MATH3000
Canonical Scattering Problems
Interim report

Veracruz Gonzalez Gomez
Student ID: 10138385
Supervised by Raphaël Assier

University of Manchester
January 2020

Contents

Abstract

The aim of this project is to better understand the scattering of acoustic waves around different objects. The starting point for this are the Navier-Stokes equations and a simplified model of acoustic wave motion in air.

The model of wave motion relies on assumptions about the nature of fluid flows, namely, that air is a barotropic, adiabatic ideal gas. These are explained in some detail in the development of the theory.

We use perturbation theory to arrive at the Linear Wave Equation from the Navier-Stokes equations. We then use this wave equation to derive the Helmholtz equation.

These equations are used to investigate our first problem; scattering of acoustic waves around a cylinder. There were two primary aims for this project: firstly, to find an expression for the resulting field and, secondly, to plot this field using Python. The plotting of the actual wave field is still pending.

Chapter 1

Theory

1.1 Governing equations

We can use the Navier-Stokes equations to model the velocity field $\mathbf{u}$ of a fluid of density ρ and viscosity μ.

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \nabla p + \mu\nabla^2\mathbf{u} \tag{1.1}$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0 \tag{1.2}$$

Where ∇p is the pressure gradient within the fluid, $\mathbf{F}$ is the external force applied onto the fluid and D/Dt is the material derivative. For our problems, we will be interested in the velocity field of air, where the viscosity is negligible. Setting $\mu = 0$ gives the Euler momentum equation (Shaughnessy 2005)

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \nabla p. \tag{1.3}$$

We will consider problems where there are no external forces acting on our fluid, so we can set $\mathbf{F}$ to zero as well:

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p. \tag{1.4}$$

This, together with the continuity equation (1.2) are the governing equations for our velocity field.

1.1.1 Perturbation of the governing equations

We will use perturbation theory to arrive at the linear wave equation from our governing equations. First, we must consider air at rest. In this case ρ, p and $\mathbf{u}$ will be constant, in particular

$$\rho = \rho_0,\ p = p_0,\ \mathbf{u} = \mathbf{0}.$$

We can therefore think of an acoustic wave as a small perturbation of this rest state. Let $\epsilon << 1$, then we can express ρ, p and $\mathbf{u}$ in this state as follows:

$$\rho = \rho_0 + \epsilon\tilde{\rho},\ p = p_0 + \epsilon\tilde{p},\ \mathbf{u} = \epsilon\tilde{\mathbf{u}}. \tag{1.5}$$

To derive our wave equation, we can input (1.5) into (1.4) and (1.2). From (1.4) we get

$$(\rho_0 + \epsilon\tilde{\rho})(\frac{\partial(\epsilon\tilde{\mathbf{u}})}{\partial t} + (\epsilon\tilde{\mathbf{u}} \cdot \nabla)(\epsilon\tilde{\mathbf{u}})) = -\nabla(p_0 + \epsilon\tilde{p}).$$

Since ϵ is small, all terms of order ϵ^2 or smaller are negligible. Hence we are left with

$$\rho_0 \frac{\partial\tilde{\mathbf{u}}}{\partial t} = -\nabla\tilde{p}. \tag{1.6}$$

From (1.2) we get

$$\frac{\partial}{\partial t}(\rho_0 + \epsilon\tilde{\rho}) + (\rho_0 + \epsilon\tilde{\rho})(\nabla \cdot (\epsilon\tilde{\mathbf{u}})) + (\epsilon\tilde{\mathbf{u}} \cdot \nabla)(\rho_0 + \epsilon\tilde{\rho}).$$

Since $\epsilon << 1$, we are left with

$$\frac{\partial\tilde{\rho}}{\partial t} + \rho_0(\nabla \cdot \tilde{\mathbf{u}}) = 0. \tag{1.7}$$

Differentiating (1.7) by t:

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}(\nabla \cdot \tilde{\mathbf{u}}) = 0$$

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} + \rho_0 \nabla \cdot \frac{\partial\tilde{\mathbf{u}}}{\partial t} = 0. \tag{1.8}$$

To continue we need to make a physical assumption about the pressure field.

1.2 The barotropic assumption

A fluid can either be barotropic or baroclinic. Naively, baroclinic fluids are those where there is high variability. For example, where there are different air masses, cold and warm fronts or weather. In the problems we are going to consider this will not be the case - our fluid will be barotropic.

Definition 1.1. *(Shames 2002) A barotropic fluid is one where ρ is expressible as a function of p only.*

$$\rho = \rho(p)$$

We can therefore also express p as a funcion of ρ only. I will call this function f for clarity.

$$p = f(\rho) \tag{1.9}$$

From (1.5), we have that $\rho = \rho_0 + \epsilon\tilde{\rho}$. Hence

$$p = f(\rho_0 + \epsilon\tilde{\rho}).$$

We can now expand this around the point ρ_0 using Taylor series. This is valid since $f(\rho)$ is a real valued function.

$$\begin{aligned}
p &= f(\rho) \\
&= f(\rho_0) + f'(\rho_0)(\rho - \rho_0) + \frac{1}{2!} f''(\rho_0)(\rho - \rho_0) + \cdots \\
&= f(\rho_0) + \epsilon \tilde{\rho} f'(\rho_0) + O(\epsilon^2)
\end{aligned}$$

Hence, since $\epsilon << 1$:

$$p = p_0 + \epsilon \tilde{\rho} f'(\rho_0) \tag{1.10}$$

But from (1.5) we have that $p = p_0 + \epsilon \tilde{p}$, and so $p_0 = p - \epsilon \tilde{p}$. Then from (1.10) we get:

$$\tilde{p} = \tilde{\rho} f'(\rho_0) \tag{1.11}$$

1.3 The linear wave equation

From equation (1.6) we have

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = -\frac{1}{\rho_0} \nabla \tilde{p},$$

and we can substitute this into (1.7) to get:

$$\begin{aligned}
\frac{\partial^2 \tilde{\rho}}{\partial t^2} + \rho_0 \nabla \cdot (\frac{-\nabla \tilde{p}}{\rho_0}) = 0 \\
\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \nabla^2 \tilde{p} = 0.
\end{aligned}$$

Now we can use 1.11 to find an expression for $\tilde{\rho}$.

