Basic definitions: Strings of non-empty sets

Work with strings $s = \alpha_1 \cdots \alpha_n$ of sets α_i, boxed. E.g., $\Box$ is $\emptyset = \{\}$ *qua* symbol.

$$(\Box\text{-drop}) \qquad d(s) \;:=\; s \text{ with each } \alpha_i = \Box \text{ deleted}$$

For any set A,

$$\begin{array}{rrl}
(A\text{-strips}) & \mathcal{L}_A & := \{d(s) \mid s \in (2^A)^*\} \;=\; (2^A - \{\Box\})^* \\
(A\text{-reduct}) & \rho_A(s) & := (\alpha_1 \cap A) \cdots (\alpha_n \cap A) \\
(A\text{-projection}) & d_A(s) & := d(\rho_A(s)) \\
(A\text{-preimage}) & [\![(A, s)]\!]_{A'} & := \{s' \in \mathcal{L}_{A'} \mid d_A(s') = s\}
\end{array}$$

Superposition as intersection with granularity

$$\begin{aligned}
s \;\&_{A,A'}\; s' &:= [\![(A, s)]\!]_{A\cup A'} \cap [\![(A', s')]\!]_{A\cup A'} \\
&= \{s'' \in \mathcal{L}_{A\cup A'} \mid d_A(s'') = s \text{ and } d_{A'}(s'') = s'\}
\end{aligned}$$

which lifts to languages (i.e. string sets) L, L'

$$\begin{aligned}
L \;\&_{A,A'}\; L' &:= \bigcup_{s\in L} \bigcup_{s'\in L'} s \;\&_{A,A'}\; s' \\
[\![(A, L)]\!]_{A'} &:= \{s' \in \mathcal{L}_{A'} \mid d_A(s') \in L\} \;=\; \bigcup_{s\in L} [\![(A, s)]\!]_{A'}
\end{aligned}$$

so that

$$\begin{aligned}
L \;\&_{A,A'}\; L' &= [\![(A, L)]\!]_{A\cup A'} \cap [\![(A', L')]\!]_{A\cup A'} \\
s \;\&_{A,A'}\; s' &= \{s\} \;\&_{A,A'}\; \{s'\}
\end{aligned}$$

and the special case $A = A'$ leads to intersection

$$\begin{aligned}
L \;\&_{A,A}\; L' &= [\![(A, L)]\!]_A \cap [\![(A, L')]\!]_A \\
&= [\![(A, L \cap L')]\!]_A \\
&= L \cap L' \text{ provided } L \cup L' \subseteq \mathcal{L}_A \\
s \;\&_{A,A}\; s' &= \begin{cases} \{s\} & \text{for } s = s' \in \mathcal{L}_A \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}$$

Different granularities $A \neq A'$ are useful in Allen interval relations, where a string s has default granularity

$$(\text{vocabulary}) \qquad voc(\alpha_1 \cdots \alpha_n) \;:=\; \bigcup_{i=1}^{n} \alpha_i$$

and often $voc(s) \neq voc(s')$ for $s \neq s'$

$$\boxed{l(i)}\boxed{r(i)} \;\&_{\{l(i),r(i)\},\{l(j),r(j)\}}\; \boxed{l(j)}\boxed{r(j)} \;=\; 13 \text{ strings, one per Allen}$$

relation; e.g. meet(i, j)

$$\boxed{l(i)}\boxed{r(i), l(j)}\boxed{r(j)}$$

Computing superposition Generate $\&_{A,A'} \subseteq \mathcal{L}_A \times \mathcal{L}_{A'} \times \mathcal{L}_{A \cup A'}$ so that

$$L \;\&_{A,A'}\; L' = \{s'' \mid (\exists s \in L)(\exists s' \in L')\; \&_{A,A'}(s, s', s'')\}$$

