

# A-005

## Regarding Relationship and Classification of Equal Coplanar Circles

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Equal coplanar circle-pairs occur frequently in tradework and engineering.

Circle-pairs of equal radii can be considered a special condition of the more general unequal radii condition.

The ten possible general coplanar circle-pair arrangements are described in a previous article.<sup>1</sup>

How these ten general cases reduce upon application of the special equal radii condition is considered. Additionally, effect of the special condition upon circle-pair tangents is investigated<sup>2</sup>.

### 1 Problem

Refer to Figure 1.

Given:

- Two coplanar circles  $A$  and  $B$ ,
- with center points  $C_A$  and  $C_B$  respectively,
- and having radii  $r_A$  and  $r_B$  respectively,
- where  $r_A = r_B$ ;

Find:

1. How the ten coplanar circle cases<sup>1</sup> degenerate when circle  $A$  is equal to circle  $B$ .

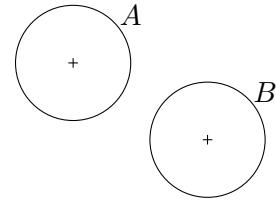


Figure 1: Equal coplanar circles  $A$  and  $B$

2. How tangent classes<sup>2</sup> are affected upon application of equal circles.
3. The point of intersection  $I$  of Inner and Outer tangent lines when circles  $A$  and  $B$  are equal.
4. Classification<sup>2</sup> of coplanar circle cases when circles are equal.

### 2 Solution

The strategy here is to adapt the circle-pair cases from prior article<sup>1</sup> for the condition of equal circles. Next, the effect of equal circles upon tangent classes is examined. Then, how  $I$  tangent intersection point is affected. Finally, classification for equal circles is performed.

#### 2.1 Effect upon Circle-Pair Cases

Refer to Figure 2 for the discussion that follows.

For each of the cases defined in prior work<sup>1</sup> set the radius of circle  $B$  to be equal to the radius of circle  $A$ , such that  $r_B = r_A$ . For each equation or inequality substitute  $r_A$  in place of all

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<sup>1</sup>A-003, Regarding Coplanar Circles, Vincent W. Finley, March 2022

<sup>2</sup>A-004, Regarding Classification of Coplanar Circle Tangents, Vincent W. Finley, April 2022

occurrences of  $r_B$ . Combine terms and simplify the equation or inequality. Each of the ten cases follow:

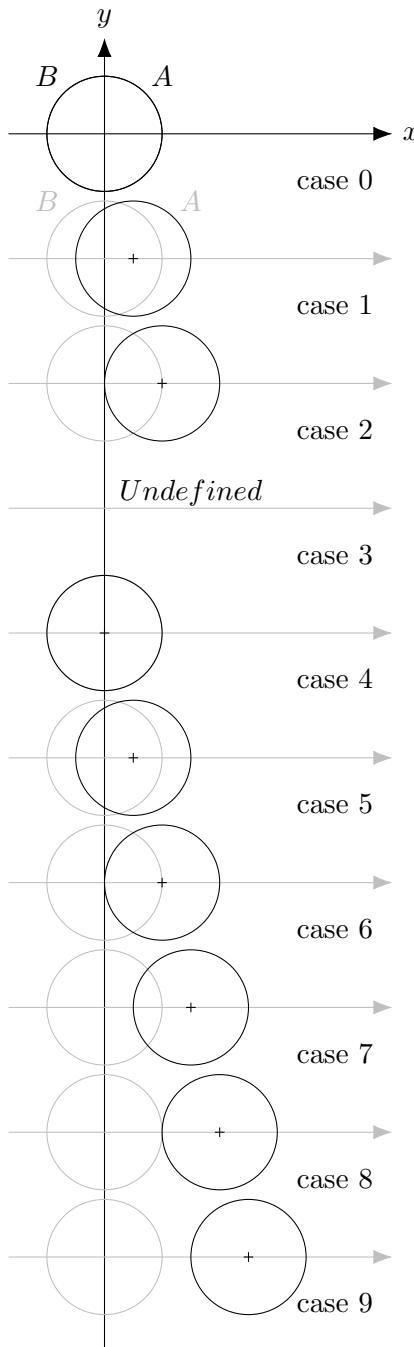


Figure 2: Locations of  $C_A$  relative to  $C_B$

- **case 0:**  $x_A = 0$

Identical to case 4. When the center of  $A$  is at the origin,  $A$  and  $B$  are the *same* circle.

