A-007
Regarding Circle Intersections and Tangent Ogives

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Study of the Tangent Ogive has useful application in Engineering, especially to aeronautics and ballistics. Previous investigation described how the Tangent Ogive may be defined by inscribing the ogive upon a circle. It also described relationship of the ogive parts.

While fundamental Tangent Ogive analysis is interesting, deeper exploration is desired. What are its properties? More pertinently, how can it be used in other mathematical solutions?

1 Problem

Given:

- Two overlapping circles of equal radius.

Find:

1. An alternate method to represent and construct tangent ogives.
2. Relationship between tangent ogives and circle intersections.
3. Formula to calculate tangent ogive cross-sectional areas.
4. Formula to calculate volume of the tangent ogive.
5. Area of circle: intersections, unions and differences.
6. Volume of revolved circle intersections.

2 Solution

2.1 Alternate Construction

An alternate tangent ogive construction method is to first draw an intersection of equal circles, then retain half of the intersection. This method is presented in figure 1. It is identical to that found in stage 3 of figure 1 in the previous article[1]. The similarity between these two figures is readily seen. Here the ogive is drawn directly in a single step, whereas, the previous article presents

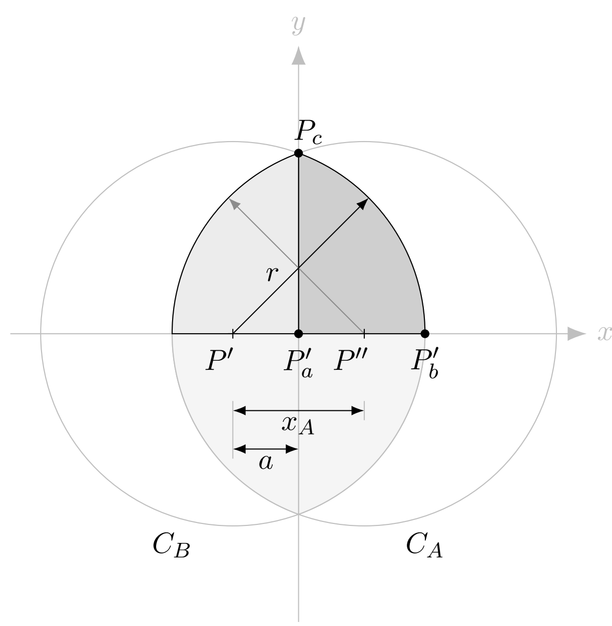

Figure 1: Tangent Ogive Alternate Construction

[1] A-006, Regarding Tangent Ogives, Vincent W. Finley, September 2022

the process in extensive detail. Each article has something useful to teach, as each approaches tangent ogive construction from a different perspective. The prior article emphasizes ogive geometry. However, this article emphasizes relationship of equal coplanar circles. Furthermore, it makes connections with another previous article[2] by describing how tangent ogives and intersections of equal circles are related.

2.2 Relationship between tangent ogives and circle intersections. Comparing to A-005 and A-006.

Figure 2 of article A-005, shows cases where equal coplanar circles will have regions of intersection. These are cases 5 through 7 specifically, with cases 1 and 2 being redundant. It is worth noting, for all cases resulting in regions of intersection, only a pair of Class C outer tangent lines is possible.

Recall the conditions for cases 5 through 7:

- Case 5: $0 < x_A < r_A$
- Case 6: $x_A = r_A$
- Case 7: $r_A < x_A < 2 \cdot r_A$

Combining the three conditions yields a general condition, which equal circle pairs must satisfy, to have a region of intersection:

$$\boxed{0 < x_A < 2 \cdot r_A} \tag{1}$$

Since circles A and B are equal, their radii $r_A = r_B = r$ are equal. Furthermore, the value x_A is simply the distance between the centers of circles C_A and C_B as described by A-005. In that article C_B was fixed at the origin and C_A was permitted to vary some distance x_A along the non-negative x-axis.

In Stage 3 on Figure 1 from article A-006, the two circles are centered about points P' and P''. These center points lie along the x-axis some distance a from the origin. This is confirmed by equations 8a and 9a in A-006, where these are given as $P' = (-a, 0)$ and $P'' = (a, 0)$ respectively.

Therefore the relationship between variable x_A in A-005 and variable a in A-006 can be described as:

$$x_A = 2 \cdot a \tag{2}$$

where x_A is the distance between centers, and:

$$\boxed{a = \frac{x_A}{2}} \tag{3}$$

This relationship bridges articles A-005 and A-006 together. Bridging of these articles is expected since both merely deal with overlapping equal coplanar circles whose centers are separated by some distance.

