Homework II

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Contents

Problem 1

1.A)

Let M be our $D > 2$ dimensional C^∞ manifold and $\phi : \mathbb{R} \times M \to M$ be a one parameter family of diffeomorphisms, that is, $\forall t \in \mathbb{R}|\ \phi_t : M \to M$ is a diffeomorphism such that,

- $\forall p \in M|\ \phi_0(p) = p$
- $\forall p \in M; \forall t, s \in \mathbb{R}|\ \phi_{t+s}(p) = (\phi_t \circ \phi_s)(p)$
- $\forall p \in M|\ \phi(p) : \mathbb{R} \to M$ is at least C^1

Then this family of diffeomorphisms define in a natural manner a vector which generate these transformations, we define at each point $p \in M$ this vector by it's action in a function $f : M \to \mathbb{R}$,

$$\xi_p(f) = \frac{\mathrm{d}}{\mathrm{d}t}((f \circ \phi_t)(p))\bigg|_{t=0}$$

And from this we define ξ as a vector field in M, this vector field has as integral curves exactly ϕ. Let's open a little bit more in some chart $x : M \to \mathbb{R}^D$,

$$\begin{aligned}
\xi_p(f) &= \frac{\mathrm{d}}{\mathrm{d}t}\big((f \circ x^{-1} \circ x \circ \phi_t)(p)\big)\bigg|_{t=0} \\
\xi_p(f) &= \frac{\mathrm{d}}{\mathrm{d}t}\big((f \circ x^{-1}) \circ (x \circ \phi_t)(p)\big)\bigg|_{t=0} \\
\xi_p(f) &= \partial_\mu\big(f \circ x^{-1}\big)\bigg|_{x\circ\phi_0(p)} \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(p))\bigg|_{t=0} \\
\xi_p^\mu \partial_\mu\big(f \circ x^{-1}\big)\bigg|_{x(p)} &= \partial_\mu\big(f \circ x^{-1}\big)\bigg|_{x(p)} \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(p))\bigg|_{t=0}
\end{aligned}$$

Where of curse ∂_μ is to be interpreted as the derivative of the μ-th component in the chart x. Here we have a clear definition of the values of the ξ vector field in a chart x,

$$\xi_p^\mu = \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(p))\bigg|_{t=0}$$

The term inside the derivative is just the pullback of the chart x — the chart can be seen as a $\mathbb{R}^D$-valued function —, which in it's own can be seen as a new chart x'_t defined by the transformations of the diffeomorphism family ϕ, that is,

$$x'_t = \phi_t^* x = x \circ \phi_t : M \to \mathbb{R}^D$$

All of this is consistent with our interpretation of the diffeomorphisms being a '*coordinate change*', in principle, with enough derivability of ϕ we can actually write,

$$\begin{aligned}
x'_t &= x'_0 + t\frac{\mathrm{d}}{\mathrm{d}t}(x'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big) \\
{x'_1}^\mu =: x'^\mu &= x^\mu + \xi^\mu + \cdots
\end{aligned}$$

We just restored the index to not confuse the components of the vector field ξ in the basis x with the vector field itself. That is, we showed that the transformation done by ϕ_1 is equivalent to a '*infinitesimal coordinate change*' by ξ^μ. Actually, all this we did is the special case of a more general type of derivative, the Lie Derivative, given a vector field ξ and it's family of integral curves ϕ, it's defined in terms of the pushforward of the object under analysis,

$$\pounds_\xi T = \frac{\mathrm{d}}{\mathrm{d}t}(\phi_{-t*}T)\bigg|_{t=0}$$

For a $(0,2)$ tensor, that is, for the metric,

$$\pounds_\xi g_{\mu\nu} = \nabla_\xi g_{\mu\nu} + g_{\mu\alpha}\nabla_\nu \xi^\alpha + g_{\alpha\nu}\nabla_\mu \xi^\alpha$$

And as the connection is metric compatible,

$$\pounds_\xi g_{\mu\nu} = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 2\nabla_{(\mu}\xi_{\nu)}$$

This amounts for the first term in an expansion in t of the diffeomorphism transformed metric $\phi_{-t*}g = g'_{t\mu\nu}$, that is,

$$\begin{aligned}\phi_{-t*}g_{\mu\nu} =: g'_{t\mu\nu} &= g_{\mu\nu} + t\pounds_\xi g_{\mu\nu} + \mathcal{O}\big(t^2\big)\\ g'_{t\mu\nu} &= g_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big)\end{aligned}$$

Which also can be interpreted as the '*infinitesimal transformation*' of the metric. Imposing that the initial and transformed metric are conformally flat,

$$\begin{aligned}\exp{(2\omega'_t)}\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big)\\ \exp\left(2\omega'_0 + 2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big)\right)\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big)\\ \exp{(2\omega)}\exp\left(2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big)\right)\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big)\\ \exp{(2\omega)}\left(1 + 2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} + \mathcal{O}\big(t^2\big)\right)\eta_{\mu\nu} &= \exp{(2\omega)}\eta_{\mu\nu} + 2t\nabla_{(\mu}\xi_{\nu)} + \mathcal{O}\big(t^2\big)\\ \exp{(2\omega)}2t\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0}\eta_{\mu\nu} &= 2t\nabla_{(\mu}\xi_{\nu)}\\ \frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} g_{\mu\nu} &= \nabla_{(\mu}\xi_{\nu)}\end{aligned}$$

The term $\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\big|_{t=0}$ is fully determined by ξ, to see this just contract both sides with the metric,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} g^{\mu\nu}g_{\mu\nu} &= g^{\mu\nu}\nabla_{(\mu}\xi_{\nu)}\\ \frac{\mathrm{d}}{\mathrm{d}t}(\omega'_t)\bigg|_{t=0} D &= \nabla_\mu \xi^\mu\end{aligned}$$

Substituting this back in the original equation,

$$\nabla_\alpha \xi^\alpha g_{\mu\nu} = D\nabla_{(\mu}\xi_{\nu)} \tag{1.1}$$

This is the condition upon ξ that ensures the diffeomorphism maintain the conformally flatness of the metric. To solve it in it's full generality is hard, so, we'll make some use of symmetry, first, let's write it again in a coordinate free form,

$$\mathrm{Div}_g(\xi) g = \frac{D}{2} \pounds_\xi g \tag{1.2}$$

Where $\mathrm{Div}_g(\xi)$ is just $\nabla_\alpha \xi^\alpha$, but with the connection ∇ defined with respect with the metric g, now, suppose we have a function $f : M \to \mathbb{R}$, define $\tilde{g} = \exp(2f) g$, thus,

$$\begin{aligned}
\pounds_\xi \tilde{g} &= \pounds_\xi(\exp(2f) g) \\
\pounds_\xi \tilde{g} &= \exp(2f)\pounds_\xi g + g\pounds_\xi \exp(2f) \\
\pounds_\xi \tilde{g} &= \exp(2f)\pounds_\xi g + g\xi(\exp(2f)) \\
\pounds_\xi \tilde{g} &= \exp(2f)\pounds_\xi g + g\xi(f)\exp(2f) \\
\pounds_\xi \tilde{g} &= \exp(2f)\pounds_\xi g + \tilde{g}\xi(2f)
\end{aligned} \tag{1.3}$$

Also,

$$\begin{aligned}
\mathrm{Div}_g(\xi) &= \partial_\alpha \xi^\alpha + \Gamma^\alpha_{\alpha\lambda}\xi^\lambda \\
\mathrm{Div}_g(\xi) &= \partial_\alpha \xi^\alpha + \frac{1}{2} g^{\alpha\beta}(\partial_\alpha g_{\beta\lambda} + \partial_\lambda g_{\beta\alpha} - \partial_\beta g_{\alpha\lambda})\xi^\lambda \\
\mathrm{Div}_g(\xi) &= \partial_\alpha \xi^\alpha + g^{\alpha\beta}(\partial_\alpha \omega g_{\beta\lambda} + \partial_\lambda \omega g_{\beta\alpha} - \partial_\beta \omega g_{\alpha\lambda})\xi^\lambda \\
\mathrm{Div}_g(\xi) &= \partial_\alpha \xi^\alpha + (\partial_\lambda \omega + D\partial_\lambda \omega - \partial_\lambda \omega)\xi^\lambda \\
\mathrm{Div}_g(\xi) &= \partial_\alpha \xi^\alpha + D\partial_\lambda \omega \xi^\lambda
\end{aligned}$$

And as $\tilde{g} = \exp(2f) g = \exp(2(f + w))\eta$,

$$\begin{aligned}
\mathrm{Div}_{\tilde{g}}(\xi) &= \partial_\alpha \xi^\alpha + D\partial_\lambda (f + \omega)\xi^\lambda \\
\mathrm{Div}_{\tilde{g}}(\xi) &= \partial_\alpha \xi^\alpha + D\partial_\lambda \omega \xi^\lambda + D\xi^\lambda \partial_\lambda f \\
\mathrm{Div}_{\tilde{g}}(\xi) &= \mathrm{Div}_g(\xi) + D\xi(f)
\end{aligned} \tag{1.4}$$

Multiplying 1.4 by g and subtracting 1.3,

$$\begin{aligned}
\mathrm{Div}_{\tilde{g}}(\xi)\tilde{g} - \frac{D}{2}\pounds_\xi \tilde{g} &= \mathrm{Div}_g(\xi)\tilde{g} + D\tilde{g}\xi(f) - \frac{D}{2}\exp(2f)\pounds_\xi g - \frac{D}{2}\tilde{g}\xi(2f) \\
\mathrm{Div}_{\tilde{g}}(\xi)\tilde{g} - \frac{D}{2}\pounds_\xi \tilde{g} &= \exp(2f)\mathrm{Div}_g(\xi) g - \exp(2f)\frac{D}{2}\pounds_\xi g + D\tilde{g}\xi(f) - D\tilde{g}\xi(f) \\
\mathrm{Div}_{\tilde{g}}(\xi)\tilde{g} - \frac{D}{2}\pounds_\xi \tilde{g} &= \exp(2f)\left[\mathrm{Div}_g(\xi) g - \frac{D}{2}\pounds_\xi g\right]
\end{aligned}$$

That is, as long as f is sufficiently well behaved, we have,

$$\mathrm{Div}_g(\xi)g = \frac{D}{2}\pounds_\xi g \Leftrightarrow \mathrm{Div}_{\tilde{g}}(\xi)\tilde{g} = \frac{D}{2}\pounds_\xi \tilde{g} \tag{1.5}$$

In other words, the vector field ξ which generates diffeomorphisms that preserves the conformally flat condition, does not depend on ω of our conformally flat metric $g = \exp(2\omega)\eta$, so, we can choose $f = -\omega$, such that $\tilde{g} = \eta$, and by 1.5, we just have to solve for,

$$\partial_\alpha \xi^\alpha \eta_{\mu\nu} = D\partial_{(\mu}\xi_{\nu)} \tag{1.6}$$

Which is a lot easier then 1.1, first, we apply ∂^ν to the both sides, relabel the index, apply ∂_ν, symmetrize $\mu \leftrightarrow \nu$ and use 1.6,

$$\begin{aligned}
\partial^\nu \partial_\alpha \xi^\alpha \eta_{\mu\nu} &= D\partial^\nu \partial_{(\mu}\xi_{\nu)} \\
\frac{2}{D}\partial_\mu \partial_\alpha \xi^\alpha &= \partial^\nu \partial_\mu \xi_\nu + \partial^\nu \partial_\nu \xi_\mu \\
\frac{2}{D}\partial_\mu \partial_\alpha \xi^\alpha &= \partial^\alpha \partial_\mu \xi_\alpha + \partial^\alpha \partial_\alpha \xi_\mu \\
\frac{2}{D}\partial_\nu \partial_\mu \partial_\alpha \xi^\alpha &= \partial_\nu \partial^\alpha \partial_\mu \xi_\alpha + \partial_\nu \partial^\alpha \partial_\alpha \xi_\mu \\
\frac{2}{D}\partial_{(\nu}\partial_{\mu)} \partial_\alpha \xi^\alpha &= \partial_{(\nu}\partial_{\mu)} \partial^\alpha \xi_\alpha + \partial^\alpha \partial_\alpha \partial_{(\nu}\xi_{\mu)} \\
\frac{2}{D}\partial_\nu \partial_\mu \partial_\alpha \xi^\alpha &= \partial_\nu \partial_\mu \partial_\alpha \xi^\alpha + \frac{\eta_{\mu\nu}}{D}\partial^\alpha \partial_\alpha \partial_\beta \xi^\beta \\
(2-D)\partial_\nu \partial_\mu \partial_\alpha \xi^\alpha &= \eta_{\mu\nu}\partial^\alpha \partial_\alpha \partial_\beta \xi^\beta
\end{aligned} \tag{1.7}$$

Contracting with $\eta^{\mu\nu}$ gives,

$$\begin{aligned}
(2-D)\partial^\mu \partial_\mu \partial_\alpha \xi^\alpha &= D\partial^\alpha \partial_\alpha \partial_\beta \xi^\beta \\
2(1-D)\partial^\mu \partial_\mu \partial_\alpha \xi^\alpha &= 0 \\
\partial^\mu \partial_\mu \partial_\alpha \xi^\alpha &= 0
\end{aligned} \tag{1.8}$$

The last step is justified by merely $D \geq 2$, so that up to now, we haven't fully used the hypothesis of being in $D > 2$. So, we'll invoke it now, by using 1.8 in equation 1.7, this is only justified if $D \neq 2$, because, for $D = 2$ the left-hand side of 1.7 is identically zero, but, for $D > 2$ the use of 1.8 in 1.7 results in another constraint,

$$\begin{aligned}
(2-D)\partial_\mu \partial_\nu \partial_\alpha \xi^\alpha &= 0 \\
\partial_\mu \partial_\nu \partial_\alpha \xi^\alpha &= 0
\end{aligned} \tag{1.9}$$

It's clear that equation 1.7 allows for much more solutions than 1.9, for example, $\partial \cdot \xi = \cos(k_\mu x^\mu)$ with $k_\mu k^\mu = 0$ is a solution for 1.8, but, isn't for 1.9. To integrate 1.9 is also a lot easier than 1.8,

$$\begin{aligned}
\partial_\mu \partial_\nu \partial_\alpha \xi^\alpha &= 0 \\
\partial_\nu \partial_\alpha \xi^\alpha &= a_\nu
\end{aligned}$$

$$\partial_\alpha \xi^\alpha = a_\nu x^\nu + b \tag{1.10}$$

With a_ν, b arbitrary constants, this is also automatically a solution of 1.8. To integrate 1.10 is not trivial, as there might be some divergenceless current contributing to ξ^α, that is, the integration gives,

$$\begin{aligned}\partial_\alpha \xi^\alpha &= a_\nu x^\nu + b \\ \xi^\alpha &= \frac{1}{2} a^\alpha{}_{\mu\nu} x^\mu x^\nu + b^\alpha{}_\mu x^\mu + c^\alpha + f^\alpha \end{aligned} \tag{1.11}$$

With of course $a^\alpha{}_{\mu\nu} = a^\alpha{}_{(\mu\nu)}, a^\alpha{}_{\alpha\nu} = a_\nu, b^\alpha{}_\alpha = b, c^\alpha$ being constants, with exception of f^α, which is some non constant divergenceless vector field $\partial_\alpha f^\alpha = 0$. To know what choice of f^α is the correct one we have to go back to equation 1.6, and apply to it ∂_λ,

$$\frac{2}{D} \partial_\lambda \partial_\alpha \xi^\alpha \eta_{\mu\nu} = \partial_\lambda \partial_\mu \xi_\nu + \partial_\lambda \partial_\nu \xi_\mu$$

Now we permute the index,

$$\begin{aligned}\frac{2}{D} \partial_\lambda \partial_\alpha \xi^\alpha \eta_{\mu\nu} &= \partial_\lambda \partial_\mu \xi_\nu + \partial_\nu \partial_\lambda \xi_\mu \\ \frac{2}{D} \partial_\mu \partial_\alpha \xi^\alpha \eta_{\nu\lambda} &= \partial_\mu \partial_\nu \xi_\lambda + \partial_\lambda \partial_\mu \xi_\nu \\ \frac{2}{D} \partial_\nu \partial_\alpha \xi^\alpha \eta_{\lambda\mu} &= \partial_\nu \partial_\lambda \xi_\mu + \partial_\mu \partial_\nu \xi_\lambda \end{aligned}$$

Sum the two first equations and subtract the third one,

$$\frac{2}{D}(\partial_\lambda \partial_\alpha \xi^\alpha \eta_{\mu\nu} + \partial_\mu \partial_\alpha \xi^\alpha \eta_{\nu\lambda} - \partial_\nu \partial_\alpha \xi^\alpha \eta_{\lambda\mu}) = \partial_\lambda \partial_\mu \xi_\nu + \partial_\nu \partial_\lambda \xi_\mu + \partial_\mu \partial_\nu \xi_\lambda + \partial_\lambda \partial_\mu \xi_\nu - \partial_\nu \partial_\lambda \xi_\mu - \partial_\mu \partial_\nu \xi_\lambda$$

By use of 1.10,

$$\frac{2}{D}(a_\lambda \eta_{\mu\nu} + a_\mu \eta_{\nu\lambda} - a_\nu \eta_{\lambda\mu}) = 2\partial_\lambda \partial_\mu \xi_\nu$$

And substituting 1.11 in the right-hand side,

$$\frac{2}{D}(a_\lambda \eta_{\mu\nu} + a_\mu \eta_{\nu\lambda} - a_\nu \eta_{\lambda\mu}) = 2a_{\nu\mu\lambda} + 2\partial_\lambda \partial_\mu f_\nu$$

We can thus use this to fix f_λ, it's trivial to carry out the integration,

$$\begin{aligned}\partial_\lambda \partial_\mu f_\nu &= \frac{1}{D}(a_\lambda \eta_{\mu\nu} + a_\mu \eta_{\nu\lambda} - a_\nu \eta_{\lambda\mu}) - a_{\nu\mu\lambda} \\ \partial_\mu f_\nu &= \frac{1}{D}(a \cdot x \eta_{\mu\nu} + a_\mu x_\nu - a_\nu x_\mu) - a_{\nu\mu\lambda} x^\lambda + A_{\nu\mu} \\ f_\nu &= \frac{1}{D}\left(a \cdot x x_\nu - \frac{1}{2} a_\nu x \cdot x\right) - \frac{1}{2} a_{\nu\mu\lambda} x^\lambda x^\mu + A_{\nu\mu} x^\mu + B_\nu \\ \partial_\mu f^\mu &= \frac{1}{D}(a \cdot x D + a \cdot x - a \cdot x) - a \cdot x + A^\mu{}_\mu = A^\mu{}_\mu \end{aligned} \tag{1.12}$$

Where we just computed also the divergence, which implies the constraint $A^{\mu}{}_{\mu} = 0$. Substituting this back in 1.11 get us,

$$\xi^{\alpha} = \frac{1}{2}a^{\alpha}{}_{\mu\nu}x^{\mu}x^{\nu} + b^{\alpha}{}_{\mu}x^{\mu} + c^{\alpha} + \frac{1}{D}\left(a\cdot x x^{\alpha} - \frac{1}{2}a^{\alpha}x\cdot x\right) - \frac{1}{2}a^{\alpha}{}_{\mu\lambda}x^{\lambda}x^{\mu} + A^{\alpha}{}_{\mu}x^{\mu} + B^{\alpha}$$
$$\xi^{\alpha} = \frac{1}{D}\left(a\cdot x x^{\alpha} - \frac{1}{2}a^{\alpha}x\cdot x\right) + \left(A^{\alpha}{}_{\mu} + b^{\alpha}{}_{\mu}\right)x^{\mu} + (B^{\alpha} + c^{\alpha})$$
$$\xi^{\alpha} = \frac{1}{D}\left(a\cdot x x^{\alpha} - \frac{1}{2}a^{\alpha}x\cdot x\right) + b^{\alpha}{}_{\mu}x^{\mu} + c^{\alpha}$$

