TRIDIAGONAL STOCHASTIC MATRICES

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Uyen Nguyen

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AUTHOR: Uyen Nguyen

DATE SUBMITTED: Summer 2017

Department of Mathematics and Statistics

Dr. Alan Krinik
Thesis Committee Chair
Mathematics & Statistics

Dr. Randall J. Swift
Mathematics & Statistics

Dr. Hubertus von Bremen
Mathematics & Statistics

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Finally, I would like to dedicate this thesis to my family who supported my decision to attend graduate school. I appreciate my parents who raised me with love and supported me in all my pursuits. And most of all for my loving, caring and encouraging husband, Mark, for his faithful support throughout my years of study.

ABSTRACT

A birth-death chain with one-step transition probability matrix P often has a dual birth-death chain with one-step transition probability matrix P^*. The same holds for birth-death processes. From Professor Kouachi's work, we are able to determine the eigenvalues of suitable matrices P and P^*. We describe the exact diagonalization of P and P^* in Chapter 1. Chapter 2 summarizes Professor Kouachi's work in determining the exact formulas for eigenvalues and eigenvectors of certain tridiagonal matrices having arbitrary large dimension.

In Chapter 3, we apply Professor Kouachi's results to diagonalize a certain class of birth-death chains and processes. We obtain exact expressions for P^n, $(P^*)^n$ and $P(t)$. Generalizations of our results to non-tridiagonal stochastic matrices are presented in Chapter 4. Final conclusions and plans for future work are given in Chapter 5. Computer programs written by collaborators by Luis Cervantes, Mark Dela and Dave Luk to calculate P^n, $(P^*)^n$ and $P(t)$ are gratefully acknowledged and used in this thesis to obtain results in higher dimensions.

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Markov chains and processes

1.1 Diagonalization

Consider the following stochastic matrix:

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \text{ where } 0 < a, b < 1.$$

In *An Introduction to Stochastic Modeling* [11], it is shown by an induction proof that for $n = 1, 2, 3, \ldots$, we have:

$$P^n = \begin{pmatrix} \dfrac{b}{a+b} + \dfrac{(1-a-b)^n}{a+b}a & \dfrac{a}{a+b} - \dfrac{(1-a-b)^n}{a+b}a \\ \dfrac{b}{a+b} - \dfrac{(1-a-b)^n}{a+b}b & \dfrac{a}{a+b} + \dfrac{(1-a-b)^n}{a+b}b \end{pmatrix} \tag{1.1}$$

To illustrate the linear algebraic method of diagonalization (which is used throughout this thesis), we establish (1.1) by finding the eigenvalues and eigenvectors of P. The matrix P may be diagonalized as $P = S \cdot D \cdot S^{-1}$ by a Key Theorem (7.14) on page 294 of

[10], where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $S = \quad \vec{v_1}, \vec{v_2} \quad$ where $\vec{v_1}, \vec{v_2}$ are the associated eigenvectors corresponding to eigenvalues λ_1, λ_2 respectively.

First, the eigenvalues are those λ for which $\det[P - \lambda I] = 0$. So

$$\begin{aligned} det[P - \lambda I] &= det \begin{pmatrix} 1-a-\lambda & a \\ b & 1-b-\lambda \end{pmatrix} \\ &= (\lambda - 1)(\lambda + a + b - 1). \end{aligned} \tag{1.2}$$

The eigenvalues of P are the solutions of the following quadratic equation

$$(\lambda - 1)(\lambda + a + b - 1) = 0, \text{ namely } \lambda_1 = 1 \text{ and } \lambda_2 = 1 - a - b.$$

Thus we will have $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-a-b \end{pmatrix}$.

Next to find the associated eigenvectors $\vec{v_i}$, we solve the equation $P \cdot \vec{v_i} = \lambda_i \cdot \vec{v_i}$ where $i = 1, 2$. The matrix S is composed of eigenvectors $\vec{v_1}, \vec{v_2}$ as its columns.

$$S = \begin{pmatrix} 1 & a \\ 1 & -b \end{pmatrix} \text{ and } S^{-1} = \frac{-1}{a+b} \begin{pmatrix} -b & -a \\ -1 & 1 \end{pmatrix}$$

Since $P = S \cdot D \cdot S^{-1}$, it follows that

$$\begin{aligned} P^2 &= S \cdot D \cdot S^{-1} \cdot S \cdot D \cdot S^{-1} \\ &= S \cdot D^2 \cdot S^{-1}. \end{aligned}$$

And, in general, for $n = 1, 2, \ldots$, we have

$$P^n = S \cdot D^n \cdot S^{-1}$$

$$= \frac{-1}{a+b} \begin{pmatrix} 1 & a \\ 1 & -b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{pmatrix} \begin{pmatrix} -b & -a \\ -1 & 1 \end{pmatrix}$$

$$= \frac{-1}{a+b} \begin{pmatrix} 1 & a \\ 1 & -b \end{pmatrix} \begin{pmatrix} -b & -a \\ -(1-a-b)^n & (1-a-b)^n \end{pmatrix}$$

$$= \frac{-1}{a+b} \begin{pmatrix} -b - a(1-a-b)^n & -a + a(1-a-b)^n \\ -b + b(1-a-b)^n & -a - b(1-a-b)^n \end{pmatrix}$$

$$= \begin{pmatrix} \frac{b}{a+b} + \frac{(1-a-b)^n}{a+b} a & \frac{a}{a+b} - \frac{(1-a-b)^n}{a+b} a \\ \frac{b}{a+b} - \frac{(1-a-b)^n}{a+b} b & \frac{a}{a+b} + \frac{(1-a-b)^n}{a+b} b, \end{pmatrix}$$

which is (1.1).

Note

$$P^n = \begin{pmatrix} \frac{b + \lambda_2^n a}{a+b} & \frac{a - \lambda_2^n a}{a+b} \\ \frac{b - \lambda_2^n b}{a+b} & \frac{a + \lambda_2^n b}{a+b}, \end{pmatrix}$$

where $\lambda_2 = 1 - a - b$.

1.2 Markov chains

In this thesis, we consider a system that has a finite number of states. Let $\mathscr{S}$ denote the set of all states which is called the *state space* of the system. Let $\mathscr{S} = \{0,1,2,...,H\}$ and suppose X_n is a random variable that denotes the state of the system at time n.

In many systems, past and present states of the system influence the probability of future states. On the other hand, there are some systems that have the property that the probability of a future state depends only on the state at the most recent known time of the system. This property is called the *Markov property*.

Definition 1.2.1. A sequence of random variables $\{X_n, n = 0,1,2,...\}$ obeys the Markov property if and only if

$$P[X_{n+1} = i_{n+1}|X_0 = i_0,...,X_n = i_n] = P[X_{n+1} = i_{n+1}|X_n = i_n]$$

for every choice of the nonnegative integer n and for all states $i_0, i_1, ..., i_{n+1}$, each in $\mathscr{S}$. A system having this property is called a *Markov chain,* see [2].

The probability of X_{n+1} being in state j given that X_n is in state i is called the *one-step transition probability* and is denoted by $P_{i,j}$, where $P_{i,j} = P[X_{n+1} = j|X_n = i]$. The numbers $P_{i,j}$ must satisfy

$$0 \leq P_{i,j} \leq 1 \quad \text{for} \;\; i,j \in \mathscr{S} \tag{1.3}$$

$$\text{and} \; \sum_{j=0}^{H} P_{i,j} = 1 \;\; \text{for} \;\; i \in \mathscr{S}. \tag{1.4}$$

In some chains the probabilities $P[X_{n+1} = j|X_n = i]$ are functions of n; however, if the probabilities do not depend upon n, then the chain is said to be *homogeneous* in time. That is,

$$P_{i,j} = P[X_1 = j|X_0 = i] = P[X_{n+1} = j|X_n = i] \text{ for all } n = 0,1,2,...$$

We assume all Markov chains in this thesis are homogeneous.

It is customary to arrange these numbers $P_{i,j}$ in a matrix:

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} & \ldots & P_{0,H} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} & \ldots & P_{1,H} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ P_{H,0} & P_{H,1} & P_{H,2} & P_{H,3} & \ldots & P_{H,H} \end{pmatrix}$$

and refer to P as the *one-step transition probability matrix* of the Markov chain, see [2].

The $(i+1)$st row of P is the probability distribution of the values of X_{n+1} under the condition that $X_n = i$. Also P is a finite square matrix with the number of rows corresponding to the number of states. Any one-step transition probability matrix P satisfies the conditions (1.3) and (1.4). Also, any square matrix that satisfies those two conditions is said to be a *stochastic matrix*.

We now define *the n-step transition probabilities* as the probability of a chain in state j after n steps given that the chain started at state i, that is, $P_{i,j}^{(n)} = P[X_n = j | X_0 = i]$ where $i, j \in \mathscr{S}$.

It is known that taking P to the n^{th} power gives the n-step transition probability matrix

$$P^n = \begin{pmatrix} P^n_{0,0} & P^n_{0,1} & P^n_{0,2} & P^n_{0,3} & \cdots & P^n_{0,H} \\ P^n_{1,0} & P^n_{1,1} & P^n_{1,2} & P^n_{1,3} & \cdots & P^n_{1,H} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ P^n_{H,0} & P^n_{H,1} & P^n_{H,2} & P^n_{H,3} & \cdots & P^n_{H,H} \end{pmatrix}$$

where $P^n_{i,j}$ are the n-step transition probabilities for $i, j \in \mathscr{S}$. The proof of this result appears as Theorem 3.1 on page 70 of Howard, see [11].

Note that P^n is also a stochastic matrix, that is

$$0 \leq P_{i,j}^{(n)} \leq 1, \quad \sum_{j=0}^{H} P_{i,j}^{(n)} = 1 \qquad \text{for each } i = 0, 1, 2, \ldots, H.$$

Example 1.1. Consider the following state transition diagram:

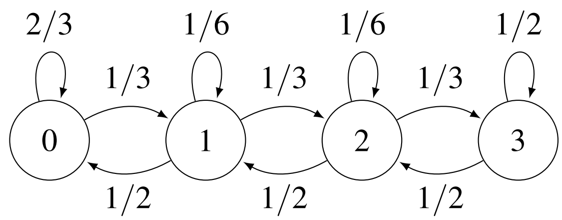


Figure 1.1

Each arrow represents different *one-step transition probabilities*. For example, if the Markov chain starts at state 1, then the probability it goes back to state 0 in one step is $\dfrac{1}{2}$, that it goes to state 2 is $\dfrac{1}{3}$ and that it stays at state 1 is $\dfrac{1}{6}$. However, it cannot go from state 1 to state 3 in one step so this transition probability is 0. Thus the sum of probabilities coming out from each state is 1; for example, we have $P_{1,0} + P_{1,1} + P_{1,2} = 1$.

We can form the transition probability matrix of this Markov chain:

$$P = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

In the same manner as stated in section 1.1, we want to diagonalize the one-step transition probability matrix P by finding its eigenvalues and eigenvectors. The eigenvalues of P are computed to be

$$\lambda_0 = \tfrac{1}{6}(1 - 2\sqrt{3}),\ \lambda_1 = \tfrac{1}{6}(1 + 2\sqrt{3}),\ \lambda_2 = \tfrac{1}{6} \text{ and } \lambda_3 = 1$$

Thus we will have

$$D = \frac{1}{6}\begin{pmatrix} 1-2\sqrt{3} & 0 & 0 & 0 \\ 0 & 1+2\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Now after solving the equation $P \cdot \vec{v}_i = \lambda_i \cdot \vec{v}_i$ where $i = 0,1,2,3$, we obtain the associated eigenvectors $\vec{v}_i$ and hence:

$$S = \begin{pmatrix} -\frac{4}{9} & -\frac{4}{9} & \frac{4}{9} & 1 \\ \frac{2}{9}(3+2\sqrt{3}) & -\frac{2}{9}(-3+2\sqrt{3}) & -\frac{2}{3} & 1 \\ -\frac{2}{3}(1+\sqrt{3}) & \frac{2}{3}(-1+\sqrt{3}) & -\frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{and } S^{-1} = \begin{pmatrix} -\frac{3}{10} & \frac{3}{10} & \frac{9}{20} & -\frac{9}{20} \\ \frac{12}{65} & \frac{8}{65} & \frac{27}{65} & \frac{18}{65} \\ \frac{3-9\sqrt{3}}{52} & \frac{15-6\sqrt{3}}{52} & \frac{-45+18\sqrt{3}}{104} & \frac{9+12\sqrt{3}}{104} \\ \frac{3+9\sqrt{3}}{52} & \frac{15+6\sqrt{3}}{52} & \frac{-45-18\sqrt{3}}{104} & \frac{9-12\sqrt{3}}{104} \end{pmatrix}.$$

Similarly, we can find P^n where $P^n = S \cdot D^n \cdot S^{-1}$ and

$$D^n = \begin{pmatrix} \left[\frac{1}{6}(1-2\sqrt{3})\right]^n & 0 & 0 & 0 \\ 0 & \left[\frac{1}{6}(1+2\sqrt{3})\right]^n & 0 & 0 \\ 0 & 0 & \left(\frac{1}{6}\right)^n & 0 \\ 0 & 0 & 0 & (1)^n \end{pmatrix}.$$

More generally, consider a Markov chain on $\mathscr{S} = \{0, 1, 2, ..., H\}$ having the following state transition diagram:

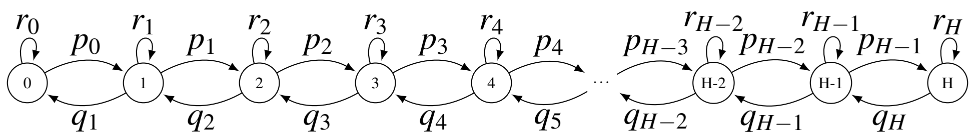


Figure 1.2

Figure 1.2 shows an important, special type of a Markov chain, which is called a *birth-death chain*. The one-step transition probabilities of birth-death chain are given by:

$$P_{i,j} = \begin{cases} r_0 & \text{for } i = 0, j = 0 \\ p_0 & \text{for } i = 0, j = 1 \\ q_i & \text{for } 1 \leq i \leq H-1, j = i-1 \\ r_i & \text{for } 1 \leq i \leq H-1, j = i \\ p_i & \text{for } 1 \leq i \leq H-1, j = i+1 \\ q_H & \text{for } i = H, j = H-1 \\ r_H & \text{for } i = H, j = H \\ 0 & \text{for all other } i, j. \end{cases} \tag{1.5}$$

We assume $0 \leq r_0, r_i < 1$, $0 < p_i, q_i < 1$, $0 \leq p_0$ and $0 \leq q_H$ such that $p_i + q_i + r_i = 1$ for $i = 1, 2, .., H-1$ and $r_0 + p_0 = 1$, $q_H + r_H = 1$.

Birth-death chains originally were used to model the changes in population of some living system. The transitions from state i to state $i+1$ correspond to a "birth" while a transition from state i to state $i-1$ corresponds to a "death". Notice that the Markov chain in Example 1.1 is a birth-death chain.

The behavior of a Markov chain depends on the structure of the one-step transition probability matrix P. So we want to classify *recurrent and transient states*. Let $X_n,\ n > 0$ be a Markov chain having state space $\mathscr{S}$. Then we denote $\rho_{i,j}$ as the probability that a Markov chain starting at state i will be in state j at some positive time. So $\rho_{i,i}$ denotes the probability that a Markov chain starting at i will eventually return to i. We say that a state i is *recurrent* if and only if $\rho_{i,i} = 1$. This statement says that a state i is recurrent if and only if the probability of returning to state i after some finite length of time is one. A state i is *transient* if $\rho_{i,i} < 1$ which implies that a Markov chain starting at i has positive probability $1-\rho_{i,i}$ of never returning to i. If $P_{i,i} = 1$ then i is called an *absorbing* state; thus an absorbing state is special kind of recurrent state, see [2] for more examples.

We now consider the *Chapman-Kolmogorov Equations* for further use in this thesis.

For $i, j \in \mathscr{S}$ and $n \in \mathbb{N}$

$$\begin{aligned}
P_{i,j}^{(n)} &= P(X_n = j | X_0 = i) \\
&= \frac{P(X_0 = i, X_n = j)}{P(X_0 = i)} \qquad\qquad \text{assuming } P(X_0 = i) \neq 0 \\
&= \sum_{k=0}^{N} \frac{P(X_0 = i, X_m = k, X_n = j)}{P(X_0 = i)} \quad \text{where } 0 < m < n \\
&= \sum_{k=0}^{N} \frac{P(X_0 = i) \cdot P(X_m = k | X_0 = i) \cdot P(X_n = j | X_0 = i, X_m = k)}{P(X_0 = i)} \\
&\text{using } P(A \cap B \cap C) = P(A) \cdot p(B|A) \cdot P(C|A \cap B) \\
&= \sum_{k=0}^{N} P(X_m = k | X_0 = i) P(X_n = j | X_m = k) \text{ using the Markov property} \\
&= \sum_{k=0}^{N} P_{i,k}^{(m)} \cdot P_{k,j}^{(n-m)}.
\end{aligned}$$

So $P_{i,j}^{(n)} = \sum_{k=0}^{N} P_{i,k}^{(m)} \cdot P_{k,j}^{(n-m)}$ for all $0 < m < n$ and for all $i, j \in \mathscr{S}$.

This result is known as the **Chapman-Kolmogorov equations**, see [4]. These equations state that in order to move from state i to state j in n steps, X_n moves to some state k in m steps and then from state k to state j in the remaining $n-m$ steps.

1.3 Dual Markov chains

Suppose an operation on an object A leads to new object A^*. If repeating this operation on A^* leads back to A then A^* is often called the dual of A.

To define the dual of a birth-death chain, we start with an original birth death chain as reproduced from Figure 1.3 below

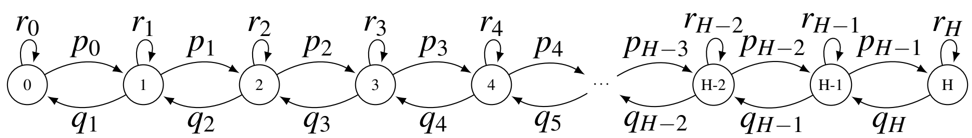


Figure 1.3

As before we assume $0 \leq r_0, r_i < 1$, $0 < p_i, q_i < 1$, $0 \leq p_0$ and $0 \leq q_H$ such that $p_i + q_i + r_i = 1$ for $i = 1, 2, .., H-1$ and $r_0 + p_0 = 1$, $q_H + r_H = 1$.

