# SCALAR PROXY

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## 1. Introduction

We will work most with the scalar proxy given by the lagrangian,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2M^2}\Box\phi\Box\phi - \frac{\kappa}{2}\Box\phi\phi^2 \tag{1.1}$$

The idea here is reintegrate the higher derivative term, in order to obtain a lower derivative term, but in terms of additional fields. This is easily done by,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \Box\phi\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box\phi\phi^2 \tag{1.2}$$

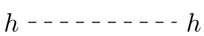
The new lagrangian has mixed propagator terms, to diagonalize it is also easy, we just open in terms of $\phi = h - \eta$,

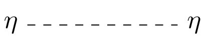
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 + \eta\Box(h-\eta) + \frac{M^2}{2}\eta^2 \tag{1.3}$$

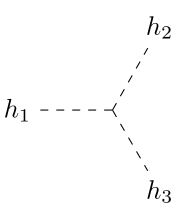
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 - \partial_\mu\eta\partial^\mu(h-\eta) + \frac{M^2}{2}\eta^2 \tag{1.4}$$

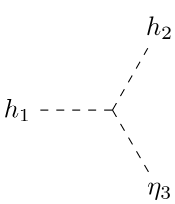
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 \tag{1.5}$$

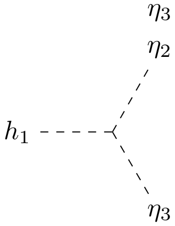
The Feynman rules are easily red as,

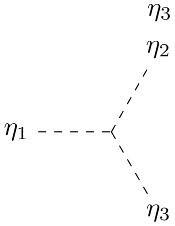
- 

$= \frac{1}{\mathrm{i}}\frac{1}{p^2}$
- 

$= -\frac{1}{\mathrm{i}}\frac{1}{p^2+M^2}$
- 

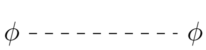
$= \mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$
- 

$= -\mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$
- 

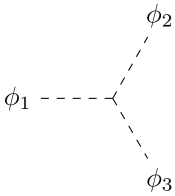
$= \mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$
- 

$= -\mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$

Which can also be seen directly from the Feynman rules of the $\phi$ field,

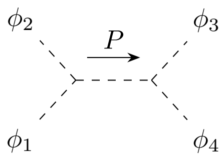
- 

$= \frac{1}{\mathrm{i}}\frac{1}{p^2+\frac{p^4}{M^2}}$

- 

$= \mathrm{i}\kappa\big(p_1^2+p_2^2+p_3^2\big) = \mathrm{i}\kappa(p_1+p_2+p_3)^2 - 2\mathrm{i}\kappa(p_1\cdot p_2 + p_2\cdot p_3 + p_3\cdot p_1) = -\mathrm{i}\kappa(\langle 12\rangle[12] + \langle 23\rangle[23] + \langle 31\rangle[31])$

So that the four point amplitude can be computed by,




$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] + \langle 2P\rangle[2P] + \langle P1\rangle[P1])(\langle 34\rangle[34] - \langle 4P\rangle[4P] - \langle P3\rangle[P3]) \tag{1.6}$$

$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] - \langle P2\rangle[2P] - \langle P1\rangle[1P])(\langle 34\rangle[34] + \langle P4\rangle[4P] + \langle P3\rangle[3P]) \tag{1.7}$$

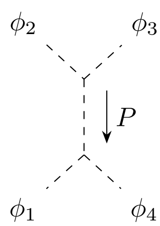
$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] + \langle P|\,1+2|P])(\langle 34\rangle[34] - \langle P|\,3+4|P]) \tag{1.8}$$

$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] - \langle P|\,P|P])(\langle 34\rangle[34] - \langle P|\,P|P]) \tag{1.9}$$