In the last line we just redefined the tensors. All of this with $a^{\alpha}, b^{\alpha}{}_{\mu}, c^{\alpha}$ arbitrary constants, but, we didn't really confirmed this is solution of 1.9 is a fully compatible with 1.6, we just checked it satisfy some derived equations from 1.6, now, let's put it to the real test,

$$\begin{aligned}
\partial_{\alpha}\xi^{\alpha}\eta_{\mu\nu} &= D\partial_{(\mu}\xi_{\nu)} \\
\left[\frac{1}{D}\left(a\cdot xD + a\cdot x - 2\frac{1}{2}a\cdot x\right) + b^{\alpha}{}_{\alpha}\right]\eta_{\mu\nu} &= D\left[\frac{1}{D}\left(a\cdot x\eta_{\mu\nu} + a_{(\mu}x_{\nu)} - a_{(\nu}x_{\mu)}\right) + b_{(\nu\mu)}\right] \\
a\cdot x\eta_{\mu\nu} + b^{\alpha}{}_{\alpha}\eta_{\mu\nu} &= a\cdot x\eta_{\mu\nu} + Db_{(\nu\mu)} \\
b^{\alpha}{}_{\alpha}\eta_{\mu\nu} &= Db_{(\nu\mu)} \\
b_{(\nu\mu)} &= \frac{1}{D}b^{\alpha}{}_{\alpha}\eta_{\mu\nu}
\end{aligned}$$

This constrains the symmetric part of $b_{\mu\nu}$ being a pure trace, so that the degrees of freedom reduces from $b_{\mu\nu}$ to $b_{[\mu\nu]}, b^{\alpha}{}_{\alpha}$. This is not the end! We haven't show all the solutions we found are in fact all the solutions from 1.6, as we obtained them from a different approach than direct integration of the equation 1.6, so let's show this, first, 1.6 is a set of first order partial differential equations, there are D^2 such equation, but, they're symmetric in exchange $\mu \leftrightarrow \nu$, so we have to account only for $\frac{D^2+D}{2}$ of them, still, we have to account for the possible constants which can be added to ξ without changing the equation, D of them, and also for another boundary condition on the divergence, 1, so that the full number of constants is,

$$\frac{D^2 + D}{2} + D + 1 = \frac{1}{2}(D+1)(D+2)$$

Now let's count the number of constants in the solution we found, namely,

$$\xi_{\alpha} = \frac{1}{D}\left(a\cdot x x_{\alpha} - \frac{1}{2}a_{\alpha}x\cdot x\right) + b_{[\alpha\mu]}x^{\mu} + \frac{1}{D}b^{\mu}{}_{\mu}x_{\alpha} + c_{\alpha}$$

Both c^{α}, a^{α} contributes with D, $b^{\mu}{}_{\mu}$ contributes with 1, and lastly $b_{[\alpha\mu]}$ contributes with $\frac{D^2-D}{2}$, hence, the total number of constants is,

$$\frac{D^2 - D}{2} + 2D + 1 = \frac{1}{2}(D+1)(D+2)$$

Matching exactly the number from the original equation, that is, we already got all the solutions, hence, the most general solution of 1.6, for $D > 2$, is given by, just for better reading we make a change of variables that does not affect at all the number of free parameters,

$$\xi_\alpha = 2a \cdot x x_\alpha - a_\alpha x \cdot x + b_{[\alpha\mu]} x^\mu + x_\alpha d + c_\alpha$$

1.B)

We have already counted for the number of independent parameters in the last item as an way of making sure we got all the solutions to the equation, so now it remains to get what is the geometric picture of the transformations ϕ generated by this ξ, first, notice that we have four different kinds of terms in our expression for ξ,

$$\xi_\alpha = 2a \cdot x x_\alpha - a_\alpha x \cdot x + b_{[\alpha\mu]} x^\mu + x_\alpha d + c_\alpha$$

A quadratic, a two linear and a constant in x. We'll treat one by one, first, let us discuss what is to get ϕ from ξ. As we pointed before,

$$\xi^\mu_p = \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(p))\bigg|_{t=0}$$

We can deform this definition a little by,

$$\begin{aligned}
\xi^\mu_{\phi_s(p)}\bigg|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_t)^\mu(\phi_s(p)))\bigg|_{t=0}\bigg|_{s=0} \\
\xi^\mu_{\phi_s(p)} &= \frac{\mathrm{d}}{\mathrm{d}t}((x \circ \phi_{t+s})^\mu(p))\bigg|_{t=0} \\
\xi^\mu_{\phi_s(p)} &= \frac{\mathrm{d}}{\mathrm{d}(t+s)}((x \circ \phi_{t+s})^\mu(p))\bigg|_{t+s=s} \frac{\mathrm{d}(t+s)}{\mathrm{d}t}\bigg|_{t=0} \\
\xi^\mu_{\phi_s(p)} &= \frac{\mathrm{d}}{\mathrm{d}s}((x \circ \phi_s)^\mu(p))
\end{aligned}$$

Seeing $\xi_{\phi_s(p)}$ as a function of the coordinates of the point $\phi_s(p)$,

$$\begin{aligned}
\xi^\mu((x \circ \phi_s)(p)) &= \frac{\mathrm{d}}{\mathrm{d}s}((x \circ \phi_s)^\mu(p)) \\
\xi^\mu(x \circ \phi_s) &= \frac{\mathrm{d}}{\mathrm{d}s}(x \circ \phi_s)^\mu \\
\xi^\mu(x'_s) &= \frac{\mathrm{d}}{\mathrm{d}s}(x'_s)^\mu
\end{aligned}$$

That is, to solve for ϕ, we just have to solve this first order differential equation for the coordinates of $\phi_s(p)$, and as we had already defined, $x \circ \phi_s(p) := x'_s$, so that, we have to solve for,

$$\frac{\mathrm{d}}{\mathrm{d}s} x'_{s\alpha} = \dot{x}'_{s\alpha} = 2a \cdot x'_s x'_{s\alpha} - a_\alpha x'_s \cdot x'_s + b_{[\alpha\mu]} x'^\mu_s + x'_{s\alpha} d + c_\alpha$$

In this form it's obvious that it is too hard to solve, so we rather solve for each parameter individually, starting by,

- c^{α}

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} &= c_{\alpha} \\
x'_{s\alpha} &= sc_{\alpha} + x'_{0\alpha} \\
x'_{s\alpha} &= sc_{\alpha} + x_{\alpha}
\end{aligned}$$

In other words, the transformation parametrized by c^{α} describe a translation of the coordinate x^{α},

- d

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} &= x'_{s\alpha}d \\
x'_{s\alpha} &= A_{\alpha}\exp\left(sd\right)
\end{aligned}$$

Imposing $x'_{0\alpha} = x_{\alpha}$,

$$x'_{s\alpha} = x_{\alpha}\exp\left(sd\right)$$

Which accounts for dilatations — rescaling — of the coordinate x_{α},

- $b_{[\alpha\mu]}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} &= b_{[\alpha\mu]}x'^{\mu}_{s} \\
x'_{s\alpha} &= \exp\left(s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)_{\alpha\mu}x^{\mu}
\end{aligned}$$

This solution may seem taken directly from a hat, but, let's record that this equation we're solving is a set of first order linear ordinary differential equations, and the solution to one of the kind $\dot{\mathbf{x}}(s) = \mathbf{A}\cdot\mathbf{x}(s)$ is always of the form $\mathbf{x}(s) = \exp\left(s\mathbf{A}\right)\mathbf{x}(0)$, which is exactly what we wrote, apart from introducing a new tensor M and a $\frac{1}{2}$ factor, why you may ask? Let us be honest, we all know what is happening here... But, we have explicit written just two of the four index in M, the other two stands outside the exponential, M is not at all a free parameter, it has to satisfy the consistency condition,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} &= \frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}{}_{\alpha}{}^{\nu}\exp\left(s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)_{\nu\mu}x^{\mu} \\
\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} &= \frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}{}_{\alpha}{}^{\nu}x'_{s\nu} \\
\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} &= \frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}{}_{\alpha\nu}x'^{\nu}_{s} \Rightarrow \frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}{}_{\alpha\nu} = b_{[\alpha\nu]}
\end{aligned}$$

This is easy to solve,

$$
\begin{aligned}
\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}{}_{\alpha\nu} &= b_{[\alpha\nu]} \\
\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}{}_{\alpha\nu} &= \frac{1}{2}b_{[\rho\sigma]}(\eta_\alpha{}^\rho\eta_\nu{}^\sigma - \eta_\nu{}^\rho\eta_\alpha{}^\sigma) \\
M^{[\rho\sigma]}{}_{[\alpha\nu]} &= (\eta_\alpha{}^\rho\eta_\nu{}^\sigma - \eta_\nu{}^\rho\eta_\alpha{}^\sigma)
\end{aligned}
$$

It's clear that these transformations are just the usual Lorentz ones, boosts plus rotations, as M is exactly the generators of such transformations — in a real form —, just for completeness, we compute,

$$
\begin{aligned}
x'^{\alpha}_{s}x'_{s\alpha} &= \exp\left(s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)^{[\alpha\nu]} x_\nu \exp\left(s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)_{[\alpha\mu]} x^\mu \\
x'^{\alpha}_{s}x'_{s\alpha} &= \exp\left(-s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)^{\nu][\alpha} x_\nu \exp\left(s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)_{[\alpha\mu]} x^\mu \\
x'^{\alpha}_{s}x'_{s\alpha} &= x_\nu \exp\left(-s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)^{[\nu\alpha]} \exp\left(s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)_{[\alpha\mu]} x^\mu
\end{aligned}
$$

As trivially $b_{[\rho\sigma]}M^{[\rho\sigma]}$ commute with itself — everything commutes with itself, as long as it doesn't have characteristic 2 —, so that BCH gives trivially,

$$
\begin{aligned}
x'^{\alpha}_{s}x'_{s\alpha} &= x_\nu \exp\left(-s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]} + s\frac{1}{2}b_{[\rho\sigma]}M^{[\rho\sigma]}\right)^{\nu}{}_{\mu} x^\mu \\
x'^{\alpha}_{s}x'_{s\alpha} &= x_\nu \exp\left(\mathbf{0}\right)^{\nu}{}_{\mu} x^\mu \\
x'^{\alpha}_{s}x'_{s\alpha} &= x_\nu \eta^{\nu}{}_{\mu} x^\mu \\
x'^{\alpha}_{s}x'_{s\alpha} &= x_\mu x^\mu
\end{aligned}
$$

Just exactly what a Lorentz transformation should do. Then, the $b_{[\alpha\mu]}$ generates the Lorentz transformations.

- a_α

$$
\dot{x}'_{s\alpha} = 2a \cdot x'_s x'_{s\alpha} - a_\alpha x'_s \cdot x'_s \tag{1.13}
$$

This is the last, hardest and most interesting one to solve, we proceed with the usual magic hat tricks and divine inspiration/intervention,

$$
\begin{aligned}
\frac{\dot{x}'_{s\alpha}}{x'_s \cdot x'_s} &= \frac{2(a \cdot x'_s)x'_{s\alpha}}{x'_s \cdot x'_s} - a_\alpha \\
\frac{\dot{x}'_{s\alpha}}{x'_s \cdot x'_s} - \frac{2(a \cdot x'_s)x'_{s\alpha}}{x'_s \cdot x'_s} &= -a_\alpha \\
\frac{\dot{x}'_{s\alpha}}{x'_s \cdot x'_s} - \frac{2(a \cdot x'_s)(x'_s \cdot x'_s)x'_{s\alpha}}{(x'_s \cdot x'_s)^2} &= -a_\alpha
\end{aligned} \tag{1.14}
$$

Now multiply equation 1.13 by x'^{α}_{s}, which gives,

$$
\begin{aligned}
x'_s \cdot \dot{x}'_s &= 2(a \cdot x'_s)(x'_s \cdot x'_s) - (a \cdot x'_s)(x'_s \cdot x'_s) \\
x'_s \cdot \dot{x}'_s &= (a \cdot x'_s)(x'_s \cdot x'_s)
\end{aligned}
$$

Substituting this back in equation 1.14,

$$
\begin{aligned}
\frac{\dot{x}'_{s\alpha}}{x'_s \cdot x'_s} - \frac{2(x'_s \cdot \dot{x}'_s)x'_{s\alpha}}{(x'_s \cdot x'_s)^2} &= -a_\alpha \\
\frac{1}{x'_s \cdot x'_s}\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} - x'_{s\alpha}\frac{1}{(x'_s \cdot x'_s)^2}\frac{\mathrm{d}}{\mathrm{d}s}(x'_s \cdot x'_s) &= -a_\alpha \\
\frac{1}{x'_s \cdot x'_s}\frac{\mathrm{d}}{\mathrm{d}s}x'_{s\alpha} + x'_{s\alpha}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{x'_s \cdot x'_s} &= -a_\alpha \\
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x'_{s\alpha}}{x'_s \cdot x'_s}\right) &= -a_\alpha \\
\frac{x'_{s\alpha}}{x'_s \cdot x'_s} &= -sa_\alpha + \frac{x_\alpha}{x \cdot x}
\end{aligned}
$$

What remains is a mere algebraic equation which is straightforward to solve, first, contract with x'^{α}_s,

$$
\begin{aligned}
\frac{x'_s \cdot x'_s}{x'_s \cdot x'_s} = 1 &= -sx'_s \cdot a + \frac{x'_s \cdot x}{x \cdot x} \\
-sx'_s \cdot a + \frac{x'_s \cdot x}{x \cdot x} &= 1
\end{aligned}
$$

Which gives one constraint, now, contract with $-sa^\alpha + \frac{x^\alpha}{x \cdot x}$,

$$
\frac{1}{x'_s \cdot x'_s}\left(-sx'_s \cdot a + \frac{x'_s \cdot x}{x \cdot x}\right) = \left(-sa^\alpha + \frac{x^\alpha}{x \cdot x}\right)\left(-sa_\alpha + \frac{x_\alpha}{x \cdot x}\right)
$$

Using the constraint derived,

$$
\frac{1}{x'_s \cdot x'_s} = \left(-sa^\alpha + \frac{x^\alpha}{x \cdot x}\right)\left(-sa_\alpha + \frac{x_\alpha}{x \cdot x}\right)
$$

Which gives the final constraint, substituting this back on our original equation gives,

$$
\frac{x'_{s\alpha}}{x'_s \cdot x'_s} = -sa_\alpha + \frac{x_\alpha}{x \cdot x} \tag{1.15}
$$

$$
\begin{aligned}
x'_{s\alpha}\left(-sa^\alpha + \frac{x^\alpha}{x \cdot x}\right)\left(-sa_\alpha + \frac{x_\alpha}{x \cdot x}\right) &= -sa_\alpha + \frac{x_\alpha}{x \cdot x} \\
x'_{s\alpha} &= \frac{-sa_\alpha + \frac{x_\alpha}{x \cdot x}}{\left(-sa^\alpha + \frac{x^\alpha}{x \cdot x}\right)\left(-sa_\alpha + \frac{x_\alpha}{x \cdot x}\right)} \\
x'_{s\alpha} &= \frac{-sa_\alpha(x \cdot x) + x_\alpha}{(-sa^\alpha(x \cdot x) + x^\alpha)\left(-sa_\alpha + \frac{x_\alpha}{x \cdot x}\right)}
\end{aligned}
$$

$$
x'_{s\alpha} = \frac{x_\alpha - sa_\alpha(x \cdot x)}{1 - 2sa \cdot x + s^2 a \cdot a(x \cdot x)} \tag{1.16}
$$

These are the special conformal transformations, from 1.15 it's easy to interpret geometrically what they do. First, apply a inversion, $x^\mu \to \frac{x^\mu}{x \cdot x}$, second, do a translation $\frac{x^\mu}{x \cdot x} \to \frac{x^\mu}{x \cdot x} - sa^\mu$, and lastly perform a second inversion, $\frac{x^\mu}{x \cdot x} - sa^\mu \to \frac{\frac{x^\mu}{x \cdot x} - sa^\mu}{\left(\frac{x^\mu}{x \cdot x} - sa^\mu\right)\left(\frac{x_\mu}{x \cdot x} - sa_\mu\right)}$. As opposed to a first impression, they're indeed well behaved at $x \cdot x = 0$, as can be seen from 1.16, but sadly, they aren't well behaved for the denominator in 1.16 be zero, that is,

$$1 - 2\tilde{s}a \cdot x + \tilde{s}^2(a \cdot a)(x \cdot x) = 0 \Rightarrow \tilde{s} = \frac{2(a \cdot x) \pm \sqrt{4(a \cdot x)^2 - 4(a \cdot a)(x \cdot x)}}{2(a \cdot a)(x \cdot x)}$$

Which only has a real solution for $x^\mu \propto a^\mu$, but, for this point away from the origin the transformation needs not to be well behaved, as the charts x, x'_s themselves doesn't need to be defined from the whole manifold, nevertheless, if we're working in a compact manifold we can make sense of this divergence $x'^\mu_s \to \infty$ as this is indeed a point of a compact manifold.