Now we also assume that

$$p_0 + q_1 \leq 1$$

$$p_1 + q_2 \leq 1$$

$$...$$

$$p_{H-1} + q_H \leq 1.$$

Then the dual birth-death chain is defined by the following state transition diagram:

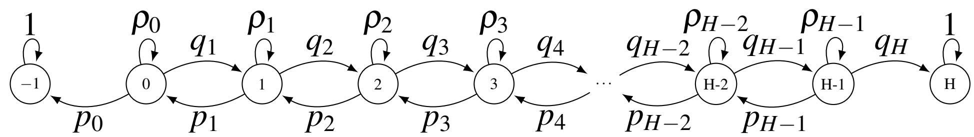


Figure 1.4

where ρ_i are nonnegative fractions less than 1 and

$$p_0 + \rho_0 + q_1 = 1$$

$$p_1 + \rho_1 + q_2 = 1$$

$$p_2 + \rho_2 + q_3 = 1$$

$$...$$

$$p_{H-1} + \rho_{H-1} + q_H = 1$$

This absorbing birth-death chain shown in Figure 1.4 represents the dual birth-death chain of the birth-death chain pictured in Figure 1.3. Since states -1 and H are absorbing, the probability it stays is 1 as we defined the recurrent state in section 1.2. Now the probability from state 0 back to state -1 is the probability from state 0 to 1 in Figure 3. Next, the probability from state 0 to 1 is the one from state 1 to 0 in Figure 1.3. Continue in this manner, all the arrows up in Figure 3 will switch down in Figure 1.4, and all the arrows going down in Figure 1.3 will become up in Figure 1.4. Notice that we do not have arrow coming from state -1 to 0 or from state H to $H-1$ since the system cannot leave.

Birth-death chains having absorbing states at the first and last states arise in gamblers ruin problems. It is of interest to compute the probability of absorption into states -1 starting from state i where $i \geq 1$. If we consider the system in the language of gambling, the probability is also known as *ruin probability*; that is, once the players enters state -1

they get broke or the game is over. The Markov chain has these absorbing states is called *dual Markov chain*.

The following theorem is from the article [7].

Duality Theorem If $P_{i,j}^{(n)}$ and $P_{i,j}^{*(n)}$ are the n-step transient probabilities of the birth-death chains corresponding to Figure 3 and 4 respectively, then

$$P_{r,s}^{(n)} = \sum_{k=r}^{H} [P_{s,k}^{*(n)} - P_{s-1,k}^{*(n)}] \text{ and } P_{r,s}^{*(n)} = \sum_{k=0}^{r} [P_{s,k}^{(n)} - P_{s+1,k}^{(n)}] \tag{1.6}$$

for $n \geq 0$ and for all states $r, s = 0, 1, 2, 3, ..., H$ with the convention $P_{-1,k}^{(n)} = 0$ if $k > -1$.

See Appendix A for the proof of this theorem.

1.4 Markov processes

We will now consider random variables that are indexed by continuous time t, that is $\{X(t), t \geq 0\}$. If $X(t)$ satisfies the *Markov property* for continuous time then we say $X(t)$ is a *Markov process*, see [2]. In other words, Markov processes are characterized by the condition that future development of the system depends only on their most current known states and not their history prior to that time. This condition is expressed in the following definition.

Definition 1.4.1. A random process $X(t)$ obeys the *Markov property* if and only if

$$P[X_{t_{k+1}} = i_{k+1} | X_{t_1} = i_1, X_{t_2} = i_2, ..., X_{t_k} = i_k] = P[X_{t_{k+1}} = i_{k+1} | X_{t_k} = i_k]$$

for all $t_1, ..., t_k, t_{k+1} \in [0, \infty)$ where $t_1 < t_2 < ... < t_k < t_{k+1}$ and for all $i_1, i_2, ..., i_k, i_{k+1} \in \mathscr{S}$. Continuous time systems with this property are called *Markov processes.*

Again, we will consider all Markov processes in this thesis to be homogeneous in time. That is,

$$P_{i,j}(t) = P[X_t = j|X_0 = i] = P[X_{k+t} = j|X_k = i]$$

for all $i, j \in \mathscr{S}$ and all $k > 0$. We denote $P_{i,j}(t)$ as the transition probability functions of the system going from state i to state j in time t.

The following initial values of $P_{i,j}(t)$ are presumed,

$$P_{i,j}(0) = \begin{cases} 0 & \text{if } i = j \\ \\ 1 & \text{if } i = j \end{cases} \qquad (i, j \in \mathscr{S})$$

Now let $P(t)$ denote the matrix of a transition probability functions $P_{i,j}(t)$ where

$$P(t) = \begin{pmatrix} P_{0,0}(t) & P_{0,1}(t) & P_{0,2}(t) & P_{0,3}(t) & \dots & P_{0,H}(t) \\ P_{1,0}(t) & P_{1,1}(t) & P_{1,2}(t) & P_{1,3}(t) & \dots & P_{1,H}(t) \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ P_{H,0}(t) & P_{H,1}(t) & P_{H,2}(t) & P_{H,3}(t) & \dots & P_{H,H}(t) \end{pmatrix}$$

As before, the entries of $P(t)$ satisfy the following conditions:

$$0 \le P_{i,j}(t) \le 1, \quad \sum_{j=0}^{H} P_{i,j}(t) = 1 \qquad \text{for each } i = 0, 1, 2, ..., H.$$

Definition 1.4.2. Consider a Markov Process $X(t)$ on $\mathscr{S}$ with probability transition functions $P_{i,j}(t)$. Suppose $P_{i,j}(t)$ is a differentiable function with respect to t (with a right-sided derivative at 0). We define the *transition rate matrix* Q to be the constant matrix whose entries are $q_{i,j} = \frac{d}{dt} \; P_{i,j}(t) \; \Big|_{t=0}$. That is,

$$Q = \begin{pmatrix} q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} & \cdots & q_{0,H} \\ q_{1,0} & q_{1,1} & q_{1,2} & q_{1,3} & \cdots & q_{1,H} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ q_{H,0} & q_{H,1} & q_{H,2} & q_{H,3} & \cdots & q_{H,H} \end{pmatrix}$$

The matrix Q has three main properties:

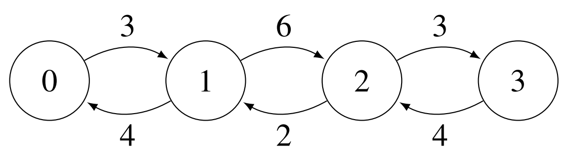
$$\sum_{j=0}^{H} q_{i,j} = 0 \text{ for all } i = 0, 1, ..., H$$

$$q_{i,i} \leq 0 \text{ for all } i = 0, 1, ..., H$$

$$q_{i,j} \geq 0 \text{ for all states } i = j$$

It turns out that $P(t)$ satisfies two systems of differential equations and the solution of thses systems is $P(t) = e^{Qt}$. This results appears in section 6.6 on page 253 of Howard, see [11].

Example 1.2. Consider the following state rate transition diagram of a birth-death Markov process:



We wish to determine $P(t) = [P_{ij}(t)]$ $i, j \in \{0, 1, 2, 3\}$. Since $P(t) = e^{Qt}$ where Q is:

$$Q = \begin{pmatrix} -3 & 3 & 0 & 0 \\ 4 & -10 & 6 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 4 & -4 \end{pmatrix}$$

From definition, $e^{Qt} = I + (Qt) + \dfrac{(Qt)^2}{2!} + ...$

Suppose Q is diagonalized where $Q = SDS^{-1}$ then

$$\begin{aligned} e^{Qt} &= SS^{-1} + (SDS^{-1}t) + \frac{(SDS^{-1}t)^2}{2!} + ... \\ &= S[I + Dt + \frac{(Dt)^2}{2!} + ...]S^{-1} = Se^{Dt}S^{-1}. \end{aligned}$$

The eigenvalues of Q are : $\lambda_0 = -13, \quad \lambda_1 = -2, \quad \lambda_2 = -7, \quad \lambda_3 = 0.$

Thus we have $D = \begin{pmatrix} -13 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Using linear algebra, the eigenvector matrix S is :

$$S = \begin{pmatrix} -9 & -9 & 9 & 1 \\ 30 & -3 & -12 & 1 \\ -9 & 2 & -12 & 1 \\ 4 & 4 & 16 & 1 \end{pmatrix}$$

and hence we can calculate and get:

$$S^{-1} = \begin{pmatrix} \frac{-4}{429} & \frac{10}{429} & \frac{-3}{143} & \frac{1}{143} \\ \frac{-4}{55} & \frac{-1}{55} & \frac{3}{55} & \frac{3}{55} \\ \frac{1}{105} & \frac{-1}{105} & \frac{-1}{35} & \frac{1}{35} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \end{pmatrix}$$

$P(t) = Se^{Dt}S^{-1}$ where $e^{Dt} = \begin{pmatrix} e^{-13t} & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{-7t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Using Luis Cervantes' program, the solution is given by:

$$P(t) = e^{-2t}\begin{pmatrix} \frac{36}{55} & \frac{9}{55} & \frac{-18}{55} & \frac{-27}{55} \\ \frac{12}{55} & \frac{3}{55} & \frac{-6}{55} & \frac{-9}{55} \\ \frac{-8}{55} & \frac{-2}{55} & \frac{4}{55} & \frac{6}{55} \\ \frac{-16}{55} & \frac{-4}{55} & \frac{-8}{55} & \frac{12}{55} \end{pmatrix} + e^{-7t}\begin{pmatrix} \frac{3}{35} & \frac{-3}{35} & \frac{-9}{35} & \frac{9}{35} \\ \frac{-4}{35} & \frac{4}{35} & \frac{12}{35} & \frac{-12}{35} \\ \frac{-4}{35} & \frac{4}{35} & \frac{12}{35} & \frac{-12}{35} \\ \frac{16}{105} & \frac{-16}{105} & \frac{-16}{35} & \frac{16}{35} \end{pmatrix}$$

$$+ e^{-13t}\begin{pmatrix} \frac{12}{143} & \frac{-30}{143} & \frac{27}{143} & \frac{-9}{143} \\ \frac{-40}{143} & \frac{100}{143} & \frac{-90}{143} & \frac{30}{143} \\ \frac{12}{143} & \frac{-30}{143} & \frac{27}{143} & \frac{-9}{143} \\ \frac{-16}{429} & \frac{40}{429} & \frac{-12}{143} & \frac{4}{143} \end{pmatrix} + e^{0}\begin{pmatrix} \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \end{pmatrix}$$

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Chapter 2

Kouachi's tridiagonal matrices

2.1 Alternating diagonal terms

The following Theorem 2.1.1 appears in Kouachi's 2008 article [6]. Professor Kouachi considers tridiagonal matrices of the form:

$$A_N = \begin{pmatrix} -\alpha+b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_1 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_2 & b_3 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & c_{N-1} \\ 0 & \cdots & \cdots & 0 & a_{N-1} & -\beta+b_N \end{pmatrix} \tag{2.1}$$

where a_j and c_j, $j = 1,...,N-1$, are complex numbers and α, β, d and b_j, $j = 1,...,N$, are also complex numbers. Professor Kouachi assumes that

$$a_j c_j = d^2, \quad \text{for } j = 1,...,N-1 \text{ where } d = 0$$

and

$$b_j = \begin{cases} b_1 & \text{if } j \text{ is odd,} \\ b_2 & \text{if } j \text{ is even,} \end{cases} \qquad (j = 1, ..., N)$$

Note that diagonal entries of A_N are assumed to alternate in general.

Theorem 2.1.1. (Eigenvalues) Suppose A_N satisfies the preceding conditions.

Case 1 If $\alpha = \beta = 0$ then the eigenvalues of λ_k of A_N are:

(A) $N = 2m$ where $m \in \mathbb{N}$.

$$\lambda_k = \begin{cases} \dfrac{(b_1+b_2) - \sqrt{(b_1-b_2)^2 + 16d^2cos^2\theta_k}}{2}, & k = 1,2,...,m \\ \dfrac{(b_1+b_2) + \sqrt{(b_1-b_2)^2 + 16d^2cos^2\theta_k}}{2}, & k = m+1,...,2m \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{k\pi}{2m+1}, & k = 1,2,...,m \\ \dfrac{(k-m)\pi}{2m+1}, & k = m+1,...,2m \end{cases}$$

(B) $N = 2m+1$ where $m \in \mathbb{N}$.

$$\lambda_k = \begin{cases} \dfrac{(b_1+b_2) - \sqrt{(b_1-b_2)^2 + 16d^2cos^2\theta_k}}{2}, & k = 1,...,m \\ \dfrac{(b_1+b_2) + \sqrt{(b_1-b_2)^2 + 16d^2cos^2\theta_k}}{2}, & k = m+1,...,2m \\ b_1, & k = N \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{k\pi}{2m+2}, & k = 1,2,...,m \\ \dfrac{(k-m)\pi}{2m+2}, & k = m+1,...,2m \end{cases}$$

<u>Case 2</u> Suppose $\alpha\beta = d^2$ and $N = 2m$ where $m \in \mathbb{N}$, then the eigenvalues of A_N are:

$$\lambda_k = \begin{cases} \dfrac{(b_1+b_2) - \sqrt{(b_1-b_2)^2+16d^2cos^2\theta_k}}{2}, & k = 1,\ldots,m-1 \\ \dfrac{(b_1+b_2) + \sqrt{(b_1-b_2)^2+16d^2cos^2\theta_k}}{2}, & k = m,\ldots,2m-2 \\ \dfrac{(b_1+b_2-\alpha-\beta) - \sqrt{(b_1-b_2)^2+(\alpha+\beta)^2-2(b_1-b_2)(\alpha-\beta)}}{2}, & k = N-1 \\ \dfrac{(b_1+b_2-\alpha-\beta) + \sqrt{(b_1-b_2)^2+(\alpha+\beta)^2-2(b_1-b_2)(\alpha-\beta)}}{2}, & k = N \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{k\pi}{2m}, & k = 1,\ldots,m-1 \\ \dfrac{(k-m+1)\pi}{2m}, & k = m,\ldots,2m-2 \end{cases}$$

Proof: See Theorem 2 and Theorem 3 from [6].

Comment: For $N = 2m+1$ in Case 2, Kouachi does not determine any eigenvalues. We will address this case in Chapter 3 for tridiagonal stochastic matrices.

Theorem 2.1.2. (Eigenvectors) For Case 2 of Theorem 2.1.1, the eigenvectors $\vec{u}^{(k)} = (u_1^{(k)}, \ldots, u_N^{(k)})^T$ of the tridiagonal matrix A_N in (2.1) may be expressed as:

For $k = 1, 2, \ldots, N-2$

$$u_1^{(k)} = (-d)^{N-1} \; (b_2 - \lambda_k - \beta)\sin \mathrm{N}\theta_k - \beta \sin(\mathrm{N\text{-}2})\theta_k$$

For $k = 1, 2, \ldots, N-2$ and $j = 2, 3, \ldots, N$.

$$u_j^{(k)} = \rho_j = (-d)^{N-j} a_1 \ldots a_{j-1} \begin{cases} (b_2 - \lambda_k - \beta)\sin(N-j+1)\theta_k - \beta \sin(N-j-1)\theta_k \\ \text{for } j \text{ odd,} \\ d\sin(N-j+2)\theta_k - \quad \dfrac{\beta(b_1-\lambda_k)}{d} - d \quad \sin(N-j)\theta_k \\ \text{for } j \text{ even} \end{cases}$$

For $k = N-1, N$

$$u_2^{(k)} = \frac{\lambda_k - b_1 + \alpha}{c_1} u_1^{(k)} \tag{2.2}$$

$$u_{i+2}^{(k)} = \frac{(\lambda_k - b_{i+1}) u_{i+1}^{(k)} - a_i u_i^{(k)}}{c_{i+1}} \tag{2.3}$$

where $u_1^{(k)} = 1$ and $i = 1, 2, ..., N-2$.

Proof: See [6], Theorem 4. Now we will show the proof for the formula (2.2) and (2.3) by using linear algebra.

To find the associated eigenvectors $u^{(N-1)}$ and $u^{(N)}$, we will solve the equation

$$A_N \cdot \vec{u}^{(k)} = \lambda_i \cdot \vec{u}^{(k)} \text{ where } k = N-1, N.$$

Thus we have

$$\begin{pmatrix} -\alpha + b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_1 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_2 & b_3 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{N-2} & b_{N-1} & c_{N-1} \\ 0 & \cdots & \cdots & 0 & a_{N-1} & -\beta + b_N \end{pmatrix} \begin{pmatrix} u_1^{(k)} \\ u_2^{(k)} \\ u_3^{(k)} \\ u_4^{(k)} \\ \vdots \\ u_{N-1}^{(k)} \\ u_N^{(k)} \end{pmatrix} = \lambda_k \begin{pmatrix} u_1^{(k)} \\ u_2^{(k)} \\ u_3^{(k)} \\ u_4^{(k)} \\ \vdots \\ u_{N-1}^{(k)} \\ u_N^{(k)} \end{pmatrix}$$

So from the top row of the equation we have

$$(-\alpha + b_1) u_1^{(k)} + c_1 u_2^{(k)} = \lambda_k u_1^{(k)}.$$

Then $c_1 u_2^{(k)} = \lambda_k u_1^{(k)} + (-b_1 + \alpha) u_1^{(k)}$.

Hence $u_2^{(k)} = \dfrac{(\lambda_k - b_1 + \alpha) u_1^{(k)}}{c_1}$, which is (2.2).

Now from the second row of the equation, we get

$$a_1 u_1^{(k)} + b_2 u_2^{(k)} + c_2 u_3^{(k)} = \lambda_k u_2^{(k)}.$$

Hence $u_3^{(k)} = \dfrac{\lambda_k u_2^{(k)} - a_1 u_1^{(k)} - b_2 u_2^{(k)}}{c_2} = \dfrac{(\lambda_k - b_2)u_2^{(k)} - a_1 u_1^{(k)}}{c_2}$.

Thus we get $u_{i+2}^{(k)} = \dfrac{(\lambda_k - b_{i+1})u_{i+1}^{(k)} - a_i u_i^{(k)}}{c_{i+1}}$ for $i = 1, 2, ..., N-2$, which is (2.3). ■

Note: Theorem 2.1.2 appears as Theorem 4 in [6]. Unfortunately, Kouachi's statement does not appear to include eigenvectors $\vec{u}^{N-1}$, $\vec{u}^{N}$. The preceding linear algebraic argument along with Kouachi's argument in Theorem 4 of [6] establishes the complete result.

2.2 Constant diagonal terms

Professor Kouachi's earlier article 2006 [5] addresses tridiagonal matrices with mainly constant diagonal terms.

Consider tridiagonal matrices of the form:

$$A_N = \begin{pmatrix} -\alpha+b & c_1 & 0 & 0 & \cdots & 0 \\ a_1 & b & c_2 & 0 & \cdots & 0 \\ 0 & a_2 & b & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & \cdots & 0 & a_{n-1} & -\beta+b \end{pmatrix}$$

where a_j and c_j, $j = 1, ..., N-1$, and α and β are complex numbers. We suppose that

$$a_j c_j = \begin{cases} d_1^2, & \text{if } j \text{ is odd} \\ d_2^2, & \text{if } j \text{ is even} \end{cases} \tag{2.4}$$

for $j = 1, 2, 3, ...$ and $d_1 d_2 = 0$ where d_1 and d_2 are complex numbers.

