CANONICAL SCATTERING PROBLEMS

THIRD YEAR PROJECT
MAY 2020

THEORY

PHYSICS

We want to represent real life problems.

Assumptions:

- no external forces

- air at room temperature and pressure, no viscosity

- air is a barotropic fluid, density can be expressed as a function of pressure

- fluid flow is adiabatic

GOVERNING EQUATIONS

Navier-Stokes Equations

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{u}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

→

Euler momentum Equation

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \nabla p.$$

↙

Linear wave equation

$$\nabla^2 \mathbf{u} = \frac{1}{c^2} \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

→

Helmholtz equation

$$\nabla^2 \Phi + k^2 \Phi = 0$$

for $\mathbf{u} = \nabla \phi = \nabla \mathrm{Re}[\Phi(x, y, z) e^{-i\omega t}]$

THE BESSEL DIFFERENTIAL EQUATION

$$z^2 \frac{d^2U}{dz^2} + z\frac{dU}{dz} + (z^2 - \nu^2)U = 0$$

Bessel functions:

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+n}}{m!\Gamma(n+m)}$$

Neumann functions:

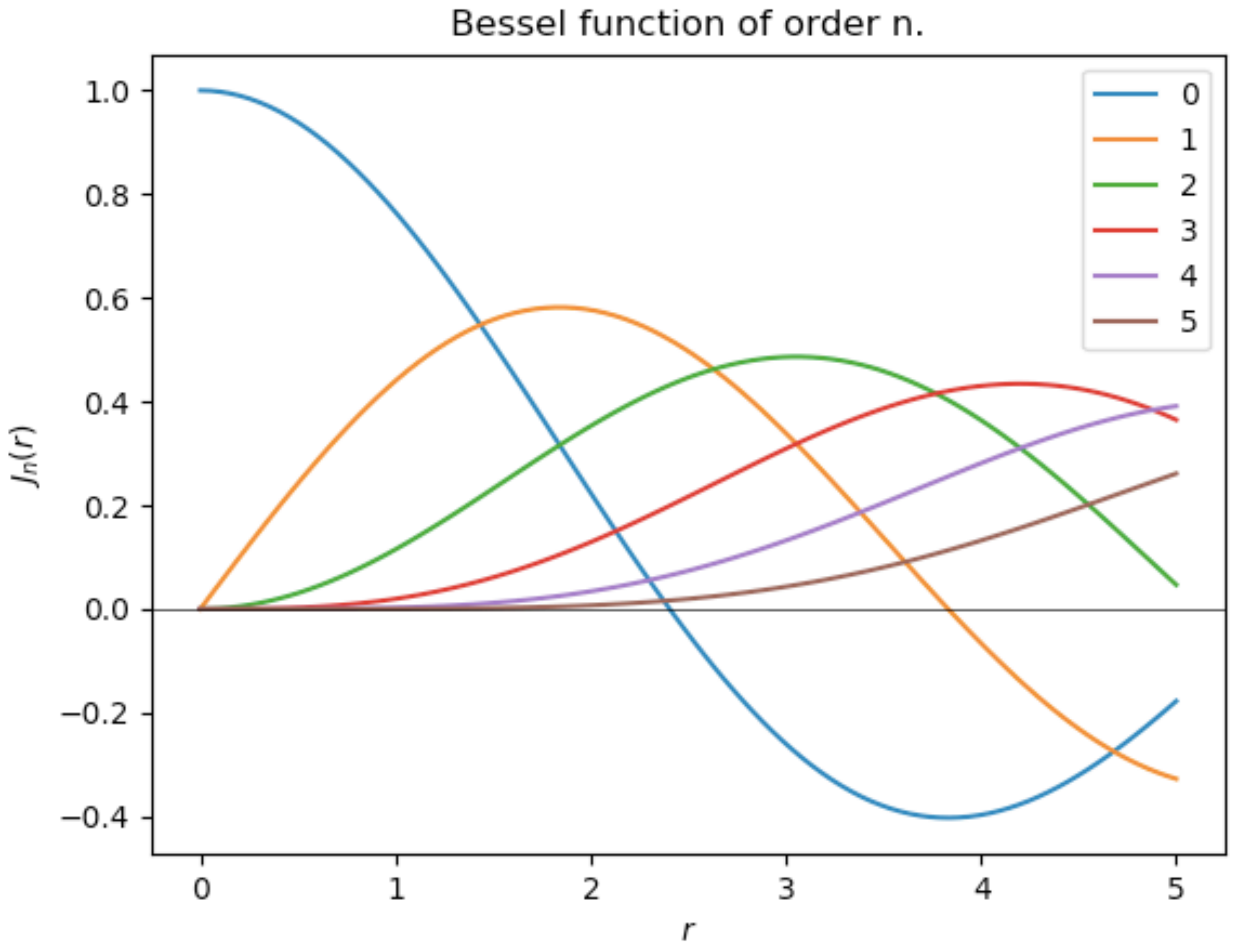
$$\begin{aligned} Y_n(z) &= \lim_{\nu\to n} \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \\ &= \lim_{\nu\to n} \frac{\partial}{\partial\nu} \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \end{aligned}$$