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \nabla^2 (\tilde{\rho} f'(\rho_0)) = 0$$

Note $f'(\rho_0)$ is a constant. Let $f'(\rho_0) = c^2$. Then:

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = c^2 \nabla^2 \tilde{\rho} \tag{1.12}$$

Which is the linear wave equation! We can do the same for $\tilde{p}$ and get

$$\frac{1}{c^2} \frac{\partial^2 \tilde{p}}{\partial t^2} = \nabla^2 \tilde{p}. \tag{1.13}$$

Similarly,

$$\nabla^2 \mathbf{u} = \frac{1}{c^2} \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{1.14}$$

Throughout this paper, it will be useful to consider the velocity field $\mathbf{u}$ represented by the scalar function ϕ, where

$$\mathbf{u}(x, y, z, t) = \nabla \phi(x, y, z, t). \tag{1.15}$$

Proposition 1.2. *The scalar function ϕ satisfies the Helmholtz equation.*

Proof.

$$\begin{aligned}
\nabla^2(\nabla\phi) &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\nabla\phi) \\
\nabla^2\left(\frac{\partial\phi}{\partial x_i}\right) &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial\phi}{\partial x_i}\right) \\
\frac{\partial}{\partial x_i}(\nabla^2\phi) &= \frac{\partial}{\partial x_i}\left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right)
\end{aligned}$$

Integrating with respect to x_i gives,

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + C$$

The constant of integration will depend on our choice of ϕ. We set it to zero without loss of generality. Hence,

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{1.16}$$

□

1.4 Laplace's hypothesis

We need to make one more physical assumption, namely that motion in air is an adiabatic process. This will become relevant shortly.

Hypothesis 1.3 (Laplace's hypothesis)**.** *Sound propagation occurs with negligible internal heat flow.*

Definition 1.4. *A process is adiabatic if it satisfies Laplace's hypothesis.*

For an adiabatic process and a gas at constant pressure and volume, with constant specific heat coefficients per unit mass, and with $p \propto \rho$ at constant temperature the following relationship holds:

$$p = K\rho^\gamma \tag{1.17}$$

where $\gamma = c_p/c_v$ the specific heat ratio, and K constant in time (Pierce 2019, §1.4.1).

1.5 Speed of sound

In 1.3 we arbitrarily set $f'(\rho_0) = c^2$. We can now show c is in fact the speed of sound in air.

By definition of f, we have

$$p = f(\rho), \text{ so } f'(\rho_0) = \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0} \tag{1.18}$$

Then, assuming motion in air is an adiabatic process,

$$\begin{aligned} c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0} = \left.\frac{\partial}{\partial \rho}(K\rho^\gamma)\right|_{\rho_0} \\ &= \left.(\gamma K \rho^{\gamma-1})\right|_{\rho_0} = \gamma \frac{K\rho_0^\gamma}{\rho_0} \\ &= \gamma \frac{p_0}{\rho_0} \end{aligned}$$

Hence, our constant c^2 depends only on our initial density and initial pressure. Additionally, it has dimensions

$$\frac{[p]}{[\rho]} = \frac{kgm^{-1}s^{-2}}{kgm^{-3}} = (ms^{-1})^2$$

since γ is a dimensionless ratio. Hence c is indeed a speed.

Law 1.5 (Ideal Gas Law)**.** *For an ideal gas with ideal gas constant R at temperature T_K measured in degrees Kelvin, the following relationship holds*

$$p = \rho R T_K.$$

We can now assume that air is an ideal gas and apply the Ideal Gas law. We hence show that our arbitrary constant c is in fact the speed of sound for an ideal gas of ideal gas constant R at temperature T_K with specific heat ratio γ:

$$c = \sqrt{\gamma R T_K} \tag{1.19}$$

1.6 The Helmholtz equation

1.6.1 Separation of variables

We can now use the linear wave equation for ϕ(1.16) to derive the Helmholtz equation.

To do this, employ a standard separation of variables argument. We propose that, since $\phi(\mathbf{x}, t)$, there exist X and T such that

$$\phi = X(\mathbf{x})T(t). \tag{1.20}$$

This expression along with (1.16) gives us:

$$\begin{aligned} T(t)\nabla^2 X(\mathbf{x}) &= \frac{1}{c^2}\frac{d^2 T}{dt^2} X(\mathbf{x}), \\ \frac{\nabla^2 X}{X} &= \frac{1}{c^2}\frac{T''}{T}. \end{aligned}$$

This can only be true if both sides are equal to the same constant, say $\hat{k}$. We therefore yield two ordinary differential equations:

$$\nabla^2 X = \hat{k}X \qquad (1.21) \qquad\qquad T'' = \hat{k}c^2 T \qquad (1.22)$$

Equation (1.21) represents a time independent form of the linear wave equation. Equation (1.22) can be solved to uncover the time dependence.

1.6.2 Time dependence

We now seek a solution to (1.22). We can rewrite (1.22) as

$$\frac{d^2T}{dt^2} - \hat{k}c^2 T = 0 \tag{1.23}$$

Since c is strictly a physical constant, namely the speed of sound in air, $c^2 \geq 0$. Therefore there are three cases to consider: $\hat{k} = 0$, $\hat{k} < 0$ and $\hat{k} > 0$.

Case 1. The trivial case where $\hat{k} = 0$ leads to a linear solution for time:

$$T(t) = A_1 t + B_1. \tag{1.24}$$

Case 2. Now we consider $\hat{k} = k^2 > 0$. This has general solution of exponential form:

$$T(t) = A_2 e^{(kc)^2 t} + B_2 e^{-(kc)^2 t} \tag{1.25}$$

Case 3. Next we consider $\hat{k} = -k^2 < 0$. This has a general solution of trigonometric form:

$$T(t) = A_3 \cos(kct) + B_3 \sin(kct) \tag{1.26}$$

We seek solutions which are periodic in time, so we can discard Case 1 and 2.

Proposition 1.6. *We can express the general solution for T as follows.*

$$T(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t) + B_n \sin(\omega_n t), \tag{1.27}$$

for $\omega = kc$.

