A-006
Regarding Tangent Ogives

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Ogives are tapered solids of revolution. They are often used for aircraft/missile nose-cones and firearm bullets.

The **tangent ogive** is one of several ogive family members. It is a simple shape, easily turned with an engine lathe.

1 Problem

Given:

- Some semi-circle of radius r,
- *and* a vertical line, of length c, inscribed on the semi-circle.

Find:

1. A procedure to construct tangent ogives.
2. Relationships among tangent ogive elements.

2 Solution

2.1 (Stage 1) Inscribing ogive section on semi-circle

Refer to Figure 1 (Stage 1). Let a radius r sweep out a semi-circle centered about a point P which coincides with the origin.

The semi-circle intersects the x-axis at a point P_b located on the x-axis at distance r away from point P.

Let c be the **inscribed** vertical line which intersects the semi-circle at some point P_c, and intersects the x-axis, at right angles, at a point P_a. Point P_a lies on the x-axis some distance a away from P. Finally, let b be the line-segment between points P_a and P_b.

Select the area bounded by sides b, c, and the arc between P_b and P_c. This area will be used in subsequent stages of the construction.

Distances between points can be listed as:

$$\begin{aligned}\overline{PP_a} &= a\\ \overline{P_aP_b} &= b\\ \overline{P_aP_c} &= c\\ \overline{PP_b} &= r\end{aligned}$$

Furthermore, locations of the points can also be listed:

$$P = (0,0) \tag{1a}$$
$$P_a = (a,0) \tag{1b}$$
$$P_b = (r,0) \tag{1c}$$
$$P_c = (a,c) \tag{1d}$$

Because points P, P_a and P_b are collinear, the distance $\overline{PP_b}$ is just the sum of $\overline{PP_a}$ and $\overline{P_aP_b}$.

$$\boxed{r = a + b} \tag{2}$$

Now since points P, P_a and P_c form a right triangle, the Pythagorean theorem can be used to relate hypotenuse r with sides a and c.

$$\boxed{r = \sqrt{a^2 + c^2}} \tag{3}$$

To find length of side a given lengths of b and c, first equate variable r in equations 2 and 3, then reduce.

$$
\begin{aligned}
a + b &= \sqrt{a^2 + c^2} \\
(a+b)^2 &= a^2 + c^2 \\
a^2 + 2ab + b^2 &= a^2 + c^2 \\
2ab + b^2 &= c^2 \\
2ab &= c^2 - b^2
\end{aligned}
$$

$$\boxed{a = \frac{c^2 - b^2}{2b}} \tag{4}$$

Next, from equation 3, it is convenient to express r^2 in terms of a^2 and c^2

$$
\begin{aligned}
r &= \sqrt{a^2 + c^2} \\
r^2 &= a^2 + c^2
\end{aligned}
$$

Then to substitute $a = r - b$ from equation 2.

$$r^2 = (r - b)^2 + c^2 \tag{5}$$

Using equation 5, b can be found in terms of r and c.

$$
\begin{aligned}
r^2 &= (r - b)^2 + c^2 \\
(r-b)^2 &= r^2 - c^2 \\
\sqrt{(r-b)^2} &= \sqrt{r^2 - c^2} \\
r - b &= \sqrt{r^2 - c^2}
\end{aligned}
$$

$$\boxed{b = r - \sqrt{r^2 - c^2}} \tag{6}$$

Next, by expanding equation 5 and simplifying, c can be found in terms of r and b.

$$
\begin{aligned}
r^2 &= (r - b)^2 + c^2 \\
r^2 &= r^2 - 2rb + b^2 + c^2 \\
c^2 &= 2rb - b^2
\end{aligned}
$$

$$\boxed{c = \sqrt{2rb - b^2}} \tag{7}$$

2.2 (Stage 2) Translate ogive half-section to y-axis

The ogive section is easily translated to the origin and y-axis by subtracting length a from the