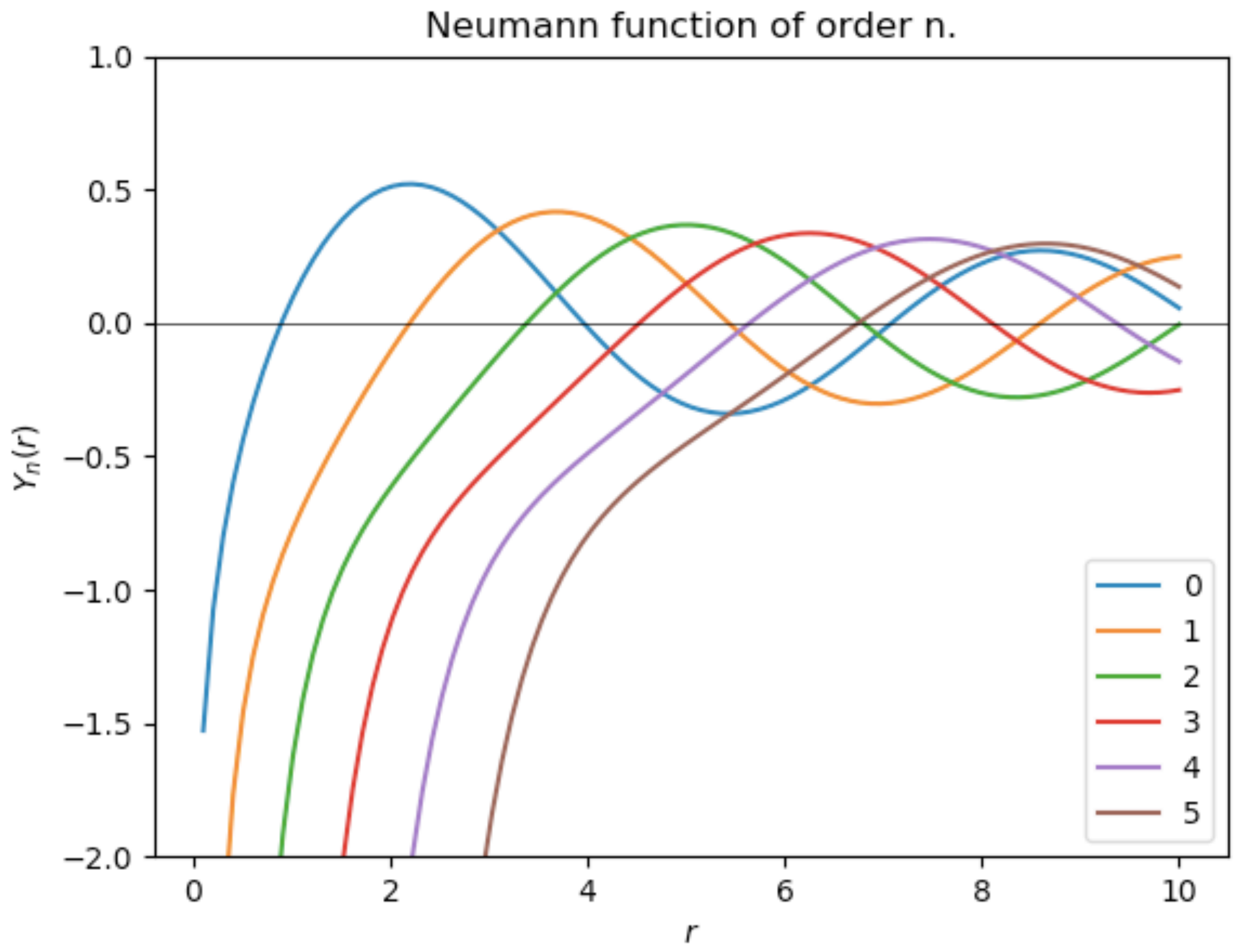
Hankel functions:

