

Notes on Fluid Mechanics

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Last edited September 3, 2020

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Chapter 1

Introduction

1.1 The Continuum Hypothesis

Proposition 1.1 (The Continuum Hypothesis). *We assume that a fluid can be treated as a continuous substance rather than a group of discrete molecules.*

Calculating the motion of all the molecules in a fluid body is prohibitively expensive and unnecessary for a wide range of applications where models based on the continuum hypothesis describe reality well. One key benefit of the continuum hypothesis is that it allows for the use of differential calculus to find solutions to the models of fluid mechanics since we consider each fluid property to be a continuous function of position and time.

It is important to note that this model represents reality *well enough* but it is not reality itself. We are making assumptions about fluid behaviour that we know break down for sufficiently small quantities, specifically when the length scale of a physical phenomenon is of the same order as molecular dimensions of practical importance. An important quantity of this kind is the average distance travelled by a molecule between successive impact, known as the *mean free path*. This is an important consideration when applying the continuum hypothesis to gases. In fluids, the mean distance between molecules is very small, so our theory has a wider range of applications.

It is always a good idea to check all the assumptions we are making before solving a problem using any given method.

The Lagrangian Description

The Lagrangian Description is a continuum description (so we can use calculus). The key idea here is that we consider a conceptual fluid particle – a shapeless blob of fluid which contains mass. We then assume our fluid is a collection of such discrete fluid particles. We must also assume these fluid particles are small enough to allow for us to treat the fluid as if it was infinitely divisible. Crucially, these are **not** molecules. We are still considering the fluid to be a continuum, except now we are cutting this continuum up into smaller fluid particles.

A classic example of the Lagrangian approach is dropping droplets of dye into a moving body of water. You would be able to follow the path of the droplet of dye as it moves within the fluid: this is a fluid particle.

This description is conceptually similar to the idea of treating bodies as point masses in classical mechanics.

The Eulerian Description

The Eulerian description is a different kind of continuum description. In this case, instead of looking at a particular fluid particle and watching how its position changes within the fluid, we fix the position and look at how the velocity field changes with time. Fluid motion in the Eulerian description has fluid properties applied to specific places in space, instead of particular fluid particles.

The Eulerian approach constitutes a field-like description of fluid motion much like we see fields electromagnetism or classical understanding of Gravity. This is particularly useful because fields are well understood mathematically. We can easily apply vector calculus to problems formulated using the Eulerian Description to find solutions to our problems.

1.2 Defining a fluid: shear

Consider a shear testing device consisting of two plates with a small piece of material between them. We apply a force F continuously on the upper plate. The shear stress τ is defined as the amount of force per unit area,

$$\tau = \frac{F}{A} \quad (1.1)$$

where A is the area of the plate in contact with the fluid.

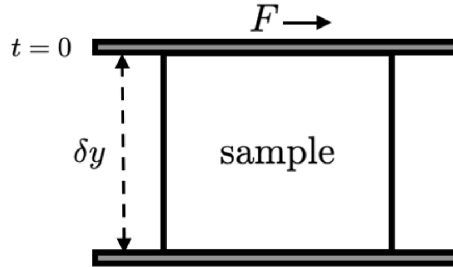


Figure 1.1: A shear testing device (depicted in 2D).

Shear in solids

Consider the sample in our shear testing device is a solid, Fig 1.2 shows the deformation after time t' . For any given shear τ , the shear deformation is finite in solids. We can define a new quantity, shear strain λ , to measure this deformation:

$$\lambda = \frac{\Delta x}{\delta y} \quad (1.2)$$

We can derive the relationship between shear stress and strain experimentally. In fact, it turns out they are related linearly. We call the constant of proportionality the *shear modulus*, G , and it is dependent on the material.

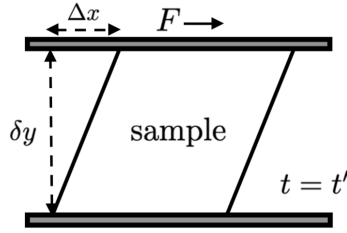


Figure 1.2: Shear deformation at time $t > 0$ for a solid.

$$\tau = G\lambda \quad (1.3)$$

Shear in fluids

In comparison to the behaviour in solids, when a continuous shear stress is applied to a fluid it continues to deform indefinitely, assuming there are no counteracting forces. We can define what a fluid is according to this behaviour.

Definition 1.2. A *fluid* is any material that is unable to prevent the deformation caused by shear stress.