Problem 2

2.A)

First, let's state a few remarks, we have at our disposal the following algebras,

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}m\big(m^2-1\big)\delta_{m,-n}\\
&[\alpha^\mu_m, \alpha^\nu_n] = mg^{\mu\nu}\delta_{m,-n}\\
&[L_m, \alpha^\nu_n] = -n\alpha^\nu_{n+m}
\end{aligned}
$$

Which should be interpreted only as elements of a Lie Algebra. But, we seek representations of this algebra as linear maps in our physical Hilbert space, as any Hilbert space, it'll contain a notion of inner product, to which can be defined adjoint of linear maps. Let's formally manipulate the adjoints of the algebra,

$$
\begin{cases}
[L_m, L_n]^\dagger &= (m-n)L^\dagger_{n+m} + \frac{c}{12}m(m^2-1)\delta_{m,-n}\\
[\alpha^\mu_m, \alpha^\nu_n]^\dagger &= mg^{\mu\nu}\delta_{m,-n}\\
[L_m, \alpha^\nu_n]^\dagger &= -n\big(\alpha^\nu_{n+m}\big)^\dagger
\end{cases}
$$

$$
\begin{cases}
\big[L^\dagger_n, L^\dagger_m\big] & = (-n-(-m))L^\dagger_{n+m} + \frac{c}{12}(-n)\big((-n)^2-1\big)\delta_{-n,-(-m)}\\
\Big[(\alpha^\nu_n)^\dagger, (\alpha^\mu_m)^\dagger\Big] & = -ng^{\mu\nu}\delta_{-n,-(-m)}\\
\Big[(\alpha^\nu_n)^\dagger, L^\dagger_m\Big] = -\Big[L^\dagger_m, (\alpha^\nu_n)^\dagger\Big] & = -n\big(\alpha^\nu_{n+m}\big)^\dagger
\end{cases}
$$

In this format it's clear to see that the algebra span by $L^\dagger_n, (\alpha^\mu_m)^\dagger$ is isomorphic to the algebra span by L_n, α^μ_m, in particular, the isomorphism is given by, $L^\dagger_n \to L_{-n}, (\alpha^\mu_m) \to \alpha^\mu_{-m}$, as can be seen directly from the algebra by making the substitutions,

$$
\begin{aligned}
&[L_{-n}, L_{-m}] = (-n-(-m))L_{-n-m} + \frac{c}{12}(-n)\big((-n)^2-1\big)\delta_{-n,-(-m)}\\
&\big[\alpha^\nu_{-n}, \alpha^\mu_{-m}\big] = -ng^{\mu\nu}\delta_{-n,-(-m)}\\
&\big[L_{-m}, \alpha^\nu_{-n}\big] = n\alpha^\nu_{-n-m}
\end{aligned}
$$

Which is exactly the right algebra. We'll suppose that in our representation we have exactly $L^\dagger_n = L_{-n}, (\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$ as hinted above. A few comments on the inner product of our Hilbert space, it is non degenerate and non negative. About the non negative part, as we have a Lorentzian metric in the target space, at least one of the components of X — the time-like one — has a wrong sign in the kinetic term in the action, that is, some of the modes will inevitably '*create*' negative norm states, but, these cannot be physical, and therefore must/can be removed. One way of doing such is fixing the gauge a priori, which isn't the case, the other way is restricting the polarization vectors to which the modes, of ∂X can couple to. That is, as example, if α^μ_n when acting in the vacuum creates a negative norm state, this by itself is not enough to guarantee that it annihilate the vacuum, we have to show that no matter what choice of ξ_μ is done, $\xi_\mu\alpha^\mu_n$ always is a negative norm state, then we can conclude α^μ_n annihilate the vacuum. Closing these initial remarks, we start by defining the vacuum state, Ψ_0, as the one for which,

$$
L_m\Psi_0 = 0, \quad m \geq -1
$$

Let's analyze the action of $\xi_\mu \alpha_n^\mu$ in it,

$$\begin{aligned}
L_m \xi_\mu \alpha_n^\mu \Psi_0 &= \xi_\mu (L_m \alpha_n^\mu - \alpha_n^\mu L_m)\Psi_0, \quad m \geq -1 \\
L_m \xi_\mu \alpha_n^\mu \Psi_0 &= \xi_\mu [L_m, \alpha_n^\mu]\Psi_0, \quad m \geq -1 \\
L_m \xi_\mu \alpha_n^\mu \Psi_0 &= -n \xi_\mu \alpha_{n+m}^\mu \Psi_0, \quad m \geq -1
\end{aligned} \tag{2.1}$$

The first step was add $0 = \xi_\mu \alpha_n^\mu L_m \Psi_0$, as L_m, $m \geq -1$ annihilate the vacuum. In the special case of $n = 0$, we have,

$$L_m \xi_\mu \alpha_0^\mu \Psi_0 = 0, \quad m \geq -1$$

That implies $\xi_\mu \alpha_0^\mu \Psi_0$ has exactly all the symmetries of the vacuum, hence, it must be proportional to it, $\xi_\mu \alpha_0^\mu \Psi_0 = \xi \Psi_0$ for some constant ξ, now we have to evoke that we do have an inner product $(\cdot, \cdot)$, so that the special case of $n = -m$ in 2.1 is,

$$\begin{aligned}
L_m \xi_\mu \alpha_{-m}^\mu \Psi_0 &= m \xi_\mu \alpha_0^\mu \Psi_0, \quad m \geq -1 \\
L_m \xi_\mu \alpha_{-m}^\mu \Psi_0 &= m \xi \Psi_0, \quad m \geq -1 \\
(\Psi_0, L_m \xi_\mu \alpha_{-m}^\mu \Psi_0) &= (\Psi_0, m\xi\Psi_0), \quad m \geq -1 \\
\left(L_m^\dagger \Psi_0, \xi_\mu \alpha_{-m}^\mu \Psi_0\right) &= m\xi(\Psi_0, \Psi_0), \quad m \geq -1 \\
(L_{-m} \Psi_0, \xi_\mu \alpha_{-m}^\mu \Psi_0) &= m\xi(\Psi_0, \Psi_0), \quad m \geq -1
\end{aligned}$$

For $m = \pm 1$,

$$\begin{aligned}
\left(L_{\mp 1} \Psi_0, \xi_\mu \alpha_{\mp 1}^\mu \Psi_0\right) &= \pm \xi (\Psi_0, \Psi_0) \\
\left(0, \xi_\mu \alpha_{\mp 1}^\mu \Psi_0\right) &= \pm \xi (\Psi_0, \Psi_0) \\
0 &= \pm \xi (\Psi_0, \Psi_0)
\end{aligned}$$

Of course $(\Psi_0, \Psi_0) \neq 0$ otherwise it would be the null state, hence, $\xi = 0 \Rightarrow \xi_\mu \alpha_0^\mu \Psi_0 = \xi \Psi_0 = 0$. As this is valid for any choice of ξ_μ, we just proved that $\alpha_0^\mu \Psi_0 = 0$.

2.B)

Problem 3

3.A)

Our action is,

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z\, \psi\bar{\partial}\psi$$

To obtain the equation of motion is simple, first we set up the path integral of a total derivative, which is zero, the argument of the total derivative we set to $\exp{(-S)}\mathcal{O}$, where $\mathcal{O}$ is any combination of local fields that does not contain $\psi(z_1, \bar{z}_1)$, then,

$$\begin{aligned}
0 &= \int \mathcal{D}\psi \frac{\delta}{\delta\psi(z_1, \bar{z}_1)}[\exp{(-S)}\mathcal{O}] \\
0 &= \int \mathcal{D}\psi \frac{\delta}{\delta\psi(z_1, \bar{z}_1)}[\exp{(-S)}]\mathcal{O}
\end{aligned}$$

As $\frac{\delta}{\delta\psi(z_1,\bar{z}_1)}\mathcal{O}$ is zero except for $\mathcal{O} = \psi(z_1, \bar{z}_1)$, to which it's $\delta^{(2)}(0)$, but, that's not the case,

$$\begin{aligned}
0 &= \int \mathcal{D}\psi \frac{\delta}{\delta\psi(z_1, \bar{z}_1)}[\exp{(-S)}]\mathcal{O} \\
0 &= -\frac{1}{4\pi}\int \mathcal{D}\psi \frac{\delta}{\delta\psi(z_1, \bar{z}_1)}\left[\int \mathrm{d}^2 z_2\, \psi(z_2, \bar{z}_2)\partial_{\bar{z}_2}\psi(z_2, \bar{z}_2)\right]\mathcal{O}\exp{(-S)} \\
0 &= -\frac{1}{4\pi}\int \mathcal{D}\psi \int \mathrm{d}^2 z_2 \left[\delta^{(2)}(z_2 - z_1)\partial_{\bar{z}_2}\psi(z_2, \bar{z}_2) - \psi(z_2, \bar{z}_2)\partial_{\bar{z}_2}\delta^{(2)}(z_2 - z_1)\right]\mathcal{O}\exp{(-S)}
\end{aligned}$$

As long as we're dealing with the closed fermion, we need not to worry about the boundary conditions, as there is no boundary, so,

$$\begin{aligned}
0 &= -\frac{1}{4\pi}\int \mathcal{D}\psi \int \mathrm{d}^2 z_2 \left[\delta^{(2)}(z_2 - z_1)\partial_{\bar{z}_2}\psi(z_2, \bar{z}_2) + \partial_{\bar{z}_2}\psi(z_2, \bar{z}_2)\delta^{(2)}(z_2 - z_1)\right]\mathcal{O}\exp{(-S)} \\
0 &= -\frac{1}{2\pi}\int \mathcal{D}\psi \partial_{\bar{z}_1}\psi(z_1, \bar{z}_1)\mathcal{O}\exp{(-S)}
\end{aligned}$$

What in the Operator formalism would account for some radially ordered expectation value under some state, which is specified with the boundary conditions on the path integral itself,

$$\partial_{\bar{z}_1}\langle\psi(z_1, \bar{z}_1)\mathcal{O}\rangle = 0$$

As both the operator $\mathcal{O}$ and the state are arbitrary, as long as there is no insertion of $\psi(z_1, \bar{z}_1)$, we conclude the equation of motion in the operator form is just,

$$\bar{\partial}\psi = 0 \Rightarrow \psi(z, \bar{z}) \equiv \psi(z)$$

That is, ψ is, at least, a meromorphic operator/function. Now, let's repeat this to obtain the two point function, this time we include in $\mathcal{O}$ a single factor of $\psi(z_1, \bar{z}_1)$ disguised as $\psi(z_2, \bar{z}_2)$, this also can be phrased as we being interested in the limit $z_2 \to z_1$,

$$
\begin{aligned}
0 &= \int \mathcal{D}\psi \frac{\delta}{\delta\psi(z_1,\bar{z}_1)}[\exp(-S)\psi(z_2,\bar{z}_2)\mathcal{O}] \\
0 &= \int \mathcal{D}\psi \frac{\delta}{\delta\psi(z_1,\bar{z}_1)}[\exp(-S)\psi(z_2,\bar{z}_2)]\mathcal{O} \\
0 &= \int \mathcal{D}\psi \left\{ \frac{\delta}{\delta\psi(z_1,\bar{z}_1)}[\exp(-S)]\psi(z_2,\bar{z}_2) + \frac{\delta}{\delta\psi(z_1,\bar{z}_1)}[\psi(z_2,\bar{z}_2)]\exp(-S) \right\}\mathcal{O} \\
0 &= \int \mathcal{D}\psi \left\{ \frac{\delta}{\delta\psi(z_1,\bar{z}_1)}[-S]\psi(z_2,\bar{z}_2) + \delta^{(2)}(z_2-z_1) \right\}\exp(-S)\mathcal{O}
\end{aligned}
$$

As we derived before, $\frac{\delta}{\delta\psi(z_1,\bar{z}_1)}[-S] = -\frac{1}{2\pi}\partial_{\bar{z}_1}\psi(z_1,\bar{z}_1)$,

$$
0 = \int \mathcal{D}\psi \left\{ -\frac{1}{2\pi}\partial_{\bar{z}_1}\psi(z_1,\bar{z}_1)\psi(z_2,\bar{z}_2) + \delta^{(2)}(z_2-z_1) \right\}\exp(-S)\mathcal{O}
$$

Which is, translating to the operator formalism,

$$
\begin{aligned}
\partial_{\bar{z}_1}\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle &= \left\langle 2\pi\delta^{(2)}(z_2-z_1)\mathcal{O}\right\rangle \\
\partial_{\bar{z}_1}\psi(z_1)\psi(z_2) &= 2\pi\delta^{(2)}(z_2-z_1), \quad |z_1|\geq|z_2|
\end{aligned}
$$

So that we can interpret this as a operator equality. The condition $|z_1| \geq |z_2|$ is due to the implicit radial ordering in the expectation value, and also notice that in the path integral formulation $\psi = \psi(z_1,\bar{z}_1)$, but in the operator formalism $\psi = \psi(z_1)$, as every operator is always '*on-shell*', while the path integral integrands aren't. Now we integrate this two point function, pick any compact closed region R in the complex plane which contains z_2 and not any other of the points of the insertions $\mathcal{O}$, with also the boundary being a continuous curve, for all $m \in \mathbb{N}$ the following is true,

$$
\begin{aligned}
\partial_{\bar{z}_1}\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle &= 2\pi\delta^{(2)}(z_2-z_1)\langle\mathcal{O}\rangle \\
(z_1-z_2)^m\partial_{\bar{z}_1}\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle &= 2\pi(z_1-z_2)^m\delta^{(2)}(z_2-z_1)\langle\mathcal{O}\rangle, \quad m\in\mathbb{N} \\
\int_R \mathrm{d}^2z_1\,(z_1-z_2)^m\partial_{\bar{z}_1}\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle &= 2\pi\int_R \mathrm{d}^2z_1\,(z_1-z_2)^m\delta^{(2)}(z_2-z_1)\langle\mathcal{O}\rangle, \quad m\in\mathbb{N} \\
\int_R \mathrm{d}^2z_1\,\partial_{\bar{z}_1}\{(z_1-z_2)^m\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle\} &= 2\pi(z_1-z_2)^m\Big|_{z_1=z_2}\langle\mathcal{O}\rangle, \quad m\in\mathbb{N}
\end{aligned}
$$

In the last line we used the fact '*trivial*' fact that $\partial_{\bar{z}_1}(z_1-z_2)^m = 0$, which is valid for $m\in\mathbb{N}$, but isn't for negative non-integer values.

$$
\begin{aligned}
\int_R \mathrm{d}^2z_1\,\partial_{\bar{z}_1}\{(z_1-z_2)^m\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle\} &= 2\pi(z_1-z_2)^m\Big|_{z_1=z_2}\langle\mathcal{O}\rangle, \quad m\in\mathbb{N} \\
-\mathrm{i}\int_{\partial R}\mathrm{d}z_1\,(z_1-z_2)^m\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle &= 2\pi\delta_{m,0}\langle\mathcal{O}\rangle, \quad m\in\mathbb{N}
\end{aligned}
$$

This is simply the complex version of the divergence theorem,

$$
\begin{aligned}
-\mathrm{i}\int_{\partial R} \mathrm{d}z_1\,(z_1-z_2)^m\,\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle &= 2\pi\delta_{m,0}\,\langle\mathcal{O}\rangle\,, \quad m\in\mathbb{N}\\
\int_{\partial R} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}}(z_1-z_2)^m\,\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle &= \delta_{m,0}\,\langle\mathcal{O}\rangle\,, \quad m\in\mathbb{N}
\end{aligned}
$$

Well, the left-hand side of this last equation picks up the pole of m-th order of the expression $\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle$ under $z_1\to z_2$, but, the right-hand side is only non-zero for $m=0$, this is telling us that $\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle$ doesn't have any pole besides the first order one with residue $\langle\mathcal{O}\rangle$, so we can write,

$$
\langle\psi(z_1)\psi(z_2)\mathcal{O}\rangle = \frac{1}{z_1-z_2}\,\langle\mathcal{O}\rangle + \text{regular}
$$

Or, in the operator formalism,

$$
\begin{aligned}
\psi(z_1)\psi(z_2) &= \frac{1}{z_1-z_2} + \text{regular}, \quad |z_1|\geq|z_2|\\
\psi(z_1)\psi(z_2) &= \frac{1}{z_1-z_2} + :\psi(z_1)\psi(z_2):, \quad |z_1|\geq|z_2|
\end{aligned}
$$

Which is the OPE.

3.B)

The energy momentum tensor has two components, a meromorphic one $T(z)$, and an anti-meromorphic one $\bar{T}(\bar{z})$, as we have at our disposal only a weight $\left(\frac{1}{2},0\right)$ field, it's impossible to construct a anti-meromorphic energy momentum tensor, but, for the meromorphic one, let us see what kind of combinations have the right weights. First, we remember that our energy momentum tensor must be normal ordered, so, all the possible normal ordered weight $(2,0)$ combinations of ψ and derivatives are,

$$
:\psi\partial\psi:(z),\ :(\partial\psi)\psi:(z),\ :\partial(\psi\psi):(z),\ :\psi\psi\psi\psi:(z)
$$

The third and fourth are zero due to the fermionic statistic, and the first and second are linear dependent also because the fermionic statistic. Hence, up to a unknown constant α, the meromorphic energy momentum tensor is,

$$
T(z) = \alpha:\psi\partial\psi:(z)
$$

Let's use this expression to compute the following,

$$
:T(z_1)T(z_2): = T(z_1)T(z_2) + \alpha^2\ \text{contractions}\{:(\psi\partial\psi)(z_1)(\psi\partial\psi)(z_2):\}, \quad |z_1|\geq|z_2|
$$

Where the contraction part means exchanging pairs of ψ with different z by $-(z_1-z_2)^{-1}$, as cause of the fermion statistic, we have to pay attention to the signs, also, we'll omit for now the $|z_1|\geq|z_2|$, but it will still be imposed,

$$
\begin{aligned}
:T(z_1)T(z_2): &= T(z_1)T(z_2) + \alpha^2 : (\psi\partial\psi)(z_1)(\psi\partial\psi)(z_2) : + \alpha^2 : (\psi\partial\psi)(z_1)(\psi\partial\psi)(z_2) : \\
&+ \alpha^2 : (\psi\partial\psi)(z_1)(\psi\partial\psi)(z_2) : + \alpha^2 : (\psi\partial\psi)(z_1)(\psi\partial\psi)(z_2) : \\
&+ \alpha^2 : (\psi\partial\psi)(z_1)(\psi\partial\psi)(z_2) : + \alpha^2 : (\psi\partial\psi)(z_1)(\psi\partial\psi)(z_2) : \\
:T(z_1)T(z_2): &= T(z_1)T(z_2) + \frac{1}{z_1 - z_2}\alpha^2 : \partial\psi(z_1)\partial\psi(z_2) : - \partial_{z_2}\left(\frac{1}{z_1 - z_2}\right)\alpha^2 : \partial\psi(z_1)\psi(z_2) : \\
&- \alpha^2\partial_{z_1}\left(\frac{1}{z_1 - z_2}\right) : \psi(z_1)\partial\psi(z_2) : + \alpha^2\partial_{z_1}\partial_{z_2}\left(\frac{1}{z_1 - z_2}\right) : \psi(z_1)\psi(z_2) : \\
&+ \alpha^2\frac{1}{z_1 - z_2}\partial_{z_1}\partial_{z_2}\left(\frac{1}{z_1 - z_2}\right) - \alpha^2\partial_{z_2}\left(\frac{1}{z_1 - z_2}\right)\partial_{z_1}\left(\frac{1}{z_1 - z_2}\right) \\
:T(z_1)T(z_2): &= T(z_1)T(z_2) + \frac{\alpha^2}{z_1 - z_2} : \partial\psi(z_1)\partial\psi(z_2) : - \frac{\alpha^2}{(z_1 - z_2)^2} : \partial\psi(z_1)\psi(z_2) : \\
&+ \frac{\alpha^2}{(z_1 - z_2)^2} : \psi(z_1)\partial\psi(z_2) : + \partial_{z_1}\left(\frac{\alpha^2}{(z_1 - z_2)^2}\right) : \psi(z_1)\psi(z_2) : \\
&+ \frac{\alpha^2}{z_1 - z_2}\partial_{z_1}\left(\frac{1}{(z_1 - z_2)^2}\right) - \frac{\alpha^2}{(z_1 - z_2)^2}\partial_{z_1}\left(\frac{1}{z_1 - z_2}\right) \\
:T(z_1)T(z_2): &= T(z_1)T(z_2) + \frac{\alpha^2}{z_1 - z_2} : \partial\psi(z_1)\partial\psi(z_2) : - \frac{\alpha^2}{(z_1 - z_2)^2} : \partial\psi(z_1)\psi(z_2) : \\
&+ \frac{\alpha^2}{(z_1 - z_2)^2} : \psi(z_1)\partial\psi(z_2) : - 2\frac{\alpha^2}{(z_1 - z_2)^3} : \psi(z_1)\psi(z_2) : \\
&- 2\frac{\alpha^2}{z_1 - z_2}\frac{1}{(z_1 - z_2)^3} + \frac{\alpha^2}{(z_1 - z_2)^2}\frac{1}{(z_1 - z_2)^2} \\
:T(z_1)T(z_2): &= T(z_1)T(z_2) + \frac{\alpha^2}{z_1 - z_2} : \partial\psi(z_1)\partial\psi(z_2) : - \frac{\alpha^2}{(z_1 - z_2)^2} : \partial\psi(z_1)\psi(z_2) : \\
&+ \frac{\alpha^2}{(z_1 - z_2)^2} : \psi(z_1)\partial\psi(z_2) : - 2\frac{\alpha^2}{(z_1 - z_2)^3} : \psi(z_1)\psi(z_2) : \\
&- \frac{\alpha^2}{(z_1 - z_2)^4}
\end{aligned}
$$

