SCALAR PROXY

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1. Conformal Toy Model

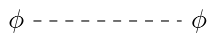
Consider the following Lagrangian,

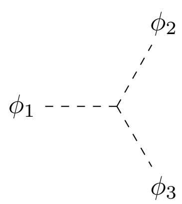
$$\mathcal{L} = -\frac{1}{2}\Box\phi\Box\phi - \frac{g}{2}\phi^2\Box\phi - \frac{g^2}{8}\phi^4 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

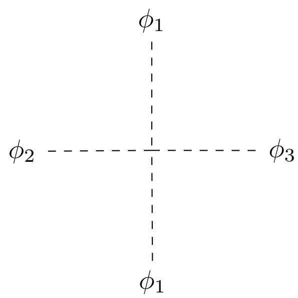
Notice the form of the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\left(\Box\phi + \frac{g}{2}\phi^2\right)^2 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

It possesses the Feynman rules,

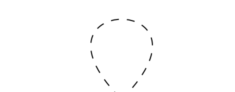
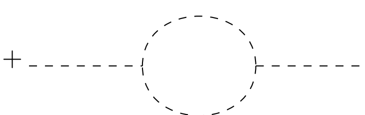
- 

$= \frac{1}{\mathrm{i}}\frac{1}{m^2p^2+p^4}$
- 

$= \mathrm{i}g\left(p_1^2 + p_2^2 + p_3^2 + m^2\right)$
- 

$= -3\mathrm{i}g^2$

Let's compute the self energy,

(1.1) $\mathrm{i}\Pi\left(p^2\right) =$  $+$  $+ \cdots$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\mathrm{i}}\frac{1}{\ell^2}\frac{1}{\ell^2+m^2} \tag{1.2}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\ell^2}\frac{1}{\ell^2+m^2} \tag{1.3}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2\frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)}\left(m^2\right)^{\frac{D}{2}-2}\frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(1)} \tag{1.4}$$

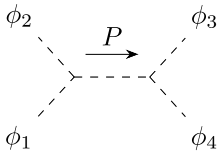
$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2\frac{\left(m^2\right)^{-\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-\epsilon)} \tag{1.5}$$

$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}(\mathrm{i}g)^2\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{\mathrm{i}^2}\frac{1}{\ell^2(\ell+p)^2}\frac{\left(m^2+\ell^2+p^2+(\ell+p)^2\right)^2}{\ell^2+m^2}\frac{1}{(\ell+p)^2+m^2} \tag{1.6}$$

For the mass renormalization we can take $p = 0$,

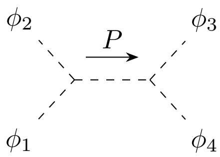
$$\mathrm{i}\Pi^{(2)} = \frac{1}{2} g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{\left(m^2 + 2\ell^2\right)^2}{\ell^4 \left(\ell^2 + m^2\right)^2} \tag{1.7}$$

Let's compute the four point amplitude for this theory,



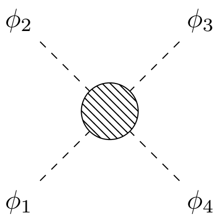
$$= (\mathrm{i}g)^2 \frac{\left(p_1^2 + p_2^2 + (p_1 + p_2)^2 + m^2\right)\left(p_3^2 + p_4^2 + (p_3 + p_4)^2 + m^2\right)}{\mathrm{i}(p_1 + p_2)^2 \left((p_1 + p_2)^2 + m^2\right)} \tag{1.8}$$

First let's consider all legs massless,

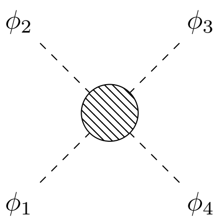


$$= \mathrm{i}g^2 \frac{\left(-s + m^2\right)\left(-s + m^2\right)}{(-s)(-s + m^2)} = -\mathrm{i}g^2 \frac{\left(-s + m^2\right)}{s} \tag{1.9}$$

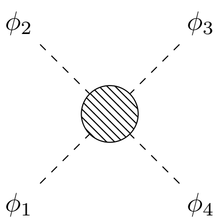
So,



$$= -\mathrm{i}g^2 \frac{\left(-s + m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t + m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u + m^2\right)}{u} - 3\mathrm{i}g^2 \tag{1.10}$$

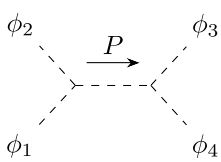


$$= -\mathrm{i}g^2 \frac{\left(-s + m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t + m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u + m^2\right)}{u} - \mathrm{i}g^2 \frac{s}{s} - \mathrm{i}g^2 \frac{t}{t} - \mathrm{i}g^2 \frac{u}{u} \tag{1.11}$$



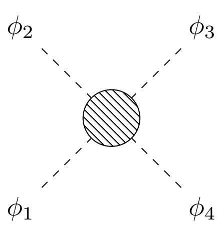
$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right) = \mathrm{i}g^2 m^2 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 14\rangle[14]} + \frac{1}{\langle 13\rangle[13]}\right) \tag{1.12}$$