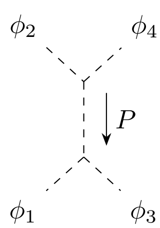
$$= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2 \frac{1}{P^2+\frac{P^4}{M^2}}\big(\langle 12\rangle[12] - 2P^2\big)\big(\langle 34\rangle[34] - 2P^2\big) \tag{1.10}$$

$$= -\mathrm{i}\frac{(\kappa M)^2}{s(M^2-s)}(\langle 12\rangle[12] + 2s)(\langle 34\rangle[34] + 2s) \tag{1.11}$$

It's trivial to read the $t$ and $u$ channels from this expression,



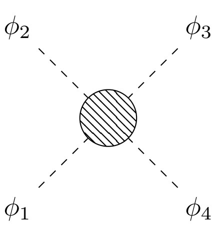

$$= -\mathrm{i}\frac{(\kappa M)^2}{t(M^2-t)}(\langle 23\rangle[23] + 2t)(\langle 41\rangle[41] + 2t) \tag{1.12}$$




$$= -\mathrm{i}\frac{(\kappa M)^2}{u(M^2-u)}(\langle 24\rangle[24] + 2u)(\langle 31\rangle[31] + 2u) \tag{1.13}$$

So that the full 4–point amplitude is,

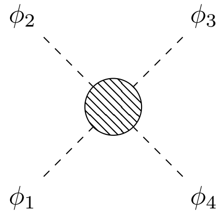
(1.14)




$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\big[(\langle 12\rangle[12] + 2s)(\langle 34\rangle[34] + 2s)tu\big(M^2-t\big)\big(M^2-u\big) + (\langle 23\rangle[23] + 2t)(\langle 41\rangle[41] + 2t$$

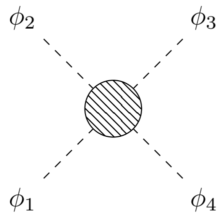
Let us specialize when $1, 2$ are massless and $3, 4$ are massive, then,

(1.15)



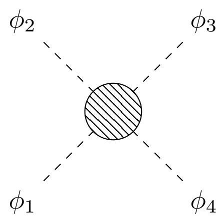

$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\big[s\big(-s+2M^2+2s\big)tu\big(M^2-t\big)\big(M^2-u\big)+\big(-t+M^2+2t\big)\big(-t+M^2+2t\big)su\big(M^2$$

(1.16)




$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\big[stu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+su\big(M^2+t\big)\big(M^2+t\big)\big(M^2-s\big)\big(M^2-u\big)+st$$

(1.17)




$$= -\mathrm{i}\frac{(\kappa M)^2}{tu(M^2-s)(M^2-t)(M^2-u)}\Big[tu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+u\big(M^2+t\big)^2\big(M^2-s\big)\big(M^2-u\big)+t\big(M^2+u\big)^2\big($$

## 2. Conformal Toy Model

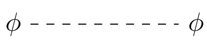
Consider the following Lagrangian,

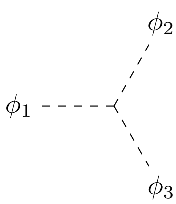
$$\mathcal{L} = -\frac{1}{2}\Box\phi\Box\phi - \frac{g}{2}\phi^2\Box\phi - \frac{g^2}{8}\phi^4 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

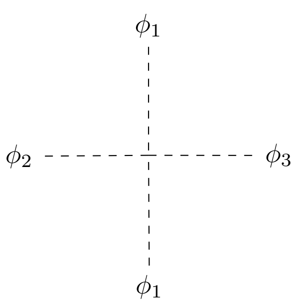
Notice the form of the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\left(\Box\phi + \frac{g}{2}\phi^2\right)^2 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

It possesses the Feynman rules,

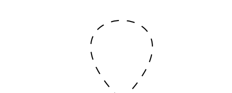
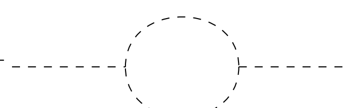
- 

