A-008
Regarding Placement of a Constrained Circle

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A circle is fully defined by its radius and center on a plane. In many design scenarios circle locations and sizes are known requirements, therefore these circles can be immediately placed on the design.

Often, circle radius and center location are initially unknown as they depend upon surrounding geometric features. A circle inscribed on a triangle is a classic example.

How to: pass a circle through a point and fit it to a tangent line; is described here. Geometric and trigonometric techniques are presented.

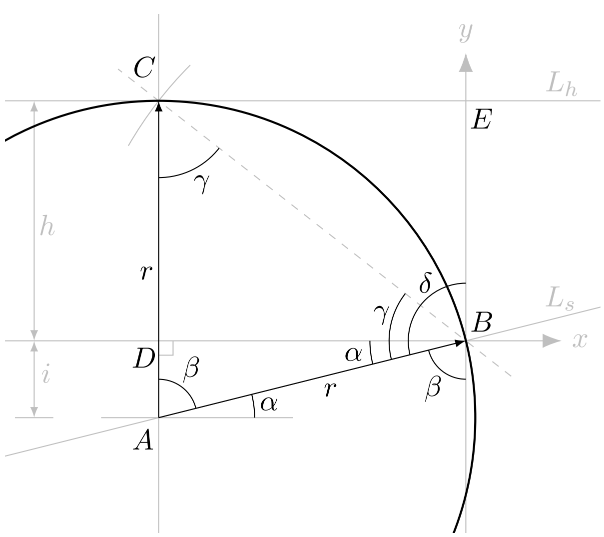

Figure 1: Hypothetical Constrained Circle

1 Problem

Given:

- A positively sloped line L_s passing through the origin,
- *and* a horizontal line L_h passing some distance h above the origin.

Find:

1. The circle that:
   - passes through the origin,
   - *and* is tangent to horizontal line L_h,
   - *and* has its center somewhere on sloped line L_s.
2. A geometric construction for fitting and placing such a circle.
3. A trigonometric method for fitting and placing such a circle.
4. Locations of the circle center and tangent point.
5. The radius of the circle.

2 Solution

Two solutions are presented here. Both are useful and address different design needs.

When drafting a design, either manually or with CAD software, graphical solutions save time. The convenient *geometric construction* described below will speed progress.

When creating: programs, scripts, spreadsheets or tabular data; equations are preferred. The *trigonometric calculation* detailed below will eliminate any guesswork.

2.1 Geometric Construction

It is desirable to find a simple geometric construction to locate the problem statement's hypothetical circle.

Neither the hypothetical circle's center nor radius is known. This makes locating it difficult as knowing one can usually be used to find the other with simple drafting tools: compass, ruler and square.

Refer to figure 1 for the following discussion. Let's define points of interest:

- Let point A be the unknown center of the hypothetical circle that lies somewhere on sloped line L_s.
- Let point B an alias for the the origin. It lies on the circle since the circle passes through it. Furthermore it also lies on sloped line L_s since L_s passes through the origin.
- Let point C be a location on the circle where it is tangent to line L_h. In other words, C is the tangent point.
- Let point D be the location where line $\overline{AC}$ intersects the x-axis at right angles.
- Let point E be the location where line L_h intersects the y-axis at right angles.

Any point that is distance r, from a point A, lies on the circle of radius r centered at A. Therefore to be on the circle of radius r centered at A, points B and C must be a distance r from A. This implies that the line from A to B is of length r. Also implied is the line from A to C is of length r, and it is equal to the length of the line from A to B, as follows:

$$\overline{AB} = \overline{AC} = r \tag{1}$$

Since $\overline{AB} = \overline{AC}$ two sides of the hypothetical triangle $\triangle ABC$ are equal; therefore $\triangle ABC$ an isosceles triangle.

The radius and center of $\triangle ABC$ are still unknown. The points A, B and C are also still unknown. To find them some algebra will be applied to triangles.

Let α be the angle L_s makes with the negative x-axis. Also, let β be the angle complimentary to α such that:

$$\alpha = \angle ABD \tag{2}$$
$$\beta = 90° - \alpha \tag{3}$$

Now β forms an alternate interior angle with $\angle CAB$ so that:

$$\angle CAB = \beta \tag{4}$$

Since $\triangle ABC$ is isosceles the other two angles, $\angle ACB$ and $\angle ABC$, must be equal. Name these other angles γ so that.

$$\angle ACB = \angle ABC = \gamma = \frac{180° - \beta}{2} \tag{5}$$

Substituting equation 3 into equation 5 it is found:

$$\gamma = \frac{90° + \alpha}{2} \tag{6}$$

By inspecting equation 6 it can be seen that angle γ on $\triangle ABC$ is half of some angle whose value is $90° + \alpha$. Let's call the angle δ so that:

$$\delta = 90° + \alpha \tag{7}$$

and,

$$\gamma = \frac{\delta}{2} \tag{8}$$

γ on the triangle is just half some angle $\delta = 90° + \alpha$, but:

$$\delta = 90° + \alpha = \angle ABE \tag{9}$$

and, it is readily observed $\angle ABE$ is just the angle between sloped line L_s and the y-axis.