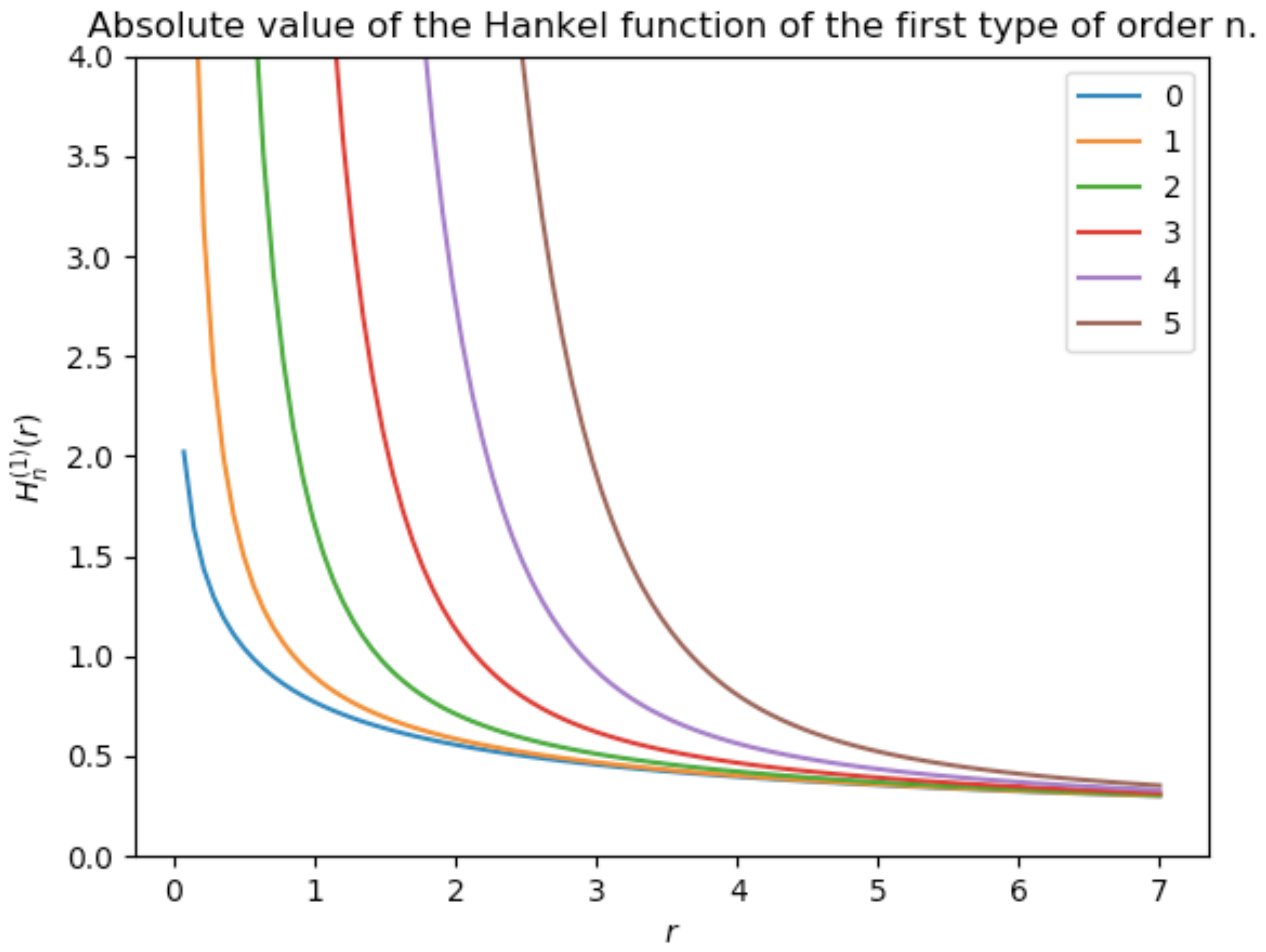
$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + iY_\nu(z) \\ H_\nu^{(2)}(z) &= J_\nu(z) - iY_\nu(z) \end{aligned}$$