$= \frac{1}{\mathrm{i}}\frac{1}{m^2p^2+p^4}$
- 

$= \mathrm{i}g\big(p_1^2+p_2^2+p_3^2+m^2\big)$
- 

$= -3\mathrm{i}g^2$

Let's compute the self energy,

(2.1) $\mathrm{i}\Pi\big(p^2\big) =$  $+$  $+\cdots$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2 \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\mathrm{i}}\frac{1}{\ell^2}\frac{1}{\ell^2+m^2} \tag{2.2}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2 \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\ell^2}\frac{1}{\ell^2+m^2} \tag{2.3}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2 \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}\left(m^2\right)^{\frac{D}{2}-2}\frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(1)} \tag{2.4}$$

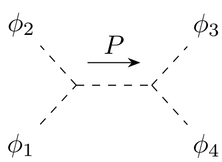
$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2 \frac{\left(m^2\right)^{-\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-\epsilon)} \tag{2.5}$$

$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}(\mathrm{i}g)^2 \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\mathrm{i}^2}\frac{1}{\ell^2(\ell+p)^2}\frac{\left(m^2+\ell^2+p^2+(\ell+p)^2\right)^2}{\ell^2+m^2}\frac{1}{(\ell+p)^2+m^2} \tag{2.6}$$

For the mass renormalization we can take $p=0$,

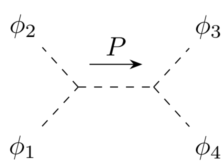
$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}g^2 \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\left(m^2+2\ell^2\right)^2}{\ell^4(\ell^2+m^2)^2} \tag{2.7}$$

Let's compute the four point amplitude for this theory,

(2.8)


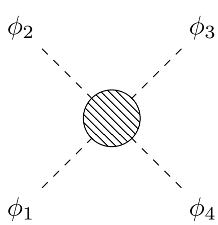

$$= (\mathrm{i}g)^2\frac{\left(p_1^2+p_2^2+(p_1+p_2)^2+m^2\right)\left(p_3^2+p_4^2+(p_3+p_4)^2+m^2\right)}{\mathrm{i}(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)}$$

First let's consider all legs massless,

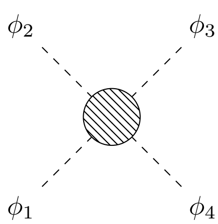
(2.9)



$$= \mathrm{i}g^2\frac{\left(-s+m^2\right)\left(-s+m^2\right)}{(-s)(-s+m^2)} = -\mathrm{i}g^2\frac{\left(-s+m^2\right)}{s}$$

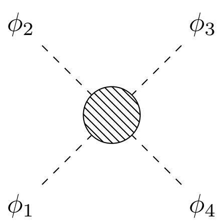
So,

(2.10)



$$= -\mathrm{i}g^2\frac{\left(-s+m^2\right)}{s} - \mathrm{i}g^2\frac{\left(-t+m^2\right)}{t} - \mathrm{i}g^2\frac{\left(-u+m^2\right)}{u} - 3\mathrm{i}g^2$$

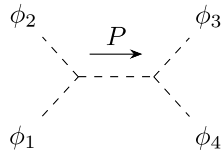
(2.11)



$$= -\mathrm{i}g^2\frac{\left(-s+m^2\right)}{s} - \mathrm{i}g^2\frac{\left(-t+m^2\right)}{t} - \mathrm{i}g^2\frac{\left(-u+m^2\right)}{u} - \mathrm{i}g^2\frac{s}{s} - \mathrm{i}g^2\frac{t}{t} - \mathrm{i}g^2\frac{u}{u}$$

(2.12)


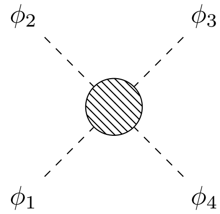

$$= -\mathrm{i}g^2m^2\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right) = \mathrm{i}g^2m^2\left(\frac{1}{\langle 12\rangle[12]}+\frac{1}{\langle 14\rangle[14]}+\frac{1}{\langle 13\rangle[13]}\right)$$

Para uma perna massiva, $\phi_4$,

(2.13)


