SCALAR PROXY

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1. Cut solutions

Of course in each amplitude we have different cut solutions. Now let us solve them,

1.1. all massless.

The cut condition is,

$$k_1^2 = k_2^2 = (3 - k_1)^2 = (3 - k_1 - k_2)^2 = (3 + 4 - k_1 - k_2)^2 = 0$$

The first and third condition enforces $k_1 = -|k_1]\langle 3|$. But the fourth and fifth conditions enforces $3 - k_1 - k_2 = n$, with $n \cdot 4 = 0$ & $n^2 = 0$. Lastly, the second condition imposes $(3 - k_1 - n)^2 = -23 \cdot n + 2k_1 \cdot n = 0$, that is,

$$[3n]\langle n3\rangle = [k_1 n]\langle n3\rangle$$

which has two solutions, $|n] = |k_1] - |3]$ & $|n\rangle = z|4\rangle$ or $|n\rangle = |3\rangle$ & $|n] = z|4]$. When working with scalar particles it's better to choose the first solution, as this avoids singularities in denominators such as $(k_1 \cdot k_2)^{-1}$. Hence, the solution we're going to choose is,

$$\begin{cases} k_1 & = -|k_1]\langle 3| \\ k_2 & = -|3]\langle 3| + |k_1]\langle 3| + z(|k_1] - |3])\langle 4| \end{cases}$$

1.2. massive legs first topology.

Our approach to massive legs is to shift the solution with massless, in order to obtain a well behaved solution in the $m^2 \to 0$ limit. For this topology the cut constrains are,

$$l_1^2 = l_2^2 = (3 - l_1)^2 = -m^2 \ \& \ (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = 0$$

The ideia here is to define, $l_i = k_i + \alpha_i q_i$ (no sum), with $q_i^2 = 0$ and $\alpha_i = -m^2 (2k_i \cdot q_i)^{-1}$, then, q_i, k_i are not allowed to have any dependence on m^2. The first and second constrains are already satisfied. The third one gives,

$$-23 \cdot l_1 = 0 \to 3 \cdot (k_1 + \alpha_1 q_1) = 0 \to 3 \cdot q_1 = 0$$

As $|q_1\rangle = |3\rangle$ is forbidden, $|q_1] = |3]$. The fourth and fifth constrains imposes,

$$\begin{cases} -n \cdot (\alpha_1 q_1 + \alpha_2 q_2) + \alpha_1 \alpha_2 q_1 \cdot q_2 & = 0 \\ 4 \cdot (\alpha_1 q_1 + \alpha_2 q_2) & = 0 \end{cases}$$

This imposes actually $q_1 \cdot q_2 = 0$, for this to be true we have to options, either $|q_2] = |3]$, or $|q_2\rangle = |q_1\rangle$. If we choose the first, we can shift k_1 by 3 such to make $|q_1\rangle = |4\rangle$, this imposes further $|q_2\rangle = |4\rangle$. Hence, a possible solution is,

$$q_1 = q_2 = -|3]\langle 4|$$

1.3. massive legs second topology.

The constrains now are slightly different,

$$l_1^2 = (3 - l_1)^2 = 0 \ \& \ l_2^2 = (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = -m^2$$

Which has as solution $q_2 = -|4]\langle 3|$

1.4. massive legs third topology.

Now the constrain is difficult to solve,

$$l_2^2 = 0 \ \& \ l_1^2 = (3 - l_1)^2 = (3 - l_1 - l_2)^2 = (3 + 4 - l_1 - l_2)^2 = -m^2$$

The second and third constrains give, $l_1 = -|k_1]\langle 3| - \alpha|3]\langle l_1|$. Now, the fourth and fifth constrains gives,

$$3 - l_1 - l_2 = -z(|k_1] - |3])\langle 4| + \beta|4]\langle n|$$

With of course $\beta = -\frac{m^2}{z\langle 4n\rangle[4|(|k_1]-|3])}$. At last the second constrain gives,

$$\begin{aligned}
l_2 &= -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n| \\
l_2^2 = 0 &= -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1] - |3]) \\
0 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\
-2m^2 &= \beta\langle 3n\rangle[4|(|3] - |k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\
-2m^2 &= -\frac{m^2}{z\langle 4n\rangle[4|(|k_1] - |3])}\langle 3n\rangle[4|(|3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]} z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43]
\end{aligned}$$

For the quadratic term in m^2 to vanish is necessary $\langle l_1 n\rangle = 0 \to |l_1\rangle \propto |n\rangle$, thus,

$$\begin{aligned} -2m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle} z\langle 4l_1\rangle \\ -2 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle} z\langle 4n\rangle \to \langle 3n\rangle = -z\langle 4n\rangle \end{aligned}$$

The best parametrization is $|n\rangle = |l_1\rangle = |4\rangle - \frac{1}{z}|3\rangle$.

1.5. **massive legs fourth topology.** Now the constrain is the hardest to solve,

$$l_1^2 = l_2^2 = (3-l_1)^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

Happily, most of the work was done already in the last solution, we just have to change:

$$\begin{aligned} l_2 &= -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1] - |3])\langle 4| - \beta|4]\langle n| \\ l_2^2 = -m^2 \to m^2 &= -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1] - |3]) \\ m^2 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\ -m^2 &= \beta\langle 3n\rangle[4|(|3] - |k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\ -m^2 &= -\frac{m^2}{z\langle 4n\rangle[4|(|k_1] - |3])}\langle 3n\rangle[4|(|3] - |k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]} z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \end{aligned}$$

Again we fix $|l_1\rangle \propto |n\rangle$. The solution then is given by,

$$\begin{aligned} -m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle} z\langle 4l_1\rangle \\ -1 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle} z\langle 4n\rangle \to \langle 3n\rangle = -z\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\right)\langle 4n\rangle = -z\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\langle 4n\rangle \end{aligned}$$