Proof. TBD □

We have now shown that our constant $\hat{k} = -k^2$. Hence we can rewrite (1.21) as follows

$$\nabla^2 X + k^2 X = 0. \tag{1.28}$$

This is the Helmholtz equation, or Helmholtz eigenvalue problem.

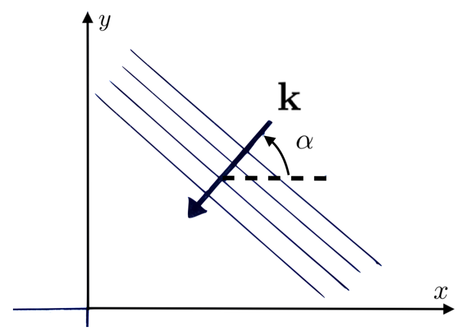

Figure 1.1: Incident wave with wave vector **k**

1.7 Physical interpretation

We can interpret the constants we have stated in this chapter physically.

Definition 1.7. *The* wave vector **k** *of a 2D plane wave is defined as*

$$\mathbf{k} = (a, b) = -(k\cos\alpha, k\sin\alpha)$$

where k is the wave number *and α is the* incident angle *of the wave as shown in Fig 1.1.*

Physically, the wave number corresponds to the number of oscillations per unit distance, and is inversely proportional to the wavelength λ:

$$k = \frac{2\pi}{\lambda}. \tag{1.29}$$

In §1.6.2 we defined $\omega = kc$, where c is the speed of sound. This has a physical interpretation.

Definition 1.8. *The frequency of a plane wave with wave vector* **k** *is $kc = \omega$.*

In the following chapters, we will be concerned with finding expressions for the velocity fields of a plane wave scattered by some object.

In §1.3 we showed that $\mathbf{u} = \nabla\phi$, and that this function ϕ satisfies the linear wave equation. Additionally, we found a general expression for the time dependency of ϕ, and we showed that its spatial dependency is determined by the Helmholtz equation (1.28).

Proposition 1.9. *We can express ϕ as follows*

$$\phi(x, y, z, t) = Re[\Phi(x, y, z)e^{-i\omega t}]. \tag{1.30}$$

Proof. Let $\Phi = \Phi_r + i\Phi_i$. Then,

$$\begin{aligned}
\Phi e^{-i\omega t} &= (\Phi_r + i\Phi_i)(\cos(\omega t) - i\sin(\omega t)) \\
&= \Phi_r\cos(\omega t) - i\Phi_r\sin(\omega t) + i\Phi_i\cos(\omega t) - (i)^2\Phi_i\sin(\omega t) \\
\therefore\ \mathrm{Re}[\Phi e^{-i\omega t}] &= \Phi_r\cos(\omega t) + \Phi_i\sin(\omega t)
\end{aligned}$$

Clearly this would work just as well for $e^{i\omega t}$, we would just need a different choice of Φ_i.

Clearly, (1.30) is a solution to the time dependent differential equation (1.22), since $\Phi_{r,i}$ do not depend on t.

We also require that ϕ satisfy the Linear Wave Equation (1.16). Then,

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t}$$
$$\mathrm{Re}[\nabla^2\Phi(x,y,z)e^{-i\omega t}] = \mathrm{Re}\left[\frac{1}{c^2}\frac{\partial^2}{\partial t}(\Phi(x,y,z)e^{-i\omega t})\right]$$

$$\mathrm{Re}\left[\nabla^2\Phi e^{-i\omega t} - \frac{1}{c^2}(-\omega^2)\Phi e^{-i\omega t}\right] = 0$$
$$\mathrm{Re}\left[e^{-i\omega t}(\nabla^2\Phi + k^2\Phi)\right] = 0.$$

Since this must hold for all t,

$$\nabla^2\Phi + k^2\Phi = 0$$

as required. □

1.8 Bessel functions

We will use Bessel functions throughout. These functions are solutions to the differential equation.

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - n^2)u = 0 \tag{1.31}$$

called the Bessel equation of order n.

Definition 1.10. *(Korenev 2002) Bessel functions, or cylindrical functions of the first kind, are solutions to the Bessel differential equation,* (1.31)*.*

$$J_n(z) = \sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m+n}}{m!(n+m)!}$$

Definition 1.11. *(ibid.) Cylindrical functions of the second kind, also called Neumann functions, are linear combinations of Bessel functions of the first kind.*

$$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}$$

where we take the limit $n \to n'$ for integer values of n (Cohen 2007).

We run into problems when we consider Neumann functions of integer argument. Since this will be our main application it is worth discussing.

Proposition 1.12. *Neumann functions are not well defined for $n \in \mathbb{Z}$.*

Proof. It is sufficient to note the following limits:

$$\lim_{n \to \text{ even int}^+} \frac{1}{\sin(n\pi)} = +\infty, \quad \lim_{n \to \text{ even int}^-} \frac{1}{\sin(n\pi)} = -\infty$$
$$\lim_{n \to \text{ odd int}^+} \frac{1}{\sin(n\pi)} = -\infty, \quad \lim_{n \to \text{ odd int}^-} \frac{1}{\sin(n\pi)} = +\infty$$
.

Hence, the limit of $1/\sin(n\pi)$ as $n \to$ integer does not exist, and therefore the Neumann functions are not well defined for $n \in \mathbb{Z}$. □

However Neumann functions are still useful in this context because they form part of the definition of Hankel functions.

Definition 1.13. *(Korenev 2002) Cylindrical functions of the third kind or Hankel functions, are linear combinations of Bessel functions of the first and second kind.*

$$H_n^{(1)}(z) = J_n(z) + iY_n(z)$$
$$H_n^{(2)}(z) = J_n(z) - iY_n(z)$$

By the principle of superpostion both Neumann and Hankel functions are solutions to the Bessel differential equation.

Chapter 2

Scattering around a circular cylinder

2.1 Introduction to the problem

For this problem we consider a plane wave propagates from infinity onto a cylinder centered at the origin and of radius σ, as depicted in Fig 2.1. We will attempt to find an expression for the velocity field of this wave as it scatters around the cylinder. We will consider two different boundary conditions, Neumann and Dirichlet, and will find expressions for both of these.

