SCALAR PROXY

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1. Introduction

We will work most with the scalar proxy given by the lagrangian,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2M^2}\Box\phi\Box\phi - \frac{\kappa}{2}\Box\phi\phi^2 \tag{1.1}$$

The idea here is reintegrate the higher derivative term, in order to obtain a lower derivative term, but in terms of additional fields. This is easily done by,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \Box\phi\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box\phi\phi^2 \tag{1.2}$$

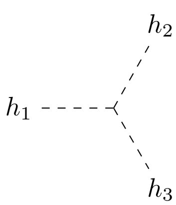
The new lagrangian has mixed propagator terms, to diagonalize it is also easy, we just open in terms of $\phi = h - \eta$,

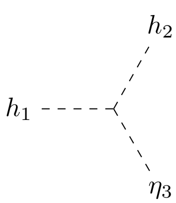
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 + \eta\Box(h-\eta) + \frac{M^2}{2}\eta^2 \tag{1.3}$$

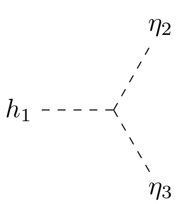
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h\partial^\mu\eta - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 - \partial_\mu\eta\partial^\mu(h-\eta) + \frac{M^2}{2}\eta^2 \tag{1.4}$$

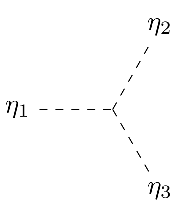
$$\mathcal{L} = -\frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{M^2}{2}\eta^2 - \frac{\kappa}{2}\Box(h-\eta)(h-\eta)^2 \tag{1.5}$$

The Feynman rules are easily red as,

- h - - - - - - - - - - h $= \frac{1}{\mathrm{i}}\frac{1}{p^2}$
- η - - - - - - - - - - η $= -\frac{1}{\mathrm{i}}\frac{1}{p^2+M^2}$
- 

$= \mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$
- 

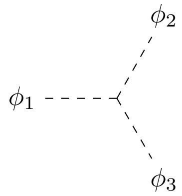
$= -\mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$
- 

$= \mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$
- 

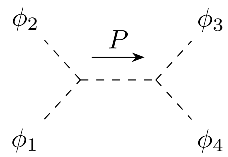
$= -\mathrm{i}\kappa\left(p_1^2 + p_2^2 + p_3^2\right)$

Which can also be seen directly from the Feynman rules of the ϕ field,

- ϕ - - - - - - - - - - ϕ $= \frac{1}{\mathrm{i}}\frac{1}{p^2+\frac{p^4}{M^2}}$

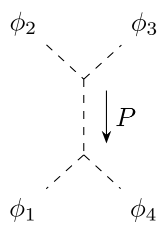
-  $$= \mathrm{i}\kappa\big(p_1^2 + p_2^2 + p_3^2\big) = \mathrm{i}\kappa(p_1 + p_2 + p_3)^2 - 2\mathrm{i}\kappa(p_1\cdot p_2 + p_2\cdot p_3 + p_3\cdot p_1) = -\mathrm{i}\kappa(\langle 12\rangle[12] + \langle 23\rangle[23] + \langle 31\rangle[31])$$

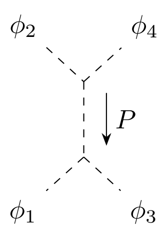
So that the four point amplitude can be computed by,



$$\begin{aligned}
&= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2\frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] + \langle 2P\rangle[2P] + \langle P1\rangle[P1])(\langle 34\rangle[34] - \langle 4P\rangle[4P] - \langle P3\rangle[P3]) && (1.6)\\
&= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2\frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] - \langle P2\rangle[2P] - \langle P1\rangle[1P])(\langle 34\rangle[34] + \langle P4\rangle[4P] + \langle P3\rangle[3P]) && (1.7)\\
&= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2\frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] + \langle P|\,1+2|P])(\langle 34\rangle[34] - \langle P|\,3+4|P]) && (1.8)\\
&= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2\frac{1}{P^2+\frac{P^4}{M^2}}(\langle 12\rangle[12] - \langle P|\,P|P])(\langle 34\rangle[34] - \langle P|\,P|P]) && (1.9)\\
&= \frac{1}{\mathrm{i}}(-\mathrm{i}\kappa)^2\frac{1}{P^2+\frac{P^4}{M^2}}\big(\langle 12\rangle[12] - 2P^2\big)\big(\langle 34\rangle[34] - 2P^2\big) && (1.10)\\
&= -\mathrm{i}\frac{(\kappa M)^2}{s(M^2-s)}(\langle 12\rangle[12] + 2s)(\langle 34\rangle[34] + 2s) && (1.11)
\end{aligned}$$

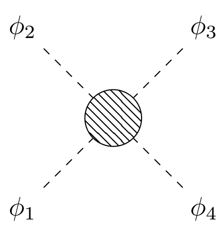
It's trivial to read the t and u channels from this expression,

(1.12)  $$= -\mathrm{i}\frac{(\kappa M)^2}{t(M^2-t)}(\langle 23\rangle[23] + 2t)(\langle 41\rangle[41] + 2t)$$

(1.13)  $$= -\mathrm{i}\frac{(\kappa M)^2}{u(M^2-u)}(\langle 24\rangle[24] + 2u)(\langle 31\rangle[31] + 2u)$$

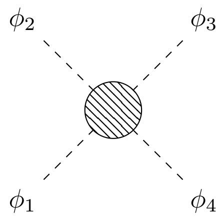
So that the full 4–point amplitude is,

(1.14)

 $$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\big[(\langle 12\rangle[12] + 2s)(\langle 34\rangle[34] + 2s)tu\big(M^2-t\big)\big(M^2-u\big) + (\langle 23\rangle[23] + 2t)(\langle 41\rangle[41] + 2t$$

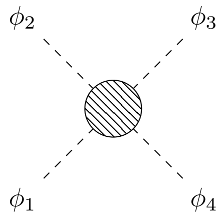
Let us specialize when $1, 2$ are massless and $3, 4$ are massive, then,

(1.15)



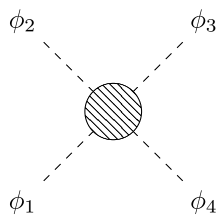
$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\Big[s\big(-s+2M^2+2s\big)tu\big(M^2-t\big)\big(M^2-u\big)+\big(-t+M^2+2t\big)\big(-t+M^2+2t\big)su\big(M^2$$

(1.16)



$$= -\mathrm{i}\frac{(\kappa M)^2}{stu(M^2-s)(M^2-t)(M^2-u)}\Big[stu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+su\big(M^2+t\big)\big(M^2+t\big)\big(M^2-s\big)\big(M^2-u\big)+st$$

(1.17)



$$= -\mathrm{i}\frac{(\kappa M)^2}{tu(M^2-s)(M^2-t)(M^2-u)}\Big[tu\big(2M^2+s\big)\big(M^2-t\big)\big(M^2-u\big)+u\big(M^2+t\big)^2\big(M^2-s\big)\big(M^2-u\big)+t\big(M^2+u\big)^2\big($$

2. Conformal Toy Model

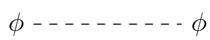
Consider the following Lagrangian,

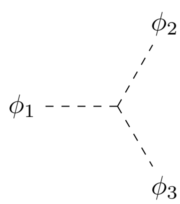
$$\mathcal{L} = -\frac{1}{2}\Box\phi\Box\phi - \frac{g}{2}\phi^2\Box\phi - \frac{g^2}{8}\phi^4 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

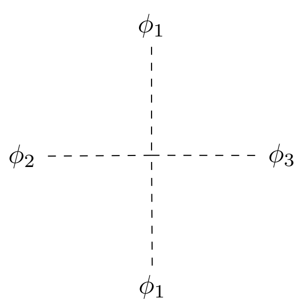
Notice the form of the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\left(\Box\phi + \frac{g}{2}\phi^2\right)^2 + m^2\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{3!}\phi^3\right)$$

