

# Lecture 5-8: Fourier Analysis and Spectral Representation of Signals

ENGR 76 lecture notes — Updated: April 19, 2024  
Ayfer Ozgur, Stanford University

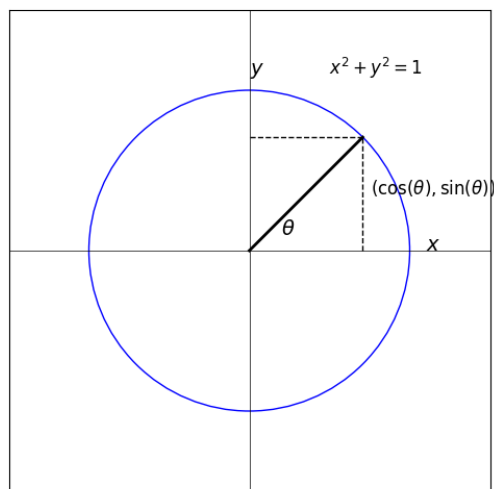
## 1 Sine and Cosine

This week, we investigate what we will (earnestly) refer to as a Major Secret of the Universe:

Every signal has a spectrum, and the spectrum determines the signal.

Learning this secret is about learning Fourier analysis. The subject is both vast and deep, and its origins going back to the French mathematician and physicist Jean-Baptiste Joseph Fourier, 1822, are far removed from where we are headed. In this course, we will see three different applications of Fourier analysis: 1) compression of images; 2) analog to digital conversion; 3) design of digital communication systems.

We start by recalling that  $\sin(\theta)$  and  $\cos(\theta)$  can be defined as the  $x$  and  $y$  coordinates of a point on the unit circle that lies at angle  $\theta$  radians from the  $x$ -axis. See the picture below.



This definition can be regarded as an extension of the more elementary definitions of sine and cosine on a right triangle that you may have first encountered in your studies. It allows us to define sine and cosine for angles beyond  $[0, \pi/2]$ , including negative angles and immediately observe the following relationships:

$$\cos(\theta + 2\pi k) = \cos(\theta) \quad \sin(\theta + 2\pi k) = \sin(\theta) \quad \text{for } k = 0, \pm 1, \pm 2, \dots$$

From this interpretation of  $\sin(\theta)$  and  $\cos(\theta)$  as giving a point on the unit circle, we move toward a dynamic picture and think of this point as moving on the circle in the counterclockwise direction. Assume the point completes  $f$  cycles around the circle in 1 second, so that the angle  $\theta$  now becomes a linear function of time:  $\theta = 2\pi ft$ . The  $x$  and  $y$  coordinates of the point are now functions of time  $t$  and are given by

$$\cos(2\pi ft) \quad \sin(2\pi ft) \quad \text{for } t \in \mathbb{R}.$$

More generally, we will allow ourselves the flexibility to start from an arbitrary initial point, specified by a phase  $\phi$ , and modify the radius of the circle to  $A$  to obtain the functions

$$A \cos(2\pi f t + \phi) \quad A \sin(2\pi f t + \phi) \quad \text{for } t \in \mathbb{R}. \quad (1)$$

The number  $A$  is called the amplitude and  $f$  is called the frequency. See the lecture slides for the graphs of these functions for different  $f$ . The frequency has units cycles per second and dimension 1/sec. This unit is called Hertz, abbreviated Hz, after Heinrich Hertz (1857-1894), who experimentally demonstrated the existence of electromagnetic waves verifying the theoretical work of James Clerk Maxwell.

Recall that a function  $f$  is called periodic if there exists a constant  $T$  such that

$$f(t + T) = f(t), \quad \forall t \in \mathbb{R}.$$

The smallest  $T > 0$  for which the function satisfies this property is called the period of the function. Note that the functions in (1) are periodic with period

$$T = \frac{1}{f}.$$

## 2 It all adds up

In the lecture slides, we see that by adding sinusoids of the form

$$\sum_{j=1}^N A_j \sin(2\pi \frac{j}{T} t + \phi_j),$$

we can approximate other interesting periodic functions with period  $T$ , such as the sawtooth or the square wave. Note that in order to obtain a periodic function of period  $T$ , we include sinusoids that complete an integer number of cycles within the interval of length  $T$ . Specifically, we start with the smallest (fundamental) frequency  $f_1 = \frac{1}{T}$ , which corresponds to one cycle in  $T$  seconds, and add the higher frequency harmonics  $f_j = \frac{j}{T}$ , where the  $j$ 'th harmonic completes  $j$  cycles within the length  $T$  interval.

