MATH3000
Canonical Scattering Problems
Interim report

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Contents

Abstract

The aim of this project is to better understand the scattering of acoustic waves around different objects. The starting point for this are the Navier-Stokes equations and a simplified model of acoustic wave motion in air.

The model of wave motion relies on assumptions about the nature of fluid flows, namely, that air is a barotropic, adiabatic ideal gas. These are explained in some detail in the development of the theory.

We use perturbation theory to arrive at the Linear Wave Equation from the Navier-Stokes equations. We then use this wave equation to derive the Helmholtz equation.

These equations are used to investigate our first problem; scattering of acoustic waves around a cylinder. There were two primary aims for this project: firstly, to find an expression for the resulting field and, secondly, to plot this field using Python. The plotting of the actual wave field is still pending.

Chapter 1

Theory

1.1 Governing equations

We can use the Navier-Stokes equations to model the velocity field $\mathbf{u}$ of a fluid of density ρ and viscosity μ.

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{u} \tag{1.1}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{1.2}$$

Where ∇p is the pressure gradient within the fluid, $\mathbf{F}$ is the external force applied onto the fluid and D/Dt is the material derivative. For our problems, we will be interested in the velocity field of air, where the viscosity is negligible. Setting $\mu = 0$ gives the Euler momentum equation (Shaughnessy, 2005)

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{F} - \nabla p. \tag{1.3}$$

We will consider problems where there are no external forces acting on our fluid, so we can set $\mathbf{F}$ to zero as well:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p. \tag{1.4}$$

This, together with the continuity equation (1.2) are the governing equations for our velocity field.

1.1.1 Perturbation of the governing equations

We will use perturbation theory to arrive at the linear wave equation from our governing equations. First, we must consider air at rest. In this case ρ, p and $\mathbf{u}$ will be constant, in particular

$$\rho = \rho_0, \ p = p_0, \ \mathbf{u} = \mathbf{0}.$$

We can therefore think of an acoustic wave as a small perturbation of this rest state. Let $\epsilon << 1$, then we can express ρ, p and $\mathbf{u}$ in this state as follows:

$$\rho = \rho_0 + \epsilon\tilde{\rho},\ \ p = p_0 + \epsilon\tilde{p},\ \ \mathbf{u} = \epsilon\tilde{\mathbf{u}}. \tag{1.5}$$

To derive our wave equation, we can input (1.5) into (1.4) and (1.2). From (1.4) we get

$$(\rho_0 + \epsilon\tilde{\rho})(\frac{\partial(\epsilon\tilde{\mathbf{u}})}{\partial t} + (\epsilon\tilde{\mathbf{u}} \cdot \nabla)(\epsilon\tilde{\mathbf{u}})) = -\nabla(p_0 + \epsilon\tilde{p}).$$

Since ϵ is small, all terms of order ϵ^2 or smaller are negligible. Hence we are left with

$$\rho_0 \frac{\partial\tilde{\mathbf{u}}}{\partial t} = -\nabla\tilde{p}. \tag{1.6}$$

From (1.2) we get

$$\frac{\partial}{\partial t}(\rho_0 + \epsilon\tilde{\rho}) + (\rho_0 + \epsilon\tilde{\rho})(\nabla \cdot (\epsilon\tilde{\mathbf{u}})) + (\epsilon\tilde{\mathbf{u}} \cdot \nabla)(\rho_0 + \epsilon\tilde{\rho}).$$

Since $\epsilon << 1$, we are left with

$$\frac{\partial\tilde{\rho}}{\partial t} + \rho_0(\nabla \cdot \tilde{\mathbf{u}}) = 0. \tag{1.7}$$

Differentiating (1.7) by t:

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} + \rho_0\frac{\partial}{\partial t}(\nabla \cdot \tilde{\mathbf{u}}) = 0$$

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} + \rho_0\nabla \cdot \frac{\partial\tilde{\mathbf{u}}}{\partial t} = 0. \tag{1.8}$$

To continue we need to make a physical assumption about the pressure field.

