A-005
Regarding Relationship and Classification of Equal Coplanar Circles

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Equal coplanar circle-pairs occur frequently in tradework and engineering.

Circle-pairs of equal radii can be considered a special condition of the more general unequal radii condition.

The ten possible general coplanar circle-pair arrangements are described in a previous article.[1]

How these ten general cases reduce upon application of the special equal radii condition is considered. Additionally, effect of the special condition upon circle-pair tangents is investigated[2].

1 Problem

Refer to Figure 1.

Given:

- Two coplanar circles A and B,
- *with* center points C_A and C_B respectively,
- *and* having radii r_A and r_B respectively,
- *where* $r_A = r_B$;

[1]A-003, Regarding Coplanar Circles, Vincent W. Finley, March 2022

[2]A-004, Regarding Classification of Coplanar Circle Tangents, Vincent W. Finley, April 2022

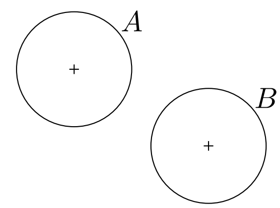

Figure 1: Equal coplanar circles A and B

Find:

1. How the ten coplanar circle cases[1] degenerate when circle A is equal to circle B.
2. How tangent classes[2] are affected upon application of equal circles.
3. The point of intersection I of Inner and Outer tangent lines when circles A and B are equal.
4. Classification[2] of coplanar circle cases when circles are equal.

2 Solution

The strategy here is to adapt the circle-pair cases from prior article[1] for the condition of equal circles. Next, the effect of equal circles upon tangent classes is examined. Then, how I tangent intersection point is affected. Finally, classification for equal circles is performed.

2.1 Effect upon Circle-Pair Cases

Refer to Figure 2 for the discussion that follows.

For each of the cases defined in prior work[1] set the radius of circle B to be equal to the radius of circle A, such that $r_B = r_A$. For each equation or inequality substitute r_A in place of all occurrences of r_B. Combine terms and simplify the equation or inequality. Each of the ten cases follow:

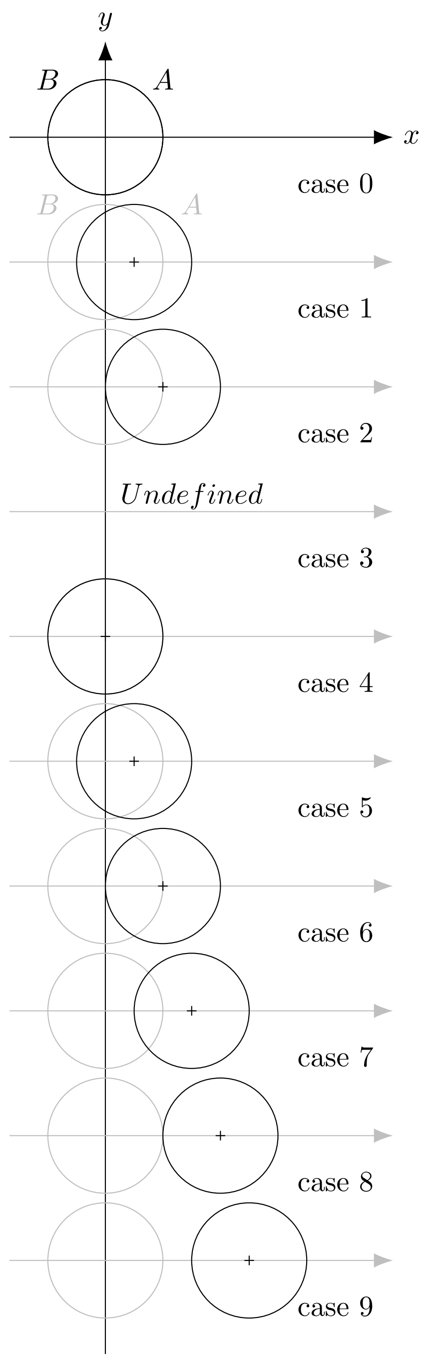

Figure 2: Locations of C_A relative to C_B

- **case 0:** $x_A = 0$

  Identical to case 4. When the center of A is at the origin, A and B are the *same* circle.