Joseph Fourier's major secret of the universe is that any (well-behaved) periodic signal  $y(t)$  with period  $T$  can be written in this form, i.e.,

$$y(t) = b_0 + \sum_{j=1}^{\infty} A_j \sin(2\pi \frac{j}{T} t + \phi_j), \quad (2)$$

where we add a constant term  $b_0$ , which can be regarded as the zero frequency component. This representation can be rewritten in the following alternative form by using the trigonometric identity  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ :

$$y(t) = b_0 + \sum_{j=1}^{\infty} A_j \cos \phi_j \sin(2\pi \frac{j}{T} t) + A_j \sin \phi_j \cos(2\pi \frac{j}{T} t), \quad (3)$$

$$= b_0 + \sum_{j=1}^{\infty} a_j \sin(2\pi \frac{j}{T} t) + b_j \cos(2\pi \frac{j}{T} t), \quad (4)$$

where we define  $a_j = A_j \cos \phi_j$  and  $b_j = A_j \sin \phi_j$ . Note that the two representations in (2) and (4) are equivalent; in both representations we have two parameters associated with each frequency component (you can think of them as knobs we can adjust to control the contribution of each frequency component). We can easily go back and forth between (2) and (4) using  $a_j = A_j \cos \phi_j$  and  $b_j = A_j \sin \phi_j$  to go to (4) from (2), and  $A_j^2 = a_j^2 + b_j^2$  and  $\tan \phi_j = b_j/a_j$  to go to (2) from (4).

How can we compute the coefficients  $a_k$  and  $b_k$  of the  $k$ 'th harmonic in (4) for a given signal  $y(t)$  ? These coefficients can be computed by the following formulas:

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt \quad k = 1, 2, 3, \dots \quad (5)$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(2\pi \frac{k}{T} t) dt. \quad k = 1, 2, \dots \quad (6)$$

$$b_0 = \frac{1}{T} \int_{-T/2}^{T/2} y(t) dt \quad (7)$$

Assuming  $y(t)$  can be written in the form (4), it is not too difficult to verify that the above formulas extract the desired coefficients. For example, plug in the expression for  $y(t)$  in (4) in the formula for  $a_k$  for some  $k = 1, 2, 3, \dots$ :

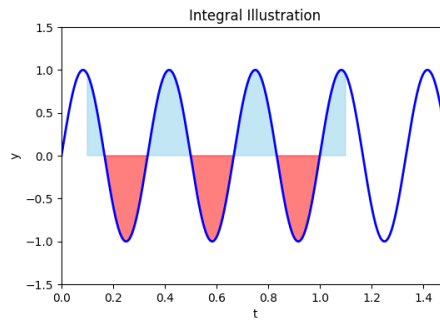
$$\frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt = \frac{2}{T} \int_{-T/2}^{T/2} \left( b_0 + \sum_{j=1}^{\infty} a_j \sin(2\pi \frac{j}{T} t) + b_j \cos(2\pi \frac{j}{T} t) \right) \sin(2\pi \frac{k}{T} t) dt \quad (8)$$

$$= \frac{2}{T} \int_{-T/2}^{T/2} b_0 \sin(2\pi \frac{k}{T} t) dt \quad (9)$$

$$+ \sum_{j=1}^{\infty} \frac{a_j}{T} \int_{-T/2}^{T/2} 2 \sin(2\pi \frac{j}{T} t) \sin(2\pi \frac{k}{T} t) dt \quad (10)$$

$$+ \sum_{j=1}^{\infty} \frac{b_j}{T} \int_{-T/2}^{T/2} 2 \cos(2\pi \frac{j}{T} t) \sin(2\pi \frac{k}{T} t) dt. \quad (11)$$

Now observe that if you take the sine or the cosine function and integrate it over one period, the integral will be equal to zero. This is because of the symmetry between the positive and negative parts of the function: the contribution to the integral from the parts where the function takes positive values will cancel exactly with the contribution from the parts where the function takes negative values. This is also true when the function is integrated over any interval of length that is an integer multiple of the period. See the picture below:



This implies that the first term (9) above is equal to zero:

$$\frac{2}{T} \int_{-T/2}^{T/2} b_0 \sin(2\pi \frac{k}{T} t) dt = 0.$$

The terms in the summation in (10) can be rewritten as

$$\frac{a_j}{T} \int_{-T/2}^{T/2} 2 \sin(2\pi \frac{j}{T} t) \sin(2\pi \frac{k}{T} t) dt = \frac{a_j}{T} \int_{-T/2}^{T/2} \cos(2\pi \frac{j-k}{T} t) dt - \frac{a_j}{T} \int_{-T/2}^{T/2} \cos(2\pi \frac{j+k}{T} t) dt,$$

where we used the trigonometric identity  $2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$ . Note that by the same argument, both of the integrals above will be equal to zero when  $j \neq k$ , since we are integrating a cosine function over an interval of length  $T$  and the function completes an integer number of cycles ( $j - k$  or  $j + k$  respectively) in this interval. The only non-zero term corresponds to  $j = k$ , in which case we have

$$\begin{aligned} \frac{a_k}{T} \int_{-T/2}^{T/2} 2 \sin(2\pi \frac{k}{T} t) \sin(2\pi \frac{k}{T} t) dt &= \frac{a_k}{T} \int_{-T/2}^{T/2} \cos(2\pi \frac{0}{T} t) dt - \frac{a_j}{T} \int_{-T/2}^{T/2} \cos(2\pi \frac{2k}{T} t) dt \\ &= \frac{a_k}{T} \int_{-T/2}^{T/2} 1 dt \\ &= a_k, \end{aligned}$$

where we used the fact that  $\cos(0) = 1$ . Similarly one can argue that all the terms in (11)

$$\begin{aligned} \frac{b_j}{T} \int_{-T/2}^{T/2} 2 \cos(2\pi \frac{j}{T} t) \sin(2\pi \frac{k}{T} t) dt \\ = \frac{b_j}{T} \int_{-T/2}^{T/2} \sin(2\pi \frac{j+k}{T} t) dt + \frac{b_j}{T} \int_{-T/2}^{T/2} \sin(2\pi \frac{j-k}{T} t) dt \\ = 0. \end{aligned}$$

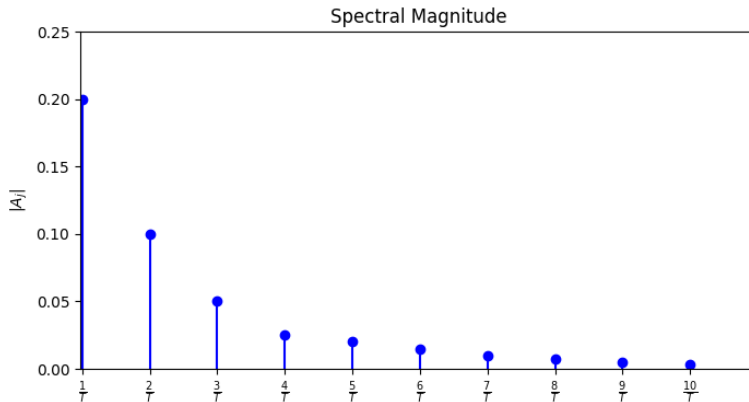
This shows that the expression in (8) indeed extracts the coefficient  $a_k$  in the representation of  $y(t)$  as desired.

We also note that  $\cos(2\pi \frac{j}{T} t) = 1$  for  $j = 0$ , and therefore

$$b_0 = \frac{1}{T} \int_{-T/2}^{T/2} y(t) dt,$$

i.e., the constant term  $b_0$  corresponds to the average value of the signal.

We can visualize the frequency series coefficients of a signal in a plot as shown below which we will refer to as the spectrum of the signal. Note that in both representations in (2) and (4), each harmonic is represented using two values (e.g., amplitude and phase), and thus each point in the plot below actually represents two values. One way to think of it is in terms of complex value, where the plot just shows the amplitudes ( $A_j$ 's).



