5. Logical Reasoning and Predicate Logic

The story of week 5

- Reasoning with Propositional Logic
  - More examples
- Introducing predicates
- The universal quantifier, ∀
- The existential quantifier, ∃
- Negating quantifiers
- Optional Extra
  - Typing predicates
  - Fermat's Last Theorem

Recap: propositional logic

- Propositions are true or false
- We have connectives: $\vee, \wedge, \longrightarrow, \neg, \longleftrightarrow$

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \longrightarrow q$	$\neg p \vee q$	$p \longleftrightarrow q$
0	0	1	0	0	1	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	0	0	0
1	1	0	1	1	1	1	1

- Tautologies, contradictions/fallacies, satisfiability, equivalence

Reasoning: modus ponens

- $(p \wedge (p \longrightarrow q)) \longrightarrow q$

This is a valid argument.
You can show it using a truth table.

In our proofs we can take these 5 rules as tautologies so we can write

$$\begin{array}{l} p \\ \underline{p \longrightarrow q} \\ \therefore q \end{array}$$

(using Modus Ponens)

Here are Five Rules of Reasoning you can use in Logical Proof

Rule1: Modus ponens: $(p \wedge (p \longrightarrow q)) \longrightarrow q$

Rule2: Syllogism: $((p \longrightarrow q) \wedge (q \longrightarrow r)) \longrightarrow (p \longrightarrow r)$

Rule3: Modus tollens: $((p \longrightarrow q) \wedge \neg q) \longrightarrow \neg p$

Rule4: Disjunctive syllogism: $((p \vee q) \wedge \neg p) \longrightarrow q$

Rule5: Reductio ad absurdum: $(\neg p \longrightarrow 0) \longrightarrow p$

(Ps. You can use truth tables to show these are all valid arguments!)

Exercise Recap from week 4

- Suppose:

  p means the screen is off

  r means processes are running

  q means the music has stopped

  s means power is being used

- And that the following are ALL true

  $\neg p \longrightarrow r$ If the screen is on then processes are running

  $r \longrightarrow s$ If processes are running then power is being used

  $p \longrightarrow q$ If the screen is off then the music has stopped

- Show that if the music has not stopped then power is being used

  $\neg q \longrightarrow s$

What do we know is true?

You know the rules you are given are true, and you are also considering the case when you know “not q” is true. So you know FOUR things and you must FIND THE CORRECT RULES to prove that the statement “s” is true. Here we go:

We KNOW the following 4 things are true

$\neg p \longrightarrow r$ $\quad$ $r \longrightarrow s$ $\quad$ $p \longrightarrow q$ $\quad$ $\neg q$

Now we need to APPLY the correct rules in the correct order to show that the truth of “s” follows logically

$\neg p \longrightarrow r$ $\quad r \longrightarrow s$ $\quad p \longrightarrow q$ $\quad \neg q$

Step 1: Using Syllogism (Rule2) we have

$\boldsymbol{\neg p \longrightarrow s}$

Step 2: Using Modus tollens (Rule 3) we have

$\boldsymbol{\neg p}$

Step 3: Using Modus ponens we then have

s

(QED! Or We are home!!)

Short hand

What we know

$\neg p \longrightarrow r$ $\quad\quad$ $\neg q$

$\underline{r \longrightarrow s}$ $\quad\quad$ $\underline{p \longrightarrow q}$

What we can deduce

$\underline{\therefore \neg p \longrightarrow s}$ (Syllogism)

$\underline{\therefore \neg p}$ (Modus tollens)

$\underline{\therefore s}$ (Modus ponens)

- This has used Syllogism, Modus tollens and Modus ponens
- QED!!

Closer look at Rule 2: The Law of Syllogism

The Law of the Syllogism

A second rule of inference is given by the logical implication

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Where **p,q,r** are any statements. In tabular form, it is written

$p \rightarrow q$
$\underline{q \rightarrow r}$
$\therefore\ p \rightarrow r$

This rule is referred to as the Law of the Syllogism. For example,

1) If it rains then I will get wet.
2) If I get wet then my clothes will be ruined.
3) Therefore, if it rains, my clothes will get ruined.

Rule 2: The Law of Syllogism

Step 1: Modus Ponens – the first two statements seem to fit a Modus Ponens rule

$\mathbf{p}$
$\underline{\mathbf{p \rightarrow \neg q}}$
$\therefore \mathbf{\neg q}$

Step 2: Syllogism – the second two statements seem to fit this law

$\mathbf{p \rightarrow \neg q}$
$\underline{\mathbf{\neg q \rightarrow \neg r}}$
$\therefore \mathbf{p \rightarrow \neg r}$

Step 3: **Modus Ponens.**

$\mathbf{p}$ **(1)**
$\underline{\mathbf{p \rightarrow \neg r}}$ **(2)**
$\therefore \mathbf{\neg r}$ **(3)**

QED! Or "We are home"!
This method of combining inference rules is called Natural Deduction.

