Logic, Automata, and Algebra, Together Again

Dakotah Lambert

23 April 2024

software

slides

Intro: Formal languages

Assume finite alphabet A

Formal language partitions words (A^+) into two classes: accepted / rejected

Intro: Formal languages and classification

Assume finite alphabet A

FORMAL LANGUAGE partitions words (A^+) into two classes:
accepted / rejected

CLASSES: sets of languages with some shared property

Intro: Applications

- Language modelling
- Machine learning and learnability
- Robotic planning and control
- Software security
  - Can’t correctly parse with something too weak
  - But too much power creates and exposes vulnerabilities

    (Sassaman *et al.*, 2013)

Intro: Some types of classes

- Logical
  - “Definable with first-order logic on strings with adjacency”
  - Prove: give the formula
  - Disprove: Ehrenfeucht-Fraïssé games, …

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- Algebraic
  - "Semigroup is finite and satisfies both $x^\omega a y^\omega b x^\omega c y^\omega = x^\omega c y^\omega b x^\omega a y^\omega$ and $x x^\omega = x^\omega$"
  - Prove: verify equations for all instantiations
  - Disprove: find an instantiation breaking an equation

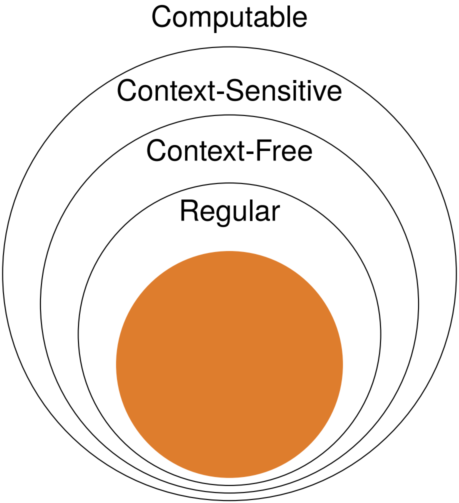
Intro: Why algebra?

Often want logical or language-theoretic descriptions.

So why algebra?

- Easy to prove and disprove membership
- Equations make subclass relations easy to show
- Unified approach for all applicable classes

Intro: The Chomsky hierarchy (Chomsky, 1959)



Semigroups

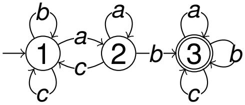
SEMIGROUP: a set S with multiplication

- xy always exists
- $(xy)z = x(yz)$ so xyz makes sense

Strings are a semigroup but infinite

Semigroups: Equivalence classes

$$A = \{a, b, c\}$$



$$L_{\text{good}} = \{w \in A^+ : w \text{ contains } ab\}$$
$$L_{\text{bad}} = A^+ - L_{\text{good}}$$

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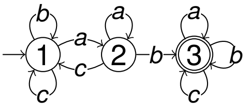

$$L_{\text{good}} = \{w \in A^{+} : w \text{ contains } ab\}$$
$$L_{\text{bad}} = A^{+} - L_{\text{good}}$$

not compatible with concatenation

$a \sim b$ but $a\,a \not\sim a\,b$

Semigroups: Equivalence classes

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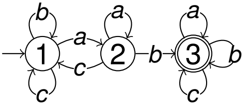

Semigroups: Equivalence classes

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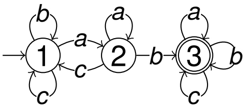

$$L_3 = \{w \in A^+ : w \text{ contains } ab\}$$
$$L_2 = \{w \in A^+ : w \text{ ends with } a\} - L_3$$
$$L_1 = A^+ - L_2 - L_3$$

Semigroups: Equivalence classes

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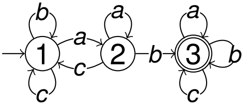

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$$L_1 = A^+ - L_2 - L_3$$

still not compatible with concatenation

$a \sim ba$ but $a\,a \not\sim a\,ba$

Semigroups: Equivalence classes

Nerode: $u \sim v$ means, for all y,
uy is accepted if and only if vy is

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Myhill: $u \sim v$ means, for all x, y,
xuy is accepted if and only if xvy is

Semigroups: Equivalence classes

Nerode: $u \sim v$ means, for all y,
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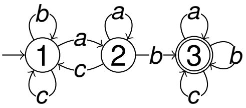
Myhill: $u \sim v$ means, for all x, y,
xuy is accepted if and only if xvy is

Myhill compatible with concatenation:
$u \sim v$ and $x \sim y$ implies $ux \sim vy$
(ux is good $\leftrightarrow$ vx is good $\leftrightarrow$ vy is good)

Semigroups: Equivalence classes

$$A = \{a, b, c\}$$

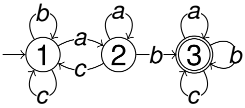


Core of Myhill equivalence:
State categorizes prefix.
How do words act on these prefixes?