THE BESSEL DIFFERENTIAL EQUATION

Limits of Bessel functions at the origin





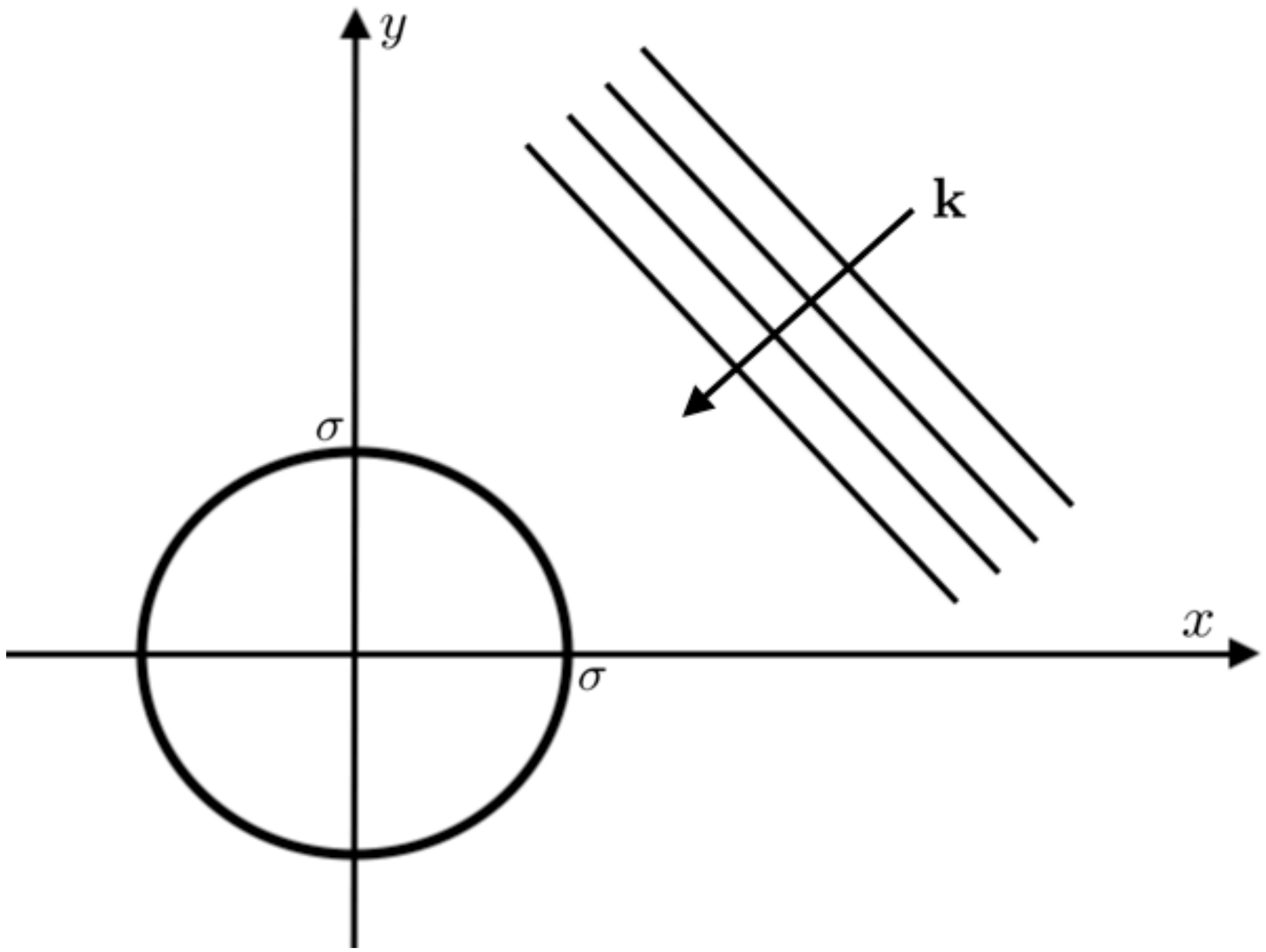


This becomes useful in the second problem, but it helps with overall understanding of these functions.

PROBLEM 1

SCATTERING OUTSIDE THE CYLINDER

INTRODUCTION



- Wave with known velocity field (pressure, density and velocity field all related)

- Incident on a cylinder with known properties

SOLUTION

- separation of variables
- angular dependence → trigonometric
- radial dependence → bessel differential equation
- Sommerfeld radiation condition → Hankel function of the first kind

$$\Phi_{sc} = \sum_{n=0}^{\infty} \epsilon_n i^n B_n H_n(kr)\{A\cos(n\theta) + B\sin(n\theta)\}$$

Solution for the scattered field

BOUNDARY CONDITIONS

DIRICHLET BOUNDARY CONDITION

- **Velocity is zero at the boundary**

$$u = 0, \ on \ r = \sigma \qquad \rightarrow \qquad B_n = \frac{-J_n(k\sigma)}{H_n(k\sigma)}$$