To proceed further we have to Taylor expand both $\psi(z_1), \partial\psi(z_1)$, of course this will generate regular terms in our expansion, as for example, the second term in the right-hand side,

$$
\frac{1}{z_1 - z_2} : \partial\psi(z_1)\partial\psi(z_2) : = \frac{1}{z_1 - z_2} : \partial\psi(z_2)\partial\psi(z_2) : + \sum_{n=1}^{\infty}\frac{1}{z_1 - z_2}\frac{1}{n!}(z_1 - z_2)^n : \partial^n\psi(z_2)\partial\psi(z_2) :
$$
$$
\frac{1}{z_1 - z_2} : \partial\psi(z_1)\partial\psi(z_2) : = \frac{1}{z_1 - z_2} : \partial\psi\partial\psi : (z_2) + \sum_{n=1}^{\infty}\frac{(z_1 - z_2)^{n-1}}{n!} : \partial^n\psi\partial\psi : (z_2)
$$

It's clear that the sum in the right-hand side is of only regular terms, so that,

$$
\frac{1}{z_1 - z_2} : \partial\psi(z_1)\partial\psi(z_2) : = \frac{1}{z_1 - z_2} : \partial\psi\partial\psi : (z_2) + \text{regular}
$$

But $:\partial\psi\partial\psi:$ is zero by statistics, hence,

$$\frac{1}{z_1 - z_2} : \partial\psi(z_1)\partial\psi(z_2) := \text{regular}$$

We do this procedure for all the terms in the expansion,

$$\begin{aligned}
: T(z_1)T(z_2) := T(z_1)T(z_2) &- \frac{\alpha^2}{(z_1 - z_2)^2} : \partial\psi\psi : (z_2) - \frac{\alpha^2}{(z_1 - z_2)} : \partial^2\psi\psi : (z_2) \\
&+ \frac{\alpha^2}{(z_1 - z_2)^2} : \psi\partial\psi : (z_2) + \frac{\alpha^2}{(z_1 - z_2)} : \partial\psi\partial\psi : (z_2) \\
&- 2\frac{\alpha^2}{(z_1 - z_2)^3} : \psi\psi : (z_2) - 2\frac{\alpha^2}{(z_1 - z_2)^2} : \partial\psi\psi : (z_2) - \frac{\alpha^2}{(z_1 - z_2)} : \partial^2\psi\psi : (z_2) \\
&- \frac{\alpha^2}{(z_1 - z_2)^4} + \text{regular}
\end{aligned}$$

Removing the terms which are zero by statistics and grouping the others,

$$\begin{aligned}
: T(z_1)T(z_2) := T(z_1)T(z_2) + \frac{4\alpha^2}{(z_1 - z_2)^2} : \psi\partial\psi : (z_2) + \frac{2\alpha^2}{(z_1 - z_2)} : \psi\partial^2\psi : (z_2) \\
- \frac{\alpha^2}{(z_1 - z_2)^4} + \text{regular}
\end{aligned}$$

As $: \partial\psi\partial\psi :\equiv 0$ we can add it as we please,

$$\begin{aligned}
: T(z_1)T(z_2) := T(z_1)T(z_2) &+ \frac{4\alpha^2}{(z_1 - z_2)^2} : \psi\partial\psi : (z_2) + \frac{2\alpha^2}{(z_1 - z_2)} : \psi\partial^2\psi + \partial\psi\partial\psi : (z_2) \\
&- \frac{\alpha^2}{(z_1 - z_2)^4} + \text{regular} \\
: T(z_1)T(z_2) := T(z_1)T(z_2) &+ \frac{4\alpha^2}{(z_1 - z_2)^2} : \psi\partial\psi : (z_2) + \frac{2\alpha^2}{(z_1 - z_2)}\partial : \psi\partial\psi : (z_2) \\
&- \frac{\alpha^2}{(z_1 - z_2)^4} + \text{regular} \\
: T(z_1)T(z_2) := T(z_1)T(z_2) &+ \frac{4\alpha}{(z_1 - z_2)^2}T(z_2) + \frac{2\alpha}{(z_1 - z_2)}\partial T(z_2) \\
&- \frac{\alpha^2}{(z_1 - z_2)^4} + \text{regular}
\end{aligned}$$

Well, $: T(z_1)T(z_2) :$ itself is regular, then,

$$T(z_1)T(z_2) = -\frac{4\alpha}{(z_1 - z_2)^2}T(z_2) - \frac{2\alpha}{(z_1 - z_2)}\partial T(z_2) + \frac{\alpha^2}{(z_1 - z_2)^4} + \text{regular}$$

But we know the general form of the TT OPE,

$$T(z_1)T(z_2) = \frac{2}{(z_1 - z_2)^2}T(z_2) + \frac{1}{(z_1 - z_2)}\partial T(z_2) + \frac{c}{2(z_1 - z_2)^4} + \text{regular}$$

From where is easy to read $\alpha = -\frac{1}{2}$, and also, $c = 2\alpha^2 = 2\frac{1}{4} = \frac{1}{2}$, that is, the meromorphic component of the energy momentum tensor, and the central charge are,

$$T(z) = -\frac{1}{2} : \psi \partial \psi : (z), \quad c = \frac{1}{2}$$

3.C)

Now let ψ have an additional index, ψ^i which takes two values $i = 1, 2$. This is a realization of a internal $SO(2)$ symmetry, to why, let $\psi'^i = \Lambda^i{}_j \psi^j$, the change in the action is,

$$\begin{aligned}
S &= \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \psi^i \bar{\partial} \psi^j \\
S' &= \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \psi'^i \bar{\partial} \psi'^j \\
S' &= \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \Lambda^i{}_k \Lambda^j{}_l \psi^k \bar{\partial} \psi^l
\end{aligned}$$

As long as $\delta_{ij} \Lambda^i{}_k \Lambda^j{}_l = \delta_{kl}$, this transformation is a symmetry. But that's exactly the definition of Λ belonging to the $O(2)$ group, but as we're just interested in elements continuously connected to the identity, we set $SO(2)$. The group definition under an infinitesimal transformation,

$$\begin{aligned}
\delta_{kl} &= \delta_{ij} \Lambda^i{}_k \Lambda^j{}_l \\
\delta_{kl} &= \delta_{ij} \big(\delta^i{}_k + \omega^i{}_k + \mathcal{O}(\omega^2)\big)\big(\delta^j{}_l + \omega^j{}_l + \mathcal{O}(\omega^2)\big) \\
\delta_{kl} &= \delta_{kl} + \omega_{lk} + \omega_{kl} + \mathcal{O}(\omega^2) \\
\omega_{lk} &= -\omega_{kl}
\end{aligned}$$

We can use this information to write,

$$\begin{aligned}
\Lambda^i{}_j &= \delta^i{}_j + \omega^i{}_j + \mathcal{O}(\omega^2) \\
\Lambda^i{}_j &= \delta^i{}_j + \frac{1}{2} \omega_{lk} \big(\delta^{il} \delta^k{}_j - \delta^{ik} \delta^l{}_j\big) + \mathcal{O}(\omega^2) \\
\Lambda^i{}_j &= \delta^i{}_j - \frac{\mathrm{i}}{2} \omega_{lk} \mathrm{i} \big(\delta^{il} \delta^k{}_j - \delta^{ik} \delta^l{}_j\big) + \mathcal{O}(\omega^2) \\
\Lambda^i{}_j &= \delta^i{}_j - \frac{\mathrm{i}}{2} \omega_{[lk]} T^{[lk]i}{}_j + \mathcal{O}(\omega^2)
\end{aligned}$$

Where we defined the generator of the group, $T^{[lk]i}{}_j = \mathrm{i}\big(\delta^{il} \delta^k{}_j - \delta^{ik} \delta^l{}_j\big)$. The sign of the definition is arbitrary. A finite transformation is then,

$$\psi^i \to \Lambda^i{}_j \psi^j = \exp\left(-\frac{\mathrm{i}}{2} \omega_{[lk]} T^{[lk]}\right)^i{}_j \psi^j$$

To obtain the conserved current of this symmetry we make a little trick, set ω as a function $\omega(z, \bar{z})$, so that the infinitesimal variation of the action is,

$$
\begin{aligned}
S &= \frac{1}{4\pi}\int \mathrm{d}^2 z\, \delta_{ij}\psi^i \bar{\partial}\psi^j \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2 z\, \delta_{ij}\psi'^i \bar{\partial}\psi'^j \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2 z\, \delta_{ij}\Lambda^i{}_k\psi^k \bar{\partial}\big(\Lambda^j{}_l\psi^l\big) \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2 z\, \delta_{ij}\Lambda^i{}_k\psi^k \big\{\Lambda^j{}_l\bar{\partial}\psi^l + \psi^l\bar{\partial}\Lambda^j{}_l\big\} \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2 z\, \delta_{ij}\Lambda^i{}_k\psi^k \Big\{\Lambda^j{}_l\bar{\partial}\psi^l - \frac{\mathrm{i}}{2}T^{[ab]m}{}_l\,\psi^l\Lambda^j{}_m\bar{\partial}\omega_{[ab]}\Big\} \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2 z\, \Lambda_{jk}\psi^k\Lambda^j{}_l\bar{\partial}\psi^l - \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2 z\, \Lambda_{jk}\psi^k T^{[ab]m}{}_l\,\psi^l\Lambda^j{}_m\bar{\partial}\omega_{[ab]} \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2 z\, \delta_{kl}\psi^k\bar{\partial}\psi^l - \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2 z\, \delta_{km}\psi^k T^{[ab]m}{}_l\,\psi^l\bar{\partial}\omega_{[ab]} \\
S' &= S - \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2 z\, \psi^k T^{[ab]}{}_{kl}\,\psi^l\bar{\partial}\omega_{[ab]} \\
\delta S &= -\frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2 z\, \bar{\partial}\Big\{\psi^k T^{[ab]}{}_{kl}\,\psi^l\omega_{[ab]}\Big\} + \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2 z\, \bar{\partial}\Big\{\psi^k T^{[ab]}{}_{kl}\,\psi^l\Big\}\omega_{[ab]}
\end{aligned}
$$

With suitable boundary conditions on $\omega_{[ab]}$, or, with no boundaries, we conclude,

$$
\delta S = \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2 z\, \bar{\partial}\Big\{\psi^k T^{[ab]}{}_{kl}\,\psi^l\Big\}\omega_{[ab]}
$$

Now back to $\omega_{[ab]}$ being a constant — so that $\delta S = 0$ —, using also that there is just one two index anti-symmetric tensor in two dimensions, the Levi-Civita ϵ_{ab}, $\epsilon_{12} = 1$, we conclude that $\omega_{[ab]} = \omega\epsilon_{ab}$,

$$
\begin{aligned}
0 &= \frac{1}{4\pi}\frac{\mathrm{i}}{2}\omega\epsilon_{ab}T^{[ab]}{}_{kl}\int \mathrm{d}^2 z\, \bar{\partial}\big\{\psi^k\psi^l\big\} \\
0 &= -\frac{\omega\epsilon_{12}}{2\pi\mathrm{i}}\frac{1}{2}T^{12}{}_{kl}\int \mathrm{d}^2 z\, \bar{\partial}\big\{\psi^k\psi^l\big\}
\end{aligned}
$$

So there is just one current, as there is just one linear independent generator. The classical current can be read from the last equation as,

$$
j(z) = \frac{1}{2}T^{12}{}_{kl}\,\psi^k\psi^l(z)
$$

We factored out the $-\frac{\omega\epsilon_{12}}{2\pi\mathrm{i}}$ as is usual to obtain an interpretation of the residue of the j OPE. The quantum version of it will require normal ordering,

$$
\begin{aligned}
j(z) &= \frac{1}{2}T^{12}{}_{kl}\, :\psi^k\psi^l:(z) \\
j(z) &= \frac{1}{2}\mathrm{i}\big(\delta_k{}^1\delta^2{}_l - \delta_k{}^2\delta^1{}_l\big) :\psi^k\psi^l:(z) \\
j(z) &= \frac{\mathrm{i}}{2}\big(:\psi^1\psi^2:(z) - :\psi^2\psi^1:(z)\big)
\end{aligned}
$$

$$j(z) = \mathrm{i} : \psi^1 \psi^2 : (z)$$

Now, to get it's OPE we proceed as usual, computing the following normal ordered operator,

$$
\begin{aligned}
: j(z_1) j(z_2) : &= j(z_1) j(z_2) - : (\psi^1 \psi^2)(z_1)(\psi^1 \psi^2)(z_2) : - : (\psi^1 \psi^2)(z_1)(\psi^1 \psi^2)(z_2) : \\
&\quad - : (\psi^1 \psi^2)(z_1)(\psi^1 \psi^2)(z_2) : \\
: j(z_1) j(z_2) : &= j(z_1) j(z_2) - \frac{1}{z_1 - z_2} : \psi^2(z_1) \psi^2(z_2) : - \frac{1}{z_1 - z_2} : \psi^1(z_1) \psi^1(z_2) : \\
&\quad - \frac{1}{(z_1 - z_2)^2} \\
: j(z_1) j(z_2) : &= j(z_1) j(z_2) - \frac{1}{z_1 - z_2} : \psi^2(z_2) \psi^2(z_2) : - \frac{1}{z_1 - z_2} : \psi^1(z_1) \psi^1(z_1) : \\
&\quad - \frac{1}{(z_1 - z_2)^2} + \text{regular}
\end{aligned}
$$

In the first to second like we used the $\psi\psi$ OPE, remembering that the $\psi^1\psi^2$ OPE has no poles, from the second line to third, we expanded every term around z_2. And now we set the terms which are zero by statistics,

$$
\begin{aligned}
: j(z_1) j(z_2) : &= j(z_1) j(z_2) - \frac{1}{(z_1 - z_2)^2} + \text{regular} \\
j(z_1) j(z_2) &= \frac{1}{(z_1 - z_2)^2} + \text{regular}
\end{aligned}
$$

A rather simple OPE.

3.D)

Much of what we did can be recycled here. First, we start with N real fermions ψ^i, $i = 1, \cdots, N$, and by similar arguments analyzing the action,

$$
\begin{aligned}
S &= \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \psi^i \bar{\partial} \psi^j \\
S' &= \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \psi'^i \bar{\partial} \psi'^j \\
S' &= \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \Lambda^i{}_k \Lambda^j{}_l \psi^k \bar{\partial} \psi^l
\end{aligned}
$$

We obtain the constraint of this being a symmetry, $\delta_{ij} \Lambda^i{}_k \Lambda^j{}_l = \delta_{kl}$, which is the defining property of the $O(N)$ group. Of course, the relevant part for generators and Lie Algebras is just the part connected to the identity, $SO(N)$, the same reasoning of infinitesimal transformations allows us to write,

$$
\begin{aligned}
\delta_{kl} &= \delta_{ij} \Lambda^i{}_k \Lambda^j{}_l \\
\delta_{kl} &= \delta_{ij} \big(\delta^i{}_k + \omega^i{}_k + \mathcal{O}(\omega^2) \big) \big(\delta^j{}_l + \omega^j{}_l + \mathcal{O}(\omega^2) \big) \\
\delta_{kl} &= \delta_{kl} + \omega_{lk} + \omega_{kl} + \mathcal{O}(\omega^2)
\end{aligned}
$$

$$\omega_{lk} = -\omega_{kl}$$

We can use this information to write,

$$\begin{aligned}
\Lambda^i{}_j &= \delta^i{}_j + \omega^i{}_j + \mathcal{O}(\omega^2) \\
\Lambda^i{}_j &= \delta^i{}_j + \frac{1}{2}\omega_{lk}\left(\delta^{il}\delta^k{}_j - \delta^{ik}\delta^l{}_j\right) + \mathcal{O}(\omega^2) \\
\Lambda^i{}_j &= \delta^i{}_j - \frac{\mathrm{i}}{2}\omega_{lk}\mathrm{i}\left(\delta^{il}\delta^k{}_j - \delta^{ik}\delta^l{}_j\right) + \mathcal{O}(\omega^2) \\
\Lambda^i{}_j &= \delta^i{}_j - \frac{\mathrm{i}}{2}\omega_{[lk]}T^{[lk]i}{}_j + \mathcal{O}(\omega^2) \\
\Lambda^i{}_j &= \exp\left(-\frac{\mathrm{i}}{2}\omega_{[lk]}T^{[lk]}\right)^i{}_j
\end{aligned}$$

To obtain the current we do again the same trick of promoting $\omega_{[lk]}$ to a function and computing the change in the action,

$$\begin{aligned}
S &= \frac{1}{4\pi}\int \mathrm{d}^2z\, \delta_{ij}\psi^i\bar{\partial}\psi^j \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2z\, \delta_{ij}\psi'^i\bar{\partial}\psi'^j \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2z\, \delta_{ij}\Lambda^i{}_k\psi^k\bar{\partial}\left(\Lambda^j{}_l\psi^l\right) \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2z\, \delta_{ij}\Lambda^i{}_k\psi^k\left\{\Lambda^j{}_l\bar{\partial}\psi^l + \psi^l\bar{\partial}\Lambda^j{}_l\right\} \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2z\, \delta_{ij}\Lambda^i{}_k\psi^k\left\{\Lambda^j{}_l\bar{\partial}\psi^l - \frac{\mathrm{i}}{2}T^{[ab]m}{}_l\,\psi^l\Lambda^j{}_m\bar{\partial}\omega_{[ab]}\right\} \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2z\, \Lambda_{jk}\psi^k\Lambda^j{}_l\bar{\partial}\psi^l - \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2z\, \Lambda_{jk}\psi^kT^{[ab]m}{}_l\,\psi^l\Lambda^j{}_m\bar{\partial}\omega_{[ab]} \\
S' &= \frac{1}{4\pi}\int \mathrm{d}^2z\, \delta_{kl}\psi^k\bar{\partial}\psi^l - \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2z\, \delta_{km}\psi^kT^{[ab]m}{}_l\,\psi^l\bar{\partial}\omega_{[ab]} \\
S' &= S - \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2z\, \psi^kT^{[ab]}{}_{kl}\,\psi^l\bar{\partial}\omega_{[ab]} \\
\delta S &= -\frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2z\, \bar{\partial}\left\{\psi^kT^{[ab]}{}_{kl}\,\psi^l\omega_{[ab]}\right\} + \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2z\, \bar{\partial}\left\{\psi^kT^{[ab]}{}_{kl}\,\psi^l\right\}\omega_{[ab]}
\end{aligned}$$

With suitable boundary conditions on $\omega_{[ab]}$, or, with no boundaries, we conclude,

$$\delta S = \frac{1}{4\pi}\frac{\mathrm{i}}{2}\int \mathrm{d}^2z\, \bar{\partial}\left\{\psi^kT^{[ab]}{}_{kl}\,\psi^l\right\}\omega_{[ab]}$$

Demoting $\omega_{[ab]}$ back to being a constant — so that $\delta S = 0$ —, we obtain,

$$0 = -\frac{\omega_{[ab]}}{2\pi\mathrm{i}}\frac{1}{2}T^{[ab]}{}_{kl}\int \mathrm{d}^2z\, \bar{\partial}\left\{\psi^k\psi^l\right\}$$

