First axiomatisation of a variant of Strategy Logic

First-Order Coalition Logic

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Motivation *Strategy Logic* (SL) is believed to be not finitely axiomatisable. We present **the first axiomatisation** of a variant of SL, where we keep the arbitrary quantifier prefixes but consider only single-step strategies. We also **reopen** the problem of unaxiomatisability of SL.

A Well-Behaved Fragment of SL

$$\text{FOCL} \ni \varphi := p \mid \neg\varphi \mid (\varphi \land \varphi) \mid (\!(t_1, ..., t_n)\!)\, \varphi \mid \forall x \varphi,$$

where term t_i is either a variable or a an explicit action.
Let $\mathfrak{G} = \langle n, \mathtt{Ac}, \mathcal{D}, S, R, \mathcal{V}\rangle$ be a a *Concurrent Game Structure* (CGS).

$\mathfrak{G}, s \models (\!(a_1, ..., a_n)\!)\, \psi$ iff $\exists t \in S$ s.t. $\langle s, a_1, ..., a_n, t\rangle \in R$ and $\mathfrak{G}, t \models \psi$
$\mathfrak{G}, s \models \forall x \psi$ iff $\forall a \in \mathtt{Ac} : \mathfrak{G}, s \models \psi[a/x]$

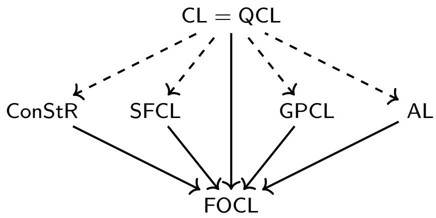
FOCL allows for arbitrary quantifier prefixes and strategy sharing. We can, for example, express Nash Equilibrium for deterministic strategies: $\exists x_1, ..., x_n (\bigwedge_{i=1}^{n} \exists y_i (\!(x_1, ..., y_i, ..., x_n)\!)\, \psi_i \to (\!(x_1, ..., x_i, ..., x_n)\!)\, \psi_i)$.

Results

1. FOCL has a **sound and complete axiomatisation**.
2. FOCL is **more expressive** than other coalition logics.
3. **Model checking** is **PSPACE-complete**, **SAT** is **undecidable**.

Details

Expressivity FOCL allows for strategy sharing. E.g. formula $\exists x (\!(x, x)\!)\, \neg p$ does not have an equivalent counterpart in many coalition logics, and hence FOCL is **strictly more expressive** than them.



Axiomatisation Compared to SL, we consider quantification over actions rather than strategies. This, together with explicit actions, allows us to provide the following **sound and complete** axiomatisation.

PC	Every propositional tautology
K	$((\!(\vec{t})\!)\, \varphi \land (\!(\vec{t})\!)\, \psi) \leftrightarrow (\!(\vec{t})\!)\, (\varphi \land \psi)$
N	$\neg(\!(\vec{t})\!)\, \varphi \leftrightarrow (\!(\vec{t})\!)\, \neg\varphi$
E	$\forall x \varphi \to \varphi[t/x]$
B	$\forall x (\!(\vec{t})\!)\, \varphi \to (\!(\vec{t})\!)\, \forall x \varphi$, s.t. $t_i \neq x$
MP	From $\varphi, \varphi \to \psi$, infer ψ
Nec	From φ, infer $(\!(\vec{t})\!)\, \varphi$
Gen	From $\varphi \to \psi[t/x]$, infer $\varphi \to \forall x \psi$, if $t \notin \varphi$

Undecidability We use a reduction from the tiling problem to show that the satisfiability problem for FOCL is **undecidable**. We have also discovered a **gap** in the Σ_1^1-hardness proof for SL **in the original** SL **papers**. This implies that the question of whether SL is finitely, or even recursively, axiomatisable is **open again**. We conjecture that SL is indeed Σ_1^1-hard, as quantification over strategies is essentially a second-order quantification over functions. However, the formal argument is **yet to be provided**.

IJCAI 2025 MONTREAL

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