It possesses the Feynman rules,

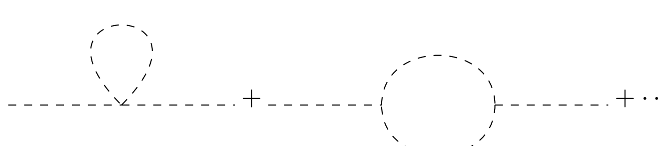
- 

$= \frac{1}{\mathrm{i}}\frac{1}{m^2p^2+p^4}$
- 

$= \mathrm{i}g\big(p_1^2+p_2^2+p_3^2+m^2\big)$
- 

$= -3\mathrm{i}g^2$

Let's compute the self energy,

(2.1) $\mathrm{i}\Pi\big(p^2\big) =$ 

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\mathrm{i}} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2} \tag{2.2}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\ell^2} \frac{1}{\ell^2 + m^2} \tag{2.3}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}g^2 \frac{\mathrm{i}}{(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)} \left(m^2\right)^{\frac{D}{2}-2} \frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(1)} \tag{2.4}$$

$$\mathrm{i}\Pi^{(1)} = -\frac{3}{2}\mathrm{i}g^2 \frac{\left(m^2\right)^{-\epsilon}\Gamma(\epsilon)\Gamma(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-\epsilon)} \tag{2.5}$$

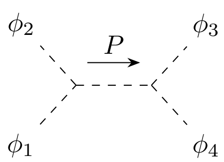
$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}(\mathrm{i}g)^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\mathrm{i}^2} \frac{1}{\ell^2(\ell+p)^2} \frac{\left(m^2+\ell^2+p^2+(\ell+p)^2\right)^2}{\ell^2+m^2} \frac{1}{(\ell+p)^2+m^2} \tag{2.6}$$

For the mass renormalization we can take $p = 0$,

$$\mathrm{i}\Pi^{(2)} = \frac{1}{2}g^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{\left(m^2+2\ell^2\right)^2}{\ell^4\left(\ell^2+m^2\right)^2} \tag{2.7}$$

Let's compute the four point amplitude for this theory,

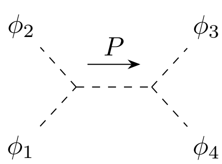
(2.8)



$$= (\mathrm{i}g)^2 \frac{\left(p_1^2+p_2^2+(p_1+p_2)^2+m^2\right)\left(p_3^2+p_4^2+(p_3+p_4)^2+m^2\right)}{\mathrm{i}(p_1+p_2)^2\left((p_1+p_2)^2+m^2\right)}$$

First let's consider all legs massless,

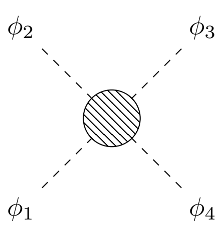
(2.9)



$$= \mathrm{i}g^2 \frac{\left(-s+m^2\right)\left(-s+m^2\right)}{(-s)(-s+m^2)} = -\mathrm{i}g^2 \frac{\left(-s+m^2\right)}{s}$$

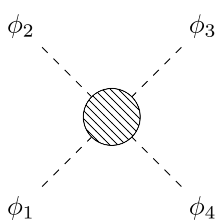
So,

(2.10)



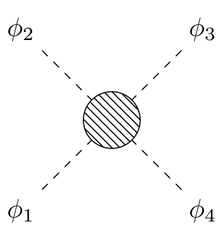
$$= -\mathrm{i}g^2 \frac{\left(-s+m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t+m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u+m^2\right)}{u} - 3\mathrm{i}g^2$$

(2.11)



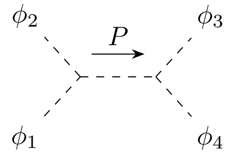
$$= -\mathrm{i}g^2 \frac{\left(-s+m^2\right)}{s} - \mathrm{i}g^2 \frac{\left(-t+m^2\right)}{t} - \mathrm{i}g^2 \frac{\left(-u+m^2\right)}{u} - \mathrm{i}g^2\frac{s}{s} - \mathrm{i}g^2\frac{t}{t} - \mathrm{i}g^2\frac{u}{u}$$

(2.12)



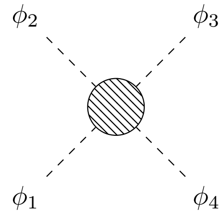
$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right) = \mathrm{i}g^2 m^2 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 14\rangle[14]} + \frac{1}{\langle 13\rangle[13]}\right)$$

Para uma perna massiva, ϕ_4,



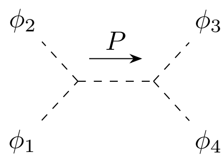
$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)(-s)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2 \tag{2.13}$$

So,

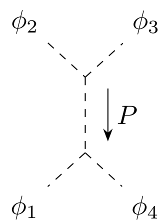


$$= \mathrm{i}g^2 + \mathrm{i}g^2 + \mathrm{i}g^2 - 3\mathrm{i}g^2 = 0 \tag{2.14}$$

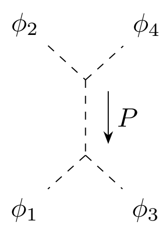
Para duas pernas massivas, $\phi_{3,4}$,



$$= (\mathrm{i}g)^2 \frac{\left(-s+m^2\right)\left(-s-m^2\right)}{\mathrm{i}(-s)(-s+m^2)} = \mathrm{i}g^2 \frac{s+m^2}{s} \tag{2.15}$$

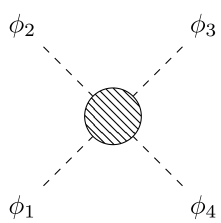


$$= (\mathrm{i}g)^2 \frac{(-t)(-t)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t}{-t+m^2} \tag{2.16}$$

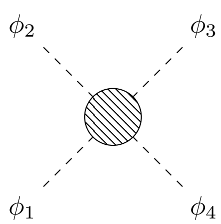


$$= -\mathrm{i}g^2 \frac{u}{-u+m^2} \tag{2.17}$$

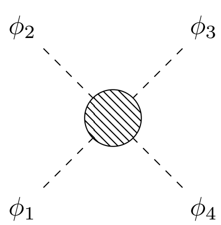
So,



$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - 3\mathrm{i}g^2 \tag{2.18}$$

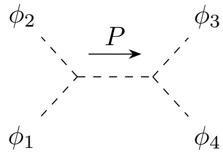


$$= \mathrm{i}g^2 \frac{s+m^2}{s} - \mathrm{i}g^2 \frac{t}{-t+m^2} - \mathrm{i}g^2 \frac{u}{-u+m^2} - \mathrm{i}g^2 \frac{s}{s} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2} \tag{2.19}$$

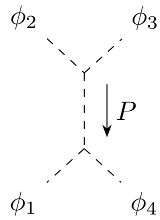


$$= -\mathrm{i}g^2 m^2 \left(-\frac{1}{s} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2}\right) = -\mathrm{i}g^2 m^2 \left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1\mathbf{4}\rangle[1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3}\rangle[1\mathbf{3}]}\right) \tag{2.20}$$

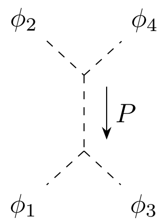
Para uma perna sem massa ϕ_1,

(2.21) 

$$= (\mathrm{i}g)^2 \frac{(-s)\big(-s-m^2\big)}{\mathrm{i}(-s)(-s+m^2)} = -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2}$$

(2.22) 

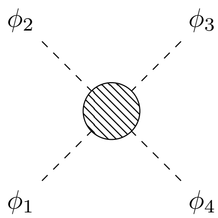
$$= (\mathrm{i}g)^2 \frac{(-t)\big(-t-m^2\big)}{\mathrm{i}(-t)(-t+m^2)} = -\mathrm{i}g^2 \frac{t+m^2}{-t+m^2}$$

(2.23) 

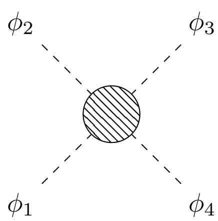
$$= (\mathrm{i}g)^2 \frac{(-u)\big(-u-m^2\big)}{\mathrm{i}(-u)(-u+m^2)} = -\mathrm{i}g^2 \frac{u+m^2}{-u+m^2}$$

(2.24)

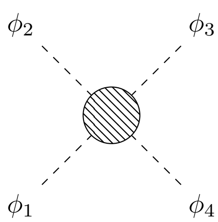
So,

(2.25) 

