A-004
Regarding Classification of Coplanar Circle Tangents

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Understanding how circles and lines join together has always been useful in: surveying, construction, civil engineering and architecture.

When a circle and a line contact at a single point the meeting point is called a "tangent point".

Multiple circles and lines can be joined, at their tangent points, into complex paths or shapes.

This article classifies how pairs of circles can be joined together by tangent lines.

1 Problem

A previous article[1] is the foundation for the discussion that follows. The reader is encouraged to refer to it frequently.

The "Given" section here is repeated from the previous. However the "Find" section here seeks answers to different, yet related, questions.

Given:

- Two coplanar circles A and B,
- *with* center points C_A and C_B respectively,
- *and* having radii r_A and r_B respectively,
- *where* $r_A < r_B$;

Find:

1. The ways A and B can be joined together by tangent lines.
2. A description of each scenario.
3. Locations of critical points.
4. A classification of the cases described in article[1].

2 Solution

As shown in Figure 1 paired circles, A and B, can be gathered into 4 classes. The classes are based upon the relationship between paired circles and their common tangent line(s), if any.

Class A: Null

No common tangent line between circles A and B is possible. This is due to two reasons.

First, any line tangent to A will contact B at two points. The points where the line contacts B will be points of intersection rather than points of tangency.

Second, a line tangent to B cannot pass through B. Since A lies entirely within B any line tangent to B cannot contact A.

Cases 0, 1, 2 and 3; from article[1] are examples of this class.

Class B: Single

Where circle A contacts B at a single point, A and B will be tangent to each other.

[1] A-003, Regarding Coplanar Circles, Vincent W. Finley, January 2022

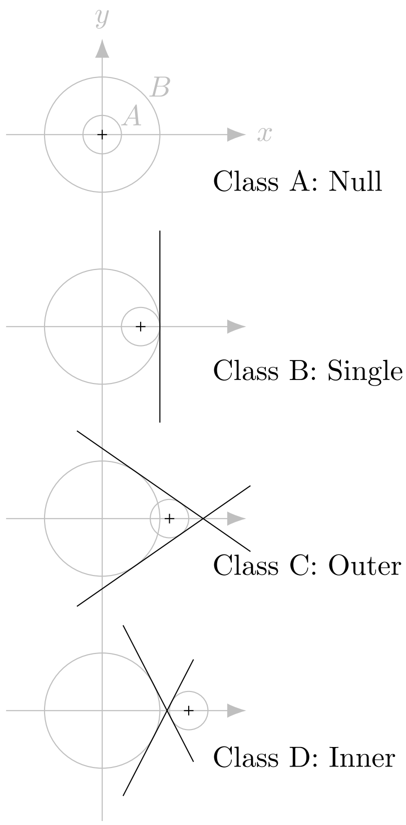

Figure 1: Tangent Classes

Any line that is tangent to B at the point of contact will also be tangent to A.

A line tangent to a circle will always intersect a radius line at right angles[2]. Any line that is tangent to B will be at right angles to the radius line from the center of B to the tangent point.

Since r_B is given as the the radius of B, the tangent point of any line tangent to B will lie a distance of r_B from the center of B. A coordinate system can be arbitrarily constructed such that the origin coincides with the center of B, and the x-axis coincides with the line between the centers of A and B. In such a construction the tangent point will lie at a point $(r_B, 0)$.

Furthermore the center of A must lie a distance r_A from tangent point $(r_B, 0)$. Therefore the center of A can lie at locations:

$$\boxed{C_A = (r_B \pm r_A, 0)} \tag{1}$$

These two positions correspond to cases 4 and 8 of article[1].

[2] ISBN 0-8311-2575-6, Machinery's Handbook, page 46

As the center of A moves from positions covered by Class C and Class D, toward positions covered by Class B, pairs of tangents degenerate into the single tangent line detailed by Class B.

Cases 4 and 8; from article[1] are examples of this class.

