$$= -\mathrm{i}g^2 \frac{\left(-s + m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t + m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u + m^2\right)}{u} - 3\mathrm{i}g^2 \tag{1.10}$$

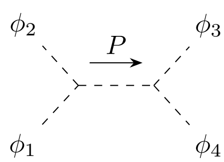


$$= -\mathrm{i}g^2 \frac{\left(-s + m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t + m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u + m^2\right)}{u} - \mathrm{i}g^2 \frac{s}{s} - \mathrm{i}g^2 \frac{t}{t} - \mathrm{i}g^2 \frac{u}{u} \tag{1.11}$$



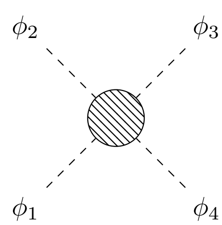
$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right) = \mathrm{i}g^2 m^2 \left(\frac{1}{\langle 12\rangle [12]} + \frac{1}{\langle 14\rangle [14]} + \frac{1}{\langle 13\rangle [13]}\right) \tag{1.12}$$

Para uma perna massiva, ϕ_4,



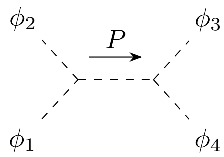
$$= (\mathrm{i}g)^2 \frac{\left(-s + m^2\right)(-s)}{\mathrm{i}(-s)(-s + m^2)} = \mathrm{i}g^2 \tag{1.13}$$

So,

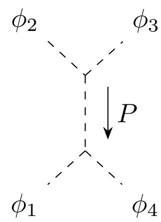


$$= \mathrm{i}g^2 + \mathrm{i}g^2 + \mathrm{i}g^2 - 3\mathrm{i}g^2 = 0 \tag{1.14}$$

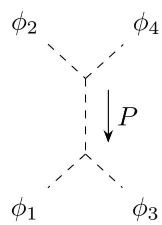
Para duas pernas massivas, $\phi_{3,4}$,



$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2 \frac{s+m^2}{s} \tag{1.15}$$

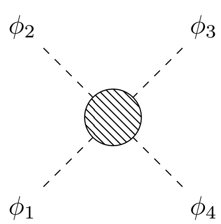


$$= (\mathrm{i}g)^2 \frac{(-t)(-t)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t}{-t+m^2} \tag{1.16}$$

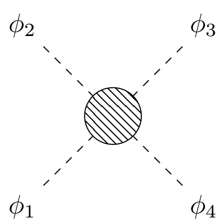


$$= -\mathrm{i}g^2 \frac{u}{-u+m^2} \tag{1.17}$$

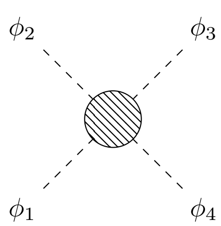
So,



$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - 3\mathrm{i}g^2 \tag{1.18}$$

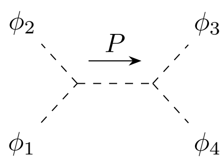


$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - \mathrm{i}g^2 \frac{s}{s} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2} \tag{1.19}$$

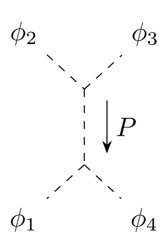


$$= -\mathrm{i}g^2 m^2 \left(-\frac{1}{s} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -\mathrm{i}g^2 m^2 \left(\frac{1}{\langle 12 \rangle [12]} + \frac{1}{\langle 1\mathbf{4} \rangle [1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3} \rangle [1\mathbf{3}]} \right) \tag{1.20}$$

Para uma perna sem massa ϕ_1,

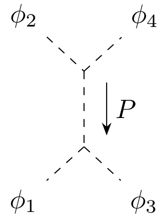


$$= (\mathrm{i}g)^2 \frac{(-s)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} \tag{1.21}$$



$$= (\mathrm{i}g)^2 \frac{(-t)\left(-t-m^2\right)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t+m^2}{-t+m^2} \tag{1.22}$$

(1.23)

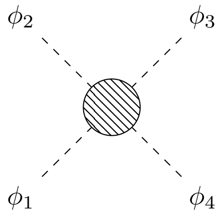


$$= (\mathrm{i}g)^2 \frac{(-u)\big(-u-m^2\big)}{\mathrm{i}(-u)(-u+m^2)} = -\mathrm{i}g^2 \frac{u+m^2}{-u+m^2}$$

(1.24)