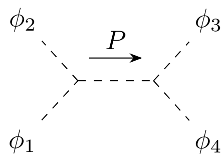
$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)(-s)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2$$

So,

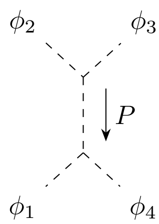
(2.14)


$$= \mathrm{i}g^2 + \mathrm{i}g^2 + \mathrm{i}g^2 - 3\mathrm{i}g^2 = 0$$

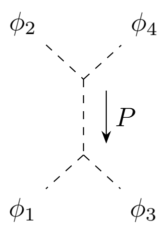
Para duas pernas massivas, $\phi_{3,4}$,

(2.15)


$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2 \frac{s+m^2}{s}$$

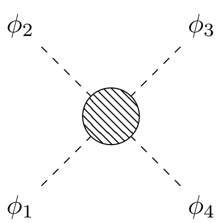
(2.16)


$$= (\mathrm{i}g)^2 \frac{(-t)(-t)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t}{-t+m^2}$$

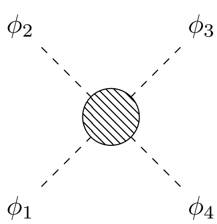
(2.17)


$$= -\mathrm{i}g^2 \frac{u}{-u+m^2}$$

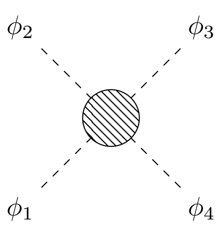
So,

(2.18)


$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - 3\mathrm{i}g^2$$

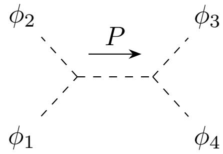
(2.19)


$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - \mathrm{i}g^2 \frac{s}{s} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2}$$

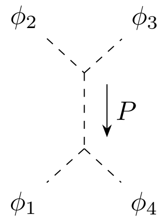
(2.20)


$$= -\mathrm{i}g^2 m^2 \left(-\frac{1}{s} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2}\right) = -\mathrm{i}g^2 m^2 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1\mathbf{4}\rangle[1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3}\rangle[1\mathbf{3}]}\right)$$

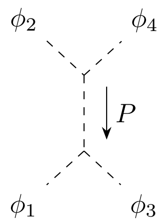
Para uma perna sem massa $\phi_1$,

(2.21)



$$= (\mathrm{i}g)^2 \frac{(-s)\big(-s-m^2\big)}{\mathrm{i}(-s)(-s+m^2)} = -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2}$$

(2.22)


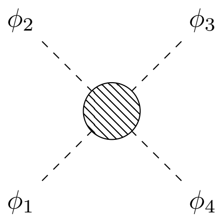

$$= (\mathrm{i}g)^2 \frac{(-t)\big(-t-m^2\big)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t+m^2}{-t+m^2}$$

(2.23)


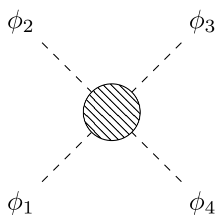

$$= (\mathrm{i}g)^2 \frac{(-u)\big(-u-m^2\big)}{\mathrm{i}(-u)(-u+m^2)} = -\mathrm{i}g^2 \frac{u+m^2}{-u+m^2}$$

(2.24)

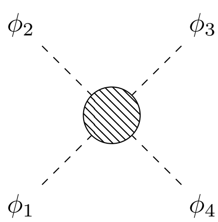
So,

(2.25)



$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - 3\mathrm{i}g^2$$

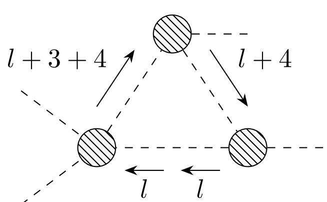
(2.26)



$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - \mathrm{i}g^2 \frac{-s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2}$$