Throughout this problem we will be concerned with finding an expression for the total velocity field around the cylinder, $\mathbf{u}$. As we showed in §1.7, it will be sufficient for us to seek $\Phi(x, y)$, since this is a 2D problem.

Let $\Phi_{\text{tot}} = \Phi_{\text{in}} + \Phi_{\text{sc}}$, where Φ_{in} is the incident field, and Φ_{sc} is the scattered field. All three of these must satisfy the Helmholtz equation.

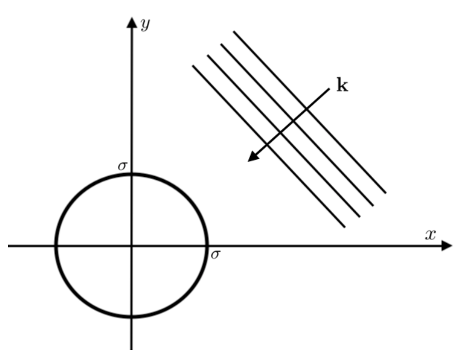

Figure 2.1: Problem 1

2.2 The incident field

First let us consider Φ_{in}, the incident field. We choose Cartesian coordinates (x, y), and polar coordinates (r, θ).

Proposition 2.1. *The incident field has the form*

$$\Phi_{in} = e^{-i(\mathbf{k}\cdot\mathbf{x})} \tag{2.1}$$

Proof. We need to show that this expression for Φ_{in} satisfies the Helmholtz equation, (1.28).

$$\begin{aligned}
\nabla^2\Phi_{\text{in}} &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] e^{-i(\mathbf{k}\cdot\mathbf{x})} \\
&= (-i)^2 a^2 e^{-i(ax+by)} + (-i)^2 b^2 e^{-i(ax+by)} \\
&= -(a^2+b^2)\Phi_{\text{in}}
\end{aligned}$$

Then,

$$\begin{gathered}
\nabla^2\Phi_{\text{in}} + k^2\Phi_{\text{in}} = -(a^2+b^2)\Phi_{\text{in}} + k^2\Phi_{\text{in}} = 0 \\
\Rightarrow k^2 - (a^2+b^2) = 0
\end{gathered}$$

which is true by definition of the wave vector $\mathbf{k}$ (see definition 1.7). □

Proposition 2.2. The Jacobi expansion.

$$e^{i\omega cos\varphi} = \sum_{n=-\infty}^{\infty} i^n J_n(\omega) e^{in\varphi} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(\omega) cos(n\varphi)$$

where ϵ_n is the Neumann factor, defined as follows.

$$\epsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \geq 1 \end{cases}$$

Proof. TBD (Martin 2006, §2.5) □

Proposition 2.3. *The incident field can be expressed as an infinite sum as follows.*

$$\Phi_{in} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr)\cos(n(\theta-\alpha))$$

Proof. Follows directly from Propositions 2.1 and 2.2. □

2.3 The scattered field

We will now attempt to find an expression for the scattered field, Φ_{sc}.

2.3.1 Separation of variables

We don't want to make any assumptions about the form of the scattered field at this point, so we pose that it will depend on both r and $theta$. We can hence employ the method of separation of variables once more:

$$\Phi_{\mathrm{sc}} = R(r)\Theta(\theta). \tag{2.2}$$

Additionally, we know that Φ_{sc} must satisfy the Helmholtz equation:

$$\nabla^2\Phi_{\mathrm{sc}} + k^2\Phi_{\mathrm{sc}} = 0. \tag{2.3}$$

Hence,

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}[R(r)\Theta(\theta)]\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[R(r)\Theta(\theta)] + k^2R(r)\Theta(\theta) = 0 \\
\Theta(\theta)\left(\frac{1}{r}\frac{dR(r)}{dr} + \frac{d^2R(r)}{dr^2}\right) + \frac{R(r)}{r^2}\left(\frac{d^2\Theta(\theta)}{d\theta^2}\right) + k^2R(r)\Theta(\theta) = 0 \\
\Theta(\frac{1}{r}R' + R'') + \frac{1}{r^2}R\Theta'' + k^2R\Theta = 0 \\
\left(r^2\frac{R''}{R} + r\frac{R'}{R} + (kr)^2\right) = -\frac{\Theta''}{\Theta} \qquad (2.4)
\end{aligned}$$

Let $\hat{\nu}$ be a constant. By the same argument as before (§1.6.1), we yield two ordinary differential equations:

$$\frac{d^2\Theta}{d\theta^2} + \hat{\nu}\Theta = 0, \text{ and} \tag{2.5}$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (k^2r^2 - \hat{\nu})R = 0. \tag{2.6}$$

2.3.2 θ-dependence

To solve equation (2.5) we need to consider three cases.

Case 1. Let $\hat{\nu} = 0$. This gives a linear solution:

$$\Theta(\theta) = A_1\theta + B_1. \tag{2.7}$$

Case 2. Let $\hat{\nu} = \nu^2 > 0$. Then we get an solution of exponential form.:

$$\Theta(\theta) = A_2e^{\nu^2\theta} + B_2e^{-\nu^2\theta}. \tag{2.8}$$

Case 3. Let $\hat{\nu} = -\nu^2 > 0$. This gives a solution of trigonometric form:

$$\Theta(\theta) = A_3\cos(\nu\theta) + B_3\sin(\nu\theta). \tag{2.9}$$

Since θ is the polar angular coordinate, we expect our solution to be 2π periodic. We can therefore discount Case 2. Case 1 is only periodic in the trivial case where $A_1 = 0$, and this is included in Case 3, when $A_3, B_3 = 0$. We can therefore assume that $\hat{nu} = -\nu^2 > 0$.