Again, the usual normalization is done by removing $-\frac{\omega_{[ab]}}{2\pi\mathrm{i}}$, so that the current is,

$$j^{[ab]}(z) = \frac{1}{2} T^{[ab]}{}_{kl} : \psi^k \psi^l : (z) \tag{3.1}$$

This is not a very orthodox way of displaying, all of our index here have the range $1, \cdots, N$, but notice the generators are indexed by an anti-symmetric pair of index, $[ab]$, so there are $\frac{N(N-1)}{2}$ such, this is only useful if we work in the fundamental representation, so, we'll change now to a more common notation, instead of labeling the generators by $\frac{N(N-1)}{2}$ anti-symmetric pairs of index, we'll label them by just one index with range $A = 1, \cdots, \frac{N(N-1)}{2}$, so that the same equation reads,

$$j^A(z) = \frac{1}{2} T^A{}_{kl} : \psi^k \psi^l : (z)$$

This is more aesthetic for Lie Algebras in general, but not always desired — Lorentz group as example —. Now we proceed by computing the OPE associated with this collection of currents,

$$\begin{aligned}
: j^A(z_1) j^B(z_2) : = j^A(z_1) j^B(z_2) &+ \frac{1}{4} T^A{}_{ij} T^B{}_{kl} : (\psi^i \psi^j)(z_1)(\psi^k \psi^l)(z_2) : \\
&+ \frac{1}{4} T^A{}_{ij} T^B{}_{kl} : (\psi^i \psi^j)(z_1)(\psi^k \psi^l)(z_2) : + \frac{1}{4} T^A{}_{ij} T^B{}_{kl} : (\psi^i \psi^j)(z_1)(\psi^k \psi^l)(z_2) : \\
&+ \frac{1}{4} T^A{}_{ij} T^B{}_{kl} : (\psi^i \psi^j)(z_1)(\psi^k \psi^l)(z_2) : + \frac{1}{4} T^A{}_{ij} T^B{}_{kl} : (\psi^i \psi^j)(z_1)(\psi^k \psi^l)(z_2) : \\
&+ \frac{1}{4} T^A{}_{ij} T^B{}_{kl} : (\psi^i \psi^j)(z_1)(\psi^k \psi^l)(z_2) :
\end{aligned}$$

As we stated before, the contraction only is non zero for equal index ψ,

$$\begin{aligned}
: j^A(z_1) j^B(z_2) : = j^A(z_1) j^B(z_2) &+ \frac{\delta^{ik}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^j(z_1) \psi^l(z_2) : \\
&- \frac{\delta^{il}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^j(z_1) \psi^k(z_2) : - \frac{\delta^{jk}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^i(z_1) \psi^l(z_2) : \\
&+ \frac{\delta^{jl}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^i(z_1) \psi^k(z_2) : + \frac{\delta^{ik} \delta^{jl}}{4(z_1 - z_2)^2} T^A{}_{ij} T^B{}_{kl} \\
&- \frac{\delta^{il} \delta^{jk}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl}
\end{aligned}$$

Expanding in Taylor everything around z_2,

$$\begin{aligned}
: j^A(z_1) j^B(z_2) : = j^A(z_1) j^B(z_2) &+ \frac{\delta^{ik}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^j(z_2) \psi^l(z_2) : \\
&- \frac{\delta^{il}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^j(z_2) \psi^k(z_2) : - \frac{\delta^{jk}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^i(z_2) \psi^l(z_2) : \\
&+ \frac{\delta^{jl}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} : \psi^i(z_2) \psi^k(z_2) : + \frac{\delta^{ik} \delta^{jl}}{4(z_1 - z_2)^2} T^A{}_{ij} T^B{}_{kl} \\
&- \frac{\delta^{il} \delta^{jk}}{4(z_1 - z_2)} T^A{}_{ij} T^B{}_{kl} + \text{regular}
\end{aligned}$$

It's clear we have two kinds of terms, with $:\psi\psi:$ and without, to group together all of them, we'll go under a massive round of relabeling, first, relabel all ψ index to be b in the right one and a in the left one,

$$
\begin{aligned}
:j^A(z_1)j^B(z_2): = j^A(z_1)j^B(z_2) &+ \frac{\delta^{ik}}{4(z_1-z_2)}T^A{}_{ia}T^B{}_{kb}\ :\psi^a\psi^b:(z_2) \\
&- \frac{\delta^{il}}{4(z_1-z_2)}T^A{}_{ia}T^B{}_{bl}\ :\psi^a\psi^b:(z_2) - \frac{\delta^{jk}}{4(z_1-z_2)}T^A{}_{aj}T^B{}_{kb}\ :\psi^a\psi^b:(z_2) \\
&+ \frac{\delta^{jl}}{4(z_1-z_2)}T^A{}_{aj}T^B{}_{bl}\ :\psi^a\psi^b:(z_2) + \frac{\delta^{ik}\delta^{jl}}{4(z_1-z_2)^2}T^A{}_{ij}T^B{}_{kl} \\
&- \frac{\delta^{il}\delta^{jk}}{4(z_1-z_2)}T^A{}_{ij}T^B{}_{kl} + \text{regular}
\end{aligned}
$$

Notice that $T^A{}_{ij} = T^{[ab]}{}_{ij} = \mathrm{i}\big(\delta_i{}^a\delta^b{}_j - \delta_i{}^b\delta^a{}_j\big)$ is antisymmetric in ij, so we reverse the order of every pair of TT to be with T^A with the index a at left and T^B with index b in the right,

$$
\begin{aligned}
:j^A(z_1)j^B(z_2): = j^A(z_1)j^B(z_2) &- \frac{\delta^{ik}}{4(z_1-z_2)}T^A{}_{ai}T^B{}_{kb}\ :\psi^a\psi^b:(z_2) \\
&- \frac{\delta^{il}}{4(z_1-z_2)}T^A{}_{ai}T^B{}_{lb}\ :\psi^a\psi^b:(z_2) - \frac{\delta^{jk}}{4(z_1-z_2)}T^A{}_{aj}T^B{}_{kb}\ :\psi^a\psi^b:(z_2) \\
&- \frac{\delta^{jl}}{4(z_1-z_2)}T^A{}_{aj}T^B{}_{lb}\ :\psi^a\psi^b:(z_2) - \frac{\delta^{ik}\delta^{jl}}{4(z_1-z_2)^2}T^A{}_{ji}T^B{}_{kl} \\
&- \frac{\delta^{il}\delta^{jk}}{4(z_1-z_2)}T^A{}_{ij}T^B{}_{kl} + \text{regular}
\end{aligned}
$$

Now it's trivial to sum the terms,

$$
\begin{aligned}
:j^A(z_1)j^B(z_2): = j^A(z_1)j^B(z_2) &- \frac{\delta^{ik}T^A{}_{ai}T^B{}_{kb}}{(z_1-z_2)}\ :\psi^a\psi^b:(z_2) - \frac{\delta^{ik}\delta^{jl}}{2(z_1-z_2)^2}T^A{}_{ji}T^B{}_{kl} \\
&+ \text{regular}
\end{aligned}
$$

Due to the statistic we have to anti-symmetrize the second term in the right-hand side in the index ab

$$
\begin{aligned}
:j^A(z_1)j^B(z_2): = j^A(z_1)j^B(z_2) &- \frac{T^A{}_a{}^iT^B{}_{ib} - T^A{}_b{}^iT^B{}_{ia}}{2(z_1-z_2)}\ :\psi^a\psi^b:(z_2) - \frac{\operatorname{Tr}\left[T^AT^B\right]}{2(z_1-z_2)^2} \\
&+ \text{regular} \\
:j^A(z_1)j^B(z_2): = j^A(z_1)j^B(z_2) &- \frac{T^A{}_a{}^iT^B{}_{ib} - T^B{}_{ai}T^{Ai}{}_b}{2(z_1-z_2)}\ :\psi^a\psi^b:(z_2) - \frac{\operatorname{Tr}\left[T^AT^B\right]}{2(z_1-z_2)^2} \\
&+ \text{regular}
\end{aligned}
$$

Well the numerator in the second term in the right-hand side is just the commutator of the T's, which is of course written in terms of the structure constants,

$$
:j^A(z_1)j^B(z_2): = j^A(z_1)j^B(z_2) - \frac{\mathrm{i}f^{AB}{}_CT^C{}_{ab}}{2(z_1-z_2)}\ :\psi^a\psi^b:(z_2) - \frac{\operatorname{Tr}\left[T^AT^B\right]}{2(z_1-z_2)^2}
$$

$$+ \text{regular}$$

$$: j^A(z_1)j^B(z_2) : = j^A(z_1)j^B(z_2) - \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}j^C(z_2) - \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2}$$

$$+ \text{regular}$$

So that, finally,

$$j^A(z_1)j^B(z_2) = \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}j^C(z_2) + \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2} + \text{regular} \tag{3.2}$$

3.E)

As ψ has conformal weight $\left(\frac{1}{2},0\right)$ it's clear that j^A will have conformal weight $(1,0)$, so that a mode expansion is,

$$j^A(z) = \sum_{n\in\mathbb{Z}} \frac{j^A_n}{z^{n+1}}$$

$$j^A_n = \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^n j^A(z)$$

Where C is any sufficiently well behaved closed curve around the origin. To obtain the algebra of the current modes, is easier to rewrite 3.2 back in the path integral formalism, from which the radial order is always manifest,

$$\left\langle j^A(z_1)j^B(z_2)\mathcal{O}\right\rangle = \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}\left\langle j^C(z_2)\mathcal{O}\right\rangle + \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2}\left\langle\mathcal{O}\right\rangle + \text{regular}$$

We now integrate this expression with two different contours, C_1 and C_2, C_1 is a contour around the origin which also encloses the point z_2, and C_2 is a contour around the origin that does not encloses z_2, hence,

$$\oint_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\left\langle j^A(z_1)j^B(z_2)\mathcal{O}\right\rangle = \oint_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}\left\langle j^C(z_2)\mathcal{O}\right\rangle + \oint_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2}\left\langle\mathcal{O}\right\rangle + \text{regular}$$

$$\oint_{C_2}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\left\langle j^A(z_1)j^B(z_2)\mathcal{O}\right\rangle = \oint_{C_2}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}\left\langle j^C(z_2)\mathcal{O}\right\rangle + \oint_{C_2}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2}\left\langle\mathcal{O}\right\rangle + \text{regular}$$

Notice that for the second contour, the order of the j inside the expectation value will change once we go to the operator formalism, as for C_2, $|z_2| > |z_1|$, we'll use this to our advantage, first, as for the expectation value the order is irrelevant, due to the implicit radial ordering, we exchange the order of the two j, as we're in the region $|z_2| > |z_1|$,

$$\oint_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\left\langle j^A(z_1)j^B(z_2)\mathcal{O}\right\rangle = \oint_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}\left\langle j^C(z_2)\mathcal{O}\right\rangle + \oint_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2}\left\langle\mathcal{O}\right\rangle + \text{regular}$$

$$\oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^B(z_2) j^A(z_1)\mathcal{O}\right\rangle = \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)} \left\langle j^C(z_2)\mathcal{O}\right\rangle + \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2} \langle\mathcal{O}\rangle + \text{regular}$$

Now we subtract the second equation from the first,

$$\begin{aligned}
\oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^A(z_1) j^B(z_2)\mathcal{O}\right\rangle - \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^B(z_2) j^A(z_1)\mathcal{O}\right\rangle = \\
+ \oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)} \left\langle j^C(z_2)\mathcal{O}\right\rangle - \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)} \left\langle j^C(z_2)\mathcal{O}\right\rangle \\
+ \oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2} \langle\mathcal{O}\rangle - \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2} \langle\mathcal{O}\rangle \\
+ \oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \text{regular} - \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \text{regular}
\end{aligned}$$

$$\begin{aligned}
\oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^A(z_1) j^B(z_2)\mathcal{O}\right\rangle - \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^B(z_2) j^A(z_1)\mathcal{O}\right\rangle = \\
+ \left[\oint_{C_1} - \oint_{C_2}\right] \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)} \left\langle j^C(z_2)\mathcal{O}\right\rangle \\
+ \left[\oint_{C_1} - \oint_{C_2}\right] \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2} \langle\mathcal{O}\rangle \\
+ \left[\oint_{C_1} - \oint_{C_2}\right] \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \text{regular}
\end{aligned}$$

The subtraction of two integrals, one contouring the origin and z_2, C_1, by other just contouring the origin, C_2, is given by an integral over a contour of z_2 but not of the origin C_3,

$$\begin{aligned}
\oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^A(z_1) j^B(z_2)\mathcal{O}\right\rangle - \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^B(z_2) j^A(z_1)\mathcal{O}\right\rangle = \\
+ \oint_{C_3} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)} \left\langle j^C(z_2)\mathcal{O}\right\rangle \\
+ \oint_{C_3} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \frac{\mathrm{Tr}\left[T^A T^B\right]}{2(z_1-z_2)^2} \langle\mathcal{O}\rangle \\
+ \oint_{C_3} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \text{regular}
\end{aligned}$$

As the regular part doesn't have poles at $z_1 = z_2$,

$$\oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^A(z_1) j^B(z_2)\mathcal{O}\right\rangle - \oint_{C_2} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} z_1^n \left\langle j^B(z_2) j^A(z_1)\mathcal{O}\right\rangle =$$

$$+\oint\limits_{C_3}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}\left\langle j^C(z_2)\mathcal{O}\right\rangle$$
$$+\oint\limits_{C_3}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{Tr}\left[T^AT^B\right]}{2(z_1-z_2)^2}\left\langle\mathcal{O}\right\rangle$$

We now can go back to the operator formalism, and, as z_1 is integrated, there is no ordering ambiguities,

$$\begin{aligned}
\oint\limits_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n j^A(z_1)j^B(z_2)-\oint\limits_{C_2}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n j^B(z_2)j^A(z_1) &= \\
&+\oint\limits_{C_3}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{i}f^{AB}{}_C}{(z_1-z_2)}j^C(z_2) \\
&+\oint\limits_{C_3}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}}z_1^n\frac{\mathrm{Tr}\left[T^AT^B\right]}{2(z_1-z_2)^2} \\
j_n^A j^B(z_2)-j^B(z_2)j_n^A &= z_2^n\mathrm{i}f^{AB}{}_C j^C(z_2)+\left(\partial_{z_1}z_1^n\right)\Big|_{z_1=z_2}\frac{\mathrm{Tr}\left[T^AT^B\right]}{2} \\
j_n^A j^B(z_2)-j^B(z_2)j_n^A &= z_2^n\mathrm{i}f^{AB}{}_C j^C(z_2)+nz_2^{n-1}\frac{\mathrm{Tr}\left[T^AT^B\right]}{2}
\end{aligned}$$

Now integrating over a contour C around the origin to get the modes of j^B,

$$\begin{aligned}
\oint\limits_{C}\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}z_2^m\left(j_n^A j^B(z_2)-j^B(z_2)j_n^A\right) &= \oint\limits_{C}\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}z_2^m z_2^n\mathrm{i}f^{AB}{}_C j^C(z_2)+\oint\limits_{C}\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}z_2^m nz_2^{n-1}\frac{\mathrm{Tr}\left[T^AT^B\right]}{2} \\
j_n^A j_m^B - j_m^B j_n^A &= \mathrm{i}f^{AB}{}_C j_{n+m}^C+\frac{n}{2}\delta_{m+n,0}\,\mathrm{Tr}\left[T^AT^B\right] \\
\left[j_n^A, j_m^B\right] &= \mathrm{i}f^{AB}{}_C j_{n+m}^C+\frac{n}{2}\delta_{m+n,0}\,\mathrm{Tr}\left[T^AT^B\right]
\end{aligned}$$

This is the algebra of the modes! We used that z_2^{n+m-1} only has a residue at $z_2=0$ for $n+m=0$. The value $\mathrm{Tr}\left[T^AT^B\right]$ is dependent of the normalization of the algebra and the representation, but usually it can be taken to be $\mathrm{Tr}\left[T^AT^B\right]=k\delta^{AB}$.

3.F)

To have a $SU(N)$ group, it's obvious that we need a transformation law $\psi'^i=\Lambda^i{}_j\psi^j$, such that,

$$\delta_{ij}=\delta_{kl}\Lambda^{*i}{}_k\Lambda^j{}_l\Leftrightarrow\Lambda^\dagger\Lambda=\mathbb{1}$$

This is only possible if we include in the action a complex fermion, and, one of them being a complex conjugate, that is, now we have a collection of $2N$ real fermions,

$$\psi^i=\mathrm{Re}\left[\psi^i\right]+\mathrm{i}\mathrm{Im}\left[\psi^i\right]$$

Which compose a N dimensional vector, while seems natural to express a complex field as a pair $\psi, \bar{\psi}$, the notation $\bar{\psi}$ is kind of misleading, does it is a meromorphic or anti-meromorphic function/operator? So, we will choose do denote the complex fermion, and it's complex conjugate by,

$$\psi^i = \mathrm{Re}\big[\psi^i\big] + \mathrm{i}\mathrm{Im}\big[\psi^i\big]$$
$$\tilde{\psi}^i = \mathrm{Re}\big[\psi^i\big] - \mathrm{i}\mathrm{Im}\big[\psi^i\big]$$

This could be seen as a change of variables from a model of $2N$ free real fermions, which of course enjoy a $O(2N)$ symmetry, and as $U(N) \subset O(2N)$, it's possible for us to have a $SU(N)$ symmetry. We impose the following transformations,

$$\psi'^i = \Lambda^i{}_j \psi^j, \quad \tilde{\psi}'^i = \Lambda^{*i}{}_j \tilde{\psi}^j$$

So that the action is,

$$S = \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \tilde{\psi}^i \bar{\partial} \psi^j$$
$$S' = \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \tilde{\psi}'^i \bar{\partial} \psi'^j$$
$$S' = \frac{1}{4\pi} \int \mathrm{d}^2 z \, \delta_{ij} \Lambda^{*i}{}_k \Lambda^j{}_l \tilde{\psi}^k \bar{\partial} \psi^l$$

From where we get our long waited $U(N)$ constraint, $\delta_{ij}\Lambda^{*i}{}_k\Lambda^j{}_l = \delta_{kl}$, but, no longer we have any problem requiring the group to be connected, as $U(N)$ is already connected, but nevertheless, there is a number of reasons why we would prefer to have only $SU(N)$, to go from $U(N)$ to $SU(N)$ is simple, just impose $\mathrm{Det}[\Lambda] = 1$.

