MATH 2418: Linear Algebra

Assignment# 6

Due :10/08, Tuesday, before 11:59pm Term : Fall 2024

[Last Name]	[First Name]	[Net ID]	[Lab Section]

Recommended Problems (do not turn in)
Sec 3.1: 1, 2, 4, 5, 9, 10, 14, 16, 17, 23, 25, 26. **Sec 3.2:** 1, 2, 3, 4, 7, 9, 11, 13, 15, 17, 23, 33, 45. **Sec 3.3:** 1, 2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 20, 26, 29.

1. Which of the following are vector spaces? Justify your answer.

   (a) The set of all polynomials of degree 5.

   (b) The set of all vectors $\mathbf{x} = (x_1, x_2, x_3)$, satisfying $5x_1 + 2x_2 - 2x_3 = 2024$.

   (c) The set of all vectors $\mathbf{x} = (x_1, x_2, x_3)$, satisfying $2418x_1 + x_2 - 3x_3 = 0$.

   (d) The set of all 3×3 matrices A such that $A\mathbf{x} = \mathbf{0}$ has a unique solution.

   (e) The set of all $n \times n$ $(n \in \mathbb{N})$ diagonal matrices where n is a fixed positive integer.

   Solution:

   (a) No. The only possible zero vector is $f(x) = 0$, but this is not a polynomial of degree 5 since the coefficient of x^5 is zero.

   (b) No. Notice that $\mathbf{0}$ does not solve the equation, so the solution space does not have a zero vector.

   (c) Yes. This is the null space of the matrix $A = \begin{bmatrix} 2418 & 1 & -3 \\ 2418 & 1 & -3 \\ 2418 & 1 & -3 \end{bmatrix}$ and the null space of a matrix is a vector space.

   (d) No. Notice that the identity matrix I satisfies $I\mathbf{x} = \mathbf{0}$ has a unique solution but $0I = 0_3$ (zero matrix) does not have a unique solution to $0_3\mathbf{x} = \mathbf{0}$, thus the set is not closed under scalar multiplication.

   (e) Yes.

      (i) The zero matrix is diagonal.

      (ii) Let $A = \operatorname{diag}(a_1, \ldots, a_n)$ be the $n \times n$ diagonal matrix with entries $a_1, \ldots, a_n$. Then $cA = \operatorname{diag}(ca_1, \ldots, ca_n)$ is also diagonal.

      (iii) Let $A = \operatorname{diag}(a_1, \ldots, a_n)$ and let $B = \operatorname{diag}(b_1, \ldots, b_n)$. Then $A + B = \operatorname{diag}(a_1 + b_1, \ldots, a_n + b_n)$ is also diagonal.

2. Let U be the set of all vectors (x_1, x_2, x_3) satisfying $x_1 - 3x_2 + 2x_3 = 0$ and V be the set of all vectors (x_1, x_2, x_3) satisfying $4x_1 + 2x_2 - x_3 = 0$.

   (a) Show that U and V are subspaces of $\mathbb{R}^3$.

   (b) Is the set $U \cup V := \{\mathbf{x} \mid \mathbf{x} \in U \text{ or } \mathbf{x} \in V\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

   (c) Is the set $U \cap V := \{\mathbf{x} \mid \mathbf{x} \in U \text{ and } \mathbf{x} \in V\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

Solution:

(a) We note that U can be expressed as the set of vectors orthogonal to $(1, -3, 2)$ and V the set of vectors orthogonal to $(4, 2, -1)$.

Let us show that U is a subspace:

i. We have that the vector $(0, 0, 0)$ is in U since $(0, 0, 0) \cdot (1, -3, 2) = 0$.

ii. For vectors $u_1 = (u_{11}, u_{12}, u_{13}), u_2 = (u_{21}, u_{22}, u_{23})$ such that $u_1, u_2 \in U$, and for a scalar $k \in \mathbb{R}$, we will show that $ku_1 + u_2 \in U$.

$$\begin{aligned}(ku_1 + u_2) \cdot (1, -3, 2) &= ku_{11} + u_{21} - 3ku_{12} - 3u_{22} + 2ku_{13} + 2u_{23} \\ &= k(u_{11} - 3u_{12} + 2u_{13}) + (u_{21} - 3u_{22} + 2u_{23}) \\ &= k(u_1 \cdot (1, -3, 2)) + (u_2 \cdot (1, -3, 2)) \\ &= k(0) + 0\end{aligned}$$

Let us show that V is a subspace:

i. We have that the vector $(0, 0, 0)$ is in V since $(0, 0, 0) \cdot (4, 2, -1) = 0$.