Proposition 2.4. *The general solution is*

$$\Theta(\theta) = \sum_{n=0}^{\infty} C_n \cos(n(\theta - \alpha)) \tag{2.10}$$

Proof. By Proposition 1.6 we get a general solution of the form

$$\Theta(\theta) = \sum_{n=0}^{\infty} A_n \cos(\nu_n \theta) + B_n \sin(\nu_n \theta). \tag{2.11}$$

Since Θ must be 2π periodic,

$$\Theta(\theta) = \Theta(\theta + 2\pi).$$

Hence we must have

$$\cos(\nu_n \theta) = \cos(\nu_n \theta + \nu_n 2\pi) \text{ and } \sin(\nu_n \theta) = \sin(\nu_n \theta + \nu_n 2\pi) \text{ for all } n \in \mathbb{Z}.$$

$$\begin{aligned} \cos(\nu_n \theta) &= \cos(\nu_n \theta + \nu_n 2\pi) \\ &= \cos(\nu_n \theta)\cos(\nu_n 2\pi) - \sin(\nu_n \theta)\sin(\nu_n 2\pi) \text{ for all } \theta \end{aligned}$$

$$\cos(\nu_n 2\pi) = 1 \text{ and } \sin(\nu_n 2\pi) = 0, \text{ so } \nu_n = n \in \mathbb{Z}$$

We can check that this works for the sine terms too:

$$\begin{aligned} \sin(\nu_n \theta + \nu_n 2\pi) &= \sin(n\theta + 2n\pi) \\ &= \sin(n\theta)\cos(2n\pi) + \sin(2n\pi)\cos(n\theta) \\ &= \sin(n\theta) = \sin(\nu_n \theta). \end{aligned}$$

This gives us a solution of the form

$$\Theta(\theta) = \sum_{n=0}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta).$$

But we know that $\sin(n\theta) = 0 \ \forall n \in \mathbb{Z}, \theta \in \mathbb{R}$. We can therefore remove the second term, and also rewrite the first as follows:

$$\begin{aligned} A_n \cos(n\theta) &= \frac{A_n}{\sin(n\alpha)}[\cos(n\theta)\cos(n\alpha) + \sin(n\theta)\sin(n\alpha)] \\ &= C_n \cos(n(\theta - \alpha)) \end{aligned}$$

Since for any given $n \in \mathbb{Z}$, $\sin(n\alpha)$ will be a constant. $\square$

2.3.3 r-dependence

We now know that $\hat{\nu} = -\nu^2$, so we can rewrite (2.6) as follows:

$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} + (k^2 r^2 + \nu^2)R = 0. \tag{2.12}$$

Proposition 2.5. *Equation* (2.12) *is a Bessel differential equation of order $i\nu$.*

Proof. Consider the substitution $r = kz$. Then

$$\frac{dR}{dr} = \frac{1}{k}\frac{dR}{dz}, \quad \frac{d^2R}{dr^2} = \frac{1}{k^2}\frac{d^2R}{dz^2}.$$

So (2.12) becomes

$$\frac{r^2}{k^2}\frac{d^2R}{dz^2} + \frac{r}{k}\frac{dR}{dz} + (k^2r^2 + \nu^2)R = 0, \tag{2.13}$$

$$z^2\frac{d^2R}{dz^2} + z\frac{dR}{dz} + (z^2 - (i\nu)^2)R = 0. \tag{2.14}$$

□

Since $R(kr)$ satisfies a Bessel differential equation, the Bessel functions are solutions, and by the superposition principle any linear superposition of these is also a solution. We will need to consider the Sommerfeld radiation condition in order specify the general solution for $R(r)$.

Definition 2.6. The Sommerfeld radiation condition. *TBD*

Proposition 2.7. *Φ_{sc} must satisfy the Sommerfeld radiation condition.*

Proof. TBD □

Proposition 2.8. *Hankel functions of the first kind satisfy the Sommerfeld radiation condition.*

Proof. TBD, see (Martin 2006, §4.2) □

From now on we refer to $H_\nu^{(1)}$ as H_ν for simplicity.

Proposition 2.9.

$$J_\nu(z) = (-1)^\nu J_\nu(z)$$

Proof. TBD □

Proposition 2.10.

$$H_{-\nu}(z) = e^{i\nu\pi} H_\nu(z)$$

Proof. TBD □

Proposition 2.11.

$$R(r) = \sum_{n=0}^{\infty} F_n H_n(kr), \text{ for } F_n \text{ constant.}$$

Proof. From Proposition 2.10

$$\begin{aligned}\sum_{n=-\infty}^{\infty} H_n(kr) &= \sum_{n=0}^{\infty} H_n(kr) + \sum_{n=0}^{\infty} e^{in\pi} H_n(kr) \\ &= \sum_{n=0}^{\infty} F_n H_n(kr)\end{aligned}$$

since $e^{in\pi}$ is a constant for any given $n \in \mathbb{Z}$. □

2.3.4 General solution

We can now combine our solutions for Θ and R to find an expression for Φ_{sc}.

Proposition 2.12. *The general solution for scattered field can be expressed as follows.*

$$\Phi_{sc} = \sum_{n=-\infty}^{\infty} \epsilon_n i^n B_n H_n(kr) \cos(n(\theta - \alpha)) \tag{2.15}$$

Proof. TBD. Need to show that:

$$\Phi_{\text{sc}} = \sum_{n=-\infty}^{\infty} C_n H_n(kr) \cos(n(\theta - \alpha))$$

with, $C_n = \epsilon_n i^n B_n$. □

2.3.5 Neumann boundary condition

We first cosider the boundary at the cylinder wall to satisfy a Neumann boundary condition.

Definition 2.13. *(Martin 2006, §1.3.2) A boundary is sound-hard if*

$$\frac{\partial u}{\partial r} = 0, \text{ on } r = \sigma.$$

Equivalently, we can express this boundary condition in terms of Φ:

$$\frac{\partial \Phi}{\partial r} = 0, \text{ on } r = \sigma.$$

We can now apply this to find an expression for the constant terms in (2.15). Differentiating this equation gives

$$\sum_{n=0}^{\infty} \epsilon_n k i^n \{J_n'(kr) + B_n H_n'(kr)\} \cos(n(\theta - \alpha)) = 0 \tag{2.16}$$

Since $\cos(n(\theta-\alpha)) \neq 0 \ \forall n, \theta$, it must be that the expression inside the braces must be zero for each n at the boundary $r = \sigma$. Hence, we get an expression for B_n:

$$B_n = \frac{J_n'(k\sigma)}{H_n'(k\sigma)}. \tag{2.17}$$

2.3.6 Dirichlet boundary condition

We now consider the Dirichlet boundary condition.