$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - 3\mathrm{i}g^2$$

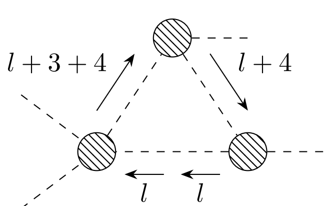
(2.26) 

$$= -\mathrm{i}g^2 \frac{s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{u+m^2}{-u+m^2} - \mathrm{i}g^2 \frac{-s+m^2}{-s+m^2} - \mathrm{i}g^2 \frac{-t+m^2}{-t+m^2} - \mathrm{i}g^2 \frac{-u+m^2}{-u+m^2}$$

(2.27) 

$$= -\mathrm{i}g^2 m^2 \left(\frac{1}{-s+m^2} + \frac{1}{-t+m^2} + \frac{1}{-u+m^2} \right) = -\mathrm{i}g^2 m^2 \left(\frac{1}{\langle 1\mathbf{2}\rangle[1\mathbf{2}]} + \frac{1}{\langle 1\mathbf{4}\rangle[1\mathbf{4}]} + \frac{1}{\langle 1\mathbf{3}\rangle[1\mathbf{3}]} \right)$$

Cut comparison, only massless legs

(2.28) 

$$= \frac{\mathrm{i}g^2 m^2 \big(\mathrm{i}g m^2\big)^2}{(\mathrm{i}m^2)^3} \left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]} \right)$$

to solve for the cuts, $l^2 = (l+3+4)^2 = (l+4)^2 = 0$,

(2.29) $$l^2 = 0 \Rightarrow l = -|l\rangle[l|$$

(2.30) $$0 = (l+4)^2 = \langle l4\rangle[l4] = 0 \Rightarrow |l] = |4]$$

(2.31) $$0 = (l+3+4)^2 = \langle lP_{34}\rangle[lP_{34}] + (3+4)^2 = \langle l|3+4|l] + \langle 34\rangle[34] = \langle l|3+4|4] + \langle 34\rangle[34]$$

(2.32) $$\langle 43\rangle[34] = -\langle l3\rangle[34] \Rightarrow |l\rangle = -|4\rangle + z|3\rangle$$

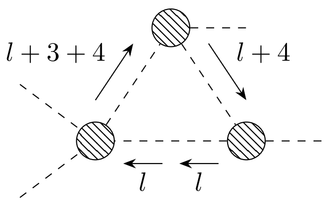
$$l = -(-|4\rangle + z|3\rangle)[4| \tag{2.33}$$

The cuts are solved by this. Hence,

$$
\begin{aligned}
&= g^4\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]}\right)\\
&= g^4\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{(-\langle 14\rangle + z\langle 13\rangle)[14]} - \frac{1}{(-\langle 24\rangle + z\langle 23\rangle)[24]}\right)
\end{aligned}
$$

Now for internal massive lines,

(2.34)



$$= \frac{-\mathrm{i}g^2m^2\left(-\mathrm{i}gm^2\right)^2}{\left(-\mathrm{i}m^2\right)^3}\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1l\rangle[1l]} - \frac{1}{\langle 2l\rangle[2l]}\right)$$

With the cuts being, $l^2 = (l+3+4)^2 = (l+4)^2 = -m^2$,

$$0 = (l+4)^2 - l^2 = 2l\cdot p_4 \tag{2.35}$$

$$0 = (l+4+3)^2 - l^2 = 2l\cdot(4+3) + (4+3)^2 = 2l\cdot p_3 + (4+3)^2 \tag{2.36}$$

As ansatz, $l = |4\rangle[4| + \alpha|4\rangle[3| + \beta|3\rangle[4|$ satisfy both conditions above. The remaining condition is,

$$l^2 = -m^2 \tag{2.37}$$

$$-\alpha\beta[43]\langle 43\rangle = -m^2 \Rightarrow \alpha = \frac{m^2}{\beta\langle 34\rangle[34]} \tag{2.38}$$

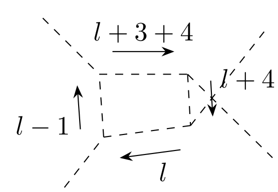
Setting now $-\beta = z$,

$$l = |4\rangle[4| - \frac{m^2}{z\langle 34\rangle[34]}|4\rangle[3| - z|3\rangle[4| \tag{2.39}$$

The value of the diagram is,

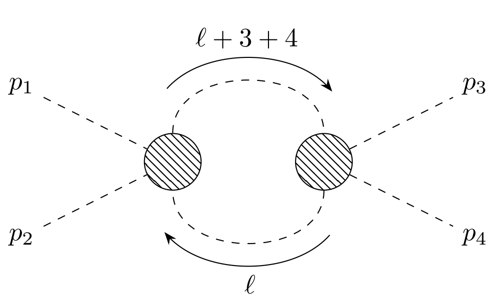
$$
\begin{aligned}
&= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{\langle 1l\rangle[l1]} + \frac{1}{\langle 2l\rangle[l2]}\right)\\
&= g^4\left(\frac{1}{\langle 12\rangle[12]} + \frac{1}{-\langle 1|l|1]} + \frac{1}{-\langle 2|l|2]}\right)\\
&= g^4\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 14\rangle[41] - \frac{m^2}{z\langle 34\rangle[34]}\langle 14\rangle[31] - z\langle 13\rangle[41]} - \frac{1}{\langle 2|l|2]}\right)
\end{aligned}
$$

The explicit cut loop amplitude is,



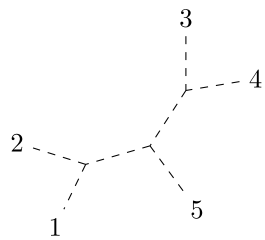
Triple cut has no improvement, what about a double cut,

(2.40)



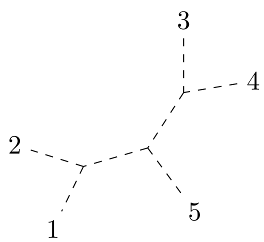
$$= \frac{\left(\mathrm{i}g^2m^2\right)^2}{\left(\mathrm{i}m^2\right)^2}\left(\frac{1}{\langle 12\rangle[12]} - \frac{1}{\langle 1\ell\rangle[1\ell]} - \frac{1}{\langle 2\ell\rangle[2\ell]}\right)\left(\frac{1}{\langle 34\rangle[34]} + \frac{1}{\langle 3\ell\rangle[3\ell]} + \frac{1}{\langle 4\ell\rangle[4\ell]}\right)$$

Five point amplitude,



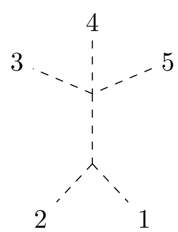
$$= \frac{(\mathrm{i}g)^3\Big(m^2+p_1^2+p_2^2+(p_1+p_2)^2\Big)\Big(m^2+p_5^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)\Big(m^2+p_3^2+p_4^2+(p_3+p_4)^2\Big)}{i^2(p_1+p_2)^2\Big((p_1+p_2)^2+m^2\Big)(p_3+p_4)^2\Big((p_3+p_4)^2+m^2\Big)}$$

Let's consider the special case of all massless,



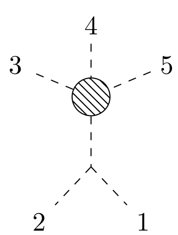
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)\Big(m^2+(p_3+p_4)^2\Big)}{(p_3+p_4)^2\Big((p_3+p_4)^2+m^2\Big)} = \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)}{(p_3+p_4)^2}$$

Combining this graph with,



$$= \frac{-\mathrm{i}3g^2\mathrm{i}g\Big(m^2+(p_1+p_2)^2\Big)}{\mathrm{i}(p_1+p_2)^2\Big((p_1+p_2)^2+m^2\Big)} = \frac{-3\mathrm{i}g^3}{(p_1+p_2)^2}$$

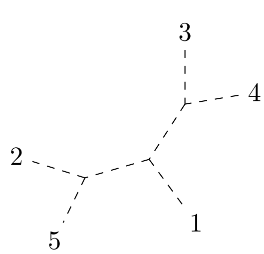
We get,



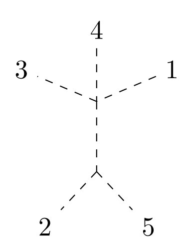
$$= \frac{\mathrm{i}g^3}{(p_1+p_2)^2}\left[\frac{\Big(m^2+(p_3+p_4)^2+(p_1+p_2)^2\Big)}{(p_3+p_4)^2} - \frac{(p_3+p_4)^2}{(p_3+p_4)^2} + \frac{\Big(m^2+(p_3+p_5)^2+(p_1+p_2)^2\Big)}{(p_3+p_5)^2} - \frac{(p_3+p_5)^2}{(p_3+p_5)^2} + \frac{\Big(m^2+(p_5}{}\right.$$

$$= \frac{\mathrm{i}g^3\Big(m^2+(p_1+p_2)^2\Big)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,