ii. For vectors $v_1 = (v_{11}, v_{12}, v_{13}), v_2 = (v_{21}, v_{22}, v_{23})$ such that $v_1, v_2 \in U$, and for a scalar $k \in \mathbb{R}$, we will show that $kv_1 + v_2 \in V$.

$$\begin{aligned}(kv_1 + v_2) \cdot (4, 2, -1) &= 4kv_{11} + 4v_{21} + 2kv_{12} + 2v_{22} - kv_{13} - v_{23} \\ &= k(4v_{11} + 2v_{12} - v_{13}) + (4v_{21} + 2v_{22} - v_{23}) \\ &= k(v_1 \cdot (4, 2, -1)) + (v_2 \cdot (4, 2, -1)) \\ &= k(0) + 0\end{aligned}$$

(b) No, take $u = (2, 0, -1)$, $v = (1, 0, 4)$, and $k = 1$. Since $u \in U$ and $v \in V$, then both are in $U \cup V$. Consider $ku + v = (3, 0, 3)$, but this vector is in not in U or V. Thus $ku + v \notin U \cup V$ and it is not a subspace.

(c) Yes, since we have the following:

i. The vector $(0, 0, 0)$ is in both U and V, thus $(0, 0, 0) \in U \cap V$.

ii. Let $w_1, w_2 \in U \cap V$ and $k \in \mathbb{R}$. We have that w_1 and w_2 are in both U and V, and since they are both subspaces, we have that $kw_1 + w_2 \in U$ and $kw_1 + w_2 \in V$. Thus, we have that $kw_1 + w_2 \in U \cap V$.

3. Which of the following are spanning sets for $\mathbb{R}^3$? Justify your answer.

   (a) $\{(2,\ 0,\ 0),\ (0,\ 3,\ 0),\ (0,\ 0,\ 2024)\}$
   (b) $\{(3,\ 0,\ 0),\ (0,\ 1,\ 1),\ (4,\ 4,\ 4)\}$
   (c) $\{(1,\ 0,\ 0),\ (0,\ 1,\ 0),\ (0,\ 0,\ 1),\ (3,\ 4,\ 5)\}$
   (d) $\{(1,\ 0,\ 0),\ (0,\ 2,\ 3),\ (4,\ 0,\ 5)\}$
   (e) $\{(2,\ 3,\ 4),\ (0,\ 0,\ 5)\}$

   Solution:

   (a) Spans $\mathbb{R}^3$. The three vectors are immediately linearly independent.
   (b) Spans $\mathbb{R}^3$. The three vectors are immediately linearly independent.
   (c) Spans $\mathbb{R}^3$. The three standard basis vectors form $\mathbb{R}^3$, and the vector $(3, 4, 5)$ is dependent on the first three.
   (d) Spans $\mathbb{R}^3$. It suffices to show that the three vectors are linearly independent. Clearly $(1, 0, 0)$ and $(0, 2, 3)$ are independent. Suppose then that $(4, 0, 5) = \alpha(1, 0, 0) + \beta(0, 2, 3) = (\alpha, 2\beta, 3\beta)$ hence $\alpha = 4$, $2\beta = 0$, and $3\beta = 5$ but the last two contradict one another therefore $(4, 0, 5)$ must be linearly independent from the other two vectors.
   (e) Does not span $\mathbb{R}^3$. There are only two vectors so at most their span would be $\mathbb{R}^2$.

4. Given

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 3 & 6 & 6 & 5 & 5 \\ 4 & 8 & 14 & 16 & 6 \end{bmatrix}$$

(a) Compute the reduced row echelon form R of A;

(b) Which column vectors of R correspond to the free variables? Write each of these vectors as a linear combination of the column vectors corresponding to the pivot variables.

Solution:

(a) The reduced row echelon form R of A can be computed as below,

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 3 & 6 & 6 & 5 & 5 \\ 4 & 8 & 14 & 16 & 6 \end{bmatrix} \xrightarrow[R_3=R_3-4R_1]{R_2=R_2-3R_1} \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 6 & 4 & 2 \end{bmatrix} \xrightarrow[R_2\rightarrow R_3]{R_3\rightarrow -R_2/4} \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 6 & 4 & 2 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \xrightarrow{R_2=R_2/6}$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \xrightarrow[R_2=R_2-\frac{2}{3}R_3]{R_1=R_1-2R_2} \begin{bmatrix} 1 & 2 & 0 & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \xrightarrow{R_1=R_1-\frac{5}{3}R_3} \begin{bmatrix} 1 & 2 & 0 & 0 & \frac{7}{6} \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}.$$

(b) 2^{nd} and 5^{th} column vectors of R correspond to the free variables. Now, each of these vectors as a linear combination of the column vectors corresponding to the pivot variables can be written as below,

$$V_2 = 2V_1,$$
$$V_5 = \frac{7}{6}V_1 + \frac{2}{3}V_3 - \frac{1}{2}V_4.$$

5. Let

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 6 & 0 & 4 & 12 \\ 0 & 3 & 6 & 15 \\ 6 & 0 & 6 & 24 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 9 \\ k \end{bmatrix}.$$

(a) Find condition on $k \in \mathbb{R}$ such that $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{R}^n$ is solvable.