- **case 1:** $0 < x_A < r_A$

  Identical to case 5. When C_A lies less than one r_A radius from the origin, A will overlap left and right regions of B.

- **case 2:** $x_A = r_A$

  Identical to case 6. When C_A lies exactly than one r_A radius from the origin, circle A will pass through center of C_B at the origin. Furthermore, C_A will lie on B.

- **case 3:** This case is UNDEFINED.

  Taking case 3 inequality from prior article[1], substituting r_A in place of r_B, and reducing yields:

  $$r_A < x_A < r_B - r_A$$
  $$r_A < x_A < r_A - r_A$$
  $$r_A < x_A < 0$$

  x_A cannot be less than zero (negative), and simultaneously greater than non-negative values of r_A. Therefore the inequality is not satisfied, and case 3 is undefined.

- **case 4:** $x_A = 0$

  Taking case 4 equation from prior article[1], substituting r_A in place of r_B, and reducing yields:

  $$x_A = r_B - r_A$$
  $$x_A = r_A - r_A$$
  $$x_A = 0$$

  Since the center of A is at the origin, A and B are the *same* circle. This is identical to case 0 above.

- **case 5:** $0 < x_A < r_A$

  Taking case 5 inequality from prior article[1], substituting r_A in place of r_B, and reducing yields:

  $$r_B - r_A < x_A < r_B$$
  $$r_A - r_A < x_A < r_A$$
  $$0 < x_A < r_A$$

This is identical to case 1 above.

- **case 6:** $x_A = r_A$

  Taking case 6 equation from prior article[1], substituting r_A in place of r_B, and reducing yields:

$$x_A = r_B$$
$$x_A = r_A$$

  This is identical to case 2 above.

- **case 7:** $r_A < x_A < 2 \cdot r_A$

  Taking case 7 inequality from prior article[1], substituting r_A in place of r_B, and reducing yields:

$$r_B < x_A < r_B + r_A$$
$$r_A < x_A < r_A + r_A$$
$$r_A < x_A < 2 \cdot r_A$$

  The center of A is beyond the edge of circle B. Circle A partially overlaps the right side of B.

- **case 8:** $x_A = 2 \cdot r_A$

  Taking case 8 equation from prior article[1], substituting r_A in place of r_B, and reducing yields:

$$x_A = r_B + r_A$$
$$x_A = r_A + r_A$$
$$x_A = 2 \cdot r_A$$

  Circles A an B are tangent to each other with no overlap.

- **case 9:** $x_A > 2 \cdot r_A$

  Taking case 9 inequality from prior article[1], substituting r_A in place of r_B, and reducing yields:

$$x_A > r_B + r_A$$
$$x_A > r_A + r_A$$
$$x_A > 2 \cdot r_A$$

  Circle A is entirely outside B. A and B are non-overlapping.

2.2 Effect upon Tangent Classes

For each tangent class defined in prior work[2] consider how tangents are affected by setting the radii of circles A and B to be equal.

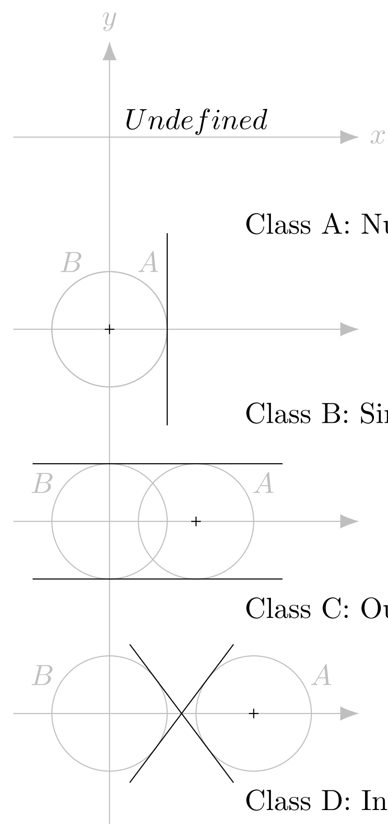

Figure 3: Tangent Classes

Class A: Null

The Null class covers those cases where circles A and B have no common tangent line, as seen in cases 0-3 in prior article[1].

