A-001
Regarding Circles and Tangent Lines

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Occasionally, one may need to locate the point where a straight line is tangent to a circle. Scenarios such as: preparing a mechanical drawing, designing a belt and pulley system, or planning a highway; come to mind.

1 Problem

Given:

- Circle C of radius r,
- *and* line L sloped at an angle α with respect to the horizontal,
- *and* point P where L is tangent to C;

Find: The location of P relative to the center of C.

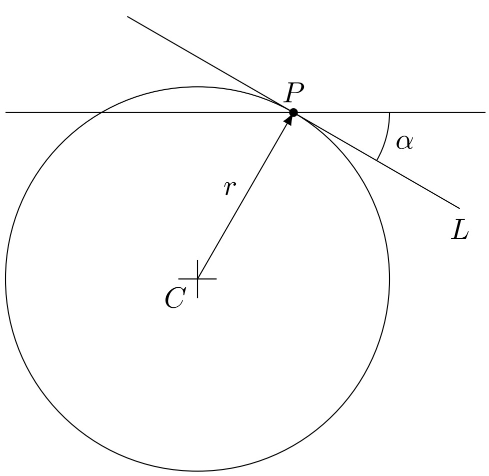

Figure 1: Line L tangent to circle C

2 Solution

1. Draw axes, placing the origin at center of C, such that point $C = (0, 0)$.
2. Project point P vertically downwards to a point I on the x-axis. Points: C, I and P; form a right-triangle CIP having sides: Δx, Δy and r.

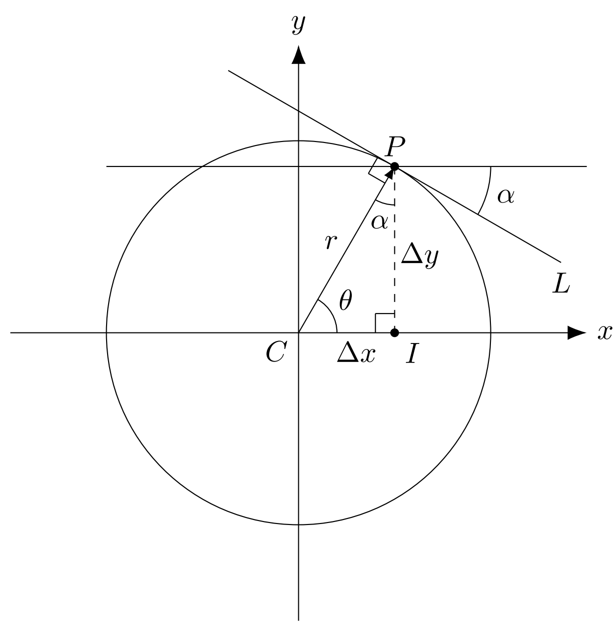

Figure 2: Point P projected onto x-axis

3. All points on a circle lie a distance r from the center, the distance r being the radius of the circle. Furthermore, a point on the circle will lie at some angle θ from the positive x-axis. In other words, point P on the circle will be located at polar coordinate $P = (r, \theta)$.

4. A line tangent to a circle will always intersect radius line r at right angles.[1]. Which means, L and r will intersect at right angles at point $P = (r, \theta)$.

5. Since L is at an angle α with respect to the x-axis (or parallel to x-axis), *and* r is right angles to L, then r will be at an angle α with respect to the y-axis (or parallel to y-axis).

6. Aside from having a right angle, triangle CIP has angles θ and α. The sum of θ and $|\alpha|$ must be $\pi/2$. However since the actual value of α can be negative with respect to the horizontal, $\theta - \alpha = \pi/2$ is used here. Furthermore $\theta = \pi/2 + \alpha$.

7. Using the fundamental trigonometric functions, polar coordinates can be easily converted to Cartesian coordinates, such that $P = P_{r,\theta} = (r, \theta)$ and $P = P_{x,y} = (\Delta x, \Delta y)$, as follows:

From:

$$\cos(\theta) = \frac{\Delta x}{r}$$
$$\sin(\theta) = \frac{\Delta y}{r}$$

it is found,

$$\Delta x = r \cdot \cos(\theta) \tag{1}$$
$$\Delta y = r \cdot \sin(\theta) \tag{2}$$

and since:

$$\theta - \alpha = \frac{\pi}{2}$$
$$\theta = \frac{\pi}{2} + \alpha \tag{3}$$

substituting equation 3 into equations 1 and 2 yields:

$$\Delta x = r \cdot \cos\left(\frac{\pi}{2} + \alpha\right)$$
$$\Delta y = r \cdot \sin\left(\frac{\pi}{2} + \alpha\right)$$

Therefore, from above:

$$P = (\Delta x, \Delta y)$$

$$\boxed{P = \left(r \cdot \cos\left(\frac{\pi}{2} + \alpha\right), r \cdot \sin\left(\frac{\pi}{2} + \alpha\right)\right)} \tag{4}$$

Q.E.D.

[1]ISBN 0-8311-2575-6, Machinery's Handbook, page 46

3 Example

Given:

- A line L tangent to circle C whose radius $r = 3.75$,

- *and* L is at an angle $\alpha = -12.5°$ from the horizontal;

Find: The polar and cartesian coordinates of tangent point P.

Steps:

1. First, α must be converted from degrees to radians:

$$\alpha_{rad} = \alpha_{deg}\left(\frac{\pi}{180°}\right)$$
$$\alpha_{rad} = -12.5°\left(\frac{\pi}{180°}\right)$$
$$\alpha_{rad} = -0.218166$$

2. Substituting values for r and α yields:

$$P_{r,\theta} = (r, \theta)$$
$$P_{r,\theta} = \left(r, \frac{\pi}{2} + \alpha\right)$$
$$P_{r,\theta} = \left(3.75, \frac{\pi}{2} + -0.218166\right)$$
$$P_{r,\theta} = \left(3.75, \frac{\pi}{2} - 0.218166\right)$$
$$P_{r,\theta} = (3.75, 1.570796 - 0.218166)$$
$$P_{r,\theta} = (3.75, 1.352630)$$

3. Substituting values into equation 4 yields:

$$P_{x,y} = \left(r \cdot \cos\left(\frac{\pi}{2} + \alpha\right), r \cdot \sin\left(\frac{\pi}{2} + \alpha\right)\right)$$
$$P_{x,y} = (r \cdot \cos\left(\frac{\pi}{2} + -0.218166\right),$$
$$r \cdot \sin\left(\frac{\pi}{2} + -0.218166\right))$$
$$P_{x,y} = (r \cdot \cos\left(\frac{\pi}{2} - 0.218166\right),$$
$$r \cdot \sin\left(\frac{\pi}{2} - 0.218166\right))$$
$$P_{x,y} = (r \cdot \cos(1.570796 - 0.218166),$$
$$r \cdot \sin(1.570796 - 0.2181662))$$
$$P_{x,y} = (r \cdot \cos(1.352630), r \cdot \sin(1.352630))$$
$$P_{x,y} = (3.75 \cdot \cos(1.352630), 3.75 \cdot \sin(1.352630))$$
$$P_{x,y} = (3.75 \cdot 0.216440, 3.75 \cdot 0.976296)$$
$$P_{x,y} = (0.811649, 3.661110)$$