

# A-003

## Regarding Coplanar Circles

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March 2022

Arrangements of coplanar circles are frequently encountered in mechanical design. They are often observed when determining centers of rotation for: wheels, geartrains and pulley systems.

In the field of astronomy: orbits of planets, solar eclipses and moon transits; can be considered in terms of coplanar circles.

The myriad examples of coplanar circles, both in nature and industry, make their study worthwhile.

### 1 Problem

Refer to Figure 1.

Given:

- Two coplanar circles  $A$  and  $B$ ,
- with center points  $C_A$  and  $C_B$  respectively,
- and having radii  $r_A$  and  $r_B$  respectively,
- where  $r_A < r_B$ ;

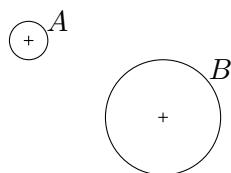


Figure 1: Coplanar circles  $A$  and  $B$

Find:

1. The critical location of  $C_A$  where  $A$  and  $B$  are concentric.
2. The critical location of  $C_A$  where  $C_B$  lies on  $A$ .
3. The critical locations of  $C_A$  where  $A$  and  $B$  are tangent to each other.
4. The critical location of  $C_A$  where  $C_A$  lies on  $B$ .
5. All locations of  $C_A$  where  $A$  otherwise overlaps  $B$ .
6. All locations of  $C_A$  where  $A$  has no overlap with  $B$ .

### 2 Solution

1. Refer to Figure 2 for the discussion that follows.
2. Since  $A$  and  $B$  are coplanar circles, center points  $C_A$  and  $C_B$  lie in the same plane with the circles.
3. Two points define the line between them, therefore a line can be drawn between  $C_A$  and  $C_B$ . This line shall be coplanar with circles  $A$  and  $B$ , and their centers.
4. A coordinate system can be drawn in the plane containing  $A$  and  $B$ , and the line between their centers. For convenience, the coordinate system origin can be arbitrarily chosen to coincide with  $C_B$ .

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5. Furthermore, for convenience, the coordinate system can be oriented such that the x-axis aligns with the line connecting  $C_B$  and  $C_A$ .
6. Since the origin is always relative to  $C_B$ ,  $C_B$  will always be at  $C_B = (x_B, y_B) = (0, 0)$ .
7. Since the x-axis is aligned with the line connecting  $C_B$  and  $C_A$ ,  $C_A$  will always lie along the x-axis, such that:  $C_A = (x_A, y_A) = (x_A, 0)$ .
8. Furthermore, the distance between  $C_A$  and  $C_B$  will always be  $\Delta x = x_A - x_B = x_A - 0 = x_A$

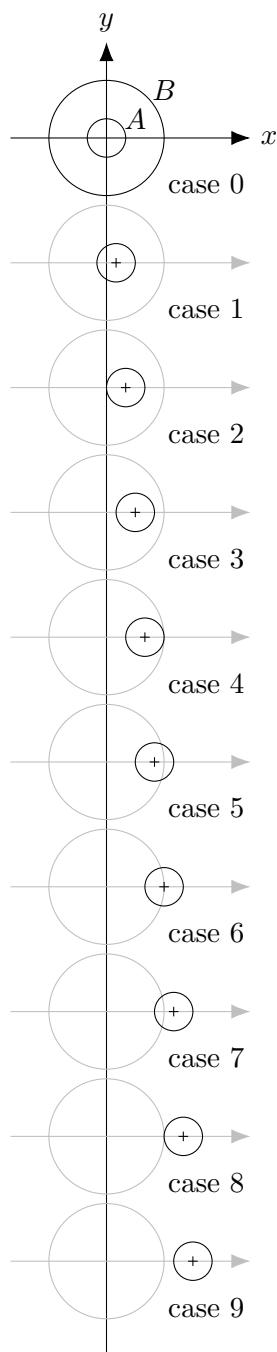


Figure 2: Locations of  $C_A$  relative to  $C_B$

9. Therefore the problem of finding critical locations of  $C_A$  simplifies into the trivial problem of finding critical values of  $x_A$ .
10. The locations of  $C_A$  relative to  $C_B$  can be generalized into the 10 cases shown in Figure 2.

- **case 0:**  $x_A = 0$

At the critical point where  $C_A$  is at the origin,  $C_A$  will be concentric with  $C_B$ .

- **case 1:**  $0 < x_A < r_A$

When  $C_A$  lies less than one  $r_A$  radius from the origin,  $A$  will overlap left and right regions of  $B$ .

- **case 2:**  $x_A = r_A$

When  $C_A$  lies exactly than one  $r_A$  radius from the origin, circle  $A$  will pass through point  $C_B$ .

- **case 3:**  $r_A < x_A < r_B - r_A$

In this case,  $C_A$  overlaps with the right side region of  $B$ .

- **case 4:**  $x_A = r_B - r_A$

When  $C_A$  lies at one  $r_A$  radius less than  $r_B$ ,  $A$  and  $B$  will share a tangent point.  $A$  will overlap with right region of  $B$ .

- **case 5:**  $r_B - r_A < x_A < r_B$

In this case,  $A$  only partially overlaps  $B$ . Furthermore it overlaps the right edge of  $B$ .

- **case 6:**  $x_A = r_B$

$C_A$ , the center of  $A$ , lies on circle  $B$ .

- **case 7:**  $r_B < x_A < r_B + r_A$

The center of  $A$  is beyond the edge of circle  $B$ . Circle  $A$  partially overlaps the right side of  $B$ .

- **case 8:**  $x_A = r_B + r_A$

Circles  $A$  and  $B$  are tangent to each other with no overlap.

- **case 9:**  $x_A > r_B + r_A$

Circle  $A$  is entirely outside  $B$ .  $A$  and  $B$  are non-overlapping.

Q.E.D.

### 3 Example

Given:

- Two circles  $A$  and  $B$  whose radii are  $r_A = 1.0$  and  $r_B = 3.0$  units respectively,
- and circle centers  $C_A$  and  $C_B$  separated by a distance of 2.5 units.

Find:

- Which solution case best matches the given arrangement?

Steps:

1. Let:

$$x_A = 2.5$$

$$r_A = 1.0$$

$$r_B = 3.0$$

2. From the solution section, use the inequality for case 5:

$$r_B - r_A < x_A < r_B$$

3. Plug in values for  $r_A$  and  $r_B$ ,

$$3.0 - 1.0 < x_A < 3.0$$

$$2.0 < x_A < 3.0$$

4. Since  $x_A = 2.5$  is between 2.0 and 3.0,  $x_A$  satisfies the inequality for case 5. Therefore the relationship between circles  $A$  and  $B$  are classified by case 5.