1.2 The barotropic assumption

A fluid can either be barotropic or baroclinic. Naively, baroclinic fluids are those where there is high variability. For example, where there are different air masses, cold and warm fronts or weather. In the problems we are going to consider this will not be the case - our fluid will be barotropic.

Definition 1.1. *(Shames, 2002) A barotropic fluid is one where ρ is expressible as a function of p only.*

$$\rho = \rho(p)$$

We can therefore also express p as a funcion of ρ only. I will call this function f for clarity.

$$p = f(\rho) \tag{1.9}$$

From (1.5), we have that $\rho = \rho_0 + \epsilon\tilde{\rho}$. Hence

$$p = f(\rho_0 + \epsilon\tilde{\rho}).$$

We can now expand this around the point ρ_0 using Taylor series. This is valid since $f(\rho)$ is a real valued function.

$$\begin{aligned}
p &= f(\rho) \\
&= f(\rho_0) + f'(\rho_0)(\rho - \rho_0) + \frac{1}{2!} f''(\rho_0)(\rho - \rho_0) + \cdots \\
&= f(\rho_0) + \epsilon\tilde{\rho} f'(\rho_0) + O(\epsilon^2)
\end{aligned}$$

Hence, since $\epsilon << 1$:

$$p = p_0 + \epsilon\tilde{\rho} f'(\rho_0) \tag{1.10}$$

But from (1.5) we have that $p = p_0 + \epsilon\tilde{p}$, and so $p_0 = p - \epsilon\tilde{p}$. Then from (1.10) we get:

$$\tilde{p} = \tilde{\rho} f'(\rho_0) \tag{1.11}$$

1.3 The linear wave equation

From equation (1.6) we have

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = -\frac{1}{\rho_0}\nabla\tilde{p},$$

and we can substitute this into (1.7) to get:

$$\begin{aligned}
\frac{\partial^2\tilde{\rho}}{\partial t^2} + \rho_0\nabla\cdot(\frac{-\nabla\tilde{p}}{\rho_0}) = 0 \\
\frac{\partial^2\tilde{\rho}}{\partial t^2} - \nabla^2\tilde{p} = 0.
\end{aligned}$$

Now we can use 1.11 to find an expression for $\tilde{\rho}$.

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} - \nabla^2(\tilde{\rho} f'(\rho_0)) = 0$$

Note $f'(\rho_0)$ is a constant. Let $f'(\rho_0) = c^2$. Then:

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} = c^2\nabla^2\tilde{\rho} \tag{1.12}$$

Which is the linear wave equation! We can do the same for $\tilde{p}$ and get

$$\frac{1}{c^2}\frac{\partial^2\tilde{p}}{\partial t^2} = \nabla^2\tilde{p}. \tag{1.13}$$

Similarly,

$$\nabla^2\mathbf{u} = \frac{1}{c^2}\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{1.14}$$

Throughout this paper, it will be useful to consider the velocity field $\mathbf{u}$ represented by the scalar function ϕ, where

$$\mathbf{u}(x, y, z, t) = \nabla\phi(x, y, z, t). \tag{1.15}$$

Proposition 1.2. *The scalar function ϕ satisfies the Helmholtz equation.*

Proof.

$$\nabla^2(\nabla\phi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\nabla\phi)$$
$$\nabla^2\left(\frac{\partial\phi}{\partial x_i}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial\phi}{\partial x_i}\right)$$
$$\frac{\partial}{\partial x_i}(\nabla^2\phi) = \frac{\partial}{\partial x_i}\left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right)$$

Integrating with respect to x_i gives,

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + C$$

The constant of integration will depend on our choice of ϕ. We set it to zero without loss of generality. Hence,

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{1.16}$$

□

1.4 Laplace's hypothesis

We need to make one more physical assumption, namely that motion in air is an adiabatic process. This will become relevant shortly.