Para uma perna massiva, ϕ_4,



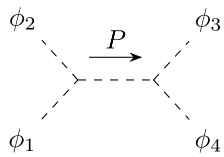
$$= (\mathrm{i}g)^2 \frac{\left(-s + m^2\right)(-s)}{\mathrm{i}(-s)(-s + m^2)} = \mathrm{i}g^2 \tag{1.13}$$

So,

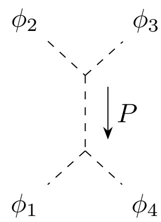


$$= \mathrm{i}g^2 + \mathrm{i}g^2 + \mathrm{i}g^2 - 3\mathrm{i}g^2 = 0 \tag{1.14}$$

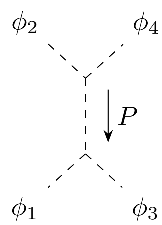
Para duas pernas massivas, $\phi_{3,4}$,



$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2 \frac{s+m^2}{s} \tag{1.15}$$

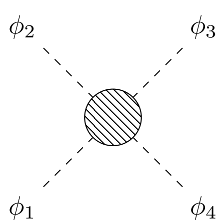


$$= (\mathrm{i}g)^2 \frac{(-t)(-t)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t}{-t+m^2} \tag{1.16}$$

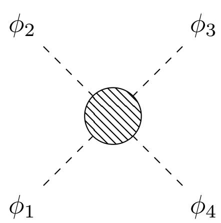


$$= -\mathrm{i}g^2 \frac{u}{-u+m^2} \tag{1.17}$$

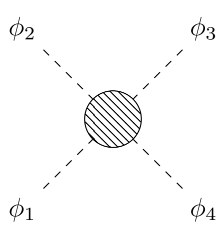
So,



$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - 3\mathrm{i}g^2 \tag{1.18}$$

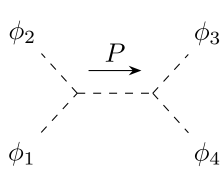


$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - \mathrm{i}g^2 \frac{s}{s} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2} \tag{1.19}$$

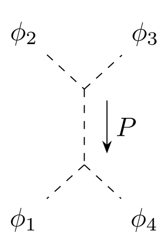


$$= -\mathrm{i}g^2 m^2 \left(-\frac{1}{s} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -\mathrm{i}g^2 m^2 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1\mathbf{4}\rangle[1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3}\rangle[1\mathbf{3}]} \right) \tag{1.20}$$

Para uma perna sem massa ϕ_1,

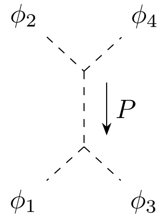


$$= (\mathrm{i}g)^2 \frac{(-s)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} \tag{1.21}$$



$$= (\mathrm{i}g)^2 \frac{(-t)\left(-t-m^2\right)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t+m^2}{-t+m^2} \tag{1.22}$$

(1.23)

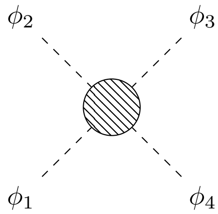


$$= (\mathrm{i}g)^2 \frac{(-u)\big(-u-m^2\big)}{\mathrm{i}(-u)(-u+m^2)} = -\mathrm{i}g^2 \frac{u+m^2}{-u+m^2}$$

(1.24)

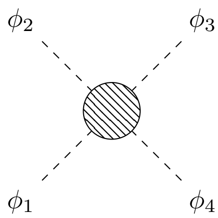
So,

(1.25)



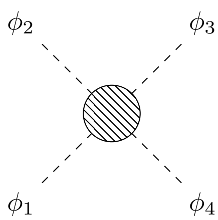
$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - 3\mathrm{i}g^2$$

(1.26)



$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - \mathrm{i}g^2 \frac{-s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2}$$

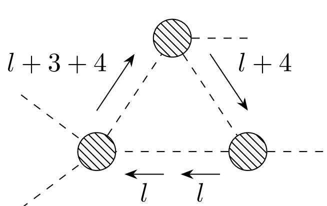
(1.27)



$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -\mathrm{i}g^2 m^2 \left(\frac{1}{\langle 1\mathbf{2}\rangle[1\mathbf{2}]} + \frac{1}{\langle 1\mathbf{4}\rangle[1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3}\rangle[1\mathbf{3}]} \right)$$

Cut comparison, only massless legs

(1.28)



$$= \frac{\mathrm{i}g^2 m^2 \big(\mathrm{i}gm^2\big)^2}{\big(\mathrm{i}m^2\big)^3} \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right)$$

to solve for the cuts, $l^2 = (l+3+4)^2 = (l+4)^2 = 0$,

$$l^2 = 0 \Rightarrow l = -|l\rangle[l| \tag{1.29}$$

$$0 = (l+4)^2 = \langle l4\rangle[l4] = 0 \Rightarrow |l] = |4] \tag{1.30}$$

$$0 = (l+3+4)^2 = \langle lP_{34}\rangle[lP_{34}] + (3+4)^2 = \langle l|3+4|l] + \langle 34\rangle[34] = \langle l|3+4|4] + \langle 34\rangle[34] \tag{1.31}$$

$$\langle 43\rangle[34] = -\langle l3\rangle[34] \Rightarrow |l\rangle = -|4\rangle + z|3\rangle \tag{1.32}$$