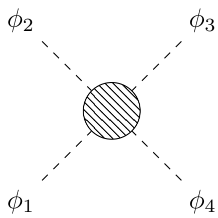
So,

(1.25)



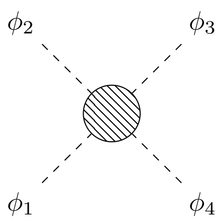
$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - 3\mathrm{i}g^2$$

(1.26)



$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - \mathrm{i}g^2 \frac{-s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2}$$

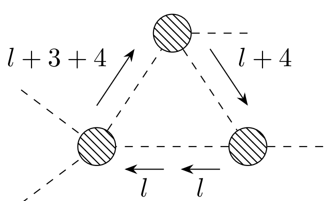
(1.27)



$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -\mathrm{i}g^2 m^2 \left(\frac{1}{\langle 1\mathbf{2}\rangle[1\mathbf{2}]} + \frac{1}{\langle 1\mathbf{4}\rangle[1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3}\rangle[1\mathbf{3}]} \right)$$

Cut comparison, only massless legs

(1.28)



$$= \frac{\mathrm{i}g^2 m^2 \big(\mathrm{i}gm^2\big)^2}{\big(\mathrm{i}m^2\big)^3} \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right)$$

to solve for the cuts, $l^2 = (l+3+4)^2 = (l+4)^2 = 0$,

$$l^2 = 0 \Rightarrow l = -|l\rangle[l| \tag{1.29}$$

$$0 = (l+4)^2 = \langle l4\rangle[l4] = 0 \Rightarrow |l] = |4] \tag{1.30}$$

$$0 = (l+3+4)^2 = \langle lP_{34}\rangle[lP_{34}] + (3+4)^2 = \langle l|3+4|l] + \langle 34\rangle[34] = \langle l|3+4|4] + \langle 34\rangle[34] \tag{1.31}$$

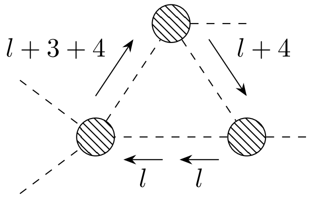
$$\langle 43\rangle[34] = -\langle l3\rangle[34] \Rightarrow |l\rangle = -|4\rangle + z|3\rangle \tag{1.32}$$

$$l = -(-|4\rangle + z|3\rangle)[4| \tag{1.33}$$

The cuts are solved by this. Hence,

$$= g^4 \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right)$$
$$= g^4 \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{(-\langle 14\rangle + z\langle 13\rangle)[14]} - \frac{1}{(-\langle 24\rangle + z\langle 23\rangle)[24]} \right)$$

Now for internal massive lines,



(1.34)
$$= \frac{-\mathrm{i}g^2m^2\left(-\mathrm{i}gm^2\right)^2}{\left(-\mathrm{i}m^2\right)^3}\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]}\right)$$

With the cuts being, $l^2 = \left(l+3+4\right)^2 = \left(l+4\right)^2 = -m^2$,

(1.35)
$$0 = (l+4)^2 - l^2 = 2l\cdot p_4$$

(1.36)
$$0 = (l+4+3)^2 - l^2 = 2l\cdot(4+3) + (4+3)^2 = 2l\cdot p_3 + (4+3)^2$$

As ansatz, $l = |4\rangle[4| + \alpha|4\rangle[3| + \beta|3\rangle[4|$ satisfy both conditions above. The remaining condition is,

(1.37)
$$l^2 = -m^2$$

(1.38)
$$-\alpha\beta[43]\langle 43\rangle = -m^2 \Rightarrow \alpha = \frac{m^2}{\beta\langle 34\rangle[34]}$$

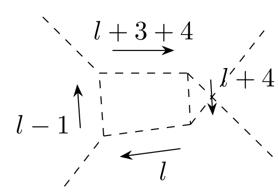
Setting now $-\beta = z$,

(1.39)
$$l = |4\rangle[4| - \frac{m^2}{z\langle 34\rangle[34]}|4\rangle[3| - z|3\rangle[4|$$