Hypothesis 1.3 (Laplace's hypothesis)**.** *Sound propagation occurs with negligible internal heat flow.*

Definition 1.4. *A process is adiabatic if it satisfies Laplace's hypothesis.*

For an adiabatic process and a gas at constant pressure and volume, with constant specific heat coefficients per unit mass, and with $p \propto \rho$ at constant temperature the following relationship holds:

$$p = K\rho^\gamma \tag{1.17}$$

where $\gamma = c_p/c_v$ the specific heat ratio, and K constant in time (Pierce, 2019, §1.4.1).

1.5 Speed of sound

In 1.3 we arbitrarily set $f'(\rho_0) = c^2$. We can now show c is in fact the speed of sound in air.

By definition of f, we have

$$p = f(\rho), \text{ so } f'(\rho_0) = \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0} \tag{1.18}$$

Then, assuming motion in air is an adiabatic process,

$$\begin{aligned} c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} = \left.\frac{\partial}{\partial \rho}(K\rho^\gamma)\right|_{\rho_0} \\ &= \left.(\gamma K \rho^{\gamma-1})\right|_{\rho_0} = \gamma \frac{K\rho_0^\gamma}{\rho_0} \\ &= \gamma \frac{p_0}{\rho_0} \end{aligned}$$

Hence, our constant c^2 depends only on our initial density and initial pressure. Additionally, it has dimensions

$$\frac{[p]}{[\rho]} = \frac{kgm^{-1}s^{-2}}{kgm^{-3}} = (ms^{-1})^2$$

since γ is a dimensionless ratio. Hence c is indeed a speed.

Law 1.5 (Ideal Gas Law)**.** *For an ideal gas with ideal gas constant R at temperature T_K measured in degrees Kelvin, the following relationship holds*

$$p = \rho R T_K.$$

We can now assume that air is an ideal gas and apply the Ideal Gas law. We hence show that our arbitrary constant c is in fact the speed of sound for an ideal gas of ideal gas constant R at temperature T_K with specific heat ratio γ:

$$c = \sqrt{\gamma R T_K} \tag{1.19}$$

1.6 The Helmholtz equation

1.6.1 Separation of variables

We can now use the linear wave equation for ϕ(1.16) to derive the Helmholtz equation.

To do this, employ a standard separation of variables argument. We propose that, since $\phi(\mathbf{x}, t)$, there exist X and T such that

$$\phi = X(\mathbf{x})T(t). \tag{1.20}$$

This expression along with (1.16) gives us:

$$\begin{aligned} T(t)\nabla^2 X(\mathbf{x}) &= \frac{1}{c^2}\frac{d^2T}{dt^2}X(\mathbf{x}), \\ \frac{\nabla^2 X}{X} &= \frac{1}{c^2}\frac{T''}{T}. \end{aligned}$$

This can only be true if both sides are equal to the same constant, say $\hat{k}$. We therefore yield two ordinary differential equations:

$$\nabla^2 X = \hat{k} X \quad (1.21) \qquad\qquad T'' = \hat{k} c^2 T \quad (1.22)$$

Equation (1.21) represents a time independent form of the linear wave equation. Equation (1.22) can be solved to uncover the time dependence.

1.6.2 Time dependence

We now seek a solution to (1.22). We can rewrite (1.22) as

$$\frac{d^2T}{dt^2} - \hat{k} c^2 T = 0 \tag{1.23}$$

Since c is strictly a physical constant, namely the speed of sound in air, $c^2 \geq 0$. Therefore there are three cases to consider: $\hat{k} = 0$, $\hat{k} < 0$ and $\hat{k} > 0$.

Case 1. The trivial case where $\hat{k} = 0$ leads to a linear solution for time:

$$T(t) = A_1 t + B_1. \tag{1.24}$$

Case 2. Now we consider $\hat{k} = k^2 > 0$. This has general solution of exponential form:

$$T(t) = A_2 e^{(kc)^2 t} + B_2 e^{-(kc)^2 t} \tag{1.25}$$

Case 3. Next we consider $\hat{k} = -k^2 < 0$. This has a general solution of trigonometric form:

$$T(t) = A_3 \cos(kct) + B_3 \sin(kct) \tag{1.26}$$

We seek solutions which are periodic in time, so we can discard Case 1 and 2.

