I. Introduction:

The signals under consideration in numerous engineering applications, including voice synthesis, analysis, radar, sonar, and telecommunications, are known to be nonstationary, meaning that their properties change over time. A typical method for power spectral density (PSD) estimate cannot be used for spectral analysis of these kinds of data. To cope with such signals, various techniques have been offered, including the time-frequency analysis technique [1].

For time-frequency analysis, one of the first techniques was the short-time Fourier transform (STFT). A portion of the signal is removed by a moving window, and the local characteristics of the signal are obtained from the Fourier transform of this portion. Nonstationary signals are analysed using the spectrogram, which is the squared magnitude of the STFT.

Another commonly used time-frequency distribution (TFD) is the Wigner–Ville distribution (WVD)

- The Wigner-Ville Distribution (WVD) is a mathematical tool used in signal processing to analyze non-stationary signals. It's named after Eugene Wigner and is a representation of a signal in the time-frequency domain. In essence, WVD provides a joint time-frequency representation of a signal, which means it shows how the frequency content of the signal changes over time. This is particularly useful for signals whose frequency components change over time, such as those encountered in communication systems, radar, sonar, and biomedical signal processing.

- Fourier-Bessel coefficients are coefficients used in expressing a function in terms of a series of orthogonal functions called Bessel functions. These coefficients play a role similar to the Fourier coefficients in expressing a function in terms of sine and cosine functions but for functions defined on a finite interval or a bounded domain, especially in cylindrical or spherical geometries.

The research project aims to develop a novel method for decomposing digital signals into time-frequency representations using a combination of the Wigner-Ville Distribution (WVD) and Fourier-Bessel Coefficients (FBC).

II. Approach:

The WVD of a signal x(t)[1]is defined in the time domain as: $W_x(t,\omega) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j\omega\tau}\,d\tau,$

where x*(t) is the complex conjugate of x(t). In the frequency domain, the WVD is defined as follows:

$$W_x(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega+\xi/2)X^*(\omega-\xi/2)e^{j\xi t}\,d\xi,$$

where X(ω) is the Fourier transform of x(t). The various desirable properties of the WVD such as preservation of time and frequency support, infinite time and frequency resolutions, and more, make the WVD a useful tool for signal analysis. The main drawback of this distribution is that it is quadratic and the method based on the WVD introduces the cross terms in the time-frequency domain making the transform space difficult to interpret. The WVD of the sum of M signals

$$x(t) = \sum_{i=1}^{M} x_i(t)$$

is given by

$$W_x(t,\omega) = \underbrace{\sum_{i=1}^{M} W_{x_i}(t,\omega)}_{\text{autocomponents}} + \underbrace{\sum_{k=1}^{M-1}\sum_{l=k+1}^{M} 2\Re\left[W_{x_k x_l}(t,\omega)\right]}_{\text{crosscomponents}}$$

these cross-component terms need to be removed from a signal.

The following is the MATLAB representation of the above formula.

```
function [tfr,t,f] = tfrwv(x,t,N,trace)
%TFRWV  Wigner-Ville time-frequency distribution.
%   [TFR,T,F]=TFRWV(X,T,N,TRACE) computes the Wigner-Ville distribution
%   of a discrete-time signal X,
%   or the cross Wigner-Ville representation between two signals.
%   X     : signal if auto-WV, or [X1,X2] if cross-WV.
%   T     : time instant(s)          (default : 1:length(X)).
%   N     : number of frequency bins (default : length(X)).
%   TRACE : if nonzero, the progression of the algorithm is shown
%                                    (default : 0).
%   TFR   : time-frequency representation. When called without
%           output arguments, TFRWV runs TFRQVIEW.
%   F     : vector of normalized frequencies
tfr= zeros (N,tcol);
if trace, disp('Wigner-Ville distribution'); end;
%Applying WVD
for icol=1:tcol,
 ti= t(icol); taumax=min([ti-1,xrow-ti,round(N/2)-1]);
 tau=-taumax:taumax; indices= rem(N+tau,N)+1;
 tfr(indices,icol) = x(ti+tau,1) .* conj(x(ti-tau,xcol));
 tau=round(N/2);
 if (ti<=xrow-tau)&(ti>=tau+1),
  tfr(tau+1,icol) = 0.5 * (x(ti+tau,1) * conj(x(ti-tau,xcol))  + ...
                          x(ti-tau,1) * conj(x(ti+tau,xcol))) ;
 end;
 if trace, disprog(icol,tcol,10); end;
end;
tfr= fft(tfr);
```