The characteristic polynomial of A_N is denoted as $\triangle_n$. Let

$$Y^2 = d_1^2 + d_2^2 + 2d_1d_2\cos\theta,$$
$$\text{where } Y = b - \lambda.$$

Theorem 2.1.3. (The characteristic determinants) Assuming (2.4), the eigenvalues of the class of matrices A_N are independent of the entries $(a_i, c_i, i = 1, ..., N-1)$ and their characteristic determinants are given by:

$$\triangle_N = (d_1d_2)^{m-1}\frac{d_1d_2(Y-\alpha-\beta)\sin(m+1)\theta + (\alpha\beta Y - \alpha d_1^2 - \beta d_2^2)\sin m\theta}{\sin\theta} \tag{2.5}$$

when $N = 2m+1$ and

$$\triangle_N = (d_1d_2)^{m-1}\frac{d_1d_2\sin(m+1)\theta + [\alpha\beta + d_2^2 - (\alpha+\beta)Y]\sin m\theta + \alpha\beta\frac{d_1}{d_2}\sin(m-1)\theta}{sin\theta} \tag{2.6}$$

when $N = 2m$.

Proof: See Kouachi's 2006 article [5].

Now when $\alpha = \beta = 0$, the formula (2.5) and (2.6) become, respectively

$$\triangle_N = (d_1d_2)^m Y\frac{\sin(m+1)\theta}{sin\theta} \tag{2.7}$$

when $N = 2m+1$ and

$$\triangle_N = (d_1d_2)^m\frac{\sin(m+1)\theta + \dfrac{d_2}{d_1}\sin m\theta}{sin\theta} \tag{2.8}$$

when $N = 2m$.

Chapter 3

Applications to birth-death chains and processes

In this chapter, Kouachi's eigenvalues and eigenvectors results in Chapter 2 are applied to certain birth-death chains and processes.

3.1 Application to birth-death chains

Theorem 3.1.1. Suppose we have a birth-death chain having one-step transition probabilities as shown in Figure 3.1 with the assumptions of (1.5).

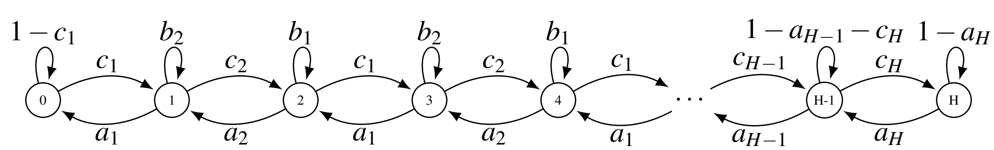


Figure 3.1

The P matrix is:

$$P = \begin{pmatrix} 1-c_1 & c_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_1 & b_2 & c_2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & a_2 & b_1 & c_1 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & a_1 & b_2 & c_2 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & a_{H-1} & 1-a_{H-1}-c_H & c_H \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & a_H & 1-a_H \end{pmatrix}$$

<u>Case 1</u> Assume $a_1c_2 = a_2c_1 = d^2$ where $0 < d < 1$, $a_1 + c_1 \leq 1$ and $a_2 + c_2 \leq 1$.

Then the eigenvalues are given below:

(A) If H is odd, $c_H = c_1,\ a_H = a_1,\ a_{H-1} = a_2$:

$$\lambda_k = \begin{cases} \dfrac{(2-a_1-a_2-c_1-c_2) - \sqrt{(a_2-a_1+c_2-c_1)^2 + 16a_1c_2cos^2\theta_k}}{2} \\ \text{for } k = 0,1,\ldots,\dfrac{H-3}{2} \\ \dfrac{(2-a_1-a_2-c_1-c_2) + \sqrt{(a_2-a_1+c_2-c_1)^2 + 16a_1c_2cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H-1}{2},\ldots,H-2 \\ 1-a_1-c_1 \quad \text{for } k = H-1 \\ 1 \qquad\qquad \text{for } k = H \end{cases}$$

$$\text{where} \quad \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+3}, & k = 0,1,\ldots,\dfrac{H-3}{2} \\ \dfrac{(2k+1-H)\pi}{2(H+3)}, & k = \dfrac{H-1}{2},\ldots,H-2 \end{cases}$$

(B) For H even, $c_H = c_2,\ a_H = a_2,\ a_{H-1} = a_1$:

$$\lambda_k = \begin{cases} \dfrac{(2-a_1-a_2-c_1-c_2) - \sqrt{(a_2-a_1+c_2-c_1)^2+16a_1c_2cos^2\theta_k}}{2} \\ \text{for } k = 0,1,\ldots,\dfrac{H}{2}-1 \\ \dfrac{(2-a_1-a_2-c_1-c_2) + \sqrt{(a_2-a_1+c_2-c_1)^2+16a_1c_2cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H}{2},\ldots,H-1 \\ 1 \qquad \text{for } k = H \end{cases}$$

$$\text{where} \quad \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+1}, & k = 0,1,\ldots,\dfrac{H}{2}-1 \\ \dfrac{(2k+2-H)\pi}{2(H+1)}, & k = \dfrac{H}{2},\ldots,H-1 \end{cases}$$

<u>Case 2</u> Assume H is odd then $c_H = c_1, a_H = a_1$ and $a_{H-1} = a_2$. Also assume $a_1c_1 = a_2c_2 = d^2$ where $0 < d < 1$. Then the eigenvalues of P are given below:

$$\lambda_k = \begin{cases} \dfrac{(2-a_1-a_2-c_1-c_2) - \sqrt{(a_1-a_2+c_2-c_1)^2+16a_2c_2cos^2\theta_k}}{2} \\ \text{for } k = 0,1,\ldots,\dfrac{H-3}{2} \\ \dfrac{(2-a_1-a_2-c_1-c_2) + \sqrt{(a_1-a_2+c_2-c_1)^2+16a_2c_2cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H-1}{2},\ldots,H-2 \end{cases}$$

$$\text{where} \quad \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+1}, & k = 0,1,\ldots,\dfrac{H-3}{2} \\ \dfrac{(2k+3-H)\pi}{2(H+1)}, & k = \dfrac{H-1}{2},\ldots,H-2 \end{cases}$$

$$\lambda_k = \frac{-\sqrt{(a_1-a_2+c_2-c_1)^2+(-a_2-c_2)^2-2(a_1-a_2+c_2-c_1)(c_2-a_2)}}{2} + \frac{2-a_1-c_1}{2} \text{ where } k = H-1.$$

$$\lambda_k = \frac{\sqrt{(a_1-a_2+c_2-c_1)^2+(-a_2-c_2)^2-2(a_1-a_2+c_2-c_1)(c_2-a_2)}}{2} + \frac{2-a_1-c_1}{2} \text{ where } k = H.$$

Proof: Assume we have a birth-death chain as in Figure 3.1.

<u>**Case 1**</u>

We assume that $a_1+c_1 \leq 1$ and $a_2+c_2 \leq 1$ to insure that the dual birth-death chain is well-defined. Then the dual birth-death chain is described as in Figure 3.2:

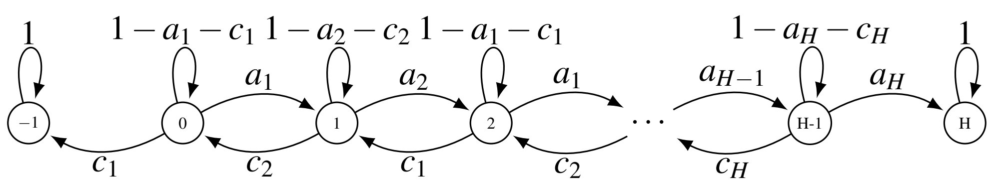


Figure 3.2

The corresponding matrix P^* is

$$P^* = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
c_1 & 1-a_1-c_1 & a_1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & c_2 & 1-a_2-c_2 & a_2 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & c_1 & 1-a_1-c_1 & a_1 & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & c_H & 1-a_H-c_H & a_H \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}$$

Let $a_1c_2 = a_2c_1 = d^2$ where $0 < d < 1$. Then the dual birth-death chains has known eigenvalues given by Theorem 2.1.1 Case 1. Notice that $1 - a_H - c_H$ can either be $1 - a_1 - c_1$ or $1 - a_2 - c_2$ since we can apply Theorem 2.1.1 Case 1 for matrices of odd or even dimension.

By the Duality Theorem (1.6), the original birth-death chain has the same eigenvalues as

$$\begin{pmatrix} 1-a_1-c_1 & a_1 & 0 & \cdots\cdots\cdots & 0 & 0 \\ c_2 & 1-a_2-c_2 & a_2 & \cdots\cdots\cdots & \vdots & \vdots \\ 0 & c_1 & 1-a_1-c_1 & a_1 \ddots \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots\ddots\ddots & 0 & a_{H-1} \\ \vdots & \ddots & \ddots & \ddots\ddots 0 & c_H & 1-a_H-c_H \end{pmatrix} \text{ and } \lambda = 1.$$

This establishes Case 1.

<u>Case 2</u> Now assume $a_1c_1 = a_2c_2 = d^2$ in matrix P where $0 < d < 1$. Since H is odd, we have an even numbers of states.

Since $b_1 - \alpha = 1 - c_1$, $\alpha = b_1 + c_1 - 1$. Similarly, since $b_2 - \beta = 1 - a_1$, we have $\beta = a_1 + b_2 - 1$.

But $a_2 + b_1 + c_1 = 1$ so $b_1 = 1 - a_2 - c_1$. Hence $\alpha = b_1 + c_1 - 1 = 1 - a_2 - c_1 + c_1 - 1 = -a_2$.

Similarly, $a_1 + b_2 + c_2 = 1$ so $\beta = a_1 + b_2 - 1 = a_1 + 1 - a_1 - c_2 - 1 = -c_2$.

So $\alpha\beta = (-a_2)(-c_2) = a_2c_2 = d^2$. Hence this Markov chain satisfies Kouachi's condition for α and β in Professor Kouachi's Theorem 2.1.1 Case 2 which gives us the eigenvalues and eigenvectors of this chain. This completes the proof.

In general, Theorem 3.1.1 depends upon probabilities a_1, a_2, c_1 and c_2. We now consider some simple examples.

Corollary 3.1.2. Consider a birth-death chain having the following state transition diagram:

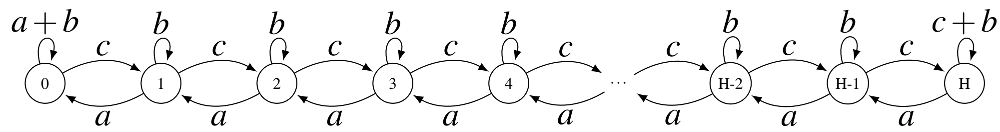


Figure 3.3

We assume that $a+b+c=1$ and $0<a,b,c<1$. Suppose P is the one-step transition matrix corresponding to Figure 3.3. Then the eigenvalues of P are:

<u>**Case 1**</u> H even

$\lambda_k = b - 2\sqrt{ac}cos\theta_k$, $k = 0,1,2,\ldots,\dfrac{H}{2}-1$

$\lambda_k = b + 2\sqrt{ac}cos\theta_k$, k = $\dfrac{H}{2},\ldots,H-1$

$\lambda_H = 1$

where

$$\theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+1}, & k = 0,1,2,\ldots,\dfrac{H}{2}-1 \\ \dfrac{(2k+2-H)\pi}{2(H+1)}, & k = \dfrac{H}{2},\ldots,H-1 \end{cases}$$

<u>**Case 2**</u> H odd

$\lambda_k = b - 2\sqrt{ac}cos\theta_k$, $k = 0,1,2,\ldots,\dfrac{H-3}{2}$

$\lambda_k = b + 2\sqrt{ac}cos\theta_k$, k = $\dfrac{H-1}{2},\ldots,H-2$

$\lambda_{H-1} = b$

$\lambda_H = 1$

where

$$\theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+1}, & k = 0,1,2,...,\dfrac{H-3}{2} \\ \dfrac{(2k+2-H)\pi}{2(H+1)}, & k = \dfrac{H-1}{2},...,H-2 \end{cases}$$

Proof: For H even, Theorem 3.1.1 Case 1 gives the result.

For H odd, Theorem 3.1.1 Case 1 or 2 gives the result.

Remark 1. For H odd, we also know the eigenvectors by Theorem 2.1.2.

For $k = 0,1,2,3,...,H-1$.

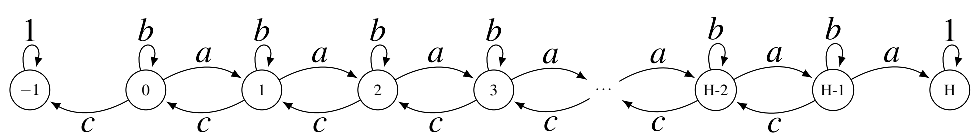
$u_1^{(k)} = a^{\,H/2} c^{\,(H+2)/2}\{a\sin[(H+2)\theta_k] + (1-c-\lambda_k)\sin[(H)\theta_k]\}$.

$u_j^{(k)} = a^{(H+j-1)/2} c^{(H-j+3)/2}\{a\sin[(H-j+3)\theta_k] + (1-c-\lambda_k)\sin[(H-j+1)\theta_k]\}$ for j odd.

$u_j^{(k)} = -a^{(H+j)/2} c^{(H-j+2)/2}\{(1-a-\lambda_k)\sin[(H-j+2)\theta_k] + c\sin[(H-j)\theta_k]\}$ for j even.

For $k = H$, we have $u^{(H)} = (1,1,...,1)$.

Remark 2. For H odd, the Duality Theorem (1.6) gives finite ruin probability for the following diagram:



We next consider the following birth-death chain having alternating probabilities where $0 < a_1 < \frac{1}{2},\ 0 < a_2 < \frac{1}{2},\ a_1 = a_2,\ a_1 a_2 = d^2$ and H odd:

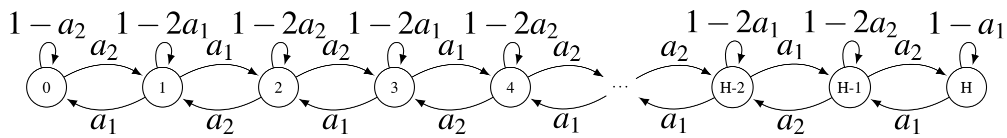


Figure 3.4

This birth-death chain satisfies Theorem 3.1.1 Case 2 and thus its eigenvalues can be exactly determined. See Appendix B for matrix P^n of this Markov chain when $H = 3$. We look at a couple of examples of this type of birth-death chain.

Example 3.1. Consider the following birth-death Markov chain:

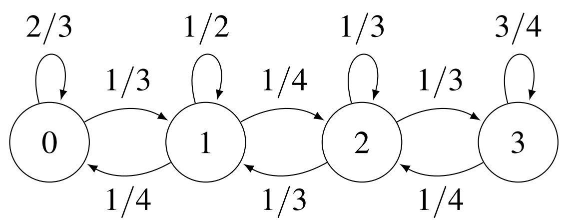


Figure 3.5

Notice that this Markov chain has $H = 3$ and $a_1 c_1 = a_2 c_2 = \frac{1}{12}$. So it follows the conditions of Theorem 3.1.1 Case 2. Therefore, we can compute the eigenvalues of this Markov chain:

$$\lambda_0 = 0, \quad \lambda_2 = \frac{5}{6}, \quad \lambda_3 = \frac{5}{12}, \quad \lambda_4 = 1.$$

So we have:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & 0 & 0 \\ 0 & 0 & \frac{5}{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} -1 & -1 & \frac{16}{9} & 1 \\ 2 & \frac{-1}{2} & \frac{-4}{3} & 1 \\ -3 & \frac{1}{3} & \frac{-4}{3} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$P^n = SD^nS^{-1} =$$

$$\begin{bmatrix} \frac{64\left(\frac{5}{12}\right)^n}{175} + \frac{9\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & -\frac{64\left(\frac{5}{12}\right)^n}{175} + \frac{6\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & -\frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & \frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{12\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} \\ -\frac{48\left(\frac{5}{12}\right)^n}{175} + \frac{9\left(\frac{5}{6}\right)^n}{50} + \frac{3}{14} & \frac{48\left(\frac{5}{12}\right)^n}{175} + \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & \frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{3\left(\frac{5}{6}\right)^n}{50} + \frac{3}{14} & -\frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{6\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} \\ -\frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & \frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{2\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & \frac{36\left(\frac{5}{12}\right)^n}{175} + \frac{\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & -\frac{36\left(\frac{5}{12}\right)^n}{175} + \frac{4\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} \\ \frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{9\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & -\frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{6\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & -\frac{27\left(\frac{5}{12}\right)^n}{175} + \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & \frac{27\left(\frac{5}{12}\right)^n}{175} + \frac{12\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} \end{bmatrix}$$

This matrix expression was obtained using a computer program written by Dave Luk, see Appendix C for code.

Example 3.2. Now we would like to see the changes in P^n when we switch probabilities a_1 and a_2 in Figure 3.5. So the Markov chain will become:

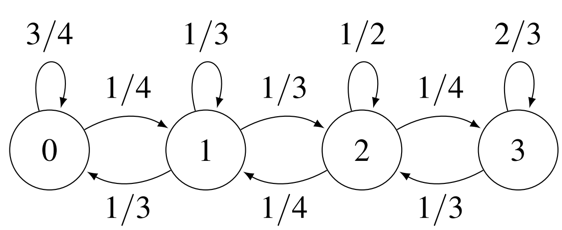


Figure 3.6

Even though we get the same eigenvalues for the birth-death chains in Figure 3.5 and 3.6, the eigenvectors are different since P matrix is changing. Thus we have the following matrices:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & 0 & 0 \\ 0 & 0 & \frac{5}{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} -1 & -1 & \frac{9}{16} & 1 \\ 3 & \frac{-1}{3} & \frac{-3}{4} & 1 \\ -2 & \frac{1}{2} & \frac{-3}{4} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

As before, Dave Luk's program gives us:

$$P^n = SD^nS^{-1} =$$

$$\begin{bmatrix} \frac{27\left(\frac{5}{12}\right)^n}{175} + \frac{12\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & -\frac{27\left(\frac{5}{12}\right)^n}{175} + \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & -\frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{6\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & \frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{9\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} \\ -\frac{36\left(\frac{5}{12}\right)^n}{175} + \frac{4\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & \frac{36\left(\frac{5}{12}\right)^n}{175} + \frac{\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & \frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{2\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & -\frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} \\ -\frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{6\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & \frac{36\left(\frac{5}{12}\right)^n}{175} - \frac{3\left(\frac{5}{6}\right)^n}{50} + \frac{3}{14} & \frac{48\left(\frac{5}{12}\right)^n}{175} + \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & -\frac{48\left(\frac{5}{12}\right)^n}{175} + \frac{9\left(\frac{5}{6}\right)^n}{50} + \frac{3}{14} \\ \frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{12\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & -\frac{48\left(\frac{5}{12}\right)^n}{175} - \frac{3\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} & -\frac{64\left(\frac{5}{12}\right)^n}{175} + \frac{6\left(\frac{5}{6}\right)^n}{25} + \frac{2}{7} & \frac{64\left(\frac{5}{12}\right)^n}{175} + \frac{9\left(\frac{5}{6}\right)^n}{25} + \frac{3}{14} \end{bmatrix}$$

Remark. The preceding results work when H is odd in the Figure 3.4 using Theorem 3.1.1 Case 2.