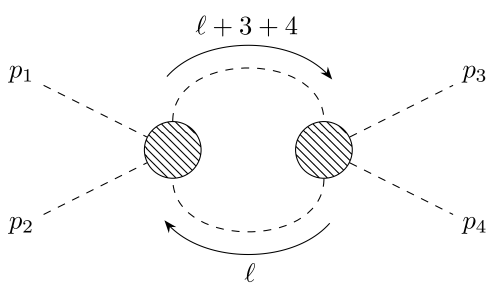
The value of the diagram is,

$$\begin{aligned}
&= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1l\rangle[l1]} + \frac{1}{\langle 2l\rangle[l2]}\right)\\
&= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{-\langle 1|l|1]} + \frac{1}{-\langle 2|l|2]}\right)\\
&= g^4\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 14\rangle[41] - \frac{m^2}{z\langle 34\rangle[34]}\langle 14\rangle[31] - z\langle 13\rangle[41]} - \frac{1}{\langle 2|l|2]}\right)
\end{aligned}$$

The explicit cut loop amplitude is,

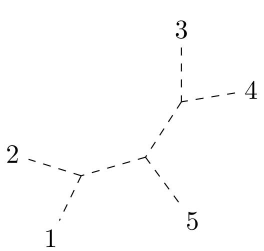


Triple cut has no improvement, what about a double cut,



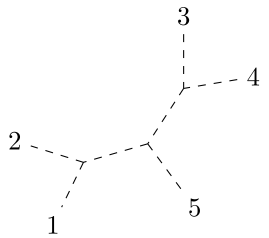
(1.40)
$$= \frac{\left(\mathrm{i}g^2m^2\right)^2}{\left(\mathrm{i}m^2\right)^2}\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1\ell\rangle[1\ell]} - \frac{1}{\langle 2\ell\rangle[2\ell]}\right)\left(\frac{1}{\langle 34\rangle[34]} + \frac{1}{\langle 3\ell\rangle[3\ell]} + \frac{1}{\langle 4\ell\rangle[4\ell]}\right)$$

Five point amplitude,



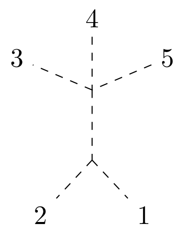
$$= \frac{(\mathrm{i}g)^3\left(m^2+p_1^2+p_2^2+(p_1+p_2)^2\right)\left(m^2+p_5^2+(p_3+p_4)^2+(p_1+p_2)^2\right)\left(m^2+p_3^2+p_4^2+(p_3+p_4)^2\right)}{i^2(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)(p_3+p_4)^2\left((p_3+p_4)^2+m^2\right)}$$

Let's consider the special case of all massless,



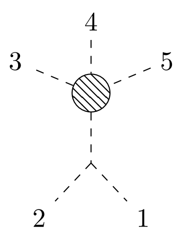
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\left(m^2+(p_3+p_4)^2+(p_1+p_2)^2\right)\left(m^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2\left((p_3+p_4)^2+m^2\right)} = \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\left(m^2+(p_3+p_4)^2+(p_1+p_2)^2\right)}{(p_3+p_4)^2}$$

Combining this graph with,



$$= \frac{-\mathrm{i}3g^2\mathrm{i}g\left(m^2+(p_1+p_2)^2\right)}{\mathrm{i}(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)} = \frac{-3\mathrm{i}g^3}{(p_1+p_2)^2}$$

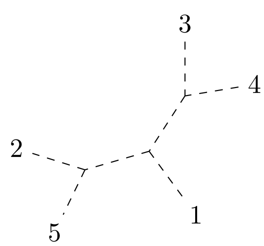
We get,



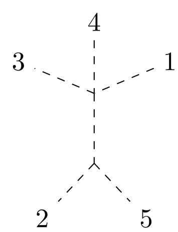
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\left[\frac{\left(m^2+(p_3+p_4)^2+(p_1+p_2)^2\right)}{(p_3+p_4)^2} - \frac{(p_3+p_4)^2}{(p_3+p_4)^2} + \frac{\left(m^2+(p_3+p_5)^2+(p_1+p_2)^2\right)}{(p_3+p_5)^2} - \frac{(p_3+p_5)^2}{(p_3+p_5)^2} + \frac{\left(m^2+(p_5\right.}{}\right.$$

$$= \frac{\mathrm{i}g^3\left(m^2+(p_1+p_2)^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,



$+$



$+$ perm

Which will be,

$$\frac{\mathrm{i}g^3}{(p_2+p_3)^2}\frac{m^2+(p_2+p_3)^2+(p_4+p_5)^2}{(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5)$$

