Chapter 1

Truss Structures

1.1 Introduction

A truss structure, also called a *pin-jointed framework*, is a structure that consists of straight, slender members, known as *bars*, connected by frictionless spherical joints. These joints allow each bar to swivel and twist, unless these motions are restricted by connections to other bars. Joints of this kind are an abstraction, as real connections are usually rigidly jointed, but in fact many rigidly-jointed structures can be usefully modelled as pin-jointed. Changing the end conditions of the bars in this way has little effect on the overall response of the structure, provided that the pin-jointed version of the actual structure is *kinematically determinate* [1].

Consider a structure whose straight members are made from a high-modulus material and are rigidly connected to one another. Imagine releasing all of the rotational degrees of freedom at the connections. This change will have little effect on the overall response of the structure if the stress distribution in the constrained rotation members is mainly axial, and so the bending and shearing stresses are a "secondary", rapidly-decaying effect near the joints. This will generally be the case for a kinematically determinate structure. The stiffness of this structure is mainly derived from its *axial mode* of action, where its members are either in uniform tension or uniform compression, as the stiffness provided by the *bending mode* of action is several orders of magnitude smaller.

Thus, the global behaviour of the structure is fully captured by the pin-jointed model of the structure, whereas its local behaviour —which is important to determine local stress concentrations, and hence to check the safety of the structure against fracture and failure— needs a detailed stress analysis of the connections.

Figure 1.1 shows three planar structures used for a statical equilibrium experiment by the engineering undergraduates at the University of Cambridge: the geometrical layout of the three structures is the same, but their construction is different. The first structure is made of thin-walled, square section steel tubes with welded connections; the second is made of extruded Aluminium-alloy profiles bolted to gusset plates; and the third is made of thin-walled carbon-fibre-reinforced-plastic (CFRP) tubes (made by winding a carbon filament coated with epoxy onto a mandrel and curing the epoxy in a furnace) bonded to Aluminium-alloy joint fittings that are connected by steel pins.

During the experiment the undergraduates measure the axial forces in the members of the trusses, by comparing the strain in each member of a truss to the strain in the

[1] This concept will be explained in Section 1.2.

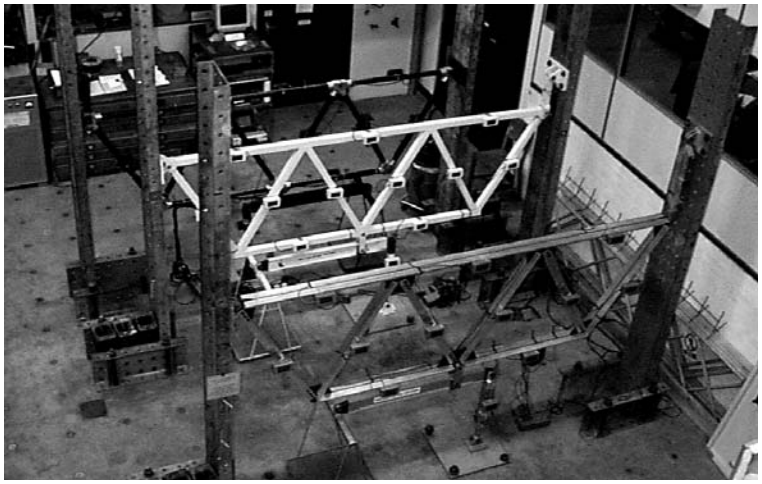


Figure 1.1: Photographs of three truss models in the Structures Laboratory at the University of Cambridge.

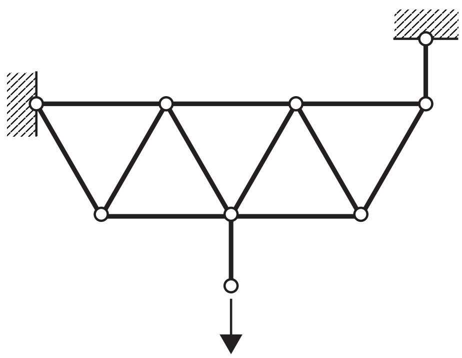


Figure 1.2: Idealised, pin-jointed model of truss structures shown in Figure 1.1.

directly-loaded, vertical link under the truss —of identical construction to the rest of the truss. They find that the axial forces in corresponding members are practically identical[2]. They also find that the values of the axial forces can be accurately estimated by analysing the truss shown in Fig. 1.2.

1.2 Rigidity Theory

The key question that we want to deal with is *whether or not a given pin-jointed framework is rigid.*[3] This straightforward question is a surprisingly difficult one to answer. Indeed, in many cases it can be answered in full only by carrying out a computer analysis, or by testing a physical model of the structure.

[2]Note that this experiment is concerned only with the linear-elastic behaviour of the three structures. If the loads were increased into the non-linear range, and up to failure, the behaviour of the three structures would no longer be identical.

[3]By rigid we mean a structure that does not deform at all if is assumed that the bars are inextensional, i.e. do not change their length.