To fortify the bridge between articles, substitute equation 3 above into the ogive equations: 2, 3 and 4; from A-006. The substitution yields the respective ogive equation in terms of circle separation distance, as follows:

$$\boxed{r = \frac{x_A}{2} + b} \tag{4}$$

$$\boxed{r = \sqrt{\left(\frac{x_A}{2}\right)^2 + c^2}} \tag{5}$$

$$\boxed{x_A = \frac{c^2 - b^2}{b}} \tag{6}$$

Note: equations: 6 and 7; from A-006 are unaffected since they are independent of a, and are therefore independent of x_A.

2.3 Finding area of ogive cross-section

The area of the ogive cross-section is found by doubling the area under the circle between points P_a and P_b in figure 1.

By remembering figure 1 here is just Stage 3 from Figure 1 in article A-006, one can work backward to Stage 1. Referring to the half-ogive shown in Stage 1 makes the following integration more apparent.

From a table of integrals[3] the definite integral: with respect to x, of the circle with having radius

[2] A-005, Regarding Relationship and Classification of Equal Coplanar Circles, Vincent W. Finley, May 2022

[3] ISBN 0-471-85045-4, Calculus, 3rd Edition, Howard Anton, Drexel University, Copyright 1988 Anton Textbooks, Inc., John Wiley & Sons, See integral #40 inside front cover.

r, over limits a and r; is found to be:

$$\begin{aligned}F(x)]_a^r &= \int_a^r \sqrt{r^2-x^2}\,dx \\ &= \frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2} sin^{-1}\left(\frac{x}{r}\right)\Bigg]_a^r \\ &= \frac{r}{2}\sqrt{r^2-r^2} + \frac{r^2}{2} sin^{-1}\left(\frac{r}{r}\right) \\ &\quad - \frac{a}{2}\sqrt{r^2-a^2} - \frac{r^2}{2} sin^{-1}\left(\frac{a}{r}\right) \\ &= \frac{\pi r^2}{4} - \frac{a}{2}\sqrt{r^2-a^2} - \frac{r^2}{2} sin^{-1}\left(\frac{a}{r}\right)\end{aligned}$$

The area A_{to} of the tangent ogive cross-section is twice the area of definite integral.

$$A_{to} = 2\left[\frac{\pi r^2}{4} - \frac{a}{2}\sqrt{r^2-a^2} - \frac{r^2}{2} sin^{-1}\left(\frac{a}{r}\right)\right]$$

$$A_{to} = \frac{\pi r^2}{2} - a\sqrt{r^2-a^2} - r^2 sin^{-1}\left(\frac{a}{r}\right) \qquad (7)$$

Recognizing $\sqrt{r^2-a^2} = c$, from A-006 equation 3, yields A_{to} in a more convenient form.

$$\boxed{A_{to} = \frac{\pi r^2}{2} - ac - r^2 sin^{-1}\left(\frac{a}{r}\right)} \qquad (8)$$

2.4 Finding volume of tangent ogive

Starting with the standard form of the equation of a circle[4], substitute in $P' = (-a, 0)$ in place of (x_0, y_0). Doing so will translate the circle center to P' as shown in figure 1 and in A-006, Figure 1, Stage 2. Solve the equation for x, thereby yielding a function of y.

$$\begin{aligned}r^2 &= (x-x_0)^2 + (y-y_0)^2 \\ r^2 &= (x-(-a))^2 + (y-0)^2 \\ r^2 &= (x+a)^2 + y^2 \\ (x+a)^2 &= r^2 - y^2 \\ x + a &= \sqrt{r^2-y^2} \\ u(y) = x &= -a + \sqrt{r^2-y^2} \qquad (9)\end{aligned}$$

Use the "Volumes by Disks Perpendicular to the y-Axis"[5] method to integrate equation 9.

$$V = \int_c^d \pi[u(y)]^2\,dy \qquad (10)$$

[4]ISBN 0-471-85045-4, Anton, page 49, equation #1.
[5]ISBN 0-471-85045-4, Anton, page 380, equation #7.