Problem 4

4.A)

Just restating the hypothesis,

$$\begin{aligned} X(z)X(w) &= -\ln{(z-w)} + \text{regular} \\ T(z) &= -\frac{1}{2} : \partial X \partial X : (z) \\ V_n(z) &=: \exp{(\mathrm{i}nX)} : (z) \end{aligned}$$

To compute the TV_n OPE, we start by computing the normal ordered product,

$$\begin{aligned} : T(z)V_n(w) : &= T(z)V_n(w) - \frac{1}{2} : (\partial X \partial X)(z) \exp(\mathrm{i}nX)(w) : \\ &\quad - \frac{1}{2} : (\partial X \partial X)(z) \exp(\mathrm{i}nX)(w) : - \frac{1}{2} : (\partial X \partial X)(z) \exp(\mathrm{i}nX)(w) : \\ : T(z)V_n(w) : &= T(z)V_n(w) - \frac{\mathrm{i}n}{2} \partial_z(\ln{(z-w)}) : \partial X(z) \exp(\mathrm{i}nX)(w) : \\ &\quad - \frac{\mathrm{i}n}{2} \partial_z(\ln{(z-w)}) : \partial X \exp(\mathrm{i}nX)(w) : \\ &\quad + \frac{\mathrm{i}n\mathrm{i}n}{2} \partial_z(\ln{(z-w)}) \partial_z(\ln{(z-w)}) : \exp(\mathrm{i}nX)(w) : \\ : T(z)V_n(w) : &= T(z)V_n(w) - \frac{\mathrm{i}n}{2(z-w)} : \partial X(z) \exp(\mathrm{i}nX)(w) : \\ &\quad - \frac{\mathrm{i}n}{2(z-w)} : \partial X(z) \exp(\mathrm{i}nX)(w) : \\ &\quad - \frac{n^2}{2(z-w)^2} : \exp(\mathrm{i}nX)(w) : \\ : T(z)V_n(w) : &= T(z)V_n(w) - \frac{\mathrm{i}n}{(z-w)} : \partial X(z) \exp(\mathrm{i}nX)(w) : - \frac{n^2}{2(z-w)^2} : \exp(\mathrm{i}nX) : (w) \\ : T(z)V_n(w) : &= T(z)V_n(w) - \frac{\mathrm{i}n}{(z-w)} : \partial X \exp(\mathrm{i}nX) : (w) - \frac{n^2 V_n(w)}{2(z-w)^2} + \text{regular} \\ : T(z)V_n(w) : &= T(z)V_n(w) - \frac{1}{(z-w)} \partial : \exp(\mathrm{i}nX) : (w) - \frac{n^2 V_n(w)}{2(z-w)^2} + \text{regular} \\ : T(z)V_n(w) : &= T(z)V_n(w) - \frac{\partial V_n(w)}{(z-w)} - \frac{n^2 V_n(w)}{2(z-w)^2} + \text{regular} \end{aligned}$$

That is, the OPE is,

$$T(z)V_n(w) = \frac{\partial V_n(w)}{(z-w)} + \frac{n^2 V_n(w)}{2(z-w)^2} + \text{regular} \tag{4.1}$$

4.B)

First, the OPE of j with itself,

$$
\begin{aligned}
: j(z)j(w) : &= j(z)j(w) - : \partial X(z)\partial X(w) : \\
: j(z)j(w) : &= j(z)j(w) - \partial_z\partial_w(\ln{(z-w)}) \\
: j(z)j(w) : &= j(z)j(w) + \partial_z\frac{1}{z-w} \\
: j(z)j(w) : &= j(z)j(w) - \frac{1}{(z-w)^2}
\end{aligned}
$$

So that,

$$
j(z)j(w) = \frac{1}{(z-w)^2} + \text{regular}
$$

Now the jV_n OPE,

$$
\begin{aligned}
: j(z)V_n(w) : &= j(z)V_n(w) + \mathrm{i} : \partial X(z)\exp(\mathrm{i}nX)(w) : \\
: j(z)V_n(w) : &= j(z)V_n(w) + \mathrm{i}\mathrm{i}n\partial_z(\ln{(z-w)}) : \exp{(\mathrm{i}nX)} : (w) \\
: j(z)V_n(w) : &= j(z)V_n(w) - \frac{n}{z-w} : \exp{(\mathrm{i}nX)} : (w) \\
: j(z)V_n(w) : &= j(z)V_n(w) - \frac{n}{z-w}V_n(w)
\end{aligned}
$$

So that the OPE is,

$$
j(z)V_n(w) = \frac{n}{z-w}V_n(w) + \text{regular}
$$

In the last problem we already derived how to obtain the algebra of modes from the OPE, we simply integrate over a contour around w with weight z^p, we need to remember that j has conformal weight $(1,0)$, so that the integration with respect to z^p around the origin gives j_p,

$$
\begin{aligned}
[j_p, j(w)] &= \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^p j(z)j(w) = \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{z^p}{(z-w)^2} + \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^p \text{regular} \\
[j_p, V_n(w)] &= \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^p j(z)V_n(w) = \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{nz^p}{z-w}V_n(w) + \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^p \text{regular}
\end{aligned}
$$

The regular part doesn't have any pole at $z = w$, hence,

$$
\begin{aligned}
[j_p, j(w)] &= \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{z^p}{(z-w)^2} = \partial_z(z^p)\Big|_{z=w} = pw^{p-1} \\
[j_p, V_n(w)] &= \oint\limits_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{nz^p}{z-w}V_n(w) = nw^pV_n(w)
\end{aligned}
\tag{4.2}
$$

We integrate over again to obtain the commutator of the modes only, this time with a contour around the origin, but, notice from 4.1, V_n has conformal weight $\left(\frac{n^2}{2}, 0\right)$, thus, when integrating around the origin with w^q, this will give the mode $V_{n\left(q+1-\frac{n^2}{2}\right)}$,

$$[j_p, j_q] = \oint\limits_C \frac{\mathrm{d}w}{2\pi\mathrm{i}} w^q [j_n, j(w)] = \oint\limits_C \frac{\mathrm{d}w}{2\pi\mathrm{i}} w^q p w^{p-1} = p\delta_{p+q,0}$$

$$\left[j_p, V_{n\left(q+1-\frac{n^2}{2}\right)}\right] = \oint\limits_C \frac{\mathrm{d}w}{2\pi\mathrm{i}} w^q [j_n, V_n(w)] = \oint\limits_C \frac{\mathrm{d}w}{2\pi\mathrm{i}} w^q n w^p V_n(w) = n V_{n\left(p+q+1-\frac{n^2}{2}\right)}$$

From the first algebra, the jj one, we get that j is the current associated with some $SO(2)$ or $U(1)$ algebra, as there is no structure constants and just one current, and the first algebra is just the Kac-Moody central extension of this $SO(2)$ or $U(1)$ algebra. The usual conserved charge obtained from a conserved current is just j_0, from where we can see that,

$$\left[j_0, V_{n\left(q+1-\frac{n^2}{2}\right)}\right] = n V_{n\left(q+1-\frac{n^2}{2}\right)}$$

Or even from 4.2,

$$[j_0, V_n(w)] = n V_n(w) \tag{4.3}$$

This is saying to us that, under transformations generated by j_0 — Which are the symmetries that originate the j current —, the modes of V_n, and of course also V_n itself, change just by a scaling of n, the correct relation between commutator with current and change of operators is,

$$[j_0, A] = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} A \bigg|_{t=0}$$

Where t is a parametrization of the transformation, in our case,

$$\frac{\mathrm{d}}{\mathrm{d}t} V_n(w) \bigg|_{t=0} = \mathrm{i} n V_n(w)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_n(w;t) = \mathrm{i} n V_n(w;t) \Rightarrow V_n(w;t) = V_n(w) \exp{(\mathrm{i} n t)}$$

This makes clear that the transformation generated by j_0 is a $U(1)$ transformation, or in particular, $V_n(w)$ transforms under a $U(1)$ representation of the symmetry generated by j_0, with '*charge*' n.

4.C)

The energy momentum tensor has to have conformal weight $(2, 0)$ and be symmetric under the $U(1)$,

$$\psi \to \exp{(\mathrm{i}t)}\psi, \quad \tilde{\psi} \to \exp{(-\mathrm{i}t)}\tilde{\psi}$$

There are just two terms which are compatible with this,

$$T_\psi(z) = \alpha :\tilde{\psi}\partial\psi: (z) + \beta :\partial\tilde{\psi}\psi: (z)$$

To fix α, β we have to compute the OPE's $T\psi, T\tilde{\psi}$,

$$
\begin{aligned}
: T_\psi(z)\psi(w) : &= T_\psi(z)\psi(w) + \alpha : (\tilde{\psi}\partial\psi)(z)\psi(w) : + \beta : (\partial\tilde{\psi}\psi)(z)\psi(w) : \\
: T_\psi(z)\psi(w) : &= T_\psi(z)\psi(w) + \alpha \frac{1}{z-w}\partial\psi(z) + \beta\partial_z\left(\frac{1}{z-w}\right)\psi(z) \\
: T_\psi(z)\psi(w) : &= T_\psi(z)\psi(w) + \alpha \frac{1}{z-w}\partial\psi(w) - \frac{\beta}{(z-w)^2}\psi(z) + \text{regular} \\
: T_\psi(z)\psi(w) : &= T_\psi(z)\psi(w) + \alpha \frac{1}{z-w}\partial\psi(w) - \frac{\beta}{(z-w)^2}\psi(w) - \frac{\beta}{z-w}\partial\psi(w) + \text{regular} \\
T_\psi(z)\psi(w) &= \frac{\beta-\alpha}{z-w}\partial\psi(w) + \frac{\beta}{(z-w)^2}\psi(z) + \text{regular}
\end{aligned}
$$

This fixes $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$, otherwise we have the wrong conformal transformations, so,

$$
T_\psi(z) = -\frac{1}{2} : \tilde{\psi}\partial\psi : (z) + \frac{1}{2} : \partial\tilde{\psi}\psi : (z)
$$

For the current, we need it to have conformal weight $(1,0)$ and to be hermitian, the only possible term is,

$$
j_\psi(z) = \alpha : \tilde{\psi}\psi : (z)
$$

With α real. To fix it we compute the $j\psi$ OPE,

$$
\begin{aligned}
: j_\psi(z)\psi(w) : &= j_\psi(z)\psi(w) + \alpha : (\tilde{\psi}\psi)(z)\psi(w) : \\
: j_\psi(z)\psi(w) : &= j_\psi(z)\psi(w) + \frac{\alpha}{z-w}\psi(z) \\
: j_\psi(z)\psi(w) : &= j_\psi(z)\psi(w) + \frac{\alpha}{z-w}\psi(w) + \text{regular} \\
j_\psi(z)\psi(w) &= -\frac{\alpha}{z-w}\psi(w) + \text{regular} \\
\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} j_\psi(z)\psi(w) &= -\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\alpha}{z-w}\psi(w) + \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \text{regular}
\end{aligned}
$$

Where C is any contour that encloses w, again, the regular terms have no poles at $z = w$,

$$
\begin{aligned}
[j_0, \psi(w)] &= \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} j_\psi(z)\psi(w) = -\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\alpha}{z-w}\psi(w) \\
[j_0, \psi(w)] &= -\oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\alpha}{z-w}\psi(w) = -\alpha\psi(w) \\
\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}[\exp(\mathrm{i}t)\psi(w)]\bigg|_{t=0} &= -\alpha\psi(w) \Rightarrow \alpha = -1
\end{aligned}
$$

In other words, α is specified once we set the transformation of ψ, in this case setting the charge of it to be 1 is enough. So,

$$j_{\psi}(z) = - :\tilde{\psi}\psi:(z)$$

And then, we compute the remaining OPE's, starting by $T_{\psi}j_{\psi}$,

$$\begin{aligned}
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &+ \frac{1}{2}:(\tilde{\psi}\partial\psi)(z)(\tilde{\psi}\psi)(w): + \frac{1}{2}:(\tilde{\psi}\partial\psi)(z)(\tilde{\psi}\psi)(w): \\
&\frac{1}{2} + :(\tilde{\psi}\partial\psi)(z)(\tilde{\psi}\psi)(w): - \frac{1}{2}:(\partial\tilde{\psi}\psi)(z)(\tilde{\psi}\psi)(w): \\
&- \frac{1}{2}:(\partial\tilde{\psi}\psi)(z)(\tilde{\psi}\psi)(w): - \frac{1}{2}:(\partial\tilde{\psi}\psi)(z)(\tilde{\psi}\psi)(w): \\
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &- \frac{1}{2}\frac{1}{z-w}:\partial\psi(z)\tilde{\psi}(w): - \frac{1}{2}\partial_z\left(\frac{1}{z-w}\right):\tilde{\psi}(z)\psi(w): \\
&- \frac{1}{2}\frac{1}{z-w}\partial_z\left(\frac{1}{z-w}\right) + \frac{1}{2}\partial_z\left(\frac{1}{z-w}\right):\psi(z)\tilde{\psi}(w): \\
&+ \frac{1}{2}\frac{1}{z-w}:\partial\tilde{\psi}(z)\psi(w): + \frac{1}{2}\frac{1}{z-w}\partial_z\left(\frac{1}{z-w}\right) \\
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &- \frac{1}{2}\frac{1}{z-w}:\partial\psi(z)\tilde{\psi}(w): + \frac{1}{2}\frac{1}{(z-w)^2}:\tilde{\psi}(z)\psi(w): \\
&+ \frac{1}{2}\frac{1}{(z-w)^3} - \frac{1}{2}\frac{1}{(z-w)^2}:\psi(z)\tilde{\psi}(w): \\
&+ \frac{1}{2}\frac{1}{z-w}:\partial\tilde{\psi}(z)\psi(w): - \frac{1}{2}\frac{1}{(z-w)^3} \\
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &- \frac{1}{2}\frac{:\partial\psi\tilde{\psi}:(w)}{z-w} + \frac{1}{2}\frac{:\tilde{\psi}\psi:(w)}{(z-w)^2} + \frac{1}{2}\frac{:\partial\tilde{\psi}\psi:(w)}{z-w} \\
&- \frac{:\psi\tilde{\psi}:(w)}{2(z-w)^2} - \frac{:\partial\psi\tilde{\psi}:(w)}{2(z-w)} + \frac{:\partial\tilde{\psi}\psi:(w)}{2(z-w)} + \text{regular} \\
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &- \frac{:\partial\psi\tilde{\psi}:(w)}{z-w} + \frac{1}{2}\frac{:\tilde{\psi}\psi:(w)}{(z-w)^2} + \frac{:\partial\tilde{\psi}\psi:(w)}{z-w} \\
&- \frac{:\psi\tilde{\psi}:(w)}{2(z-w)^2} + \text{regular} \\
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &+ \frac{:\tilde{\psi}\partial\psi:(w)}{z-w} + \frac{:\partial\tilde{\psi}\psi:(w)}{z-w} + \frac{1}{2}\frac{:\tilde{\psi}\psi:(w)}{(z-w)^2} \\
&+ \frac{:\tilde{\psi}\psi:(w)}{2(z-w)^2} + \text{regular} \\
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &+ \frac{\partial:\tilde{\psi}\psi:(w)}{z-w} + \frac{:\tilde{\psi}\psi:(w)}{(z-w)^2} + \text{regular} \\
:T_{\psi}(z)j_{\psi}(w): = T_{\psi}(z)j_{\psi}(w) &- \frac{\partial j_{\psi}(w)}{z-w} - \frac{j_{\psi}(w)}{(z-w)^2} + \text{regular}
\end{aligned}$$

So that the OPE is,

$$T_{\psi}(z)j_{\psi}(w) = \frac{\partial j_{\psi}(w)}{z-w} + \frac{j_{\psi}(w)}{(z-w)^2} + \text{regular}$$

Which has the correct values for weight $(1,0)$. Now jj OPE,

$$
\begin{aligned}
:j_\psi(z)j_\psi(w):&=j_\psi(z)j_\psi(w)+:(\tilde{\psi}\psi)(z)(\tilde{\psi}\psi)(w):+:(\tilde{\psi}\psi)(z)(\tilde{\psi}\psi)(w):\\
&\qquad+:(\tilde{\psi}\psi)(z)(\tilde{\psi}\psi)(w):\\
:j_\psi(z)j_\psi(w):&=j_\psi(z)j_\psi(w)-\frac{1}{z-w}:\psi(z)\tilde{\psi}(w):-\frac{1}{z-w}:\tilde{\psi}(z)\psi(w):\\
&\qquad-\frac{1}{(z-w)^2}\\
:j_\psi(z)j_\psi(w):&=j_\psi(z)j_\psi(w)-\frac{1}{z-w}:\psi\tilde{\psi}:(w)-\frac{1}{z-w}:\tilde{\psi}\psi:(w)\\
&\qquad-\frac{1}{(z-w)^2}+\text{regular}\\
:j_\psi(z)j_\psi(w):&=j_\psi(z)j_\psi(w)+\frac{1}{z-w}:\tilde{\psi}\psi:(w)-\frac{1}{z-w}:\tilde{\psi}\psi:(w)\\
&\qquad-\frac{1}{(z-w)^2}+\text{regular}\\
:j_\psi(z)j_\psi(w):&=j_\psi(z)j_\psi(w)-\frac{1}{(z-w)^2}+\text{regular}
\end{aligned}
$$

So that the OPE is,

$$
j_\psi(z)j_\psi(w)=\frac{1}{(z-w)^2}+\text{regular}
$$

At last, the TT OPE,

$$
\begin{aligned}
:T_\psi(z)T_\psi(w):&=T_\psi(z)T_\psi(w)+\frac{1}{4}:(\tilde{\psi}\partial\psi)(z)(\tilde{\psi}\partial\psi)(w):+\frac{1}{4}:(\tilde{\psi}\partial\psi)(z)(\tilde{\psi}\partial\psi)(w):\\
&\qquad+\frac{1}{4}:(\tilde{\psi}\partial\psi)(z)(\tilde{\psi}\partial\psi)(w):-\frac{1}{4}:(\tilde{\psi}\partial\psi)(z)(\partial\tilde{\psi}\psi)(w):\\
&\qquad-\frac{1}{4}:(\tilde{\psi}\partial\psi)(z)(\partial\tilde{\psi}\psi)(w):-\frac{1}{4}:(\tilde{\psi}\partial\psi)(z)(\partial\tilde{\psi}\psi)(w):\\
&\qquad-\frac{1}{4}:(\partial\tilde{\psi}\psi)(z)(\tilde{\psi}\partial\psi)(w):-\frac{1}{4}:(\partial\tilde{\psi}\psi)(z)(\tilde{\psi}\partial\psi)(w):\\
&\qquad-\frac{1}{4}:(\partial\tilde{\psi}\psi)(z)(\tilde{\psi}\partial\psi)(w):+\frac{1}{4}:(\partial\tilde{\psi}\psi)(z)(\partial\tilde{\psi}\psi)(w):\\
&\qquad+\frac{1}{4}:(\partial\tilde{\psi}\psi)(z)(\partial\tilde{\psi}\psi)(w):+\frac{1}{4}:(\partial\tilde{\psi}\psi)(z)(\partial\tilde{\psi}\psi)(w):\\
:T_\psi(z)T_\psi(w):&=T_\psi(z)T_\psi(w)-\frac{1}{4}\partial_w\left(\frac{1}{z-w}\right):\partial\psi(z)\tilde{\psi}(w):-\frac{1}{4}\partial_z\left(\frac{1}{z-w}\right):\tilde{\psi}(z)\partial\psi(w):\\
&\qquad-\frac{1}{4}\partial_z\left(\frac{1}{z-w}\right)\partial_w\left(\frac{1}{z-w}\right)+\frac{1}{4}\frac{1}{z-w}:\partial\psi(z)\partial\tilde{\psi}(w):\\
&\qquad+\frac{1}{4}\partial_z\partial_w\left(\frac{1}{z-w}\right):\tilde{\psi}(z)\psi(w):+\frac{1}{4}\partial_z\partial_w\left(\frac{1}{z-w}\right)\frac{1}{z-w}\\
&\qquad+\frac{1}{4}\partial_z\partial_w\left(\frac{1}{z-w}\right):\psi(z)\tilde{\psi}(w):+\frac{1}{4}\frac{1}{z-w}:\partial\tilde{\psi}(z)\partial\psi(w):
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\partial_z\partial_w\left(\frac{1}{z-w}\right)\frac{1}{z-w}-\frac{1}{4}\partial_z\left(\frac{1}{z-w}\right):\psi(z)\partial\tilde{\psi}(w):\\
&-\frac{1}{4}\partial_w\left(\frac{1}{z-w}\right):\partial\tilde{\psi}(z)\psi(w):-\frac{1}{4}\partial_z\left(\frac{1}{z-w}\right)\partial_w\left(\frac{1}{z-w}\right)
\end{aligned}
$$

$$
\begin{aligned}
:T_\psi(z)T_\psi(w):=T_\psi(z)T_\psi(w)&-\frac{:\partial\psi(z)\tilde{\psi}(w):}{4(z-w)^2}+\frac{:\tilde{\psi}(z)\partial\psi(w):}{4(z-w)^2}\\
&+\frac{1}{4}\frac{1}{(z-w)^4}+\frac{:\partial\psi(z)\partial\tilde{\psi}(w):}{4(z-w)}\\
&-\frac{1}{2}\frac{:\tilde{\psi}(z)\psi(w):}{(z-w)^3}-\frac{1}{2}\frac{1}{(z-w)^4}\\
&-\frac{1}{2}\frac{:\psi(z)\tilde{\psi}(w):}{(z-w)^3}+\frac{:\partial\tilde{\psi}(z)\partial\psi(w):}{4(z-w)}\\
&-\frac{1}{2}\frac{1}{(z-w)^4}+\frac{1}{4}\frac{:\psi(z)\partial\tilde{\psi}(w):}{(z-w)^2}\\
&-\frac{1}{4}\frac{:\partial\tilde{\psi}(z)\psi(w):}{(z-w)^2}+\frac{1}{4}\frac{1}{(z-w)^4}
\end{aligned}
$$