Summing all the contributions we have,

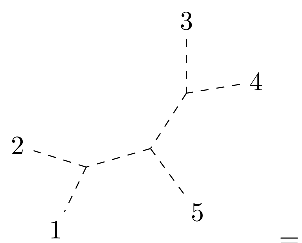
$$= \frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right] + (2 \leftrightarrow 3,4,5)$$

$$+\,\mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_2+p_4)^2}+\frac{1}{(p_2+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_2)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_2)^2}+\cdots\right.$$

$$+ \frac{\mathrm{i}g^3m^2}{(p_2+p_3)^2(p_4+p_5)^2} - \frac{2\mathrm{i}g^3}{(p_2+p_3)^2} - \frac{2\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5)$$

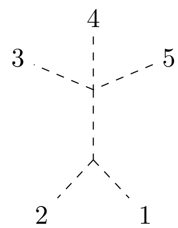
$$=\frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]+(2\leftrightarrow 3,4,5)$$
$$+\frac{\mathrm{i}g^3m^2}{(p_2+p_3)^2(p_4+p_5)^2}+\frac{\mathrm{i}g^3m^2}{(p_2+p_4)^2(p_3+p_5)^2}+\frac{\mathrm{i}g^3m^2}{(p_2+p_5)^2(p_4+p_3)^2}$$

By residue, any amplitude with just one massive external on-shell leg is zero. For two massive external on-shell legs, let's take as massive $1,2$,



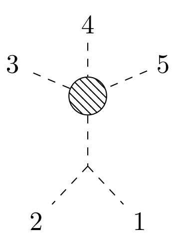
$$=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}$$

Combining this graph with,



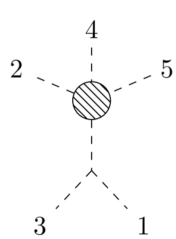
$$=-3\mathrm{i}g^3\frac{(p_1+p_2)^2-m^2}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}$$

so,



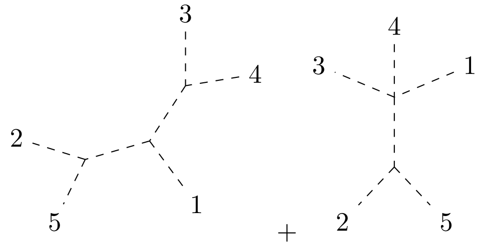
$$=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\left[\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}-\frac{(p_3+p_4)^2}{(p_3+p_4)^2}+(5\leftrightarrow 3,4)\right]$$
$$=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

The other contributions are,



$$=\frac{\mathrm{i}g^3(p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{(p_4+p_5)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,



$$+ \quad + \text{ perm}$$

which are,

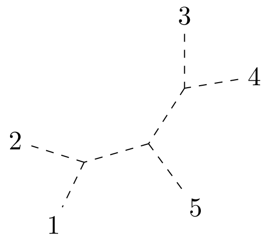
$$= \mathrm{i}g^3 \frac{(p_2+p_3)^2 + (p_4+p_5)^2}{\Big(m^2+(p_2+p_3)^2\Big)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5)$$

So, summing all the contributions,

$$\begin{aligned}
&= \mathrm{i}g^3 \frac{\Big((p_1+p_2)^2 - m^2\Big)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \frac{\mathrm{i}g^3 (p_1+p_3)^2}{\Big(m^2+(p_1+p_3)^2\Big)}\left[\frac{1}{m^2+(p_2+p_4)^2} + \frac{1}{m^2+(p_2+p_5)^2} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad + \mathrm{i}g^3 \frac{(p_2+p_3)^2 + (p_4+p_5)^2}{\Big(m^2+(p_2+p_3)^2\Big)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5) \\
&= -\mathrm{i}g^3 \frac{m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad - \mathrm{i}g^3 \frac{m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad + \mathrm{i}g^3 \frac{-m^2 + 2p_2\cdot p_3 + (p_4+p_5)^2}{(2p_2\cdot p_3)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{2p_2\cdot p_3} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5) \\
&= -\frac{\mathrm{i}g^3 m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2} + \frac{1}{2p_2\cdot p_3} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_3+p_5)^2} + \frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_3} + \frac{1}{(p_4+p_3)^2}\right] \\
&\quad - \frac{\mathrm{i}g^3 m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad - \frac{\mathrm{i}g^3 m^2}{(2p_2\cdot p_3)(p_4+p_5)^2} + (3 \leftrightarrow 4,5) \\
&\quad - \frac{2\mathrm{i}g^3}{2p_2\cdot p_3} - \frac{2\mathrm{i}g^3}{(p_4+p_5)^2} - \frac{2\mathrm{i}g^3}{2p_2\cdot p_4} - \frac{2\mathrm{i}g^3}{(p_3+p_5)^2} - \frac{2\mathrm{i}g^3}{2p_2\cdot p_5} - \frac{2\mathrm{i}g^3}{(p_4+p_3)^2} \\
&= -\frac{\mathrm{i}g^3 m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad - \frac{\mathrm{i}g^3 m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)
\end{aligned}$$