### 3 Connection to Linear Algebra

Assume the function  $y(t)$  is zero-mean, i.e. the constant term  $b_0$  in the representation in (4) is equal to zero. Then the Fourier series representation reduces to:

$$y(t) = \sum_{k=0}^{\infty} a_k \cos(2\pi \frac{k}{T}t) + b_k \sin(2\pi \frac{k}{T}t), \quad (12)$$

with the coefficients in this representation given by the formula

$$\begin{aligned} a_k &= \frac{2}{T} \int_0^T y(t) \sin(2\pi \frac{k}{T}t) dt & k = 1, 2, 3, \dots \\ b_k &= \frac{2}{T} \int_0^T y(t) \cos(2\pi \frac{k}{T}t) dt. & k = 1, 2, \dots \end{aligned}$$

If you are familiar with the notions of an inner product space and orthonormal basis from linear algebra, one way to interpret (12) is by thinking of the set of functions  $\{e_1(t) = \sin(2\pi \frac{1}{T}t), e_2(t) = \cos(2\pi \frac{1}{T}t), e_3(t) = \cos(2\pi \frac{2}{T}t), e_4(t) = \sin(2\pi \frac{2}{T}t), \dots\}$  in (12) as forming an orthonormal basis for the inner product space of (well-behaved) periodic functions with period  $T$ , with the inner product between any two functions in this space defined as:

$$\langle v(t), w(t) \rangle = \frac{2}{T} \int_0^T v(t)w(t)dt. \quad (13)$$

With this interpretation, (12) simply states that any function  $y(t)$  in this space can be expressed as a linear combination of the basis functions  $\{e_1(t), e_2(t), e_3(t), \dots\}$ . In order to find the coefficients in this representation we need to compute the inner product of the function  $y(t)$  with the basis functions  $\langle y(t), e_k(t) \rangle$  for  $k = 1, 2, \dots$  according to (13), the definition of inner product in this space. This is analogous to how we can represent a vector  $\mathbf{v} \in \mathbb{R}^n$  in terms of an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  in the form:

$$\mathbf{v} = \sum_{k=1}^n \langle \mathbf{v}, \mathbf{e}_k \rangle \mathbf{e}_k,$$

i.e.  $\mathbf{v}$  is decomposed as a sum of vectors in the directions of the orthonormal basis vectors, and the components are given by the (usual) inner product of  $\mathbf{v}$  with the basis vectors.

### 4 Discrete Cosine Series

In this section, we will investigate what happens when we apply the Fourier series decomposition in (4) to an even function. Recall that a function  $f(t)$  is called even if  $f(-t) = f(t)$  for all  $t \in \mathbb{R}$ . For example the cosine function is even since  $\cos(-t) = \cos(t)$ . In contrast, a function  $h(t)$  is called odd if  $h(-t) = -h(t)$  for all  $t \in \mathbb{R}$ . The sine function is odd since  $\sin(-t) = -\sin(t)$ . Note that the product of an even function  $f(t)$  and an odd function  $h(t)$  is odd since

$$f(-t)h(-t) = f(t)(-h(t)) = -f(t)h(t).$$

What happens when we compute the Fourier series decomposition in (4) for an even function? Assume  $y(t)$  is even. Using the formula in (5), coefficients  $a_k$  are given by

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{k}{T}t) dt.$$

Note that the product function  $y(t) \sin(2\pi \frac{k}{T}t)$  is odd because it is the product of an even function  $y(t)$  and an odd function  $\sin(2\pi \frac{k}{T}t)$ . Note that the integral of an odd function over an interval of the form

$[-T/2, T/2]$  is always zero because the integral over the  $[-T/2, 0]$  cancels the integral over the duration  $[0, T/2]$ . Formally,

$$\begin{aligned}
 a_k &= \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt \\
 &= \frac{2}{T} \int_{-T/2}^0 y(t) \sin(2\pi \frac{k}{T} t) dt + \frac{2}{T} \int_0^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt \\
 &= -\frac{2}{T} \int_0^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt + \frac{2}{T} \int_0^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt \\
 &= 0.
 \end{aligned}$$