(2.27)



$$= -\mathrm{i}g^2 m^2 \left( \frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -\mathrm{i}g^2 m^2 \left( \frac{1}{\langle 1\mathbf{2}\rangle[1\mathbf{2}]} + \frac{1}{\langle 1\mathbf{4}\rangle[1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3}\rangle[1\mathbf{3}]} \right)$$

Cut comparison, only massless legs

(2.28)



$$= \frac{\mathrm{i}g^2 m^2 \big(\mathrm{i}gm^2\big)^2}{(\mathrm{i}m^2)^3} \left( \frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right)$$

to solve for the cuts, $l^2 = (l+3+4)^2 = (l+4)^2 = 0$,

(2.29) $$l^2 = 0 \Rightarrow l = -|l\rangle[l|$$

(2.30) $$0 = (l+4)^2 = \langle l4\rangle[l4] = 0 \Rightarrow |l] = |4]$$

(2.31) $$0 = (l+3+4)^2 = \langle lP_{34}\rangle[lP_{34}] + (3+4)^2 = \langle l|3+4|l] + \langle 34\rangle[34] = \langle l|3+4|4] + \langle 34\rangle[34]$$

(2.32) $$\langle 43\rangle[34] = -\langle l3\rangle[34] \Rightarrow |l\rangle = -|4\rangle + z|3\rangle$$

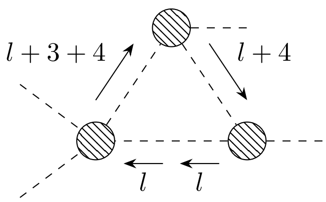
$$l=-(-|4\rangle+z|3\rangle)[4| \tag{2.33}$$

The cuts are solved by this. Hence,

$$\begin{aligned}&=g^4\left(\frac{1}{\langle 12\rangle[12]}-\frac{1}{\langle 1l\rangle[1l]}-\frac{1}{\langle 2l\rangle[2l]}\right)\\&=g^4\left(\frac{1}{\langle 12\rangle[12]}-\frac{1}{(-\langle 14\rangle+z\langle 13\rangle)[14]}-\frac{1}{(-\langle 24\rangle+z\langle 23\rangle)[24]}\right)\end{aligned}$$

Now for internal massive lines,

(2.34)




$$=\frac{-\mathrm{i}g^2m^2\left(-\mathrm{i}gm^2\right)^2}{\left(-\mathrm{i}m^2\right)^3}\left(\frac{1}{\langle 12\rangle[12]}-\frac{1}{\langle 1l\rangle[1l]}-\frac{1}{\langle 2l\rangle[2l]}\right)$$

With the cuts being, $l^2=(l+3+4)^2=(l+4)^2=-m^2$,

$$0=(l+4)^2-l^2=2l\cdot p_4 \tag{2.35}$$

$$0=(l+4+3)^2-l^2=2l\cdot(4+3)+(4+3)^2=2l\cdot p_3+(4+3)^2 \tag{2.36}$$

As ansatz, $l=|4\rangle[4|+\alpha|4\rangle[3|+\beta|3\rangle[4|$ satisfy both conditions above. The remaining condition is,

$$l^2=-m^2 \tag{2.37}$$

$$-\alpha\beta[43]\langle 43\rangle=-m^2\Rightarrow\alpha=\frac{m^2}{\beta\langle 34\rangle[34]} \tag{2.38}$$

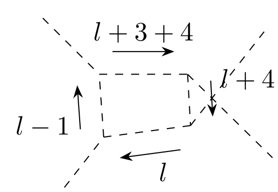
Setting now $-\beta=z$,

$$l=|4\rangle[4|-\frac{m^2}{z\langle 34\rangle[34]}|4\rangle[3|-z|3\rangle[4| \tag{2.39}$$

