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EIGENVALUES AND EIGENVECTORS OF SOME TRIDIAGONAL MATRICES

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1. Introduction

Let n, k be fixed natural numbers, $1 \leqq k \leqq n$, and denote by $M_{n,k} = M_{n,k}(v,a,b,s,t)$ the $(n+1)\times(n+1)$ matrix

$$(1) \qquad \begin{pmatrix}
a+v & & & t & & & & & \\
& \ddots & & & \cdot & & & & \\
& & a+v & & & \cdot & & & \\
s & & & v & & & \cdot & & \\
& \cdot & & & \ddots & & & \cdot & \\
& & \cdot & & & v & & & t \\
& & & \cdot & & & v+b & & \\
& & & & \cdot & & & \ddots & \\
& & & & & s & & & v+b
\end{pmatrix}$$

where $v, a, b, s, t \in \mathbf{C}$. The entries $m_{j\ell}$ $(j, \ell = 0, \ldots, n)$ of $M_{n,k}$ are

$$m_{jj} = \begin{cases} a+v & \text{if} \quad j = 0, \ldots, k-1 \\ v & \text{if} \quad j = k, \ldots, n-k \\ b+v & \text{if} \quad j = n-k+1, \ldots, n \end{cases} \qquad (n+1-2k \geqq 0),$$

$$m_{jj} = \begin{cases} a+v & \text{if} \quad j = 0, \ldots, n-k \\ a+b+v & \text{if} \quad j = n-k+1, \ldots, k-1 \\ b+v & \text{if} \quad j = k, \ldots, n \end{cases} \quad (n+1-2k < 0)$$

and

$$m_{j\ell} = \begin{cases} s & \text{if} \quad j = \ell + k,\ \ell = 0, \ldots, n-k \\ 0 & \text{if} \quad 0 < |j-\ell| \neq k \\ t & \text{if} \quad \ell = j+k,\ j = 0, \ldots, n-k. \end{cases}$$

In [7] we factorized $\det M_{n,k}$ if $s = t = 1$ and used this result to find the best constants in some quadratic inequalities. Here we apply another

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method to determine the eigenvalues of $M_{n,k}$. This method enables us to find the eigenvectors of $M_{n,k}$ too. In possession of the eigenvectors we can complement Theorem 4 of [6] by giving the cases of equality. Moreover, we can find some new discrete quadratic inequalities of Wirtinger type.

We remark that multidiagonal matrices (i.e. matrices which have the same entries in the diagonals — except possibly the main diagonal) appear in different areas of mathematics [2], [3], [4], [9]. Thus the investigation of $M_{n,k}$ is of interest in itself too. Throughout the paper $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$ denote the set of complex, real, and integer numbers respectively.

2. Evaluation of $\det M_{n,k}$

Denote by $D_{n,k} = D_{n,k}(v,a,b,s,t)$ the determinant of $M_{n,k}(v,a,b,s,t)$, and let $D_{0,1}(v,a,b,s,t) = a+b+v$.

THEOREM 1. *Let $n+1 = kq+r$ $(0 \leqq r < k)$. Then we have*

$$D_{n,k}(v,a,b,s,t) = D_{q,1}(v,a,b,s,t)^r D_{q-1,1}(v,a,b,s,t)^{k-r}. \tag{2}$$

PROOF. Let us rearrange both the rows and columns of $M_{n,k}$ in the order of indices

$$0,k,\dots,qk;1,k+1,\dots,qk+1;\dots;r-1,k+r-1,\dots,qk+r-1; \tag{3}$$

$$r,k+r,\dots,(q-1)k+r;r+1,k+r+1,\dots,(q-1)k+r+1;\dots \tag{4}$$
$$\dots;k-1,2k-1,\dots,(q-1)k+k-1.$$

We remark that throughout the paper $\{0,1,\dots,n\}$ is used as the index set of rows and columns of $M_{n,k}$. This index set is more convenient for our purpose than the usual set $\{1,2,\dots,n+1\}$.

(3) contains r groups of $q+1$ indices while (4) has $k-r$ groups of q indices. If $r=0$, (3) is empty and all indices are contained in (4) .

The rearranged matrix has r blocks of $M_{q,1}(v,a,b,s,t)$ and $k-r$ blocks of $M_{q-1,1}(v,a,b,s,t)$ in the "diagonal". Hence (2) follows. □

A similar rearrangement has been applied by Egerváry and Szász [2] to find $M_{n,k}(-\lambda,0,0,1,1)$. Next we find $D_{q,1}(v,a,b,s,t)$. We may suppose that $st \neq 0$ since if $s=0$ or $t=0$ then $D_{q,1}$ is a triangular determinant whose value is the product of its diagonal elements.