As we can see FFT of the TFR has been taken. We aim to bring down cross terms by considering the Fourier Bessel coefficients of the 2D TFR.

```
function [tfr,t,f] = tfrwv_fb(x,t,N,trace)
%TFRWV  Wigner-Ville time-frequency distribution.
%   [TFR,T,F]=TFRWV(X,T,N,TRACE) computes the Wigner-Ville distribution
%   of a discrete-time signal X,
%   or the cross Wigner-Ville representation between two signals.
%
%   X     : signal if auto-WV, or [X1,X2] if cross-WV.
%   T     : time instant(s)          (default : 1:length(X)).
%   N     : number of frequency bins (default : length(X)).
%   TRACE : if nonzero, the progression of the algorithm is shown
%                                    (default : 0).
%   TFR   : time-frequency representation. When called without
%           output arguments, TFRWV runs TFRQVIEW.
%   F     : vector of normalized frequencies.

tfr= zeros (N,tcol);
if trace, disp('Wigner-Ville distribution'); end;
for icol=1:tcol,
 ti= t(icol); taumax=min([ti-1,xrow-ti,round(N/2)-1]);
 tau=-taumax:taumax; indices= rem(N+tau,N)+1;
 tfr(indices,icol) = x(ti+tau,1) .* conj(x(ti-tau,xcol));
 tau=round(N/2);
 if (ti<=xrow-tau)&(ti>=tau+1),
  tfr(tau+1,icol) = 0.5 * (x(ti+tau,1) * conj(x(ti-tau,xcol))  + ...
                           x(ti-tau,1) * conj(x(ti+tau,xcol))) ;
 end;
 if trace, disprog(icol,tcol,10); end;
end;
tfr= fb_new(tfr);
```

Here we considered a new tfrwv_fb function which uses the fb_new function to calculate the TFR.

```
function a3 = fb_new(s1)
    % 2D Fourier-Bessel coefficients computation

    [rows, cols] = size(s1); % Get the dimensions of the input matrix
    a3 = zeros(rows, cols);  % Initialize the output matrix

    for row = 1:rows
        % Compute Fourier-Bessel coefficients for each row

        a3(row, :) = fb1_row(s1(row, :));
    end
end
```

fb_new utilizes fb1_row on each row of our TFR matrix to calculate the Bessel coefficients.

```
function a3_row = fb1_row(s_row)
    % Compute Fourier-Bessel coefficients for a single row

    MM = length(s_row);  % Order of FB expansion
    x = 2;               % Initial value for computing Bessel function roots
    alfa = zeros(1, MM); % Array to store Bessel function roots

    % Compute Bessel function roots
    for i = 1:MM
        ex = 1;
        while abs(ex) > 0.00001
            ex = -besselj(0, x) / besselj(1, x);
            x = x - ex;
        end
        alfa(i) = x;
        x = x + pi;
    end

    % Compute Fourier-Bessel coefficients
    N = length(s_row);
    nb = 1:N;
    a = N;
    a3_row = zeros(1, MM);

    for m1 = 1:MM
        a3_row(m1) = (2 / (a^2 * (besselj(1, alfa(m1)))^2)) ...
            * sum(nb .* s_row .* besselj(0, alfa(m1) / a * nb));
    end
end
```

With a tolerance of 0.0001 the bessel function roots are computed using the Newton-Rphson method applied to the ratio of the functions $J_0(x)$ and $J_1(x)$ and stored in an array.

Further, the Fourier-bessel coefficients are computed utilizing Bessel function roots using the below-mentioned formula.

$$a_{m1} = \frac{2}{a^2 \cdot (J_1(\alpha_{m1}))^2} \cdot \sum_{n=1}^{N} n \cdot s(n) \cdot J_0\left(\frac{\alpha_{m1}}{a} \cdot n\right)$$

III. Efforts along the way:

Given below is a summary of all the steps taken in the entire semester to reach the final product:

Tasks Completed:

- **Testing Wigner-Ville Plot Code:**
  - Successfully implemented a MATLAB code to compute the Wigner-Ville plot of a chirp signal with random Gaussian noise.
  - Conducted testing to validate the accuracy and reliability of the Wigner-Ville plot computation under varying signal and noise conditions.
  - Obtained valuable insights into the behaviour of the Wigner-Ville Distribution in the presence of noise, laying the groundwork for further experimentation.

- **Integration of Fourier-Bessel Coefficients:**
  - Utilized Fourier-Bessel coefficients to enhance the computation of the Wigner-Ville Distribution.
  - Developed a method to replace the Fast Fourier Transform (FFT) step in traditional WVD computation by multiplying Fourier-Bessel coefficients with a complex exponential term that shows promise in enhancing the accuracy and efficiency of time-frequency representation.
  - Implemented the shifting of frequency components to desired frequency bins and multiplication with the complex conjugate of the coefficient to obtain the magnitude of the Wigner-Ville Distribution. Used Fourier-Bessel coefficients to compute WVD instead of FFT by multiplying the Fourier-Bessel coefficients with a complex exponential term and shifting the frequency component to the desired frequency bin. Finally, obtain the magnitude of the WVD by multiplying the result by the complex conjugate of the coefficient.