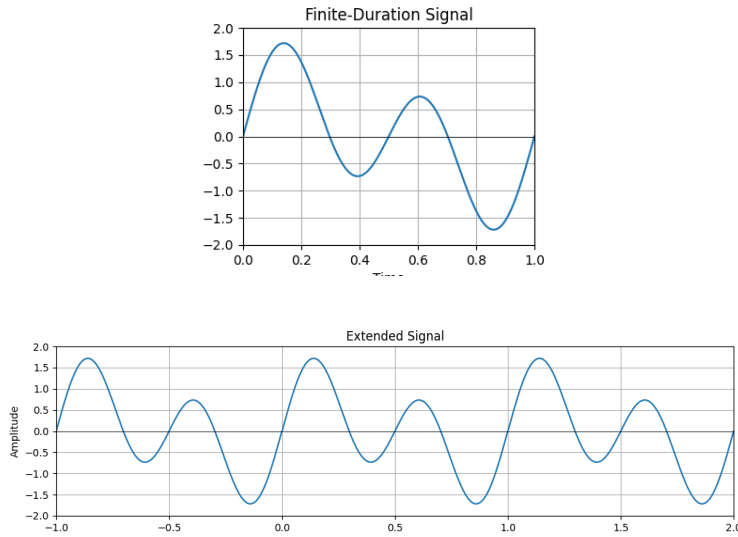
This shows that the Fourier series representation of an even function  $y(t)$  has the following simpler form which contains only cosine terms:

$$y(t) = b_0 + \sum_{j=1}^{\infty} b_j \cos(2\pi \frac{j}{T} t).$$

This is sometimes referred to as the discrete cosine series representation of an even function. This notion will be useful in the next chapter, when we look into developing a Fourier series representation for finite duration signals.

## 5 Finite-duration Signals

How about when the signal is not periodic? Many of the signals we will encounter in practice will be finite-duration signals which are only defined in a finite interval of duration  $T$ . For such finite duration signals, we can construct their so-called periodic extension, which is a periodic signal with period  $T$  equal to the finite-duration signal, and compute the Fourier series representation of the periodic extension. See the figure below.

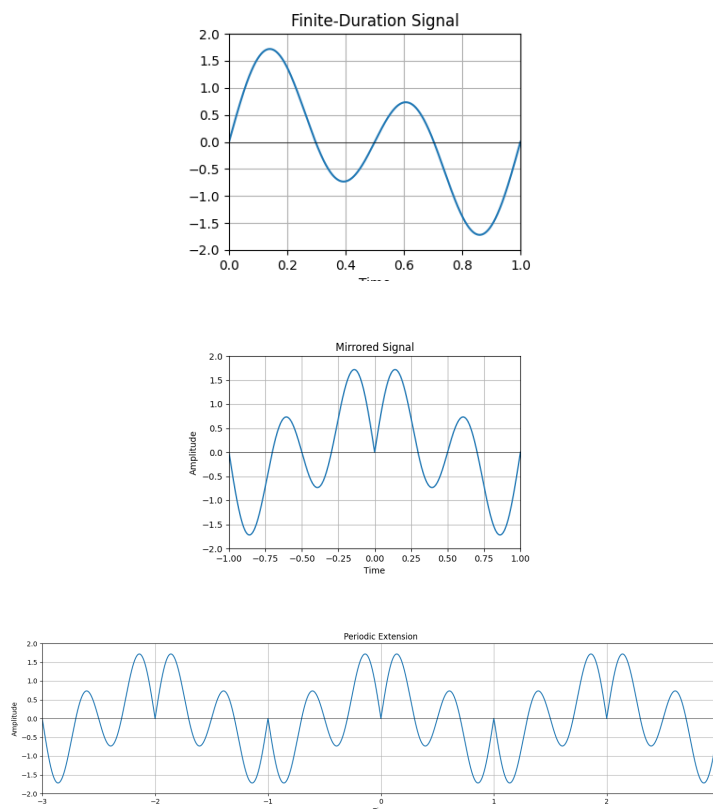


The periodic extension in the lower picture has a Fourier series representation of the form

$$y_p(t) = b_0 + \sum_{j=1}^{\infty} a_j \sin(2\pi \frac{j}{T} t) + b_j \cos(2\pi \frac{j}{T} t).$$

We can take that to be the Fourier series representation of the finite-duration signal.

There are multiple ways in which we can construct a periodic signal from a finite-duration signal. For example, we could also do the following. We can first take the mirror image of the finite-duration signal with respect to the y-axis to create an even signal of duration  $2T$ . We can then construct the periodic extension of this even signal of duration  $2T$ . See the figure below.



The periodic signal in the third image is called the even periodic extension of the finite duration signal in the first image. Note that because the even periodic extension of the signal is even and periodic with period  $2T$ , it has a Fourier series representation of the form

$$y(t) = b_0 + \sum_{j=1}^{\infty} b_j \cos(2\pi \frac{j}{2T} t).$$

This is desirable since the Fourier representation is now simpler and only contains cosine terms. Moreover, the even periodic extension of the signal always yields a continuous signal while the periodic extension can be discontinuous when the signal does not start and end with the same value.