Proposition 1.6. *We can express the general solution for T as follows.*

$$T(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t) + B_n \sin(\omega_n t), \tag{1.27}$$

for $\omega = kc$.

Proof. TBD □

We have now shown that our constant $\hat{k} = -k^2$. Hence we can rewrite (1.21) as follows

$$\nabla^2 X + k^2 X = 0. \tag{1.28}$$

This is the Helmholtz equation, or Helmholtz eigenvalue problem.

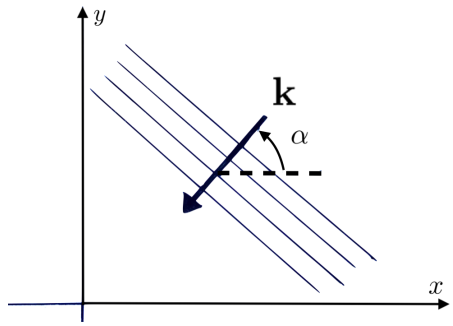

Figure 1.1: Incident wave with wave vector $\mathbf{k}$

1.7 Physical interpretation

We can interpret the constants we have stated in this chapter physically.

Definition 1.7. *The* wave vector $\mathbf{k}$ *of a 2D plane wave is defined as*

$$\mathbf{k} = (a, b) = -(k\cos\alpha, k\sin\alpha)$$

where k is the wave number *and α is the* incident angle *of the wave as shown in Fig 1.1.*

Physically, the wave number corresponds to the number of oscillations per unit distance, and is inversely proportional to the wavelength λ:

$$k = \frac{2\pi}{\lambda}. \tag{1.29}$$

In §1.6.2 we defined $\omega = kc$, where c is the speed of sound. This has a physical interpretation.

Definition 1.8. *The frequency of a plane wave with wave vector $\mathbf{k}$ is $kc = \omega$.*

In the following chapters, we will be concerned with finding expressions for the velocity fields of a plane wave scattered by some object.

In §1.3 we showed that $\mathbf{u} = \nabla\phi$, and that this function ϕ satisfies the linear wave equation. Additionally, we found a general expression for the time dependency of ϕ, and we showed that its spatial dependency is determined by the Helmholtz equation (1.28).

Proposition 1.9. *We can express ϕ as follows*

$$\phi(x, y, z, t) = Re[\Phi(x, y, z)e^{-i\omega t}]. \tag{1.30}$$

Proof. Let $\Phi = \Phi_r + i\Phi_i$. Then,

$$\begin{aligned}
\Phi e^{-i\omega t} &= (\Phi_r + i\Phi_i)(\cos(\omega t) - i\sin(\omega t)) \\
&= \Phi_r\cos(\omega t) - i\Phi_r\sin(\omega t) + i\Phi_i\cos(\omega t) - (i)^2\Phi_i\sin(\omega t) \\
\therefore\ \mathrm{Re}[\Phi e^{-i\omega t}] &= \Phi_r\cos(\omega t) + \Phi_i\sin(\omega t)
\end{aligned}$$

Clearly this would work just as well for $e^{i\omega t}$, we would just need a different choice of Φ_i.

Clearly, (1.30) is a solution to the time dependent differential equation (1.22), since $\Phi_{r,i}$ do not depend on t.