Null tangent class arises when circle A is wholly contained by circle B, and there is no contact between the circles. This can only occur when $r_A < r_B$.

However, when radii of circles A and B are equal, there can be no case where circle A is wholly contained by circle B. In fact, even when $r_A = r_B$ and the centers $C_A = C_B$ coincide, circle A will not be contained by circle B. This is because for this condition A and B will actually be the *same* circle.

When, A and B are the *same* circle they will have an infinite number of common tangents.

Since A can never be fully contained by B when their radii are equal, the Null tangent class does not apply (is undefined) for $r_A = r_B$.

Class B: Single

When there is a tangent at the point of contact between circles A and B, the Single tangent class is satisfied.

Inspecting Figure 2, shows case 8 has a single tangent at the point where A and B contact.

Furthermore when $r_A = r_B$, cases 0 and 4 degenerate from case 4 in prior article[1]. The point of contact for case 4 in prior article[1] is $x = r_B$. For equal circles, the point of contact is also equivalent to $x = r_A$. Therefore A and B will have contact at $x = r_A$ since they are the *same* circle.

In cases 0 and 4, A and B have infinite contact points and tangent lines since they are the *same* circle. Moreover, for cases 0 and 4, circles A and B reduce to the trivial case of a single circle with tangent lines at every point on the circle, essentially infinite.

Class C: Outer

Outer tangents only occur when the centers of A and B are separated. When $r_A = r_B$ separation will occur for cases: 1, 2, 5, 6, 7, 8, 9.

See Figure 3. Because $r_A = r_B$ are equal, the Outer tangents will always be parallel to the x-axis. Furthermore, they will always be parallel to each other and never intersect.

This can be confirmed by attempting to calculate the Outer tangent intersection point I from equations 3 and 4 in prior work[2].

$$x_I = \frac{r_B \cdot x_A}{r_B - r_A}$$
$$x_I = \frac{r_A \cdot x_A}{r_A - r_A}$$
$$x_I = \frac{r_A \cdot x_A}{0}$$
$$x_I = \infty$$

Substituting x_I into equation 4 from prior work[2] yields:

$$I = (x_I, 0)$$
$$\boxed{I = (\infty, 0)} \tag{1}$$

Therefore the intersection point of the Outer tangent lines is at somewhere at infinity along the x-axis. In other words, the Outer tangent lines are parallel because they will never intersect.

Class D: Inner

See Figure 3. Inner tangents only occur for case 9. This is due to the fact that inner tangents will intersect each other along the x-axis, while not contacting circles A or B more than once. Case 9 is the only to satisfy this condition for Inner class tangents.

Since the x-axis connects centers of A and B, the point of intersection I will occur along the x-axis at the midpoint between centers C_A and C_B. This can be confirmed by applying equation 4 and 9 from prior article[1].

$$x_I = \frac{r_B \cdot x_{A'}}{r_B + r_A}$$
$$x_I = \frac{r_A \cdot x_{A'}}{r_A + r_A}$$
$$x_I = \frac{r_A \cdot x_{A'}}{2 \cdot r_A}$$

$$\boxed{x_I = \frac{x_{A'}}{2}} \tag{2}$$

Substituting equation 2 here into equation 4 from prior work[2] yields:

$$\boxed{I = \left(\frac{x_{A'}}{2}, 0\right)} \tag{3}$$

Since the center of circle B is at the origin, it can be seen from equation 3 that I is located at the midpoint between A and B.