Class C: Outer

The point of intersection for a pair of tangents lays outside the line segment connecting two circle centers, thus the name of the class. It should be noted: the arrangement for this tangent class is identical to the arrangement for fillets[3].

However, there are two important difference between fillets and outer tangents. First, when examining fillets the point and angle of intersection, I and θ, are usually known, while the center point of the fillet, C, is sought. The opposite is true for outer tangents. Second, a fillet has only one circle. Whereas the outer tangent class involves tangents common to two circles.

Due to these differences a geometric approach, based on similar triangles, is taken here.

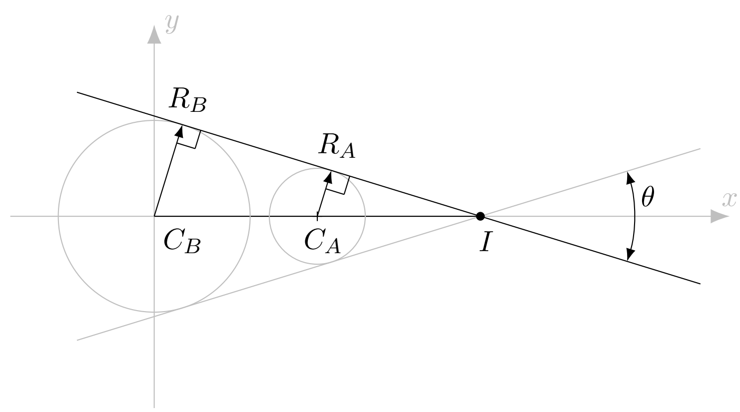

Figure 2: Arrangement of Outer Tangents

Similar triangles are are scaled versions of each other. They are triangles whose corresponding angles are the congruent, yet whose corresponding sides are proportional. In other words the respective angles of two similar triangles are equal. However, the respective sides are multiples of

[3] A-002, Regarding Fillets, Vincent W. Finley, February 2022

each other. That is, the ratio of corresponding sides will always be equal.

If Circles A and B are arranged as shown in Figure 2, a pair of tangent lines can be drawn to connect them. The tangent lines will be at right angles to radius lines from the circle center points to the tangent points.

The upper tangent line will contact A and B at tangent points R_A and R_B respectively. It will lie at right angles to the respective radius lines.

Each tangent line is a reflection of the other. Together they are symmetric about the x-axis. Both intersect the x-axis at some point $I = (x_I, 0)$. Included angle θ, whose vertex is point I, results from the intersection.

Triangles $\triangle IC_AR_A$ and $\triangle IC_BR_B$ are similar triangles, where side $\overline{IC_A}$ corresponds to $\overline{IC_B}$, and side $\overline{C_AR_A}$ corresponds to $\overline{C_BR_B}$.

Let r_A be the length of $\overline{C_AR_A}$, such that $r_A = C_AR_A$. Likewise: let $r_B = C_BR_B$ and let $x_I = IC_B$. Moreover, the center of A, $C_A = (x_A, 0)$, and length of $\overline{IC_A}$ is $IC_A = x_I - x_A$.

Since triangles $\triangle IC_AR_A$ and $\triangle IC_BR_B$ are similar, the ratio of their corresponding sides are equal. Therefore:

$$\frac{IC_A}{IC_B} = \frac{C_AR_A}{C_BR_B} \tag{2}$$

$$\begin{aligned} \frac{x_I - x_A}{x_I} &= \frac{r_A}{r_B} \\ r_B(x_I - x_A) &= r_A \cdot x_I \\ r_B \cdot x_I - r_B \cdot x_A &= r_A \cdot x_I \\ r_B \cdot x_I - r_A \cdot x_I &= r_B \cdot x_A \\ x_I(r_B - r_A) &= r_B \cdot x_A \end{aligned}$$

$$\boxed{x_I = \frac{r_B \cdot x_A}{r_B - r_A}} \tag{3}$$

If the radii of two circles are known, and the distance that separates circle centers is known, then Equation 3 above can be used to find the location where the tangent lines will intersect at point:

$$\boxed{I = (x_I, 0)} \tag{4}$$

By recognizing the fact that two tangent lines together with a circle form a fillet, the angle of intersection θ can then be found by rearranging Equation 4 from article[3], and substituting variables for triangle $\triangle IC_BR_B$ as follows:

$$\theta = 2 \cdot \arcsin\left(\frac{r}{\Delta x}\right)$$

$$\boxed{\theta = 2 \cdot \arcsin\left(\frac{r_B}{x_I}\right)} \tag{5}$$

The tangent points: R_A, R_B, and their reflections; can likewise be found using Equations 7 and 8 from article[3].

Cases 5, 6, 7, 8 and 9; from article[1] are examples of this class.

Class D: Inner

This class is characterized by the point of intersection laying inside the line segment connecting two circle centers.

By inspecting Figures 2 and 3 the relationship between Inner tangents and Outer tangents becomes immediately obvious.

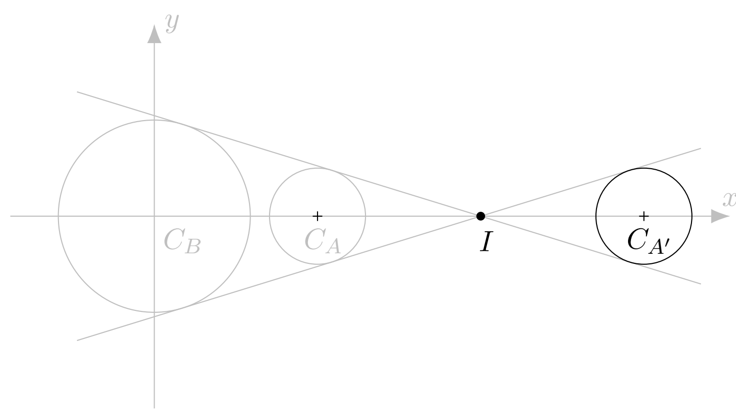

Figure 3: Arrangement of Inner Tangents

When I lays between two circles, the arrangement is as if circle A were reflected about point I to a new circle A'. Let the center location of A' be named $C_{A'}$. Meanwhile radius of the reflection, A', remains equal to r_A, the radius of A.

The location $C_{A'}$ is simply the interval between $C_{A'}$ and I added to the location of I, as

Case	Number of Tangents					Description
	(A) Null	(B) Single	(C) Outer	(D) Inner	Total	
0-3	0	-	-	-	0	No tangents possible
4	-	1	-	-	1	Circles tangent to each other
5-7	-	-	2	-	2	Two outer tangents
8	-	1	2	-	3	Circles tangent to each other, and 2 outer tangents
9	-	-	2	2	4	Two outer and two inner tangents

Table 1: Classification of Coplanar Circle Cases

follows:

$$x_{A'} = (x_I - x_A) + x_I \tag{6}$$
$$x_{A'} = 2x_I - x_A \tag{7}$$

Given two circles, the strategy of finding I for a pair of inner tangents is as follows.

1. Let B be the larger circle.
2. Let A' be the smaller circle.
3. Assume A' has a reflection thru I named A.
4. Use Equation 7 to find the location of reflection A.
5. Substitute the location of A into Equation 3.
6. Rearrange terms and solve for x_I.

Solving Equation 7 for x_A yields:

$$\boxed{x_A = 2x_I - x_{A'}} \tag{8}$$

Substituting Equation 8 into Equation 3 yields:

$$x_I = \frac{r_B \cdot (2x_I - x_{A'})}{r_B - r_A}$$
$$x_I = \frac{2r_B \cdot x_I - r_B \cdot x_{A'}}{r_B - r_A}$$
$$x_I \cdot (r_B - r_A) = 2r_B \cdot x_I - r_B \cdot x_{A'}$$

$$x_I \cdot r_B - x_I \cdot r_A - 2r_B \cdot x_I = -r_B \cdot x_{A'}$$
$$x_I \cdot (r_B - r_A - 2r_B) = -r_B \cdot x_{A'}$$
$$x_I \cdot (-r_B - r_A) = -r_B \cdot x_{A'}$$