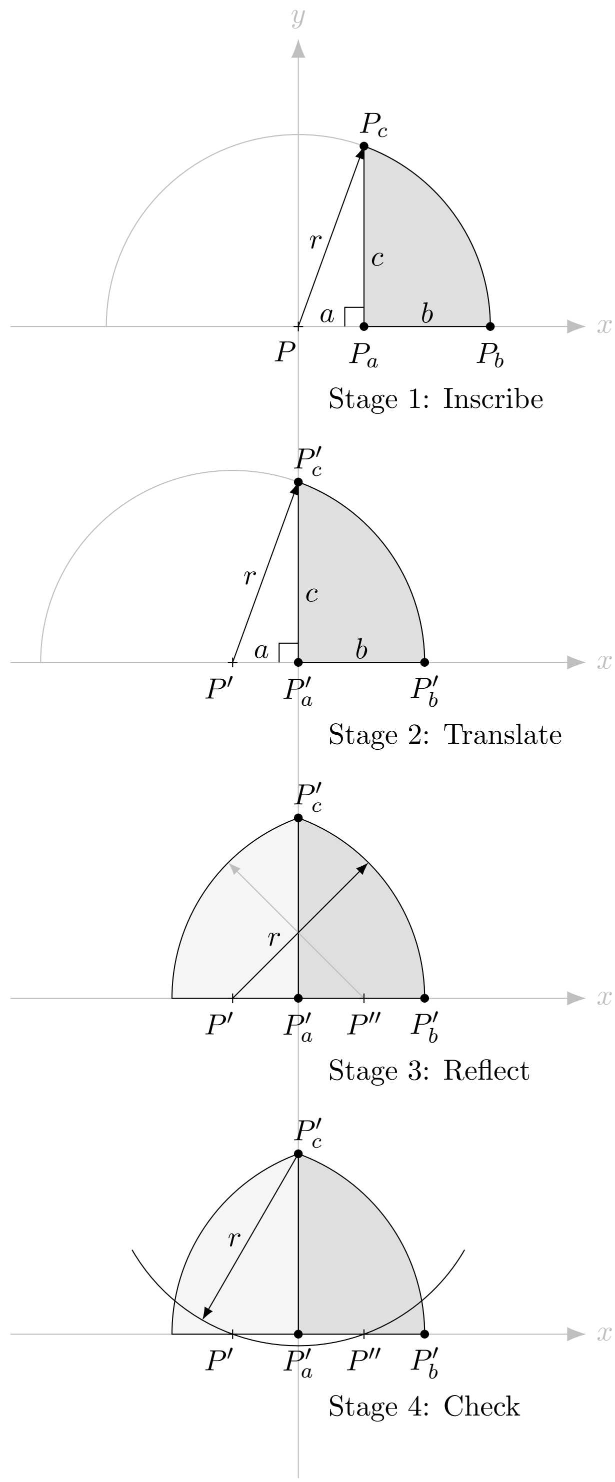

Figure 1: Construction of Tangent Ogive

x-coordinates in points P through P_c from equations 1a through 1d, as follows:

$$P' = (-a, 0) \tag{8a}$$
$$P'_a = (0, 0) \tag{8b}$$
$$P'_b = (b, 0) \tag{8c}$$
$$P'_c = (0, c) \tag{8d}$$

Prime notation is used here to show how the new points are derived from the original points.

2.3 (Stage 3) Reflect ogive half-section about y-axis

Reflecting the right ogive half-section across the y-axis is easily accomplished by applying a negative to all x-coordinates in points P' through P'_c from equations 8a through 8d, as follows:

$$P'' = (a, 0) \tag{9a}$$
$$P''_a = (0, 0) \tag{9b}$$
$$P''_b = (-b, 0) \tag{9c}$$
$$P''_c = (0, c) \tag{9d}$$

Double-prime notation is used here to show how the new points are derived from the original points. Reflecting the half-section about y-axis yields a full ogive cross-section.

Note: points P''_a, P''_b and P''_c are omitted for clarity in Figure 1 (Stage 3). However point P'' is shown.

2.4 (Stage 4) Check ogive cross-section for accuracy

Once the ogive cross-section has been calculated and constructed, checking for correctness and accuracy is recommended.

Checking the completed cross-section is a simple matter of recognizing the relationship between: P'_c, P' and P''. Refer to Figure 1 (Stage 3). The lines centered at either P' or P'', drawn to point P'_c, have length r and sweep arcs of radius r. The arcs meet at a common point P'_c.

Refer to Figure 1 (Stage 4). Lines centered at P'_c drawn back, in the opposite direction, to either P' or P'' will also have length r. So, an arc of radius r, centered at point P'_c, will include P' and P''.