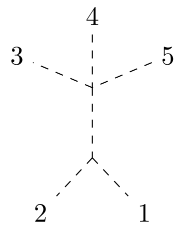
$$-\frac{\mathrm{i}g^3m^2}{(2p_2\cdot p_3)(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

Is almost the same of the all massless, but with a different denominator in the $1,2$ channel. Now with three massive legs, being $3,4,5$,



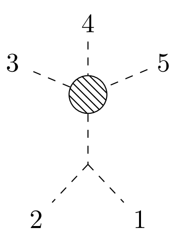
$$=\mathrm{i}g^3\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_1+p_2)^2(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}$$

With,



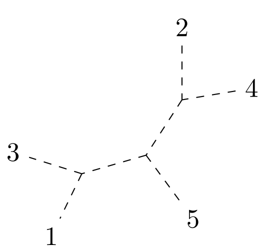
$$=\frac{-\mathrm{i}3g^2\mathrm{i}g\left(m^2+(p_1+p_2)^2\right)}{\mathrm{i}(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)}=\frac{-3\mathrm{i}g^3}{(p_1+p_2)^2}$$

So,

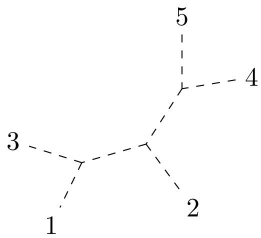


$$\begin{aligned}
&=\mathrm{i}g^3\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_1+p_2)^2(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}-\frac{3\mathrm{i}g^3}{(p_1+p_2)^2}\\
&=\frac{\mathrm{i}g^3}{(p_1+p_2)^2}\left[\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}-\frac{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}\right]+(5\leftrightarrow 3,4)\\
&=\frac{\mathrm{i}g^3}{(p_1+p_2)^2}\left[\frac{(p_1+p_2)^2\left(-m^2+(p_3+p_4)^2\right)-2m^2(p_3+p_4)^2}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}\right]+(5\leftrightarrow 3,4)
\end{aligned}$$

The other topology is,



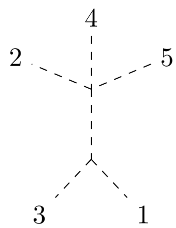
$$\begin{aligned}
&=\mathrm{i}g^3\frac{\left(m^2+p_3^2+(p_1+p_3)^2\right)\left(m^2+p_5^2+(p_1+p_3)^2+(p_2+p_4)^2\right)\left(m^2+p_4^2+(p_2+p_4)^2\right)}{(p_1+p_3)^2\left(m^2+(p_1+p_3)^2\right)(p_2+p_4)^2\left(m^2+(p_2+p_4)^2\right)}\\
&=\mathrm{i}g^3\frac{\left((p_1+p_3)^2+(p_2+p_4)^2\right)}{\left(m^2+(p_1+p_3)^2\right)\left(m^2+(p_2+p_4)^2\right)}
\end{aligned}$$



$$= \mathrm{i}g^3 \frac{\left(m^2 + p_3^2 + (p_1+p_3)^2\right)\left(m^2 + (p_1+p_3)^2 + (p_5+p_4)^2\right)\left(m^2 + p_5^2 + p_4^2 + (p_5+p_4)^2\right)}{(p_1+p_3)^2\left(m^2+(p_1+p_3)^2\right)(p_5+p_4)^2\left(m^2+(p_5+p_4)^2\right)}$$

$$= \mathrm{i}g^3 \frac{\left(m^2 + (p_1+p_3)^2 + (p_5+p_4)^2\right)\left(-m^2 + (p_5+p_4)^2\right)}{\left(m^2+(p_1+p_3)^2\right)(p_5+p_4)^2\left(m^2+(p_5+p_4)^2\right)}$$

