第二章作业.

一. 证:

(1) $(A\cup B)\times(C\cup D)$

$=(A\times C)\cup(A\times D)\cup(B\times C)\cup(B\times D)$

$\supseteq(A\times C)\cup(B\times D)$

(2) $\forall\langle x,y\rangle$,

$\langle x,y\rangle\in(A-B)\times(C-D)$

$\Leftrightarrow x\in A-B\wedge y\in C-D$

$\Leftrightarrow x\in A\wedge x\in\sim B\wedge y\in C\wedge y\in\sim D$

$\Leftrightarrow(x\in A\wedge y\in C)\wedge(x\notin B\wedge y\notin D)$

$\Rightarrow\langle x,y\rangle\in A\times C\wedge\langle x,y\rangle\notin B\times D$

$\Leftrightarrow\langle x,y\rangle\in(A\times C)\cap\sim(B\times D)$

$\Leftrightarrow\langle x,y\rangle\in(A\times C)-(B\times D)$

即 $(A-B)\times(C-D)\subseteq A\times C-B\times D$.

二. 证:

(1) $\forall\langle x,y\rangle$

$\langle x,y\rangle\in(A\times C)-(B\times C)$

$\Leftrightarrow\langle x,y\rangle\in(A\times C)\cap\sim(B\times C)$

$\Leftrightarrow\langle x,y\rangle\in A\times C\wedge\langle x,y\rangle\notin(B\times C)$

$\Leftrightarrow(x\in A\wedge y\in C)\wedge(x\notin B\vee y\notin C)$

$\Leftrightarrow(x\in(A\cap\sim B)\wedge y\in C)\vee(x\in A\wedge\phi)$

$\Leftrightarrow x\in(A\cap\sim B)\wedge y\in C$

$\Leftrightarrow\langle x,y\rangle\in(A-B)\times C$.

$\therefore(A\times C)-(B\times C)=(A-B)\times C$.

(2) $(A\oplus B)\times C$

$=((A-B)\cup(B-A))\times C$

$=((A-B)\times C)\cup((B-A)\times C)$

由(1)知:

上式 $=(A\times C-B\times C)\cup(B\times C-A\times C)$

$=(A\times C)\oplus(B\times C)$

三. 解:(1) $R_1\cup R_2=\{\langle a,b\rangle,\langle b,d\rangle,\langle c,c\rangle,\langle c,d\rangle,\langle a,c\rangle,\langle d,b\rangle,\langle d,d\rangle\}$

$R_1\cap R_2=\{\langle b,d\rangle\}$

$R_1\oplus R_2=(R_1-R_2)\cup(R_2-R_1)=\{\langle a,b\rangle,\langle c,c\rangle,\langle c,d\rangle,\langle a,c\rangle,\langle d,b\rangle,\langle d,d\rangle\}$

(2) $\mathrm{dom}R_1=\{a,b,c\}$

$\mathrm{dom}R_2=\{a,b,d\}$

$\mathrm{dom}(R_1\cup R_2)=\mathrm{dom}R_1\cup\mathrm{dom}R_2=\{a,b,c,d\}$.

(3) $\mathrm{ran}R_1=\{b,c,d\}$

$\mathrm{ran}R_2=\{b,c,d\}$

$\mathrm{ran}R_1\cap\mathrm{ran}R_2=\{b,c,d\}$.

(4) $R_1\upharpoonright A=\{\langle a,b\rangle,\langle c,c\rangle,\langle c,d\rangle\}$

$R_1\upharpoonright\{c\}=\{\langle c,c\rangle,\langle c,d\rangle\}$

$(R_1\cup R_2)\upharpoonright A=\{\langle a,b\rangle,\langle c,c\rangle,\langle c,d\rangle,\langle a,c\rangle\}$

$R_2\upharpoonright A=\{\langle a,c\rangle\}$

(5) $R_1[A]=\{b,c,d\}$

$R_2[A]=\{c\}$

$(R_1\cap R_2)[A]=\phi$

(6) $R_1\circ R_2=\{\langle a,c\rangle,\langle a,d\rangle,\langle d,d\rangle\}$

$R_2\circ R_1=\{\langle a,d\rangle,\langle b,b\rangle,\langle b,d\rangle,\langle c,b\rangle,\langle c,d\rangle\}$