We next consider a similar state transition diagram to Figure 3.4 but when H is even. An example for $H = 4$ is shown below in Figure 3.7:

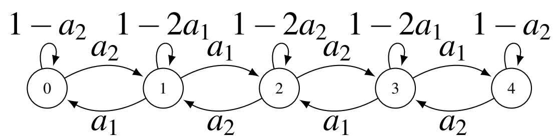


Figure 3.7

where we assume $0 \leq a_1, a_2 \leq \frac{1}{2}$ and $a_1 = a_2$. Then the corresponding P-matrix is:

$$P = \begin{pmatrix} 1-a_2 & a_2 & 0 & 0 & 0 \\ a_1 & 1-2a_1 & a_1 & 0 & 0 \\ 0 & a_2 & 1-2a_2 & a_2 & 0 \\ 0 & 0 & a_1 & 1-2a_1 & a_1 \\ 0 & 0 & 0 & a_2 & 1-a_2 \end{pmatrix}$$

Unfortunately, this birth-death chain does not satisfy Theorem 3.1.1 (nor Kouachi's results).

Now consider the dual of P:

$$P^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 1-a_1-a_2 & a_1 & 0 & 0 & 0 \\ 0 & a_1 & 1-a_1-a_2 & a_2 & 0 & 0 \\ 0 & 0 & a_2 & 1-a_1-a_2 & a_1 & 0 \\ 0 & 0 & 0 & a_1 & 1-a_1-a_2 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where P^*_{inside}= $\begin{pmatrix} 1-a_1-a_2 & a_1 & 0 & 0 \\ a_1 & 1-a_1-a_2 & a_2 & 0 \\ 0 & a_2 & 1-a_1-a_2 & a_1 \\ 0 & 0 & a_1 & 1-a_1-a_2 \end{pmatrix}$

Again, this matrix does not satisfy our previous theorems in Chapter 3. However it does satisfy Theorem 2.1.3. To illustrate our approach, consider the birth-death chain with 5 states having transition probabilities diagrammed in Figure 3.8 and its dual in Figure 3.9.

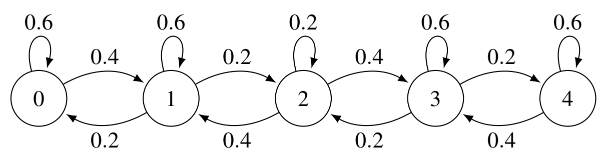


Figure 3.8

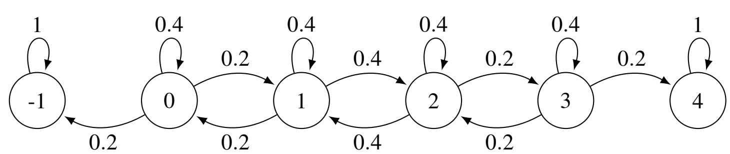


Figure 3.9

The corresponding matrix for P^* is:

$$P^*=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .4 & .4 & .2 & 0 & 0 & 0 \\ 0 & .2 & .4 & .4 & 0 & 0 \\ 0 & 0 & .4 & .4 & .2 & 0 \\ 0 & 0 & 0 & .2 & .4 & .4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } P^*_{inside}=\begin{pmatrix} .4 & .2 & 0 & 0 \\ .2 & .4 & .4 & 0 \\ 0 & .4 & .4 & .2 \\ 0 & 0 & .2 & .4 \end{pmatrix}$$

We know by the Duality Theorem (1.6) that the dual matrix P^* has the same eigenvalues as the original matrix P with the only difference of having a repeated eigenvalue (or a double root) of $\lambda_5 = \lambda_6 = 1$. However, as we can observe the 4×4 matrix inside of the dual P^* is a tridiagonal matrix satisfying condition of Theorem 2.1.3, so we can retrieve the other eigenvalues from (2.8) which is very exciting since we can get the eigenvalues of the original P matrix which itself does not satisfy the theorem's assumptions but the dual of the matrix does.

Since $a_1c_1 = (.2)^2 = d_1^2$ and $a_2c_2 = (.4)^2 = d_2^2$, we have $d_1 = .2$ and $d_2 = .4$.

Also we have $H = 2m = 4$ so $m = 2$, $\alpha = \beta = 0$ and $b = 0.4$.

In order to find eigenvalues we need to determine the angle θ by setting $\triangle_N = 0$ in formula 2.8 which gives us

$$\sin(m+1)\theta + \frac{d_2}{d_1}\sin m\theta = 0 \tag{3.1}$$

Since $m = 2$ and $\dfrac{d_2}{d_1} = 2$, we get $\sin(3\theta) + 2\sin(2\theta) = 0$.

Ali Oudich used trigonometry formula to solve for the equation as shown below:

$$\sin 3\theta + 2\sin 2\theta = 0.$$

$$\sin 2\theta \cos\theta + \cos 2\theta \sin\theta + 2\sin 2\theta = 0.$$

$$2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta + 4\cos\theta\sin\theta = 0.$$

$$2\cos^2\theta + 2\cos^2\theta - 1 + 4\cos\theta = 0.$$

$$4\cos^2\theta + 4\cos\theta - 1 = 0.$$

Using the quadratic formula, we get $\cos\theta = \dfrac{-1 \pm \sqrt{2}}{2}$.

Once we know $\cos\theta$, we can substitute into $Y^2 = d_1^2 + d_2^2 + 2d_1d_2\cos\theta$ and use $Y = b - \lambda$ to find eigenvalues.

So the eigenvalue for this P^*_{inside} matrix are $-0.0828,\ 0.31716,\ 0.48284,\ 0.882843$.

Thus the eigenvalue for P-matrix corresponding to Figure 3.8 are:

$$-0.0828,\ 0.31716,\ 0.48284,\ 0.882843 \text{ and } 1.$$

We can also determine the eigenvalues by different approach using Chebyshev polynomials.

From [8], we have formula for $\sin(nx)$ as follow:

$$\sin 1x = (\sin x)(1)$$

$$\sin 2x = (\sin x)(2\cos x)$$

$$\sin 3x = (\sin x)(-1 + 4\cos^2 x)$$

$$\sin 4x = (\sin x)(-4\cos x + 8\cos^3 x)$$

$$\sin 5x = (\sin x)(1 - 12\cos^2 x + 16\cos^4 x)$$

$$\sin 6x = (\sin x)(6\cos x - 32\cos^3 x + 32\cos^5 x)$$

$$\sin 7x = (\sin x)(-1 + 24\cos^2 x - 80\cos^4 x + 64\cos^6 x)$$

$$\sin 8x = (\sin x)(-8\cos x + 80\cos^3 x - 192\cos^5 x + 128\cos^7 x)$$

$$\sin 9x = (\sin x)(1 - 40\cos^2 x + 240\cos^4 x - 448\cos^6 x + 256\cos^8 x)$$

$$\sin 10x = (\sin x)(10\cos x - 160\cos^3 x + 672\cos^5 x - 1024\cos^7 x + 512\cos^9 x)$$

Using these Chebyshev polynomials we have:

$$\sin 3\theta + 2\sin 2\theta = 0.$$
$$(\sin\theta)(-1+4\cos^2\theta) + 2(\sin\theta)(2\cos\theta) = 0$$
$$\sin\theta(4\cos^2\theta + 4\cos\theta - 1) = 0$$
$$4\cos^2\theta + 4\cos\theta - 1 = 0,$$

which will give the same result for $\cos\theta$ as before.

Now if we have $H = 6$ which is corresponding to $m = 3$, then using (3.1) we have:

$$\sin(4\theta) + 2\sin(3\theta) = 0.$$
$$(\sin\theta)(-4\cos\theta + 8\cos^3\theta) + 2(\sin\theta)(-1+4\cos^2\theta) = 0$$
$$8\cos^3\theta + 8\cos^2\theta - 4\cos\theta - 2 = 0$$

Using Wolfram we get value of $\cos\theta$ as complex numbers.

If $H = 8$ and $m = 4$, we have $\sin(5\theta) + 2\sin(4\theta) = 0$.

Thus $16\cos^4\theta + 16\cos^3\theta - 12\cos^2\theta - 8\cos\theta + 1 = 0$.

Then we have exact value for $\cos\theta$ in this case, which are:

$\cos\theta_1 = \frac{1}{4}(-1 - \sqrt{9-4\sqrt{3}})$

$\cos\theta_2 = \frac{1}{4}(-1 + \sqrt{9-4\sqrt{3}})$

$\cos\theta_3 = \frac{1}{4}(-1 - \sqrt{9+4\sqrt{3}})$

$\cos\theta_4 = \frac{1}{4}(-1 + \sqrt{9+4\sqrt{3}})$

Knowing $\cos\theta$, we can determine the eigenvalues when $H = 8$.

Now when m=7, we have $\sin(8\theta) + 2\sin(7\theta) = 0$.

Wolfram give us the following values of $\cos\theta$:

$\cos\theta_1 = -\frac{1}{3} - \frac{5(1+i\sqrt{3})}{6\sqrt[3]{1+3i\sqrt{111}}} - \frac{1}{12}(1-i\sqrt{3})\sqrt[3]{1+3i\sqrt{111}}$

$$\cos\theta_2 = -\frac{1}{3} - \frac{5(1-i\sqrt{3})}{6\sqrt[3]{1+3i\sqrt{111}}} - \frac{1}{12}(1+i\sqrt{3})\sqrt[3]{1+3i\sqrt{111}}$$

$$\cos\theta_3 = -\frac{1}{3} + \frac{1}{6}\left(\frac{10}{\sqrt[3]{1+3i\sqrt{111}}} + \sqrt[3]{1+3i\sqrt{111}}\right)$$

which again are complex numbers.

Remark We can determine the exact eigenvalue when $m = 2, 4$ corresponding to $H = 4, 8$ respectively.

Our next result is a generalization and hybrid of the two cases given in Theorem 3.1.1 (when H is even).

Theorem 3.1.3. Consider the following birth-death chain satisfying (1.5) when H is even.

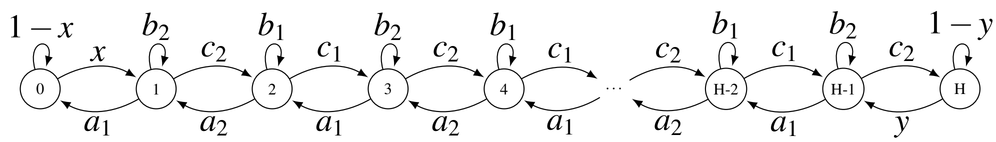


Figure 3.10

Assume $a_1c_2 = a_2c_1$ and $a_1 + x \leq 1$, $a_2 + c_2 \leq 1$, $a_1 + c_1 \leq 1$ and $c_2 + y \leq 1$. Also assume that $\frac{c_1}{1-a_2} \leqslant x \leqslant 1$ and $y = \frac{a_2 x}{x - c_1}$ then the eigenvalues of the P matrix corresponding to Figure 3.10 are:

$$\lambda_k = \begin{cases} \dfrac{(2-a_1-a_2-c_1-c_2) - \sqrt{(a_2-a_1+c_2-c_1)^2 + 16a_2c_1cos^2\theta_k}}{2} \\ \text{for } k = 0, 1, ..., \dfrac{H-4}{2} \\ \dfrac{(2-a_1-a_2-c_1-c_2) + \sqrt{(a_2-a_1+c_2-c_1)^2 + 16a_2c_1cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H-2}{2}, ..., H-3 \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H}, & k=0,1,...,\dfrac{H-4}{2} \\ \dfrac{(2k+4-H)\pi}{2H}, & k=\dfrac{H-2}{2},...,H-3 \end{cases}$$

$$\lambda_{H-2} = \frac{(2-a_1-a_2-c_1-c_2)-\sqrt{(a_2-a_1+c_2-c_1)^2+(x-c_1+y-a_2)^2-2(a_2-a_1+c_2-c_1)(x-y-c_1+a_2)}}{2}$$

$$\lambda_{H-1} = \frac{(2-a_1-a_2-c_1-c_2)+\sqrt{(a_2-a_1+c_2-c_1)^2+(x-c_1+y-a_2)^2-2(a_2-a_1+c_2-c_1)(x-y-c_1+a_2)}}{2}$$

$$\lambda_H = 1.$$

Proof: The dual birth-death chain exist and looks like:

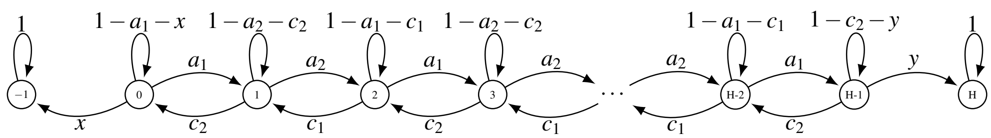


Figure 3.11

The corresponding matrix P^* is:

$$P^* = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots\cdots\cdots & 0 & 0 & 0 \\ x & 1-a_1-x & a_1 & 0 & \cdots\cdots\cdots & 0 & 0 & 0 \\ 0 & c_2 & 1-a_2-c_2 & a_2 & \cdots\cdots\cdots & \vdots & \vdots & \vdots \\ 0 & 0 & c_1 & 1-a_1-c_1 & a_1 \ddots \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots\ddots\ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots\ddots\ 0 & c_2 & 1-c_2-y & y \\ 0 & 0 & \cdots & \cdots & \cdots\cdots\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we proceed to find the eigenvalues of the matrix inside P^*:

$$\begin{pmatrix} 1-a_1-x & a_1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ c_2 & 1-a_2-c_2 & a_2 & \cdots & \cdots & \cdots & \vdots & \vdots \\ 0 & c_1 & 1-a_1-c_1 & a_1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & c_2 & 1-c_2-y \end{pmatrix}.$$

Since $1-a_1-c_1-\alpha = 1-a_1-x,$ therefore $\alpha = x-c_1$.

Also since $1-a_2-c_2-\beta = 1-c_2-y,$ we obtain $\beta = y-a_2$.

$$\begin{aligned} \text{Hence } \alpha\beta &= (x-c_1)(y-a_2) \\ &= (x-c_1)y - a_2x + a_2c_1 \\ &= (x-c_1)\frac{a_2x}{x-c_1} - a_2x + a_2c_1 \\ &= a_2c_1. \end{aligned}$$

Thus the conditions of Theorem 2.1.1 Case 2 are met and therefore, the eigenvalues are known.

$$\lambda_k = \begin{cases} \dfrac{(2-a_1-a_2-c_1-c_2) - \sqrt{(a_2-a_1+c_2-c_1)^2+16a_2c_1cos^2\theta_k}}{2} \\ \text{for } k = 0,1,\ldots,\dfrac{H-4}{2} \\ \dfrac{(2-a_1-a_2-c_1-c_2) + \sqrt{(a_2-a_1+c_2-c_1)^2+16a_2c_1cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H-2}{2},\ldots,H-3 \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H}, & k = 0,1,\ldots,\dfrac{H-4}{2} \\ \dfrac{(2k+4-H)\pi}{2H}, & k = \dfrac{H-2}{2},\ldots,H-3 \end{cases}$$

$$\lambda_{H-2} = \frac{(2-a_1-a_2-c_1-c_2)-\sqrt{(a_2-a_1+c_2-c_1)^2+(x-c_1+y-a_2)^2-2(a_2-a_1+c_2-c_1)(x-y-c_1+a_2)}}{2}$$

$$\lambda_{H-1} = \frac{(2-a_1-a_2-c_1-c_2)+\sqrt{(a_2-a_1+c_2-c_1)^2+(x-c_1+y-a_2)^2-2(a_2-a_1+c_2-c_1)(x-y-c_1+a_2)}}{2}$$

$\lambda_H = 1.$

Interpretation: For probability x where $\dfrac{c_1}{1-a_2} \leq x \leq 1$, the following graph shows probability y is on a segment of a hyperbola.

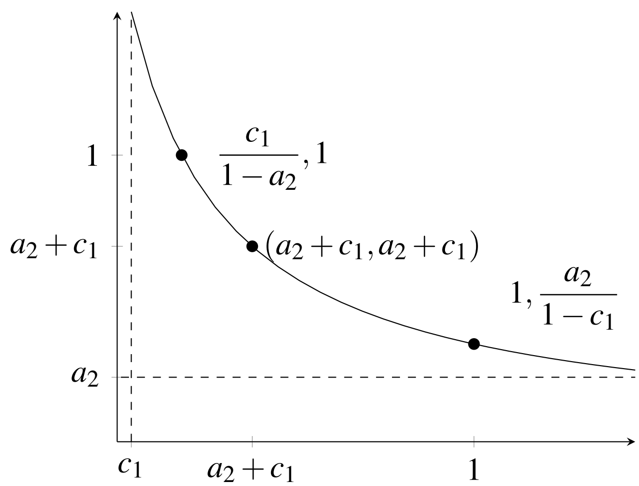


Figure 3.12

The following Figure 3.13 is an example of Theorem 3.1.3.

Example 3.3. Consider the following birth-death chain when $H = 4$:

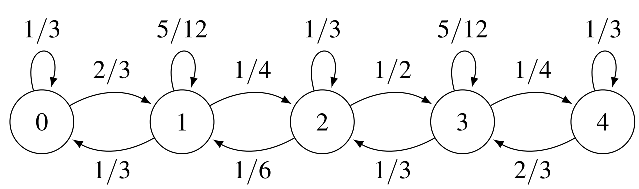


Figure 3.13

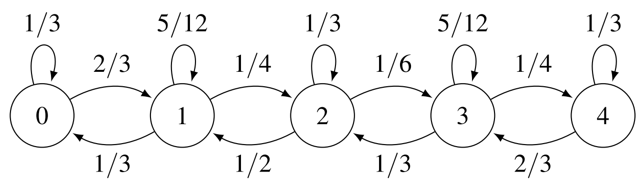
Notice that $\dfrac{1}{3} \cdot \dfrac{1}{4} = \dfrac{1}{6} \cdot \dfrac{1}{2}$, which means we have $a_1 c_2 = a_2 c_1$.

Also since $x = \dfrac{2}{3}$, we get $y = \dfrac{a_2 x}{x - c_1} = \dfrac{2}{3}$.