In two dimensions, i.e. for the case of structures that lie in a plane, the easiest way of constructing a rigid truss is by arranging the bars to form a sequence of triangles. A triangle consisting of pin-jointed bars is the simplest two-dimensional rigid structure, and two triangles with a side in common also form a rigid structure in two dimensions (but not in three dimensions, as one of the triangles can move out of plane by rotating about the common side).

To extend this approach to three dimensions, i.e. to structures in a three-dimensional (Euclidean) space, one can use the simplest three-dimensional structure that is rigid, i.e. the tetrahedron (there will be more on this later); or alternatively one can form a closed surface that is completely triangulated. These approaches are often followed in the design of practical structures, but there are also many rigid structures that are not triangulated. Hence, it is of great importance to have a general way of telling whether or not a general three-dimensional structure is rigid. A general method to find the answer computationally will be given in Section 1.5.

A simpler question, that can be answered much more directly, is whether *a pin-jointed structure contains a sufficient number of members to be rigid*. The answer is to count the total number of degrees of freedom of its joints and to subtract the number of degrees of freedom suppressed by applying kinematic constraints, i.e. foundation constraints, to the joints, and by connecting pairs of joints by means of bars.

In two dimensions, each joint has two degrees of freedom, i.e. two independent translation components, and hence for a structure with j joints the total number of degrees of freedom is $2j$. Denoting by k the total number of kinematic constraints, where —for example— connecting a joint to a foundation counts as two because it suppresses both translation components, and by b the total number of pin-jointed bars —each bar counts one as it imposes a single "distance" constraint between the joints it connects— we require that

$$2j - k - b \leq 0 \tag{1.1}$$

This is known as Maxwell's equation (Maxwell, 1864). Consider, for example, the structure shown in Fig. 1.3(a). It consists of four triangles, the first of which is connected to a foundation, and hence it is obviously a rigid structure. Substituting $j = 6$, $k = 4$, $b = 8$ (obviously, there is no need for a bar between the two foundation joints) into Eq. 1.1 we obtain

$$2 \times 6 - 4 - 8 = 0$$

Hence, we conclude that this structure has (just) enough bars to be rigid.

It is important to realize that a structure that has enough bars to be rigid may not, in fact, be rigid, as its bars may be "incorrectly" placed. For example, if in Fig. 1.3(a) we re-locate the bar bracing the left-hand square, so that the right-hand square is now doubly-braced, as shown in Fig. 1.3(b), we obtain a structure that still satisfies Eq. 1.1 and yet is clearly not rigid. In this case we have a single-degree-of-freedom mechanism, Fig. 1.4(a). A structure that admits no mechanisms is called *kinematically determinate*.

Note that the doubly-braced square on the right-hand side of the structure in Fig. 1.3(b) admits a state of self-stress, i.e. there is a set of non-zero bar forces that are in equilibrium with zero external forces, as shown in Fig. 1.4(b). A structure that admits no states of self-stress is called *statically determinate*.

Denoting by m the number of independent mechanisms of a structure, and by s the number of states of independent states of self-stress, for the structure of Fig. 1.3(a) we have $s = 0$ and $m = 0$ (statically and kinematically determinate), whereas for the structure

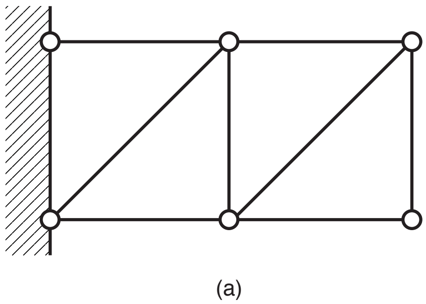


Figure 1.3: Examples of two-dimensional pin-jointed structures that are (a) fully triangulated and hence rigid, (b) a mechanism.

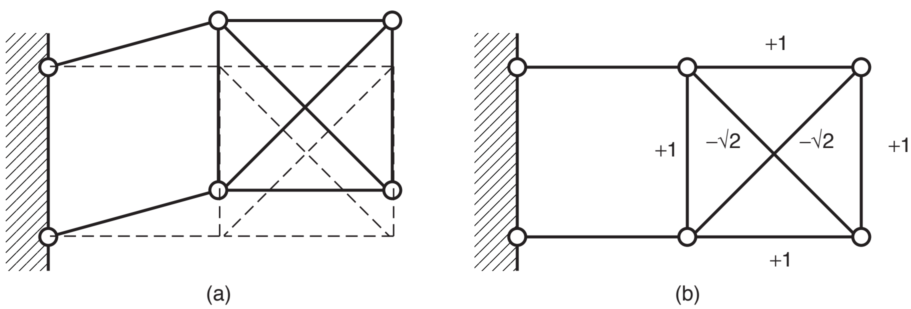


Figure 1.4: Mechanism (exaggerated amplitude of a small-amplitude motion) and state of self-stress of structure shown in Fig. 1.3(b).

of Fig. 1.3(b) we have $s = 1$ and $m = 1$ (statically and kinematically indeterminate). Here, by *independent* we mean that if any mechanism is represented by a vector, whose components correspond to the tangent motions of the joint, and any state of self-stress by a vector whose components correspond to the bar forces, it is not possible to obtain one of the vectors as a linear combination of the others.