THEOREM 2. *Let* $q = 0, 1, \dots; v, a, b, s, t \in \mathbf{C}$, $st \neq 0$, $\sigma = \sqrt{st}$. *If* $v \neq \pm 2\sigma$ *then*

(5) $$D_{q,1}(v,a,b,s,t) = \frac{\sigma^{q+1}}{\sin\vartheta}\left[\sin(q+2)\vartheta + \frac{a+b}{\sigma}\sin(q+1)\vartheta + \frac{ab}{\sigma^2}\sin q\vartheta\right]$$

where $\vartheta \in \mathbf{C}$ *is such that* $v = 2\sigma\cos\vartheta$ $(\sin\vartheta \neq 0)$. *If* $v = \pm 2\sigma$ *then*
(6)
$$D_{q,1}(v,a,b,s,t) = \sigma^{q+1}\left[(\pm 1)^{q+2}(q+2) + \frac{a+b}{\sigma}(\pm 1)^{q+1}(q+1) + \frac{ab}{\sigma^2}(\pm 1)^q q\right]$$

i.e. in this case $D_{q,1}$ *can be obtained as the limit of the right hand side of* (5) *as* $\vartheta \to m\pi$ *where* $v = (-1)^m 2\sigma$, $m \in \mathbf{Z}$.

$D_{q,1}$ *can also be expressed by help of the Chebychev polynomials* U_j *of the second kind as*

(7) $$D_{q,1}(v,a,b,s,t) = \sigma^{q+1}\left[U_{q+1}\left(\frac{v}{2\sigma}\right) + \frac{a+b}{\sigma}U_q\left(\frac{v}{2\sigma}\right) + \frac{ab}{\sigma^2}U_{q-1}\left(\frac{v}{2\sigma}\right)\right]$$

where U_j *is defined as the extension of the polynomial*

(8) $$U_j(\cos\vartheta) = \frac{\sin(j+1)\vartheta}{\sin\vartheta} \quad (j = -1, 0, 1 \dots).$$

PROOF. Expanding $D_{q,1}(v,a,b,s,t)$ by the zeroth row and also expanding the cofactor of t by the zeroth column we get

(9) $$D_{q,1}(v,a,b,s,t) = (a+v)D_{q-1,1}(v,0,b,s,t) - tsD_{q-2,1}(v,0,b,s,t)$$

if $q \geqq 2$. (9) shows that

(10) $$d_q := D_{q,1}(v,0,b,s,t)$$

satisfies the linear homogeneous difference equation

(11) $$d_{\ell+2} - vd_{\ell+1} + tsd_\ell = 0 \quad (\ell = 0, 1, \dots)$$

with the initial conditions

(12) $$d_0 = b + v, \quad d_1 = (b+v)v - ts.$$

Since $\sigma \neq 0$ we can always find $\vartheta \in \mathbf{C}$ such that $v = 2\sigma\cos\vartheta$ (ϑ is unique if we require that $-\pi \leqq \operatorname{Re}\vartheta < \pi$, $\operatorname{Im}\vartheta > 0$ or $-\pi \leqq \operatorname{Re}\vartheta \leqq 0$, $\operatorname{Im}\vartheta = 0$ hold).

Then the roots of the characteristic equation

$$\lambda^2 - (2\sigma\cos\vartheta)\lambda + \sigma^2 = 0$$

of (11) are $\lambda_{1,2} = \sigma e^{\pm i\vartheta}$. $\lambda_1 \neq \lambda_2$ if and only if $\vartheta \neq m\pi$, $m \in \mathbf{Z}$. Hence

$$d_\ell = \begin{cases} \sigma^\ell(c_1 e^{i\ell\vartheta} + c_2 e^{-i\ell\vartheta}) & \text{if} \quad \vartheta \neq m\pi \ (\lambda_1 \neq \lambda_2) \\ \sigma^\ell(c_1 e^{i\ell\vartheta} + c_2 \ell e^{i\ell\vartheta}) & \text{if} \quad \vartheta = m\pi \ (\lambda_1 = \lambda_2) \end{cases}$$

for $\ell = 0, 1, \ldots$ where c_1, c_2 are constants to be determined from (12). Calculating c_1, c_2 we obtain that

$$\text{(13)} \quad d_\ell = \begin{cases} \dfrac{\sigma^\ell}{\sin\vartheta}[\sigma\sin(\ell+2)\vartheta + b\sin(\ell+1)\vartheta] & \text{if} \quad \vartheta \neq m\pi \\ \sigma^\ell\left[\sigma(\ell+2)(-1)^{m(\ell+2)} + b(\ell+1)(-1)^{m(\ell+1)}\right] & \text{if} \quad \vartheta = m\pi. \end{cases}$$

From (9), (10), (13) by applying the formula $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ several times we get exactly (5) and (6). (7) follows from (5) and (8) taking into consideration that $D_{q,1}$ is a polynomial of its variables. □

REMARKS. In (5)–(13) any fixed value of the square root $\sqrt{st} = \sigma$ can be used.