## 6 Infinite-duration non-periodic signals

What happens when the duration of the signal  $T \rightarrow \infty$ ? This gives a infinite duration signal that is not necessarily periodic. Note that the gap between consecutive frequencies in the spectrum of a periodic signal is  $\frac{1}{T}$ . If we consider an infinite duration signal with  $T \rightarrow \infty$ , then we can expect the frequency domain representation to be no longer discrete, and instead include a continuum of frequencies.

With a certain amount of bravado and disregard for mathematical politeness we can see what happens in the limit  $T \rightarrow \infty$  with the following argument. Write  $\frac{1}{T} = \Delta$  and the points  $\frac{j}{T} = j\Delta$ . Then the Fourier series representation of a periodic signal becomes

$$\begin{aligned} y(t) &= b_0 + \sum_{j=1}^{\infty} \left( \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{j}{T} t) dt \right) \sin(2\pi \frac{j}{T} t) + \left( \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(2\pi \frac{j}{T} t) dt \right) \cos(2\pi \frac{j}{T} t), \\ &= b_0 + \sum_{j=1}^{\infty} \Delta \left( \int_{-T/2}^{T/2} 2y(t) \sin(2\pi(j\Delta)t) dt \right) \sin(2\pi(j\Delta)t) + \Delta \left( \int_{-T/2}^{T/2} 2y(t) \cos(2\pi(j\Delta)t) dt \right) \cos(2\pi(j\Delta)t) \end{aligned}$$

Looks like a Riemann sum of the form  $\sum_{j=1}^{\infty} \Delta g(j\Delta) \rightarrow \int_0^{\infty} g(f) df$  when  $\Delta \rightarrow 0$ . Looks like when  $T \rightarrow \infty$ , the expression becomes

$$y(t) = b_0 + \int_0^{\infty} \left( \frac{2}{T} \left( \int_{-\infty}^{\infty} y(t) \sin(2\pi f t) dt \right) \sin(2\pi f t) + \left( \frac{2}{T} \int_{-\infty}^{\infty} y(t) \cos(2\pi f t) dt \right) \cos(2\pi f t) \right) df.$$

We will turn this into a definition. For any (well-behaved) non-periodic signal  $y(t)$ , we define its Fourier transform to be given by the following two functions

$$\begin{aligned} A(f) &= 2 \int_{-\infty}^{\infty} y(t) \sin(2\pi f t) dt, \\ B(f) &= 2 \int_{-\infty}^{\infty} y(t) \cos(2\pi f t) dt. \end{aligned}$$

The integration is with respect to  $t$  but the integrand involves both the variables  $t$  and  $f$ . After integration what remains is a function of  $f$ . The formula looks similar to that for the Fourier coefficients but unlike the discrete Fourier coefficients for a periodic signal, the Fourier transform of a general (non-periodic) signal is now a (transformed) function of a continuous real variable  $f$ . We still think of  $f$  as a frequency variable, and the Fourier transform functions  $A(f)$  and  $B(f)$  as the frequency domain representation of the signal  $y(t)$ , but instead of discrete frequencies as in the periodic case we have a continuum of frequencies in the non-periodic case. The spectrum of  $y(t)$  is the set of real numbers  $f$  where  $A^2(f) + B^2(f) \neq 0$  (or equivalently either  $A(f)$  or  $B(f)$  is non-zero). The bandwidth of the signal is defined as

$$B = f_{max} - f_{min},$$

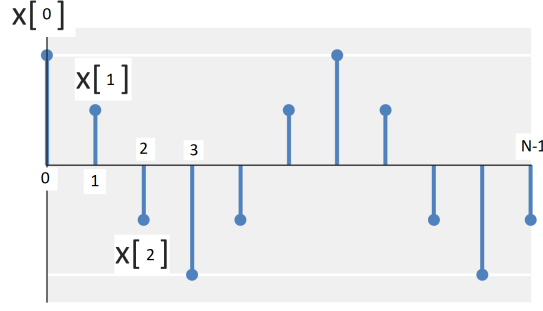
where  $f_{max}$  is largest frequency  $f$  for which  $A^2(f) + B^2(f) \neq 0$  and  $f_{min}$  is smallest frequency  $f$  for which  $A^2(f) + B^2(f) \neq 0$ . The signal  $y(t)$  can be written in terms of its spectrum by taking an integral over all frequencies  $f$ :

$$y(t) = b_0 + \int_0^{\infty} A(f) \sin(2\pi f t) + B(f) \cos(2\pi f t) df.$$

## 7 Discrete Fourier Transform (DFT)

We next consider a signal in discrete time of duration  $N$ , with values  $X[n]$  for  $n = 0, \dots, N-1$ , as shown below.





