

ADTGR SEMINAR

VICENTE V. FIGUEIRA

1. INTRODUCTION

Einstein-Hilbert action:

$$S_{\text{EH}} = \frac{1}{2\kappa} \int_M d^D x \sqrt{|g|} g^{ab} R_{cb}{}^c{}_a$$

The Vierbein/Tetrad formalism:

$$\begin{aligned}\eta_{\mu\nu} &= \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) \\ g_{ab} &= e^\mu{}_a e^\nu{}_b \eta_{\mu\nu} \\ \eta_{\mu\nu} &= e_\mu{}^a e_\nu{}^b g_{ab}\end{aligned}$$

A connection is defined with respect to a vector basis:

$$\begin{aligned}\nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= \omega(\mathbf{X})^\mu{}_\nu \mathbf{e}_\mu \\ \nabla_{\mathbf{X}}(\mathbf{e}_\nu) &= X^a \omega_a{}^\mu{}_\nu \mathbf{e}_\mu \\ \mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) &= X^a \omega_a{}^\mu{}_\nu \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\mu) \\ \mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) + \mathbf{g}(\mathbf{e}_\nu, \nabla_{\mathbf{X}}(\mathbf{e}_\alpha)) &= X^a \omega_a{}^\mu{}_\nu \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\mu) + X^a \omega_a{}^\mu{}_\alpha \mathbf{g}(\mathbf{e}_\nu, \mathbf{e}_\mu) \\ \mathbf{g}(\mathbf{e}_\alpha, \nabla_{\mathbf{X}}(\mathbf{e}_\nu)) + \mathbf{g}(\nabla_{\mathbf{X}}(\mathbf{e}_\alpha), \mathbf{e}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha} \\ \nabla_{\mathbf{X}}(\mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\nu)) - \nabla_{\mathbf{X}}(\mathbf{g})(\mathbf{e}_\alpha, \mathbf{e}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha} \\ -\nabla_{\mathbf{X}}(\mathbf{g})(\mathbf{e}_\alpha, \mathbf{e}_\nu) &= X^a \omega_{a\alpha\nu} + X^a \omega_{a\nu\alpha} \\ -\omega_{a\nu\alpha} &= \omega_{a\alpha\nu}, \quad \text{Metric compatibility}\end{aligned}\tag{1.1}$$

Riemann curvature tensor, $\mathbf{Riem}(\mathbf{X}, \mathbf{Y}) : \mathfrak{X} \rightarrow \mathfrak{X}$:

$$\begin{aligned}
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= (\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X}, \mathbf{Y}]})\mathbf{e}_\mu \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= \left(\nabla_{\mathbf{X}}(Y^b \omega_b^\nu{}_\mu \mathbf{e}_\nu) - \nabla_{\mathbf{Y}}(X^a \omega_a^\nu{}_\mu \mathbf{e}_\nu) - [\mathbf{X}, \mathbf{Y}]^b \omega_b^\nu{}_\mu \mathbf{e}_\nu \right) \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= \left(\nabla_{\mathbf{X}}(Y^b \omega_b^\nu{}_\mu \mathbf{e}_\nu) - \nabla_{\mathbf{Y}}(X^a \omega_a^\nu{}_\mu \mathbf{e}_\nu) - [\mathbf{X}, \mathbf{Y}]^b \omega_b^\nu{}_\mu \mathbf{e}_\nu \right) \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= (Y^b \nabla_{\mathbf{X}}(\omega_b^\nu{}_\mu \mathbf{e}_\nu) - X^a \nabla_{\mathbf{Y}}(\omega_a^\nu{}_\mu \mathbf{e}_\nu) \\
&\quad + \nabla_{\mathbf{X}}(Y^b) \omega_b^\nu{}_\mu \mathbf{e}_\nu - \nabla_{\mathbf{Y}}(X^b) \omega_b^\nu{}_\mu \mathbf{e}_\nu - [\mathbf{X}, \mathbf{Y}]^b \omega_b^\nu{}_\mu \mathbf{e}_\nu) \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= (X^a Y^b \nabla_a(\omega_b^\nu{}_\mu \mathbf{e}_\nu) - X^a Y^b \nabla_b(\omega_a^\nu{}_\mu \mathbf{e}_\nu) \\
&\quad + (\nabla_{\mathbf{X}}(Y^b) - \nabla_{\mathbf{Y}}(X^b)) \omega_b^\nu{}_\mu \mathbf{e}_\nu - [\mathbf{X}, \mathbf{Y}]^b \omega_b^\nu{}_\mu \mathbf{e}_\nu) \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= (X^a Y^b \nabla_a(\omega_b^\nu{}_\mu \mathbf{e}_\nu) + X^a Y^b \omega_b^\nu{}_\mu \omega_a^\alpha{}_\nu \mathbf{e}_\alpha - X^a Y^b \nabla_b(\omega_a^\nu{}_\mu \mathbf{e}_\nu) - X^a Y^b \omega_a^\nu{}_\mu \omega_b^\alpha{}_\nu \mathbf{e}_\alpha) \\
&\quad + (X^a \partial_a Y^b + X^a \Gamma_a{}^b{}_c Y^c - Y^a \partial_a X^b - Y^a \Gamma_a{}^b{}_c X^c) \omega_b^\nu{}_\mu \mathbf{e}_\nu - (X^a \partial_a Y^b - Y^a \partial_a X^b) \omega_b^\nu{}_\mu \mathbf{e}_\nu) \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= X^a Y^b (\nabla_a(\omega_b^\nu{}_\mu) + \omega_b^\alpha{}_\mu \omega_a^\nu{}_\alpha - \nabla_b(\omega_a^\nu{}_\mu) - \omega_a^\alpha{}_\mu \omega_b^\nu{}_\alpha + \Gamma_a{}^c{}_b \omega_c^\nu{}_\mu - \Gamma_b{}^c{}_a \omega_c^\nu{}_\mu) \mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= X^a Y^b (\partial_a \omega_b^\nu{}_\mu - \partial_b \omega_a^\nu{}_\mu + \omega_a^\nu{}_\alpha \omega_b^\alpha{}_\mu - \omega_b^\nu{}_\alpha \omega_a^\alpha{}_\mu) \mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= X^a Y^b (d\omega + \omega \wedge \omega)_{ab}{}^\nu{}_\mu \mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})\mathbf{e}_\mu &= X^a Y^b R_{ab}{}^\nu{}_\mu \mathbf{e}_\nu \\
\mathbf{Riem}(\mathbf{X}, \mathbf{Y})(e_\mu{}^e \partial_e) &= X^a Y^b R_{ab}{}^\nu{}_\mu e_\nu{}^c \partial_c \\
e_\mu{}^e \mathbf{Riem}(\mathbf{X}, \mathbf{Y})\partial_e &= X^a Y^b R_{ab}{}^\nu{}_\mu e_\nu{}^c \partial_c \\
e_\mu{}^e X^a Y^b R_{ab}{}^c{}_e \partial_c &= X^a Y^b R_{ab}{}^\nu{}_\mu e_\nu{}^c \partial_c \\
e^\mu{}_d e_\mu{}^e R_{ab}{}^c{}_e &= e^\mu{}_d R_{ab}{}^\nu{}_\mu e_\nu{}^c \\
R_{ab}{}^c{}_d &= e^\mu{}_d R_{ab}{}^\nu{}_\mu e_\nu{}^c
\end{aligned}$$

We would like to write the Einstein-Hilbert action as only a function of the Vierbein and the spin connection, for this, let us explicit write the Riemann tensor as a $\text{End}(\text{TM})$ -valued 2-form,

$$\begin{aligned}
\mathbf{R}^\nu{}_\mu &= \frac{1}{2} R_{ab}{}^\nu{}_\mu dx^a \wedge dx^b \\
\mathbf{R}^\nu{}_\mu &= \frac{1}{2} R_{ab}{}^\nu{}_\mu e_\alpha{}^a e^\alpha{}_c e_\beta{}^b e^\beta{}_d dx^c \wedge dx^d \\
\mathbf{R}^\nu{}_\mu &= \frac{1}{2} R_{ab}{}^\nu{}_\mu e_\alpha{}^a e_\beta{}^b \tilde{\mathbf{e}}^\alpha \wedge \tilde{\mathbf{e}}^\beta
\end{aligned}$$