Finally x_I is found to be:

$$\boxed{x_I = \frac{r_B \cdot x_{A'}}{r_B + r_A}} \tag{9}$$

Substituting Equation 9 into Equation 4 gives the location of I. Because the location of point I has been found, Equation 5 can be used to find the angle θ. Equations 7 and 8 from article[3] can be used to find the tangent points.

Case 9 from article[1] is an example of this class.

Classification

Now that four tangent classes have been defined, classifying the ten coplanar circle-pairs (identified by article[1]) is desired. Table 1 summarizes which tangent classes apply to each of the ten circle-pair cases.

Either one or two tangent classes many apply to a circle-pair case. Generally, the number of total tangents increases as a circle A retreats from circle B. Depending upon which tangent classes apply, each case may have a total of zero to four possible tangents.

The classification of coplanar circle-pairs described here provides a convenient framework for future planar-geometry discussion.

Q.E.D.

3 Example

Given:

- Two circles A and B whose radii are $r_A = 1.0$ and $r_B = 3.0$ units respectively,
- *and* circle centers C_A and C_B separated by a distance of 6.5 units.

Find:

- Which coplanar circle-pair case best matches the given arrangement.
- Which tangent classes apply to the circle-pair case.
- The location(s) where tangent lines intersect.
- The angle of intersection between tangent lines.

Steps:

1. Let:

$$r_A = 1.0$$
$$r_B = 3.0$$
$$x_A = 6.5$$

2. From the solution section of article[1], use the inequality for case 9:

$$x_A > r_B + r_A$$

3. Plug in values for r_A and r_B,

$$6.5 > 3.0 + 1.0$$

$$6.5 > 4.0$$

4. Since $x_A = 6.5$ is greater than the sum of the radii, 3.0 and 1.0, the inequality for circle-pair case 9 is satisfied. Therefore the relationship between circles A and B are classified by the case 9 circle-pair.

5. From table 1 tangent classes (C) outer and (D) inner apply to circle-pair case 9. Therefore circles A and B are connected by two outer tangent lines and two inner tangent lines.

6. Outer tangents:
The point of intersection I_{outer} for the outer tangent pair is found by using Equation 3 to get the x coordinate. Substituting values from above yields.

$$x_I = \frac{3.0 \cdot 6.5}{3.0 - 1.0}$$
$$x_I = 9.75$$

Then x_I can be substituted into Equation 4 to find the point I_{outer}.

$$\boxed{I_{outer} = (9.75, 0)}$$

The included angle between the outer tangent pair can be calculated from Equation 5.

$$\theta_{outer} = 2 \cdot \arcsin\left(\frac{3.0}{9.75}\right)$$
$$\theta_{outer} = 0.62553\ rads$$

$$\boxed{\theta_{outer} = 35.84°}$$

7. Inner tangents:
The point of intersection I_{inner} for the inner tangent pair is found by using Equation 9 to get the x coordinate. Taking x_A to be $x_{A'}$ in this context, and substituting values from above yields.

$$x_I = \frac{3.0 \cdot 6.5}{3.0 + 1.0}$$
$$x_I = 4.875$$

Then x_I can be substituted into Equation 4 to find the point I_{inner}.

$$\boxed{I_{inner} = (4.875, 0)}$$

The included angle between the inner tangent pair can be calculated from Equation 5.

$$\theta_{inner} = 2 \cdot \arcsin\left(\frac{3.0}{4.875}\right)$$
$$\theta_{inner} = 1.32575\ rads$$

$$\boxed{\theta_{inner} = 75.96°}$$

8. Finally, all tangent points can be found by referring to Equations 7 and 8 from article[3]. This is left as an exercise for the reader.