The value of the diagram is,

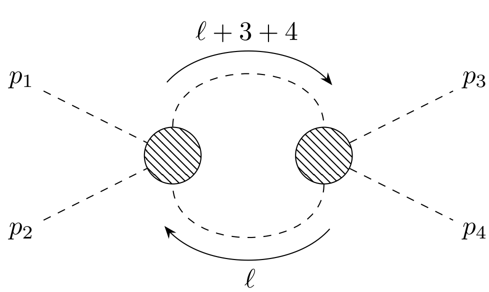
$$\begin{aligned}&=g^4\left(\frac{1}{\langle 12\rangle[12]}+\frac{1}{\langle 1l\rangle[l1]}+\frac{1}{\langle 2l\rangle[l2]}\right)\\&=g^4\left(\frac{1}{\langle 12\rangle[12]}+\frac{1}{-\langle 1|l|1]}+\frac{1}{-\langle 2|l|2]}\right)\\&=g^4\left(\frac{1}{\langle 12\rangle[12]}-\frac{1}{\langle 14\rangle[41]-\frac{m^2}{z\langle 34\rangle[34]}\langle 14\rangle[31]-z\langle 13\rangle[41]}-\frac{1}{\langle 2|l|2]}\right)\end{aligned}$$

The explicit cut loop amplitude is,



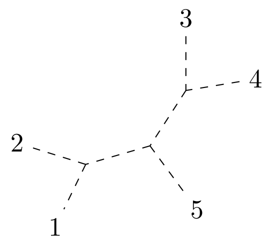

Triple cut has no improvement, what about a double cut,

(2.40)



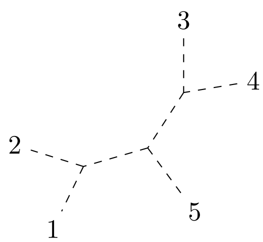

$$=\frac{\left(\mathrm{i}g^2m^2\right)^2}{\left(\mathrm{i}m^2\right)^2}\left(\frac{1}{\langle 12\rangle[12]}-\frac{1}{\langle 1\ell\rangle[1\ell]}-\frac{1}{\langle 2\ell\rangle[2\ell]}\right)\left(\frac{1}{\langle 34\rangle[34]}+\frac{1}{\langle 3\ell\rangle[3\ell]}+\frac{1}{\langle 4\ell\rangle[4\ell]}\right)$$

Five point amplitude,



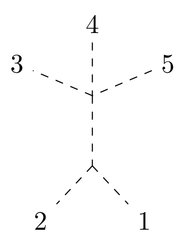

$$= \frac{(\mathrm{i}g)^3\Big(m^2+p_1^2+p_2^2+(p_1+p_2)^2\Big)\Big(m^2+p_5^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)\Big(m^2+p_3^2+p_4^2+(p_3+p_4)^2\Big)}{i^2(p_1+p_2)^2\Big((p_1+p_2)^2+m^2\Big)(p_3+p_4)^2\Big((p_3+p_4)^2+m^2\Big)}$$

Let's consider the special case of all massless,



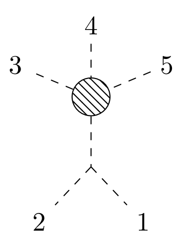

$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)\Big(m^2+(p_3+p_4)^2\Big)}{(p_3+p_4)^2\Big((p_3+p_4)^2+m^2\Big)} = \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)}{(p_3+p_4)^2}$$

Combining this graph with,




$$= \frac{-\mathrm{i}3g^2\mathrm{i}g\Big(m^2+(p_1+p_2)^2\Big)}{\mathrm{i}(p_1+p_2)^2\Big((p_1+p_2)^2+m^2\Big)} = \frac{-3\mathrm{i}g^3}{(p_1+p_2)^2}$$

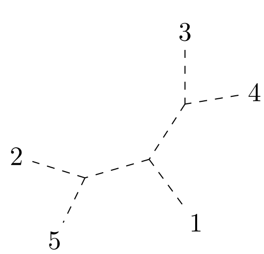
We get,




$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\Bigg[\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)}{(p_3+p_4)^2} - \frac{(p_3+p_4)^2}{(p_3+p_4)^2} + \frac{\Big(m^2+(p_3+p_5)^2+(p_1+p_2)^2\Big)}{(p_3+p_5)^2} - \frac{(p_3+p_5)^2}{(p_3+p_5)^2} + \frac{\Big(m^2+(p_5}{}$$

$$= \frac{\mathrm{i}g^3\Big(m^2+(p_1+p_2)^2\Big)}{(p_1+p_2)^2}\Bigg[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\Bigg]$$