$R_2\circ R_1=\{\langle a,d\rangle,\langle c,d\rangle,\langle c,c\rangle\}$

四. 解:

(1) $R^{-1}=\{\langle\{\phi,\{\phi\}\},\phi\rangle,\langle\phi,\{\phi\}\rangle,\langle\phi,\phi\rangle\}$

(2) $R\circ R=\{\langle\{\phi\},\{\phi,\{\phi\}\}\rangle,\langle\{\phi\},\phi\rangle,\langle\phi,\{\phi,\{\phi\}\}\rangle,\langle\phi,\phi\rangle\}$

(3) $R\upharpoonright\phi=\phi$

$R\upharpoonright\{\phi\}=\{\langle\phi,\{\phi,\{\phi\}\}\rangle,\langle\phi,\phi\rangle\}$

$R\upharpoonright\{\{\phi\}\}=\{\langle\{\phi\},\phi\rangle\}$

$R\upharpoonright\{\phi,\{\phi\}\}=\{\langle\phi,\{\phi,\{\phi\}\}\rangle,\langle\{\phi\},\phi\rangle,\langle\phi,\phi\rangle\}=R$.

(4) $R[\phi]=\phi$

$R[\{\phi\}]=\{\{\phi,\{\phi\}\},\phi\}$

$R[\{\{\phi\}\}]=\{\phi\}$

$R[\{\phi,\{\phi\}\}]=\{\{\phi,\{\phi\}\},\phi\}$

(5) $\mathrm{dom}R=\{\phi,\{\phi\}\}$

$\mathrm{ran}R=\{\{\phi,\{\phi\}\},\phi\}$

$\mathrm{fld}R=\mathrm{dom}R\cup\mathrm{ran}R=\{\phi,\{\phi\},\{\phi,\{\phi\}\}\}$

五. 解:

(1) $R=\{\langle 0,10\rangle,\langle 1,9\rangle,\langle 2,8\rangle,\langle 3,7\rangle,\langle 4,6\rangle,\langle 5,5\rangle,$

$\langle 6,4\rangle,\langle 7,3\rangle,\langle 8,2\rangle,\langle 9,1\rangle,\langle 10,0\rangle\}$

$S=\{\langle 0,4\rangle,\langle 3,3\rangle,\langle 6,2\rangle,\langle 9,1\rangle,\langle 12,0\rangle\}$

(2) $\exists x(x\in A\wedge\langle x,x\rangle\notin R)\wedge\exists x(x\in A\wedge\langle x,x\rangle\in R)$

$\therefore$ R不具有自反性或反自反性

$\forall\langle x,y\rangle(\langle x,y\rangle\in R\rightarrow\langle y,x\rangle\in R)$, R是对称的

$\exists x,y,z(\langle x,y\rangle\in R\wedge\langle y,z\rangle\in R\wedge\langle x,z\rangle\notin R\}$, R不具有传递性

综上, R是对称的.

$\exists x(x\in A\wedge\langle x,x\rangle\notin S)\wedge\exists x(x\in A\wedge\langle x,x\rangle\in S)$

$\therefore$ S不具有自反性或反自反性

$\forall\langle x,y\rangle(\langle x,y\rangle\in S\wedge\langle y,x\rangle\in S\rightarrow x=y)$

$\therefore$ S是反对称的

$\exists x,y,z(\langle x,y\rangle\in S\wedge\langle y,z\rangle\in S\wedge\langle x,z\rangle\notin S)$

$\therefore$ S不具有传递性

综上, S是反对称的,

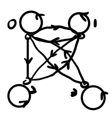
六解: $R=I_A\cup\{\langle 0,1\rangle,\langle 0,2\rangle,\langle 0,3\rangle,\langle 1,0\rangle,\langle 1,2\rangle,\langle 2,0\rangle,\langle 2,1\rangle,\langle 3,0\rangle\}$

$$M(R)=\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&1&0\\1&0&0&1\end{bmatrix}$$

R是自反、对称的

$(I_A\leq M(R),\ M^T(R)=M(R))$

G(R) =



由于 $\langle 3,0\rangle,\langle 0,2\rangle\in R\wedge\langle 3,2\rangle\notin R$.