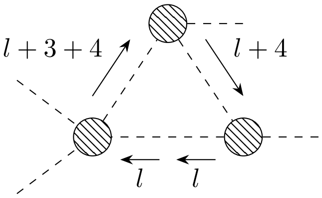
$$l = -(-|4\rangle + z|3\rangle)[4| \tag{1.33}$$

The cuts are solved by this. Hence,

$$\begin{aligned} &= g^4 \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right) \\ &= g^4 \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{(-\langle 14\rangle + z\langle 13\rangle)[14]} - \frac{1}{(-\langle 24\rangle + z\langle 23\rangle)[24]} \right) \end{aligned}$$

Now for internal massive lines,

(1.34)



$$= \frac{-\mathrm{i}g^2m^2\left(-\mathrm{i}gm^2\right)^2}{\left(-\mathrm{i}m^2\right)^3}\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]}\right)$$

With the cuts being, $l^2 = \left(l+3+4\right)^2 = \left(l+4\right)^2 = -m^2$,

$$0 = (l+4)^2 - l^2 = 2l\cdot p_4 \tag{1.35}$$

$$0 = (l+4+3)^2 - l^2 = 2l\cdot(4+3) + (4+3)^2 = 2l\cdot p_3 + (4+3)^2 \tag{1.36}$$

As ansatz, $l = |4\rangle[4| + \alpha|4\rangle[3| + \beta|3\rangle[4|$ satisfy both conditions above. The remaining condition is,

$$l^2 = -m^2 \tag{1.37}$$

$$-\alpha\beta[43]\langle 43\rangle = -m^2 \Rightarrow \alpha = \frac{m^2}{\beta\langle 34\rangle[34]} \tag{1.38}$$

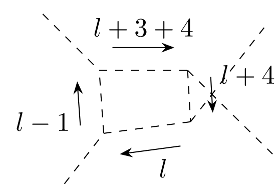
Setting now $-\beta = z$,

$$l = |4\rangle[4| - \frac{m^2}{z\langle 34\rangle[34]}|4\rangle[3| - z|3\rangle[4| \tag{1.39}$$

The value of the diagram is,

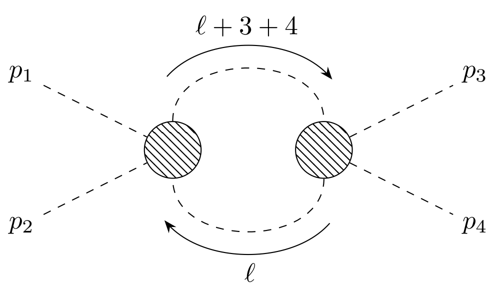
$$\begin{aligned}
&= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1l\rangle[l1]} + \frac{1}{\langle 2l\rangle[l2]}\right)\\
&= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{-\langle 1|l|1]} + \frac{1}{-\langle 2|l|2]}\right)\\
&= g^4\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 14\rangle[41] - \frac{m^2}{z\langle 34\rangle[34]}\langle 14\rangle[31] - z\langle 13\rangle[41]} - \frac{1}{\langle 2|l|2]}\right)
\end{aligned}$$

The explicit cut loop amplitude is,



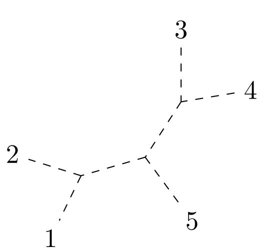
Triple cut has no improvement, what about a double cut,

(1.40)



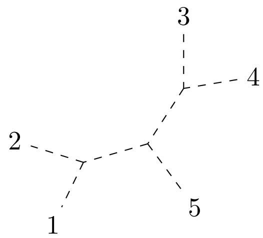
$$= \frac{\left(\mathrm{i}g^2m^2\right)^2}{\left(\mathrm{i}m^2\right)^2}\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1\ell\rangle[1\ell]} - \frac{1}{\langle 2\ell\rangle[2\ell]}\right)\left(\frac{1}{\langle 34\rangle[34]} + \frac{1}{\langle 3\ell\rangle[3\ell]} + \frac{1}{\langle 4\ell\rangle[4\ell]}\right)$$

Five point amplitude,



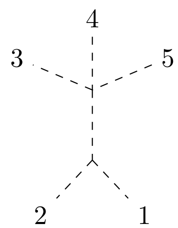
$$= \frac{(\mathrm{i}g)^3\left(m^2+p_1^2+p_2^2+(p_1+p_2)^2\right)\left(m^2+p_5^2+(p_3+p_4)^2+(p_1+p_2)^2\right)\left(m^2+p_3^2+p_4^2+(p_3+p_4)^2\right)}{i^2(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)(p_3+p_4)^2\left((p_3+p_4)^2+m^2\right)}$$