$+$



$+$ perm

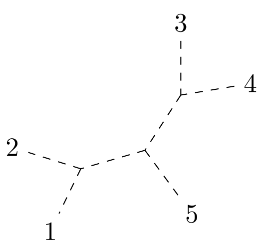
Which will be,

$$\frac{\mathrm{i}g^3}{(p_2+p_3)^2}\frac{m^2+(p_2+p_3)^2+(p_4+p_5)^2}{(p_4+p_5)^2}-\frac{3\mathrm{i}g^3}{(p_2+p_3)^2}-\frac{3\mathrm{i}g^3}{(p_4+p_5)^2}+(3\leftrightarrow 4,5)$$

Summing all the contributions we have,

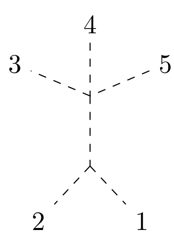
$$\begin{aligned}
=&\frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]+(2\leftrightarrow 3,4,5)\\
&+\mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_2+p_4)^2}+\frac{1}{(p_2+p_5)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_2)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_2)^2}+\frac{}{(p}\right.\\
&+\frac{\mathrm{i}g^3m^2}{(p_2+p_3)^2(p_4+p_5)^2}-\frac{2\mathrm{i}g^3}{(p_2+p_3)^2}-\frac{2\mathrm{i}g^3}{(p_4+p_5)^2}+(3\leftrightarrow 4,5)\\
=&\frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_3+p_5)^2}+\frac{1}{(p_5+p_4)^2}\right]+(2\leftrightarrow 3,4,5)\\
&+\frac{\mathrm{i}g^3m^2}{(p_2+p_3)^2(p_4+p_5)^2}+\frac{\mathrm{i}g^3m^2}{(p_2+p_4)^2(p_3+p_5)^2}+\frac{\mathrm{i}g^3m^2}{(p_2+p_5)^2(p_4+p_3)^2}
\end{aligned}$$

By residue, any amplitude with just one massive external on-shell leg is zero. For two massive external on-shell legs, let's take as massive $1,2$,



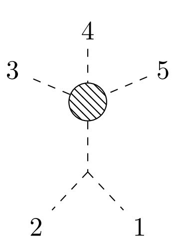
$$=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}$$

Combining this graph with,



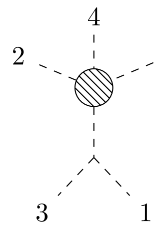
$$=-3\mathrm{i}g^3\frac{(p_1+p_2)^2-m^2}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}$$

so,



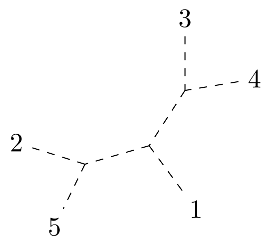
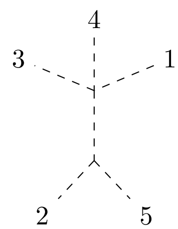
$$\begin{aligned}
&=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2\left(m^2+(p_1+p_2)^2\right)}\left[\frac{\left(m^2+(p_1+p_2)^2+(p_3+p_4)^2\right)}{(p_3+p_4)^2}-\frac{(p_3+p_4)^2}{(p_3+p_4)^2}+(5\leftrightarrow 3,4)\right]\\
&=\mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]
\end{aligned}$$

The other contributions are,



$$= \frac{\mathrm{i}g^3(p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{(p_4+p_5)^2}\right]$$

Now we have to sum the contributions of 1 being in the middle,

 +  + perm

which are,

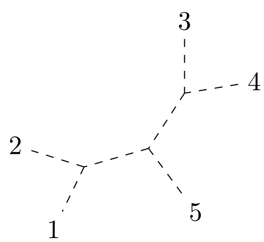
$$= \mathrm{i}g^3\frac{(p_2+p_3)^2+(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3\leftrightarrow 4,5)$$

So, summing all the contributions,

$$\begin{aligned}
&= \mathrm{i}g^3\frac{\left((p_1+p_2)^2-m^2\right)}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]\\
&\quad + \frac{\mathrm{i}g^3(p_1+p_3)^2}{\left(m^2+(p_1+p_3)^2\right)}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)\\
&\quad + \mathrm{i}g^3\frac{(p_2+p_3)^2+(p_4+p_5)^2}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3\leftrightarrow 4,5)\\
&= -\mathrm{i}g^3\frac{m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]\\
&\quad + \mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]\\
&\quad + \mathrm{i}g^3\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)\\
&\quad - \mathrm{i}g^3\frac{m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)\\
&\quad + \mathrm{i}g^3\frac{-m^2+2p_2\cdot p_3+(p_4+p_5)^2}{(2p_2\cdot p_3)(p_4+p_5)^2} - \frac{3\mathrm{i}g^3}{2p_2\cdot p_3} - \frac{3\mathrm{i}g^3}{(p_4+p_5)^2} + (3\leftrightarrow 4,5)\\
&= -\frac{\mathrm{i}g^3m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]\\
&\quad + \mathrm{i}g^3\left[\frac{1}{(p_3+p_4)^2}+\frac{1}{(p_5+p_4)^2}+\frac{1}{(p_3+p_5)^2}\right]\\
&\quad + \mathrm{i}g^3\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}+\frac{1}{2p_2\cdot p_3}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_3+p_5)^2}+\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_3}+\frac{1}{(p_4+p_3)^2}\right]\\
&\quad - \frac{\mathrm{i}g^3m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4}+\frac{1}{2p_2\cdot p_5}+\frac{1}{(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)
\end{aligned}$$

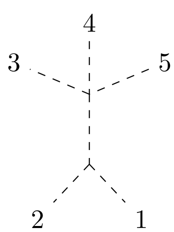
$$
\begin{aligned}
&- \frac{ig^3 m^2}{(2p_2\cdot p_3)(p_4+p_5)^2} + (3 \leftrightarrow 4,5) \\
&- \frac{2ig^3}{2p_2\cdot p_3} - \frac{2ig^3}{(p_4+p_5)^2} - \frac{2ig^3}{2p_2\cdot p_4} - \frac{2ig^3}{(p_3+p_5)^2} - \frac{2ig^3}{2p_2\cdot p_5} - \frac{2ig^3}{(p_4+p_3)^2} \\
= &-\frac{ig^3 m^2}{(p_1+p_2)^2}\left[\frac{1}{(p_3+p_4)^2} + \frac{1}{(p_5+p_4)^2} + \frac{1}{(p_3+p_5)^2}\right] \\
&- \frac{ig^3 m^2}{(2p_1\cdot p_3)}\left[\frac{1}{2p_2\cdot p_4} + \frac{1}{2p_2\cdot p_5} + \frac{1}{(p_4+p_5)^2}\right] + (3 \leftrightarrow 4,5) \\
&- \frac{ig^3 m^2}{(2p_2\cdot p_3)(p_4+p_5)^2} + (3 \leftrightarrow 4,5)
\end{aligned}
$$

Is almost the same of the all massless, but with a different denominator in the $1,2$ channel. Now with three massive legs, being $3,4,5$,



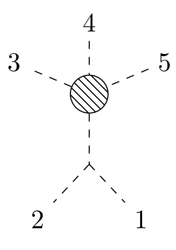
$$
= ig^3 \frac{\left((p_1+p_2)^2 + (p_3+p_4)^2\right)\left(-m^2 + (p_3+p_4)^2\right)}{(p_1+p_2)^2 (p_3+p_4)^2 \left(m^2 + (p_3+p_4)^2\right)}
$$

With,



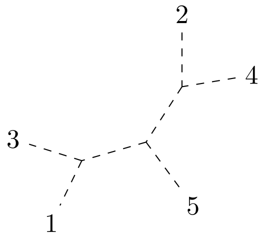
$$
= \frac{-i3g^2 ig\left(m^2 + (p_1+p_2)^2\right)}{i(p_1+p_2)^2\left((p_1+p_2)^2 + m^2\right)} = \frac{-3ig^3}{(p_1+p_2)^2}
$$