Therefore, the **Geometric Construction Procedure** to find circle center and radius is:

1. Bisect angle $\angle ABE$ (see figure 1 dashed line),
2. *next*, extend the bisection line from B, at the origin, and intersect line L_h,
3. *next*, label the intersection of the bisection line and L_h as point C, (C is the tangent point)
4. *next*, draw a vertical line down from C,
5. *next*, label the intersection of the vertical line with L_s as point A, (A is the circle center)
6. *next*, the value of radius r can be found by measuring length of either $\overline{AB}$ or $\overline{AC}$,
7. *next*, verify $\overline{AB} = \overline{AC} = r$. They should be equal, otherwise an error occurred while applying the construction procedure.

2.2 Trigonometric Calculation

Application of trigonometry makes discovery of the circle radius and center a trivial exercise.

Figure 1 shows radius r as the hypotenuse of $\triangle ABD$. Recall from equation 2 the value of angle of $\angle ABD$ is α.

Unknown length i is just the vertical radius distance r less the distance h:

$$i = r - h \tag{10}$$

The sine of α is the ratio of length i over radius r. From this relationship, the length of r can be found in terms of h and α.

$$\begin{aligned} \sin\alpha &= \frac{i}{r} \\ i &= r \cdot \sin\alpha \\ r - h &= r \cdot \sin\alpha \\ r - r \cdot \sin\alpha &= h \\ r(1 - \sin\alpha) &= h \end{aligned}$$

$$\boxed{r = \frac{h}{1 - \sin\alpha}} \tag{11}$$

Line L_s is a given by the problem statement. Its angle α with the x-axis, can be: measured directly (equation 2) or calculated from the slope m of L_s as follows:

$$\alpha = \tan^{-1}(m) \tag{12}$$

Substituting values for α and h into equation 11 yields the length of r.

The location of point A, the circle center, now can be calculated. Use equation 10 to find y_A, and use the cosine function to find x_A.

$$x_A = \overline{BD} = -r \cdot \cos\alpha \tag{13}$$

Substitute to find A.

$$A = (x_A, y_A) \tag{14}$$
$$A = (\overline{BD}, -i) \tag{15}$$
$$A = (-r \cdot \cos\alpha, h - r) \tag{16}$$

Point B is already known since it is just an alias for the origin.

$$B = (0, 0) \tag{17}$$

Point C is then easily found since h was given in the problem statement and x_A has already been calculated in equation 13.

$$C = (x_A, h) \tag{18}$$

Point D also uses the value of x_A from equation 13.

$$D = (x_A, 0) \tag{19}$$

Point E is simply distance h above the origin.

$$E = (0, h) \tag{20}$$

Angles β, γ and δ are easily calculated from α and equations: 3, 6 and 7.

Q.E.D.

3 Example

Given:

- Line L_s, with slope $m = 0.268$, passing through the origin,
- *and* horizontal line L_h passing distance $h = 13\,units$ above the origin.

Find:

- The circle that:
  - passes through the origin,
  - and is tangent to horizontal line L_h,
  - and has its center somewhere on sloped line L_s.
- A geometric construction locating the circle on L_s, tangent to L_h, and through the origin.
- The radius of the circle.
- Values for all angles and critical points.

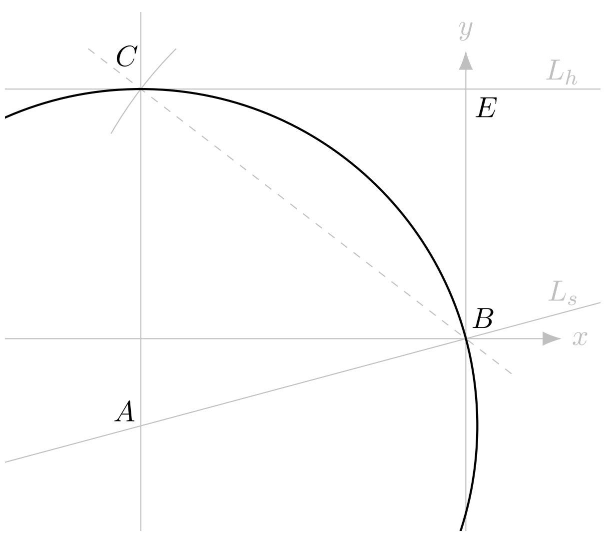

Figure 2: Example Circle

Steps:

1. Follow the "Geometric Construction Procedure" from the end of subsection 2.1. Then, on figure 2:
   - bisect $\angle ABE$,
   - label the intersection of the bisection line and L_h,
   - draw a vertical line downward to intersect with L_s,
   - draw a circle: centered at A the intersection of the vertical line and L_s, through point B the origin, and through C the intersection of the bisection line with L_h.
2. To get the angle α from slope m apply equation 12:

$$\begin{aligned}\alpha &= \tan^{-1}(m)\\ \alpha &= \tan^{-1}(0.268)\\ \alpha &= 15^\circ\end{aligned}$$

3. From the given values, m and h, all other values can be calculated using the numbered equations as follows:

Variable	Value	Equation
m	0.268	
h	13.0	
α	15°	12
r	17.540	11
x_A	-16.942	13
A	(-16.942, -4.540)	16
B	(0.0, 0.0)	17
C	(-16.942, 13.0)	18
D	(-16.942, 0.0)	19
E	(0.0, 13.0)	20
β	75.0°	3
γ	52.5°	8
δ	105.0°	7