Let's consider the special case of all massless,



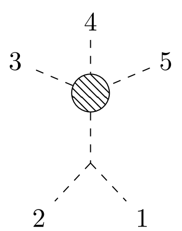
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)\Big(m^2+(p_3+p_4)^2\Big)}{(p_3+p_4)^2\Big((p_3+p_4)^2+m^2\Big)} = \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)}{(p_3+p_4)^2}$$

Combining this graph with,



$$= \frac{-\mathrm{i}3g^2\mathrm{i}g\Big(m^2+(p_1+p_2)^2\Big)}{\mathrm{i}(p_1+p_2)^2\Big((p_1+p_2)^2+m^2\Big)} = \frac{-3\mathrm{i}g^3}{(p_1+p_2)^2}$$

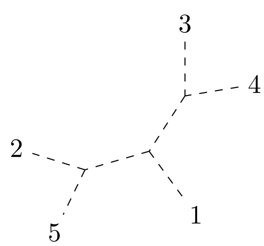
We get,



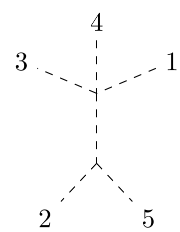
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\Bigg[\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)}{(p_3+p_4)^2} - \frac{(p_3+p_4)^2}{(p_3+p_4)^2} + \frac{\Big(m^2+(p_3+p_5)^2+(p_1+p_2)^2\Big)}{(p_3+p_5)^2} - \frac{(p_3+p_5)^2}{(p_3+p_5)^2} + \frac{\Big(m^2+(p_5}{}$$

$$= \frac{\mathrm{i}g^3\Big(m^2+(p_1+p_2)^2\Big)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,



$+$



$+$ perm

Which will be,

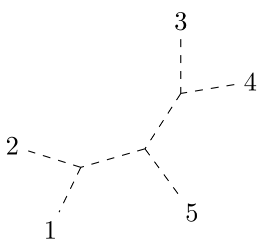
$$\frac{\mathrm{i}g^3}{(p_2+p_3)^2}\frac{m^2+(p_2+p_3)^2+(p_4+p_5)^2}{(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3\leftrightarrow 4,5)$$

Summing all the contributions we have,

$$\begin{aligned}
=&\frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right] + (2\leftrightarrow 3,4,5)\\
&+\mathrm{i}g^3\Bigg[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_2+p_4)^2}+\frac{1}{(p_2+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_2)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_2)^2}+\cdots\\
&+\frac{\mathrm{i}g^3m^2}{(p_2+p_3)^2(p_4+p_5)^2} - \frac{2\mathrm{i}g^3}{(p_2+p_3)^2} - \frac{2\mathrm{i}g^3}{(p_4+p_5)^2} + (3\leftrightarrow 4,5)
\end{aligned}$$

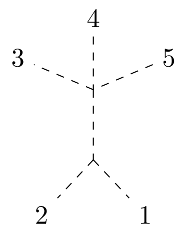
$$=\frac{ig^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]+(2\leftrightarrow 3,4,5)$$
$$+\frac{ig^3m^2}{(p_2+p_3)^2(p_4+p_5)^2}+\frac{ig^3m^2}{(p_2+p_4)^2(p_3+p_5)^2}+\frac{ig^3m^2}{(p_2+p_5)^2(p_4+p_3)^2}$$

By residue, any amplitude with just one massive external on-shell leg is zero. For two massive external on-shell legs, let's take as massive $1,2$,



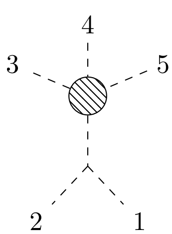
$$=ig^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}$$

Combining this graph with,



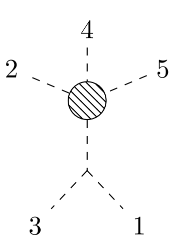
$$=-3ig^3\frac{(p_1+p_2)^2-m^2}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}$$

so,



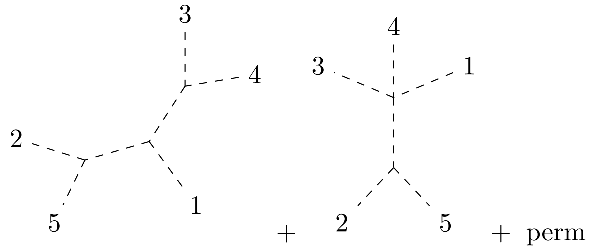
$$=ig^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\left[\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}-\frac{(p_3+p_4)^2}{(p_3+p_4)^2}+(5\leftrightarrow 3,4)\right]$$
$$=ig^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]$$

The other contributions are,



$$=\frac{ig^3(p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{(p_4+p_5)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,



which are,

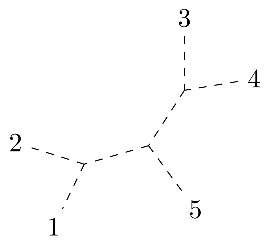
$$= \mathrm{i}g^3 \frac{(p_2+p_3)^2 + (p_4+p_5)^2}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5)$$