So, Maxwell's equation in the form of Eq. 1.1 is only a *necessary condition* for the kinematic determinacy of pin-jointed structures, but not a *sufficient condition*. It will be shown in Section 1.5 that the general, and most useful way, of writing Maxwell's equation is

$$dj - b - k = m - s \tag{1.2}$$

where $d = 2$, or 3 depending on the dimensions of the (Euclidean) space in which the structure is considered.

Consider the three-dimensional structures, $d = 3$, shown in Fig. 1.5. The tripod structure in Fig. 1.5(a) has a single free joint plus three fully constrained joints; so $j = 4$ and $k = 9$. The unconstrained joint is connected by three non-coplanar bars, $b = 3$, to the foundation joints. It has no states of self-stress, $s = 0$, as the condition for the joint to be in equilibrium in three different directions without external forces requires that the bar forces be zero. Substituting into Eq. 1.2 gives

$$3 \times 4 - 3 - 9 = 0 = m - 0 \tag{1.3}$$

from which the number of mechanisms is $m = 0$.

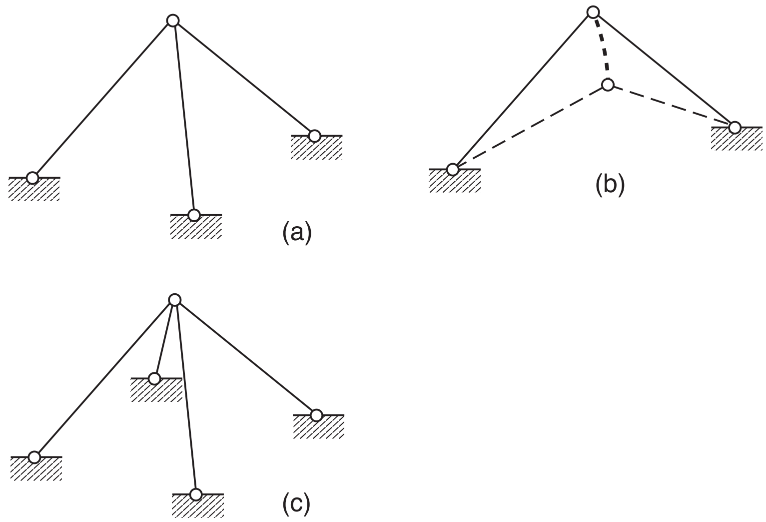


Figure 1.5: Examples of simple three-dimensional trusses.

Having established that $s = 0$ for the structure of Fig. 1.5(a), obviously s will remain unchanged if a bar is removed, Fig. 1.5(b). Hence, for this structure $j = 3$, $k = 6$, and $b = 2$. Substituting into Maxwell's equation

$$3 \times 3 - 2 - 6 = m - 0 \tag{1.4}$$

which gives $m = 1$. The mechanism involves a rotation of the two bars about an axis passing through the two foundation joints, as shown in Fig. 1.5(b).

By an analogous argument, the structure of Fig. 1.5(c), which is obtained by adding a bar to the structure of Fig. 1.5(a), has $m = 0$ and, from Maxwell's equation, $s = 1$.

Figure 1.6 shows two examples of pin-jointed structures that are topologically identical to the structure in Fig. 1.5(a), i.e. they have the same numbers of joints, bars, and constraints; *but now the bars are coplanar*. These structures admit a state of self-stress, e.g. a tension in the two inclined members equilibrated by a compression in the vertical member.

Since the left-hand side of Eq. 1.3 is unchanged, but now $s = 1$, here $m = 1$. In both structures the mechanism is identical to that shown in Fig. 1.5(b), but whereas Fig. 1.6(a) is a *finite mechanism*, in Fig. 1.6(b) only a small-amplitude motion of the mechanism is possible. This is because the central foundation joint is aligned with the other two in Fig. 1.6(a) but not in Fig. 1.6(b).

The truss structure in Fig. 1.6(b) is a simple example of an *infinitesimal mechanism*. If a structure of this kind is made with infinitely rigid members and perfectly fitting joints, it would admit only an infinitesimal motion of its mechanism. In practice, of course, its members will be elastic and there will be some tolerance in the joints; hence, the stiffness of the structure will be of a "lower order" than that of a normal, kinematically determinate structure.

Note that in the mathematics literature on structural rigidity a rigid structure is any structure that is either kinematically determinate or indeterminate but with mechanisms that are only infinitesimal (Connelly, 1993). Engineers tend to use the definition of rigid structures adopted here, which includes the smaller class that admit no mechanisms at all.