Substituting equation 9 into 10, integrating from $y = 0$ to $y = c$ (A-006, Figure 1, Stage 2), and recognizing $\sqrt{r^2-c^2} = a$ (A-006, Equation 3); yields:

$$\begin{aligned}V_{to} &= \int_0^c \pi\left[-a + \sqrt{r^2-y^2}\right]^2\,dy \\ &= \pi\int_0^c \left[a^2 - 2a\sqrt{r^2-y^2} + r^2 - y^2\right]dy \\ &= \pi\left[a^2 y - ay\sqrt{r^2-y^2} - ar^2 sin^{-1}\left(\frac{y}{r}\right) + r^2 y - \frac{y^3}{3}\right]_0^c \\ &= \pi\left[a^2 c - ac\sqrt{r^2-c^2} - ar^2 sin^{-1}\left(\frac{c}{r}\right) + r^2 c - \frac{c^3}{3}\right] \\ &= \pi\left[a^2 c - a^2 c - ar^2 sin^{-1}\left(\frac{c}{r}\right) + r^2 c - \frac{c^3}{3}\right]\end{aligned}$$

Further reduction yields the volume V_{to} of the tangent ogive in terms of a, c and r:

$$\boxed{V_{to} = \pi\left[r^2 c - \frac{c^3}{3} - ar^2 sin^{-1}\left(\frac{c}{r}\right)\right]} \qquad (11)$$

Substituting in $c = \sqrt{r^2-a^2}$ and multiplying through by π; gives the ogive volume V only in terms of distance a and radius r.

$$\begin{aligned}V_{to} = \pi r^2\sqrt{r^2-a^2} &- \frac{\pi(r^2-a^2)^{3/2}}{3} \\ &- \pi ar^2 sin^{-1}\left(\frac{\sqrt{r^2-a^2}}{r}\right)\end{aligned} \qquad (12)$$

However, describing V_{to} in terms of the two variables in equation 12 is computationally intensive. In practice, first calculating c from a and r, then substituting the value of c into equation 11 requires less redundant computation.

2.5 Area of circle intersection

Upon inspecting figure 1 one can see similarities between intersecting circles and Venn diagrams. While Venn diagrams concern themselves with set membership, figure 1 is concerned with regions on a plane. However, if one considers all points in some bounded region to be members of an infinite point set for that region, then figure 1 becomes a Venn diagram representing point membership. Thinking of figure 1 in this manner relates: tangent ogive area and intersectional circle area; to set theory and set operations.

Start by realizing the intersectional circle area A_{ci} of two equal coplanar circles is simply twice the area of the tangent ogive cross-section. This is due to the reflection of the tangent ogive cross-section about the x-axis.

$$\boxed{A_{ci} = 2 \cdot A_{to}} \tag{13}$$

Now let Q and R be two equal overlapping circles, such that the area of each circle is:

$$A_Q = A_R = \pi r^2$$

Set operators could be defined as operations upon areas of Q and R. For example, define the: intersection, union, set difference and symmetric difference; of Q and R as:

$$Q \cap R = A_{ci} \tag{14a}$$

$$\begin{aligned} Q \cup R &= A_Q + A_R - A_{ci} \\ &= \pi r^2 + \pi r^2 - A_{ci} \\ &= 2\pi r^2 - A_{ci} \end{aligned} \tag{14b}$$

$$\begin{aligned} Q - R &= A_Q - A_{ci} \\ &= \pi r^2 - A_{ci} \end{aligned} \tag{14c}$$

$$\begin{aligned} Q \ominus R &= A_Q + A_R - 2 \cdot A_{ci} \\ &= \pi r^2 + \pi r^2 - 2 \cdot A_{ci} \\ &= 2\pi r^2 - 2 \cdot A_{ci} \end{aligned} \tag{14d}$$

A complete collection of set operators could be defined in a similar manner. Mapping set operators to Boolean operators could also be useful.

2.6 Volume of circle intersection

The volume, V_{ci}, of the circle intersection area revolved around the y-axis is simply twice the volume of the solid tangent ogive. This is due to the fact that the revolved circle intersection is just the solid tangent ogive and its reflection about the x-axis.

$$\boxed{V_{ci} = 2 \cdot V_{to}} \tag{15}$$

Note: V_{ci} is not the same as the volume of the intersection of 2 equal spheres.

Q.E.D.

3 Example

Given:

- Two equal circles A and B whose radii are 4.5 units, and whose centers are separated by 7.7356318424 units. This is the same circle pair from the example section in A-006.

Find:

- The range of distances between centers where intersections are defined.
- The case of overlap, tangent classes and number of common tangent lines.
- The values of a, b and c for both circles.
- A tangent ogive cross-sectional area formed by the circles.
- The area of intersection for both circles.
- The symmetric difference operation $A \ominus B$.
- The volume of the tangent ogive.
- Volume of rotation for the intersection region.