We also require that ϕ satisfy the Linear Wave Equation (1.16). Then,

$$\begin{aligned}\nabla^2\phi &= \frac{1}{c^2}\frac{\partial^2\phi}{\partial t} \\ \mathrm{Re}[\nabla^2\Phi(x,y,z)e^{-i\omega t}] &= \mathrm{Re}\left[\frac{1}{c^2}\frac{\partial^2}{\partial t}(\Phi(x,y,z)e^{-i\omega t})\right]\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\left[\nabla^2\Phi e^{-i\omega t} - \frac{1}{c^2}(-\omega^2)\Phi e^{-i\omega t}\right] &= 0 \\ \mathrm{Re}\left[e^{-i\omega t}(\nabla^2\Phi + k^2\Phi)\right] &= 0.\end{aligned}$$

Since this must hold for all t,

$$\nabla^2\Phi + k^2\Phi = 0$$

as required. $\square$

Chapter 2

Problem 1: scattering around a circular cylinder

2.1 Introduction to the problem

For this problem we consider a plane wave propagates from infinity onto a cylinder centered at the origin and of radius σ, as depicted in Fig 2.1. We will attempt to find an expression for the velocity field of this wave as it scatters around the cylinder. We will consider two different boundary conditions, Neumann and Dirichlet, and will find expressions for both of these.

Throughout this problem we will be concerned with finding an expression for the total velocity field around the cylinder, $\mathbf{u}$. As we showed in §1.7, it will be sufficient for us to seek $\Phi(x, y)$, since this is a 2D problem.

Let $\Phi_{\text{tot}} = \Phi_{\text{in}} + \Phi_{\text{sc}}$, where Φ_{in} is the incident field, and Φ_{sc} is the scattered field. All three of these must satisfy the Helmholtz equation.

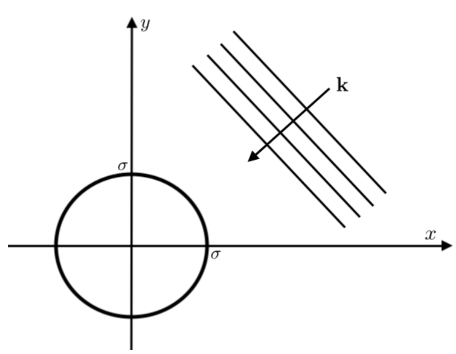

Figure 2.1: Problem 1

2.1.1 Introducing Bessel functions

Throughout this chapter we will use Bessel functions. These functions are solutions to the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - n^2)u = 0 \tag{2.1}$$

called the Bessel equation of order n, where $n \in \mathbb{C}$.

Definition 2.1. *(Korenev, 2002) Bessel functions, or cylindrical functions of the first kind, are solutions to the Bessel differential equation,* (2.1)*.*

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n}}{2^{2m+n} m!(n+m)!}$$

Definition 2.2. *(Korenev, 2002) Cylindrical functions of the second kind, also called Neumann functions, are linear combinations of Bessel functions of the first kind.*

$$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}$$

Definition 2.3. *(Korenev, 2002) Cylindrical functions of the third kind or Hankel functions, are linear combinations of Bessel functions of the first and second kind.*

$$H_n^{(1)}(z) = J_n(z) + iY_n(z)$$
$$H_n^{(2)}(z) = J_n(z) - iY_n(z)$$

By the principle of superpostion both Neumann and Hankel functions are solutions to the Bessel differential equation.

2.2 The incident field

First let us consider Φ_{in}, the incident field. We choose Cartesian coordinates (x, y), and polar coordinates (r, θ).

Proposition 2.4. *The incident field has the form*

$$\Phi_{in} = e^{-i(\mathbf{k}\cdot\mathbf{x})} \tag{2.2}$$

Proof. We need to show that this expression for Φ_{in} satisfies the Helmholtz equation, (1.28).

$$\begin{aligned}\nabla^2\Phi_{\text{in}} &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] e^{-i(\mathbf{k}\cdot\mathbf{x})} \\ &= (-i)^2 a^2 e^{-i(ax+by)} + (-i)^2 b^2 e^{-i(ax+by)} \\ &= -(a^2 + b^2)\Phi_{\text{in}}\end{aligned}$$