A closer look at Rule 3: Modus Tollens

Modus Tollens (from the Latin – meaning “method of denying”).

This follows from the logical implication:

$$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$$

We can see using a truth table that it is a valid argument because we get a tautology!

p	**q**	**¬p**	**¬q**	**p → q**	**(p→q)∧¬q**	**[(p→q)∧¬q]→¬p**
0	0	1	1	1	1	1
0	1	1	0	1	0	1
1	0	0	1	0	0	1
1	1	0	0	1	0	1

Example

1) If it rains today (**p**), then I will travel by tube (**q**).
2) I didn’t travel by tube.
3) Therefore, it didn’t rain today.

A closer look at Rule 3: Modus Tollens

Beware the other way round!

Beware of invalid arguments.

1) If it rains today, then I will travel by tube.
2) I travelled by tube.
3) Therefore, it rained today.

This may be expressed as:

$$[(p \to q) \land q] \to p$$

Let's examine this using truth tables. It is not a tautology and so NOT a valid argument.

p	q	¬p	¬q	p → q	(p→q)∧¬q	[(p→q)∧¬q]→¬p	(p→q)∧q	[(p→q)∧q]→p
0	0	1	1	1	1	1	0	1
0	1	1	0	1	0	1	1	0
1	0	0	1	0	0	1	0	1
1	1	0	0	1	0	1	1	1

A closer look at Rule 4: **Disjunctive Syllogism**

The Rule of Disjunctive Syllogism

The rule of disjunctive syllogism might be called applied common sense, symbolically is may be stated as:

$$[(p \vee q) \wedge \neg p] \rightarrow q$$

Example

1) My passport is either in my bag or in my pocket.
2) My passport is not in my bag.
3) Therefore, it must be in my pocket.

A closer look at Rule 5:
The Rule of Contradiction (Reductio ad absurdum)

This rule may be stated as:

$$\frac{\neg \mathbf{p} \rightarrow \mathbf{0}}{\therefore \mathbf{p}}$$

This rule tells us that if **¬p** leads to a falsehood (which is the opposite of a tautology), then **p** must be true.
We can see this by creating a truth table.

p	**¬p**	**0**	**¬p → 0**	**(¬p → *F*) → p**
1	0	0	1	1
0	1	0	0	1

A closer look at Rule 5: **The Rule of Contradiction (Reductio ad absurdum)**

Frequently used as a method of proof in mathematics.

Assume that what we are trying to validate is false.

Use this in order to prove a contradiction (or impossible situation).

Once this has been derived we can establish that the statement given was true.

A closer look at Rule 5: **The Rule of Contradiction (Reductio ad absurdum) Example**

If **n** is an integer such that $\mathbf{n^2}$ is a multiple of 3. Prove that **n** is also a multiple of 3.

Proof: Suppose that $\mathbf{n^2}$ is a multiple of 3 BUT **n** is not a multiple of 3.

So, we divide **n** by 3, we get a remainder of either 1 or 2

So either n=1 +3k or **n=2+3k** for some integer **k** or

Case 1 $\mathbf{n^2=(1+3k)^2 = 1 + 6k + 9k^2 = 1 +3(2k + 3k^2)}$

But this means that $\mathbf{n^2}$ is 1 more than a multiple of 3, which is false, as we are given that $\mathbf{n^2}$ is a multiple of 3.

Case 2 $\mathbf{n^2=(2+3k)^2 = 4 + 12k + 9k^2 = 1 +3(1+4k + 3k^2)}$

which again is false as $\mathbf{n^2}$ is a multiple of 3 so cannot be one more than a multiple of 3!

We have shown that assuming **n** is not a multiple of 3 leads to a false statement. Hence we have demonstrated that **n** is a multiple of 3, by reductio ad absurdum.