(b) Find all solutions when condition in a) holds.

Solution:

(a) Consider the augmented matrix $[A|b]$ and reduce it into an upper triangular form as follows :

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 1 \\ 6 & 0 & 4 & 12 & 4 \\ 0 & 3 & 6 & 15 & 9 \\ 6 & 0 & 6 & 24 & k \end{array}\right] \xrightarrow[R_2-6R_1]{R_4-6R_1} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 1 \\ 0 & -18 & -26 & -30 & -2 \\ 0 & 3 & 6 & 15 & 9 \\ 0 & -18 & -24 & -18 & k-6 \end{array}\right] \cdots$$

$$\cdots \xrightarrow[R_3+\frac{1}{6}R_2]{R_4-R_2} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 1 \\ 0 & -18 & -26 & -30 & -2 \\ 0 & 0 & \frac{5}{3} & 10 & \frac{26}{3} \\ 0 & 0 & 2 & 12 & k-4 \end{array}\right] \xrightarrow{R_4-\frac{6}{5}R_3} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 1 \\ 0 & -18 & -26 & -30 & -2 \\ 0 & 0 & \frac{5}{3} & 10 & \frac{26}{3} \\ 0 & 0 & 0 & 0 & k-\frac{72}{5} \end{array}\right]$$

Thus, $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in \mathbb{R}^n$ is solvable if $k - \dfrac{72}{5} = 0 \implies k = \dfrac{72}{5}$.

(b) When condition in (a) holds, that is $k = \dfrac{72}{5}$, then we have :

$$[U|c] = \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 1 \\ 0 & -18 & -26 & -30 & -2 \\ 0 & 0 & \frac{5}{3} & 10 & \frac{26}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\frac{3}{5}R_3]{-\frac{1}{18}R_2} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 1 \\ 0 & 1 & \frac{13}{9} & \frac{5}{3} & \frac{1}{9} \\ 0 & 0 & 1 & 6 & \frac{26}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \cdots$$

$$\cdots \xrightarrow[R_1-3R_2]{R_2-\frac{13}{9}R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 2 & \frac{2}{3} \\ 0 & 1 & 0 & -7 & -\frac{37}{5} \\ 0 & 0 & 1 & 6 & \frac{26}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-\frac{2}{3}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -\frac{14}{5} \\ 0 & 1 & 0 & -7 & -\frac{37}{5} \\ 0 & 0 & 1 & 6 & \frac{26}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the free variable is x_4, so that $x_p = (-\frac{14}{5}, -\frac{37}{5}, \frac{26}{5}, 0)$ for $x_4 = 0$.
And for $x_4 = 1$, we get the special solution $s_1 = (2, 7, -6, 1)$.
Thus,

$$x_{\text{complete}} = \begin{bmatrix} -\frac{14}{5} \\ -\frac{37}{5} \\ \frac{26}{5} \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 7 \\ -6 \\ 1 \end{bmatrix}, d \in \mathbb{R}$$

6. (a) Construct a matrix B whose null space consists of all linear combinations of vectors (2, 0, 4, 8) and (0, 2, 8, −2).

   (b) Express matrix B as a sum of two rank one matrices.

Solution:

(a) We require that the null space of B, denoted $N(B)$, consists of all linear combinations of two linearly independent vectors in $\mathbb{R}^4$. This will be the case if B has dimension 2×4, has rank 2, and if $N(B)$ contains each of the selected vectors. This is because $\text{rank}(B) = \dim(\mathbb{R}^4) - \dim(N(B))$, which forces $\dim(N(B)) = 2$. When $\dim(N(B)) = 2$ and $N(B)$ contains the linearly independent set $\{(2,0,4,8),(0,2,8,-2)\}$, we may conclude that the nullspace of B is precisely the set of linear combinations of these vectors.