So, summing all the contributions,

$$\begin{aligned}
&= \mathrm{i}g^3 \frac{\left((p_1+p_2)^2 - m^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \frac{\mathrm{i}g^3 (p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)}\left[\frac{1}{m^2+(p_2+p_4)^2} + \frac{1}{m^2+(p_2+p_5)^2} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad + \mathrm{i}g^3 \frac{(p_2+p_3)^2 + (p_4+p_5)^2}{\left(m^2 + (p_2+p_3)^2\right)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5) \\
&= -\mathrm{i}g^3 \frac{m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad - \mathrm{i}g^3 \frac{m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad + \mathrm{i}g^3 \frac{-m^2 + 2p_2\cdot p_3 + (p_4+p_5)^2}{(2p_2\cdot p_3)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{2p_2\cdot p_3} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3 \leftrightarrow 4,5) \\
&= -\frac{\mathrm{i}g^3 m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad + \mathrm{i}g^3\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2} + \frac{1}{2p_2\cdot p_3} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_3+p_5)^2} + \frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_3} + \frac{1}{(p_4+p_3)^2}\right] \\
&\quad - \frac{\mathrm{i}g^3 m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&\quad - \frac{\mathrm{i}g^3 m^2}{(2p_2\cdot p_3)(p_4+p_5)^2} + (3 \leftrightarrow 4,5) \\
&\quad - \frac{2\mathrm{i}g^3}{2p_2\cdot p_3} - \frac{2\mathrm{i}g^3}{(p_4+p_5)^2} - \frac{2\mathrm{i}g^3}{2p_2\cdot p_4} - \frac{2\mathrm{i}g^3}{(p_3+p_5)^2} - \frac{2\mathrm{i}g^3}{2p_2\cdot p_5} - \frac{2\mathrm{i}g^3}{(p_4+p_3)^2} \\
&= -\frac{\mathrm{i}g^3 m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&\quad - \frac{\mathrm{i}g^3 m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5)
\end{aligned}$$

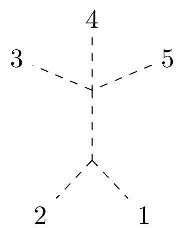
$$
-\frac{\mathrm{i}g^3m^2}{(2p_2\cdot p_3)(p_4+p_5)^2}+(3\leftrightarrow 4,5)
$$

Is almost the same of the all massless, but with a different denominator in the $1,2$ channel. Now with three massive legs, being $3,4,5$,



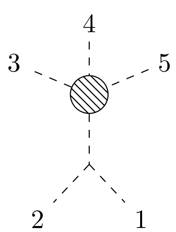
$$
=\mathrm{i}g^3\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_1+p_2)^2(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}
$$

With,



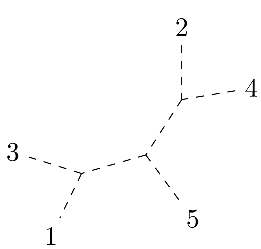
$$
=\frac{-\mathrm{i}3g^2\mathrm{i}g\left(m^2+(p_1+p_2)^2\right)}{\mathrm{i}(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)}=\frac{-3\mathrm{i}g^3}{(p_1+p_2)^2}
$$

So,

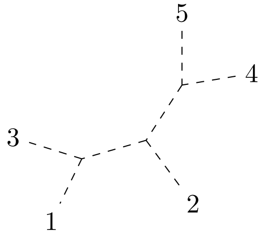


$$
\begin{aligned}
&=\mathrm{i}g^3\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_1+p_2)^2(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}-\frac{3\mathrm{i}g^3}{(p_1+p_2)^2}\\
&=\frac{\mathrm{i}g^3}{(p_1+p_2)^2}\left[\frac{\left((p_1+p_2)^2+(p_3+p_4)^2\right)\left(-m^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}-\frac{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}\right]+(5\leftrightarrow 3,4)\\
&=\frac{\mathrm{i}g^3}{(p_1+p_2)^2}\left[\frac{(p_1+p_2)^2\left(-m^2+(p_3+p_4)^2\right)-2m^2(p_3+p_4)^2}{(p_3+p_4)^2\left(m^2+(p_3+p_4)^2\right)}\right]+(5\leftrightarrow 3,4)
\end{aligned}
$$

The other topology is,



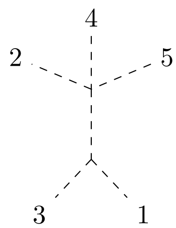
$$
\begin{aligned}
&=\mathrm{i}g^3\frac{\left(m^2+p_3^2+(p_1+p_3)^2\right)\left(m^2+p_5^2+(p_1+p_3)^2+(p_2+p_4)^2\right)\left(m^2+p_4^2+(p_2+p_4)^2\right)}{(p_1+p_3)^2\left(m^2+(p_1+p_3)^2\right)(p_2+p_4)^2\left(m^2+(p_2+p_4)^2\right)}\\
&=\mathrm{i}g^3\frac{\left((p_1+p_3)^2+(p_2+p_4)^2\right)}{\left(m^2+(p_1+p_3)^2\right)\left(m^2+(p_2+p_4)^2\right)}
\end{aligned}
$$



$$= \mathrm{i}g^3 \frac{\left(m^2+p_3^2+(p_1+p_3)^2\right)\left(m^2+(p_1+p_3)^2+(p_5+p_4)^2\right)\left(m^2+p_5^2+p_4^2+(p_5+p_4)^2\right)}{(p_1+p_3)^2\left(m^2+(p_1+p_3)^2\right)(p_5+p_4)^2\left(m^2+(p_5+p_4)^2\right)}$$

$$= \mathrm{i}g^3 \frac{\left(m^2+(p_1+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_1+p_3)^2\right)(p_5+p_4)^2\left(m^2+(p_5+p_4)^2\right)}$$