$\therefore$ R不是传递的.

七. 证: 依题知:

$\forall x(x\in A\rightarrow\langle x,x\rangle\in R)$

$\forall x,y,z(\langle x,y\rangle\in R\wedge\langle y,z\rangle\in R\rightarrow\langle x,z\rangle\in R)$

$\forall\langle x,y\rangle$

$\langle x,y\rangle\in R\circ R$

$\Leftrightarrow\exists z(\langle x,z\rangle\in R\wedge\langle z,y\rangle\in R)$

$\Rightarrow\langle x,y\rangle\in R$. (R传递)

$\therefore R\circ R\subseteq R$.

$\forall\langle x,y\rangle$

$\langle x,y\rangle\in R$

$\Rightarrow\langle x,x\rangle\in R\wedge\langle x,y\rangle\in R$. (R自反)

$\Rightarrow\langle x,y\rangle\in R\circ R$

$\therefore R\subseteq R\circ R$

$\therefore R\circ R=R$.

下证其逆不真

$\forall\langle x,y\rangle\in R\circ R$

$\Leftrightarrow\exists z(\langle x,z\rangle\in R\wedge\langle z,y\rangle\in R)$

$\Rightarrow\exists z(\langle x,z\rangle\in R\wedge\langle z,y\rangle\in R\wedge\langle x,y\rangle\in R)$

$\Rightarrow$ R传递

(无法推出R自反, 故逆命题为假.)

证毕.

八. 证: 用数学归纳法证明:

$m=0$. $(R_1\cup R_2)^m=R_1^m\cup R_2^m=I_A$.

$m=1$. $(R_1\cup R_2)^m=R_1\cup R_2=R_1^m\cup R_2^m$

$\therefore$ 当 $m=0,1$ 时结论成立.

假设当 $m=n(n\geq 1)$ 时有 $(R_1\cup R_2)^n=R_1^n\cup R_2^n$.

则当 $m=n+1$ 时

$(R_1\cup R_2)^{n+1}=(R_1\cup R_2)^n\circ(R_1\cup R_2)$

$=(R_1^n\cup R_2^n)\circ(R_1\cup R_2)$

$=R_1^{n+1}\cup(R_2^n\circ R_1\ \cup(R_1^n\circ R_2)\cup(R_2^{n+1})$

$fldR_1\cap fldR_2$

$=(domR_1\cup ranR_1)\cap(domR_2\cup ranR_2)$

$=(domR_1\cap domR_2)\cup(domR_1\cap ranR_2)\cup(ranR_1\cap domR_2)$

$\cup(ranR_1\cap ranR_2)$

$\because fldR_1\cap fldR_2=\phi$.

$\therefore domR_1\cap domR_2=domR_1\cap ranR_2=$

$ranR_1\cap domR_2=ranR_1\cap ranR_2=\phi$.

$\therefore R_2^n\circ R_1=R_2^{n-1}\circ R_2\circ R_1=\phi$ $(ranR_1\cap domR_2=\phi)$

$R_1^n\circ R_2=R_1^{n-1}\circ R_1\circ R_2=\phi$ $(ranR_2\cap domR_1=\phi)$

$\therefore(R_1\cup R_2)^{n+1}=R_1^{n+1}\cup R_2^{n+1}\cup\phi\cup\phi$

$=R_1^{n+1}\cup R_2^{n+1}$

$\therefore$ 当 $m=n+1$ 时结论成立

综上, $(R_1\cup R_2)^m=R_1^m\cup R_2^m\ (m\geq 0)$

九. 解: (1) $r(R)=I_A\cup R=\{\langle a,a\rangle,\langle b,b\rangle,\langle c,c\rangle,\langle d,d\rangle,\langle a,b\rangle,\langle c,d\rangle\}$

(2) $s(R)=R\cup R^{-1}=\{\langle a,a\rangle,\langle b,b\rangle,\langle a,b\rangle,\langle b,a\rangle,\langle c,d\rangle,\langle d,c\rangle\}$

(3) $t(R)=\{\langle a,a\rangle,\langle a,b\rangle,\langle b,b\rangle,\langle c,d\rangle\}=R$.