Let us start by writing the volume form in terms of the Vierbein,

$$\begin{aligned}
d^D x \sqrt{|g|} &= \sqrt{|\text{Det}[g_{ab}]|} dx^0 \wedge \cdots \wedge dx^{D-1} \\
d^D x \sqrt{|g|} &= \sqrt{|\text{Det}[e^\mu{}_a \eta_{\mu\nu} e^\nu{}_b]|} dx^0 \wedge \cdots \wedge dx^{D-1} \\
d^D x \sqrt{|g|} &= \sqrt{|\text{Det}[e^\mu{}_a] \text{Det}[\eta_{\mu\nu}] \text{Det}[e^\nu{}_b]|} dx^0 \wedge \cdots \wedge dx^{D-1} \\
d^D x \sqrt{|g|} &= \sqrt{(\text{Det}[e^\mu{}_a])^2} dx^0 \wedge \cdots \wedge dx^{D-1} \\
d^D x \sqrt{|g|} &= \text{Det}[e^\mu{}_a] dx^0 \wedge \cdots \wedge dx^{D-1} \\
d^D x \sqrt{|g|} &= \epsilon_{\mu_0 \cdots \mu_{D-1}} e^{\mu_0}{}_0 \cdots e^{\mu_{D-1}}{}_{D-1} dx^0 \wedge \cdots \wedge dx^{D-1} \\
d^D x \sqrt{|g|} &= \epsilon_{\mu_0 \cdots \mu_{D-1}} e^{\mu_0}{}_0 dx^0 \wedge \cdots \wedge e^{\mu_{D-1}}{}_{D-1} dx^{D-1} \\
d^D x \sqrt{|g|} &= \frac{1}{D!} \epsilon_{\mu_0 \cdots \mu_{D-1}} e^{\mu_0}{}_{a_0} dx^{a_0} \wedge \cdots \wedge e^{\mu_{D-1}}{}_{a_{D-1}} dx^{a_{D-1}} \\
d^D x \sqrt{|g|} &= \frac{1}{D!} \epsilon_{\mu_0 \cdots \mu_{D-1}} \mathbf{e}^{\mu_0} \wedge \cdots \wedge \mathbf{e}^{\mu_{D-1}}
\end{aligned}$$

And now we express the Ricci scalar,

$$\begin{aligned}
R &= g^{ab} R_{cb}{}^c{}_a \\
R &= e_\rho{}^a e^{\rho b} R_{cbda} e_\alpha{}^c e^{\alpha d} \\
R &= \eta^{\rho\sigma} \eta^{\alpha\beta} e_\rho{}^a e_\sigma{}^b R_{cbda} e_\alpha{}^c e_\beta{}^d \\
R &= \eta^{\rho\sigma} \eta^{\alpha\beta} R_{\alpha\sigma\beta\rho} \\
R &= \frac{1}{2} (\eta^{\rho\sigma} \eta^{\alpha\beta} - \eta^{\rho\alpha} \eta^{\sigma\beta}) R_{\alpha\sigma\beta\rho} \\
R &= \frac{1}{2(D-2)!} \epsilon^{\nu_0 \dots \nu_{D-3} \beta \rho} \epsilon_{\nu_0 \dots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho}
\end{aligned}$$