Then,

$$\nabla^2 \Phi_{\text{in}} + k^2 \Phi_{\text{in}} = -(a^2+b^2)\Phi_{\text{in}} + k^2\Phi_{\text{in}} = 0$$
$$\Rightarrow k^2 - (a^2+b^2) = 0$$

which is true by definition of the wave vector $\mathbf{k}$ (see definition 1.7). $\square$

Proposition 2.5. The Jacobi expansion.

$$e^{i\omega cos\varphi} = \sum_{n=-\infty}^{\infty} i^n J_n(\omega) e^{in\varphi} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(\omega) cos(n\varphi)$$

where ϵ_n is the Neumann factor, defined as follows.

$$\epsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \geq 1 \end{cases}$$

Proof. TBD (Martin, 2006, §2.5) $\square$

Proposition 2.6. *The incident field can be expressed as an infinite sum as follows.*

$$\Phi_{in} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr)\cos(n(\theta-\alpha))$$

Proof. Follows directly from Propositions 2.4 and 2.5. $\square$

2.3 The scattered field

We will now attempt to find an expression for the scattered field, Φ_{sc}.

2.3.1 Separation of variables

We don't want to make any assumptions about the form of the scattered field at this point, so we pose that it will depend on both r and $theta$. We can hence employ the method of separation of variables once more:

$$\Phi_{\text{sc}} = R(r)\Theta(\theta). \tag{2.3}$$

Additionally, we know that Φ_{sc} must satisfy the Helmholtz equation:

$$\nabla^2\Phi_{\text{sc}} + k^2\Phi_{\text{sc}} = 0. \tag{2.4}$$

Hence,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}[R(r)\Theta(\theta)]\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[R(r)\Theta(\theta)] + k^2R(r)\Theta(\theta) = 0$$
$$\Theta(\theta)\left(\frac{1}{r}\frac{dR(r)}{dr} + \frac{d^2R(r)}{dr^2}\right) + \frac{R(r)}{r^2}\left(\frac{d^2\Theta(\theta)}{d\theta^2}\right) + k^2R(r)\Theta(\theta) = 0$$
$$\Theta(\frac{1}{r}R' + R'') + \frac{1}{r^2}R\Theta'' + k^2R\Theta = 0$$

$$\left(r^2\frac{R''}{R} + r\frac{R'}{R} + (kr)^2\right) = -\frac{\Theta''}{\Theta} \tag{2.5}$$

Let $\hat{\nu}$ be a constant. By the same argument as before (§1.6.1), we yield two ordinary differential equations:

$$\frac{d^2\Theta}{d\theta^2} + \hat{\nu}\Theta = 0, \text{ and} \tag{2.6}$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (k^2r^2 - \hat{\nu})R = 0. \tag{2.7}$$

2.3.2 θ-dependence

To solve equation (2.6) we need to consider three cases.

Case 1. Let $\hat{\nu} = 0$. This gives a linear solution:

$$\Theta(\theta) = A_1\theta + B_1. \tag{2.8}$$

Case 2. Let $\hat{\nu} = \nu^2 > 0$. Then we get an solution of exponential form.:

$$\Theta(\theta) = A_2e^{\nu^2\theta} + B_2e^{-\nu^2\theta}. \tag{2.9}$$

Case 3. Let $\hat{\nu} = -\nu^2 > 0$. This gives a solution of trigonometric form:

$$\Theta(\theta) = A_3\cos(\nu\theta) + B_3\sin(\nu\theta). \tag{2.10}$$

Since θ is the polar angular coordinate, we expect our solution to be 2π periodic. We can therefore discount Case 2. Case 1 is only periodic in the trivial case where $A_1 = 0$, and this is included in Case 3, when $A_3, B_3 = 0$. We can therefore assume that $\hat{nu} = -\nu^2 > 0$.