Definition 2.14. *(Martin 2006, §1.3.2) A body is sound-soft if*

$$u = 0, \text{ on } r = \sigma$$

Hence, $\Phi_{sc} = -\Phi_{in}$ on $r = \sigma$,

$$\sum_{n=0}^{\infty} \epsilon_n i^n B_n H_n(kr) \cos(n((\theta-\alpha))) = -\sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \cos(n((\theta-\alpha)))$$
$$\text{hence for all } n, \ B_n H_n(k\sigma) = -J_n(k\sigma)$$

Hence the Dirichlet boundary condition specifies B_n as follows.

$$B_n = \frac{-J_n(k\sigma)}{H_n(k\sigma)} \tag{2.18}$$

2.4 Total field

Throughout this chapter we have been searching for an expression for the resultant field of a plane wave with wave vector $\mathbf{k}$ incident on a cylinder radius σ, $\phi(x, y, z, t; \sigma, \mathbf{k})$.

Proposition 2.15.

$$\Phi_{tot}(r, \theta; \mathbf{k}, \sigma) = \sum_{n=0}^{\infty} \epsilon_n i^n \cos(n(\theta-\alpha))\{B_n H_n(kr) + J_n(kr)\}$$

where B_n is specified by the boundary conditions outlined in §2.3.6 and §2.3.5.

Chapter 3

Scattering inside the circular cylinder

3.1 Introduction

We now want to consider what happens inside a cylinder of different density to the outside medium. We consider our same plane wave inciding on this cylinder see Fig 3.1. The goal in this section is to find an expression for the velocity field inside the cylinder, combine this with our result in Chapter 2 to plot the complete field over the entire domain.

This is referred to as a *transmission problem* in the literature, see for example Martin 2006, §1.3.3 since the wave is not bounded at the cylinder walls but transmitted through.

The statement of the problem is as follows. We are given an incident field, Φ_{in} for which we need to find the field at the cylinder wall, $\Phi_{r=\sigma}$ and the field inside the cylinder wall $\Phi_{r<\sigma}$. The velocity field inside the cylinder will be subject to the same physics as the one outside the cylinder. We can therefore find the scattered field inside the cylinder in the same way we found the scattered field outside the cylinder in section 2.3.

$$\Phi_{\text{inside}} = \sum_{n=0}^{\infty} \epsilon_n i^n \hat{A} \text{ Bessel}(kr) \cos(n(\theta - \alpha)) \tag{3.1}$$

$$\Phi_{\text{outside}} = \sum_{n=0}^{\infty} \epsilon_n i^n \hat{B} H(kr) \cos(n(\theta - \alpha)) \tag{3.2}$$

In 2.3 we made our choice of cylindrical function to be used by describing the far field, and realising it must satisfy the Sommerfeld Radiation Condition (definition 2.6). In the same way here we consider the field as we approach the origin. Outside the cylinder the Bessel function must still be of the third kind since it must still satisfy the Sommerfeld Radiation Condition.

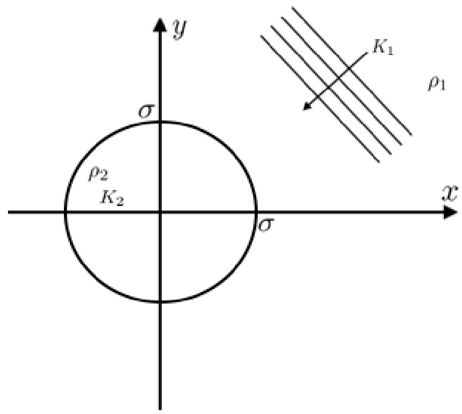

Figure 3.1: The transmission problem

3.2 Boundary conditions

The complete velocity field must be

1. continuous at the boundary,
2. well defined as $r \to \infty$,
3. and well defined as $r \to 0$.

These conditions will be sufficient to find an expression for the total field, which as before will be defined as

$$\Phi_{\text{tot}} = \Phi_{\text{in}} + \Phi_{\text{out}}. \tag{3.3}$$

Condition 2 is immediatelly satisfied by our solution to the first problem, since that solution was chosen to satisfy the Sommerfeld Radiation Condition.

We can restate Condition 3 as follows.

$$\left.\frac{\partial \Phi_{\text{out}}}{\partial r}\right|_{r=\sigma} = \left.\frac{\partial \Phi_{\text{in}}}{\partial r}\right|_{r=\sigma}. \tag{3.4}$$

Condition 1 will be satisfied by our choice of Bessel function.

Proposition 3.1. *Neumann functions are singular at the origin.*

Proof. □

Proposition 3.2. *Bessel functions of the first kind of integer order are well defined at the origin. In particular,*

$$J_n(x) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \quad \textit{for } x \to 0. \tag{3.5}$$

Proof. This is immediate from the definiton of Bessel functions (definition 1.10).

$$\begin{aligned}
J_n(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(n+m)!} \\
&= \frac{(-1)^0 (x/2)^n}{0!n!} + \frac{(-1)^1 (x/2)^{n+2}}{1!(n+1)!} + \frac{(-1)^2 (x/2)^{n+4}}{2!(n+2)!} + \cdots \\
&= \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{(x/2)^2}{(n+1)!} + \frac{(x/2)^4}{2(n+2)!} + \cdots\right] \\
\therefore J_0(x) &= 1 - \frac{(x/2)^2}{2} + \frac{(x/2)^4}{12} + \cdots
\end{aligned}$$

Hence the function is well defined at $x = 0$, and 3.5 holds for all $n \in \mathbb{Z}$. $\square$

Chapter 4

Scattering around multiple cylinders

4.1 Introduction

We will now consider scattering around two cylinders. To do this we first need to consider the relative positions of these two cylinders in relation to each other. To do this we consider their centers as their respective origins, O_1 and O_2, and the relative position vectors r_1 and r_2 from these origins to a point P. This is illustrated in Fig 4.1, which is taken from the Cambridge Encyclopedia of Mathematics (Martin 2006).