Putting everything together,

$$\begin{aligned}
S_{\text{EH}} &= \frac{1}{2\kappa} \int_M d^D x \sqrt{|g|} R \\
S_{\text{EH}} &= \frac{1}{4D!(D-2)! \kappa} \int_M \mathbf{e}^{\mu_0} \wedge \dots \wedge \mathbf{e}^{\mu_{D-1}} \epsilon_{\mu_0 \dots \mu_{D-1}} \epsilon^{\nu_0 \dots \nu_{D-3} \beta \rho} \epsilon_{\nu_0 \dots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\text{EH}} &= \frac{1}{4(D-2)! \kappa} \int_M \mathbf{e}^{\mu_0} \wedge \dots \wedge \mathbf{e}^{\mu_{D-1}} \eta_{\mu_0}{}^{[\nu_0} \dots \eta_{\mu_{D-3}}{}^{\nu_{D-3}} \eta_{\mu_{D-2}}{}^\beta \eta_{\mu_{D-1}}{}^{\rho]} \epsilon_{\nu_0 \dots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\text{EH}} &= \frac{1}{4(D-2)! \kappa} \int_M \mathbf{e}^{\nu_0} \wedge \dots \wedge \mathbf{e}^{\nu_{D-3}} \wedge \mathbf{e}^\beta \wedge \mathbf{e}^\rho \epsilon_{\nu_0 \dots \nu_{D-3}}{}^{\alpha\sigma} R_{\alpha\sigma\beta\rho} \\
S_{\text{EH}} &= \frac{1}{2\kappa} \int_M \frac{1}{2(D-2)!} R_{\alpha\sigma\beta\rho} \epsilon^{\alpha\sigma}{}_{\nu_0 \dots \nu_{D-3}} \mathbf{e}^{\nu_0} \wedge \dots \wedge \mathbf{e}^{\nu_{D-3}} \wedge \mathbf{e}^\beta \wedge \mathbf{e}^\rho \\
S_{\text{EH}} &= \frac{1}{2\kappa} \int_M \star \mathbf{R}_{\beta\rho} \wedge \mathbf{e}^\beta \wedge \mathbf{e}^\rho \\
S_{\text{EH}} &= \frac{1}{2\kappa} \int_M \mathbf{R}_{\beta\rho} \wedge \star (\mathbf{e}^\beta \wedge \mathbf{e}^\rho) \\
S_{\text{EH}} &= \frac{1}{2\kappa} \int_M \frac{1}{(D-2)!} \epsilon^{\beta\rho}{}_{\alpha_0 \dots \alpha_{D-3}} \mathbf{R}_{\beta\rho} \wedge \mathbf{e}^{\alpha_0} \wedge \dots \wedge \mathbf{e}^{\alpha_{D-3}} \\
S_{\text{EH}} &= \frac{1}{2(D-2)! \kappa} \int_M \epsilon_{\alpha_0 \dots \alpha_{D-1}} \mathbf{e}^{\alpha_0} \wedge \dots \wedge \mathbf{e}^{\alpha_{D-3}} \wedge \mathbf{R}^{\alpha_{D-2} \alpha_{D-1}}
\end{aligned}$$

The most interesting case here is $D = 3$,

$$S_{\text{EH}}[\mathbf{e}, \omega] = \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta}$$

Equations of motion are,

$$\begin{aligned}
S_{\text{EH}}[\mathbf{e} + \delta\mathbf{e}, \boldsymbol{\omega}] - S_{\text{EH}}[\mathbf{e}, \boldsymbol{\omega}] &= \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \delta\mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} = 0 \\
0 &= -\frac{1}{2} \epsilon_{\mu\alpha\beta} \mathbf{R}^{\alpha\beta} \\
0 &= -\frac{1}{2} \epsilon_{\mu\alpha\beta} \frac{1}{2} R_{\rho\sigma}^{\alpha\beta} \mathbf{e}^\rho \wedge \mathbf{e}^\sigma \\
0 &= -\frac{1}{4} \epsilon_{\mu\alpha\beta} R_{\rho\sigma}^{\alpha\beta} \star (\mathbf{e}^\rho \wedge \mathbf{e}^\sigma) \\
0 &= -\frac{1}{4} \epsilon_{\mu\alpha\beta} R_{\rho\sigma}^{\alpha\beta} \epsilon^{\rho\sigma}{}_\kappa \mathbf{e}^\kappa \\
0 &= -\frac{1}{4} \epsilon_{\mu\alpha\beta} \epsilon^{\rho\sigma\kappa} R_{\rho\sigma}^{\alpha\beta} \mathbf{e}_\kappa \\
0 &= -\frac{1}{4} \eta_\mu^{[\rho} \eta_\alpha^{\sigma} \eta_\beta^{\kappa]} R_{\rho\sigma}^{\alpha\beta} \mathbf{e}_\kappa \\
0 &= -\frac{1}{4} R_{\rho\sigma}^{[\sigma\kappa} \eta_\mu^{\rho]} \mathbf{e}_\kappa \\
0 &= -\frac{1}{4} (R_{\rho\sigma}^{\sigma\kappa} \eta_\mu^{\rho} + R_{\rho\sigma}^{\kappa\rho} \eta_\mu^{\sigma} + R_{\rho\sigma}^{\rho\sigma} \eta_\mu^{\kappa} - R_{\rho\sigma}^{\rho\kappa} \eta_\mu^{\sigma} - R_{\rho\sigma}^{\kappa\sigma} \eta_\mu^{\rho} - R_{\rho\sigma}^{\sigma\rho} \eta_\mu^{\kappa}) \mathbf{e}_\kappa \\
0 &= -\frac{1}{4} (-R_\mu^{\kappa} - R_\mu^{\kappa} + R\eta_\mu^{\kappa} - R_\mu^{\kappa} - R_\mu^{\kappa} + R\eta_\mu^{\kappa}) \mathbf{e}_\kappa \\
0 &= -\frac{1}{4} (-4R_\mu^{\kappa} + 2R\eta_\mu^{\kappa}) \mathbf{e}_\kappa \\
0 &= \left(R_{\mu\kappa} - \frac{1}{2} R\eta_{\mu\kappa} \right) \mathbf{e}^\kappa
\end{aligned}$$