$D_{q,1}(v, a, b, 1, 1)$ has been evaluated by Rutherford [10].

3. Eigenvalues and eigenvectors of $M_{q,1}$

Let λ be an eigenvalue of $M_{q,1}(v, a, b, s, t)$ and consider the system of equations

$$\text{(14)} \qquad M_{q,1}(v - \lambda, a, b, s, t)y = 0$$

where $y = (y_0, \ldots, y_q)^T$ and T denotes transposition. (14) can be written as

$$\text{(15)} \qquad \begin{cases} (a + v - \lambda)y_0 + ty_1 = 0 \\ sy_0 + (v - \lambda)y_1 + ty_2 = 0 \\ \qquad \vdots \\ sy_{q-2} + (v - \lambda)y_{q-1} + ty_q = 0 \\ sy_{q-1} + (b + v - \lambda)y_q = 0. \end{cases}$$

Let $\sigma = \sqrt{st} \neq 0$ and assuming $v - \lambda \neq \pm 2\sigma$ substitute $v - \lambda = 2\sigma\cos\vartheta$, $\vartheta \in \mathbf{C}$. From the first equation of (15)

$$y_1 = -\frac{1}{t}(a + 2\sigma\cos\vartheta)y_0 = -\frac{\sigma\sin 2\vartheta + a\sin\vartheta}{t\sin\vartheta}y_0.$$

We easily obtain by induction that

$$\text{(16)} \qquad y_\ell = (-1)^\ell\,\frac{\sigma^\ell\sin(\ell+1)\vartheta + a\sigma^{\ell-1}\sin\ell\vartheta}{t^\ell\sin\vartheta}y_0 \qquad (\ell = 1, 2, \ldots, q).$$

Substituting y_q and y_{q-1} into the last equation of (15) we get after some calculations that

$$\frac{(-1)^q}{t^q} D_{q,1}(v-\lambda, a, b, s, t) y_0 = 0.$$

Since $D_{q,1}(v-\lambda, a, b, s, t) = 0$, y_0 is arbitrary and $y_1, \dots, y_q$ are given by (16). If $v - \lambda = 2(-1)^m \sigma$, $m \in \mathbf{Z}$, then in (16) the limit of the right hand side has to be taken as $\vartheta \to m\pi$.

It can be seen that the dimension of the subspace spanned by the eigenvectors corresponding to the eigenvalue λ (i.e. the geometric multiplicity of λ, see [5], § 50) is one. $\square$

Hence we have proved

THEOREM 3. *Let λ be an eigenvalue of $M_{q,1}(v, a, b, s, t)$ ($st \neq 0$, $v, a, b, s, t \in \mathbf{C}$). Then the eigenvectors $y = (y_0, y_1, \dots, y_q)^T$ corresponding to λ are given by*

$$(17) \qquad y_\ell = \left(-\frac{\sigma}{t}\right)^\ell \left[\sin(\ell+1)\vartheta + \frac{a}{b} \sin \ell\vartheta\right] C \qquad (\ell = 0, 1, \dots, q)$$

if $v - \lambda = 2\sigma \cos \vartheta$ ($\vartheta \neq m\pi$, $m \in \mathbf{Z}$) $\sigma = \sqrt{st}$ and by

(18)

$$y_\ell = \left(-\frac{\sigma}{t}\right)^\ell \left[(\ell+1)(-1)^{m\ell} + \frac{a}{\sigma} - \ell(-1)^{m(\ell-1)}\right] C \qquad (\ell = 0, 1, \dots, q)$$

if $v - \lambda = 2\sigma \cos m\pi$, $m \in \mathbf{Z}$ where $C \neq 0$ is an arbitrary constant. Each eigenvalue is of geometric multiplicity 1.

If $\frac{a}{\sigma}, \frac{b}{\sigma} \in \{0, 1, -1\}$ then the eigenvalues of $M_{q,1}$ can be explicitly given.

THEOREM 4. *The following identities hold:*

$$(19) \qquad D_{q,1}(v-\lambda, 0, 0, s, t) = \prod_{j=0}^{q} \left(v - \lambda - 2\sigma \cos \frac{(j+1)\pi}{q+2}\right),$$