Theorem 4.1. Graf's addition theorem.
For $m \in \mathbb{Z}$:

$$\begin{aligned} J_m(kr_2)e^{im\theta_2} &= \sum_{n=-\infty}^{\infty} J_n(kb)e^{in\beta} J_{m-n}(kr_1)e^{i(m-n)\theta_1} \\ &= \sum_{n=-\infty}^{\infty} J_{m-n}(kb)e^{i(m-n)\beta} J_n(kr_1)e^{in\theta_1} \end{aligned}$$

Graf's original addition theorem gave an expansion for $J_\nu(kr)e^{i\nu\theta}$ where $\nu \in \mathbb{C}$, but we are only interested in $\nu \in \mathbb{Z}$, and this makes the proof simpler. The proof for this modified was allegedly first given by G.T. Walker in a now defunct journal called The Messenger of Mathematics, vol.25, of which it seems there are no digitalised copies! The proof is reproduced in the Cambridge Encyclopedia of Mathematics (ibid.).

Proof. TBD, possibly omitted. □

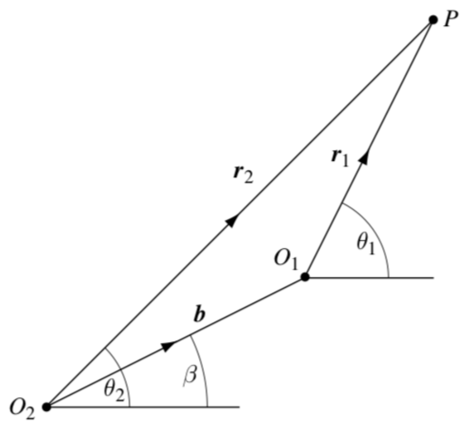

Figure 4.1: Multiple cylinder problem statement

Chapter 5

Plotter tool

5.1 Introduction

The goal of this tool was to plot the scattered field. I took Prof. Guettel's Introduction to Python in 2nd Year and was interested in learning more about using the language.

I started off with a text file on my laptop but quickly realised I would benefit from version controlling it, so I created a repository on Bitbucket. You can access it here. It also helped me version control this document, since it became quite complex quickly.

In Prof. Guettel's introduction to Python we exclusively used functions to build a game of Othello that ran in the command line. This is very different to what I have done here, where I would need the programme to output the file of the plotted field. Additionally, this tool became many orders of magnitude more complex because of the nature of the problem and I thought it appropriate to employ an object oriented approach.

TODO: I need to credit Nathan but I'm not sure how to go about it.

I used four different libraries for this project, namely `numpy`, `matplotlib.pyplot`, `scipy.special` and `mpmath`.

5.2 Approach

My tool is set up in two files: `plotter.py` and `fields.py`. These can be found in the Bitbucket repository above.

The `plotter.py` file holds three classes: `Main`, `Wave` and `Graphics`. The `fields.py` file holds subclasses which are instantiations of the `Wave` class, the most important of which is `CylinderField(Wave)`, which provides the fields calculated in Chapter 2.

Within the `Main` class I defined the `run` class, which runs the tool when it is compiled as a python script from the command line. I also define different functions for different fields I want to plot in this class, so that it is easy to call them. For example, for the cylinder scattering problem I call

```
self.create_field_around_cylinder(self.graph)
```

where I have defined

```
def create_field_around_cylinder(self, graph):
    field = CylinderField()
    graph.heat_map(field)
```

and `self.graph = Graphics` was set within the `__init__` constructor method.

5.3 The Graphics class

The `Graphics` class is equipped with two different types of plots: `heat_map` and `contour`. Both of these are constructed in the same way:

```
def contour(self, wave, xlabel='x', ylabel='y'):
    plt.contour(wave.get_Z(), extent=wave.get_extent())
    self.label_plot(wave, xlabel, ylabel)
    self.draw_plot()
```

where `contour` is replaced by `imshow` for the heat map.

Here the specific wave instantiation, `wave`, is fed through to the `contour` function in `matplotlib.pyplot` which returns the set of contour lines for the heights given by `wave.get_Z`. I then add labels to my plot, and the `draw_plot` function draws it.

The `label_plot` function looks for the title I have set for my wave instantiation using the `wave.get_name` function, along with labels for the axes.

The plotting function brings it all together

```
def draw_plot(self, wave):
    self.draw_disk_overlay(wave)
    plt.colorbar()
    plt.show()
```

where the `draw_disk_overlay` function creates a solid disk in the plot to block out the area where the field should not be calculated inside the cylinder. The size of this disk is pulled from the specific wave instantiation using `get_cylinder_radius` and plotted with `pyplot`.

```
def draw_disk_overlay(self, wave):
    r = wave.get_cylinder_radius()
    plt.gca().add_patch(plt.Circle((0,0),r, fc='#36859F'))
```

5.4 The Wave class

This is fairly big class but its functions can be divided up into three main categories: plot information, constants, and mathematical functions. It makes sense to keep them all within one class because all the specific wave instantiations will make use of these functions.

5.4.1 Plot information

Throughout this project I have attempted to name my functions to best represent the action they perform. For instance `get_wavevector` returns the wavevector, `self.wavevector`, whereas `set_wavevector(k)` sets `k = self.wavevector`. This is especially useful when setting and retrieving these constants within the specific wave instantiations.

Following this convention I have a series of functions that set or retrieve information about the specific plot.

The functions `set/get_name`, set and retrieve the name of the plot; `get_X`, `get_Y` and `get_Z` retrieve the three coordinate arrays; `set/get_ axis_ length` and set and retrieve the range of `x` and `y`; `set/get_ axis _delta` set and retrieve the granularity of the plot.

The axis length functions are additionally also used to specify an 'extent' function, `get_extent`, which is used in the `Graphics` class to make sure the axis labels in the plot are over the correct range (see 5.3).

```
def get_extent(self):
    return [-self.get_axis_length(), self.get_axis_length(),
        -self.get_axis_length(), self.get_axis_length()]
```

Finally, the `Wave` class is initalised by setting the arrays that will constitute our domain: `self.X, self.Y = self.get_xy_series()`, where this function is defined as follows.

```
def get_xy_series(self):
    x = np.linspace(-self.get_axis_length(),
        self.get_axis_length(), self.get_axis_delta())
    y = x
    return np.meshgrid(x, y)
```

TODO: need to fix how x and y are defined.

Now that we have defined the cartesian coordinates x and y, we can use them to define r and θ. We define r as follows, using the `numpy` library:

```
def get_r(self):
    return np.sqrt( self.get_X()*self.get_X()
        + self.get_Y()*self.get_Y() )
```

TODO: explain why $x * x$ instead of x^2.