Now we have to sum the contributions of 1 being in the middle,

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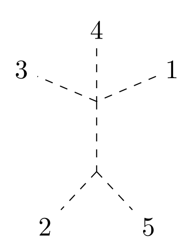



$+$ perm

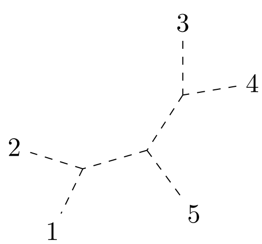
Which will be,

$$\frac{\mathrm{i}g^3}{(p_2+p_3)^2}\frac{m^2+(p_2+p_3)^2+(p_4+p_5)^2}{(p_4+p_5)^2}-\frac{3\mathrm{i}g^3}{(p_2+p_3)^2}-\frac{3\mathrm{i}g^3}{(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

Summing all the contributions we have,

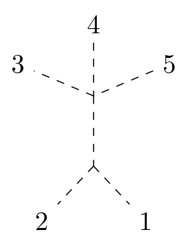
$$\begin{aligned}
=&\frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]+(2\leftrightarrow 3,4,5)\\
&+\mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_2+p_4)^2}+\frac{1}{(p_2+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_2)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_2)^2}+\right.\\
&+\frac{\mathrm{i}g^3m^2}{(p_2+p_3)^2(p_4+p_5)^2}-\frac{2\mathrm{i}g^3}{(p_2+p_3)^2}-\frac{2\mathrm{i}g^3}{(p_4+p_5)^2}+(3\leftrightarrow 4,5)\\
=&\frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]+(2\leftrightarrow 3,4,5)\\
&+\frac{\mathrm{i}g^3m^2}{(p_2+p_3)^2(p_4+p_5)^2}+\frac{\mathrm{i}g^3m^2}{(p_2+p_4)^2(p_3+p_5)^2}+\frac{\mathrm{i}g^3m^2}{(p_2+p_5)^2(p_4+p_3)^2}
\end{aligned}$$

By residue, any amplitude with just one massive external on-shell leg is zero. For two massive external on-shell legs, let's take as massive $1,2$,



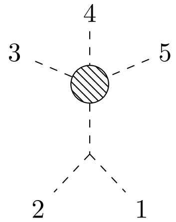

$$=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}$$

Combining this graph with,



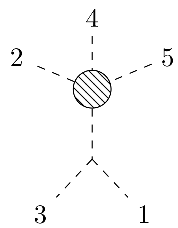

$$=-3\mathrm{i}g^3\frac{(p_1+p_2)^2-m^2}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}$$

so,



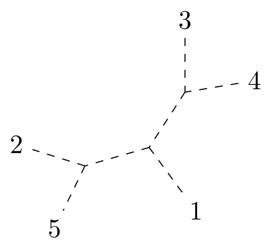

$$\begin{aligned}
&=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\left[\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}-\frac{(p_3+p_4)^2}{(p_3+p_4)^2}+(5\leftrightarrow 3,4)\right]\\
&=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]
\end{aligned}$$

The other contributions are,

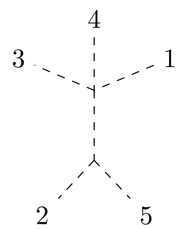



$$= \frac{\mathrm{i}g^3 (p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)} \left[ \frac{1}{m^2+(p_2+p_4)^2} + \frac{1}{m^2+(p_2+p_5)^2} + \frac{1}{(p_4+p_5)^2} \right]$$

Now we have to sum the contributions of 1 being in the middle,




$+$




$+$ perm

which are,

$$= \mathrm{i}g^3 \frac{(p_2+p_3)^2+(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5)$$