- **case 1:**  $0 < x_A < r_A$

Identical to case 5. When  $C_A$  lies less than one  $r_A$  radius from the origin,  $A$  will overlap left and right regions of  $B$ .

- **case 2:**  $x_A = r_A$

Identical to case 6. When  $C_A$  lies exactly than one  $r_A$  radius from the origin, circle  $A$  will pass through center of  $C_B$  at the origin. Furthermore,  $C_A$  will lie on  $B$ .

- **case 3:** This case is UNDEFINED.

Taking case 3 inequality from prior article<sup>1</sup>, substituting  $r_A$  in place of  $r_B$ , and reducing yields:

$$r_A < x_A < r_B - r_A$$

$$r_A < x_A < r_A - r_A$$

$$r_A < x_A < 0$$

$x_A$  cannot be less than zero (negative), and simultaneously greater than non-negative values of  $r_A$ . Therefore the inequality is not satisfied, and case 3 is undefined.

- **case 4:**  $x_A = 0$

Taking case 4 equation from prior article<sup>1</sup>, substituting  $r_A$  in place of  $r_B$ , and reducing yields:

$$x_A = r_B - r_A$$

$$x_A = r_A - r_A$$

$$x_A = 0$$

Since the center of  $A$  is at the origin,  $A$  and  $B$  are the *same* circle. This is identical to case 0 above.

- **case 5:**  $0 < x_A < r_A$

Taking case 5 inequality from prior article<sup>1</sup>, substituting  $r_A$  in place of  $r_B$ , and reducing yields:

$$r_B - r_A < x_A < r_B$$

$$r_A - r_A < x_A < r_A$$

$$0 < x_A < r_A$$

This is identical to case 1 above.

- **case 6:**  $x_A = r_A$

Taking case 6 equation from prior article<sup>1</sup>, substituting  $r_A$  in place of  $r_B$ , and reducing yields:

$$\begin{aligned}x_A &= r_B \\x_A &= r_A\end{aligned}$$

This is identical to case 2 above.

- **case 7:**  $r_A < x_A < 2 \cdot r_A$

Taking case 7 inequality from prior article<sup>1</sup>, substituting  $r_A$  in place of  $r_B$ , and reducing yields:

$$\begin{aligned}r_B < x_A &< r_B + r_A \\r_A < x_A &< r_A + r_A \\r_A &< x_A < 2 \cdot r_A\end{aligned}$$

The center of  $A$  is beyond the edge of circle  $B$ . Circle  $A$  partially overlaps the right side of  $B$ .

- **case 8:**  $x_A = 2 \cdot r_A$

Taking case 8 equation from prior article<sup>1</sup>, substituting  $r_A$  in place of  $r_B$ , and reducing yields:

$$\begin{aligned}x_A &= r_B + r_A \\x_A &= r_A + r_A \\x_A &= 2 \cdot r_A\end{aligned}$$

Circles  $A$  and  $B$  are tangent to each other with no overlap.

- **case 9:**  $x_A > 2 \cdot r_A$

Taking case 9 inequality from prior article<sup>1</sup>, substituting  $r_A$  in place of  $r_B$ , and reducing yields:

$$\begin{aligned}x_A &> r_B + r_A \\x_A &> r_A + r_A \\x_A &> 2 \cdot r_A\end{aligned}$$

Circle  $A$  is entirely outside  $B$ .  $A$  and  $B$  are non-overlapping.

## 2.2 Effect upon Tangent Classes

For each tangent class defined in prior work<sup>2</sup> consider how tangents are affected by setting the radii of circles  $A$  and  $B$  to be equal.

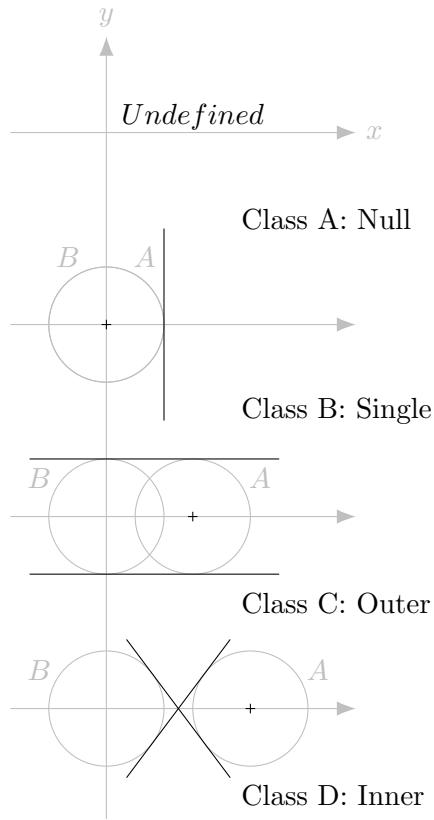


Figure 3: Tangent Classes

### Class A: Null

The Null class covers those cases where circles  $A$  and  $B$  have no common tangent line, as seen in cases 0-3 in prior article<sup>1</sup>.