(20)

$$D_{q,1}(v-\lambda, 0, \sigma, s, t) = D_{q,1}(v-\lambda, \sigma, 0, s, t) = \prod_{j=0}^{q} \left(v - \lambda - 2\sigma \cos \frac{2(j+1)\pi}{2q+3}\right),$$

$$(21) \qquad D_{q,1}(v-\lambda, 0, -\sigma, s, t) = D_{q,1}(v-\lambda, -\sigma, 0, s, t) =$$

$$= \prod_{j=0}^{q} \left(v - \lambda - 2\sigma \cos \frac{(2j+1)\pi}{2q+3}\right),$$

$$D_{q,1}(v-\lambda,\sigma,\sigma,s,t)=\prod_{j=0}^{q}\left(v-\lambda-2\sigma\cos\frac{(j+1)\pi}{q+1}\right), \tag{22}$$

$$D_{q,1}(v-\lambda,\sigma,-\sigma,s,t)=D_{q,1}(v-\lambda,-\sigma,\sigma,s,t)= \tag{23}$$
$$=\prod_{j=0}^{q}\left(v-\lambda-2\sigma\cos\frac{(2j+1)\pi}{2q+2}\right),$$

$$D_{q,1}(v-\lambda,-\sigma,-\sigma,s,t)=\prod_{j=0}^{q}\left(v-\lambda-2\sigma\cos\frac{j\pi}{q+1}\right), \tag{22}$$

where $\sigma=\sqrt{st}$.

PROOF. The proof is analogous to that of [7] hence it is omitted (we remark that in [7] from (26) a factor -1 and from (28) a factor $\sin\vartheta$ is missing). □

From the above formulae one can see that the eigenvalues of e.g. $M_{q,1}(v,\sigma,\sigma,s,t)$ are $\lambda_j=v-2\sigma\cos\frac{(j+1)\pi}{q+1}$ $(j=0,\dots,q)$.

4. Eigenvalues and eigenvectors of $M_{n,k}$

By Theorem 1 the eigenvalues of $M_{n,k}(v,a,b,s,t)$ are the zeros λ of the polynomials $D_{q,1}(v-\lambda,a,b,s,t)$ and $D_{q-1,1}(v-\lambda,a,b,s,t)$. First we consider the possibility of these polynomials to have common zeros.

THEOREM 5. *The polynomials* $D_{q,1}(v-\lambda,a,b,s,t)$ *and* $D_{q-1,1}(v-\lambda,a,b,s,t)$ *with* $st\neq 0$ *have a common zero* λ *if and only if*

$$ab=st \tag{25}$$

holds. In this case the common zero is $\lambda=v+a+b$.

In particular (25) *holds if*

$$\frac{a}{\sigma}=\frac{b}{\sigma}=1 \quad or \quad \frac{a}{\sigma}=\frac{b}{\sigma}=-1 \tag{26}$$

where $\sigma=\sqrt{st}$.

PROOF. By Theorem 2, $D_{q,1}(v-\lambda,a,b,s,t)=D_{q-1,1}(v-\lambda,a,b,s,t)=0$ holds if and only if *either* $\sin\vartheta\neq 0$ and $v-\lambda=2\sigma\cos\vartheta$ satisfies the equations

$$\sin(q+2)\vartheta+\frac{a+b}{\sigma}\sin(q+1)\vartheta+\frac{ab}{\sigma^2}\sin q\vartheta=0, \tag{27}$$

$$\sin(q+1)\vartheta + \frac{a+b}{\sigma}\sin q\vartheta + \frac{ab}{\sigma^2}\sin(q-1)\vartheta = 0, \tag{28}$$

or if $\sin\vartheta = 0$, $\vartheta = m\pi$, $m \in \mathbf{Z}$, $v - \lambda = 2(-1)^m\sigma$ and the derivatives of the left hand sides of (27), (28) vanish at $\vartheta = m\pi$.

In the first case by the addition theorem of the sine function and by (28) we can rewrite (27) as

$$\cos(q+1)\vartheta + \frac{a+b}{\sigma}\cos q\vartheta + \frac{ab}{\sigma^2}\cos(q-1)\vartheta = 0. \tag{29}$$

Multiplying (28) by i and adding it to (29) we get

$$e^{i(q-1)\vartheta}\left(e^{i\vartheta} + \frac{a}{\sigma}\right)\left(e^{i\vartheta} + \frac{b}{\sigma}\right) = 0 \tag{30}$$

thus

$$\frac{a}{\sigma} = -e^{i\vartheta} \quad \text{or} \quad \frac{b}{\sigma} = -e^{i\vartheta}. \tag{31}$$

If e.g. $\frac{a}{\sigma} = -e^{i\vartheta}$ then

$$\sin \ell\vartheta = \frac{1}{2i}\left(e^{i\ell\vartheta} - e^{-i\ell\vartheta}\right) = \frac{i}{2}(-1)^{\ell+1}\left[\left(\frac{a}{\sigma}\right)^{\ell} - \left(\frac{\sigma}{a}\right)^{\ell}\right] \qquad (\ell = 1, 2, \dots).$$