So,



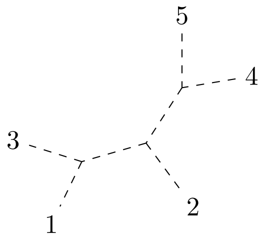
$$
\begin{aligned}
&= ig^3 \frac{\left((p_1+p_2)^2 + (p_3+p_4)^2\right)\left(-m^2 + (p_3+p_4)^2\right)}{(p_1+p_2)^2 (p_3+p_4)^2 \left(m^2 + (p_3+p_4)^2\right)} - \frac{3ig^3}{(p_1+p_2)^2} \\
&= \frac{ig^3}{(p_1+p_2)^2}\left[\frac{\left((p_1+p_2)^2 + (p_3+p_4)^2\right)\left(-m^2 + (p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2 + (p_3+p_4)^2\right)} - \frac{(p_3+p_4)^2\left(m^2 + (p_3+p_4)^2\right)}{(p_3+p_4)^2\left(m^2 + (p_3+p_4)^2\right)}\right] + (5 \leftrightarrow 3,4) \\
&= \frac{ig^3}{(p_1+p_2)^2}\left[\frac{(p_1+p_2)^2\left(-m^2 + (p_3+p_4)^2\right) - 2m^2 (p_3+p_4)^2}{(p_3+p_4)^2\left(m^2 + (p_3+p_4)^2\right)}\right] + (5 \leftrightarrow 3,4)
\end{aligned}
$$

The other topology is,



$$= \mathrm{i}g^3 \frac{\Big(m^2 + p_3^2 + (p_1+p_3)^2\Big)\Big(m^2 + p_5^2 + (p_1+p_3)^2 + (p_2+p_4)^2\Big)\Big(m^2 + p_4^2 + (p_2+p_4)^2\Big)}{(p_1+p_3)^2\Big(m^2+(p_1+p_3)^2\Big)(p_2+p_4)^2\Big(m^2+(p_2+p_4)^2\Big)}$$

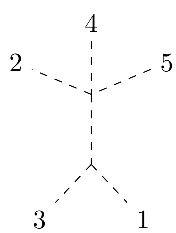
$$= \mathrm{i}g^3 \frac{\Big((p_1+p_3)^2 + (p_2+p_4)^2\Big)}{\Big(m^2+(p_1+p_3)^2\Big)\Big(m^2+(p_2+p_4)^2\Big)}$$



$$= \mathrm{i}g^3 \frac{\Big(m^2 + p_3^2 + (p_1+p_3)^2\Big)\Big(m^2 + (p_1+p_3)^2 + (p_5+p_4)^2\Big)\Big(m^2 + p_5^2 + p_4^2 + (p_5+p_4)^2\Big)}{(p_1+p_3)^2\Big(m^2+(p_1+p_3)^2\Big)(p_5+p_4)^2\Big(m^2+(p_5+p_4)^2\Big)}$$

$$= \mathrm{i}g^3 \frac{\Big(m^2 + (p_1+p_3)^2 + (p_5+p_4)^2\Big)\Big(-m^2 + (p_5+p_4)^2\Big)}{\Big(m^2+(p_1+p_3)^2\Big)(p_5+p_4)^2\Big(m^2+(p_5+p_4)^2\Big)}$$

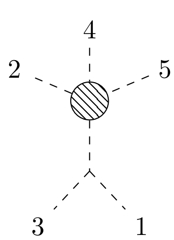
Also,



$$= -3\mathrm{i}g^2 g \frac{\Big(m^2 + p_3^2 + (p_3+p_1)^2\Big)}{(p_3+p_1)^2\Big(m^2+(p_3+p_1)^2\Big)}$$

$$= -3\mathrm{i}g^3 \frac{1}{\Big(m^2+(p_3+p_1)^2\Big)}$$

So,



$$= \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{\Big((p_1+p_3)^2+(p_2+p_4)^2\Big) - m^2 - (p_2+p_4)^2}{\Big(m^2+(p_2+p_4)^2\Big)} + \frac{\Big((p_1+p_3)^2+(p_2+p_5)^2\Big) - m^2 - (p_2+p_5)^2}{\Big(m^2+(p_2+p_5)^2\Big)}\right]$$

$$+ \frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{\Big(m^2+(p_1+p_3)^2+(p_4+p_5)^2\Big)\Big(-m^2+(p_4+p_5)^2\Big) - (p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}{(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}\right]$$

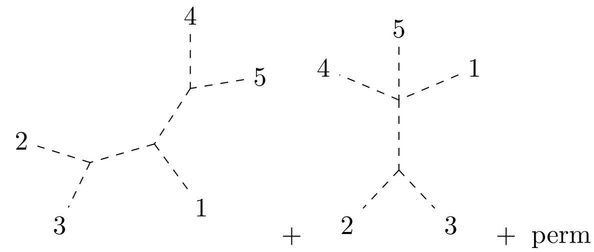
$$= \frac{\mathrm{i}g^3\Big((p_1+p_3)^2 - m^2\Big)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2} + \frac{1}{m^2+(p_2+p_5)^2}\right]$$

$$+\frac{\mathrm{i}g^3}{m^2+(p_1+p_3)^2}\left[\frac{-m^4-m^2(p_1+p_3)^2+(p_1+p_3)^2(p_4+p_5)^2-m^2(p_4+p_5)^2}{(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}\right]$$

$$=\frac{\mathrm{i}g^3\Big((p_1+p_3)^2-m^2\Big)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$-\frac{\mathrm{i}g^3m^2}{(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

Now, with 1 in the middle,



Which is,

$$=\frac{\mathrm{i}g^3\Big(m^2+p_3^2+(p_2+p_3)^2\Big)\Big(m^2+p_1^2+(p_2+p_3)^2+(p_5+p_4)^2\Big)\Big(m^2+p_5^2+p_4^2+(p_5+p_4)^2\Big)}{(p_2+p_3)^2\Big(m^2+(p_2+p_3)^2\Big)(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}-\frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2}-\frac{3\mathrm{i}g^3\Big(-m^2+(p_4+}{(p_4+p_5)^2\Big(m^2+(p_4}$$

$$=\frac{\mathrm{i}g^3\Big(m^2+(p_2+p_3)^2+(p_5+p_4)^2\Big)\Big(-m^2+(p_5+p_4)^2\Big)}{\Big(m^2+(p_2+p_3)^2\Big)(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}-\frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2}-\frac{3\mathrm{i}g^3\Big(-m^2+(p_4+p_5)^2\Big)}{(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

Summing all the contributions we have,

$$\mathrm{i}g^3\frac{\Big(-m^2+(p_5+p_4)^2\Big)}{(p_5+p_4)^2\Big(m^2+(p_5+p_4)^2\Big)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\Big(m^2+(p_5+p_4)^2\Big)}+(3\leftrightarrow 4,5)$$

$$+\frac{\mathrm{i}g^3\Big((p_1+p_3)^2-m^2\Big)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$-\frac{\mathrm{i}g^3m^2}{(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

$$+\frac{\mathrm{i}g^3\Big(m^2+(p_2+p_3)^2+(p_5+p_4)^2\Big)\Big(-m^2+(p_5+p_4)^2\Big)}{\Big(m^2+(p_2+p_3)^2\Big)(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

$$-\frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2}-\frac{3\mathrm{i}g^3\Big(-m^2+(p_4+p_5)^2\Big)}{(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\Big(m^2+(p_5+p_4)^2\Big)}+(3\leftrightarrow 4,5)$$

$$+\frac{\mathrm{i}g^3\Big((p_1+p_3)^2-m^2\Big)}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$-\frac{\mathrm{i}g^3m^2}{(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