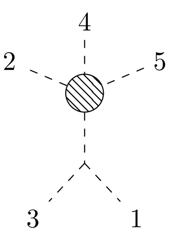
Also,



$$= -3\mathrm{i}g^2 g \frac{\left(m^2+p_3^2+(p_3+p_1)^2\right)}{(p_3+p_1)^2\left(m^2+(p_3+p_1)^2\right)}$$

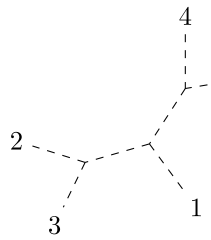
$$= -3\mathrm{i}g^3 \frac{1}{\left(m^2+(p_3+p_1)^2\right)}$$

So,

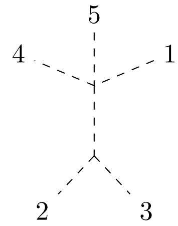


$$\begin{aligned}
&= \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{\left((p_1+p_3)^2+(p_2+p_4)^2\right)-m^2-(p_2+p_4)^2}{\left(m^2+(p_2+p_4)^2\right)}+\frac{\left((p_1+p_3)^2+(p_2+p_5)^2\right)-m^2-(p_2+p_5)^2}{\left(m^2+(p_2+p_5)^2\right)}\right] \\
&\quad + \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{\left(m^2+(p_1+p_3)^2+(p_4+p_5)^2\right)\left(-m^2+(p_4+p_5)^2\right)-(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}\right] \\
&= \frac{\mathrm{i}g^3\left((p_1+p_3)^2-m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}\right] \\
&\quad + \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{-m^4-m^2(p_1+p_3)^2+(p_1+p_3)^2(p_4+p_5)^2-m^2(p_4+p_5)^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}\right] \\
&= \frac{\mathrm{i}g^3\left((p_1+p_3)^2-m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5) \\
&\quad - \frac{\mathrm{i}g^3 m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)
\end{aligned}$$

Now, with 1 in the middle,



$+$



$+$ perm

Which is,

$$= \frac{ig^3\left(m^2+p_3^2+(p_2+p_3)^2\right)\left(m^2+p_1^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(m^2+p_5^2+p_4^2+(p_5+p_4)^2\right)}{(p_2+p_3)^2\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{3ig^3}{m^2+(p_2+p_3)^2} - \frac{3ig^3\left(-m^2+(p_4+\right.}{(p_4+p_5)^2\left(m^2+(p_4\right.}$$

$$= \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{3ig^3}{m^2+(p_2+p_3)^2} - \frac{3ig^3\left(-m^2+(p_4+p_5)^2\right)}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)$$

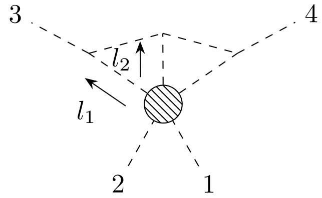
Summing all the contributions we have,

$$\begin{aligned}
& ig^3\frac{\left(-m^2+(p_5+p_4)^2\right)}{(p_5+p_4)^2\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5) \\
&\quad - \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left((p_1+p_3)^2-m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5) \\
&\quad - \frac{ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5) \\
&\quad - \frac{3ig^3}{m^2+(p_2+p_3)^2} - \frac{3ig^3\left(-m^2+(p_4+p_5)^2\right)}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5) \\
& - \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left((p_1+p_3)^2-m^2\right)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5) \\
&\quad - \frac{ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5) \\
&\quad + \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5) \\
&\quad - \frac{3ig^3}{m^2+(p_2+p_3)^2} + \frac{2ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{2ig^3}{\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5) \\
& - \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{2ig^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)\\
&\quad + ig^3\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]\\
&\quad + ig^3\left[\frac{1}{m^2+(p_2+p_3)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_3+p_5)^2}\right]\\
&\quad + ig^3\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_3)^2}+\frac{1}{m^2+(p_4+p_3)^2}\right]\\
&\quad + \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad - \frac{3ig^3}{m^2+(p_2+p_3)^2}+\frac{ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}-\frac{2ig^3}{\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&- \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad - \frac{2ig^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)\\
&\quad + \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad - \frac{ig^3}{m^2+(p_2+p_3)^2}+\frac{ig^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}-\frac{ig^3}{\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&- \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad - \frac{2ig^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)\\
&\quad + \frac{ig^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad + ig^3\frac{-(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)+m^2\left(m^2+(p_2+p_3)^2\right)-(p_4+p_5)^2\left(m^2+(p_2+p_3)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&- \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad - \frac{2ig^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)\\
&\quad + ig^3\frac{-m^4+m^2(p_4+p_5)^2-m^2(p_2+p_3)^2+(p_2+p_3)^2(p_4+p_5)^2-m^2(p_5+p_4)^2+(p_4+p_5)^4}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad + ig^3\frac{-(p_4+p_5)^2m^2-(p_4+p_5)^4+m^4+m^2(p_2+p_3)^2-(p_4+p_5)^2m^2-(p_2+p_3)^2(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)}+(3\leftrightarrow 4,5)\\
&- \frac{2ig^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)}+(3\leftrightarrow 4,5)\\
&\quad - \frac{2ig^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)
\end{aligned}
$$