Substituting this into (28) we obtain after some calculations that (28) holds if and only if

$$(\sigma^2 - a^2)(\sigma^2 - ab) = 0$$

i.e. if

$$a^2 = \sigma^2 \quad \text{or} \quad ab = \sigma^2. \tag{32}$$

$a^2 = \sigma^2$ implies $e^{2i\vartheta} = 1$, $\sin\vartheta = 0$ which was excluded. Thus $ab = \sigma^2 = st$ is necessary for $D_{q,1}$ and $D_{q-1,1}$ to have common zero. It is sufficient too since (25) ensures that (28) holds while by (31) the equation (29) and hence (27), too, hold.

$$\cos\vartheta = \frac{1}{2}\left(e^{i\vartheta} + e^{-i\vartheta}\right) = -\left(\frac{a}{\sigma} + \frac{\sigma}{a}\right) = -\frac{a+b}{\sigma}$$

thus $\lambda = v - 2\sigma\cos\vartheta = v + a + b$ is the common zero.

Starting with the root $\frac{b}{\sigma} = -e^{i\vartheta}$ of (31) we obtain the same result.

In the second case $\vartheta = m\pi$, $m \in \mathbf{Z}$ and the derivatives of (27), (28) give

$$(q+2)(-1)^{m(q+2)} + \frac{a+b}{\sigma}(q+1)(-1)^{m(q+1)} + \frac{ab}{\sigma^2}q(-1)^{mq} = 0, \tag{33}$$

$$(34)\qquad (q+1)(-1)^{m(q+1)} + \frac{a+b}{\sigma} q(-1)^{mq} + \frac{ab}{\sigma^2}(q-1)(-1)^{m(q-1)} = 0.$$

If m is even then the subtraction of (34) from (33) leads to

$$\left(1+\frac{a}{\sigma}\right)\left(1+\frac{b}{\sigma}\right) = 0$$

hence $\frac{a}{\sigma} = -1$ or $\frac{b}{\sigma} = -1$. If $\frac{a}{\sigma} = -1$ then from (33) we get $\frac{b}{\sigma} = -1$ and conversely $\frac{b}{\sigma} = -1$ and (33) imply $\frac{a}{\sigma} = -1$. Thus

$$(35)\qquad \frac{a}{\sigma} = \frac{b}{\sigma} = -1, \quad \lambda = v - 2\sigma\cos\vartheta = v - 2\sigma = v + a + b.$$

If m is odd then the difference of (33) and (34) can be written as

$$(-1)^q\left(1-\frac{a}{\sigma}\right)\left(1-\frac{b}{\sigma}\right) = 0.$$

Thus $\frac{a}{\sigma} = 1$ or $\frac{a}{\sigma} = 1$. If e.g. $\frac{a}{\sigma} = 1$ then from (33) we get $\frac{b}{\sigma} = 1$ and conversely. Hence

$$(36)\qquad \frac{a}{\sigma} = \frac{b}{\sigma} = 1, \quad \lambda = v - 2\sigma\cos\vartheta = v + 2\sigma = v + a + b.$$

Since (35), (36) are particular cases of (25) we proved that (25) is a necessary condition. It is sufficient too since with $\vartheta = m\pi$ and with (35) or (36) both (27), (28) and (33), (34) are satisfied. □

THEOREM 6. *The eigenvectors $x = (x_0, x_1, \dots, x_n)^T$ of $M_{n,k}(v,a,b,s,t)$ corresponding to the eigenvalue $\lambda = v - 2\sigma\cos\vartheta$ are given by*

$$(37)\qquad x_{u+hk} = \left(-\frac{\sigma}{t}\right)^h \left[\sin(h+1)\vartheta + \frac{a}{\sigma}\sin h\vartheta\right] C_u$$

if $\vartheta \neq m\pi$, $m \in \mathbf{Z}$, and by

$$(38)\qquad x_{u+hk} = \left(-\frac{\sigma}{t}\right)^h \left[(h+1)(-1)^{mh} + \frac{a}{\sigma} h(-1)^{m(h-1)}\right] C_u$$

if $\vartheta = m\pi$, $m \in \mathbf{Z}$. In (37), (38) $u = 0, 1, \dots, r-1$; $h = 0, 1, \dots, q$ and $u = r, r+1, \dots, k-1$; $h = 0, 1, \dots, q-1$. $C_0, C_1, \dots, C_{k-1} \in \mathbf{C}$ are constants such that

(i) *$C_0, C_1, \dots C_{r-1}$ are arbitrary constants not all zero, $C_r = C_{r+1} = \dots \cdots = C_{k-1} = 0$ if $D_{q,1}(v-\lambda,a,b,s,t) = 0 \neq D_{q-1,1}(v-\lambda,a,b,s,t)$,*

(ii) *$C_0 = C_1 = \cdots = C_{r-1} = 0$ and $C_r, C_{r+1}, \dots, C_{k-1}$ are arbitrary constants not all zero if $D_{q,1}(v-\lambda,a,b,s,t) \neq 0 = D_{q-1,1}(v-\lambda,a,b,s,t)$,*