Steps:

1. Let:

$$\begin{aligned} r &= 4.5 \\ x_A &= 7.7356318424 \end{aligned}$$

2. Use inequality 3 to verify the circle pair has a region of intersection.

$$\begin{aligned} 0 &< x_A < 2 \cdot r_A \\ 0 &< x_A < 2 \cdot 4.5 \\ 0 &< x_A < 9.0 \end{aligned}$$

Intersections will be defined if the circles centers are separated by less than 9.0 units.

$$0 < 7.7356318424 < 9.0$$

The inequality is satisfied for 7.7356318424 therefore the circle pair intersects.

3. From A-005 the circle pair satisfies the condition for case 7 overlap.

$$\begin{aligned} r_A &< x_A < 2 \cdot r_A \\ 4.5 &< x_A < 2 \cdot 4.5 \\ 4.5 &< x_A < 9.0 \\ 4.5 &< 7.7356318424 < 9.0 \end{aligned}$$

A-005 Table 1 states that this circle pair can only support two outer Class C common tangent lines.

4. Use equation 3 to find a.

$$\begin{aligned} a &= \frac{x_A}{2} \\ &= \frac{7.7356318424}{2} \\ &= 3.8678159212\, units \end{aligned}$$

5. Use equation 2 from A-006 to find b.

$$\begin{aligned} b &= r - a \\ &= 4.5 - 3.8678159212 \\ &= 0.6321840788\, units \end{aligned}$$

6. Use equation 3 from A-006 to find c.

$$\begin{aligned} c &= \sqrt{r^2 - a^2} \\ &= \sqrt{(4.5)^2 - (3.8678159212)^2} \\ &= \sqrt{20.25 - 14.96} \\ &= 2.3\, units \end{aligned}$$

7. Use equation 8 to get the tangent ogive cross-sectional area.

$$\begin{aligned} A_{to} &= \frac{\pi r^2}{2} - ac - r^2 sin^{-1}\left(\frac{a}{r}\right) \\ &= \frac{\pi(4.5)^2}{2} - (3.8678159212)(2.3) \\ &\quad - (4.5)^2 sin^{-1}\left(\frac{3.8678159212}{4.5}\right) \\ &= 10.125\pi - 8.8959766185 \\ &\quad - 20.25 \cdot sin^{-1}(0.8595146492) \\ &= 31.8086256176 - 8.8959766185 \\ &\quad - 20.25 \cdot 1.0343193134 \\ &= 31.8086256176 - 8.8959766185 \\ &\quad - 20.9449660964 \\ &= 1.9676829028\, units^2 \end{aligned}$$

8. Use equation 13 and the value of A_{to} above to find the area of the circle intersection.

$$\begin{aligned} A_{ci} &= 2 \cdot A_{to} \\ A_{ci} &= 2 \cdot 1.9676829028 \\ A_{ci} &= 3.9353658055\, units^2 \end{aligned}$$

9. From equation 14d the symmetric difference between the circles is found.

$$\begin{aligned} A \ominus B &= 2\pi r^2 - 2 \cdot A_{ci} \\ &= 2\pi(4.5)^2 - 2 \cdot 3.9353658055 \\ &= 40.5 \cdot \pi - 2 \cdot 3.9353658055 \\ &= 127.2345024704 - 7.870731611 \\ &= 119.3637708594\, units^2 \end{aligned}$$

10. The volume of a tangent ogive between the circles is calculated using equation 11.

$$\begin{aligned} V_{to} &= \pi\left[r^2 c - \frac{c^3}{3} - ar^2 sin^{-1}\left(\frac{c}{r}\right)\right] \\ &= \pi\left[(4.5)^2 \cdot 2.3 - \frac{(2.3)^3}{3}\right. \\ &\quad \left. -3.8678159212(4.5)^2 sin^{-1}\left(\frac{2.3}{4.5}\right)\right] \\ &= \pi\left[20.25 \cdot 2.3 - \frac{12.167}{3}\right. \\ &\quad \left. -3.8678159212 \cdot 20.25 \cdot sin^{-1}(0.5111111111)\right] \\ &= \pi\,[46.575 - 4.0556666667 \\ &\quad -3.8678159212 \cdot 20.25 \cdot 0.5364770134] \\ &= \pi\,[46.575 - 4.0556666667 - 42.0186352592] \\ &= \pi\,[0.5006980741] \\ &= 1.5729893913\, units^3 \end{aligned}$$

11. Use equation 15 to calculate the volume of the circle intersection from V_{to} above.

$$\begin{aligned} V_{ci} &= 2 \cdot V_{to} \\ &= 2 \cdot 1.5729893913 \\ &= 3.1459787825\, units^3 \end{aligned}$$