BOUNDARY CONDITIONS

NEUMANN BOUNDARY CONDITION

- **Normal velocity is zero at the boundary.**

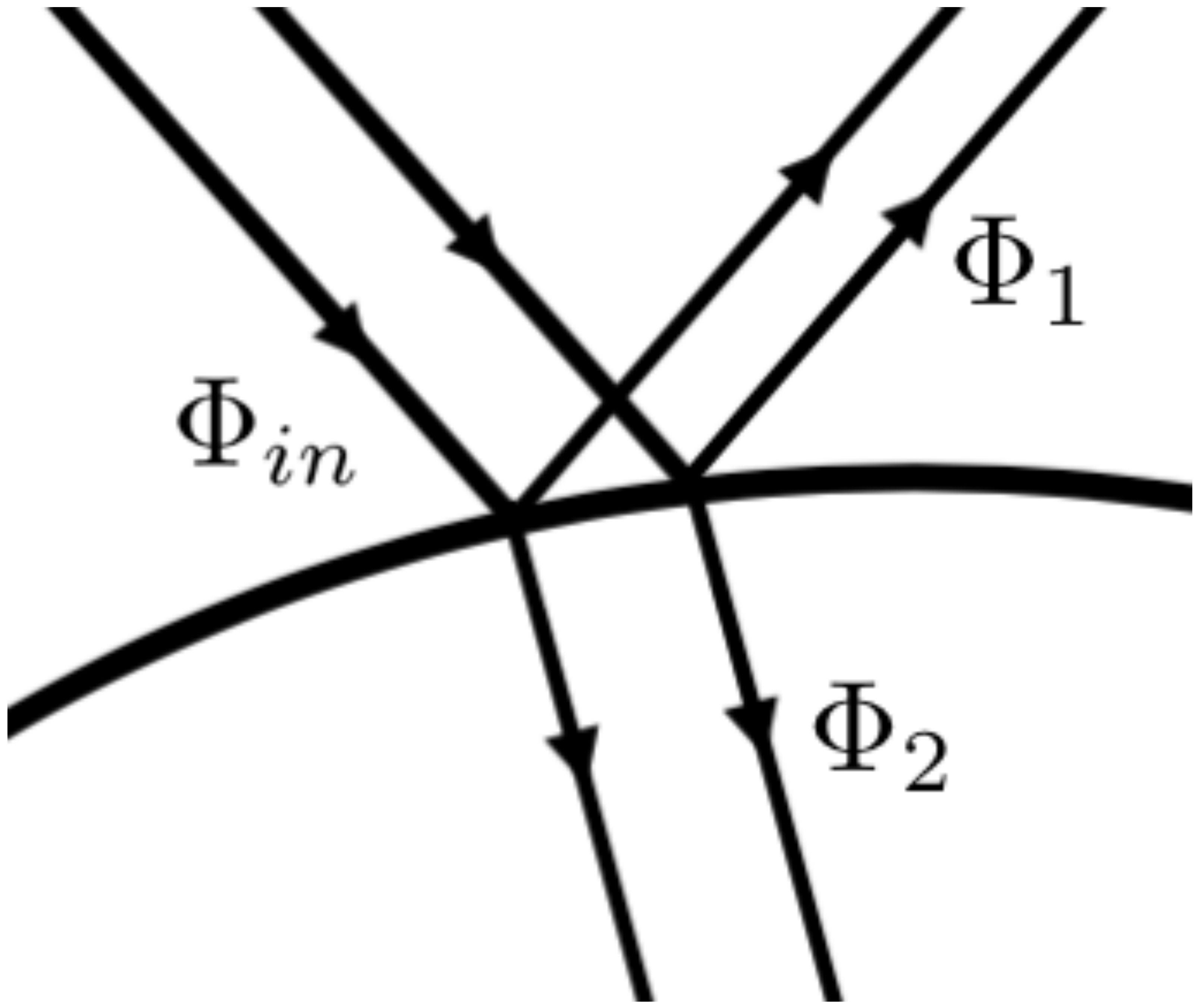
$$\frac{\partial u}{\partial r} = 0,\ on\ r = \sigma \quad \rightarrow \quad B_n = \frac{J_n'(k\sigma)}{H_n'(k\sigma)}$$

PROBLEM 2

SCATTERING INSIDE THE CYLINDER

INTRODUCTION

- Now there is transmission of sound through the boundary
- Two different mediums



SOLUTION

- Solution outside stays the same
- Solution inside must be well defined at the origin

Scattered outside

$$\Phi_1 = \sum_{n=0}^{\infty} \epsilon_n i^n B_1 H_n(kr) \cos(n\theta).$$

Inside

$$\Phi_2 = \sum_{n=0}^{\infty} \epsilon_n i^n B_2 J_n(kr) \cos(n\theta)$$

- There is also the incident field outside.

BOUNDARY CONDITIONS

- continuity of velocity and of pressure
- two boundary conditions for two unknowns