十. 证: 只需证R是传递、对称的即可

$\forall\langle x,y\rangle$,

$\langle x,y\rangle\in R$

$\Rightarrow\langle x,y\rangle\in R\wedge\langle x,x\rangle\in R$

$\Rightarrow\langle y,x\rangle\in R$

$\therefore$ R是对称的

$\forall x,y,z\in A$

$\langle x,y\rangle\in R\wedge\langle y,z\rangle\in R$

$\Rightarrow\langle y,x\rangle\in R\wedge\langle y,z\rangle\in R$

$\Rightarrow\langle x,z\rangle\in R$

$\therefore$ R是传递的,

故R是A上的等价关系

十一. 解:

(1) $R_\pi = I_A \cup \{<1,2>,<2,1>,<1,3>,<3,1>,<2,3>,<3,2>\}$

$A/R_\pi = \{\{1,2,3\},\{4\}\} = \pi$.

(2) π的所有加细为 π,

$\pi_1 = \{\{1\},\{2,3\},\{4\}\}$

$\pi_2 = \{\{1,3\},\{2\},\{4\}\}$

$\pi_3 = \{\{1,2\},\{3\},\{4\}\}$

$\pi_4 = \{\{1\},\{2\},\{3\},\{4\}\}$.

对应的A上的等价关系为

$R_1 = I_A \cup \{<2,3>,<3,2>\}$

$R_2 = I_A \cup \{<1,3>,<3,1>\}$

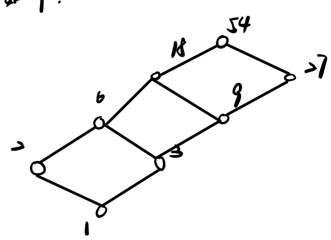
$R_3 = I_A \cup \{<1,2>,<2,1>\}$

$R_4 = I_A$. $R_\pi = I_A \cup \{<1,2>,<2,1>,<1,3>,<3,1>,<2,3>,<3,2>\}$

十二. 解:

依题知: $A=\{1,2,3,6,9,18,27,54\}$.

∴哈斯图如下:



A中最长链长为5. 共4条

2. ∵A中最长链长度为5.

∴至少可以划分为5个互不相交的反链

3. A中有8个元素,则至多可以划分成8个互不相交的反链 $\{1\},\{2\},\{3\},\{6\},\{9\},\{18\},\{27\},\{54\}$

十三. 证: $\forall x\in A, y\in B$.

$\Rightarrow <x,x>\in R_1 \wedge <y,y>\in R_2$

$\Leftrightarrow <<x,y>,<x,y>>\in R$

∴ R是自反的

$\forall <x_1,y_1>,<x_2,y_2>\in A\times B$

$<<x_1,y_1>,<x_2,y_2>>\in R \wedge <<x_2,y_2>,<x_1,y_1>>\in R$

$\Leftrightarrow <x_1,x_2>\in R_1 \wedge <y_1,y_2>\in R_2 \wedge <x_2,x_1>\in R_1 \wedge <y_2,y_1>\in R_2$

$\Rightarrow x_1=x_2 \wedge y_1=y_2$

$\Rightarrow <x_1,y_1> = <x_2,y_2>$

∴ R是反对称的.

$\forall <x_1,y_1>,<x_2,y_2>,<x_3,y_3>\in A\times B$.

$<<x_1,y_1>,<x_2,y_2>>\in R \wedge <<x_2,y_2>,<x_3,y_3>>\in R$.

$\Leftrightarrow <x_1,x_2>\in R_1 \wedge <y_1,y_2>\in R_2 \wedge <x_2,x_3>\in R_1 \wedge <y_2,y_3>\in R_2$

$\Rightarrow <x_1,x_3>\in R_1 \wedge <y_1,y_3>\in R_2$

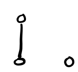
$\Leftrightarrow <<x_1,y_1>,<x_3,y_3>>\in R$.

∴ R有传递性

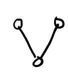
综上, R是 $A\times B$ 上的偏序关系

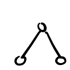
十四. 解: 考虑A上所有可能的哈斯图.

1种


$A_3^2 = 6$种


$A_3^3 = 6$种


$A_3^1 = 3$种


$A_3^1 = 3$种

由于哈斯图与偏序关系是一一对应的

所以A上共有 $1+6+6+3+3=19$种偏序关系