$$
\begin{aligned}
:T_\psi(z)T_\psi(w):=T_\psi(z)T_\psi(w)&-\frac{:\partial\psi\tilde{\psi}:(w)}{4(z-w)^2}-\frac{:\partial^2\psi\tilde{\psi}:(w)}{4(z-w)}+\frac{:\tilde{\psi}\partial\psi:(w)}{4(z-w)^2}+\frac{:\partial\tilde{\psi}\partial\psi:(w)}{4(z-w)}\\
&+\frac{:\partial\psi\partial\tilde{\psi}:(w)}{4(z-w)}-\frac{1}{2}\frac{1}{(z-w)^4}\\
&-\frac{1}{2}\frac{:\tilde{\psi}\psi:(w)}{(z-w)^3}-\frac{1}{2}\frac{:\partial\tilde{\psi}\psi:(w)}{(z-w)^2}-\frac{1}{2}\frac{:\partial^2\tilde{\psi}\psi:(w)}{2(z-w)}\\
&-\frac{1}{2}\frac{:\psi\tilde{\psi}:(w)}{(z-w)^3}-\frac{1}{2}\frac{:\partial\psi\tilde{\psi}:(w)}{(z-w)^2}-\frac{1}{2}\frac{:\partial^2\psi\tilde{\psi}:(w)}{2(z-w)}+\frac{:\partial\tilde{\psi}\partial\psi:(w)}{4(z-w)}\\
&+\frac{1}{4}\frac{:\psi\partial\tilde{\psi}:(w)}{(z-w)^2}+\frac{1}{4}\frac{:\partial\psi\partial\tilde{\psi}:(w)}{(z-w)}\\
&-\frac{1}{4}\frac{:\partial\tilde{\psi}\psi:(w)}{(z-w)^2}-\frac{1}{4}\frac{:\partial^2\tilde{\psi}\psi:(w)}{(z-w)}+\text{regular}
\end{aligned}
$$

$$
\begin{aligned}
:T_\psi(z)T_\psi(w):=T_\psi(z)T_\psi(w)&+\frac{:\tilde{\psi}\partial\psi:(w)}{4(z-w)^2}+\frac{:\tilde{\psi}\partial^2\psi:(w)}{4(z-w)}+\frac{:\tilde{\psi}\partial\psi:(w)}{4(z-w)^2}+\frac{:\partial\tilde{\psi}\partial\psi:(w)}{4(z-w)}\\
&-\frac{:\partial\tilde{\psi}\partial\psi:(w)}{4(z-w)}-\frac{1}{2}\frac{1}{(z-w)^4}\\
&-\frac{1}{2}\frac{:\tilde{\psi}\psi:(w)}{(z-w)^3}-\frac{1}{2}\frac{:\partial\tilde{\psi}\psi:(w)}{(z-w)^2}-\frac{1}{2}\frac{:\partial^2\tilde{\psi}\psi:(w)}{2(z-w)}\\
&+\frac{1}{2}\frac{:\tilde{\psi}\psi:(w)}{(z-w)^3}+\frac{1}{2}\frac{:\tilde{\psi}\partial\psi:(w)}{(z-w)^2}+\frac{1}{2}\frac{:\tilde{\psi}\partial^2\psi:(w)}{2(z-w)}+\frac{:\partial\tilde{\psi}\partial\psi:(w)}{4(z-w)}\\
&-\frac{1}{4}\frac{:\partial\tilde{\psi}\psi:(w)}{(z-w)^2}-\frac{1}{4}\frac{:\partial\tilde{\psi}\partial\psi:(w)}{(z-w)}\\
&-\frac{1}{4}\frac{:\partial\tilde{\psi}\psi:(w)}{(z-w)^2}-\frac{1}{4}\frac{:\partial^2\tilde{\psi}\psi:(w)}{(z-w)}+\text{regular}
\end{aligned}
$$

$$
\begin{aligned}
:T_\psi(z)T_\psi(w): = T_\psi(z)T_\psi(w) &+ \frac{:\tilde{\psi}\partial\psi:(w)}{(z-w)^2} + \frac{:\tilde{\psi}\partial^2\psi:(w)}{2(z-w)} \\
&- \frac{1}{2}\frac{1}{(z-w)^4} - \frac{:\partial\tilde{\psi}\psi:(w)}{(z-w)^2} - \frac{:\partial^2\tilde{\psi}\psi:(w)}{2(z-w)} \\
&+ \text{regular} \\
:T_\psi(z)T_\psi(w): = T_\psi(z)T_\psi(w) &- \frac{2}{(z-w)^2}\left(\frac{1}{2}:\partial\tilde{\psi}\psi:(w) - \frac{1}{2}:\tilde{\psi}\partial\psi:(w)\right) \\
&- \frac{1}{2}\frac{1}{(z-w)^4} \\
&- \frac{1}{z-w}\left(\frac{1}{2}:\partial^2\tilde{\psi}\psi:(w) + \frac{1}{2}:\tilde{\psi}\partial^2\psi:(w) + \frac{1}{2}:\partial\tilde{\psi}\partial\psi:(w) - \frac{1}{2}:\partial\tilde{\psi}\partial\psi:(w)\right) \\
&+ \text{regular} \\
:T_\psi(z)T_\psi(w): = T_\psi(z)T_\psi(w) &- \frac{2}{(z-w)^2}T_\psi(w) \\
&- \frac{1}{2}\frac{1}{(z-w)^4} \\
&- \frac{1}{z-w}\partial\left(\frac{1}{2}:\partial\tilde{\psi}\psi:(w) + \frac{1}{2}:\tilde{\psi}\partial\psi:(w)\right) \\
&+ \text{regular} \\
:T_\psi(z)T_\psi(w): = T_\psi(z)T_\psi(w) &- \frac{2}{(z-w)^2}T_\psi(w) \\
&- \frac{1}{2}\frac{1}{(z-w)^4} \\
&- \frac{1}{z-w}\partial T_\psi(w) \\
&+ \text{regular}
\end{aligned}
$$

So that the OPE is of the canonical form,

$$
T_\psi(z)T_\psi(w) = \frac{2}{(z-w)^2}T_\psi(w) + \frac{1}{2}\frac{1}{(z-w)^4} + \frac{1}{z-w}\partial T_\psi(w) + \text{regular}
$$

4.D)

Let's compute the OPE V_1V_{-1},

$$
::\exp(\mathrm{i}X)(z)::\exp(-\mathrm{i}X)(w):: =:\exp(\mathrm{i}X)(z)::\exp(-\mathrm{i}X)(w): + :\contraction{\exp(\mathrm{i}}{X}{(z)\exp(-\mathrm{i}}{X}\exp(\mathrm{i}X)(z)\exp(-\mathrm{i}X)(w):
$$

This contraction is a lot more complicated, thus, we'll have to use a definition of normal ordering somewhat more robust, it still is the same procedure we have doing until now, but, it's far more general in it's use, one difference is that it reverses the order we have been using for computing, instead of computing the normal ordered product through the contractions, in this form we'll be computing the OPE from the normal ordered product,

$$
:\exp(\mathrm{i}X)(z)::\exp(-\mathrm{i}X)(w):=
$$

$$
\begin{aligned}
&=: \exp\left(-\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \ln(z_1 - z_2) \frac{\delta}{\delta Y(z_1)} \frac{\delta}{\delta Z(z_2)}\right) : \exp(\mathrm{i}Y)(z) :: \exp(-\mathrm{i}Z)(w) :: \Big|_{Y=Z=X} \\
&: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) := \\
&=:: \exp(\mathrm{i}Y) : (z) \exp\left(-\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \ln(z_1 - z_2) \mathrm{i}\delta^{(2)}(z - z_1) \frac{\delta}{\delta Z(z_2)}\right) : \exp(-\mathrm{i}Z)(w) :: \Big|_{Y=Z=X} \\
&: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) := \\
&=:: \exp(\mathrm{i}Y)(z) :: \exp(-\mathrm{i}Z)(w) : \exp\left(-\mathrm{i}\int \mathrm{d}^2 z_2 \ln(z - z_2)(-\mathrm{i})\delta^{(2)}(w - z_2)\right) : \Big|_{Y=Z=X} \\
&: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) := \\
&=:: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) : \exp(-\ln(z - w)) : \\
&: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) := \frac{1}{z - w} :: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) :: \\
&: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) := \frac{1}{z - w} :: \exp(\mathrm{i}X)(w) :: \exp(-\mathrm{i}X)(w) :: + \text{regular} \\
&: \exp(\mathrm{i}X)(z) :: \exp(-\mathrm{i}X)(w) := \frac{1}{z - w} + \text{regular} \\
&V_1(z) V_{-1}(w) = \frac{1}{z - w} + \text{regular} \qquad (4.4)
\end{aligned}
$$

That is, V_1, V_{-1} seems to have a fermionic statistic, thats due the OPE being an odd power, this is clearer if we compute,

$$
\begin{aligned}
&: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) := \\
&=: \exp\left(-\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \ln(z_1 - z_2) \frac{\delta}{\delta Y(z_1)} \frac{\delta}{\delta Z(z_2)}\right) : \exp(-\mathrm{i}Y)(z) :: \exp(\mathrm{i}Z)(w) :: \Big|_{Y=Z=X} \\
&: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) := \\
&=:: \exp(-\mathrm{i}Y) : (z) \exp\left(-\int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \ln(z_1 - z_2)(-\mathrm{i})\delta^{(2)}(z - z_1) \frac{\delta}{\delta Z(z_2)}\right) : \exp(\mathrm{i}Z)(w) :: \Big|_{Y=Z=X} \\
&: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) := \\
&=:: \exp(-\mathrm{i}Y)(z) :: \exp(\mathrm{i}Z)(w) : \exp\left(\mathrm{i}\int \mathrm{d}^2 z_2 \ln(z - z_2)\mathrm{i}\delta^{(2)}(w - z_2)\right) : \Big|_{Y=Z=X} \\
&: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) := \\
&=:: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) : \exp(-\ln(z - w)) : \\
&: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) := \frac{1}{z - w} :: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) :: \\
&: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) := \frac{1}{z - w} :: \exp(-\mathrm{i}X)(w) :: \exp(\mathrm{i}X)(w) :: + \text{regular} \\
&: \exp(-\mathrm{i}X)(z) :: \exp(\mathrm{i}X)(w) := \frac{1}{z - w} + \text{regular} \\
&V_{-1}(z) V_1(w) = \frac{1}{z - w} + \text{regular} \qquad (4.5)
\end{aligned}
$$

What means,

$$
V_{-1}(z) V_1(w) = \frac{1}{z - w} + \text{regular}
$$

$$
\begin{aligned}
V_{-1}(z)V_1(w) &= \frac{1}{z-w} :: \exp{(-\mathrm{i}X)(z)} :: \exp{(\mathrm{i}X)(w)} :: \\
V_{-1}(z)V_1(w) &= \frac{1}{z-w} :: \exp{(\mathrm{i}X)(w)} :: \exp{(-\mathrm{i}X)(z)} :: \\
V_{-1}(z)V_1(w) &= -\frac{1}{w-z} :: \exp{(\mathrm{i}X)(w)} :: \exp{(-\mathrm{i}X)(z)} :: \\
V_{-1}(z)V_1(w) &= -V_1(w)V_{-1}(z)
\end{aligned}
$$

This can be traced down back to X being chiral, in other words, the XX OPE is,

$$
X(z)X(w) = -\frac{1}{2}\ln{(z-w)^2} + \text{regular}
$$

Instead of,

$$
X(z,\bar{z})X(w,\bar{w}) = -\frac{1}{2}\ln{|z-w|^2} + \text{regular}
$$

What gives rise to,

$$
V_1(z)V_{-1}(w) = \frac{1}{z-w} + \text{regular}
$$

Instead of,

$$
V_1(z,\bar{z})V_{-1}(w,\bar{w}) = \frac{1}{|z-w|} + \text{regular}
$$

Besides the OPE 4.5 being very similar to the $\tilde{\psi}\psi$ OPE, as well as both V_1, ψ and V_{-1}, ψ having charge the same charge, by 4.3, and all of them having the same conformal weight, by 4.1. This suggest the following identification,

$$
\begin{aligned}
V_1(z) &\to \alpha\psi(z) \\
V_{-1}(z) &\to \alpha^{-1}\tilde{\psi}(z)
\end{aligned}
$$

With α any nonzero number.

4.E)

Let's recall some general results derived before, from 4.3 and 4.1 we know that V_n has charge n and has conformal weight $\left(\frac{n^2}{2}, 0\right)$, the only possible building blocks to construct it are $\partial, \psi, \tilde{\psi}$, they have respective weights $1, \frac{1}{2}, \frac{1}{2}$ and charges $0, 1, -1$, so, a generic combination is given by,

$$
\partial^a \psi^b \tilde{\psi}^c
$$

Which of course has to have weight $a + \frac{b+c}{2} = \frac{n^2}{2}$ and charge $b - c = n$, in the special case of $n = 2$ we have to solve the linear equations,

$$
\begin{cases}
a + \frac{b+c}{2} &= 2 \\
b - c &= 2
\end{cases}
$$

$$\begin{cases} a+\frac{2+c+c}{2} &= 2 \\ b-c &= 2 \end{cases} \Rightarrow \begin{cases} a+c &= 1 \\ b-c &= 2 \end{cases} \Rightarrow \begin{cases} a &= 1-c \\ b &= 2+c \end{cases}$$

Of course this system is undetermined, but, we have solutions for different values of c, and as $a, b, c \geq 0$, there is a finite set of them which we can just write down for different values of c,

- $c = 0 \Rightarrow a = 1, b = 2$

There are only one possible combination, as the others are zero by statistics,

$$: \psi \partial \psi :$$

- $c = 1 \Rightarrow a = 0, b = 3$

Which gives no possible combination, as it isn't possible to make any non zero combination of more than one ψ without derivatives. Thus,

$$V_2(z) = \alpha : \psi \partial \psi : (z)$$

As no other values of c are allowed due to $a, b, c \geq 0$. Now for $n = -2$,

$$\begin{cases} a+\frac{b+c}{2} &= 2 \\ b-c &= -2 \end{cases}$$
$$\begin{cases} a+\frac{-2+c+c}{2} &= 2 \\ b-c &= -2 \end{cases} \Rightarrow \begin{cases} a+c &= 3 \\ b-c &= -2 \end{cases} \Rightarrow \begin{cases} a &= 3-c \\ b &= -2+c \end{cases}$$

- $c = 2 \Rightarrow a = 1, b = 0$

By statistics, the only non zero combination is,

$$: \tilde{\psi} \partial \tilde{\psi} :$$

- $c = 3 \Rightarrow a = 0, b = 1$

Again, no combination is possible due to statistics, and no other values of c are allowed due to $a, b, c \geq 0$. Thus,

$$V_{-2}(z) = \beta : \tilde{\psi} \partial \tilde{\psi} : (z)$$

Which being in principle possible to compute α, β from matching of the VV OPE's, but we'll not do that here.

Problem 5

5.A)

Until now we have been using a notion of normal ordering, $: A :$ which is heavily dependent on knowing the equation of motion of the operators to construct the normal ordered form, but, of course, any ordering scheme which gives a finite operator is equally good, and as all of them give finite operators they differ only by finite parts. Another possible notion of ordering, different from what we have been using, is a definition which makes no reference of the equation of motion, and relies solely in removing the divergence from the operator,

$$\mathop{}^{*}_{*}\, \mathcal{O}_1\mathcal{O}_2 \,\mathop{}^{*}_{*}(z) = \oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\mathcal{O}_1(w)\mathcal{O}_2(z)}{w-z}$$

It's clear that the contour integral picks up just the finite part of $\mathcal{O}_1(w)\mathcal{O}_2(z)$ evaluated at $z = w$. But, it has two drawbacks, it can only define the normal ordering at equal points, while our other definition can do it at different points, and it requires the knowledge of another OPE $\mathcal{O}_1(w)\mathcal{O}_2(z)$ to be computed, which has to be calculated from other arguments. But still it provides a classy way of writing the normal ordering. Now focusing in the problem at hand, we have already constructed the current for general $SO(N)$ groups, 3.1, which we recall here,

$$j^{[ab]}(z) = \frac{1}{2} T^{[ab]}{}_{kl} : \psi^k \psi^l : (z)$$

For the $N = 3$, $SO(3)$ case we have only three currents,

$$\begin{aligned} j^{12} = j^{[12]}(z) &= \frac{1}{2} T^{[12]}{}_{kl} : \psi^k \psi^l : (z) \\ j^{23} = j^{[23]}(z) &= \frac{1}{2} T^{[23]}{}_{kl} : \psi^k \psi^l : (z) \\ j^{31} = j^{[31]}(z) &= \frac{1}{2} T^{[31]}{}_{kl} : \psi^k \psi^l : (z) \end{aligned}$$

The generators in the fundamental representation are easily written in terms of the Levi-Civita, $\epsilon_{123} = 1$,

$$T^{[ab]}{}_{kl} = \mathrm{i}\epsilon^{abc}\epsilon_{ckl} \Rightarrow \begin{cases} T^{[12]}{}_{kl} & = \mathrm{i}\epsilon_{3kl} \\ T^{[23]}{}_{kl} & = \mathrm{i}\epsilon_{1kl} \\ T^{[31]}{}_{kl} & = \mathrm{i}\epsilon_{2kl} \end{cases}$$

So that,

$$\begin{aligned} j^{12} &= \mathrm{i}\frac{1}{2}\epsilon_{3kl} : \psi^k \psi^l : (z) = \mathrm{i} : \psi^1 \psi^2 : (z) \\ j^{23} &= \mathrm{i}\frac{1}{2}\epsilon_{1kl} : \psi^k \psi^l : (z) = \mathrm{i} : \psi^2 \psi^3 : (z) \\ j^{31} &= \mathrm{i}\frac{1}{2}\epsilon_{2kl} : \psi^k \psi^l : (z) = \mathrm{i} : \psi^3 \psi^1 : (z) \end{aligned}$$

One might worry about these currents being defined through the *old* normal ordering ::, but actually, the expression for the currents need not ordering prescription, as they're totally finite without normal ordering due to the OPE of two different fermions not having poles, as they do not mix kinetic terms, so there is no problem here in mixing the two kinds of orderings.