$$
\begin{aligned}
&\quad + \frac{\mathrm{i}g^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} + \frac{2\mathrm{i}g^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{2\mathrm{i}g^3}{\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&- \frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad - \frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)\\
&\quad + \mathrm{i}g^3\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]\\
&\quad + \mathrm{i}g^3\left[\frac{1}{m^2+(p_2+p_3)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_3+p_5)^2}\right]\\
&\quad + \mathrm{i}g^3\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_3)^2}+\frac{1}{m^2+(p_4+p_3)^2}\right]\\
&\quad + \frac{\mathrm{i}g^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad - \frac{3\mathrm{i}g^3}{m^2+(p_2+p_3)^2} + \frac{\mathrm{i}g^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{2\mathrm{i}g^3}{\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&- \frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad - \frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)\\
&\quad + \frac{\mathrm{i}g^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad - \frac{\mathrm{i}g^3}{m^2+(p_2+p_3)^2} + \frac{\mathrm{i}g^3m^2}{(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} - \frac{\mathrm{i}g^3}{\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&- \frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad - \frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)\\
&\quad + \frac{\mathrm{i}g^3\left(m^2+(p_2+p_3)^2+(p_5+p_4)^2\right)\left(-m^2+(p_5+p_4)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad + \mathrm{i}g^3\frac{-(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)+m^2\left(m^2+(p_2+p_3)^2\right)-(p_4+p_5)^2\left(m^2+(p_2+p_3)^2\right)}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)\\
&- \frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\left(m^2+(p_5+p_4)^2\right)} + (3\leftrightarrow 4,5)\\
&\quad - \frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right] + (3\leftrightarrow 4,5)\\
&\quad + \mathrm{i}g^3\frac{-m^4+m^2(p_4+p_5)^2-m^2(p_2+p_3)^2+(p_2+p_3)^2(p_4+p_5)^2-m^2(p_5+p_4)^2+(p_4+p_5)^4}{\left(m^2+(p_2+p_3)^2\right)(p_4+p_5)^2\left(m^2+(p_4+p_5)^2\right)} + (3\leftrightarrow 4,5)
\end{aligned}
$$

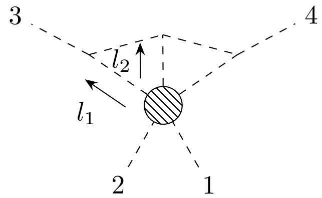
$$+\,\mathrm{i}g^3\frac{-(p_4+p_5)^2m^2-(p_4+p_5)^4+m^4+m^2(p_2+p_3)^2-(p_4+p_5)^2m^2-(p_2+p_3)^2(p_4+p_5)^2}{\Big(m^2+(p_2+p_3)^2\Big)(p_4+p_5)^2\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{(p_1+p_2)^2}\frac{1}{\Big(m^2+(p_5+p_4)^2\Big)}+(3\leftrightarrow 4,5)$$

$$-\frac{2\mathrm{i}g^3m^2}{m^2+(p_1+p_3)^2}\left[\frac{1}{m^2+(p_2+p_4)^2}+\frac{1}{m^2+(p_2+p_5)^2}+\frac{1}{m^2+(p_4+p_5)^2}\right]+(3\leftrightarrow 4,5)$$

$$-\,2\mathrm{i}g^3m^2\frac{1}{\Big(m^2+(p_2+p_3)^2\Big)\Big(m^2+(p_4+p_5)^2\Big)}+(3\leftrightarrow 4,5)$$

Now, let's do the cuts, consider a two loop four point amplitude with five cuts,



$$=-\frac{g^3}{m^4}[\mathcal{A}_5(1,2,l_{1h},l_{2h},l_{3h})-\mathcal{A}_5(1,2,l_{1h},l_{2\eta},l_{3\eta})-\mathcal{A}_5(1,2,l_{1\eta},l_{2h},l_{3\eta})-\mathcal{A}_5(1,2,l_{1\eta},l_{2\eta},l_{3h})+2\mathcal{A}_5(1,2,l_{1\eta}$$

3. Cut solutions

Of course in each amplitude we have different cut solutions. Now let us solve them,

3.1. **all massless.**

The cut condition is,

$$k_1^2=k_2^2=(3-k_1)^2=(3-k_1-k_2)^2=(3+4-k_1-k_2)^2=0$$

The first and third condition enforces $k_1=-|k_1]\langle 3|$. But the fourth and fifth conditions enforces $3-k_1-k_2=n$, with $n\cdot 4=0$ & $n^2=0$. Lastly, the second condition imposes $(3-k_1-n)^2=-23\cdot n+2k_1\cdot n=0$, that is,

$$[3n]\langle n3\rangle=[k_1n]\langle n3\rangle$$

which has two solutions, $|n]=|k_1]-|3]$ & $|n\rangle=z|4\rangle$ or $|n\rangle=|3\rangle$ & $|n]=z|4]$. When working with scalar particles it's better to choose the first solution, as this avoids singularities in denominators such as $(k_1\cdot k_2)^{-1}$. Hence, the solution we're going to choose is,

$$\begin{cases} k_1 &= -|k_1]\langle 3| \\ k_2 &= -|3]\langle 3|+|k_1]\langle 3|+z(|k_1]-|3])\langle 4| \end{cases}$$

3.2. **massive legs first topology.**

Our approach to massive legs is to shift the solution with massless, in order to obtain a well behaved solution in the $m^2\to 0$ limit. For this topology the cut constrains are,

$$l_1^2=l_2^2=(3-l_1)^2=-m^2\ \&\ (3-l_1-l_2)^2=(3+4-l_1-l_2)^2=0$$

The ideia here is to define, $l_i=k_i+\alpha_i q_i$ (no sum), with $q_i^2=0$ and $\alpha_i=-m^2(2k_i\cdot q_i)^{-1}$, then, q_i,k_i are not allowed to have any dependence on m^2. The first and second constrains are already satisfied. The third one gives,

$$-23\cdot l_1=0\to 3\cdot(k_1+\alpha_1q_1)=0\to 3\cdot q_1=0$$

As $|q_1\rangle=|3\rangle$ is forbidden, $|q_1]=|3]$. The fourth and fifth constrains imposes,

$$\begin{cases} -n\cdot(\alpha_1q_1+\alpha_2q_2)+\alpha_1\alpha_2q_1\cdot q_2 &= 0 \\ 4\cdot(\alpha_1q_1+\alpha_2q_2) &= 0 \end{cases}$$

This imposes actually $q_1\cdot q_2=0$, for this to be true we have to options, either $|q_2]=|3]$, or $|q_2\rangle=|q_1\rangle$. If we choose the first, we can shift k_1 by 3 such to make $|q_1\rangle=|4\rangle$, this imposes further $|q_2\rangle=|4\rangle$. Hence, a possible solution is,

$$q_1=q_2=-|3]\langle 4|$$

3.3. **massive legs second topology.**

The constrains now are slightly different,

$$l_1^2=(3-l_1)^2=0\ \&\ l_2^2=(3-l_1-l_2)^2=(3+4-l_1-l_2)^2=-m^2$$

Which has as solution $q_2=-|4]\langle 3|$

3.4. **massive legs third topology.** Now the constrain is difficult to solve,

$$l_2^2 = 0 \ \& \ l_1^2 = (3-l_1)^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

The second and third constrains give, $l_1 = -|k_1]\langle 3| - \alpha|3]\langle l_1|$. Now, the fourth and fifth constrains gives,

$$3 - l_1 - l_2 = -z(|k_1] - |3])\langle 4| + \beta|4]\langle n|$$

With of course $\beta = -\frac{m^2}{z\langle 4n\rangle[4|(|k_1]-|3])}$. At last the second constrain gives,

$$\begin{aligned}
l_2 &= -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1]-|3])\langle 4| - \beta|4]\langle n| \\
l_2^2 = 0 &= -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1]-|3]) \\
0 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\
-2m^2 &= \beta\langle 3n\rangle[4|(|3]-|k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\
-2m^2 &= -\frac{m^2}{z\langle 4n\rangle[4|(|k_1]-|3])}\langle 3n\rangle[4|(|3]-|k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]}z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43]
\end{aligned}$$

For the quadratic term in m^2 to vanish is necessary $\langle l_1 n\rangle = 0 \to |l_1\rangle \propto |n\rangle$, thus,

$$\begin{aligned}
-2m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle}z\langle 4l_1\rangle \\
-2 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle}z\langle 4n\rangle \to \langle 3n\rangle = -z\langle 4n\rangle
\end{aligned}$$

The best parametrization is $|n\rangle = |l_1\rangle = |4\rangle - \frac{1}{z}|3\rangle$.