And,

$$\begin{aligned}
S_{\text{EH}}[\mathbf{e}, \boldsymbol{\omega} + \delta\boldsymbol{\omega}] - S_{\text{EH}}[\mathbf{e}, \boldsymbol{\omega}] &= \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \mathbf{e}^\mu \wedge (\text{d}\delta\boldsymbol{\omega} + \delta\boldsymbol{\omega} \wedge \boldsymbol{\omega} + \boldsymbol{\omega} \wedge \delta\boldsymbol{\omega})^{\alpha\beta} = 0 \\
0 &= \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \left(-\text{d}(\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta}) + \text{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} + \mathbf{e}^\mu \wedge (\delta\boldsymbol{\omega} \wedge \boldsymbol{\omega})^{\alpha\beta} + \mathbf{e}^\mu \wedge (\boldsymbol{\omega} \wedge \delta\boldsymbol{\omega})^{\alpha\beta} \right) \\
0 &= \frac{1}{2} \epsilon_{\mu\alpha\beta} \int_M (\text{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} + \mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\gamma} \wedge \boldsymbol{\omega}_\gamma^\beta + \mathbf{e}^\mu \wedge \boldsymbol{\omega}^{\alpha\gamma} \wedge \delta\boldsymbol{\omega}_\gamma^\beta) \\
0 &= \frac{1}{2} \epsilon_{\mu\alpha\beta} \int_M (\text{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} - \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\gamma^\beta \wedge \delta\boldsymbol{\omega}^{\alpha\gamma} + \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\alpha_\gamma \wedge \delta\boldsymbol{\omega}^{\gamma\beta}) \\
0 &= \frac{1}{2} \epsilon_{\mu\alpha\beta} \int_M \text{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} - \frac{1}{2} \epsilon_{\mu\alpha\beta} \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\gamma^\beta \wedge \delta\boldsymbol{\omega}^{\alpha\gamma} + \frac{1}{2} \epsilon_{\mu\alpha\beta} \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\alpha_\gamma \wedge \delta\boldsymbol{\omega}^{\gamma\beta} \\
0 &= \frac{1}{2} \epsilon_{\mu\alpha\beta} \int_M \text{d}\mathbf{e}^\mu \wedge \delta\boldsymbol{\omega}^{\alpha\beta} - \frac{1}{2} \epsilon_{\mu\alpha\gamma} \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta^\gamma \wedge \delta\boldsymbol{\omega}^{\alpha\beta} + \frac{1}{2} \epsilon_{\mu\gamma\beta} \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma_\alpha \wedge \delta\boldsymbol{\omega}^{\alpha\beta} \\
0 &= \frac{1}{2} \int_M \left(\epsilon_{\mu\alpha\beta} \text{d}\mathbf{e}^\mu - \epsilon_{\mu\alpha\gamma} \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta^\gamma + \epsilon_{\mu\gamma\beta} \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma_\alpha \right) \wedge \delta\boldsymbol{\omega}^{\alpha\beta} \\
0 &= \frac{1}{2} \epsilon_{\mu\alpha\beta} \text{d}\mathbf{e}^\mu - \frac{1}{2} \epsilon_{\mu\alpha\gamma} \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta^\gamma + \frac{1}{2} \epsilon_{\mu\gamma\beta} \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma_\alpha \\
0 &= \frac{1}{2} \epsilon^{\alpha\beta\nu} \epsilon_{\mu\alpha\beta} \text{d}\mathbf{e}^\mu - \frac{1}{2} \epsilon^{\alpha\beta\nu} \epsilon_{\mu\alpha\gamma} \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta^\gamma + \frac{1}{2} \epsilon^{\alpha\beta\nu} \epsilon_{\mu\gamma\beta} \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma_\alpha \\
0 &= \text{d}\mathbf{e}^\nu - \eta_\gamma^{[\beta} \eta_\mu^{\nu]} \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta^\gamma + \eta_\mu^{[\nu} \eta_\gamma^{\alpha]} \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma_\alpha \\
0 &= \text{d}\mathbf{e}^\nu - \frac{1}{2} (\eta_\gamma^\beta \eta_\mu^\nu - \eta_\gamma^\nu \eta_\mu^\beta) \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta^\gamma + \frac{1}{2} (\eta_\mu^\nu \eta_\gamma^\alpha - \eta_\mu^\alpha \eta_\gamma^\nu) \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma_\alpha \\
0 &= \text{d}\mathbf{e}^\nu + \frac{1}{2} \eta_\gamma^\nu \eta_\mu^\beta \mathbf{e}^\mu \wedge \boldsymbol{\omega}_\beta^\gamma - \frac{1}{2} \eta_\mu^\alpha \eta_\gamma^\nu \mathbf{e}^\mu \wedge \boldsymbol{\omega}^\gamma_\alpha \\
0 &= \text{d}\mathbf{e}^\nu + \frac{1}{2} \mathbf{e}^\beta \wedge \boldsymbol{\omega}_\beta^\nu - \frac{1}{2} \mathbf{e}^\alpha \wedge \boldsymbol{\omega}^\nu_\alpha \\
0 &= \text{d}\mathbf{e}^\nu + \boldsymbol{\omega}^\nu_\alpha \wedge \mathbf{e}^\alpha
\end{aligned}$$