Hence this chain satisfies conditions of Theorem 3.1.3, so the eigenvalues are known and given below:

$$\lambda_0 = \frac{-1}{12},\ \lambda_1 = \frac{5}{6},\ \lambda_2 = \frac{-1}{4},\ \lambda_3 = \frac{1}{3} \text{ and } \lambda_4 = 1.$$

Remark: Note that exchanging the a and c probabilities that emanate from various states in the birth-death state transition diagram in section 3.1 also produce birth-death chains that have eigenvalues given by Kouachi's articles. For example, here is one such birth-death chain from figure 3.13.



3.2 Application to birth-death processes

The theorems for the birth-death chains of section 3.1 translate to corresponding results for birth-death processes.

Theorem 3.2.1. Suppose we have the following state rate transition diagram of a birth-death process:

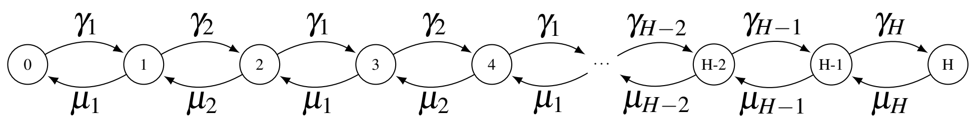


Figure 3.14

<u>**Case 1**</u> Assume $\mu_1\gamma_2 = \mu_2\gamma_1 = d^2$ where $0 < d < 1$. Then the eigenvalues are given below:

(A) For H is odd, $\gamma_H = \gamma_1,\ \mu_H = \mu_1,\ \mu_{H-1} = \mu_2$, then

$$\lambda_k = \begin{cases} \dfrac{-(\gamma_1+\gamma_2+\mu_1+\mu_2) - \sqrt{(\gamma_2-\gamma_1+\mu_2-\mu_1)^2 + 16\gamma_2\mu_1\cos^2\theta_k}}{2} \\ \text{for } k = 0, 1, \ldots, \dfrac{H-3}{2} \\ \dfrac{-(\gamma_1+\gamma_2+\mu_1+\mu_2) + \sqrt{(\gamma_2-\gamma_1+\mu_2-\mu_1)^2 + 16\gamma_2\mu_1\cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H-1}{2}, \ldots, H-2 \\ -(\gamma_1+\mu_1) \text{ for } k = H-1 \\ 0 \qquad \text{for } k = H \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+3}, & k = 0, 1, ..., \dfrac{H-3}{2} \\ \dfrac{(2k+1-H)\pi}{2(H+3)}, & k = \dfrac{H-1}{2}, ..., H-2 \end{cases}$$

(B) For H is even, $\gamma_H = \gamma_2$, $\mu_H = \mu_2$ and $\mu_H - 1 = \mu_1$, then

$$\lambda_k = \begin{cases} \dfrac{-(\gamma_1+\gamma_2+\mu_1+\mu_2) - \sqrt{(\gamma_2-\gamma_1+\mu_2-\mu_1)^2 + 16\gamma_2\mu_1\cos^2\theta_k}}{2} \\ \text{for } k = 0, ..., \dfrac{H}{2} - 1 \\ \dfrac{-(\gamma_1+\gamma_2+\mu_1+\mu_2) + \sqrt{(\gamma_2-\gamma_1+\mu_2-\mu_1)^2 + 16\gamma_2\mu_1\cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H}{2}, .., H-1 \\ 0, \quad k = H \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+1}, & k = 0, 1, ..., \dfrac{H}{2} - 1 \\ \dfrac{(2k+2-H)\pi}{2(H+1)}, & k = \dfrac{H}{2}, .., H-1 \end{cases}$$

<u>**Case 2**</u> Assume $\gamma_1\mu_1 = \gamma_2\mu_2 = d^2$ where $0 < d < 1$.

For H is odd, $\gamma_H = \gamma_1$, $\mu_H = \mu_1$, $\mu_{H-1} = \mu_2$, then the eigenvalues are given below:

$$\lambda_k = \begin{cases} \dfrac{-(\gamma_1+\gamma_2+\mu_1+\mu_2) - \sqrt{(\gamma_2-\gamma_1+\mu_1-\mu_2)^2 + 16\gamma_2\mu_2\cos^2\theta_k}}{2} \\ \text{for } k = 0, ..., \dfrac{H-3}{2} \\ \dfrac{-(\gamma_1+\gamma_2+\mu_1+\mu_2) + \sqrt{(\gamma_2-\gamma_1+\mu_1-\mu_2)^2 + 16\gamma_2\mu_2\cos^2\theta_k}}{2} \\ \text{for } k = \dfrac{H-1}{2}, ..., H-2 \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H+1}, & k = 0,1,\ldots,\dfrac{H-3}{2} \\ \dfrac{(2k+3-H)\pi}{2(H+1)}, & k = \dfrac{H-1}{2},\ldots,H-2 \end{cases}$$

$$\lambda_k = \frac{-\sqrt{(\gamma_2-\gamma_1+\mu_1-\mu_2)^2+(-\gamma_2-\mu_2)^2-2(\gamma_2-\gamma_1+\mu_1-\mu_2)^2)(\gamma_2-\mu_2)}}{2}$$

$$\frac{-(\gamma_1+\mu_1)}{2} \text{ where } k = H-1.$$

$$\lambda_k = \frac{+\sqrt{(\gamma_2-\gamma_1+\mu_1-\mu_2)^2+(-\gamma_2-\mu_2)^2-2(\gamma_2-\gamma_1+\mu_1-\mu_2)^2)(\gamma_2-\mu_2)}}{2}$$

$$\frac{-(\gamma_1+\mu_1)}{2} \text{ where } k = H.$$

Consider the following birth-death process:

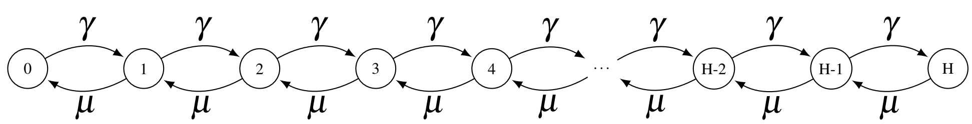


Figure 3.15

The corresponding transition rate matrix Q is:

$$Q = \begin{pmatrix} -\gamma & \gamma & 0 & \cdots & \cdots & 0 \\ \mu & -(\gamma+\mu) & \mu & \cdots & \cdots & 0 \\ 0 & \mu & -(\gamma+\mu) & \gamma & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \mu & -(\gamma+\mu) & \gamma \\ 0 & 0 & \cdots & \cdots & \mu & -\mu \end{pmatrix}$$

If we assume H is odd, then the dimension of Q is even which is $H+1 = 2s$ where s is a natural number.

Since $-\alpha+[-(\gamma+\mu)]=-\gamma,\ \alpha=-\mu$ and since $-\beta+[-(\gamma+\mu)]=-\mu, \beta=-\gamma$. Thus $\alpha\beta=(-\mu)(-\gamma)=\mu\gamma=d^2=a_jc_j$ where $j=1,2,...,N-1$.

This satisfies the condition of Theorem 3.2.1 Case 2. Thus, we obtain the following eigenvalues:

$\lambda_k=-\mu-\gamma-2\sqrt{\mu\gamma}cos\theta_k,\quad k=0,1,2,...,\dfrac{H-3}{2}$.

$\lambda_k=-\mu-\gamma+2\sqrt{\mu\gamma}cos\theta_k,\quad$ k = $\dfrac{H-1}{2},...,H-2$.

$\lambda_{H-1}=-\gamma-\mu$.

$\lambda_H=0$.

where

$$\theta_k=\begin{cases}\dfrac{(k+1)\pi}{2s}, & k=0,1,2,...,\dfrac{H-3}{2}\\ \dfrac{(k-s+2)\pi}{2s}, & k=\dfrac{H-1}{2},...,H-2\end{cases}$$

Using Theorem 2.1.2, we get the associated eigenvectors of Q matrix:

For $k=0,1,2,...,H-2$:

$u_1^{(k)}=-\mu^{\frac{n-1}{2}}\gamma^{\frac{n-1}{2}}\{-(\mu+\lambda_k)\sin(n\theta_k)+\gamma\sin[(n-2)\theta_k]\}$.

$u_j^{(k)}=-\mu^{\frac{n+j-2}{2}}\gamma^{\frac{n-j}{2}}\{-(\gamma+\lambda_k)\sin[(n-j+1)\theta_k]+\gamma\sin[(n-j-1)\theta_k]\}$ for j odd.

$u_j^{(k)}=\mu^{\frac{n+j-1}{2}}\gamma^{\frac{n-j+1}{2}}\{\sin[(n-j+2)\theta_k]-\quad\dfrac{\gamma+\lambda_k}{\mu}+2\quad\sin[(n-j)\theta_k]\}$ for j even.

For $k=H-1,H$ using formula (2.2) and (2.3) we have:

$u^{(H-1)}=\quad 1,-\dfrac{\mu}{\gamma},-\dfrac{\mu}{\gamma},\dfrac{\mu^2}{\gamma^2},\dfrac{\mu^2}{\gamma^2},,...,(-1)^{s-1}\dfrac{\mu^{s-1}}{\gamma^{s-1}},(-1)^s\dfrac{\mu^s}{\gamma^s}$

and $u^{(H)}=(1,1,...,1)$.

Since we know eigenvalues and eigenvectors of the Markov process, we can find $P(t)=S\cdot e^{Dt}\cdot S^{-1}$.

Now if we assume H is even and $H = 2t$ where t is a natural number, then the eigenvalues are known by using Theorem 3.2.1 Case 1.

$\lambda_k = -\mu - \gamma - 2\sqrt{\mu\gamma} cos\theta_k,\quad k = 0, 1, 2, ..., \frac{H}{2} - 1.$

$\lambda_k = -\mu - \gamma + 2\sqrt{\mu\gamma} cos\theta_k,\quad \text{k} = \frac{H}{2}, ..., H - 1.$

$\lambda_H = 0.$

where

$$\theta_k = \begin{cases} \dfrac{(k+1)\pi}{2t}, & k = 0, 1, 2, ..., \dfrac{H}{2} - 1 \\ \dfrac{(k-t+2)\pi}{2t}, & k = \dfrac{H}{2}, ..., H - 1 \end{cases}$$

The following example is reproduced from 1.2 page 16.

Example 3.4. Consider the following state rate transition diagram of a birth-death process:

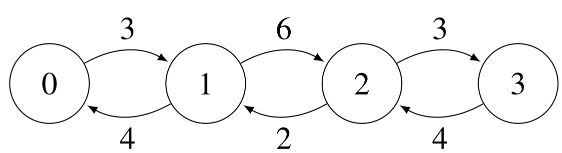


Figure 3.16

$$\text{and the corresponding matrix } Q = \begin{pmatrix} -3 & 3 & 0 & 0 \\ 4 & -10 & 6 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 4 & -4 \end{pmatrix}$$

Since this diagram has even number of states and $\gamma_1\mu_1 = \gamma_2\mu_2 = ... = 12$, it satisfies the conditions of Theorem 3.2.1 Case 2.

Hence, the eigenvalues of Q are :

$\lambda_0 = -13$

$\lambda_1 = -2$

$\lambda_2 = -7$

$\lambda_3 = 0.$

and $P(t) = e^{-2t}\begin{pmatrix} \frac{36}{55} & \frac{9}{55} & \frac{-18}{55} & \frac{-27}{55} \\ \frac{12}{55} & \frac{3}{55} & \frac{-6}{55} & \frac{-9}{55} \\ \frac{-8}{55} & \frac{-2}{55} & \frac{4}{55} & \frac{6}{55} \\ \frac{-16}{55} & \frac{-4}{55} & \frac{-8}{55} & \frac{12}{55} \end{pmatrix} + e^{-7t}\begin{pmatrix} \frac{3}{35} & \frac{-3}{35} & \frac{-9}{35} & \frac{9}{35} \\ \frac{-4}{35} & \frac{4}{35} & \frac{12}{35} & \frac{-12}{35} \\ \frac{-4}{35} & \frac{4}{35} & \frac{12}{35} & \frac{-12}{35} \\ \frac{16}{105} & \frac{-16}{105} & \frac{-16}{35} & \frac{16}{35} \end{pmatrix}$

$$+\, e^{-13t}\begin{pmatrix} \frac{12}{143} & \frac{-30}{143} & \frac{27}{143} & \frac{-9}{143} \\ \frac{-40}{143} & \frac{100}{143} & \frac{-90}{143} & \frac{30}{143} \\ \frac{12}{143} & \frac{-30}{143} & \frac{27}{143} & \frac{-9}{143} \\ \frac{-16}{429} & \frac{40}{429} & \frac{-12}{143} & \frac{4}{143} \end{pmatrix} + \quad e^{0}\begin{pmatrix} \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \end{pmatrix}.$$

Theorem 3.2.2. Consider the following birth-death process when H is even.

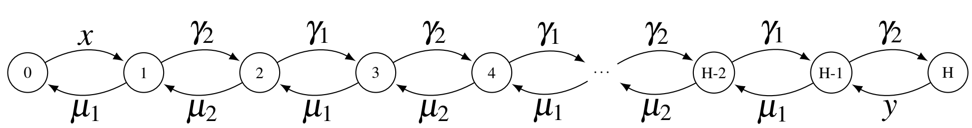


Figure 3.17

Assume $\mu_1\gamma_2 = \mu_2\gamma_1$, $x > \gamma_1$ and $y = \dfrac{\mu_2 x}{x-\gamma_1}$ then the eigenvalues of the Q matrix corresponding to Figure 3.14 are:

$$\lambda_k = \begin{cases} \dfrac{(-\mu_1-\mu_2-\gamma_1-\gamma_2) - \sqrt{(\mu_2-\mu_1+\gamma_2-\gamma_1)^2+16\mu_2\gamma_1 cos^2\theta_k}}{2} \\ \text{for } k=0,1,...,\dfrac{H-4}{2} \\ \dfrac{(-\mu_1-\mu_2-\gamma_1-\gamma_2) + \sqrt{(\mu_2-\mu_1+\gamma_2-\gamma_1)^2+16\mu_2\gamma_1 cos^2\theta_k}}{2} \\ \text{for } k=\dfrac{H-2}{2},...,H-3 \end{cases}$$

$$\text{where } \theta_k = \begin{cases} \dfrac{(k+1)\pi}{H}, & k=0,1,...,\dfrac{H-4}{2} \\ \dfrac{(2k+4-H)\pi}{2H}, & k=\dfrac{H-2}{2},...,H-3 \end{cases}$$

$$\lambda_{H-2} = \frac{(-\mu_1-\mu_2-\gamma_1-\gamma_2)-\sqrt{(\mu_2-\mu_1+\gamma_2-\gamma_1)^2+(x-\gamma_1+y-\mu_2)^2-2(\mu_2-\mu_1+\gamma_2-\gamma_1)(x-y-\gamma_1+\mu_2)}}{2}$$

$$\lambda_{H-1} = \frac{(-\mu_1-\mu_2-\gamma_1-\gamma_2)+\sqrt{(\mu_2-\mu_1+\gamma_2-\gamma_1)^2+(x-\gamma_1+y-\mu_2)^2-2(\mu_2-\mu_1+\gamma_2-\gamma_1)(x-y-\gamma_1+\mu_2)}}{2}$$

$\lambda_H = 0.$

Proof: The dual birth-death process exist and looks like:

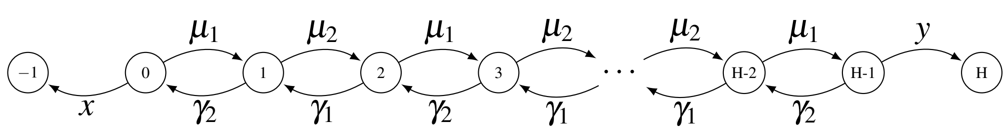


Figure 3.18

The corresponding matrix Q^* is:

$$Q^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots\cdots\cdots & 0 & 0 & 0 \\ x & -\mu_1 - x & \mu_1 & 0 & \cdots\cdots\cdots & 0 & 0 & 0 \\ 0 & \gamma_2 & -\mu_2-\gamma_2 & \mu_2 & \cdots\cdots\cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \gamma_1 & -\mu_1-\gamma_1 & \mu_1 \ddots \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots \ddots \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \ddots \; 0 & \gamma_2 & -\gamma_2 - y & y \\ 0 & 0 & \cdots & \cdots & \cdots\cdots \; 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we proceed to find the eigenvalues of the matrix inside Q^*:

$$\begin{pmatrix} -\mu_1 - x & \mu_1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \gamma_2 & -\mu_2-\gamma_2 & \mu_2 & \cdots & \cdots & \cdots & \vdots & \vdots \\ 0 & \gamma_1 & -\mu_1-\gamma_1 & \mu_1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \gamma_2 & -\gamma_2 - y \end{pmatrix}.$$

Since $-\mu_1 - \gamma_1 - \alpha = -\mu_1 - x$, therefore $\alpha = x - \gamma_1$.

Also since $-\mu_2 - \gamma_2 - \beta = -\gamma_2 - y$, we obtain $\beta = y - \mu_2$.

$$\begin{aligned} \text{Hence } \alpha\beta &= (x-\gamma_1)(y-\mu_2) \\ &= (x-\gamma_1)y - \mu_2 x + \mu_2\gamma_1 \\ &= (x-\gamma_1)\frac{\mu_2 x}{x-\gamma_1} - \mu_2 x + \mu_2\gamma_1 \\ &= \mu_2\gamma_1. \end{aligned}$$

Thus the conditions of Theorem 2.1.1 Case 2 are met and therefore, the eigenvalues are known.

The following Figure 3.19 is an example of Theorem 3.2.2.

Example 3.5. Consider the following birth-death process:

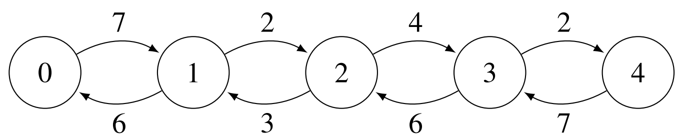


Figure 3.19

Notice that $6 \cdot 2 = 3 \cdot 4 = 12$, which means we have $\mu_1 \gamma_2 = \mu_2 \gamma_1 = 12$.

Also since $x = 7$, we get $y = \dfrac{\mu_2 x}{x - \gamma_1} = 7$.