Null tangent class arises when circle  $A$  is wholly contained by circle  $B$ , and there is no contact between the circles. This can only occur when  $r_A < r_B$ .

However, when radii of circles  $A$  and  $B$  are equal, there can be no case where circle  $A$  is wholly contained by circle  $B$ . In fact, even when  $r_A = r_B$  and the centers  $C_A = C_B$  coincide, circle  $A$  will not be contained by circle  $B$ . This is because for this condition  $A$  and  $B$  will actually be the *same* circle.

When,  $A$  and  $B$  are the *same* circle they will have an infinite number of common tangents.

Since  $A$  can never be fully contained by  $B$  when their radii are equal, the Null tangent class does not apply (is undefined) for  $r_A = r_B$ .

#### Class B: Single

When there is a tangent at the point of contact between circles  $A$  and  $B$ , the Single tangent class is satisfied.

Inspecting Figure 2, shows case 8 has a single tangent at the point where  $A$  and  $B$  contact.

Furthermore when  $r_A = r_B$ , cases 0 and 4 degenerate from case 4 in prior article<sup>1</sup>. The point of contact for case 4 in prior article<sup>1</sup> is  $x = r_B$ . For equal circles, the point of contact is also equivalent to  $x = r_A$ . Therefore  $A$  and  $B$  will have contact at  $x = r_A$  since they are the *same* circle.

In cases 0 and 4,  $A$  and  $B$  have infinite contact points and tangent lines since they are the *same* circle. Moreover, for cases 0 and 4, circles  $A$  and  $B$  reduce to the trivial case of a single circle with tangent lines at every point on the circle, essentially infinite.

#### Class C: Outer

Outer tangents only occur when the centers of  $A$  and  $B$  are separated. When  $r_A = r_B$  separation will occur for cases: 1, 2, 5, 6, 7, 8, 9.

See Figure 3. Because  $r_A = r_B$  are equal, the Outer tangents will always be parallel to the x-axis. Furthermore, they will always be parallel to each other and never intersect.

This can be confirmed by attempting to calculate the Outer tangent intersection point  $I$  from equations 3 and 4 in prior work<sup>2</sup>.

$$\begin{aligned}x_I &= \frac{r_B \cdot x_A}{r_B - r_A} \\x_I &= \frac{r_A \cdot x_A}{r_A - r_A} \\x_I &= \frac{r_A \cdot x_A}{0} \\x_I &= \infty\end{aligned}$$

Substituting  $x_I$  into equation 4 from prior work<sup>2</sup>

yields:

$$\boxed{\begin{aligned}I &= (x_I, 0) \\I &= (\infty, 0)\end{aligned}} \quad (1)$$

Therefore the intersection point of the Outer tangent lines is at somewhere at infinity along the x-axis. In other words, the Outer tangent lines are parallel because they will never intersect.

#### Class D: Inner

See Figure 3. Inner tangents only occur for case 9. This is due to the fact that inner tangents will intersect each other along the x-axis, while not contacting circles  $A$  or  $B$  more than once. Case 9 is the only to satisfy this condition for Inner class tangents.

Since the x-axis connects centers of  $A$  and  $B$ , the point of intersection  $I$  will occur along the x-axis at the midpoint between centers  $C_A$  and  $C_B$ . This can be confirmed by applying equation 4 and 9 from prior article<sup>1</sup>.

$$\begin{aligned}x_I &= \frac{r_B \cdot x_{A'}}{r_B + r_A} \\x_I &= \frac{r_A \cdot x_{A'}}{r_A + r_A} \\x_I &= \frac{r_A \cdot x_{A'}}{2 \cdot r_A} \\x_I &= \frac{x_{A'}}{2}\end{aligned} \quad (2)$$

Substituting equation 2 here into equation 4 from prior work<sup>2</sup> yields:

$$\boxed{I = \left(\frac{x_{A'}}{2}, 0\right)} \quad (3)$$

Since the center of circle  $B$  is at the origin, it can be seen from equation 3 that  $I$  is located at the midpoint between  $A$  and  $B$ .