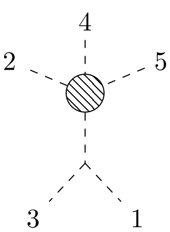
Also,



$$= -3\mathrm{i}g^2 g \frac{\left(m^2 + p_3^2 + (p_3+p_1)^2\right)}{(p_3+p_1)^2\left(m^2+(p_3+p_1)^2\right)}$$

$$= -3\mathrm{i}g^3 \frac{1}{\left(m^2+(p_3+p_1)^2\right)}$$

So,



$$= \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{\left((p_1+p_3)^2+(p_2+p_4)^2\right) - m^2 - (p_2+p_4)^2}{\left(m^2+(p_2+p_4)^2\right)} + \frac{\left((p_1+p_3)^2+(p_2+p_5)^2\right) - m^2 - (p_2+p_5)^2}{\left(m^2+(p_2+p_5)^2\right)}\right]$$

$$+ \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{\left(m^2+(p_1+p_3)^2+(p_4+p_5)^2\right)\left(-m^2+(p_4+p_5)^2\right) - (p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}\right]$$

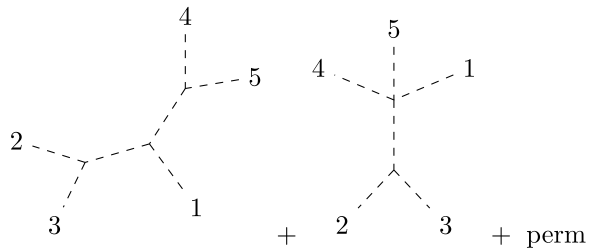
$$= \frac{\mathrm{i}g^3\left((p_1+p_3)^2 - m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2} + \frac{1}{m^2+(p_2+p_5)^2}\right]$$

$$+ \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{-m^4 - m^2(p_1+p_3)^2 + (p_1+p_3)^2(p_4+p_5)^2 - m^2(p_4+p_5)^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}\right]$$

$$= \frac{\mathrm{i}g^3\left((p_1+p_3)^2 - m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2} + \frac{1}{m^2+(p_2+p_5)^2} + \frac{1}{m^2+(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)$$

$$- \frac{\mathrm{i}g^3 m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

Now, with 1 in the middle,



Which is,

$$= \frac{ig^3\left(m^2+p_3^2+(p_2+p_3)^2\right)\left(m^2+p_1^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(m^2+p_5^2+p_4^2+(p_5+p_4)^2\right)}{(p_2+p_3)^2\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{3ig^3}{m^2+(p_2+p_3)^2} - \frac{3ig^3\left(-m^2+(p_4+\right.}{(p_4+p_5)^2\left(m^2+(p_4\right.}$$

$$= \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{3ig^3}{m^2+(p_2+p_3)^2} - \frac{3ig^3\left(-m^2+(p_4+p_5)^2\right)}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

Summing all the contributions we have,

$$\begin{aligned}
& ig^3\frac{\left(-m^2+(p_5+p_4)^2\right)}{(p_5+p_4)^2\left(m^2+(p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5) \\
&\quad - \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left((p_1+p_3)^2-m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad - \frac{ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5) \\
&\quad - \frac{3ig^3}{m^2+(p_2+p_3)^2} - \frac{3ig^3\left(-m^2+(p_4+p_5)^2\right)}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5) \\
& - \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left((p_1+p_3)^2-m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad - \frac{ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5) \\
&\quad - \frac{3ig^3}{m^2+(p_2+p_3)^2} + \frac{2ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{2ig^3}{\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5) \\
& - \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3 \leftrightarrow 4,5)
\end{aligned}$$

$$-\frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$+\mathrm{i}g^3\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]$$

$$+\mathrm{i}g^3\left[\frac{1}{m^2+(p_2+p_3)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_3+p_5)^2}\right]$$

$$+\mathrm{i}g^3\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_3)^2}+\frac{1}{m^2+(p_4+p_3)^2}\right]$$

$$+\frac{\mathrm{i}g^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2}+\frac{\mathrm{i}g^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}-\frac{2\mathrm{i}g^3}{\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$+\frac{\mathrm{i}g^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{\mathrm{i}g^3}{m^2+(p_2+p_3)^2}+\frac{\mathrm{i}g^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}-\frac{\mathrm{i}g^3}{\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$+\frac{\mathrm{i}g^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

$$+\mathrm{i}g^3\frac{-(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)+m^2\left(m^2+(p_2+p_3)^2\right)-(p_4+p_5)^2\left(m^2+(p_2+p_3)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$+\mathrm{i}g^3\frac{-m^4+m^2(p_4+p_5)^2-m^2(p_2+p_3)^2+(p_2+p_3)^2(p_4+p_5)^2-m^2(p_5+p_4)^2+(p_4+p_5)^4}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