(iii) $C_0, C_1, \dots, C_{k-1}$ *are arbitrary constants not all zero if* $D_{q,1}(v-\lambda, a, b, s, t) = 0 = D_{q-1,1}(v-\lambda, a, b, s, t)$ *(by Theorem* 5 *this case occurs if and only if* $ab = st$*).*

The geometric multiplicity of the eigenvalue λ *in the cases* (i), (ii), (iii) *is* r, $k-r$, k *respectively.*

PROOF. We have to solve the system of equations

$$M_{n,k}(v-\lambda, a, b, s, t)x = 0. \tag{39}$$

Rearranging the equations of (39) according to (3), (4) one can recognize that (39) decomposes to the following systems:

$$M_{q,1}(v-\lambda, a, b, s, t)z_u = 0 \tag{40}$$

where $z_u = (x_u, x_{u+k}, \dots, x_{u+qk})^T$, $u = 0, 1, \dots, r-1$ and

$$M_{q-1,1}(v-\lambda, a, b, s, t)w_u = 0 \tag{41}$$

where $w_u = (x_u, x_{u+k}, \dots, x_{u+(q-1)k})^T$, $u = r, r+1, \dots, k-1$.

In the case (i) Theorem 3 gives the solution (40) for each $u = 0, 1, \dots, r-1$. This justifies (37) and (38) for $u = 0, 1, \dots, r-1$; $h = 0, 1, \dots, q$. (41) has trivial solution only for each $u = r, r+1, \dots, k-1$ since the determinant of the system $D_{q-1,1}(v-\lambda, a, b, s, t)$ is not zero. These trivial solutions are included in (37) and (38) by requiring $C_r = C_{r+1} = \dots = C_{k-1} = 0$.

In the case (ii) (40) has trivial solutions only (therefore $C_0 = C_1 = \dots = C_{r-1} = 0$) and the solutions of (41) are (by Theorem 3) given by (37), (38) for $u = r, r+1, \dots, k-1$; $h = 0, 1, \dots, q-1$.

In the case (iii) both systems (40), (41) have nontrivial solutions which are given by (37), (38).

The statement concerning the geometric multiplicity of the eigenvalues is obvious. □

5. Applications

Here we apply our results to study some discrete quadratic inequalities of Wirtinger type. Let A be an Hermitian matrix of order $n+1$ with eigenvalues $\lambda_0 \geqq \lambda_1 \geqq \dots \geqq \lambda_n$ and let $x^{(0)}, x^{(1)}, \dots, x^{(n)}$ be the corresponding linearly independent eigenvectors. Then

$$\lambda_n\langle x, x\rangle \leqq \langle Ax, x\rangle \leqq \lambda_0\langle x, x\rangle \tag{42}$$

holds for every vector $x \in \mathbf{C}^{n+1}$ where $\langle \cdot, \cdot \rangle$ is the usual inner product

$$\langle x, y\rangle = \sum_{j=0}^{n} x_j \overline{y}_j$$

for $x = (x_0, \dots, x_n)^T$, $y = (y_0, \dots, y_n)^T$. The equality $\lambda_n \langle x, x\rangle = \langle Ax, x\rangle$ holds if and only if $x = 0$ or x is an eigenvector corresponding to λ_n (if $\lambda_n < \lambda_{n-1}$, then x is a scalar multiple of $x^{(1)}$). Similarly $\langle Ax, x\rangle = \lambda_0\langle x, x\rangle$ holds if and only if $x = 0$ or x is an eigenvector corresponding to λ_i (see e.g. [1]).

Let now $v, a, b \in \mathbf{R}$, $\alpha, \beta \in \mathbf{C} \setminus \{0\}$, $t = \alpha\overline{\beta}$, $s = \overline{t} = \overline{\alpha}\beta$, then $M_{n,k}(v, a, b, \alpha\overline{\beta}, \overline{\alpha}\beta)$ is an Hermitian matrix and

$$\langle M_{n,k}(v, a, b, \overline{\alpha}\beta, \alpha\overline{\beta})x, x\rangle = \tag{43}$$

$$= \sum_{j=0}^{n-k} \left[|\alpha x_{j+k} + \beta x_j|^2 - (a + |\alpha|^2)|x_{j+k}|^2 - \right.$$

$$\left. -(b + |\beta|^2)|x_j|^2\right] + (v + a + b)\sum_{j=0}^{n} |x_j|^2.$$

We are going to formulate inequalities of the form (42) in the cases when the best constants (the least and greatest eigenvalues) can explicitly be given.

We have seen that this is the case if $\frac{a}{\sigma} = \varepsilon$, $\frac{b}{\sigma} = \rho$, $\varepsilon, \rho = 0, \pm 1$. The possible values of ε, ρ are listed in the next table.