$$B_1 = \frac{(\rho_2 - \rho_1)J_n(k\sigma)J'n(k\sigma)}{R_1 - R_2}$$

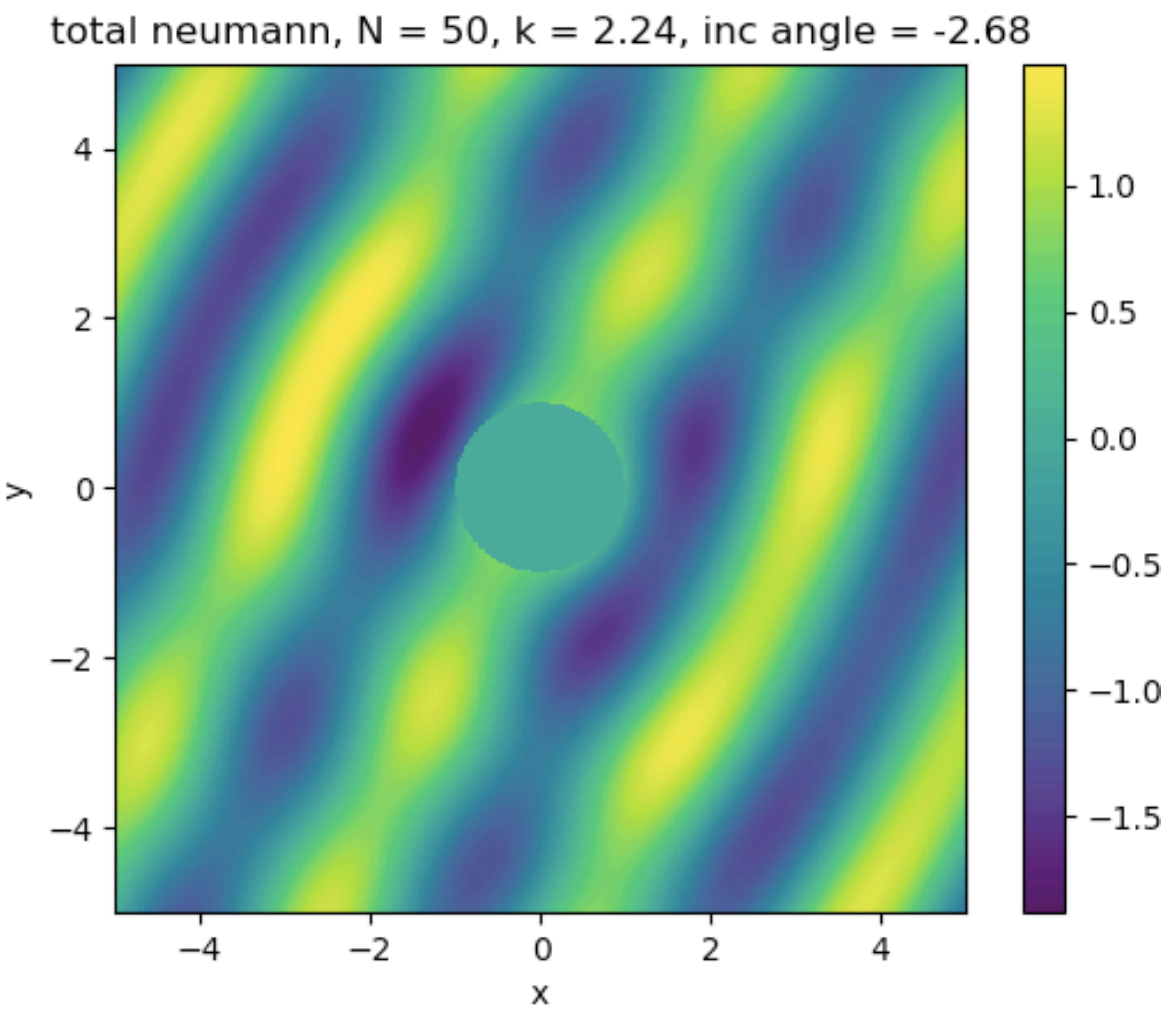
$$B_2 = \frac{\rho_1(R_1/\rho_1 - R_2/\rho_2)}{R_1 - R_2} = \frac{\rho_2 R_1 - \rho_1 R_2}{\rho_2(R_1 - R_2)}$$