Therefore, a quick algebraic check for correctness is to measure the distance of lines $\overline{P'_cP'}$ and $\overline{P'_cP''}$. Both should equal r.

A corresponding geometric check is to sweep an arc of radius r centered at P'_c. If points P' and P'' lie on the arc, then the ogive cross-section is correctly constructed.

2.5 Restrictions and valid values

The three most important variables in equations 2 through 7 are: a, b and r. They serve as inputs to the ogive geometry and are subject to some restrictions. These equations are undefined for values outside the valid ranges.

Restrictions:

- $r > 0$: Otherwise the ogive is undefined.
- $0 < a < r$: Otherwise the ogive is undefined
- $0 < b < r$: Otherwise the ogive is undefined
- $0 < c < r$: Otherwise the ogive is undefined
- Equation 2 must be satisfied. Otherwise, the ogive is undefined.

Q.E.D.

3 Example

Given:

- A tangent ogive whose radius is 4.5 units and height is 2.3 units.

Find:

- Values for a and b.
- Locations of points P through P_c.
- Locations of points P' through P'_c for the right half.
- Locations of points P'' through P''_c for the left half.
- Distances of $\overline{P'_cP'}$ and $\overline{P'_cP''}$.

Steps:

1. Let:

$$r = 4.5$$
$$c = 2.3$$

2. From Equation 6, line-segment b is found by substituting values for r and c:

$$b = r - \sqrt{r^2 - c^2}$$
$$b = 4.5 - \sqrt{4.5^2 - 2.3^2}$$
$$b = 4.5 - \sqrt{20.25 - 5.29}$$
$$b = 4.5 - \sqrt{14.96}$$
$$b = 4.5 - 3.8678159212$$

$$\boxed{b = 0.6321840788}$$

3. The value for a is easily found by rearranging Equaton 2, solving for a, and substituting values for r and b.

$$r = a + b$$
$$a = r - b$$
$$a = 4.5 - 0.6321840788$$

$$\boxed{a = 3.8678159212}$$

4. Points P through P_c are found by substituting values of a, b, c and r into Equations 1a through 1d.

$$P = (0, 0)$$
$$P_a = (3.8678159212, 0)$$
$$P_b = (4.5, 0)$$
$$P_c = (3.8678159212, 2.3)$$

5. Points P' through P'_c are found by substituting values of a, b, c and r into Equations 8a through 8d.

$$P' = (-3.8678159212, 0)$$
$$P'_a = (0, 0)$$
$$P'_b = (0.6321840788, 0)$$
$$P'_c = (0, 2.3)$$

6. Points P'' through P''_c are found by substituting values of a, b, c and r into Equations 9a through 9d.

$$P'' = (3.8678159212, 0)$$
$$P''_a = (0, 0)$$
$$P''_b = (-0.6321840788, 0)$$
$$P''_c = (0, 2.3)$$

7. Use the method of Section 2.4 to check the correctness of the calculations.

   The distance of $\overline{P'_cP'}$ is just the distance between point P'_c and point P'.

   From above:

$$P'_c = (0, 2.3)$$
$$P' = (-3.8678159212, 0)$$

   By using the Pythagorean theorem, the distance is found to be:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\overline{P'_cP'} = \sqrt{(P'_x - P'_{cx})^2 + (P'_y - P'_{cy})^2}$$
$$\overline{P'_cP'} = \sqrt{(-3.8678159212 - 0)^2 + (0 - 2.3)^2}$$
$$\overline{P'_cP'} = \sqrt{14.96 + 5.29}$$
$$\overline{P'_cP'} = \sqrt{20.25}$$
$$\overline{P'_cP'} = 4.5$$

   As expected, the distance $\overline{P'_cP'}$ is equal to r thereby confirming correctness of calculations.

   Furthermore, absolute values $|P'_x| = |P''_x|$, and P'_x or P''_x are squared when either is substituted.

$$\overline{P'_cP'} = \overline{P'_cP''} = r = 4.5$$

8. Use Equation 2 as another check.

$$r = a + b$$
$$r = 3.8678159212 + 0.6321840788$$
$$r = 4.5$$

   Since all restrictions of Section 2.5 are satisfied, correctness of the calculations is confirmed.