We can think of this as a finite length signal or equivalently in terms of its periodic extension (with  $X[n+N] = X[n]$  for all  $n$ ). Similar to the continuous case, we can consider the frequency representation:

$$X[n] = b_0 + \sum_k a_k \sin[2\pi \frac{k}{N}n] + b_k \cos[2\pi \frac{k}{N}n]$$

where the  $k$ 'th term in the summation represents the  $k$ 'th harmonic that completes  $k$  cycles within  $0, \dots, N-1$ . Recall that in continuous time,  $k$  could be arbitrarily large. However, in the discrete time case we observe that for  $l = k + N$ ,

$$\begin{aligned} \cos\left[2\pi \frac{l}{N}n\right] &= \cos\left[2\pi \frac{k+N}{N}n\right] \\ &= \cos\left[2\pi \frac{k}{N}n + 2\pi n\right] \\ &= \cos\left[2\pi \frac{k}{N}n\right] \\ &= \cos[2\pi \frac{k}{N}n] \end{aligned}$$

where the second last equality holds because of the periodicity of the cosine function. The same argument holds for the sine function. We note that the harmonics repeat after every  $N$  values of  $k$ , so it's sufficient to focus on  $k$  being in a set of  $N$  consecutive values. By convention, we consider the values from  $k = 1$  to  $N$ , and assume for simplicity that  $N$  is odd. We can in fact reduce the number of coefficients further by observing that the harmonics for  $k = \frac{N+1}{2}, \dots, N-1$  coincide with the harmonics for  $k = 1, \dots, \frac{N-1}{2}$ . To see this let  $k = j + \frac{N-1}{2}$  for  $j = 1, \dots, \frac{N-1}{2}$

$$\begin{aligned} \cos\left[2\pi \frac{k}{N}n\right] &= \cos\left[2\pi \frac{j + \frac{N-1}{2}}{N}n\right] \\ &= \cos\left[2\pi \frac{j + \frac{N-1}{2}}{N}n - 2\pi n\right] \\ &= \cos\left[2\pi \frac{j - \frac{N+1}{2}}{N}n\right] \\ &= \cos[2\pi \frac{\frac{N+1}{2} - j}{N}n] \end{aligned}$$

This shows that the  $j + \frac{N-1}{2}$ 'th harmonic coincides with the  $\frac{N+1}{2} - j$ 'th harmonic for  $j = 1, \dots, \frac{N-1}{2}$ . A similar conclusion holds for the sine function. Moreover, the  $N$ 'th harmonic coincides with the constant term  $b_0$  since  $\sin(2\pi \frac{N}{N}n) = 0$  and  $\cos(2\pi \frac{N}{N}n) = 1$ . Hence all these terms can be combined together to give

the simplified representation:

$$X[n] = b_0 + \sum_{k=1}^{\frac{N-1}{2}} a_k \sin[2\pi \frac{k}{N} n] + b_k \cos[2\pi \frac{k}{N} n].$$

Note that the above expression has  $1 + 2 \times \frac{N-1}{2} = N$  coefficients or degrees of freedom, which exactly matches with the fact that we have  $N$  equations for  $X[n]$  with  $n = 0, \dots, N-1$ . Thus, we can obtain the coefficients by solving the set of  $N$  linear equations in  $N$  variables, with the process of going from  $(X[0], X[1], \dots, X[N-1])$  to  $\{a_k\}, \{b_k\}$  referred to as the **Discrete Fourier Transform** or the **DFT**.

The actual computation of the DFT can be done more efficiently than just computing the inverse of a matrix. This is because the matrix is highly structured with the entries being sines and cosines. This efficient algorithm which is heavily used in practice is called the **Fast Fourier Transform** or **FFT**.