3.5. **massive legs fourth topology.** Now the constrain is the hardest to solve,

$$l_1^2 = l_2^2 = (3-l_1)^2 = (3-l_1-l_2)^2 = (3+4-l_1-l_2)^2 = -m^2$$

Happily, most of the work was done already in the last solution, we just have to change:

$$\begin{aligned}
l_2 &= -|3]\langle 3| + |k_1]\langle 3| + \alpha|3]\langle l_1| + z(|k_1]-|3])\langle 4| - \beta|4]\langle n| \\
l_2^2 = -m^2 \to m^2 &= -z\langle 34\rangle[k_1 3] + \beta\langle 3n\rangle[43] + \alpha\langle 3l_1\rangle[3k_1] - z\langle 34\rangle[3k_1] - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] - \beta z\langle 4n\rangle[4|(|k_1]-|3]) \\
m^2 &= \beta\langle 3n\rangle[43] + m^2 - \beta\langle 3n\rangle[4k_1] + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] + m^2 \\
-m^2 &= \beta\langle 3n\rangle[4|(|3]-|k_1]) + \alpha z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43] \\
-m^2 &= -\frac{m^2}{z\langle 4n\rangle[4|(|k_1]-|3])}\langle 3n\rangle[4|(|3]-|k_1]) + \frac{m^2}{\langle 3l_1\rangle[3k_1]}z\langle l_1 4\rangle[k_1 3] - \alpha\beta\langle l_1 n\rangle[43]
\end{aligned}$$

Again we fix $|l_1\rangle \propto |n\rangle$. The solution then is given by,

$$\begin{aligned}
-m^2 &= \frac{m^2}{z\langle 4n\rangle}\langle 3n\rangle + \frac{m^2}{\langle 3l_1\rangle}z\langle 4l_1\rangle \\
-1 &= \frac{1}{z\langle 4n\rangle}\langle 3n\rangle + \frac{1}{\langle 3n\rangle}z\langle 4n\rangle \to \langle 3n\rangle = -z\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\right)\langle 4n\rangle = -z\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\langle 4n\rangle
\end{aligned}$$

4. Amplitudes evaluated in the cuts

4.1. **all massless.** The expression for the amplitude is,

$$\begin{aligned}
=&\frac{\mathrm{i}g^3m^2}{2p_1\cdot p_2}\left[\frac{1}{2k_1\cdot k_2} + \frac{1}{2k_1\cdot k_3} + \frac{1}{2k_2\cdot k_3}\right] \\
&+ \frac{\mathrm{i}g^3m^2}{2p_1\cdot k_1}\left[\frac{1}{2p_2\cdot k_2} + \frac{1}{2p_2\cdot k_3} + \frac{1}{2k_3\cdot k_2}\right] \\
&+ \frac{\mathrm{i}g^3m^2}{2p_1\cdot k_2}\left[\frac{1}{2k_1\cdot p_2} + \frac{1}{2k_1\cdot k_3} + \frac{1}{2k_3\cdot p_2}\right] \\
&+ \frac{\mathrm{i}g^3m^2}{2p_1\cdot k_3}\left[\frac{1}{2k_1\cdot k_2} + \frac{1}{2k_1\cdot p_2} + \frac{1}{2p_2\cdot k_2}\right] \\
&+ \frac{\mathrm{i}g^3m^2}{2p_2\cdot k_1 2k_2\cdot k_3} + \frac{\mathrm{i}g^3m^2}{2p_2\cdot k_2 2k_1\cdot k_3} + \frac{\mathrm{i}g^3m^2}{2p_2\cdot k_3 2k_2\cdot k_1}
\end{aligned}$$

4.2. **two massive.**

4.2.1. l_1 *and* l_2 *massive.*

$$
\begin{aligned}
&= -\frac{\mathrm{i}g^3m^2}{-2m^2+2l_1\cdot l_2}\left[\frac{1}{2k_3\cdot p_1}+\frac{1}{2p_2\cdot p_1}+\frac{1}{2k_3\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_1\cdot p_1)}\left[\frac{1}{2l_2\cdot k_3}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2k_3\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_1\cdot p_2)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2l_2\cdot k_3}+\frac{1}{2p_1\cdot k_3}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_1\cdot k_3)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2p_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{2l_2\cdot k_3 2p_1\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_1 2k_3\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_2 2p_1\cdot k_3}\\
&= -\frac{\mathrm{i}g^3m^2}{2k_1\cdot k_2}\left[\frac{1}{2k_3\cdot p_1}+\frac{1}{2p_2\cdot p_1}+\frac{1}{2k_3\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_1\cdot p_1)}\left[\frac{1}{2k_2\cdot k_3}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2k_3\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_1\cdot p_2)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2k_2\cdot k_3}+\frac{1}{2p_1\cdot k_3}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2k_1\cdot k_3)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2p_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{2k_2\cdot k_3 2p_1\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_1 2k_3\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_2 2p_1\cdot k_3}
\end{aligned}
$$

We used that $l_3 = k_3, l_2\cdot k_3 = k_2\cdot k_3, l_1\cdot k_3 = k_1\cdot k_3, 2l_1\cdot l_2 = 2k_1\cdot k_2 + 2m^2$.

4.2.2. l_2 *and* l_3 *massive.*

$$
\begin{aligned}
&= -\frac{\mathrm{i}g^3m^2}{-2m^2+2l_3\cdot l_2}\left[\frac{1}{2k_1\cdot p_1}+\frac{1}{2p_2\cdot p_1}+\frac{1}{2k_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_3\cdot p_1)}\left[\frac{1}{2l_2\cdot k_1}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2k_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_3\cdot p_2)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2l_2\cdot k_1}+\frac{1}{2p_1\cdot k_1}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_3\cdot k_1)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2p_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{2l_2\cdot k_1 2p_1\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_1 2k_1\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_2 2p_1\cdot k_1}\\
&= -\frac{\mathrm{i}g^3m^2}{2k_3\cdot k_2}\left[\frac{1}{2k_1\cdot p_1}+\frac{1}{2p_2\cdot p_1}+\frac{1}{2k_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2k_3\cdot p_1)}\left[\frac{1}{2k_2\cdot k_1}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2k_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2k_3\cdot p_2)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2k_2\cdot k_1}+\frac{1}{2p_1\cdot k_1}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2k_3\cdot k_1)}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2l_2\cdot p_2}+\frac{1}{2p_1\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{2k_2\cdot k_1 2p_1\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_1 2k_1\cdot p_2}-\frac{\mathrm{i}g^3m^2}{2l_2\cdot p_2 2p_1\cdot k_1}
\end{aligned}
$$

We used that $l_1 = k_1, l_2\cdot k_1 = k_2\cdot k_1, l_3\cdot k_1 = k_3\cdot k_1, 2l_2\cdot l_3 = 2k_2\cdot k_3 + 2m^2$.

4.2.3. l_1 *and* l_3 *massive.*

$$
\begin{aligned}
&= -\frac{\mathrm{i}g^3m^2}{-2m^2+2l_3\cdot l_1}\left[\frac{1}{2l_2\cdot p_1}+\frac{1}{2p_2\cdot p_1}+\frac{1}{2l_2\cdot p_2}\right]\\
&\quad -\frac{\mathrm{i}g^3m^2}{(2l_3\cdot p_1)}\left[\frac{1}{2l_2\cdot l_1}+\frac{1}{2l_1\cdot p_2}+\frac{1}{2l_1\cdot p_2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{ig^3m^2}{(2l_3 \cdot p_2)}\left[\frac{1}{2l_2 \cdot p_1} + \frac{1}{2l_2 \cdot l_1} + \frac{1}{2p_1 \cdot l_1}\right] \\
&- \frac{ig^3m^2}{(2l_3 \cdot l_2)}\left[\frac{1}{2l_1 \cdot p_1} + \frac{1}{2l_1 \cdot p_2} + \frac{1}{2p_1 \cdot p_2}\right] \\
&- \frac{ig^3m^2}{2l_1 \cdot l_2 2p_1 \cdot p_2} - \frac{ig^3m^2}{2l_1 \cdot p_1 2l_2 \cdot p_2} - \frac{ig^3m^2}{2l_1 \cdot p_2 2p_1 \cdot l_1}
\end{aligned}
$$