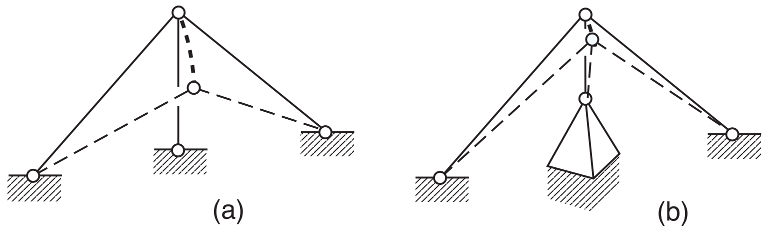


Figure 1.6: Examples of pin-jointed structures that are both statically and kinematically indeterminate. In (a) the three bars are coplanar *and* the three foundation joints are collinear.

The existence of structures with infinitesimal mechanisms was first discovered by J. Clerk Maxwell (1864), but it was only more recently that it was realised that they can be given a first-order (geometric) stiffness through a state of prestress (Calladine, 1986). This property has been successfully exploited in the design of prestressed cable nets, see Section **??**, and Tensegrity structures, see Section 4.3.

1.2.1 Polyhedral trusses

Figure 1.7 shows five trusses based on the five platonic polyhedra, more details of which can be found in Appendix A.2.

The simplest of these structures is the tetrahedral truss; from Table A.1 $j = 4$, $b = 6$, and $k = 0$. Hence, Maxwell's equation gives

$$3 \times 4 - 6 - 0 = 6 = m - s \tag{1.5}$$

Because there are only three non-coplanar bars meeting at each joint, for which three equations of equilibrium can be written, the bar forces have to be equal zero if the external loads are zero. Therefore, $s = 0$ and so, from Eq. 1.5, $m = 6$. Because the truss has six rigid-body mechanisms as a free body in three-dimensional space, i.e. three independent translations and three rotations, these are the only mechanisms of the truss. Denoting by m' the number of independent *internal mechanisms*, we have $m' = 0$ for the tetrahedral truss, i.e. it is *internally rigid*.

Next, consider the cubic truss, Fig. 1.7(b). From Table A.1, $j = 8$, $b = 12$, and $k = 0$; Maxwell's equation gives

$$3 \times 8 - 12 - 0 = 12 = m - s \tag{1.6}$$

Because $s = 0$, which can be shown by the same argument as for the tetrahedral truss, Eq. 1.6 gives

$$m = 12$$

of which six are rigid-body motions, as above, and the remaining six are internal mechanisms. For example, six independent mechanisms are obtained by deforming each square of the truss into a rhombus.

Repeating the same analysis for the remaining trusses it is found that $m - s = 6$ for the octahedral and icosahedral trusses, but $m - s = 30$ for the dodecahedral truss. Then, since it can be shown that $s = 0$ for all of them —although the proof is not straightforward for the octahedral and icosahedral trusses— it can be concluded that the octahedral and

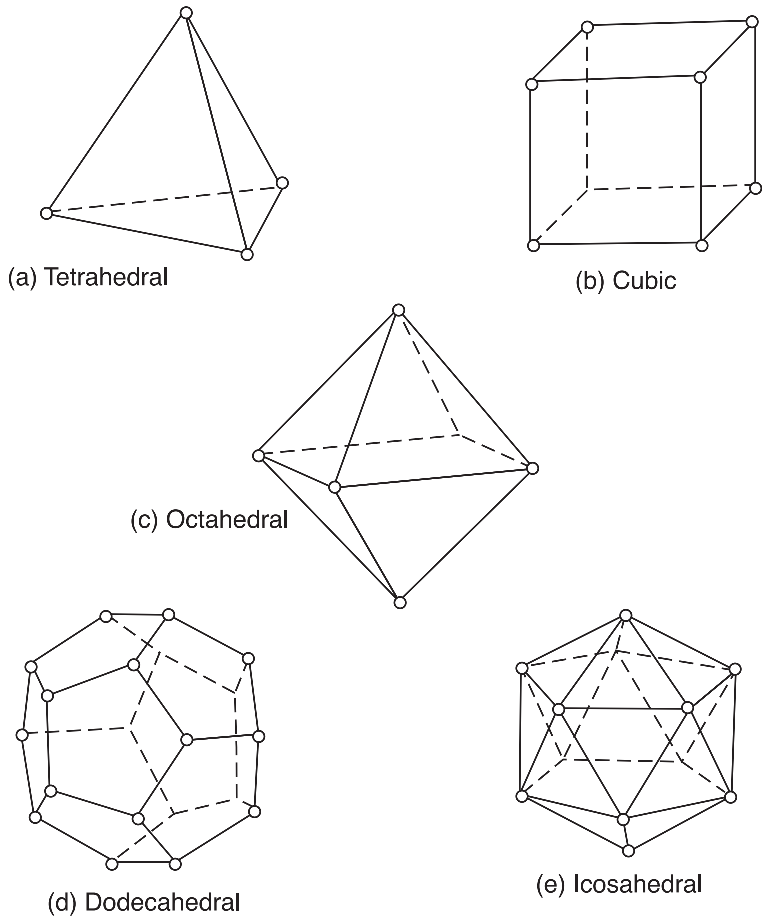


Figure 1.7: Regular polyhedral trusses.