It's not really feasible to give this a gauge theory approach, only if we're in 2+1, in this case there is an isomorphism, $\mathbf{e}^\mu \rightarrow \mathbf{e}_{\alpha\beta} = \epsilon_{\mu\alpha\beta} \mathbf{e}^\mu$, so that,

$$\begin{aligned} S_{\text{EH}} &= \frac{1}{2\kappa} \epsilon_{\mu\alpha\beta} \int_M \mathbf{e}^\mu \wedge \mathbf{R}^{\alpha\beta} \\ S_{\text{EH}} &= -\frac{1}{2\kappa} \int_M \mathbf{e}_{\beta\alpha} \wedge \mathbf{R}^{\alpha\beta} \\ S_{\text{EH}} &= -\frac{1}{2\kappa} \int_M \text{Tr} [\mathbf{e} \wedge \mathbf{R}] \\ S_{\text{EH}} &= -\frac{1}{2\kappa} \int_M \text{Tr} \left[\mathbf{e} \wedge \left(d\boldsymbol{\omega} + \frac{1}{2} [\boldsymbol{\omega} \frown \boldsymbol{\omega}] \right) \right] \end{aligned}$$

This is a lot similar to Chern-Simons theory,

$$S_{\text{CS}}[\mathbf{A}] = k \int_M \text{Tr} \left[\mathbf{A} \wedge d\mathbf{A} + \frac{1}{3} \mathbf{A} \wedge [\mathbf{A} \frown \mathbf{A}] \right]$$