Take B to be of the form:

$$B = \begin{bmatrix} -4 & c & 0 & 1 \\ d & -4 & 1 & 0 \end{bmatrix}$$

For any choice of c, d, the matrix B has rank 2. We only require now that B sends both $(2,0,4,8)$ and $(0,2,8,-2)$ to zero. This gives two equations, where "$\bullet$" is the dot product:

$$\begin{cases} (-4,c,0,1) \bullet (0,2,8,-2) & = 2c - 2 = 0 \\ (d,-4,1,0) \bullet (2,0,4,8) & = 2d + 4 = 0 \end{cases}$$

Now $c = 1$ and $d = -2$, and one may check that:

$$B = \begin{bmatrix} -4 & 1 & 0 & 1 \\ -2 & -4 & 1 & 0 \end{bmatrix}$$

is a solution to the given problem.

(b) To express B as a sum of rank one matrices, consider the below CR-factorization:

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & -4 & 1 & 0 \\ -4 & 1 & 0 & 1 \end{bmatrix}$$

It then follows that:

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 & -4 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -4 & 1 & 0 & 1 \end{bmatrix}$$

which exhibits B as a sum of two rank one matrices.

7. For which conditions on $\mathbf{b} = (b_1, b_2, b_3, b_4)$ do there exist solution(s) for the linear system $A\mathbf{x} = \mathbf{b}$?

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 6 & 12 & 12 & 10 \\ 4 & 8 & 14 & 16 \\ 0 & 2 & 6 & 14 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 0 & 6 & 6 & 0 & 12 & 12 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}$

Solution(a)

Row Echelon Form Reduction

Given the matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 6 & 12 & 12 & 10 \\ 4 & 8 & 14 & 16 \\ 0 & 2 & 6 & 14 \end{bmatrix}$$

Step 1: Eliminate the first column elements below the pivot

$$R2 = R2 - 6R1 \quad \text{and} \quad R3 = R3 - 4R1$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -6 & 4 \\ 0 & 0 & 2 & 12 \\ 0 & 2 & 6 & 14 \end{bmatrix}$$

Step 2: Swap the second and fourth rows

$$R2 \leftrightarrow R4$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & -6 & 4 \end{bmatrix}$$

Step 3: Make the second row's second element a pivot

$$R2 = \frac{1}{2}R2$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & -6 & 4 \end{bmatrix}$$

Step 4: Eliminate the second column elements below the pivot

$$R3 = R3 - 3R2 \quad \text{and} \quad R4 = R4 - 6R2$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & -6 & 4 \end{bmatrix}$$

Step 5: Make the third row's third element a pivot

$$R3 = -R3$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & -6 & 4 \end{bmatrix}$$

Step 6: Eliminate the third column elements below the pivot

$$R4 = R4 + 6R3$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 58 \end{bmatrix}$$

Step 7: Eliminate the third column elements above the pivot

$$R2 = R2 - 3R3 \quad \text{and} \quad R1 = R1 - 3R3$$

$$\begin{bmatrix} 1 & 2 & 0 & -26 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 58 \end{bmatrix}$$

Since the rank of A is 4 which implies we have C(A) is 4 dimensional space. Therefore b belongs to C(A) for all b

Solution (b)

Given the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & | & b_1 \\ 2 & 2 & 4 & 4 & 6 & 6 & | & b_2 \\ 0 & 6 & 6 & 0 & 12 & 12 & | & b_3 \\ 4 & 4 & 4 & 4 & 4 & 4 & | & b_4 \end{bmatrix}$$

Step 1: Eliminate the first column elements below the pivot

$$R2 = R2 - 2R1 \quad \text{and} \quad R4 = R4 - 4R1$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & | & b_1 \\ 0 & 0 & 2 & 2 & 4 & 4 & | & b_2 - 2b_1 \\ 0 & 6 & 6 & 0 & 12 & 12 & | & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & b_4 - 4b_1 \end{bmatrix}$$

Step 2: Make the second row's second element a pivot

$$R2 = \frac{1}{2}R2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & | & b_1 \\ 0 & 0 & 1 & 1 & 2 & 2 & | & \frac{b_2 - 2b_1}{2} \\ 0 & 6 & 6 & 0 & 12 & 12 & | & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & b_4 - 4b_1 \end{bmatrix}$$

Step 3: Eliminate the second column elements below the pivot

$$R3 = R3 - 6R2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & | & b_1 \\ 0 & 0 & 1 & 1 & 2 & 2 & | & \frac{b_2-2b_1}{2} \\ 0 & 6 & 0 & -6 & 0 & 0 & | & b_3 - 6\left(\frac{b_2-2b_1}{2}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & | & b_4 - 4b_1 \end{bmatrix}$$