Table 1.1: Static and (internal) kinematic determinacy of polyhedral trusses.

Shape	s	m'
Tetrahedron	0	0
Cube	0	6
Octahedron	0	0
Dodecahedron	0	24
Icosahedron	0	0

icosahedral trusses are internally rigid, but not the dodecahedral truss. These results are summarised in Table 1.1.

Note that the five trusses based on the platonic polyhedra can all be regarded as tessellations of triangles, squares and pentagons on a sphere. Also note that only the tessellations of triangles have turned out to be rigid; the cube and the dodecahedron —consisting of tessellations of squares and pentagons, respectively— have many mechanisms. This result was to be expected, in light of the earlier comment, in Section 1.1, on the rigidity of triangulated surfaces.

1.2.2 Cauchy's theorem

The rigidity of a truss consisting of a tessellation of triangles that lie on a sphere follows from a theorem proved by Cauchy, together with several other theorems for polygons and polyhedra. Theorem 13 of Cauchy (1813) states that:

In a convex polyhedron with invariable faces the angles at the edges are also invariable, so that with the same faces one can build only a polyhedron symmetrical to the first one.

Thus, every convex polyhedron with rigid faces will be rigid and, since the simplest way of forming a rigid face with pin-jointed bars is to use a triangle, Cauchy's theorem can also be stated in the specialised form:

Every convex polyhedral surface is rigid if all of its faces are triangles.

An example of a truss whose rigidity follows from Cauchy's theorem is shown in Fig. 1.8. This structure has been obtained by considering arcs of great circles that join the vertices of an icosahedron —which by definition lie on a sphere— and by locating an additional joint at the mid-point of each arc. Then, each joint has been connected with a bar to all of its neighbours. The resulting truss structure has $j = 42$, as 12 joints coincide with the vertices of the icosahedron, plus there are 30 joints at the mid-points of the great circle arcs. The number of bars is equal to twice the number of edges of the icosahedron, E, plus three times the number of faces, F, whose values are given in Table A.1. Hence $b = 2E + 3F = 120$. Maxwell's equation gives

$$m - s = 6$$

and, since $s = 0$, the only mechanisms are the six rigid motions.

Despite the restriction in Cauchy's theorem, that the surface should be convex, mathematicians had conjectured for over 150 years that in fact all surfaces consisting of triangles are rigid, even those surfaces that are not convex[4]. This was known as the "rigidity conjecture", which was finally proven to be wrong by a counter-example devised by Connelly (1978).

Since it took so long to find a counter-example, we can safely state that "almost all" triangulated surfaces are rigid. This means that one is very unlikely to ever encounter a triangulated surface of any shape that is not rigid, unless one uses very special tools to look for it.

[4]Note that the surfaces that we are considering here are *simply connected*, i.e. topologically identical to a sphere. Toroidal surfaces, for example, are excluded.

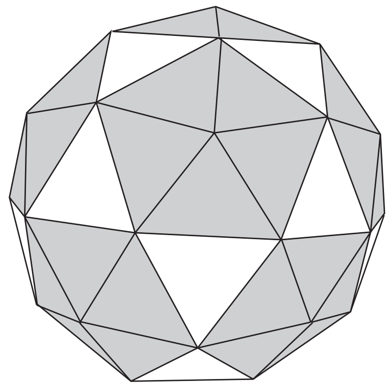


Figure 1.8: Truss structure obtained by adding a series of mid-arc nodes to an icosahedron. The nodes of the original icosahedron are at the centre of the pentagons.

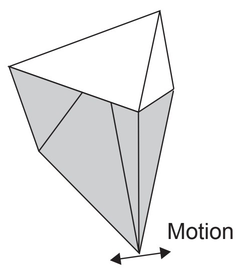


Figure 1.9: Perspective view of Connelly-Steffen "flexible sphere" (from Dewdney, 1991) made from cutting pattern in Fig. 1.10.

1.2.3 Flexible "sphere"

Several examples of concave triangulated structures that admit an infinitesimal motion were found over the years, but none whose motion was finite. Connelly's discovery of a counter-example to the "rigidity conjecture", which he called a *flexible sphere*, led to the subsequent discovery of several such structures by other authors. One of these examples is shown next.

Figure 1.9 shows a model, made from the cutting pattern in Fig. 1.10: the pattern is meant to be scaled up on a photocopying machine so that the edge numbers should be lengths in centimetres. This gives a size that is easy to work with. On the pattern, curved arrows indicate pairs of edges that should be attached, e.g. by leaving a tab on one side and gluing it under the other side.

After making your own model, try rotating the upper triangle relative to the lower one (not shown in Fig. 1.9): it will move without any resistance until two internal triangles come into contact. Note that while the structure moves, there is no sign of it stiffening up, as would happen in an infinitesimal mechanism. This indicates that the structure is a *finite mechanism*.