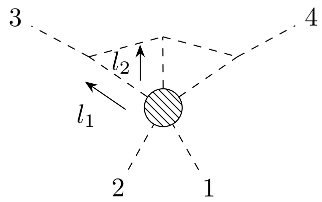
$$+\mathrm{i}g^3\frac{-(p_4+p_5)^2m^2-(p_4+p_5)^4+m^4+m^2(p_2+p_3)^2-(p_4+p_5)^2m^2-(p_2+p_3)^2(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$- 2ig^3m^2 \frac{1}{\left(m^2 + (p_2+p_3)^2\right)\left(m^2+(p_4+p_5)^2\right)} + (3 \leftrightarrow 4,5)$$

Now, let's do the cuts, consider a two loop four point amplitude with five cuts,



$$= -\frac{g^3}{m^4}[\mathcal{A}_5(1,2,l_{1h},l_{2h},l_{3h}) - \mathcal{A}_5(1,2,l_{1h},l_{2\eta},l_{3\eta}) - \mathcal{A}_5(1,2,l_{1\eta},l_{2h},l_{3\eta}) - \mathcal{A}_5(1,2,l_{1\eta},l_{2\eta},l_{3h}) + 2\mathcal{A}_5(1,2,l_{1\eta}$$

2. Cut solutions

Of course in each amplitude we have different cut solutions. Now let us solve them,

2.1. all massless. The cut condition is,

$$k_1^2 = k_2^2 = (3-k_1)^2 = (3-k_1-k_2)^2 = (3+4-k_1-k_2)^2 = 0$$

The first and third condition enforces $k_1 = -|k_1]\langle 3|$. But the fourth and fifth conditions enforces $3-k_1-k_2 = n$, with $n\cdot 4 = 0$ & $n^2 = 0$. Lastly, the second condition imposes $(3-k_1-n)^2 = -23\cdot n + 2k_1\cdot n = 0$, that is,

$$[3n]\langle n3\rangle = [k_1 n]\langle n3\rangle$$

which has two solutions, $|n] = |k_1] - |3]$ & $|n\rangle = z|4\rangle$ or $|n\rangle = |3\rangle$ & $|n] = z|4]$. When working with scalar particles it's better to choose the first solution, as this avoids singularities in denominators such as $(k_1\cdot k_2)^{-1}$. Hence, the solution we're going to choose is,

$$\begin{cases} k_1 &= -|k_1]\langle 3| \\ k_2 &= -|3]\langle 3| + |k_1]\langle 3| + z(|k_1] - |3])\langle 4| \end{cases}$$

2.2. massive legs first topology. Our approach to massive legs is to shift the solution with massless, in order to obtain a well behaved solution in the $m^2 \to 0$ limit. For this topology the cut constrains are,

$$l_1^2 = l_2^2 = (3-l_1)^2 = -m^2 \ \& \ (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = 0$$

The ideia here is to define, $l_i = k_i + \alpha_i q_i$ (no sum), with $q_i^2 = 0$ and $\alpha_i = -m^2(2k_i\cdot q_i)^{-1}$, then, q_i, k_i are not allowed to have any dependence on m^2. The first and second constrains are already satisfied. The third one gives,

$$-23\cdot l_1 = 0 \to 3\cdot(k_1+\alpha_1 q_1) = 0 \to 3\cdot q_1 = 0$$

As $|q_1\rangle = |3\rangle$ is forbidden, $|q_1] = |3]$. The fourth and fifth constrains imposes,

$$\begin{cases} -n\cdot(\alpha_1 q_1 + \alpha_2 q_2) + \alpha_1\alpha_2 q_1\cdot q_2 &= 0 \\ 4\cdot(\alpha_1 q_1 + \alpha_2 q_2) &= 0 \end{cases}$$

This imposes actually $q_1\cdot q_2 = 0$, for this to be true we have to options, either $|q_2] = |3]$, or $|q_2\rangle = |q_1\rangle$. If we choose the first, we can shift k_1 by 3 such to make $|q_1\rangle = |4\rangle$, this imposes further $|q_2\rangle = |4\rangle$. Hence, a possible solution is,

$$q_1 = q_2 = -|3]\langle 4|$$

2.3. massive legs second topology. The constrains now are slightly different,

$$l_1^2 = (3-l_1)^2 = 0 \ \& \ l_2^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

Which has as solution $q_2 = -|4]\langle 3|$

2.4. massive legs third topology. Now the constrain is difficult to solve,

$$l_2^2 = 0 \ \& \ l_1^2 = (3-l_1)^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

