A-003
Regarding Coplanar Circles

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Arrangements of coplanar circles are frequently encountered in mechanical design. They are often observed when determining centers of rotation for: wheels, geartrains and pulley systems.

In the field of astronomy: orbits of planets, solar eclipses and moon transits; can be considered in terms of coplanar circles.

The myriad examples of coplanar circles, both in nature and industry, make their study worthwhile.

1 Problem

Refer to Figure 1.

Given:

- Two coplanar circles A and B,
- *with* center points C_A and C_B respectively,
- *and* having radii r_A and r_B respectively,
- *where* $r_A < r_B$;

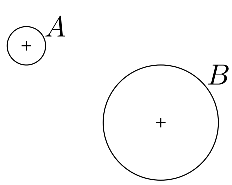

Figure 1: Coplanar circles A and B

Find:

1. The critical location of C_A where A and B are concentric.
2. The critical location of C_A where C_B lies on A.
3. The critical locations of C_A where A and B are tangent to each other.
4. The critical location of C_A where C_A lies on B.
5. All locations of C_A where A otherwise overlaps B.
6. All locations of C_A where A has no overlap with B.

2 Solution

1. Refer to Figure 2 for the discussion that follows.
2. Since A and B are coplanar circles, center points C_A and C_B lie in the same plane with the circles.
3. Two points define the line between them, therefore a line can be drawn between C_A and C_B. This line shall be coplanar with circles A and B, and their centers.
4. A coordinate system can be drawn in the plane containing A and B, and the line between their centers. For convenience, the coordinate system origin can be arbitrarily chosen to coincide with C_B.

5. Furthermore, for convenience, the coordinate system can be oriented such that the x-axis aligns with the line connecting C_B and C_A.

6. Since the origin is always relative to C_B, C_B will always be at $C_B = (x_B, y_B) = (0, 0)$.

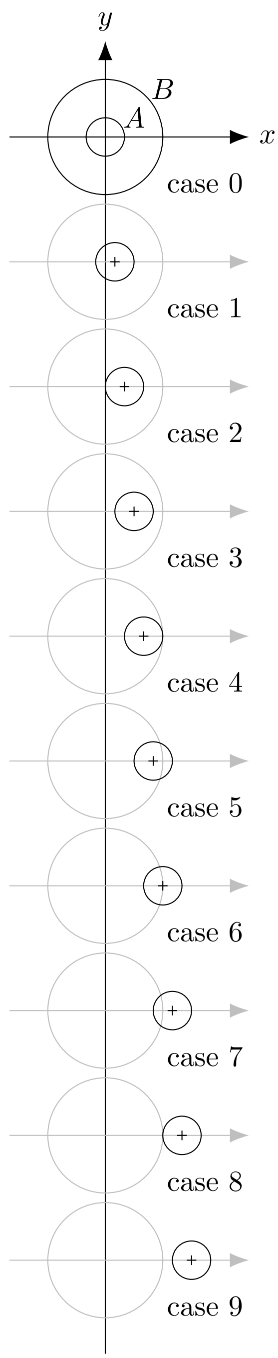

Figure 2: Locations of C_A relative to C_B

7. Since the x-axis is aligned with the line connecting C_B and C_A, C_A will always lie along the x-axis, such that: $C_A = (x_A, y_A) = (x_A, 0)$.

8. Furthermore, the distance between C_A and C_B will always be $\Delta x = x_A - x_B = x_A - 0 = x_A$

9. Therefore the problem of finding critical locations of C_A simplifies into the trivial problem of finding critical values of x_A.

10. The locations of C_A relative to C_B can be generalized into the 10 cases shown in Figure 2.

    - **case 0:** $x_A = 0$
      At the critical point where C_A is at the origin, C_A will be concentric with C_B.
    - **case 1:** $0 < x_A < r_A$
      When C_A lies less than one r_A radius from the origin, A will overlap left and right regions of B.
    - **case 2:** $x_A = r_A$
      When C_A lies exactly than one r_A radius from the origin, circle A will pass through point C_B.
    - **case 3:** $r_A < x_A < r_B - r_A$
      In this case, C_A overlaps with the right side region of B.
    - **case 4:** $x_A = r_B - r_A$
      When C_A lies at one r_A radius less than r_B, A and B will share a tangent point. A will overlap with right region of B.
    - **case 5:** $r_B - r_A < x_A < r_B$
      In this case, A only partially overlaps B. Furthermore it overlaps the right edge of B.
    - **case 6:** $x_A = r_B$
      C_A, the center of A, lies on circle B.
    - **case 7:** $r_B < x_A < r_B + r_A$
      The center of A is beyond the edge of circle B. Circle A partially overlaps the right side of B.

- **case 8:** $x_A = r_B + r_A$
  Circles A an B are tangent to each other with no overlap.
- **case 9:** $x_A > r_B + r_A$
  Circle A is entirely outside B. A and B are non-overlapping.

Q.E.D.

3 Example

Given:

- Two circles A and B whose radii are $r_A = 1.0$ and $r_B = 3.0$ units respectively,
- *and* circle centers C_A and C_B separated by a distance of 2.5 units.

Find:

- Which solution case best matches the given arrangement?

Steps:

1. Let:

$$\begin{aligned} x_A &= 2.5 \\ r_A &= 1.0 \\ r_B &= 3.0 \end{aligned}$$

2. From the solution section, use the inequality for case 5:

$$r_B - r_A < x_A < r_B$$

3. Plug in values for r_A and r_B,

$$3.0 - 1.0 < x_A < 3.0$$

$$2.0 < x_A < 3.0$$

4. Since $x_A = 2.5$ is between 2.0 and 3.0, x_A satisfies the inequality for case 5. Therefore the relationship between circles A and B are classified by case 5.