1.3 Rod-like trusses

The static and kinematic properties of a truss are very important in the design of high-performance structures. As an illustration, here we will use the ideas of kinematic and

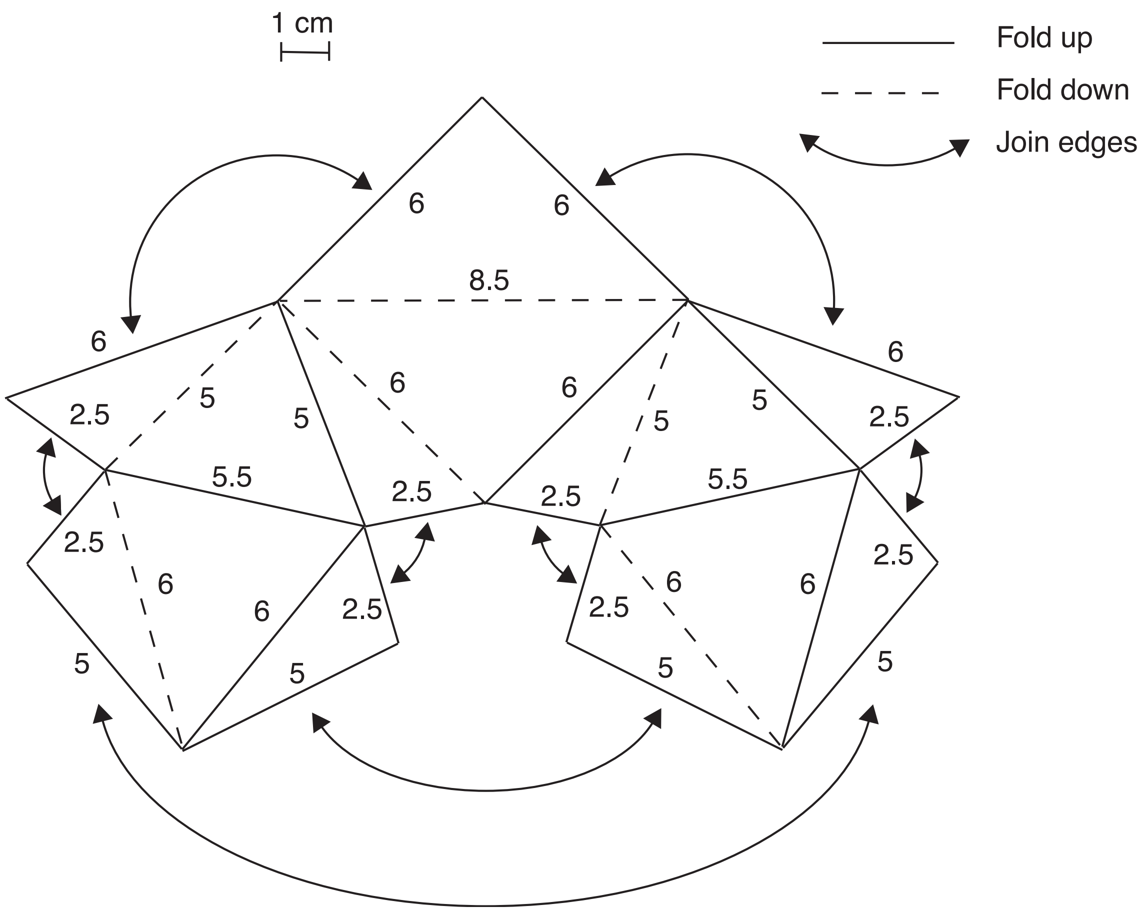


Figure 1.10: Cutting pattern for Connelly-Steffen "flexible sphere" (from Dewdney, 1991).

static determinacy to guide the design of a rod-like truss; we will return to this topic in Section 9.4 when we discuss adaptive structures.

Several different kinds of rod-like trusses can be obtained by interconnecting polyhedral trusses of the type described in Section 1.2.1.

The first requirement will be that the structure is rigid ($m = 0$), as this guarantees that the structure can carry any load in its axial mode (stiff), which will hence lead to a low-mass design. The second requirement will be that the structure is statically determinate ($s = 0$). There are two main reasons for this, first the force distribution in a statically determinate structure is uniquely determined by equilibrium considerations, and so disturbances such as thermal gradients, or manufacturing imperfections will not lead to over-stressing and, second, ease of assembly. Static determinacy guarantees that none of the bars is redundant and so, if the structure is assembled, say, by joining together a series of prefabricated elements, each time we attach a new member the distance between the end joints is not already determined by the pre-existing members.

Another advantage of statically determinate truss structures is that their shape can be adjusted, and indeed varied by large amounts by modifying the length of one member at a time, without inducing any self-stress in the structure. This feature will be exploited in Section 9.4.

Substituting the above values for m and s into Maxwell's equation, Eq. 1.2, and assuming that $k = 6$ kinematic constraints will be introduced after the design of the structure has been completed, to locate the structure with respect to a foundation

$$3j - b - 6 = 0$$

Re-arranging

$$j = \frac{b}{3} - 2$$

and, assuming j and b to be large

$$j \approx \frac{b}{3} \tag{1.7}$$

It follows that we need to consider designs where the ratio between the number of bars and the number of joints is equal to three. Therefore, because each bar is attached to two joints, *on average there should be six bars connected to each joint*. This result is not restricted to rod-like trusses, but holds for any three-dimensional truss.