Hence this process satisfies condition of Theorem 3.2.2 and therefore the eigenvalues are known and give below:

$$\lambda_0 = -15,\ \lambda_1 = -13,\ \lambda_2 = -7,\ \lambda_3 = -2 \text{ and } \lambda_4 = 0.$$

Thus we have $D = \begin{pmatrix} -15 & 0 & 0 & 0 & 0 \\ 0 & -13 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $e^{Dt} = \begin{pmatrix} e^{-15t} & 0 & 0 & 0 & 0 \\ 0 & e^{-13t} & 0 & 0 & 0 \\ 0 & 0 & e^{-7t} & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Using Dave's program, we get

$$P(t) = e^{-15t} \begin{bmatrix} \frac{63}{260} & -\frac{21}{65} & \frac{49}{260} & -\frac{28}{195} & \frac{7}{195} \\ -\frac{18}{65} & \frac{24}{65} & -\frac{14}{65} & \frac{32}{195} & -\frac{8}{195} \\ \frac{63}{260} & -\frac{21}{65} & \frac{49}{260} & -\frac{28}{195} & \frac{7}{195} \\ -\frac{18}{65} & \frac{24}{65} & -\frac{14}{65} & \frac{32}{195} & -\frac{8}{195} \\ \frac{63}{260} & -\frac{21}{65} & \frac{49}{260} & -\frac{28}{195} & \frac{7}{195} \end{bmatrix}$$

$$
+\,e^{-13t}\begin{bmatrix} \frac{28}{143} & -\frac{28}{143} & -\frac{56}{429} & \frac{28}{143} & -\frac{28}{429} \\ -\frac{24}{143} & \frac{24}{143} & \frac{16}{143} & -\frac{24}{143} & \frac{8}{143} \\ -\frac{24}{143} & \frac{24}{143} & \frac{16}{143} & -\frac{24}{143} & \frac{8}{143} \\ \frac{54}{143} & -\frac{54}{143} & -\frac{36}{143} & \frac{54}{143} & -\frac{18}{143} \\ -\frac{63}{143} & \frac{63}{143} & \frac{42}{143} & -\frac{63}{143} & \frac{21}{143} \end{bmatrix} + e^{-7t}\begin{bmatrix} \frac{1}{20} & 0 & -\frac{7}{60} & 0 & \frac{1}{15} \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{20} & 0 & \frac{7}{20} & 0 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{9}{20} & 0 & -\frac{21}{20} & 0 & \frac{3}{5} \end{bmatrix}
$$

$$
+e^{-2t}\begin{bmatrix} \frac{168}{715} & \frac{28}{143} & -\frac{112}{715} & -\frac{28}{143} & -\frac{56}{715} \\ \frac{24}{143} & \frac{20}{143} & -\frac{16}{143} & -\frac{20}{143} & -\frac{8}{143} \\ -\frac{144}{715} & -\frac{24}{143} & \frac{96}{715} & \frac{24}{143} & \frac{48}{715} \\ -\frac{54}{143} & -\frac{45}{143} & \frac{36}{143} & \frac{45}{143} & \frac{18}{143} \\ -\frac{378}{715} & -\frac{63}{143} & \frac{252}{715} & \frac{63}{143} & \frac{126}{715} \end{bmatrix}
$$

$$
+e^{0}\begin{bmatrix} \frac{366}{715} & \frac{371}{715} & \frac{42}{715} & -\frac{112}{2145} & -\frac{16}{429} \\ \frac{318}{715} & \frac{331}{715} & \frac{74}{715} & \frac{8}{2145} & -\frac{32}{2145} \\ \frac{54}{715} & \frac{111}{715} & \frac{50}{143} & \frac{668}{2145} & \frac{232}{2145} \\ -\frac{72}{715} & \frac{6}{715} & \frac{334}{715} & \frac{983}{2145} & \frac{358}{2145} \\ -\frac{36}{143} & -\frac{84}{715} & \frac{406}{715} & \frac{1253}{2145} & \frac{466}{2145} \end{bmatrix}.
$$

If $t = 2$ then we have $P(2) =$

$$\begin{pmatrix} .2812266532 & .3266632020 & .2125154952 & .1400034646 & .03959118488 \\ .2799970303 & .3256385508 & .2133353131 & .1410281157 & .04000098989 \\ .2732342081 & .3200029696 & .2178440691 & .146663696 & .04225505607 \\ .2700066818 & .3173132604 & .2199955454 & .1493534061 & .04333110605 \\ .2672404979 & .3150077954 & .2218390443 & .1516588712 & .04425379099 \end{pmatrix}$$

If $t = 4$ then we have $P(4) =$

$$\begin{pmatrix} .27700189891 & .32314260806 & .21533206739 & .14352405859 & .040999367029 \\ .27697937834 & .32312384092 & .21534708110 & .14354282573 & .041006873885 \\ .27685551521 & .32302062165 & .21542965652 & .14364604500 & .041048161593 \\ .27679639872 & .32297135791 & .21546906751 & .14369530875 & .041067867090 \\ .27674572745 & .32292913184 & .21550284836 & .14373753481 & .041084757517 \end{pmatrix}$$

The preceding results were produced using Dave Luk's computer program which allow symbolic representation.

Chapter 4

Applications to non birth-death chains

4.1 Eigenvalues of non-tridiagonal matrices

In this chapter, we are able to determine the eigenvalues of certain non-tridiagonal stochastic matrices by using duality and Kouachi's results summarized in Chapter 2.

Example 4.1. Consider the following matrix:

$$P_1 = \begin{pmatrix} .7 & .1 & 0 & .2 \\ .4 & .3 & .1 & .2 \\ .2 & .2 & .3 & .3 \\ .2 & 0 & .2 & .6 \end{pmatrix}$$

To obtain the eigenvalues of P_1, we consider the matrix P_1^* corresponding to the dual of P_1 and the matrix $P^*_{1\ inside}$ as shown below:

$$P_1^{*}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.2 & 0 & 0.2 \\ 0.2 & 0.1 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0 & 0.1 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_{inside}^{*}=\begin{pmatrix} 0.3 & 0.2 & 0 \\ 0.1 & 0.3 & 0.2 \\ 0 & 0.1 & 0.3 \end{pmatrix}$$

Note that the P_1 matrix is not tridiagonal. However, it turns out that $P_{1\ inside}^{*}$ satisfies the conditions of Theorem 2.1.1 Case 1 where N is odd.

Thus, the eigenvalues of the original P_1 matrix:

$$\lambda_k = \{0.5, 0.1, 0.3, 1\} \text{ for } k = 1, ..., 4.$$

This example can be generalized to the following proposition.

Proposition 4.1.1. Consider the 4x4 stochastic matrix P below where a, b, c are fixed probabilities with $0 \leq a+b+c \leq 1$ then:

$$P = \begin{pmatrix} b+c+q_1 & a+q_2-q_1 & -c+q_3-q_2 & 1-a-b-q_3 \\ c+q_1 & b+q_2-q_1 & a-c+q_3-q_2 & 1-a-b-q_3 \\ q_1 & c+q_2-q_1 & b-c+q_3-q_2 & 1-b-q_3 \\ q_1 & q_2-q_1 & q_3-q_2 & 1-q_3 \end{pmatrix}$$

has the same set of eigenvalues for different probabilities q_1, q_2, q_3 satisfying

$$q_1 \leq q_2 \leq q_3 \leq 1-a-b \text{ and } c \leq q_3 - q_2.$$

The eigenvalues are $\{1,\ b,\ b+\sqrt{2ac},\ b-\sqrt{2ac}\}$.

The proof consist of noticing that the dual of P is $\begin{pmatrix} b & c & 0 \\ a & b & c \\ 0 & a & b \end{pmatrix}$ as in the previous example.

Remark. Consider the following matrix:

$$P_2 = \begin{pmatrix} .7 & .1 & .1 & .1 \\ .4 & .3 & .2 & .1 \\ .2 & .2 & .4 & .2 \\ .2 & 0 & .3 & .5 \end{pmatrix}$$

We get P_2 by adding $q_1 = 0.1$ to column 3 and subtracting $q_1 = 0.1$ from column 4 of P_1. P_2 has the same $P^*_{2\ inside}$ as $P^*_{1\ inside}$. Hence P_1 and P_2 have the same eigenvalues. This leads to the following Proposition.

Proposition 4.1.2. The eigenvalues of a stochastic matrix does not change if a constant k is added to one column and is subtracted from another column.

We first illustrate the explanation for a $2x2$ stochastic matrix with $a+b=1$ and $c+d=1$, $0 < a,b,c,d < 1$. We use elementary row operations to verify this result.

We claim that $\det \begin{pmatrix} a+k-\lambda & b-k \\ c+k & d-k-\lambda \end{pmatrix} = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

Proof: $\det\begin{pmatrix} a+k-\lambda & b-k \\ c+k & d-k-\lambda \end{pmatrix}$

$= \det\begin{pmatrix} 1-\lambda & b-k \\ 1-\lambda & d-k-\lambda \end{pmatrix}$ R1'=R1+R2

interpreted as the new column 1 is the sum of previous column 1 and 2

$= \det\begin{pmatrix} 1-\lambda & b \\ 1-\lambda & d-\lambda \end{pmatrix}$ R2'=$\dfrac{k}{1-\lambda}$R1+R2

$= \det\begin{pmatrix} a+b-\lambda & b \\ c+d-\lambda & d-\lambda \end{pmatrix}$ since $a+b=1$

$= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$ R1'=R1-R2

Similarly, we get the same conclusion for a $3x3$ stochastic matrix as shown in the proof below:

$$\det\begin{pmatrix} a_{11}+k-\lambda & a_{12} & a_{13}-k \\ a_{21}+k & a_{22}-\lambda & a_{23}-k \\ a_{31}+k & a_{32} & a_{33}-k-\lambda \end{pmatrix}$$

$= \det\begin{pmatrix} 1-\lambda & a_{12} & a_{13}-k \\ 1-\lambda & a_{22}-\lambda & a_{23}-k \\ 1-\lambda & a_{32} & a_{33}-k-\lambda \end{pmatrix}$ R1'=R1+R2+R3

$$= \det \begin{pmatrix} 1-\lambda & a_{12} & a_{13} \\ 1-\lambda & a_{22}-\lambda & a_{23} \\ 1-\lambda & a_{32} & a_{33}-\lambda \end{pmatrix} \text{R3'}=\frac{k}{1-\lambda}\text{R1+R3}$$

$$= \det \begin{pmatrix} a_{11}+a_{12}+a_{13}-\lambda & a_{12} & a_{13} \\ a_{21}+a_{22}+a_{23}-\lambda & a_{22}-\lambda & a_{23} \\ a_{31}+a_{32}+a_{33}-\lambda & a_{32} & a_{33}-\lambda \end{pmatrix}$$

$$= \det \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix} \text{R1'=R1-R2-R3}$$

Example 4.2. Consider the following Markov chain:

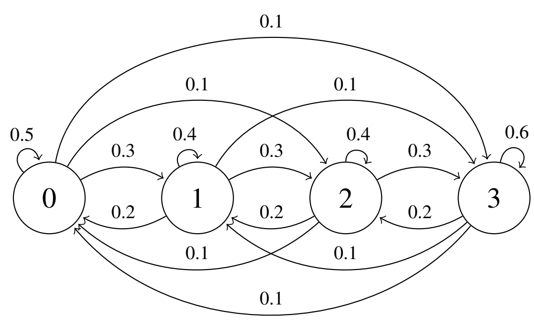


Figure 4.1

The dual Markov chain exist and looks like:

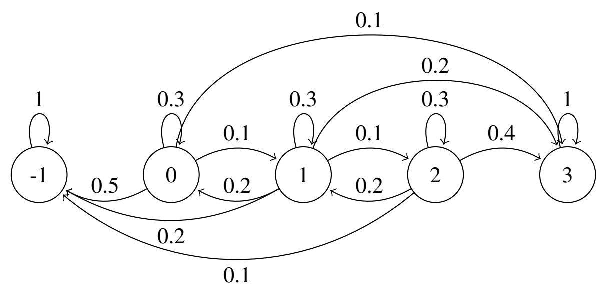


Figure 4.2

Notice that this Markov chain is not a birth-death chain since we have more than one arrow coming from one state to any other states. For example, if the Markov chain start at state 0, then the probability it goes to state 1 is 0.3, goes to state 2 is 0.1 and goes to state 3 is 0.1.

The corresponding P-matrix to Figure 4.1 is:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{5} & \frac{2}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

Since the P matrix is not tridiagonal, it does not satisfy Kouachi's theorem.

Now consider the P^* matrix corresponding to Figure 4.2:

$$P^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{10} & \frac{1}{10} & 0 & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{10} & 0 & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } P^*_{inside} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} & 0 \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{10} \\ 0 & \frac{1}{5} & \frac{3}{10} \end{pmatrix}$$

The P^*_{inside} matrix satisfy condition of Theorem 2.1.1 Case 1 where $N = 3$. So we can use Kouachi's formula to find exact eigenvalue for this matrix and again eigenvalues for the original Markov chain are also determined.

The eigenvalues for P matrix are $\frac{1}{10}, \frac{3}{10}, \frac{1}{2}$ and 1.

So we have:

$$S^* = \begin{pmatrix} 0 & 0 & 0 & \frac{683}{973} & 0 \\ \frac{1}{3} & \frac{1292}{2889} & -\frac{1}{3} & \frac{639}{1147} & \frac{262}{1669} \\ -\frac{2}{3} & 0 & -\frac{2}{3} & \frac{739}{1895} & \frac{401}{1186} \\ \frac{2}{3} & -\frac{2584}{2889} & -\frac{2}{3} & \frac{731}{3453} & \frac{246}{463} \\ 0 & 0 & 0 & 0 & \frac{690}{907} \end{pmatrix} \text{ and } D^* = \begin{pmatrix} \frac{1}{10} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, we can find the ruin probability of this Markov chain after 5 steps using Luis Cervantes program:

$$(P^*)^5 = S^* \cdot (D^*)^5 \cdot (S^*)^{-1}$$

$$= \frac{1}{100000}\begin{pmatrix} 100000 & 0 & 0 & 0 & 0 \\ 78115 & 903 & 781 & 330 & 19871 \\ 53212 & 1562 & 1563 & 781 & 42882 \\ 27971 & 1320 & 1562 & 903 & 68244 \\ 0 & 0 & 0 & 0 & 100000 \end{pmatrix}$$

The matrix tells us that the ruin probability of state 0 is 0.78115 , ruin probability of state 1 is 0.53212 and ruin probability of state 3 is 0.27971.

Chapter 5

Conclusions

Kouachi's results on eigenvalues and eigenvectors of certain tridiagonal matrices have direct consequences for stochastic, tridiagonal matrices P. For example, we can find explicit formula for P^n for $n \in N$. When combined with known results for dual Markov chains, we obtain matrices P^* that have the same set of eigenvalues as P. Furthermore, the duality theorem provides explicit expressions for $(P^*)^n$ for $n \in N$ in terms of P^n which gives us new results for finding ruin probabilities. Kouachi's articles on tridiagonal matrices have implications for determining eigenvalues of certain non-tridiagonal stochastic matrices. Kouachi's results on eigenvalues and eigenvectors have consequences for determining transient probability functions of Markov Processes.

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Appendices

Appendix A

Proof of Duality Theorem

Proof of Duality Theorem:

Use mathematical induction on n.

For $n = 0$, $P_{r,s}^{(0)} = \sum_{k=r}^{H}[P_{s,k}^{*(0)} - P_{s-1,k}^{*(0)}]$ is seen to hold by substituting initial conditions that $P_{r,s}^{(0)} = \delta_{r,s} = P_{r,s}^{*(0)}$ for all states $r,s = 0,1,2,\ldots,H$ where $\delta_{r,s}$ is the Kronecker delta and using the convention $P_{-1,k}^{(n)} = 0$ if $k > -1$. Suppose $2 \leq s \leq H-1$. As the induction hypothesis, assume $P_{r,s}^{(n)} = \sum_{k=r}^{H}[P_{s,k}^{*(n)} - P_{s-1,k}^{*(n)}]$ holds and show $P_{r,s}^{(n+1)} = \sum_{k=r}^{H}[P_{s,k}^{*(n+1)} - P_{s-1,k}^{*(n+1)}]$. But

$$P_{r,s}^{(n+1)} = \sum_{j=0}^{H} P_{r,j}^{(n)} P_{j,s}^{(1)}$$

by the Chapman-Kolmogorov equations for the birth-death chain of Figure 3. This simplifies to

$$P_{r,s}^{(n+1)} = P_{r,s-1}^{(n)} P_{s-1,s}^{(1)} + P_{r,s}^{(n)} P_{s,s}^{(1)} + P_{r,s+1}^{(n)} P_{s+1,s}^{(1)}$$

since the chain of Figure 3 is a birth-death chain and therefore all 1-step transition probabilities are zero except for possibly the above noted three transitions. More specifically

$$P_{r,s}^{(n+1)} = P_{r,s-1}^{(n)} P_{s-1} + P_{r,s}^{(n)} r_s + P_{r,s+1}^{(n)} q_{s+1}$$

again, see Figure 3. However

$$P_{r,s}^{(n+1)} = \sum_{k=r}^{H} \; P_{s-1,k}^{*(n)} - P_{s-2,k}^{*(n)} \;\; p_{s-1} + \sum_{k=r}^{H} \; P_{s,k}^{*(n)} - P_{s-1,k}^{*(n)} \;\; r_s + \sum_{k=r}^{H} \; P_{s+1,k}^{*(n)} - P_{s,k}^{*(n)} \;\; q_{s+1}$$

by the induction hypothesis. Replacing r_s by $[1-p_s-q_s]$ and rearranging terms produces

$$\begin{gathered} P_{r,s}^{(n+1)} = \\ \sum_{k=r}^{H} \; P_{s-1,k}^{*(n)} - P_{s-2,k}^{*(n)} \;\; p_{s-1} + \sum_{k=r}^{H} \; P_{s,k}^{*(n)} - P_{s-1,k}^{*(n)} \;\; [1-p_s-q_s] + \sum_{k=r}^{H} \; P_{s+1,k}^{*(n)} - P_{s,k}^{*(n)} \;\; q_{s+1} \end{gathered}$$

$$\begin{aligned} &= \sum_{k=r}^{H} \;\; p_s P_{s-1,k}^{*(n)} + [1-p_s-q_{s+1}] P_{s,k}^{*(n)} + q_{s+1} P_{s+1,k}^{*(n)} \quad - \\ &\sum_{k=r}^{H} \;\; p_{s-1} P_{s-2,k}^{*(n)} + [1-p_{s-1}-q_s] P_{s-1,k}^{*(n)} + q_s P_{s,k}^{*(n)} \end{aligned}$$

By the definition of ρ_s from Figure 4

$$\begin{gathered} P_{r,s}^{(n+1)} = \\ \sum_{k=r}^{H} \;\; p_s P_{s-1,k}^{*(n)} + \rho_s P_{s,k}^{*(n)} + q_{s+1} P_{s+1,k}^{*(n)} \quad - \sum_{k=r}^{H} \;\; p_{s-1} P_{s-2,k}^{*(n)} + \rho_{s-1} P_{s-1,k}^{*(n)} + q_s P_{s,k}^{*(n)} \end{gathered}$$

Substituting in the transition probabilities of the chain of Figure 4 and writing this last result in Chapman-Kolmogorov equation form gives

$$\begin{aligned} P_{r,s}^{(n+1)} &= \sum_{k=r}^{H} \;\; P_{s,s-1}^{*(1)} P_{s-1,k}^{*(n)} + P_{s,s}^{*(1)} P_{s,k}^{*(n)} + P_{s,s+1}^{*(1)} P_{s+1,k}^{*(n)} \quad - \\ &\sum_{k=r}^{H} \;\; P_{s-1,s-2}^{*(1)} P_{s-2,k}^{*(n)} + P_{s-1,s-1}^{*(1)} P_{s-1,k}^{*(n)} + P_{s-1,s}^{*(1)} P_{s,k}^{*(n)} \\ &= \sum_{k=r}^{H} \sum_{j=0}^{H} P_{s,j}^{*(1)} P_{j,k}^{*(n)} - \sum_{k=r}^{H} \sum_{j=0}^{H} P_{s-1,j}^{*(1)} P_{j,k}^{*(n)}. \end{aligned}$$

Therefore

$$P_{r,s}^{(n+1)} = \sum_{k=r}^{H} \; P_{s,k}^{*(n+1)} - P_{s-1,k}^{*(n+1)} \quad .$$

by the Chapman-Kolmogorov equations of the birth-death chain of Figure 4. This completes the induction step and establishes the first equality in Theorem 1.4.1 whenever $2 \leq s \leq H-1$. If $s = 0, 1$ or H, the preceding argument may be suitable modified to establish the desired result.