The angular coordinate is particularly interesting because it needs work in all four quadrants of our plotted field. `Numpy` offers two versions of the $\arctan(\theta)$ function, `numpy.arctan` and `numpy. arctan2`. The first returns the angle $\theta \in [\frac{\pi}{2}, \frac{\pi}{2}]$, that is, (x, y) in the rightmost quadrants. We don't expect our field to be symmetric around $x = 0$, so we need to use the latter function, which returns $\theta \in [\pi, -\pi]$.

```
def get_theta(self):
    return np.arctan2(self.get_X(), self.get_Y())
```

5.4.2 Wave constants

The second set of functions defined within the `Wave` class consist of the physical constants of the problem. These are determined by the wavenumber, which is set and retrieved using `get/set_wavevector` for each wave instantiation. The wavevector is an array length two, such that $\mathbf{k} = (a, b) = (kcos\alpha, ksin\alpha)$.

From this wave vector we can calculate the wavenumber, k,

```
def get_wavenumber(self):
    return np.sqrt(self.wavevector[0]*self.wavevector[0]
        + self.wavevector[1]*self.wavevector[1])
```

the incident angle, α, again using `arctan2`:

```
def get_incident_angle(self):
    return np.arctan2(self.wavevector[0], self.wavevector[1])
```

For the purposes of this project, I set the speed of sound to 343 ms^{-1}. Now we can calculate $\omega = kc$, the frequency of the wave (see Definition 1.8).

```
def get_omega(self):
    return self.get_speed_of_sound()*self.get_wavenumber()
```

Our solution deals with sums to infinity, so we must set a truncation numer. This is done with `get/set_truncation`. Similarly, we need to specify the radius of the circle we are concerned with. This is set and retrieved using `get/set_cylinder_radius`.

5.4.3 Mathematical functions

Finally, we describe the Neumann factor, as defined in 2.2. I added an error message to make sure the function behaves as expected.

```
def get_neumann_factor(self, n):
      if n==0:
          return 1
      elif n > 0:
          return 2
      else:
          print('ERROR: Invalid n for Neumann factor')
```

5.5 Particular `Wave` instantiations

All the classes in this file are subclasses of `Wave`. They are initialised with the superconstructor referring back to the parent class so that they all inherit the functions and variables of `Wave()`.

```
def __init__(self):
    print('[Field_name] started...')

    self.set_parameters()
    self.set_name("Example name")

    super(Field_name, self).__init__()
```

```python

    self.Z = self.get_z_series()
```

Note that the superconstructor needs to go stricly *after* the parameters are set. This is to override any defaults I have set in `Wave`. The `set_paramaters` function sets the specific values for the wavevector, truncation number, radius, and the range and granularity of the grid.

The array `self.Z` is what `Graphics` will plot. Each value in this array will be calculated by a (truncated) infinite sum depending on the coordinates r and θ.

```python
def get_z_series(self):
    return self.get_sum(self.get_r(), self.get_theta())
```

We can now exploit the fact that our solutions for this field are separable to write

$$\text{total field} = \text{constant terms} \cdot \text{angular terms} \cdot \text{radial terms}. \tag{5.1}$$

```python
def get_sum(self, r, theta):
    z = 0       #Initialising
    for n in range(self.truncation):
        z += self.get_constant_term(n) * self.get_angular_term(n,
    theta) * self.get_radial_term(n, r)
    return z.real
```

We can now define what these constant, angular and radial terms are for each specific field.

5.5.1 The scattered field around a cylinder

Constant term

In this case, we have two types of constant terms: Neumann or Dirichlet. One could choose to define two separate functions `get_dirichlet_constants` and `get_neumann_constants`, but I've simply defined my function to take a `type` argument which is set to Neumann by default. I added an error message to make sure the function behaves as expected.

```python
def get_constant_term(self, n, type='neumann'):
    if type == 'neumann':
        return self.get_neumann_factor(n) * np.power(1j,n) * self.
    get_neumann_bc(n)
    elif type == 'dirichlet':
        return self.get_neumann_factor(n) * np.power(1j,n) * self.
    get_dirichlet_bc(n)
    else:
        print('ERROR: Invalid type in CylinderField.
    get_constant_term')
```

Where `get_neumann_bc` and `get_dirichlet_bc` are defined as follows.

```python
def get_neumann_bc(self, n):
    return sp.jvp(n, self.get_wavenumber() * self.
    get_cylinder_radius()) / sp.h1vp(n, self.get_wavenumber() *
    self.get_cylinder_radius())
```

```python
def get_dirichlet_bc(self, n):
    return None #TBD
```

The function `sp.jvp(v, z, m=1)` computes the m^{th} derivative of the Bessel function of the first kind of order v and argument z. It is provided by the `scipy.special` library, imported as `sp` here. Similarly `sp.h1vp(v, z, m=1)` computes the m^{th} derivative of the Hankel function of the first kind of order v and argument z.

Radial dependence

```python
def get_radial_term(self, n, r):
    return ( sp.hankel1(n, self.get_wavenumber() * r) + sp.jv(n,
    self.get_wavenumber() * r))
```

Angular dependence

```python
def get_angular_term(self, n, theta):
    return np.cos(n * (theta - self.get_incident_angle()))
```

5.6 Example plots

Bibliography

Cohen, H. (2007). *Number Theory Volume II: Analytic and Modern Tools*. Vol. 240. Graduate texts in Mathematics. New York: Springer.

Korenev, B. G. (2002). *Bessel functions and their applications*. Vol. 8. Analytical methods and special functions. London: Taylor & Francis.

Martin, P. A. (2006). *Multiple scattering: interaction of time-harmonic waves with N obstacles*. Vol. 107. Encyclopedia of mathematics and its applications. Cambridge: Cambridge University Press.

Pierce, A. D. (2019). *Acoustics: an introduction to its physical principles and applications*. 3rd. Cham, Switzerland: Springer.

Shames, I. H. (2002). *Mechanics of fluids*. 4th. New York: McGraw-Hill.

Shaughnessy, E. J. et al (2005). *Introduction to fluid mechanics*. New York: Oxford University Press.