- **Implementation of the hybrid Time-Frequency representation method:**
  - Implemented the Wigner-Ville distribution computation using Fourier Bessel coefficients with the aim of reducing cross terms from the former method.
  - Results of the proposed method are given in the next section.

IV. Results/ Conclusion:

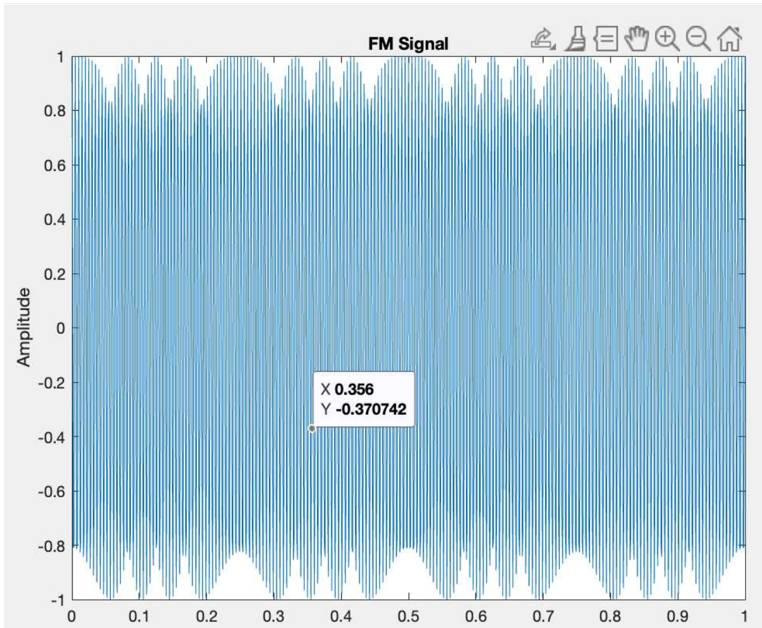
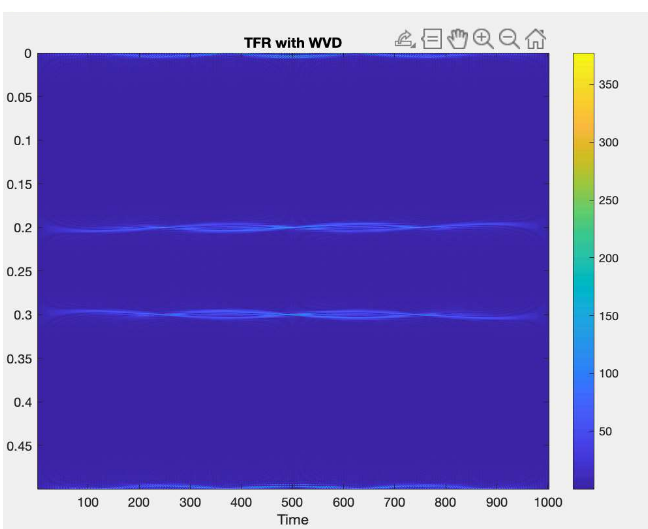
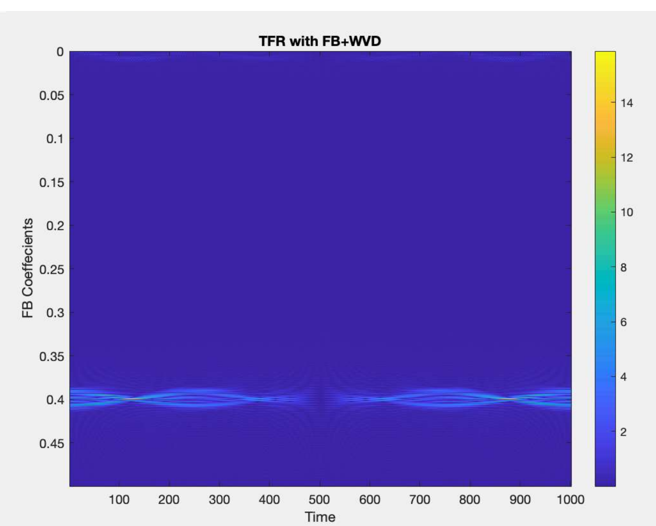


Fig 1: Input (FM) Signal:Carrier Frequency=200,Modulation frequency=2, Sampling frequency=1000,modulation signal=sin(2*pi*f*t)





Signal obtained after using tfrwv function.

Final output after WVD and Fourier-Bessel

The TFRWV function plot 2 similar curves centred around frequency=0.25

The new TFRW function using bessel coefficients plots fourier bessel coefficients at 0.4 and removes the erroneous mirror images caused by the original function