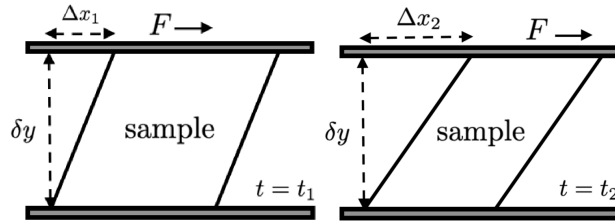


Figure 1.3: Shear deformation for $t > 0$ in a fluid.

Experimentally, we find that

$$\tau \propto \frac{d\lambda}{dt}. \quad (1.4)$$

In particular, for Newtonian fluids we find that this relationship is linear. We call the constant of proportionality the *absolute* or *dynamic viscosity*, μ .

$$\tau = \mu \frac{d\lambda}{dt} \quad (1.5)$$

This is known as Newton's Law of Viscosity.

Proposition 1.3. For a fluid with velocity $\mathbf{u} = (u, v, w)$, where u is the velocity component in the direction of the shear stress,

$$\tau = \mu \frac{du}{dy}.$$

Proof. First recall the definition of shear strain λ (1.2), so we have

$$\tau = \mu \frac{d}{dt} \frac{\Delta x}{\delta y}. \quad (1.6)$$

We know that δy is a constant, so

$$\tau = \mu \frac{1}{\delta y} \Delta \left(\frac{dx}{dt} \right) = \mu \frac{\Delta u}{\delta y} = \mu \frac{du}{dy}. \quad (1.7)$$

A more formal proof can be given by considering the limit of this expression as $\delta y \rightarrow 0$. \square

Appendix A

Dimensions and units

A.1 An important aside on dimensions and units

But what is the difference between a dimension and a unit?

Definition A.1. *A dimension is a physical variable used to specific some charactersitics of a system. For example, length, mass and temperature are all dimensions.*

Definition A.2. *A unit is a particular amount of a physical quantity or dimension. For example, leagues is a (pretty bad) unit of length, micrograms is a unit of mass and degrees Centigrade is a unit of temperature.*

The dimensions of length $[L]$, time $[t]$ and temperature $[T]$ are considered to be base dimensions and are included in all dimension sets. A dimension set of the form $[MLtT]$ or $[FLtT]$ describe most problems in fluid mechanics. Other fields might require more dimensions, for example charge.

What unit set we choose will dictate how we understand the laws of physics. Consider

$$\mathbf{F} \propto M\mathbf{a} \quad (\text{A.1})$$

in the usual notation. The proportionality symbol is required because there is no a priori reason to assume that the units of measure will automatically be consistent with $\mathbf{F} = M\mathbf{a}$. More generally we write

$$\mathbf{F} = \frac{M\mathbf{a}}{g_c}, \quad (\text{A.2})$$

where g_c will be determined by our choice of units. However the underlying dimensionsonal relationship *is* true a priori:

$$[F] = [M][Lt^{-2}]. \quad (\text{A.3})$$

Système International d'Unités (SI)

This is an $[MLtT]$ system, meaning units for mass, length, time and temperature are all indepedently defined:

- mass: kilogram (kg),

- length: metre (m),
- time: second (s),
- temperature: Kelvin (K).

The SI system also has a composite unit for force, the Newton:

$$1 \text{ N} = 1 \text{ kg m s}^{-2}.$$

The SI system has been defined so that g_c and many other proportionality constants (including Fourier's Law and Gauss' Law) are dimensionless constants of unit magnitude. You are welcome.

Centimetre-gram-second system (cgs)

This is a ripoff of the SI system but for smaller stuff which wouldn't make sense to measure in such large units as the metre and the kilogram:

- mass: gram (g),
- length: centimetre (cm),
- time: second (s),
- temperature: Kelvin (K).

British Gravitational Unit System (BG)

This is a $[FLtT]$ system:

- force: pound force (lb_f),
- length: foot (ft),
- time: second (s),
- temperature: degree Rankine (°R).

The BG system also has a composite unit, but this time for mass, which is defined in such a way that allows for $g_c = 1$:

$$1 \text{ slug} = 1 \text{ lb}_f \text{ s}^2 \text{ ft}^{-1},$$

and the weight of 1 slug is 32.2 lb_f, which is in no way a silly thing to name a unit.

English Engineering Unit System (EE)

This is an $[FMLtT]$ system and where good, sane, ways of measuring things to die:

- force: pound force (lb_f),
- force: pound mass (lb_m),
- length: foot (ft),

- time: second (s),
- temperature: degree Rankine ($^{\circ}\text{R}$).

The force unit is of course arbitrarily defined so that 1 lb_m weights 1 lb_f when acted upon by gravity at sea level. In this case we have $g_c = 32.2$. Fun fact: none of this makes any sense.