Below: $\alpha \subseteq A$, $\alpha' \subseteq A'$ (making the side conditions vacuous if $A \cap A' = \emptyset$[1])

$$\frac{}{\&_{A,A'}(\epsilon, \epsilon, \epsilon)} \tag{s0}$$

$$\frac{\&_{A,A'}(s, s', s'')}{\&_{A,A'}(\alpha s, \alpha' s', (\alpha \cup \alpha')s'')} \;\; \alpha \cap A' \subseteq \alpha' \quad \alpha' \cap A \subseteq \alpha \tag{s1}$$

$$\frac{\&_{A,A'}(s, s', s'')}{\&_{A,A'}(\alpha s, s', \alpha s'')} \;\; \alpha \cap A' = \emptyset \tag{d1}$$

$$\frac{\&_{A,A'}(s, s', s'')}{\&_{A,A'}(s, \alpha' s', \alpha' s'')} \;\; \alpha' \cap A = \emptyset \tag{d2}$$

Python implementations superp.py and for Allen relations, allenD.py

```
>>> from allenD import *
>>> test()
Allen relations from [{'l0'}, {'r0'}] superposed with [{'l1'}, {'r1'}]
  b [{'l0'}, {'r0'}, {'l1'}, {'r1'}]
  m [{'l0'}, {'l1', 'r0'}, {'r1'}]
  o [{'l0'}, {'l1'}, {'r0'}, {'r1'}]
  fi [{'l0'}, {'l1'}, {'r1', 'r0'}]
  di [{'l0'}, {'l1'}, {'r1'}, {'r0'}]
  s [{'l0', 'l1'}, {'r0'}, {'r1'}]
  eq [{'l0', 'l1'}, {'r1', 'r0'}]
  si [{'l0', 'l1'}, {'r1'}, {'r0'}]
  d [{'l1'}, {'l0'}, {'r0'}, {'r1'}]
  f [{'l1'}, {'l0'}, {'r1', 'r0'}]
  oi [{'l1'}, {'l0'}, {'r1'}, {'r0'}]
  mi [{'l1'}, {'l0', 'r1'}, {'r0'}]
  bi [{'l1'}, {'r1'}, {'l0'}, {'r0'}]
Allen transitivity table tt(r1,r2)
e.g. tt("b",d") = ['b', 'm', 'o', 's', 'd'] by superposing
  [{'l0'}, {'r0'}, {'l1'}, {'r1'}] (depicting 0 b 1) and
  [{'l2'}, {'l1'}, {'r1'}, {'r2'}] (depicting 1 d 2)
   0 b 2 from  [{'l0'}, {'r0'}, {'l2'}, {'l1'}, {'r1'}, {'r2'}]
   0 m 2 from  [{'l0'}, {'l2', 'r0'}, {'l1'}, {'r1'}, {'r2'}]
   0 o 2 from  [{'l0'}, {'l2'}, {'r0'}, {'l1'}, {'r1'}, {'r2'}]
   0 s 2 from  [{'l2', 'l0'}, {'r0'}, {'l1'}, {'r1'}, {'r2'}]
   0 d 2 from  [{'l2'}, {'l0'}, {'r0'}, {'l1'}, {'r1'}, {'r2'}]
>>>
```

[1] In which case $s \;\&_{A,A'}\; s'$ is $\bigcup_{n \geq 0}\{d(s_1 \sqcup s_2) \mid s_1 \in \mathcal{L}_{A,n}(s)$ and $s_2 \in \mathcal{L}_{A',n}(s')\}$ where $\mathcal{L}_{B,n}(x)$ is $\{y \in (2^B)^* \mid \text{length}(y) = n$ and $d(y) = x\}$ and $\sqcup$ is componentwise union

$$\alpha_1 \cdots \alpha_n \sqcup \alpha'_1 \cdots \alpha'_n \;:=\; (\alpha_1 \cup \alpha'_1) \cdots (\alpha_n \cup \alpha'_n).$$