$$- 2ig^3 m^2 \frac{1}{\left(m^2+(p_2+p_3)^2\right)\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)$$

Now, let's do the cuts, consider a two loop four point amplitude with five cuts,



$$= -\frac{g^3}{m^4}[\mathcal{A}_5(1,2,l_{1h},l_{2h},l_{3h}) - \mathcal{A}_5(1,2,l_{1h},l_{2\eta},l_{3\eta}) - \mathcal{A}_5(1,2,l_{1\eta},l_{2h},l_{3\eta}) - \mathcal{A}_5(1,2,l_{1\eta},l_{2\eta},l_{3h}) + 2\mathcal{A}_5(1,2,l_{1\eta}$$

2. DF^2 Theory

The $(DF)^2 + \mathrm{YM}$ theory is given by the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2} D_\mu F^{a\mu\nu} D_\alpha F_a{}^\alpha{}_\nu + \frac{1}{3} f_{abc} F^a{}_\mu{}^\nu F^b{}_\nu{}^\alpha F^c{}_\alpha{}^\mu - \frac{1}{2} D_\mu \phi^I D^\mu \phi_I + \frac{g}{2} C^{Iab} \phi_I F_{a\mu\nu} F_b{}^{\mu\nu} + \frac{g}{6} d^{IJK} \phi_I \phi_J \phi_K - \frac{m^2}{2} \phi_I \phi^I - \frac{m^2}{4} F_{a\mu\nu} F^{a\mu\nu}$$

Where of course,

$$\begin{aligned} F^a{}_{\mu\nu} &= \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + g f^a{}_{bc} A^b{}_\mu A^c{}_\nu \\ (D_\alpha F^{\mu\nu})^a &= \partial_\alpha F^a{}_{\mu\nu} + g f^a{}_{bc} A^b{}_\alpha F^c{}_{\mu\nu} \\ (D_\alpha \phi)^I &= \partial_\alpha \phi^I - ig \mathrm{T}_{\mathrm{R}}^a{}^I{}_J A_{a\alpha} \phi^J \end{aligned}$$

We also have to incorporate the gauge fixing part,

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2} \partial_\mu A^{a\mu} (-\Box + m^2) \partial_\nu A_a{}^\nu$$

2.1. quadratic piece.

Let us collect all the quadratic pieces,

$$\begin{aligned} -\frac{1}{2} D_\mu F^{a\mu\nu} D_\alpha F_a{}^\alpha{}_\nu &= -\frac{1}{2}\left(\partial_\mu F^{a\mu\nu} + g f^a{}_{bc} A^b{}_\mu F^{c\mu\nu}\right)\left(\partial_\alpha F_a{}^\alpha{}_\nu + g f_{ade} A^d{}_\alpha F^{e\alpha}{}_\nu\right) \\ &= -\frac{1}{2}(\partial_\mu F^{a\mu\nu})(\partial_\alpha F_a{}^\alpha{}_\nu) \\ &= -\frac{1}{2}\left(\partial_\mu\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g f^a{}_{bc} A^{b\mu} A^{c\nu}\right)\right)\left(\partial_\alpha\left(\partial^\alpha A_{a\nu} - \partial_\nu A_a{}^\alpha + g f_{ade} A^d{}_\alpha A^e{}_\nu\right)\right) \\ &= -\frac{1}{2}(\partial_\mu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}))(\partial_\alpha(\partial^\alpha A_{a\nu} - \partial_\nu A_a{}^\alpha)) \\ &= -\frac{1}{2}(\Box A^{a\nu} - \partial^\nu \partial_\mu A^{a\mu})(\Box A_{a\nu} - \partial_\nu \partial_\alpha A_a{}^\alpha) \\ &= -\frac{1}{2}\Box A^{a\nu}\Box A_{a\nu} + \frac{1}{2}\Box A^{a\nu}\partial_\nu\partial_\alpha A_a{}^\alpha + \frac{1}{2}\partial^\nu\partial_\mu A^{a\mu}\Box A_{a\nu} - \frac{1}{2}\partial^\nu\partial_\mu A^{a\mu}\partial_\nu\partial_\alpha A_a{}^\alpha \\ &= -\frac{1}{2}\Box A^{a\nu}\Box A_{a\nu} + \Box A^{a\nu}\partial_\nu\partial_\alpha A_a{}^\alpha - \frac{1}{2}\Box A^{a\mu}\partial_\mu\partial_\alpha A_a{}^\alpha \\ &= -\frac{1}{2}\Box A^{a\nu}\Box A_{a\nu} + \frac{1}{2}\Box A^{a\nu}\partial_\nu\partial_\alpha A_a{}^\alpha \\ &= \frac{1}{2} A^{a\mu}\delta_{ab}\left(-\eta_{\mu\nu}\Box^2 + \partial_\mu\partial_\nu\Box\right)A^{b\nu} \end{aligned}$$