The angle  $\theta$  between the inner tangents can be found by taking equation 5 from prior work<sup>2</sup>, and substituting  $r_A$  in place of  $r_B$ :

$$\theta = 2 \cdot \arcsin\left(\frac{r_B}{x_I}\right) \quad (4)$$

$$\theta = 2 \cdot \arcsin\left(\frac{r_A}{x_I}\right) \quad (5)$$

Case	Number of Tangents					Description
	(A) Null	(B) Single	(C) Outer	(D) Inner	Total	
0 (4)	-	1	-	-	1	Same circle
1 (5)	-	-	2	-	2	Parallel outer tangents
2 (6)	-	-	2	-	2	Parallel outer tangents
3	-	-	-	-	x	Undefined case
4 (0)	-	1	-	-	1	Same circle
5 (1)	-	-	2	-	2	Parallel outer tangents
6 (2)	-	-	2	-	2	Parallel outer tangents
7	-	-	2	-	2	Parallel outer tangents
8	-	1	2	-	3	Circles tangent to each other, and parallel outer tangents
9	-	-	2	2	4	Parallel outer tangents and two inner tangents

Table 1: Classification of Equal Coplanar Circle Cases

Now, equation 2 can be substituted into equation 5 to find the angle between inner tangents.

$$\theta = 2 \cdot \arcsin \left( \frac{2 \cdot r_A}{x_{A'}} \right) \quad (6)$$

### 2.3 Effect upon Classification

Increasing the radius of  $A$  to be equal with radius of  $B$  eliminates the possibility of circle  $A$  being wholly contained by circle  $B$ . As the center of  $A$  moves along the positive x-axis from the origin, the rightmost edge of  $A$  will not contact any part of  $B$ .

Cases 0-2 become redundant with cases 4-6. Moreover, case 3 becomes undefined.

There is no Null tangent class defined when circles  $A$  and  $B$  are equal, this is because there will always be at least one tangent between the circles.

Single tangents are defined for cases 0 and 4 because all points of  $A$  and  $B$  are in contact. Single tangent is also defined for case 8 because  $A$  and  $B$  have one point of contact.

Pairs of Outer tangents will always be parallel to each other. Pairs of Inner tangents will always intersect halfway between  $C_A$  and  $C_B$ .

Q.E.D.

## 3 Example

Given:

- Two circles  $A$  and  $B$  whose radii are  $r_A = r_B = 3.0$  units,
- and circle centers  $C_A$  and  $C_B$  separated by a distance of 9.0 units.

Find:

- Which solution case and tangent class best matches the given arrangement.
- All tangent lines and any point of intersection.
- Any angle of intersection between tangent lines.

Steps:

- Let:

$$\begin{aligned} x_A &= 9.0 \\ r_A &= 3.0 \\ r_B &= 3.0 \end{aligned}$$

- From the solution section, use the inequality for case 9:

$$x_A > 2 \cdot r_A$$

3. Plug in values for  $r_A$ ,

$$9.0 > 2 \cdot 3.0$$

$$9.0 > 6.0$$

4. Since  $x_A = 9.0$  is greater than 6.0,  $x_A$  satisfies the inequality for case 9. Therefore the relationship between circles A and B are classified by case 9.

5. From Table 1 it is found that case 9 will have two Outer tangents and two Inner tangents.

6. Outer class:

The upper and lower horizontal tangent lines will lie a distance  $r_A = 3.0$  above and below the x-axis. Tangents will be parallel and will not intersect.

7. Inner class:

Substituting in 9.0 for  $x_{A'}$  into equation 3 yields:

$$\begin{aligned} I &= \left( \frac{x_{A'}}{2}, 0 \right) \\ I &= \left( \frac{9.0}{2}, 0 \right) \end{aligned}$$

$$I = (4.5, 0)$$

Substituting for  $x_{A'}$  and  $r_A$  into equation 6 yields:

$$\begin{aligned} \theta &= 2 \cdot \arcsin \left( \frac{2 \cdot r_A}{x_{A'}} \right) \\ \theta &= 2 \cdot \arcsin \left( \frac{2 \cdot 3.0}{9.0} \right) \\ \theta &= 2 \cdot \arcsin \left( \frac{6.0}{9.0} \right) \\ \theta &= 1.45946 \text{ rads} \end{aligned}$$

$$\theta = 83.62^\circ$$