Proposition 2.7. *The general solution is*

$$\Theta(\theta) = \sum_{n=0}^{\infty} C_n\cos(n(\theta - \alpha)) \tag{2.11}$$

Proof. By Proposition 1.6 we get a general solution of the form

$$\Theta(\theta) = \sum_{n=0}^{\infty} A_n\cos(\nu_n\theta) + B_n\sin(\nu_n\theta). \tag{2.12}$$

Since Θ must be 2π periodic,

$$\Theta(\theta) = \Theta(\theta + 2\pi).$$

Hence we must have

$$\cos(\nu_n\theta) = \cos(\nu_n\theta + \nu_n 2\pi) \text{ and } \sin(\nu_n\theta) = \sin(\nu_n\theta + \nu_n 2\pi) \text{ for all } n \in \mathbb{Z}.$$

$$\begin{aligned}\cos(\nu_n\theta) &= \cos(\nu_n\theta + \nu_n 2\pi)\\ &= \cos(\nu_n\theta)\cos(\nu_n 2\pi) - \sin(\nu_n\theta)\sin(\nu_n 2\pi) \text{ for all } \theta\end{aligned}$$

$$\cos(\nu_n 2\pi) = 1 \text{ and } \sin(\nu_n 2\pi) = 0, \text{ so } \nu_n = n \in \mathbb{Z}$$

We can check that this works for the sine terms too:

$$\begin{aligned}\sin(\nu_n\theta + \nu_n 2\pi) &= \sin(n\theta + 2n\pi)\\ &= \sin(n\theta)\cos(2n\pi) + \sin(2n\pi)\cos(n\theta)\\ &= \sin(n\theta) = \sin(\nu_n\theta).\end{aligned}$$

This gives us a solution of the form

$$\Theta(\theta) = \sum_{n=0}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta).$$

But we know that $\sin(n\theta) = 0 \ \forall n \in \mathbb{Z}, \theta \in \mathbb{R}$. We can therefore remove the second term, and also rewrite the first as follows:

$$\begin{aligned}A_n \cos(n\theta) &= \frac{A_n}{\sin(n\alpha)}[\cos(n\theta)\cos(n\alpha) + \sin(n\theta)\sin(n\alpha)]\\ &= C_n \cos(n(\theta - \alpha))\end{aligned}$$

Since for any given $n \in \mathbb{Z}$, $\sin(n\alpha)$ will be a constant. $\square$

2.3.3 r-dependence

We now know that $\hat{\nu} = -\nu^2$, so we can rewrite (2.7) as follows:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (k^2r^2 + \nu^2)R = 0. \tag{2.13}$$

Proposition 2.8. *Equation* (2.13) *is a Bessel differential equation of order $i\nu$.*

Proof. Consider the substitution $r = kz$. Then

$$\frac{dR}{dr} = \frac{1}{k}\frac{dR}{dz}, \quad \frac{d^2R}{dr^2} = \frac{1}{k^2}\frac{d^2R}{dz^2}.$$

So (2.13) becomes

$$\frac{r^2}{k^2}\frac{d^2R}{dz^2} + \frac{r}{k}\frac{dR}{dz} + (k^2r^2 + \nu^2)R = 0, \tag{2.14}$$

$$z^2\frac{d^2R}{dz^2} + z\frac{dR}{dz} + (z^2 - (i\nu)^2)R = 0. \tag{2.15}$$

$\square$

Since $R(kr)$ satisfies a Bessel differential equation, the Bessel functions are solutions, and by the superposition principle any linear superposition of these is also a solution. We will need to consider the Sommerfeld radiation condition in order specify the general solution for $R(r)$.

Definition 2.9. The Sommerfeld radiation condition. *TBD*

Proposition 2.10. Φ_{sc} *must satisfy the Sommerfeld radiation condition.*

Proof. TBD □

Proposition 2.11. *Hankel functions of the first kind satisfy the Sommerfeld radiation condition.*

Proof. TBD, see (Martin, 2006, §4.2) □

From now on we refer to $H_\nu^{(1)}$ as H_ν for simplicity.