$$-\frac{1}{2}\partial_\mu A^{a\mu}(-\Box + m^2)\partial_\nu A_a{}^\nu = \frac{1}{2} A^{a\mu}\delta_{ab}(-\Box + m^2)\partial_\mu\partial_\nu A^{b\nu}$$

$$\begin{aligned} -\frac{m^2}{4} F_{a\mu\nu} F^{a\mu\nu} &= -\frac{m^2}{4}\left(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g f_{abc} A^b{}_\mu A^c{}_\nu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g f^a{}_{de} A^{d\mu} A^{e\nu}\right) \\ &= -\frac{m^2}{4}\left(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu}\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right) \\ &= \frac{1}{2} A^{a\mu}\delta_{ab}\left(\eta_{\mu\nu} m^2 \Box - m^2 \partial_\nu\partial_\mu\right)A^{b\nu} \end{aligned}$$

Summing all the contributions, the quadratic piece of the Lagrangian is,

$$\mathcal{L} \supset \frac{1}{2}A^{a\mu}\delta_{ab}\big(-\eta_{\mu\nu}\Box^2 + \partial_\mu\partial_\nu\Box\big)A^{b\nu} + \frac{1}{2}A^{a\mu}\delta_{ab}\big(-\Box + m^2\big)\partial_\mu\partial_\nu A^{b\nu} + \frac{1}{2}A^{a\mu}\delta_{ab}\big(\eta_{\mu\nu}m^2\Box - m^2\partial_\nu\partial_\mu\big)A^{b\nu}$$

$$\mathcal{L} \supset \frac{1}{2}A^{a\mu}\delta_{ab}\eta_{\mu\nu}\big(-\Box^2 + m^2\Box\big)A^{b\nu}$$

2.2. **cubic piece.** Now, the cubic piece,

$$\begin{aligned}
-\frac{1}{2}D_\mu F^{a\mu\nu}D_\alpha F_a{}^\alpha{}_\nu &= -\frac{1}{2}(\Box A^{a\nu} - \partial^\nu\partial\cdot A^a)gf_{ade}\partial^\alpha\big(A^d{}_\alpha A^e{}_\nu\big) \\
&\quad - \frac{1}{2}gf^a{}_{bc}\partial_\mu\big(A^{b\mu}A^{c\nu}\big)(\Box A_{a\nu} - \partial_\nu\partial\cdot A_a) \\
&\quad - \frac{1}{2}(\Box A^{a\nu} - \partial^\nu\partial\cdot A^a)gf_{ade}A^d{}_\alpha(\partial^\alpha A^e{}_\nu - \partial_\nu A^{e\alpha}) \\
&\quad - \frac{1}{2}gf^a{}_{bc}A^b{}_\mu(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu})(\Box A_{a\nu} - \partial_\nu\partial\cdot A_a) \\
&= -gf^a{}_{bc}\partial_\mu\big(A^{b\mu}A^{c\nu}\big)(\Box A_{a\nu} - \partial_\nu\partial\cdot A_a) \\
&\quad - gf^a{}_{bc}A^b{}_\mu(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu})(\Box A_{a\nu} - \partial_\nu\partial\cdot A_a)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{3}f_{abc}F^a{}_\mu{}^\nu F^b{}_\nu{}^\alpha F_\alpha{}^\mu &= \frac{1}{3}f_{abc}\big(\partial_\mu A^{a\nu} - \partial^\nu A^a{}_\mu\big)\big(\partial_\nu A^{b\alpha} - \partial^\alpha A^a{}_\nu\big)(\partial_\alpha A^{c\mu} - \partial^\mu A^c{}_\alpha) \\
&= \frac{2}{3}f_{abc}\big(\partial_\mu A^{a\nu} - \partial^\nu A^a{}_\mu\big)\big(\partial_\nu A^{b\alpha} - \partial^\alpha A^a{}_\nu\big)\partial_\alpha A^{c\mu}
\end{aligned}$$

$$\begin{aligned}
-\frac{m^2}{4}F_{a\mu\nu}F^{a\mu\nu} &= -\frac{m^2}{4}\big(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu}\big)gf^a{}_{bc}A^{b\mu}A^{c\nu} \\
&\quad - \frac{m^2}{4}gf_{abc}A^b{}_\mu A^c{}_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\
&= -m^2 gf_{abc}A^b{}_\mu A^c{}_\nu\partial^\mu A^{a\nu}
\end{aligned}$$