The third requirement is that the truss should consist of identical modules, because a repetitive design requires a smaller number of different parts and, hence, is simpler and cheaper to produce.

Of the three rigid polyhedral trusses that have been identified in Section 1.2.1 only the tetrahedron and the octahedron lend themselves to forming a rod-like structure, and the resulting trusses are shown in Fig. 1.11. Both of these structures are rigid, because their "building blocks" are rigid and are connected to each other through a common triangle, which does not allow any relative translation or rotation between the building blocks.

Of the two structures, the stack of octahedra, Fig. 1.11(b), is easiest to visualize, because the common triangles lie in parallel planes. It has 3-fold rotational symmetry about a central, vertical axis, and it has also mirror symmetry through three vertical planes.

The stack of tetrahedra, Fig. 1.11(a), is more difficult to visualize, as the triangles that are common to consecutive modules do not lie in parallel planes. The resulting structure, called *tetrahelix* by Buckminster Fuller (1975), has helical symmetry.

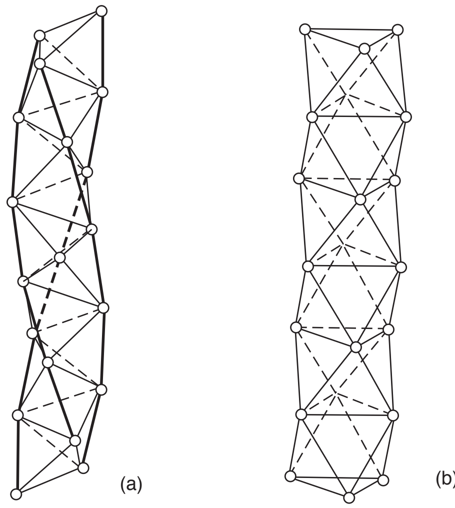


Figure 1.11: Stacks of tetrahedral and octahedral trusses.

In practice, these two "designs" are not used in the present form. Instead, since it is often desirable for structures to have straight, continuous longitudinal members —e.g. to reduce the axial forces in the members due to overall axial or bending loads, and thus increase the stiffness and strength of the structure— the forms that are normally used are the topologically equivalent structures shown in Fig. 1.12. Their members have different geometrical lengths.

1.4 Examples

In this section we discuss a series of three-dimensional trusses with interesting properties. These structures illustrate the type of behaviour that can be encountered when designing trusses, and also form a useful "catalogue" of unexpected effects that one should watch for.

Figure 1.13 shows a simple "space frame", of which some larger scale examples can be found in Section 2.2, consisting of pin-jointed bars —all of equal length— that form a *skew cube* with surface diagonals. It has $j = 8$, $b = 18$, and $k = 0$ and, substituting these values into Maxwell's equation

$$3 \times 8 - 18 - 0 = 6 = m - s \tag{1.8}$$

which gives $m - s = 6$. This truss is rigid, because it consists of an octahedral truss —a rigid structure— rigidly connected to two tetrahedra —also rigid—. Therefore, its total number of mechanisms is $m = 6$; these mechanisms are all rigid-body mechanisms. From Eq. 1.8, $s = 0$.

Figure 1.14 shows a more complex space frame, also consisting of bars of equal length a. As the previous example, this structure is also a module for a commonly-used space frame; more details will be provided in Section 2.2.

This truss consists of two unbraced squares, plus two identical squares in a plane parallel to the first set, and at a distance $a/\sqrt{2}$. The joints of the upper square are connected by diagonal members to the joints of the lower square. For this structure

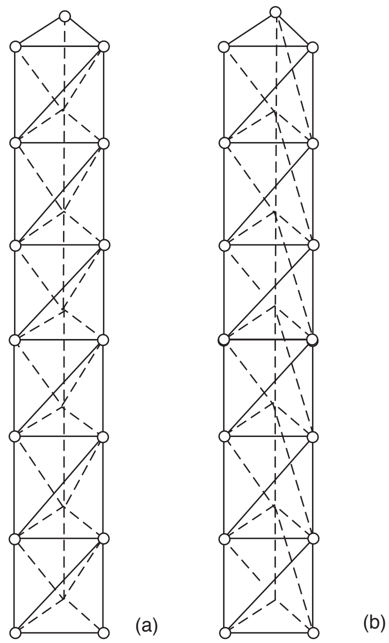


Figure 1.12: Rod-like trusses topologically equivalent to those in Fig. 1.11.

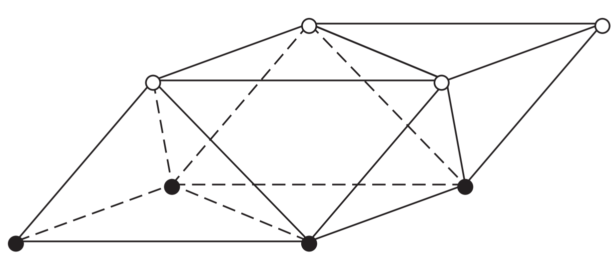


Figure 1.13: Space frame module consisting of an octahedron and two tetrahedra; the outer joints lie on six planar faces.