PYTHON

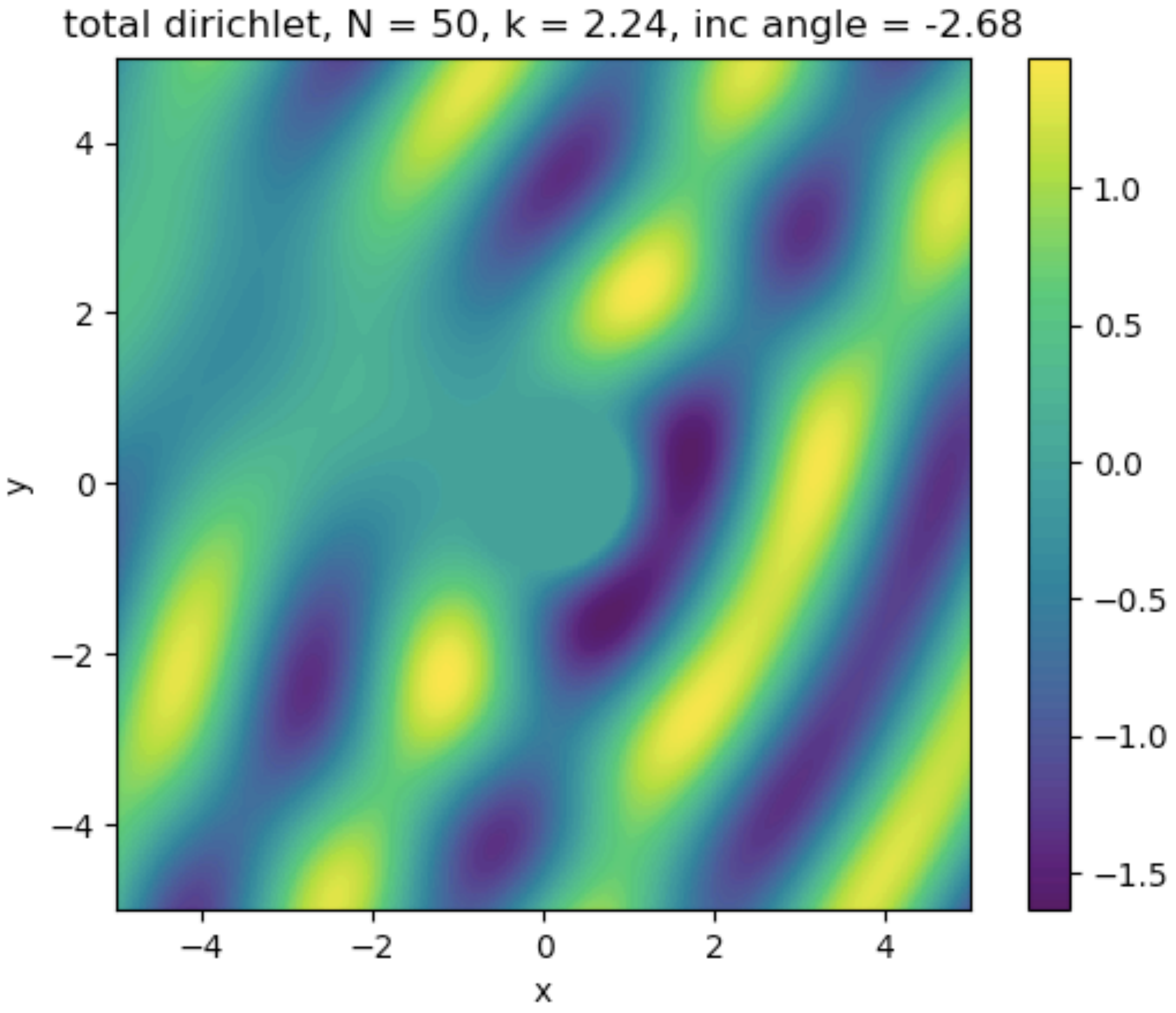
WHY?

- Help visualising
- Learn the language to a higher level
- Learn object oriented architecture
- Learn version control and using tools like git
- Get more comfortable with maths libraries (scipy.special, numpy)

RESULTS



total neumann, N = 50, k = 2.24, inc angle = -2.68
y
x
4
2
0
−2
−4
1.0
0.5
0.0
−0.5
−1.0
−1.5



total dirichlet, N = 50, k = 2.24, inc angle = -2.68
y
x
4
2
0
−2
−4
1.0
0.5
0.0
−0.5
−1.0
−1.5

PYTHON — CONCLUSION

- Not ideal to handle so many things I don’t understand at once (git, python, OOP, bessel functions, …)
- Very valuable learning experience

QUESTIONS