Proposition 2.12.

$$J_\nu(z) = (-1)^\nu J_\nu(z)$$

Proof. TBD □

Proposition 2.13.

$$H_{-\nu}(z) = e^{i\nu\pi} H_\nu(z)$$

Proof. TBD □

Proposition 2.14.

$$R(r) = \sum_{n=0}^{\infty} F_n H_n(kr), \text{ for } F_n \text{ constant.}$$

Proof. From Proposition 2.13

$$\begin{aligned}\sum_{n=-\infty}^{\infty} H_n(kr) &= \sum_{n=0}^{\infty} H_n(kr) + \sum_{n=0}^{\infty} e^{in\pi} H_n(kr) \\ &= \sum_{n=0}^{\infty} F_n H_n(kr)\end{aligned}$$

since $e^{in\pi}$ is a constant for any given $n \in \mathbb{Z}$. □

2.3.4 General solution

We can now combine our solutions for Θ and R to find an expression for Φ_{sc}.

Proposition 2.15. *The general solution for scattered field can be expressed as follows.*

$$\Phi_{sc} = \sum_{n=-\infty}^{\infty} \epsilon_n i^n B_n H_n(kr)\cos(n(\theta-\alpha)) \tag{2.16}$$

Proof. TBD. Need to show that:

$$\Phi_{\text{sc}} = \sum_{n=-\infty}^{\infty} C_n H_n(kr)\cos(n(\theta-\alpha))$$

with, $C_n = \epsilon_n i^n B_n$. $\square$

2.3.5 Neumann boundary condition

We first cosider the boundary at the cylinder wall to satisfy a Neumann boundary condition.

Definition 2.16. *(Martin, 2006, §1.3.2) A boundary is sound-hard if*

$$\frac{\partial u}{\partial r} = 0, \text{ on } r = \sigma.$$

Equivalently, we can express this boundary condition in terms of Φ:

$$\frac{\partial \Phi}{\partial r} = 0, \text{ on } r = \sigma.$$

We can now apply this to find an expression for the constant terms in (2.16). Differentiating this equation gives

$$\sum_{n=0}^{\infty} \epsilon_n k i^n \{J_n'(kr) + B_n H_n'(kr)\}\cos(n(\theta-\alpha)) = 0 \tag{2.17}$$

Since $\cos(n(\theta-\alpha)) \neq 0 \ \forall n, \theta$, it must be that the expression inside the braces must be zero for each n at the boundary $r = \sigma$. Hence, we get an expression for B_n:

$$B_n = \frac{J_n'(k\sigma)}{H_n'(k\sigma)}. \tag{2.18}$$

2.3.6 Dirichlet boundary condition

We now consider the Dirichlet boundary condition.

Definition 2.17. *(Martin, 2006, §1.3.2) A body is sound-soft if*

$$u = 0, \text{ on } r = \sigma$$

Hence, $\Phi_{sc} = -\Phi_{in}$ on $r = \sigma$,

$$\sum_{n=0}^{\infty} \epsilon_n i^n B_n H_n(kr) \cos(n((\theta - \alpha))) = -\sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \cos(n((\theta - \alpha)))$$
$$\text{hence for all } n,\ B_n H_n(k\sigma) = -J_n(k\sigma)$$

Hence the Dirichlet boundary condition specifies B_n as follows.

$$B_n = \frac{-J_n(k\sigma)}{H_n(k\sigma)} \tag{2.19}$$

2.4 Total field

Throughout this chapter we have been searching for an expression for the resultant field of a plane wave with wave vector $\mathbf{k}$ incident on a cylinder radius σ, $\phi(x, y, z, t; \sigma, \mathbf{k})$.

Proposition 2.18.

$$\Phi_{tot} = \sum_{n=0}^{\infty} \epsilon_n i^n \cos(n(\theta - \alpha))\{B_n H_n(kr) + J_n(kr)\}$$

where B_n is specified by the boundary conditions outlined in §2.3.6 and §2.3.5.

Chapter 3

Plotter tool

Access the repo on Bitbucket here.

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