Example: $f_a = (1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3)$

Semigroups: Equivalence classes

$$A = \{a, b, c\}$$



$$f_a = (1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3)$$

Semigroups: Equivalence classes

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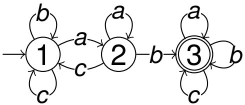

$$f_a = (1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3)$$
$$f_b = (1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 3)$$

Semigroups: Equivalence classes

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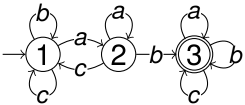

$$f_a = (1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3)$$
$$f_b = (1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 3)$$
$$f_c = (1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 3)$$

Semigroups: Equivalence classes

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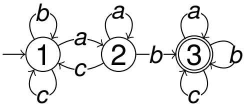

$$\begin{aligned}
f_a &= (1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3) \\
f_b &= (1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 3) \\
f_c &= (1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 3) \\
f_{ab} &= (1 \mapsto 3, 2 \mapsto 3, 3 \mapsto 3) \\
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\end{aligned}$$

Semigroups: Equivalence classes

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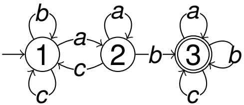

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\end{aligned}$$

others: $f_{aa} = f_{ca} = f_a, \quad f_{ac} = f_{cb} = f_{cc} = f_c, \quad f_{bb} = f_{bc} = f_b$

Semigroups: Equivalence classes

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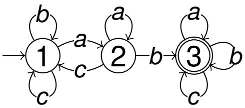

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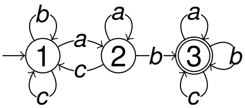
	a	*b*	*c*	*ab*	*ba*
a		*ab*			
b	*ba*				
c					
ab					
ba					

Semigroups: Equivalence classes

$A = \{a, b, c\}$



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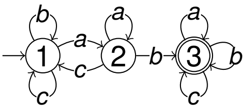
	a	b	c	ab	ba
a	a	ab	c		
b	ba	b	b		
c	a	c	c		
ab					
ba					

Semigroups: Equivalence classes

$A = \{a, b, c\}$



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	a	*b*	*c*	*ab*	*ba*
a	*a*	*ab*	*c*	*ab*	*ab*
b	*ba*	*b*	*b*	*ab*	*ba*
c	*a*	*c*	*c*	*ab*	*a*
ab	*ab*	*ab*	*ab*	*ab*	*ab*
ba	*ba*	*ab*	*b*	*ab*	*ab*

Bridging logic and algebra: Definiteness

> “*We shall first restrict ourselves to events which refer to a fixed period of time, consisting of the* χ ($\geqslant 1$) *moments* $p - \chi + 1, \ldots, p$ *ending with the present.*” *(Kleene, 1951)*

k-**DEFINITE** languages:
last *k*-many symbols determine acceptability

Bridging logic and algebra: 1-definiteness

If u and v have the same last letter,
then xuy and xvy have the same last letter.

If the last letter determines acceptability:

$$u \sim v$$

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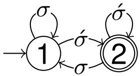
Semigroup elements are (equivalences of) nonempty words

So $sx \sim x$ for all s and x

Bridging logic and algebra: 1-definiteness

A linguistic pattern: **STRESS-FINAL**

$$A = \{\sigma, \acute{\sigma}\}$$



	σ	$\acute{\sigma}$
σ	σ	$\acute{\sigma}$
$\acute{\sigma}$	σ	$\acute{\sigma}$

$sx \sim x$ for all s and x

Bridging logic and algebra: 2-definiteness

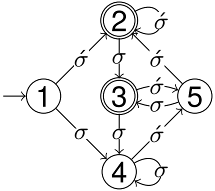
If last two letters determine acceptability
and both u and v have the same last two letters,
$u \sim v$

$sx_1x_2 \sim x_1x_2$ for all s, x_1 and x_2

Bridging logic and algebra: 2-definiteness

A linguistic pattern: **STRESS-PENULT**

$A = \{\sigma, \acute{\sigma}\}$



	σ	$\acute{\sigma}$	$\sigma\sigma$	$\sigma\acute{\sigma}$	$\acute{\sigma}\sigma$	$\acute{\sigma}\acute{\sigma}$
σ	σ	$\acute{\sigma}$	$\sigma\sigma$	$\sigma\acute{\sigma}$	$\acute{\sigma}\sigma$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}$	σ	$\acute{\sigma}$	$\sigma\sigma$	$\sigma\acute{\sigma}$	$\acute{\sigma}\sigma$	$\acute{\sigma}\acute{\sigma}$
$\vdots$			$\ddots$			

$sx_1x_2 \sim x_1x_2$ for all s and x

Bridging logic and algebra: Definiteness

k-definite: $sx_1 \ldots x_k \sim x_1 \ldots x_k$

Bridging logic and algebra: Definiteness

k-definite: $sx_1 \ldots x_k \sim x_1 \ldots x_k$

But what if we don't know k?
Do we have to try all of them?