We next verify that

$$P^{*(n)}_{r,s} = \sum_{k=0}^{r} \; P^{(n)}_{s,k} - P^{(n)}_{s+1,k} \; .$$

Consider $\sum_{k=0}^{r} \; P^{(n)}_{s,k} - P^{(n)}_{s+1,k}$. By what we have just proved we know

$$P^{(n)}_{s,k} = \sum_{\alpha=s}^{H} \; P^{*(n)}_{k,\alpha} - P^{*(n)}_{k-1,\alpha} \quad \text{and } P^{(n)}_{s+1,k} = \sum_{\alpha=s+1}^{H} \; P^{*(n)}_{k,\alpha} - P^{*(n)}_{k-1,\alpha} \; .$$

So by substitution,

$$\sum_{k=0}^{r} \; P^{(n)}_{s,k} - P^{(n)}_{s+1,k} \; = \sum_{k=0}^{r} \; \sum_{\alpha=s}^{H} \; P^{*(n)}_{k,\alpha} - P^{*(n)}_{k-1,\alpha} \; - \sum_{\alpha=s+1}^{H} \; P^{*(n)}_{k,\alpha} - P^{*(n)}_{k-1,\alpha} \; .$$

However, by canceling terms, this last equation simplifies to the following telescoping series

$$\sum_{k=0}^{r} \; P^{(n)}_{s,k} - P^{(n)}_{s+1,k} \; = \sum_{k=0}^{r} \; P^{*(n)}_{k,s} - P^{*(n)}_{k-1,s}$$

which in turn reduces to $P^{*(n)}_{r,s} - P^{*(n)}_{-1,s}$ which equals $P^{*(n)}_{r,s}$ since $P^{*(n)}_{-1,s} = 0$ for $0 \leq s \leq H$ because state -1 is an absorbing state in the birth-death chain of Figure 4. Thus, we have shown that

$$\sum_{k=0}^{r} \; P^{(n)}_{s,k} - P^{(n)}_{s+1,k} \; = P^{*(n)}_{r,s}$$

which is the second equality of Duality Theorem. This completes the proof.

Appendix B

A 4 x 4 n-step transition probability matrix

The following matrix is P^n of Example 3.2. Since we have 4 x 4 matrix, each page represents one column of matrix P^n.

$$\lambda_0 = -a_1 - a_2 - \sqrt{a_1^2 + a_2^2} + 1$$

$$\lambda_1 = -a_1 - a_2 + \sqrt{a_1^2 + a_2^2} + 1$$

$$\lambda_2 = -a_1 - a_2 + 1$$

$$\frac{1}{4(a_1^3+a_1^2a_2+a_1a_2^2+a_2^3)} \quad a_1^3\lambda_0^n+a_1^3\lambda_1^n+2a_1^3+2a_1^2a_2\lambda_0^n+2a_1^2a_2\lambda_1^n-a_1^2\lambda_0^n \sqrt{a_1^2+a_2^2}$$
$$+a_1^2\lambda_1^n \sqrt{a_1^2+a_2^2}+a_1a_2^2\lambda_0^n+a_1a_2^2\lambda_1^n+2a_1a_2^2-a_1a_2\lambda_0^n \sqrt{a_1^2+a_2^2}+a_1a_2\lambda_1^n \sqrt{a_1^2+a_2^2}$$
$$+4a_2^3\lambda_2^n$$

$$\frac{a_1}{4a_1^3+4a_1^2a_2+4a_1a_2^2+4a_2^3} \quad -a_1^2\lambda_0^n-a_1^2\lambda_1^n+2a_1^2-a_1\lambda_0^n \sqrt{a_1^2+a_2^2}+a_1\lambda_1^n \sqrt{a_1^2+a_2^2}$$
$$+a_2^2\lambda_0^n+a_2^2\lambda_1^n-4a_2^2\lambda_2^n+2a_2^2-a_2\lambda_0^n \sqrt{a_1^2+a_2^2}+a_2\lambda_1^n \sqrt{a_1^2+a_2^2}$$

$$\frac{a_1}{4a_1^3+4a_1^2a_2+4a_1a_2^2+4a_2^3} \quad -a_1^2\lambda_0^n-a_1^2\lambda_1^n+2a_1^2+a_1\lambda_0^n \sqrt{a_1^2+a_2^2}-a_1\lambda_1^n \sqrt{a_1^2+a_2^2}$$
$$+a_2^2\lambda_0^n+a_2^2\lambda_1^n-4a_2^2\lambda_2^n+2a_2^2+a_2\lambda_0^n \sqrt{a_1^2+a_2^2}-a_2\lambda_1^n \sqrt{a_1^2+a_2^2}$$

$$\frac{-a_1}{4(a_1^3+a_1^2a_2+a_1a_2^2+a_2^3)} \quad a_1^2\lambda_0^n+a_1^2\lambda_1^n-2a_1^2+2a_1a_2\lambda_0^n+2a_1a_2\lambda_1^n-4a_1a_2\lambda_2^n$$
$$-a_1\lambda_0^n \sqrt{a_1^2+a_2^2}+a_1\lambda_1^n \sqrt{a_1^2+a_2^2}+a_2^2\lambda_0^n+a_2^2\lambda_1^n-2a_2^2-a_2\lambda_0^n \sqrt{a_1^2+a_2^2}$$
$$+a_2\lambda_1^n \sqrt{a_1^2+a_2^2}$$

$$
\begin{aligned}
&\frac{a_2}{4a_1^3+4a_1^2a_2+4a_1a_2^2+4a_2^3} \quad -a_1^2\lambda_0^n-a_1^2\lambda_1^n+2a_1^2-a_1\lambda_0^n\sqrt{a_1^2+a_2^2}+a_1\lambda_1^n\sqrt{a_1^2+a_2^2}\\
&+a_2^2\lambda_0^n+a_2^2\lambda_1^n-4a_2^2\lambda_2^n+2a_2^2-a_2\lambda_0^n\sqrt{a_1^2+a_2^2}+a_2\lambda_1^n\sqrt{a_1^2+a_2^2}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4(a_1^3+a_1^2a_2+a_1a_2^2+a_2^3)} \quad 2a_1^3\lambda_0^n+2a_1^3\lambda_1^n+a_1^2a_2\lambda_0^n+a_1^2a_2\lambda_1^n+2a_1^2a_2+2a_1^2\lambda_0^n\\
&\sqrt{a_1^2+a_2^2}-2a_1^2\lambda_1^n\sqrt{a_1^2+a_2^2}+4a_1a_2^2\lambda_2^n+a_1a_2\lambda_0^n\sqrt{a_1^2+a_2^2}-a_1a_2\lambda_1^n\sqrt{a_1^2+a_2^2}\\
&+a_2^3\lambda_0^n+a_2^3\lambda_1^n+2a_2^3-a_2^2\lambda_0^n\sqrt{a_1^2+a_2^2}+a_2^2\lambda_1^n\sqrt{a_1^2+a_2^2}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{-a_2}{4(a_1^3+a_1^2a_2+a_1a_2^2+a_2^3)} \quad a_1^2\lambda_0^n+a_1^2\lambda_1^n-2a_1^2+2a_1a_2\lambda_0^n+2a_1a_2\lambda_1^n-4a_1a_2\lambda_2^n\\
&+a_1\lambda_0^n\sqrt{a_1^2+a_2^2}-a_1\lambda_1^n\sqrt{a_1^2+a_2^2}+a_2^2\lambda_0^n+a_2^2\lambda_1^n-2a_2^2+a_2\lambda_0^n\sqrt{a_1^2+a_2^2}\\
&-a_2\lambda_1^n\sqrt{a_1^2+a_2^2}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{a_2}{4a_1^3+4a_1^2a_2+4a_1a_2^2+4a_2^3} \quad a_1^2\lambda_0^n+a_1^2\lambda_1^n-4a_1^2\lambda_2^n+2a_1^2+a_1\lambda_0^n\sqrt{a_1^2+a_2^2}\\
&-a_1\lambda_1^n\sqrt{a_1^2+a_2^2}-a_2^2\lambda_0^n-a_2^2\lambda_1^n+2a_2^2+a_2\lambda_0^n\sqrt{a_1^2+a_2^2}-a_2\lambda_1^n\sqrt{a_1^2+a_2^2}
\end{aligned}
$$

$$\frac{a_1}{4(-a_1(a_1^2+a_2^2)^{\frac{3}{2}}+a_2^2(a_1^2+a_2^2)-a_2\sqrt{a_1^2+a_2^2}(a_1^2+a_2(a_2+\sqrt{a_1^2+a_2^2})))}\quad 4a_2^2\lambda_2^n$$

$$\sqrt{a_1^2+a_2^2}-\lambda_0^n(2a_1a_2^2+2a_2^2(-a_1+\sqrt{a_1^2+a_2^2}\,)+(a_1^2+a_2^2)(a_1+a_2-\sqrt{a_1^2+a_2^2}))$$

$$+\lambda_1^n(2a_1a_2^2-2a_2^2(a_1+\sqrt{a_1^2+a_2^2})+(a_1^2+a_2^2)(a_1+a_2+\sqrt{a_1^2+a_2^2}))-2(a_1^2+a_2^2)^{\frac{3}{2}}$$

$$\frac{a_1}{4a_2(-a_1(a_1^2+a_2^2)^{\frac{3}{2}}+a_2^2(a_1^2+a_2^2)-a_2\sqrt{a_1^2+a_2^2}(a_1^2+a_2(a_2+\sqrt{a_1^2+a_2^2})))}\quad -4a_1a_2^2$$

$$\lambda_2^n\sqrt{a_1^2+a_2^2}-2a_2(a_1^2+a_2^2)^{\frac{3}{2}}+\lambda_0^n(a_1+\sqrt{a_1^2+a_2^2})(2a_1a_2^2+2a_2^2(-a_1+\sqrt{a_1^2+a_2^2})$$

$$+(a_1^2+a_2^2)(a_1+a_2-\sqrt{a_1^2+a_2^2}))-\lambda_1^n(a_1-\sqrt{a_1^2+a_2^2})(2a_1a_2^2-2a_2^2(a_1+\sqrt{a_1^2+a_2^2})$$

$$+(a_1^2+a_2^2)(a_1+a_2+\sqrt{a_1^2+a_2^2}))$$

$$\frac{1}{4(-a_1(a_1^2+a_2^2)^{\frac{3}{2}}+a_2^2(a_1^2+a_2^2)-a_2\sqrt{a_1^2+a_2^2}(a_1^2+a_2(a_2+\sqrt{a_1^2+a_2^2})))}(-4a_1^2a_2\lambda_2^n$$

$$\sqrt{a_1^2+a_2^2}-2a_1(a_1^2+a_2^2)^{\frac{3}{2}}-\lambda_0^n(a_2+\sqrt{a_1^2+a_2^2})(2a_1a_2^2+2a_2^2(-a_1+\sqrt{a_1^2+a_2^2})$$

$$+(a_1^2+a_2^2)(a_1+a_2-\sqrt{a_1^2+a_2^2}))+\lambda_1^n(a_2-\sqrt{a_1^2+a_2^2})(2a_1a_2^2-2a_2^2(a_1+\sqrt{a_1^2+a_2^2})$$

$$+(a_1^2+a_2^2)(a_1+a_2+\sqrt{a_1^2+a_2^2}))$$

$$\frac{a_1}{4(-a_1(a_1^2+a_2^2)^{\frac{3}{2}}+a_2^2(a_1^2+a_2^2)-a_2\sqrt{a_1^2+a_2^2}(a_1^2+a_2(a_2+\sqrt{a_1^2+a_2^2})))}\quad 4a_1^2\lambda_2^n$$

$$\sqrt{a_1^2+a_2^2}+\lambda_0^n(2a_1a_2^2+2a_2^2(-a_1+\sqrt{a_1^2+a_2^2})+(a_1^2+a_2^2)(a_1+a_2-\sqrt{a_1^2+a_2^2}))$$

$$-\lambda_1^n(2a_1a_2^2-2a_2^2(a_1+\sqrt{a_1^2+a_2^2})+(a_1^2+a_2^2)(a_1+a_2+\sqrt{a_1^2+a_2^2}))-2(a_1^2+a_2^2)^{\frac{3}{2}}$$

$$
-\frac{a_2}{4(a_1^3+a_1^2a_2+a_1a_2^2+a_2^3)} \quad a_1^2\lambda_0^n+a_1^2\lambda_1^n-2a_1^2+2a_1a_2\lambda_0^n+2a_1a_2\lambda_1^n-4a_1a_2\lambda_2^n
$$

$$
-a_1\lambda_0^n \sqrt{a_1^2+a_2^2}+a_1\lambda_1^n \sqrt{a_1^2+a_2^2}+a_2^2\lambda_0^n+a_2^2\lambda_1^n-2a_2^2-a_2\lambda_0^n \sqrt{a_1^2+a_2^2}
$$

$$
+a_2\lambda_1^n \sqrt{a_1^2+a_2^2}
$$

$$
\frac{a_2}{4a_1^3+4a_1^2a_2+4a_1a_2^2+4a_2^3} \quad a_1^2\lambda_0^n+a_1^2\lambda_1^n-4a_1^2\lambda_2^n+2a_1^2+a_1\lambda_0^n \sqrt{a_1^2+a_2^2}-a_1\lambda_1^n
$$

$$
\sqrt{a_1^2+a_2^2}-a_2^2\lambda_0^n-a_2^2\lambda_1^n+2a_2^2+a_2\lambda_0^n \sqrt{a_1^2+a_2^2}-a_2\lambda_1^n \sqrt{a_1^2+a_2^2}
$$

$$
\frac{a_2}{4a_1^3+4a_1^2a_2+4a_1a_2^2+4a_2^3} \quad a_1^2\lambda_0^n+a_1^2\lambda_1^n-4a_1^2\lambda_2^n+2a_1^2-a_1\lambda_0^n \sqrt{a_1^2+a_2^2}+a_1\lambda_1^n
$$

$$
\sqrt{a_1^2+a_2^2}-a_2^2\lambda_0^n-a_2^2\lambda_1^n+2a_2^2-a_2\lambda_0^n \sqrt{a_1^2+a_2^2}+a_2\lambda_1^n \sqrt{a_1^2+a_2^2}
$$

$$
\frac{1}{4(a_1^3+a_1^2a_2+a_1a_2^2+a_2^3)} \quad 4a_1^3\lambda_2^n+a_1^2a_2\lambda_0^n+a_1^2a_2\lambda_1^n+2a_1^2a_2+2a_1a_2^2\lambda_0^n+2a_1a_2^2\lambda_1^n
$$

$$
-a_1a_2\lambda_0^n \sqrt{a_1^2+a_2^2}+a_1a_2\lambda_1^n \sqrt{a_1^2+a_2^2}+a_2^3\lambda_0^n+a_2^3\lambda_1^n+2a_2^3-a_2^2\lambda_0^n \sqrt{a_1^2+a_2^2}
$$

$$
+a_2^2\lambda_1^n \sqrt{a_1^2+a_2^2}
$$

Appendix C

Dave Luk's PYTHON code

New Kouachi's

August 4, 2017

0.1 Setup

```
In [ ]: from sympy import *
        from sympy.parsing.sympy_parser import parse_expr
        import numpy as np
        init_printing()
```

0.2 Parameters

```
In [ ]: n = 4
        a_1 = symbols('a_1', postive=True, real=True)
        a_2 = symbols('a_2', postive=True, real=True)
        #a_1 = parse_expr("1/4")
        #a_2 = parse_expr("1/3")
        alpha = -a_2
        beta = -a_1
        c_s = []
        a_s = []
        for i in range(0,n):
            c_s.append(a_2 if (i % 2) == 0 else a_1)
            a_s.append(a_1 if (i % 2) == 0 else a_2)

        m = int(n/2)
        b_1 = 1-2*a_2
        b_2 = 1-2*a_1
        d_sq = alpha*beta
```

```
In [ ]: n=4
        a = parse_expr('1/2')#symbols('a', positive=True)
        b = parse_expr('1/6')#symbols('b', positive=True)
        c = parse_expr('1/3')#symbols('c', positive=True)
        c_s = []
        a_s = []

        alpha = -a
        beta = -c

        for i in range(0,n):
```

```
        a_s.append(a)
        c_s.append(c)

    m = int(n/2)
    b_1 = b_2 = b
    d_sq = alpha*beta
```

```
In [ ]: n=4
        c_s = [parse_expr('1/3'),parse_expr('1/6'),parse_expr('1/3')]
        a_s = [parse_expr('1/4'),parse_expr('1/2'),parse_expr('1/4')]

        b_1 = parse_expr('1/6')
        b_2 = parse_expr('7/12')

        alpha = parse_expr('1/6')
        beta = parse_expr('1/2')

        d_sq = alpha*beta
        m = int(n/2)
```