Table 1

ℓ	ε_ℓ	ρ_ℓ
1	0	0
2	0	1
3	0	−1
4	1	−1
5	1	1
6	−1	−1
7	1	0
8	−1	0
9	−1	1

With $a = \varepsilon_\ell\sigma = \varepsilon_\ell|\alpha\beta|$, $b = \rho_\ell\sigma = \rho_\ell|\alpha\beta|$, $v = -a - b = -|\alpha\beta|(\varepsilon_\ell + \rho_\ell)$ in (43) we have

$$\langle A_{n,k}^{(\ell)}x, x\rangle = \sum_{j=0}^{n-k} \left[|\alpha x_{j+k} + \beta x_j|^2 - (\varepsilon_\ell|\alpha\beta| + |\alpha|^2)|x_{j+k}|^2 - \right. \tag{44}$$

$$\left. -(\rho_\ell|\alpha\beta| + |\beta|^2)|x_j|^2\right]$$

where

$$A_{n,k}^{(\ell)} = M_{n,k}(-|\alpha\beta|(\varepsilon_\ell + \rho_\ell), \varepsilon_\ell|\alpha\beta|, \rho_\ell|\alpha\beta|, \overline{\alpha}\beta, \alpha\overline{\beta}) \tag{45}$$

for $\ell = 1, \dots, 9$.

The eigenvalues of $A_{n,k}^{(\ell)}$ are of the form

$$\lambda = v - 2\sigma\cos\vartheta = -|\alpha\beta|(\varepsilon_\ell + \rho_\ell + 2\cos\vartheta). \tag{46}$$

Based on Theorem 4 it is easy to find the eigenvalues of $A_{n,k}^{(\ell)}$. By (46) Table 2 gives the values of ϑ corresponding to the eigenvalues λ of $A_{n,k}^{(\ell)}$ and also gives $m(\lambda)$, the algebraic multiplicity of λ (which is now equal to the geometric multiplicity).

Table 2

ℓ	ϑ	$(m(\lambda) = r)$	ϑ	$(m(\lambda) = k - r)$	ϑ $(m(\lambda) = k)$
1	$\frac{(j+1)\pi}{q+2}$	$(j = 0, \dots, q)$	$\frac{(j+1)\pi}{q+1}$	$(j = 0, \dots, q-1)$	–
2	$\frac{2(j+1)\pi}{q+3}$	$(j = 0, \dots, q)$	$\frac{2(j+1)\pi}{2q+1}$	$(j = 0, \dots, q-1)$	–
3	$\frac{(2j+1)\pi}{2q+3}$	$(j = 0, \dots, q)$	$\frac{(2j+1)\pi}{2q+1}$	$(j = 0, \dots, q-1)$	–
4	$\frac{(2j+1)\pi}{2q+2}$	$(j = 0, \dots, q)$	$\frac{(2j+1)\pi}{2q}$	$(j = 0, \dots, q-1)$	–
5	$\frac{(j+1)\pi}{q+1}$	$(j = 0, \dots, q-1)$	$\frac{(j+1)\pi}{q}$	$(j = 0, \dots, q-2)$	π
6	$\frac{j\pi}{q+1}$	$(j = 1, \dots, q)$	$\frac{j\pi}{q}$	$(j = 1, \dots, q-1)$	0
7	same as $\ell = 2$				
8	same as $\ell = 3$				
9	same as $\ell = 4$				

In Table 3 we collected $\vartheta_{\min}^{(\ell)}$, $\vartheta_{\max}^{(\ell)}$ such that for each $\ell = 1, \dots, 6$, (46) gives the minimal and maximal eigenvalue of $A_{n,k}^{(\ell)}$ if $\vartheta = \vartheta_{\min}^{(\ell)}$ and $\vartheta = \vartheta_{\max}^{(\ell)}$ respectively. Here we used the relation

$$\left[\frac{n}{k}\right] = \begin{cases} q & \text{if } r > 0 \\ q-1 & \text{if } r = 0 \end{cases}$$

where $[x]$ denotes the greatest integer not exceeding x. We omitted $\ell = 7, 8, 9$ since by Table 2 $\vartheta_{\min}^{(\ell+5)} = \vartheta_{\min}^{(\ell)}$, $\vartheta_{\max}^{(\ell+5)} = \vartheta_{\max}^{(\ell)}$ for $\ell = 2, 3, 4$.

Table 3

ℓ	$\vartheta_{\min}^{(\ell)}$	$\vartheta_{\max}^{(\ell)}$
1	$\dfrac{\pi}{\left[\frac{n}{k}\right]+2}$	$\pi-\vartheta_{\min}^{(1)}$
2	$\dfrac{2\pi}{2\left[\frac{n}{k}\right]+3}$	$\pi-\frac{1}{2}\vartheta_{\min}^{(2)}$
3	$\dfrac{\pi}{2\left[\frac{n}{k}\right]+3}$	$\pi-2\vartheta_{\min}^{(3)}$
4	$\dfrac{\pi}{2\left[\frac{n}{k}\right]+2}$	$\pi-\vartheta_{\min}^{(4)}$
5	$\dfrac{\pi}{\left[\frac{n}{k}\right]+2}$	π
6	0	$\pi-\dfrac{\pi}{\left[\frac{n}{k}\right]+1}$

Finally we formulate the main result of this section.