The second and third constrains give, $l_1 = -|k_1]\langle 3| - \alpha|3]\langle l_1|$. Now, the fourth and fifth constrains gives,

$$3 - l_1 - l_2 = -z(|k_1] - |3])\langle 4| + \beta|4]\langle n|$$

With of course $\beta = -\frac{m^2}{z\langle 4n\rangle[4|(|k_1]-|3])}$. At last the second constrain gives,

$$\begin{aligned}
l_2 &= -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n| \\
l_2^2 = 0 &= -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1] - |3]) \\
0 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\
-2m^2 &= \beta\langle 3n\rangle[4|(|3] - |k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\
-2m^2 &= -\frac{m^2}{z\langle 4n\rangle[4|(|k_1] - |3])}\langle 3n\rangle[4|(|3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]} z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43]
\end{aligned}$$

For the quadratic term in m^2 to vanish is necessary $\langle l_1 n\rangle = 0 \to |l_1\rangle \propto |n\rangle$, thus,

$$\begin{aligned}-2m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle} z\langle 4l_1\rangle \\ -2 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle} z\langle 4n\rangle \to \langle 3n\rangle = -z\langle 4n\rangle\end{aligned}$$

The best parametrization is $|n\rangle = |l_1\rangle = |4\rangle - \frac{1}{z}|3\rangle$.

2.5. **massive legs fourth topology.** Now the constrain is the hardest to solve,

$$l_1^2 = l_2^2 = (3-l_1)^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

Happily, most of the work was done already in the last solution, we just have to change:

$$l_2 = -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n|$$

$$l_2^2 = -m^2 \to m^2 = -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1] - |3])$$

$$\begin{aligned}m^2 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\ -m^2 &= \beta\langle 3n\rangle[4|(|3] - |k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\ -m^2 &= -\frac{m^2}{z\langle 4n\rangle[4|(|k_1] - |3])}\langle 3n\rangle[4|(|3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]} z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43]\end{aligned}$$

Again we fix $|l_1\rangle \propto |n\rangle$. The solution then is given by,

$$\begin{aligned}-m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle} z\langle 4l_1\rangle \\ -1 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle} z\langle 4n\rangle \to \langle 3n\rangle = -z\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\right)\langle 4n\rangle = -z\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\langle 4n\rangle\end{aligned}$$

3. Amplitudes evaluated in the cuts

3.1. **all massless.** The expression for the amplitude is,

$$\begin{aligned}&= \frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(l_1+l_2)^2} + \frac{1}{(l_1+l_3)^2} + \frac{1}{(l_3+l_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_1+l_1)^2}\left[\frac{1}{(p_2+l_2)^2} + \frac{1}{(p_2+l_3)^2} + \frac{1}{(l_3+l_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_1+l_2)^2}\left[\frac{1}{(l_1+p_2)^2} + \frac{1}{(l_1+l_3)^2} + \frac{1}{(l_3+p_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_1+l_3)^2}\left[\frac{1}{(l_1+l_2)^2} + \frac{1}{(l_1+p_2)^2} + \frac{1}{(p_2+l_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_2+l_1)^2(l_2+l_3)^2} + \frac{\mathrm{i}g^3m^2}{(p_2+l_2)^2(l_1+l_3)^2} + \frac{\mathrm{i}g^3m^2}{(p_2+l_3)^2(l_2+l_1)^2} \\ &= \frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{z\langle 34\rangle[k_1 3]} + \frac{1}{\langle 34\rangle(z[k_1 3] + [k_1 4])} + \frac{1}{(l_3+l_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_1+l_1)^2}\left[\frac{1}{(p_2+l_2)^2} + \frac{1}{(p_2+l_3)^2} + \frac{1}{(l_3+l_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_1+l_2)^2}\left[\frac{1}{(l_1+p_2)^2} + \frac{1}{(l_1+l_3)^2} + \frac{1}{(l_3+p_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_1+l_3)^2}\left[\frac{1}{(l_1+l_2)^2} + \frac{1}{(l_1+p_2)^2} + \frac{1}{(p_2+l_2)^2}\right] \\ &\quad + \frac{\mathrm{i}g^3m^2}{(p_2+l_1)^2(l_2+l_3)^2} + \frac{\mathrm{i}g^3m^2}{(p_2+l_2)^2(l_1+l_3)^2} + \frac{\mathrm{i}g^3m^2}{(p_2+l_3)^2(l_2+l_1)^2}\end{aligned}$$