0.3 Main Code

```
In [ ]: def case1():
            if n%2 == 0:
                for k in range(1,n+1):
                    if k <= m:
                        theta_k = '{}*pi/{}'.format(k,2*m+1)
                    else:
                        theta_k = '{}*pi/{}'.format((k-m), 2*m+1)
                    theta_ks.append(parse_expr(theta_k))
            else:
                for k in range(1,n+1):
                    if k <= m:
                        theta_k = '{}*pi/{}'.format(k,2*m+2)
                    else:
                        theta_k = '{}*pi/{}'.format((k-m), 2*m+2)
                    theta_ks.append(parse_expr(theta_k))

            for k in range(1,n+1):
                if k <= m:
                    lambda_k = ((b_1+b_2)-sqrt((b_1-b_2)**2 \
                                + 16*d_sq*(cos(theta_ks[k-1])**2)))/2
                elif k <= 2*m:
                    lambda_k = ((b_1+b_2)+sqrt((b_1-b_2)**2 \
                                + 16*d_sq*(cos(theta_ks[k-1])**2)))/2
                else:
                    lambda_k = b_1
                lambda_ks.append(factor(lambda_k)
```

```
                         .subs(sqrt(a_1**2+2*a_1*a_2+a_2**2), (a_1+a_2)))

def case2():
    for k in range(1,n+1):
        if k <= m-1:
            theta_k = '{}*pi/{}'.format(k,2*m)
        elif k<= n-2:
            theta_k = '{}*pi/{}'.format((k-m+1), 2*m)
        else:
            theta_k = '0'
        theta_ks.append(parse_expr(theta_k))
    for k in range(1,n+1):
        if k <= m-1:
            lambda_k = ((b_1 + b_2) - sqrt((b_1-b_2)**2 \
                        + 16* d_sq *cos(theta_ks[k-1])**2))/2
        elif k <= n-2:
            lambda_k = ((b_1 + b_2) + sqrt((b_1-b_2)**2 \
                        + 16* d_sq *cos(theta_ks[k-1])**2))/2
        elif k == n-1:
            lambda_k = ((b_1 + b_2 - alpha - beta) - sqrt((b_1-b_2)**2 \
                        + (alpha+beta)**2 - 2*(b_1-b_2)*(alpha-beta)))/2
        else:
            lambda_k = ((b_1 + b_2 - alpha - beta) + sqrt((b_1-b_2)**2 \
                        + (alpha+beta)**2 - 2*(b_1-b_2)*(alpha-beta)))/2
        lambda_ks.append(factor(expand(lambda_k))
                         .subs(sqrt(a_1**2+2*a_1*a_2+a_2**2), (a_1+a_2)))

def case3():
    if n%2 == 0:
        for k in range(1,n+1):
            if k <= m:
                theta_k = '{}*pi/{}'.format(k,2*m+1)
            else:
                theta_k = '{}*pi/{}'.format((k-m), 2*m+1)
            theta_ks.append(parse_expr(theta_k))
    else:
        for k in range(1,n+1):
            if k <= m:
                theta_k = '{}*pi/{}'.format(k,2*m+2)
            else:
                theta_k = '{}*pi/{}'.format((k-m), 2*m+2)
            theta_ks.append(parse_expr(theta_k))
    for k in range(1,n+1):
        if k <= m:
            lambda_k = b - 2*sqrt(a_s[0]*c_s[0])*cos(theta_ks[k-1])
        elif k <= 2*m:
            lambda_k = b + 2*sqrt(a_s[0]*c_s[0])*cos(theta_ks[k-1])
        else:
```

```
            lambda_k = b_1
        lambda_ks.append(simplify(factor(expand(lambda_k))
                                ).subs(sqrt(a_1**2+2*a_1*a_2+a_2**2), (a_1+a_2)))

theta_ks = []
lambda_ks = []

if alpha == 0 and beta == 0:
    print("Case 1")
    case1()
elif n%2 == 0:
    print("Case 2")
    case2()
elif b_1 == b_2 and c_s.sum(c_s-c_s[0]) == 0 and a_s.sum(a_s-a_s[0]) == 0:
    print("Case 3")
    case3()

ev = Matrix.zeros(n,n)
for r in range(1,n+1):
    rho_r = ((-sqrt(d_sq))**(n-r))*(np.prod(a_s[:(max(1,r-1))]))
    for c in range(1,n+1):
        zeta_r_c = []
        zeta_r_c.append(b_1 - lambda_ks[c-1])
        zeta_r_c.append(b_2 - lambda_ks[c-1])
        if r == 1:
            if c > n-2:
                ev[r-1,c-1] = 1
            else:
                fT = ((zeta_r_c[1]-beta)*sin(n*theta_ks[c-1]))
                sT = (beta*sin((n-2)*theta_ks[c-1]))
                ev[r-1,c-1] = fT- sT
                ev[r-1,c-1] *= (-sqrt(d_sq))**(n-1)
        elif r % 2 == 1: #if row is odd
            if n % 2 == 1: # if odd matr
                ev[r-1,c-1] = (d_sq*sin((n-r+2)*theta_ks[c-1])) \
                -((beta*zeta_r_c[1]-d_sq)*sin((n-r)*theta_ks[c-1]))
                ev[r-1,c-1] *= rho_r
            else: # even matr
                ev[r-1,c-1] = ((zeta_r_c[1]-beta)*sin((n-r+1)*theta_ks[c-1])) \
                -(beta*sin((n-r-1)*theta_ks[c-1]))
                ev[r-1,c-1] *= rho_r
            if c > n-2:
                if r == 2:
                    ev[r-1,c-1] = (-zeta_r_c[0]+alpha)/c_s[r-2]
                else:
                    ev[r-1,c-1] = ((-zeta_r_c[r%2]*ev[r-2,c-1]) \
                                   -(a_s[r-3]*ev[r-3,c-1]))/c_s[r-2]
        else: #row is even
```

```
                if n % 2 == 1: #if odd matr
                    ev[r-1,c-1] = ((zeta_r_c[0]-beta)*\
                                    sqrt(d_sq)*sin((n-r+1)*theta_ks[c-1])) \
                    -(beta*sqrt(d_sq)*sin((n-r-1)*theta_ks[c-1]))
                    ev[r-1,c-1] *= rho_r
                else: # if even matr
                    ev[r-1,c-1] = sqrt(d_sq)*sin((n-r+2)*theta_ks[c-1]) \
                    -(beta*zeta_r_c[0]/sqrt(d_sq)-sqrt(d_sq))*sin((n-r)*theta_ks[c-1])
                    ev[r-1,c-1] *= rho_r
                if c > n-2:
                    if r == 2:
                        ev[r-1,c-1] = (-zeta_r_c[0]+alpha)/c_s[r-2]
                    else:
                        ev[r-1,c-1] = ((-zeta_r_c[r%2]*ev[r-2,c-1]) \
                                       -(a_s[r-3]*ev[r-3,c-1]))/c_s[r-2]

        for r in range(0,n):
            for c in range(0,n):
                ev[r,c] /= ev[n-1,c]

        comp=simplify(ev)
```

0.3.1 Eigenvalues

```
In [ ]: lambda_ks
```

```
In [ ]: D = Matrix.diag(lambda_ks)
        D
```

```
In [ ]: lambda_symbs = []
        for i in range(0,n):
            if i == n-1:
                lambda_symbs.append(1)
            else:
                lambda_symbs.append(symbols('t_'+str(i)))
        D_2 = Matrix.diag(lambda_symbs)
        D_2
```

```
In [ ]: numLambda_ks = []
        for l in lambda_ks:
            numLambda_ks.append(l.evalf())
        numLambda_ks
```

0.3.2 Exact Values

```
In [ ]: comp.evalf()
```

0.3.3 P

```
In [ ]: S_inv = comp.inv()
        P = comp*D*S_inv
        simplify(P)
```

0.3.4 P^n

```
In [ ]: n_power = symbols('n', positive=True)
        P_n = comp*(D**n_power)*S_inv
        P_n = simplify(P_n)
```

```
In [ ]: P_n
```

```
In [ ]: P_n.subs(n_power, 10).evalf()
```

```
In [ ]: with open('complete_symbolic_P_n_example2.txt','w') as f:
            f.write(latex(P_n))
```

0.4 Computation

0.4.1 Values

```
In [ ]: a_1subs = "1/4"
        a_2subs = "1/3"
```

```
In [ ]: partial = P_n
        for i in range(0,n):
            partial = partial.subs(lambda_symbs[i],lambda_ks[i])
        v1 = partial.subs(a_1,parse_expr(a_1subs)).subs(a_2,parse_expr(a_2subs))
        v2 = partial.subs(a_2,parse_expr(a_1subs)).subs(a_1,parse_expr(a_2subs))
```

```
In [ ]: with open('compare_swapped_numeric.txt','w') as f:
            f.write(latex(v1))
            f.write("\n\n")
            f.write(latex(v2))
```

0.4.2 Close form

```
In [ ]: exp = factor(comp.subs(a_1,parse_expr(a_1subs)).subs(a_2,parse_expr(a_2subs)))
        exp
```

```
In [ ]: exp.inv()
```

0.4.3 Approximate Form

```
In [ ]: exp = factor(comp.subs(a_1,eval(a_1subs)).subs(a_2,eval(a_2subs)))
        exp.evalf()
```

```
In [ ]: exp.inv()
```

Appendix D

Mark Dela's MATLAB code

D.1 Code for finding eigenvalues

```
function [eigenvalues] = getEigenvalues (A)


%---- Description ----
%
% This takes a Kouachi matrix satifying the conditions of Theorem 3
% and returns a vector of its eigenvalues. The function will also
% check if the conditions of Theorem 3 are satisfied, throwing an
% error if conditions are not met.
%
%---- Requires ----
%
% extractor (extractor.m), eigenvalue_k (eigenvalue_k.m)
%
```

```
%---- Inputs ----
%
% A: Kouachi matrix satifying the conditions of Theorem 3.
%
%---- Outputs ----
%
% eigenvalues: a vector of eigenvalues in the order of prescribed
% by Kouachi.

%-------------------
%---- Main code ----
%-------------------

%---- Validation ----
% First derive parameters from matrix A.
[mdiag, supdiag, subdiag, alpha, beta, d, n] = extractor (A);

% Check if matrix A is of even size.
if ~(mod(n, 2) == 0)


    n
    error('Matrix is not of even size.');

end
```

```
% Check if alpha * beta = d^2.
product = alpha * beta;
dsq = d^2;
if ~(abs(product - dsq) < eps)

    product
    dsq
    error('Condition alpha*beta = d^2 is not satisfied.');

end

%---- Logic ----

% Construct vector to hold eigenvalues.
eigenvalues = zeros(1, n);

for k = 1:n

    eigenvalues(k) = eigenvalue_k(k, n, alpha, beta, mdiag(1),. . .
 mdiag(2), d);

end

end
function [eigenvalues eigenvectors] = getEigenValVec (Ab, a, c,. . .
```

```
alpha, beta)

%---- Description ----
%
% This function serves as a wrapper for the functions
% getEigenvalues and getEigenvectors. See documentation of these
% two functions for more information.
%
%---- Requires ----
%
% getEigenvalues (getEigenvalues.m),
% getEigenvectors (getEigenvectors.m)
%
%---- Inputs ----
%
% Ab: a square matrix satisfying conditions specified in Theorem 3
% of Kouachi
%
% or
%
% vector for main diagonal.
% a: vector for sub-diagonal.
%
% c: vector for super-diagonal.
%
```

```
% alpha: paramater to be subtracted from b(1).
%
% beta: parameter to be subtracted from b(n).
%
%---- Outputs ----
%
% eigenvalues: the eigenvalues associated with matrix A.
%
% eigenvectors: the eigenvectors associated with matrix A.



%-------------------
%---- Main code ----
%-------------------

%---- Validation ----
%
% Handled by the validators in functions getEigenvalues and
% getEigenvectors.

%---- Logic ----

if nargin == 1
```

```
    eigenvalues = getEigenvalues(Ab);
    eigenvectors = getEigenvectors(Ab);

else

    P = constructor(Ab, a, c, alpha, beta);

    eigenvalues = getEigenvalues(P);
    eigenvectors = getEigenvectors(P);

end

end
```

D.2 Code for finding eigenvectors

```
function [eigenvectors] = getEigenvectors (A)

%---- Description ----
%
% Outputs a matrix of eigenvectors associated with a Kouachi matrix
% satisfying Theorem 3 on page 114.
%
%---- Requires ----
%
```

```
% extractor (extractor.m),
% getEigenvalues (getEigenvalues.m),
% eigenvector1_k (eigenvector1_k.m),
% eigenvectorl_k (eigenvectorl_k.m),
% eigenvectorRecur_k (eigenvectorRecur_k)
%
%---- Inputs ----
%
% A: Kouachi matrix satifying the conditions of Theorem 3.
%
%---- Outputs ----
%
% eigenvectors: a matrix where the columns are the eigenvectors for
% Kouachi matrix A. Eigenvectors are ordered in the Kouachi sense.



%-------------------
%---- Main code ----
%-------------------


%---- Validation ----
% First derive parameters from matrix A.
[mdiag, supdiag, subdiag, alpha, beta, d, n] = extractor (A);
```

```
% Check if matrix A is of even size.
if ~(mod(n, 2) == 0)

    n
    error('Matrix is not of even size.');

end

% Check if alpha * beta = d^2.
product = alpha * beta;
dsq = d^2;
if ~(abs(product - dsq) < eps)

    product
    dsq
    error('Condition alpha*beta = d^2 is not satisfied.');

end

%---- Logic ----

% Obtain eigenvalues.
eigenvalues = getEigenvalues(A);

% Construct matrix to hold eigenvalues.
```

```
eigenvectors = zeros(n);

% Calculating the first n - 2 eigenvectors.
for i = 1:n-2

    % First component.
    eigenvectors(1, i) = eigenvector1_k (i, n, mdiag, alpha, beta, ...
                                         d);
    % Remaining components
    for l = 2:n

            eigenvectors(l, i) = eigenvectorl_k (l, i, n, mdiag,
subdiag, alpha, beta, d);

    end
end




% Calculating the final 2 eigenvectors.
n1 = n - 1;



% **** Change n - 1 to just 1 if you want the function to just use
```

```
% **** the recursion.
for i = n - 1:n

    lambda_k = eigenvalues(i);

    % Transpose needs to be taken here.
    eigenvectors(:, i) = eigenvectorRecur_k(mdiag, supdiag,
 subdiag, alpha, beta, lambda_k, lastComponent = 0);

end

% Normalize the final matrix.
for i = 1:n

    eigenvectors(:, i) = eigenvectors(:, i) / eigenvectors(n, i);

end

end

function [eigenvectorRecur_k] = eigenvectorRecur_k(mdiag, supdiag,..
 subdiag, alpha, beta, lambda_k, lastComponent = 0)

%---- Description ----
%
% This function will find the kth eigenvector recursively provided
```

```
% we know entries in the main diagonal, super-diagonal,
% subdiagonal, alpha, beta, and the kth eigenvalue. This function
% is needed since, as far as we understand, Kouachi's formulas
% for the last two eigenvectors depend on theta_{n - 1} and
% theta_{n}, but these don't exist.
%
% The recursion has a variant, in that the last component can be
% calculated a different way. The function allows for this by
% setting the last argument 'lastComponent' equal to True.
%
%---- Requires ----
%
% NA
%
%---- Inputs ----
%
% mdiag: main diagonal of Kouachi matrix.
%
% supdiag: super-diagonal of Kouachi matrix.
%
% subdiag: sub-diagonal of Kouachi matrix.
%
% alpha: parameter subtracted from mdiag(1).
%
% beta: parameter subtracted from mdiag(n).
```

```
%
% lambda_k: the kth eigenvector of Kouachi matrix.
%
% lastComponent: a Boolean automatically set to False (0).

%  Set to True (1) to apply the other formula of the last component.
%
%---- Outputs ----
%
% The kth eigenvector. Note that this will give the entire
% eigenvector, not just individual components.

%-------------------
%---- Main code ----
%-------------------

%---- Validation ----

%---- Logic ----
n = length(mdiag);
eigenvectorRecur_k = zeros(n, 1);

% Set first component to 1.
eigenvectorRecur_k(1) = 1;
```

```
eigenvectorRecur_k(2) = ((lambda_k - mdiag(1)) + alpha) / supdiag(1);

for i = 3:n

    i1 = i - 1;
    i2 = i - 2;

    ui1 = eigenvectorRecur_k(i1);
    ui2 = eigenvectorRecur_k(i2);

    nmrtr = ((lambda_k - mdiag(i1)) * ui1) - (subdiag(i2) * ui2);

    eigenvectorRecur_k(i) = nmrtr / supdiag(i1);

end

if lastComponent == 1

    n1 = n - 1;
    un1 = eigenvectorRecur_k(n1);

    eigenvectorRecur_k(n) = (subdiag(n1) * un1) /
(lambda_k - mdiag(n) + beta);

end
```

```
end
```

Appendix E

Luis Cervantes's MATLAB code

Matlab Code:

```
function [D,v]=koachibeta(A)
syms a_1
syms a_2
a_1=A(2,1);
a_2=A(3,2);
[~,n]=size(A);
m=n/2;
b_1=1-2*a_2;
b_2=1-2*a_1;
alpha=-a_2;
beta=-a_1;
dsqrd=a_1*a_2;
k=zeros(n,1);
D=sym(zeros(n));
k=sym(k);
v=sym(zeros(n));

this program is written by cases using Said Kouachi's Theorem for even matrices.

for theta=1:m-1  (case 1)

theta1=sym((theta*pi)/(2*m));
k(theta,1)=sym(theta1);
lambda=sym((((b_1+b_2)-sqrt((b_1-b_2)^2+16*dsqrd*cos(theta1)^2))/2));
lambda=sym(simplifyFraction(lambda));
D(theta,theta)=sym(lambda);
veceq =sym( A - (D(theta,theta)*eye(n)) ); (eigenvector corresponding to eigenvalue)
vecreal=sym(null(veceq));
v(:,theta)=sym(vecreal);
end

for theta=m:(2*m-2)  (case 2)
theta1=sym(((theta-m+1)*pi)/(2*m));
k(theta,1)=sym(theta1);
lambda=((b_1+b_2)+sqrt((b_1-b_2)^2+16*dsqrd*cos(theta1)^2))/2;
lambda=simplifyFraction(lambda);
D(theta,theta)=lambda;
```

```
veceq =sym( A - (D(theta,theta)*eye(n)) ); (eigenvector corresponding to ↩
    eigenvalue)
vecreal=sym(null(veceq));
v(:,theta)=sym(vecreal);
end

for theta=n-1  (case 3)
theta1=0;
k(theta,1)=sym(theta1);
lambda=simplify(((b_1+b_2-alpha-beta)-sqrt((b_1-b_2)^2+(alpha+beta)^2-2*(↩
    b_1-b_2)*(alpha-beta)))/2,'IgnoreAnalyticConstraints', true);
D(theta,theta)=sym(lambda);
veceq =sym( A - (D(theta,theta)*eye(n)) ); (eigenvector corresponding to ↩
    eigenvalue)
vecreal=sym(null(veceq));
v(:,theta)=sym(vecreal);
end

for theta=n (case 4)
theta1=0 ;
k(theta,1)=sym(theta1);
lambda=simplify(((b_1+b_2-alpha-beta)+sqrt((b_1-b_2)^2+(alpha+beta)^2-2*(↩
    b_1-b_2)*(alpha-beta)))/2,'IgnoreAnalyticConstraints', true);
D(theta,theta)=sym(lambda);
veceq =sym( A - (D(theta,theta)*eye(n)) ); (eigenvector corresponding to ↩
    eigenvalue )
vecreal=sym(null(veceq));
v(:,theta)=sym(vecreal);

end



end
```