Bridging logic and algebra: Definiteness

If there is some k: $sx_1 \ldots x_k \sim x_1 \ldots x_k$

Specifically for $s = x_1 \ldots x_k$:
$(x_1 \ldots x_k)(x_1 \ldots x_k) \sim x_1 \ldots x_k$

And if $x \sim xx$, then $x \sim xx \sim \cdots \sim x^k$

Bridging logic and algebra: Definiteness

Consider: $x, xx, \ldots$

Bridging logic and algebra: Definiteness

Consider: $x, xx, \ldots, x^n, \ldots, x^{n+m} = x^n$

Bridging logic and algebra: Definiteness

Consider: $x, xx, \ldots, x^n, \ldots, x^{n+m} = x^n$

$$x^{mn} = x^{mn}x^{mn}$$

$$x^{\omega} = \lim_{i\to\infty} x^{i!} = x^{mn}$$

Bridging logic and algebra: Definiteness

Consider: $x, xx, \ldots, x^n, \ldots, x^{n+m} = x^n$

$$x^{mn} = x^{mn}x^{mn}$$

$$x^\omega = \lim_{i\to\infty} x^{i!} = x^{mn}$$

Definite: $sx^\omega \sim x^\omega$ for all s and x

(see, among other sources, Straubing, 1985)

Ignoring letters: Culminativity

words do not contain $\acute{\sigma} \ldots \acute{\sigma}$

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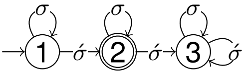

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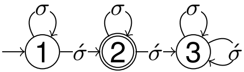

	σ	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
σ	σ	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}$	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$

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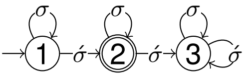


	σ	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
σ	σ	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}$	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$

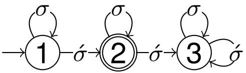
$\sigma^{\omega} = \sigma$ but $\acute{\sigma}\sigma^{\omega} \not\approx \sigma^{\omega}$

Ignoring letters: Culminativity

Tier projection:
delete letters you don't care about

Ignoring letters: Culminativity

Tier projection:
delete letters you don't care about



Ignoring letters: Culminativity

TIER PROJECTION:
delete letters you don't care about

Ignoring letters: Culminativity

Tier projection:
delete letters you don't care about

	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$

Ignoring letters: Culminativity

TIER PROJECTION:
delete letters you don't care about

	$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$
$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$	$\acute{\sigma}\acute{\sigma}$

$$sx^{\omega} \sim x^{\omega}$$

Definite after projection

"TIER" DEFINITE

Igoring letters: Multiple tiers

Culminativity: ignore σ
Stress-final: nothing can be ignored

Words that satisfy both: **MULTITIER DEFINITE**

Igoring letters: Multiple tiers

Culminativity: ignore σ
Stress-final: nothing can be ignored

Words that satisfy both: **MULTITIER DEFINITE**

Multitier definite: $s(xsy)^{\omega} \sim (xsy)^{\omega}$

Related classes

Multitier definite: attend to (projected) suffixes

Multitier reverse definite: attend to (projected) prefixes

Multitier generalized definite: attend to both

and many more

Outcomes

- StressTyp2 stress pattern database (Goedemans *et al.*, 2015)
  - 107 distinct patterns
  - All but two definable by (projected) factors
  - The remaining two are contested
    and would be local with slight modification
- Harmony and tone: need further catalogue,
  but equally simple.

Extensions

- Functions (processes)
  - Transducers are handled the same way
  - Input strictly local (Chandlee, 2014) are
    - Definite (Lambert and Heinz, 2023)
    - Quantifier-free logic on strings with adjacency (Chandlee and Lindell, to appear)
  - Attested nonlocal processes are also simple (Lambert, 2022)
- Syntactic structures
  - Under Myhill's congruence, regular $\leftrightarrow$ finite (Rabin and Scott, 1959)
    study the infinite semigroups?
  - Another congruence:
    operator-precedence language $\leftrightarrow$ finite (Henziger *et al.*, 2023)
    study these finite semigroups?

Thank You

Questions?

software

slides

```
$ cabal install language-toolkit
```

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