5.B)

With the remarks being made, we make use of the full OPE,

$$\psi^i(z)\psi^j(w) = \frac{\delta^{ij}}{z-w} + :\psi^i(z)\psi^j(w):$$

And start by computing just ${}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z)$, we'll neglect the normal orderings, because, as already mentioned, in this case they aren't really needed,

$$\begin{aligned}
{}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z) &= \oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{j^{12}(w)j^{23}(z)}{w-z} \\
{}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\psi^1(w)\psi^2(w)\psi^2(z)\psi^3(z)}{w-z} \\
{}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\psi^1(w)\psi^3(z)\psi^2(w)\psi^2(z)}{w-z}
\end{aligned}$$

Where we used that for $i \neq j$, $\{\psi^i(z), \psi^j(w)\} = 0$. Now we expand in Taylor the regular term $\psi^1(w)\psi^3(z)$,

$${}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z) = -\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\psi^1(z)\psi^3(z) + (w-z)\partial\psi^1(z)\psi^3(z) + \mathcal{O}\big((w-z)^2\big)}{w-z}\psi^2(w)\psi^2(z)$$

As the $\psi^2(w)\psi^2(z)$ OPE has at most a single pole, the contour integral of the $\mathcal{O}\big((w-z)^2\big)$ contribution gives identically zero, as we have,

$$\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\mathcal{O}\big((w-z)^2\big)}{(w-z)^2} = \oint \frac{\mathrm{d}w}{2\pi \mathrm{i}}\mathcal{O}(1) = 0$$

As it has no pole, so, we can in fact neglect it, rewriting the integral without this term and already opening the OPE,

$$\begin{aligned}
{}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\psi^1(z)\psi^3(z) + (w-z)\partial\psi^1(z)\psi^3(z)}{w-z}\left(\frac{1}{w-z} + :\psi^2(w)\psi^2(z):\right) \\
{}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}}\left\{\frac{\psi^1(z)\psi^3(z)}{(w-z)^2} + \frac{\partial\psi^1(z)\psi^3(z)}{w-z} + \frac{\psi^1(z)\psi^3(z):\psi^2(w)\psi^2(z):}{w-z}\right\} \\
&\quad - \oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \partial\psi^1(z)\psi^3(z):\psi^2(w)\psi^2(z):
\end{aligned}$$

The first term in the right-hand side has only a double pole, so, the integral vanishes, as the integral captures just single poles, so do the last term in the right-hand side, as it's regular at $w = z$, the second term has a single pole, which contributes as 1, and for the third term we have to expand it in Taylor,

$${}^{*}_{*}\, j^{12}j^{23}\, {}^{*}_{*}(z) = -\partial\psi^1(z)\psi^3(z) - \psi^1(z)\psi^3(z)\oint \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{:\psi^2(z)\psi^2(z): + (w-z):\partial\psi^2(z)\psi^2(z):}{w-z}$$

The first term of the integral is zero by statistics, $:\psi^2(z)\psi^2(z):=0$, and the second one is zero due to being regular,

$$\mathop{*}\limits^{*} j^{12} j^{23} \mathop{*}\limits^{*}(z) = -\partial\psi^1(z)\psi^3(z)$$

Now for the other component,

$$\begin{aligned}
\mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= \oint \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{j^{23}(w) j^{12}(z)}{w-z} \\
\mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\psi^2(w)\psi^3(w)\psi^1(z)\psi^2(z)}{w-z} \\
\mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\psi^3(w)\psi^1(z)}{w-z}\psi^2(w)\psi^2(z) \\
\mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\psi^3(z)\psi^1(z) + (w-z)\partial\psi^3(z)\psi^1(z)}{w-z}\left(\frac{1}{w-z} + :\psi^2(w)\psi^2(z):\right) \\
\mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= -\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}} \partial\psi^3(z)\psi^1(z)\left(\frac{1}{w-z} + :\psi^2(w)\psi^2(z):\right) \\
\mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= -\partial\psi^3(z)\psi^1(z)
\end{aligned}$$

Where we used the same arguments of before, keeping only the single poles, and the terms which does not vanishes by statistics. Hence,

$$\begin{aligned}
\mathop{*}\limits^{*} j^{12} j^{23} \mathop{*}\limits^{*}(z) - \mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= -\partial\psi^1(z)\psi^3(z) + \partial\psi^3(z)\psi^1(z) \\
\mathop{*}\limits^{*} j^{12} j^{23} \mathop{*}\limits^{*}(z) - \mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= \psi^3(z)\partial\psi^1(z) + \partial\psi^3(z)\psi^1(z) \\
\mathop{*}\limits^{*} j^{12} j^{23} \mathop{*}\limits^{*}(z) - \mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= \partial\big[\psi^3(z)\psi^1(z)\big] \\
\mathop{*}\limits^{*} j^{12} j^{23} \mathop{*}\limits^{*}(z) - \mathop{*}\limits^{*} j^{23} j^{12} \mathop{*}\limits^{*}(z) &= -\mathrm{i}\partial j^{31}(z)
\end{aligned}$$

Problem 6

6.A)

We have already done this for general N real fermions in problem 3.D). Let us just cite here the results,

$$\begin{aligned}
j^{[ab]}(z) &= \frac{1}{2}T^{[ab]}{}_{kl} :\psi^k\psi^l: (z) \\
j^{A}(z) &= \frac{1}{2}T^{A}{}_{kl} :\psi^k\psi^l: (z) \\
j^{A}(z_1)j^{B}(z_2) &= \frac{\mathrm{i}f^{AB}{}_{C}}{(z_1-z_2)}j^{C}(z_2) + \frac{\mathrm{Tr}\left[T^AT^B\right]}{2(z_1-z_2)^2} + \text{regular}
\end{aligned}$$

Where in this case $T^{[ab]}{}_{kl} = T^{A}{}_{kl}$ are the generators of $SO(4)$, and $f^{AB}{}_{C}$ are the structure constants such that $a,b,k,l = 1,2,3,4$ and $A,B,C = 1,2,3,4,5,6$. That ψ^k transforms in the fundamental representation is a trivial fact, due to they consisting of a real — the Grassmannian nature of them do not interfere — vector of four components, exactly the representation on which the $4x4$ orthogonal, determinant one matrices act. If it form a $SO(4)$ Kac-Moody algebra can be already seem from our derivation of a generic $SO(N)$ Kac-Moody algebra, which we cited above.

6.B)

To keep the conventions already established in problems 3.C),4.C), we're going to choose the following for grouping the four fermions into a pair of complex ones,

$$\begin{aligned}
\Psi^{\dot{1}} &= \frac{1}{\sqrt{2}}\psi^1 - \frac{\mathrm{i}}{\sqrt{2}}\psi^2 \\
\tilde{\Psi}^{\dot{1}} &= \frac{1}{\sqrt{2}}\psi^1 + \frac{\mathrm{i}}{\sqrt{2}}\psi^2 \\
\Psi^{\dot{2}} &= \frac{1}{\sqrt{2}}\psi^3 - \frac{\mathrm{i}}{\sqrt{2}}\psi^4 \\
\tilde{\Psi}^{\dot{2}} &= \frac{1}{\sqrt{2}}\psi^3 + \frac{\mathrm{i}}{\sqrt{2}}\psi^4
\end{aligned}$$

This choice makes the following,

$$\begin{aligned}
-:\tilde{\Psi}^{\dot{1}}\Psi^{\dot{1}}: &= -\frac{1}{2}:\psi^1\psi^1 + \mathrm{i}\psi^2\psi^1 - \psi^1\mathrm{i}\psi^2 + \psi^2\psi^2: \\
-:\tilde{\Psi}^{\dot{1}}\Psi^{\dot{1}}: &= \mathrm{i}:\psi^1\psi^2:
\end{aligned}$$

As outcome of problems 3.C),4.C) this should be seen as a good relation to hold. To bosonize each pair $\tilde{\Psi}^{\dot{a}}\Psi^{\dot{a}}$ — We'll use $\dot{a},\dot{b} = 1,2$ for the complex fermions we leave $j,k,l = 1,2,3,4$ for the real ones —, we follow the ideas of problem 4, to each pair of complex fermions we attribute a chiral boson $X^{\dot{a}}(z)$, then we should have, as was already argued in problem 4.D), the following correspondence,

$$:\exp\left(\mathrm{i}X^{\dot{a}}\right): = \Psi^{\dot{a}} \tag{6.1a}$$

$$: \exp\left(-\mathrm{i}X^{\dot{a}}\right) := \tilde{\Psi}^{\dot{a}} \tag{6.1b}$$

We choose a particular way of bosonizing, a more generic choice could be $: \exp\left(\mathrm{i}X^{\dot{a}}\right) := \alpha\Psi^{\dot{a}}, : \exp\left(-\mathrm{i}X^{\dot{a}}\right) := \alpha^{-1}\tilde{\Psi}^{\dot{a}}$. But what matters is that now we have a description of the fermionic theory by a bosonic one, consisting of two compactified chiral bosons $X^{\dot{a}}(z) = X^{\dot{a}}(z) + 2\pi$.

6.C)

The symmetry of the bosonic theory is,

$$X^{\dot{a}}(z) \rightarrow X^{\dot{a}}(z) + t^{\dot{a}} \pmod{2\pi}$$

What from 6.1 can be seen as a realization of a $U(1)$ symmetry for each index $\dot{a}$, as there is two of them, and the symmetries are disconnected, this means a $U(1) \times U(1)$ symmetry,

$$\begin{aligned}
\Psi^{\dot{a}} =: \exp\left(\mathrm{i}X^{\dot{a}}\right) : \rightarrow: \exp\left(\mathrm{i}X^{\dot{a}} + \mathrm{i}t^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi) \\
\Psi^{\dot{a}} \rightarrow \exp\left(\mathrm{i}t^{\dot{a}}\right) : \exp\left(\mathrm{i}X^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi) \\
\Psi^{\dot{a}} \rightarrow \exp\left(\mathrm{i}t^{\dot{a}}\right)\Psi^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi)
\end{aligned}$$

From the last line is clear that we have a $U(1)$ symmetry for each index, specially due to the chiral boson being compactified. Of course, this transformation above implies the transformation of the complex one also,

$$\begin{aligned}
\tilde{\Psi}^{\dot{a}} =: \exp\left(-\mathrm{i}X^{\dot{a}}\right) : \rightarrow: \exp\left(-\mathrm{i}X^{\dot{a}} - \mathrm{i}t^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi) \\
\tilde{\Psi}^{\dot{a}} \rightarrow \exp\left(-\mathrm{i}t^{\dot{a}}\right) : \exp\left(-\mathrm{i}X^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi) \\
\tilde{\Psi}^{\dot{a}} \rightarrow \exp\left(-\mathrm{i}t^{\dot{a}}\right)\tilde{\Psi}^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi)
\end{aligned}$$

So now it's clear that $\tilde{\Psi}^{\dot{a}}$ have charge -1 with the respective $U(1)$'s, and $\Psi^{\dot{a}}$ have charge $+1$ with the respective $U(1)$'s. What we have shown is that $U(1) \times U(1) \subset SO(4)$, so it's possible for us to work out the action of each of these $U(1)$'s over the vector representation, let's name $U(1) \times U(1)$ by $U(1)_{\dot{1}} \times U(1)_{\dot{2}}$, the nomenclature should be self evident. With a little help from group theory, it's known that **locally** $SO(4)$ is isomorphic to $SU(2) \times SU(2)$, that is not true globally, as the two groups have different topologies. But nevertheless, under a quantization, any global classical symmetry has to under go a double cover, here is no different, our symmetry group $SO(4)$ has to under go to a double cover, which is by definition the group $Spin(4)$, which luckily is globally isomorphic to $SU(2) \times SU(2)$. Just recalling everything, our classical theory of four fermions enjoyed a $SO(4)$ symmetry, that is, generically if we stayed at classical analysis, we would be interested in operators which transform under representations of $SO(4)$, but, as ultimately we're interested in the quantum theory, this enlargers the symmetry group, and we actually have to search for operators which transform in representations of the double cover of $SO(4)$, that is $Spin(2) \simeq SU(2) \times SU(2)$, while $SO(4)$ just allows for integer charges/spins, it's double cover allows for half-integers charges/spins, this is a well known fact, as $SU(2)$ possesses representations which have a half integer quadratic Casimir operator, the fundamental representation as example. But, we just found an explicit representation of a subgroup of $SO(4)$ by the bosonization of the fermionic theory, $U(1)_{\dot{1}} \times U(1)_{\dot{2}} \subset SO(4) \Rightarrow U(1)_{\dot{1}} \times U(1)_{\dot{2}} \subset SU(2) \times SU(2)$, let us just stress some facts here, we have found **two abelian $\mathbf{U(1)}$ subgroups of $\mathbf{SU(2) \times SU(2)}$**, of course $U(1)$ by itself is an abelian group, but, they're abelian **among**

themselves, this is what the $\times$ is saying, and this can be seen both from the bosonization as well from the fermionic representation, the transformations for $X^{\dot{a}} \to X^{\dot{a}} + t^{\dot{a}} \pmod{2\pi}$ are totally independent for each index, as well as,

$$\begin{cases} \Psi^{\dot{a}} & \to \exp\left(\mathrm{i}t^{\dot{a}}\right)\Psi^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi) \\ \tilde{\Psi}^{\dot{a}} & \to \exp\left(-\mathrm{i}t^{\dot{a}}\right)\tilde{\Psi}^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi) \end{cases}$$

are independent for each index. This raises an eyebrow, as certainly $SO(4)$ is not abelian, an neither is $SU(2)\times SU(2)$, but, the double cover has naturally an $\times$ in the definition, this means it's composed of two copies of $SU(2)$ which are abelian among themselves! It's clear that $SU(2)$ itself is not abelian, and also is clear that $U(1) \subset SU(2)$, thus, the only possible way we could extract two abelian copies of $U(1)$ from $SU(2) \times SU(2)$ is if we extracted one copy of each, this means we have a bigger identification of the symmetry $U(1)_{\dot{1}} \times U(1)_{\dot{2}}$ as being,

$$U(1)_{\dot{1}} \times U(2)_{\dot{2}} \subset SU(2)_{\dot{1}} \times SU(2)_{\dot{2}}$$

We just identified which $U(1)$ came from which $SU(1)$, this particular $U(1)_{\dot{a}} \subset SU(2)_{\dot{a}}$ we picked up can be seen to be represented in a diagonal form in $\Psi^{\dot{a}}, \tilde{\Psi}^{\dot{a}}$, as at in each $\mathfrak{su}(2)_{\dot{a}}$ we can diagonalize at maximum one generator and the single Casimir — which will have positive half-integer eigenvalues for the Casimir j, and $m = -j, -j+1, \cdots, j$ for the chosen generator —, it's clear that

$$\begin{cases} \Psi^{\dot{a}} & \to \exp\left(\mathrm{i}jt^{\dot{a}}\right)\Psi^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi), \quad j = 1 \\ \tilde{\Psi}^{\dot{a}} & \to \exp\left(-\mathrm{i}jt^{\dot{a}}\right)\tilde{\Psi}^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi), \quad j = 1 \end{cases}$$

implies they transform in the spin 1 representation of each particular $SU(2)$. After all these remarks, we finally know how to ask what we want, we want operators that transform under the spin $\frac{1}{2}$ representation of the double cover of $SO(4)$, but now we know what this means, this means we want operators that transform as,

$$\begin{cases} \Phi^{\dot{a}} & \to \exp\left(\mathrm{i}jt^{\dot{a}}\right)\Phi^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi) \\ \tilde{\Phi}^{\dot{a}} & \to \exp\left(-\mathrm{i}jt^{\dot{a}}\right)\tilde{\Phi}^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi) \end{cases}$$

for $j = \frac{1}{2}$. As for each index this is a spin $\frac{1}{2}$ representation of $SU(2)_{\dot{a}}$. But we have a natural way of analyzing the transformations being done by the diagonal generator of $SU(2)_{\dot{a}}$, the symmetry in the bosonic theory! $X^{\dot{a}} \to X^{\dot{a}} + t^{\dot{a}} \pmod{2\pi}$ correspond to a transformation generated by the diagonal generator of each $SU(2)_{\dot{a}}$, thus, is easy to get the desired result, consider,

$$\Phi^{\dot{a}} =: \exp\left(\frac{\mathrm{i}}{2}X^{\dot{a}}\right) : \to : \exp\left(\frac{\mathrm{i}}{2}X^{\dot{a}} + \frac{\mathrm{i}}{2}t^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi)$$
$$\Phi^{\dot{a}} \to \exp\left(\frac{\mathrm{i}}{2}t^{\dot{a}}\right) : \exp\left(\frac{1}{2}\mathrm{i}X^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi)$$
$$\Phi^{\dot{a}} \to \exp\left(\frac{\mathrm{i}}{2}t^{\dot{a}}\right)\Phi^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi)$$

This is exactly the desired transformation with $j = \frac{1}{2}$, what means $\Phi^{\dot{a}}$ transform as the spin $\frac{1}{2}$ representation of the double cover of $SO(4)$, the same can be done for,

$$\tilde{\Phi}^{\dot{a}} =: \exp\left(-\frac{\mathrm{i}}{2}X^{\dot{a}}\right) : \to : \exp\left(-\frac{\mathrm{i}}{2}X^{\dot{a}} - \frac{\mathrm{i}}{2}t^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi)$$

$$\tilde{\Phi}^{\dot{a}} \to \exp\left(-\frac{\mathrm{i}}{2}t^{\dot{a}}\right) : \exp\left(-\frac{\mathrm{i}}{2}X^{\dot{a}}\right) :, \quad t^{\dot{a}} \in [0, 2\pi)$$
$$\tilde{\Phi}^{\dot{a}} \to \exp\left(-\frac{\mathrm{i}}{2}t^{\dot{a}}\right)\tilde{\Phi}^{\dot{a}}, \quad t^{\dot{a}} \in [0, 2\pi)$$

These have not a polinomial definition on the fermionic theory, but, the bosonic theory provides a good representation of these spin $\frac{1}{2}$ operators,

$$: \exp\left(\frac{\mathrm{i}}{2}X^{\dot{a}}\right) :, \quad : \exp\left(-\frac{\mathrm{i}}{2}X^{\dot{a}}\right) :$$

A Bosonic CFT

Here we're going to derive some results of a bosonic CFT of X^μ,

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\partial_a X^\mu \partial^a X_\mu$$

Where it's understood a flat '*world-sheet*' metric δ_{ab}, we make a change of variables from σ^1, σ^2 to $\sigma^z \equiv z = \sigma^1 + \mathrm{i}\sigma^2, \sigma^{\bar{z}} \equiv \bar{z} = \sigma^1 - \mathrm{i}\sigma^2$, so that the new metric reads,

$$\begin{aligned}
g^{zz} &= \frac{\partial z}{\partial \sigma^1}\frac{\partial z}{\partial \sigma^1}\delta^{11} + \frac{\partial z}{\partial \sigma^2}\frac{\partial z}{\partial \sigma^2}\delta^{22} \\
g^{zz} &= 1 - 1 = 0 \\
g^{z\bar{z}} &= \frac{\partial z}{\partial \sigma^1}\frac{\partial \bar{z}}{\partial \sigma^1}\delta^{11} + \frac{\partial z}{\partial \sigma^2}\frac{\partial \bar{z}}{\partial \sigma^2}\delta^{22} \\
g^{z\bar{z}} &= 1 + 1 = 2 = g^{\bar{z}z} \\
g^{\bar{z}\bar{z}} &= \frac{\partial \bar{z}}{\partial \sigma^1}\frac{\partial \bar{z}}{\partial \sigma^1}\delta^{11} + \frac{\partial \bar{z}}{\partial \sigma^2}\frac{\partial \bar{z}}{\partial \sigma^2}\delta^{22} \\
g^{\bar{z}\bar{z}} &= 1 - 1 = 0
\end{aligned}$$

So that or action in these new coordinates is,

$$\begin{aligned}
S &= \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma^1\,\mathrm{d}\sigma^2\,\sqrt{|\mathrm{Det}[\delta]|}\partial_a X^\mu \partial^a X_\mu \\
S &= \frac{1}{4\pi\alpha'}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\sqrt{|\mathrm{Det}[g]|}\partial_a X^\mu \partial^a X_\mu \\
S &= \frac{1}{4\pi\alpha'}\frac{1}{2}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,(g^{z\bar{z}}\partial_z X^\mu \partial_{\bar{z}} X_\mu + g^{z\bar{z}}\partial_{\bar{z}} X^\mu \partial_z X_\mu) \\
S &= \frac{1}{4\pi\alpha'}\frac{4}{2}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\partial_z X^\mu \partial_{\bar{z}} X_\mu \\
S &= \frac{1}{2\pi\alpha'}\int \mathrm{d}^2 z\,\partial X^\mu \bar{\partial} X_\mu
\end{aligned}$$

Where we defined $\partial_z = \partial, \partial_{\bar{z}} = \bar{\partial}$ and $\mathrm{d}^2 z = \mathrm{d}z\,\mathrm{d}\bar{z}$