THEOREM 7. *Let n,k be fixed natural numbers, $1\leqq k\leqq n$, $n+1=kq+r$ $(0\leqq r<k)$. For every $x=(x_0,\dots,x_n)^T\in\mathbf{C}^{n+1}$ and $\ell=1,\dots,6$ the inequality*

$$-|\alpha\beta|\left(\varepsilon_\ell+\rho_\ell+2\cos\vartheta_{\min}^{(\ell)}\right)\langle x,x\rangle\leqq\langle A_{n,k}^{(\ell)}x,x\rangle\leqq \tag{47}$$

$$\leqq-|\alpha\beta|\left(\varepsilon_\ell+\rho_\ell+2\cos\vartheta_{\max}^{(\ell)}\right)\langle x,x\rangle$$

holds where

$$\langle A_{n,k}^{(\ell)}x,x\rangle=\sum_{j=0}^{n-k}\left[|\alpha x_{j+k}+\beta x_j|^2-(\varepsilon_\ell|\alpha\beta|+|\alpha|^2)|x_{j+k}|^2-\right.$$

$$\left.-(\rho_\ell|\alpha\beta|+|\beta|^2)|x_j|^2\right],$$

$\alpha,\beta\in\mathbf{C}\setminus\{0\}$ and ε_ℓ, ρ_ℓ; $\vartheta_{\min}^{(\ell)}$, $\vartheta_{\max}^{(\ell)}$ are given by Tables 1 and 3 respectively.

Equality on the left hand side of (47) occurs for $\ell=1,2,3,4,5$ if and only if

$$x_{u+hk}=\left(-\frac{|\alpha\beta|}{\alpha\bar{b}}\right)^h\left[\sin(h+1)\vartheta_{\min}^{(\ell)}+\varepsilon_\ell\sin h\vartheta_{\min}^{(\ell)}\right]C_u \tag{48}$$

holds for

$$u = 0, 1, \ldots, r-1; \quad h = 0, 1, \ldots, q \quad \textit{and} \tag{49}$$

$$u = r, r+1, \ldots, k-1; \quad h = 0, 1, \ldots, q-1$$

while for $\ell = 6$ *if and only if*

$$x_{u+hk} = \left(-\frac{|\alpha\beta|}{\alpha\overline{\beta}}\right)^h D_u \tag{50}$$

holds for the subscripts (49).

Equality on the right hand side of (47) *is valid for* $\ell = 1, 2, 3, 4, 6$ *if and only if*

$$x_{u+hk} = \left(-\frac{|\alpha\beta|}{\alpha\overline{\beta}}\right)^h \left[\sin(h+1)\vartheta_{\max}^{(\ell)} + \varepsilon_\ell \sin h\vartheta_{\max}^{(\ell)}\right] C_u \tag{51}$$

holds for the indices (49) *while for* $\ell = 5$ *if and only if*

$$x_{u+hk} = \left(\frac{|\alpha\beta|}{\alpha\overline{\beta}}\right) D_u \tag{52}$$

is fulfilled for the subscripts (49).

Here $C_0, \ldots, C_{r-l}$ *are arbitrary constants,* $C_r = C_{r+1} = \cdots = C_{k-1} = 0$ *(if* $r = 0$ *then all* C_u*'s are arbitrary) and* $D_0, \ldots, D_{k-1}$ *are arbitrary constants.*

PROOF. The statement of Theorem 7 follows from the general result (42) taking into consideration Theorem 6 and the calculations of Section 5 concerning $A_{n,k}^{(\ell)}$ and its "parameters". □

The cases $\ell = 7, 8, 9$ can be obtained from $\ell = 2, 3, 4$ by exchanging ε_ℓ, α to ρ_ℓ, β respectively.

Several special cases of (47) are known. If $k = 1$, $\alpha = \beta = 1$, (47) has been proved by Fan, Taussky and Todd [3]. Their inequalities are discrete analogues of Wirtinger's inequality; see e.g. Hardy, Littlewood and Pólya [6], p. 184. For $\beta = 1$, $\alpha = \pm 1$, $\ell = 1, 3, 6$, Theorem 7 (without the equality clause) has been proved by the author [7] (the cases (i), (ii), (iii), (iv) of [7] can easily be rewritten to the cases $\ell = 6, 3, 3, 1$ respectively). Concerning inequalities related to (47) we refer to [2], [8].

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