4.2.4. *l_1, l_2 and l_3 massive.*

5. DF^2 Theory

The $(DF)^2 + \mathrm{YM}$ theory is given by the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}D_\mu F^{a\mu\nu}D_\alpha F_a{}^\alpha{}_\nu + g\frac{1}{3}f_{abc}F^a{}_\mu{}^\nu F^b{}_\nu{}^\alpha F^c{}_\alpha{}^\mu - \frac{1}{2}D_\mu\phi^I D^\mu\phi_I + \frac{g}{2}C^{Iab}\phi_I F_{a\mu\nu}F_b{}^{\mu\nu} + \frac{g}{6}d^{IJK}\phi_I\phi_J\phi_K - \frac{m^2}{2}\phi_I\phi^I - \frac{m^2}{4}F_{a\mu\nu}F^{a\mu\nu}$$

Where of course,

$$
\begin{aligned}
F^a{}_{\mu\nu} &= \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + gf^a{}_{bc}A^b{}_\mu A^c{}_\nu \\
(D_\alpha F^{\mu\nu})^a &= \partial_\alpha F^a{}_{\mu\nu} + gf^a{}_{bc}A^b{}_\alpha F^c{}_{\mu\nu} \\
(D_\alpha\phi)^I &= \partial_\alpha\phi^I - \mathrm{i}g\mathrm{T}_{\mathrm{R}}^{a}{}^I{}_J A_{a\alpha}\phi^J
\end{aligned}
$$

We also have to incorporate the gauge fixing part,

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2}\partial_\mu A^{a\mu}(-\Box + m^2)\partial_\nu A_a{}^\nu$$

5.1. **quadratic piece.** Let us collect all the quadratic pieces,

$$
\begin{aligned}
-\frac{1}{2}D_\mu F^{a\mu\nu}D_\alpha F_a{}^\alpha{}_\nu &= -\frac{1}{2}\left(\partial_\mu F^{a\mu\nu} + gf^a{}_{bc}A^b{}_\mu F^{c\mu\nu}\right)\left(\partial_\alpha F_a{}^\alpha{}_\nu + gf_{ade}A^d{}_\alpha F^{e\alpha}{}_\nu\right) \\
&= -\frac{1}{2}(\partial_\mu F^{a\mu\nu})(\partial_\alpha F_a{}^\alpha{}_\nu) \\
&= -\frac{1}{2}\left(\partial_\mu\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + gf^a{}_{bc}A^{b\mu}A^{c\nu}\right)\right)\left(\partial_\alpha\left(\partial^\alpha A_{a\nu} - \partial_\nu A_a{}^\alpha + gf_{ade}A^d{}_\alpha A^e{}_\nu\right)\right) \\
&= -\frac{1}{2}(\partial_\mu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}))(\partial_\alpha(\partial^\alpha A_{a\nu} - \partial_\nu A_a{}^\alpha)) \\
&= -\frac{1}{2}(\Box A^{a\nu} - \partial^\nu\partial_\mu A^{a\mu})(\Box A_{a\nu} - \partial_\nu\partial_\alpha A_a{}^\alpha) \\
&= -\frac{1}{2}\Box A^{a\nu}\Box A_{a\nu} + \frac{1}{2}\Box A^{a\nu}\partial_\nu\partial_\alpha A_a{}^\alpha + \frac{1}{2}\partial^\nu\partial_\mu A^{a\mu}\Box A_{a\nu} - \frac{1}{2}\partial^\nu\partial_\mu A^{a\mu}\partial_\nu\partial_\alpha A_a{}^\alpha \\
&= -\frac{1}{2}\Box A^{a\nu}\Box A_{a\nu} + \Box A^{a\nu}\partial_\nu\partial_\alpha A_a{}^\alpha - \frac{1}{2}\Box A^{a\mu}\partial_\mu\partial_\alpha A_a{}^\alpha \\
&= -\frac{1}{2}\Box A^{a\nu}\Box A_{a\nu} + \frac{1}{2}\Box A^{a\nu}\partial_\nu\partial_\alpha A_a{}^\alpha \\
&= \frac{1}{2}A^{a\mu}\delta_{ab}\left(-\eta_{\mu\nu}\Box^2 + \partial_\mu\partial_\nu\Box\right)A^{b\nu}
\end{aligned}
$$

$$-\frac{1}{2}\partial_\mu A^{a\mu}(-\Box + m^2)\partial_\nu A_a{}^\nu = \frac{1}{2}A^{a\mu}\delta_{ab}(-\Box + m^2)\partial_\mu\partial_\nu A^{b\nu}$$

$$
\begin{aligned}
-\frac{m^2}{4}F_{a\mu\nu}F^{a\mu\nu} &= -\frac{m^2}{4}\left(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + gf_{abc}A^b{}_\mu A^c{}_\nu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + gf^a{}_{de}A^{d\mu}A^{e\nu}\right) \\
&= -\frac{m^2}{4}(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu})(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\
&= \frac{1}{2}A^{a\mu}\delta_{ab}\left(\eta_{\mu\nu}m^2\Box - m^2\partial_\nu\partial_\mu\right)A^{b\nu}
\end{aligned}
$$

Summing all the contributions, the quadratic piece of the Lagrangian is,

$$
\begin{aligned}
&\mathcal{L} \supset \frac{1}{2}A^{a\mu}\delta_{ab}\left(-\eta_{\mu\nu}\Box^2 + \partial_\mu\partial_\nu\Box\right)A^{b\nu} + \frac{1}{2}A^{a\mu}\delta_{ab}(-\Box + m^2)\partial_\mu\partial_\nu A^{b\nu} + \frac{1}{2}A^{a\mu}\delta_{ab}\left(\eta_{\mu\nu}m^2\Box - m^2\partial_\nu\partial_\mu\right)A^{b\nu} \\
&\mathcal{L} \supset \frac{1}{2}A^{a\mu}\delta_{ab}\eta_{\mu\nu}\left(-\Box^2 + m^2\Box\right)A^{b\nu}
\end{aligned}
$$

5.2. **cubic piece.** Now, the cubic piece,

$$
\begin{aligned}
-\frac{1}{2}D_\mu F^{a\mu\nu} D_\alpha F_a{}^\alpha{}_\nu &= -\frac{1}{2}(\Box A^{a\nu} - \partial^\nu \partial\cdot A^a) g f_{ade} \partial^\alpha \big(A^d{}_\alpha A^e{}_\nu\big) \\
&\quad - \frac{1}{2} g f^a{}_{bc} \partial_\mu \big(A^{b\mu} A^{c\nu}\big)(\Box A_{a\nu} - \partial_\nu \partial\cdot A_a) \\
&\quad - \frac{1}{2}(\Box A^{a\nu} - \partial^\nu \partial\cdot A^a) g f_{ade} A^d{}_\alpha (\partial^\alpha A^e{}_\nu - \partial_\nu A^{e\alpha}) \\
&\quad - \frac{1}{2} g f^a{}_{bc} A^b{}_\mu (\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu})(\Box A_{a\nu} - \partial_\nu \partial\cdot A_a) \\
&= -g f^a{}_{bc} \partial_\mu \big(A^{b\mu} A^{c\nu}\big)(\Box A_{a\nu} - \partial_\nu \partial\cdot A_a) \\
&\quad - g f^a{}_{bc} A^b{}_\mu (\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu})(\Box A_{a\nu} - \partial_\nu \partial\cdot A_a)
\end{aligned}
$$

$$
\begin{aligned}
g\frac{1}{3} f_{abc} F^a{}_\mu{}^\nu F^b{}_\nu{}^\alpha F_\alpha{}^\mu &= g\frac{1}{3} f_{abc}\big(\partial_\mu A^{a\nu} - \partial^\nu A^a{}_\mu\big)\big(\partial_\nu A^{b\alpha} - \partial^\alpha A^a{}_\nu\big)\big(\partial_\alpha A^{c\mu} - \partial^\mu A^c{}_\alpha\big) \\
&= g\frac{2}{3} f_{abc}\big(\partial_\mu A^{a\nu} - \partial^\nu A^a{}_\mu\big)\big(\partial_\nu A^{b\alpha} - \partial^\alpha A^a{}_\nu\big)\partial_\alpha A^{c\mu}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{m^2}{4} F_{a\mu\nu} F^{a\mu\nu} &= -\frac{m^2}{4}\big(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu}\big) g f^a{}_{bc} A^{b\mu} A^{c\nu} \\
&\quad - \frac{m^2}{4} g f_{abc} A^b{}_\mu A^c{}_\nu (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\
&= -m^2 g f_{abc} A^b{}_\mu A^c{}_\nu \partial^\mu A^{a\nu}
\end{aligned}
$$