Step 4: Make the third row's third element a pivot

$$R3 = \frac{1}{6}R3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & | & b_1 \\ 0 & 0 & 1 & 1 & 2 & 2 & | & \frac{b_2-2b_1}{2} \\ 0 & 1 & 0 & -1 & 0 & 0 & | & \frac{b_3-3b_2+6b_1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & | & b_4 - 4b_1 \end{bmatrix}$$

Step 5: Eliminate the third column elements above the pivot

$$R2 = R2 - R3 \quad \text{and} \quad R1 = R1 - R3$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 1 & | & b_1 - \frac{b_3-3b_2+6b_1}{6} \\ 0 & 0 & 1 & 2 & 2 & 2 & | & \frac{b_2-2b_1}{2} - \frac{b_3-3b_2+6b_1}{6} \\ 0 & 1 & 0 & -1 & 0 & 0 & | & \frac{b_3-3b_2+6b_1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & | & b_4 - 4b_1 \end{bmatrix}$$

Step 6: Eliminate the second column elements above the pivot

$$R1 = R1 - 2R2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -3 & -3 & | & b_1 - \frac{b_3-3b_2+6b_1}{6} - 2\left(\frac{b_2-2b_1}{2} - \frac{b_3-3b_2+6b_1}{6}\right) \\ 0 & 0 & 1 & 2 & 2 & 2 & | & \frac{b_2-2b_1}{2} - \frac{b_3-3b_2+6b_1}{6} \\ 0 & 1 & 0 & -1 & 0 & 0 & | & \frac{b_3-3b_2+6b_1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & | & b_4 - 4b_1 \end{bmatrix}$$

for solution to exist, b4-4b1=0, therefore we need b4=4b1 as a necessary condition for solution to exist.

8. Find the complete solution $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$ to the system $\begin{bmatrix} 6 & 10 & 12 \\ 2 & 4 & 2 \\ 4 & 10 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 14 \\ 4 \\ 16 \end{bmatrix}$.

Solution:

To solve the system of equations:

$$\begin{bmatrix} 6 & 10 & 12 \\ 2 & 4 & 2 \\ 4 & 10 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 16 \end{bmatrix}$$

Let's solve the system below by performing row reduction to determine the particular solution X_p.

$$\begin{bmatrix} 6 & 10 & 12 & | & 14 \\ 2 & 4 & 2 & | & 4 \\ 4 & 10 & 8 & | & 16 \end{bmatrix}$$

Subtract $\frac{1}{3}$ times the second row from the first row and $\frac{2}{3}$ times the second row from the third row to eliminate the first column entries below the first element:

$$\begin{bmatrix} 6 & 10 & 12 & | & 14 \\ 0 & 2/3 & -2 & | & -2/3 \\ 0 & 10/3 & 0 & | & 20/3 \end{bmatrix}$$

Subtract 5 times the second row from the third row:

$$\begin{bmatrix} 6 & 10 & 12 & | & 14 \\ 0 & 2/3 & -2 & | & -2/3 \\ 0 & 0 & 10 & | & 10 \end{bmatrix}$$

From the system above, we have

$$10x_3 = 10 \quad (1)$$

$$\frac{2x_2}{3} - 2x_3 = \frac{-2}{3} \quad (2)$$

$$6x_1 + 10x_2 + 12x_3 = 14 \quad (3)$$

Solving equations 1, 2 and 3 gives the particular solution $X_p = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$.

Since the above matrix is full rank, $X_n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence

$$X = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

9. Construct a matrix B whose column space contains $\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\\2\\0\end{bmatrix}$ and whose nullspace contains $\begin{bmatrix}3\\2\\4\end{bmatrix}$.

Solution: Since $\begin{bmatrix}3\\2\\4\end{bmatrix}$ is in the nullspace of B, we know that B has three columns v_1, v_2, v_3 satisfying

$$3v_1 + 2v_2 + 4v_3 = 0 \tag{4}$$

To ensure $\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$, $\begin{bmatrix}2\\1\\2\\0\end{bmatrix}$ are in the column space of B, we can set

$$v_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, v_2 = \begin{bmatrix}2\\1\\2\\0\end{bmatrix}. \tag{5}$$

Then we need only solve for v_3 using Equation (4),

$$v_3 = \frac{-3v_1 - 2v_2}{4} \tag{6}$$

$$= \frac{1}{4}\begin{bmatrix}-7\\-5\\-4\\-3\end{bmatrix}. \tag{7}$$

The desired B is then

$$B = \begin{bmatrix}1 & 2 & -7/4\\1 & 1 & -5/4\\0 & 2 & -1\\1 & 0 & -3/4\end{bmatrix}. \tag{8}$$