The angle θ between the inner tangents can be found by taking equation 5 from prior work[2], and substituting r_A in place of r_B:

$$\theta = 2 \cdot \arcsin\left(\frac{r_B}{x_I}\right) \tag{4}$$

$$\theta = 2 \cdot \arcsin\left(\frac{r_A}{x_I}\right) \tag{5}$$

Case	Number of Tangents					Description
	(A) Null	(B) Single	(C) Outer	(D) Inner	Total	
0 (4)	-	1	-	-	1	Same circle
1 (5)	-	-	2	-	2	Parallel outer tangents
2 (6)	-	-	2	-	2	Parallel outer tangents
3	-	-	-	-	x	Undefined case
4 (0)	-	1	-	-	1	Same circle
5 (1)	-	-	2	-	2	Parallel outer tangents
6 (2)	-	-	2	-	2	Parallel outer tangents
7	-	-	2	-	2	Parallel outer tangents
8	-	1	2	-	3	Circles tangent to each other, and parallel outer tangents
9	-	-	2	2	4	Parallel outer tangents and two inner tangents

Table 1: Classification of Equal Coplanar Circle Cases

Now, equation 2 can be substituted into equation 5 to find the angle between inner tangents.

$$\boxed{\theta = 2 \cdot \arcsin\left(\frac{2 \cdot r_A}{x_{A'}}\right)} \tag{6}$$

2.3 Effect upon Classification

Increasing the radius of A to be equal with radius of B eliminates the possibility of circle A being wholly contained by circle B. As the center of A moves along the positive x-axis from the origin, the rightmost edge of A will not contact any part of B.

Cases 0-2 become redundant with cases 4-6. Moreover, case 3 becomes undefined.

There is no Null tangent class defined when circles A and B are equal, this is because there will always be at least one tangent between the circles.

Single tangents are defined for cases 0 and 4 because all points of A and B are in contact. Single tangent is also defined for case 8 because A and B have one point of contact.

Pairs of Outer tangents will always be parallel to each other. Pairs of Inner tangents will always intersect halfway between C_A and C_B.

Q.E.D.

3 Example

Given:

- Two circles A and B whose radii are $r_A = r_B = 3.0$ units,
- *and* circle centers C_A and C_B separated by a distance of 9.0 units.

Find:

- Which solution case and tangent class best matches the given arrangement.
- All tangent lines and any point of intersection.
- Any angle of intersection between tangent lines.

Steps:

1. Let:

$$x_A = 9.0$$
$$r_A = 3.0$$
$$r_B = 3.0$$

2. From the solution section, use the inequality for case 9:

$$x_A > 2 \cdot r_A$$

3. Plug in values for r_A,

$$9.0 > 2 \cdot 3.0$$

$$9.0 > 6.0$$

4. Since $x_A = 9.0$ is greater than 6.0, x_A satisfies the inequality for case 9. Therefore the relationship between circles A and B are classified by case 9.

5. From Table 1 it is found that case 9 will have two Outer tangents and two Inner tangents.

6. Outer class:

   The upper an lower horizontal tangent lines will lie a distance $r_A = 3.0$ above and below the x-axis. Tangents will be parallel and will not intersect.

7. Inner class:

   Substituting in 9.0 for $x_{A'}$ into equation 3 yields:

$$I = \left(\frac{x_{A'}}{2}, 0\right)$$
$$I = \left(\frac{9.0}{2}, 0\right)$$

$$\boxed{I = (4.5, 0)}$$

   Substituting for $x_{A'}$ and r_A into equation 6 yields:

$$\theta = 2 \cdot \arcsin\left(\frac{2 \cdot r_A}{x_{A'}}\right)$$
$$\theta = 2 \cdot \arcsin\left(\frac{2 \cdot 3.0}{9.0}\right)$$
$$\theta = 2 \cdot \arcsin\left(\frac{6.0}{9.0}\right)$$
$$\theta = 1.45946\ rads$$

$$\boxed{\theta = 83.62^\circ}$$