Predicates

- A predicate is a statement about an object or set of objects (expressed as variables) which will maps to true or false once the variable has been instantiated
- Can be a statement about a single object or a relationship between objects from different sets
- For example,
  - blue(light)
  - red(x)
  - colour(apple, green)
  - colour(x,y)

Universal quantifier, $\forall$

- $\forall$, pronounced as "**for all**"
- A statement about all members of a set
- "All lemons are yellow"
- $\forall x: (\text{lemon}(x) \longrightarrow \text{yellow}(x))$

Exercise

- Express the following using universal quantification

1. “Every planet orbits a star”
2. “For every animal, if it is a mammal and it can fly then it is a bat”
3. “Every amphibian is either a frog or a toad or a newt”

Exercise

1. $\forall x$: (planet(x) $\longrightarrow$ orbitstar(x))
2. $\forall x$: ((mammal(x) $\wedge$ fly(x)) $\longrightarrow$ bat(x))
3. $\forall x$: (amphibian(x) $\longrightarrow$
   (frog(x) $\vee$ toad(x) $\vee$ newt(x)))

Existential quantifier, ∃

- ∃, pronounced as “**there exists**”
- States the existence of an element from a set satisfying a property
- “There exists a duck that cannot quack”
- $\exists x: (duc(x) \wedge \neg\ quack(x))$

First-order reasoning

$\forall x: (lemon(x) \longrightarrow yellow(x))$

$\underline{lemon(jiffy)}$

$\therefore yellow(jiffy)$

Negating quantifiers

- $\neg \forall x: p(x) \equiv \exists x: \neg p(x)$
- $\neg \exists x: p(x) \equiv \forall x: \neg p(x)$

Examples

- Consider the statement: **All birds can fly**. Its negation reads: **It is not the case that all birds can fly** – and *NOT* **No birds can fly**

- We can also phrase **It is not the case that all birds can fly** as **There exists at least one bird that cannot fly.**

- Hence we can say: $\neg\forall \mathbf{x} : \mathbf{fly(x)} \leftrightarrow \exists \mathbf{x} : \neg\mathbf{fly(x)}$

Typing Predicates

Set of People P

Set of Times T

Predicate "Fool (p, t)" is true if person p is fooled at

Time T

You can fool some of the people some of the time

$\exists$ p : P, t : T . Fool (p, t)

You can't fool all of the people all of the time

$\neg \forall$ p : P, t : T . Fool (p, t)

Transitive Relations

- We can define what it is for a relationship to be transitive
  - $\forall a,b,c \in A: ((a,b) \in \boldsymbol{R} \land (b,c) \in \boldsymbol{R}) \rightarrow (a,c) \in \boldsymbol{R}$

- Negating this defines non-transitive relations

(hold onto your hats!)

- $\neg\forall a,b,c \in A: (\ (a,b) \in \boldsymbol{R} \land (b,c) \in \boldsymbol{R}\ ) \rightarrow (a,c) \in \boldsymbol{R}$
- $\exists a,b,c \in A: \neg(\ (\ (a,b) \in \boldsymbol{R} \land (b,c) \in \boldsymbol{R}\ ) \rightarrow (a,c) \in \boldsymbol{R}\ )$
- $\exists a,b,c \in A: \neg(\ \neg\ (\ (a,b) \in \boldsymbol{R} \land (b,c) \in \boldsymbol{R}\ ) \lor (a,c) \in \boldsymbol{R})$
- $\exists a,b,c \in A: (\ (a,b) \in \boldsymbol{R} \land (b,c) \in \boldsymbol{R}\ ) \land\ \neg(a,c) \in \boldsymbol{R}$
- $\exists a,b,c \in A: (\ (a,b) \in \boldsymbol{R} \land (b,c) \in \boldsymbol{R} \land (a,c) \notin \boldsymbol{R})$

Reflexive, Symmetric and Anti-Symmetric Relations over the natural numbers

- **Reflexive**
  - $\forall \mathbf{a} : \mathrm{N}\ .\ \ (\mathbf{a},\mathbf{a}) \in \boldsymbol{R}$
- **Symmetric**
  - $\forall \mathbf{a}, \mathbf{b} : \mathrm{N}\ .\ \ (\mathbf{a},\mathbf{b}) \in \boldsymbol{R} \longrightarrow (\mathbf{b},\mathbf{a}) \in \boldsymbol{R}$
- **Anti-Symmetric**
  - $\forall \mathbf{a}, \mathbf{b} : \mathrm{N}\ .\ \ (\mathbf{a},\mathbf{b}) \in \boldsymbol{R} \wedge (\mathbf{b},\mathbf{a}) \in \boldsymbol{R} \longrightarrow a = b$

Extra Topic: Fermats Last Theroem

Assessed MCQ 2

- The second assessed MCQ is next week
- It's about logic, covering this week and next week
- As before, 40mins, open book in your own time and on your own
- +4 for correct, -1 for incorrect, 0 for leave blank
- The quiz opens at 2pm Thursday 31st and closes 8pm Sunday 3rd (ie start by 7:20pm)

Assessed MCQ 2

- Let's watch the videos with my blue tooth speaker so you can hear this time!