Let's try, $\mathbf{A}^x = \boldsymbol{\omega} + x\mathbf{e}$,

$$\begin{aligned} S_{\text{CS}}[\mathbf{A}^x] &= k \int_M \text{Tr} \left[(\boldsymbol{\omega} + x\mathbf{e}) \wedge d(\boldsymbol{\omega} + x\mathbf{e}) + \frac{1}{3} (\boldsymbol{\omega} + x\mathbf{e}) \wedge [\boldsymbol{\omega} + x\mathbf{e} \frown \boldsymbol{\omega} + x\mathbf{e}] \right] \\ S_{\text{CS}}[\mathbf{A}^x] &= k \int_M \text{Tr} \left[\boldsymbol{\omega} \wedge d\boldsymbol{\omega} + x\boldsymbol{\omega} \wedge d\mathbf{e} + x\mathbf{e} \wedge d\boldsymbol{\omega} + x^2 \mathbf{e} \wedge d\mathbf{e} \right. \\ &\quad \left. + \frac{2}{3} (\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e}) \wedge (\boldsymbol{\omega} + x\mathbf{e}) \right] \\ S_{\text{CS}}[\mathbf{A}^x] &= k \int_M \text{Tr} \left[\boldsymbol{\omega} \wedge d\boldsymbol{\omega} + x d\mathbf{e} \wedge \boldsymbol{\omega} + x\mathbf{e} \wedge d\boldsymbol{\omega} - \frac{1}{2} x^2 d(\mathbf{e} \wedge \mathbf{e}) \right. \\ &\quad \left. + \frac{2}{3} \boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\omega} + 2x\boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + 2x^2 \mathbf{e} \wedge \mathbf{e} \wedge \boldsymbol{\omega} + \frac{2}{3} x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e} \right] \\ S_{\text{CS}}[\mathbf{A}^x] &= k \int_M \text{Tr} \left[\boldsymbol{\omega} \wedge \left(d\boldsymbol{\omega} + \frac{2}{3} \boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\omega} \right) + x d(\mathbf{e} \wedge \boldsymbol{\omega}) + x\mathbf{e} \wedge d\boldsymbol{\omega} + x\mathbf{e} \wedge (d\boldsymbol{\omega} + 2\boldsymbol{\omega} \wedge \boldsymbol{\omega}) + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} \right. \\ &\quad \left. + \frac{2}{3} x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e} \right] \\ S_{\text{CS}}[\mathbf{A}^x] &= S_{\text{CS}}[\boldsymbol{\omega}] + k \int_M \text{Tr} \left[x d(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x\mathbf{e} \wedge (d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}) + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3} x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e} \right] \\ S_{\text{CS}}[\mathbf{A}^x] &= S_{\text{CS}}[\boldsymbol{\omega}] + k \int_M \text{Tr} \left[x d(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x\mathbf{e} \wedge \mathbf{R} + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3} x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e} \right] \\ S_{\text{CS}}[\mathbf{A}^x] &= S_{\text{CS}}[\boldsymbol{\omega}] - 4xk\kappa S_{\text{EH}} + k \int_M \text{Tr} \left[x d(\mathbf{e} \wedge \boldsymbol{\omega}) + 2x^2 \mathbf{e} \wedge \boldsymbol{\omega} \wedge \mathbf{e} + \frac{2}{3} x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e} \right] \end{aligned}$$

We can simplify if we sum two contributions,

$$\begin{aligned} S_{\text{CS}}[\mathbf{A}^x] - S_{\text{CS}}[\mathbf{A}^{-x}] &= k \int_M \text{Tr} \left[x\boldsymbol{\omega} \wedge (d\mathbf{e} + \boldsymbol{\omega} \wedge \mathbf{e}) + x\mathbf{e} \wedge \mathbf{R} + \frac{2}{3} x^3 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e} \right] \\ S_{\text{CS}}[\mathbf{A}^x] - S_{\text{CS}}[\mathbf{A}^{-x}] &= -2\kappa k x S_{\text{EH}} + k x \int_M \text{Tr} \left[\boldsymbol{\omega} \wedge (d\mathbf{e} + \boldsymbol{\omega} \wedge \mathbf{e}) + \frac{2}{3} x^2 \mathbf{e} \wedge \mathbf{e} \wedge \mathbf{e} \right] \end{aligned}$$