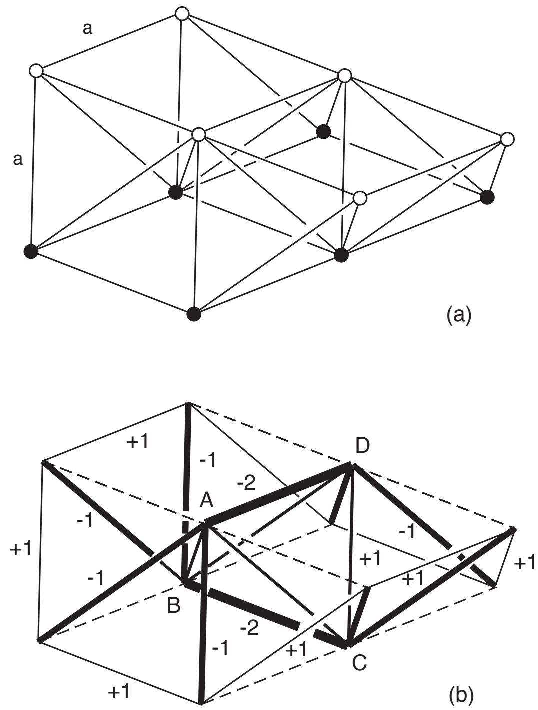


Figure 1.14: Space frame module consisting of five tetrahedra and four half-octahedra (the six nodes in the upper plane are denoted by hollow circles whereas the six nodes in the bottom plane are denoted by solid circles). (b) Shows the state of self-stress, where members carrying compression are drawn thicker.

$j = 12$, $b = 30$, and $k = 0$ and, substituting into Maxwell's equation

$$3 \times 12 - 30 - 0 = 6 = m - s \tag{1.9}$$

which gives $m - s = 6$, as before. This structure, however, is not statically determinate. It has $s = 1$, and Fig. 1.14(b) shows a self-equilibrated set of bar forces.

It can be verified by inspection that these axial forces form a state of self-stress. The eight members on the outer edge of the structure form a continuous "tension loop" carrying a force of one unit; at every kink in the loop equilibrium is ensured by a compression of two units in the member that is locally coplanar with the tension loop, whereas the fourth member —which is not coplanar with the other three— carries no force. The equilibrium of the innermost four joints, e.g. joint A, is ensured by tensions of two units in the diagonal members, e.g. AB and AC, and a compression of four units in, e.g., AD.

Having shown that this truss has $s = 1$, it follows from Eq. 1.9 that $m = 7$. Hence, in addition to the six rigid-body mechanisms, there is an internal mechanism in which the whole truss twists as the squares distort out of plane.

The next example is the cubic truss shown in Fig. 1.15(a), with members of length a lying on the edges of a cube plus four diagonal bracing members on the side faces, of length $a\sqrt{2}$. The four joints at the bottom are fully constrained. This truss has $j = 8$, $b = 12$, and $k = 12$; substituting these values into Maxwell's equation

$$3 \times 8 - 12 - 12 = 0 = m - s \tag{1.10}$$

giving $m - s = 0$. It can be shown by the matrix method of Section 1.5 that this truss is both statically and kinematically determinate ($m = s = 0$).

Now, consider varying the shape of this truss by rotating the upper square in an anti-clockwise sense, without translating. Obviously, the lengths of both the diagonal members and of the members that were originally vertical, e.g. AF and AE respectively, vary during this process.

Tarnai (1980) has shown that the coefficient matrix of the system of equilibrium equations —further details on this matrix are given in Section 1.5— has full rank, equal to 12, normally. However, the matrix becomes rank deficient, with rank of 11, in four special configurations. Of these, the configurations that are of greatest practical interest are those that are obtained for a rotation of the upper square through 45°, Fig. 1.15(b), and 135°, Fig. 1.15(c).

In each of these configurations the static and kinematic properties of the structure change from $m = s = 0$ to $m = s = 1$. The states of self-stress and mechanisms for these configurations are shown in Figs 1.16 and 1.17. In Fig. 1.16 note that the bar forces in the top square are alternatively positive and negative as one goes round the square, whereas in Fig. 1.17 the bar forces in the top square are all of the same sign. There is an important difference between these two special configurations: in the first one the mechanism allows a finite amplitude distortion of the structure, whereas the mechanism of the second configuration allows only an infinitesimal motion.

In practice, imposing a state of prestress on the first structure, e.g. by varying the length of one of its members with a turn-buckle, is impossible, as the structure will change shape instead of becoming self-stressed[5]. On the other hand, the structure of Fig. 1.15(c) has the same type of behaviour of the structure in Fig. 1.6(b). It is an example of a Tensegrity structure, and will be discussed further in Section 4.3.

Tarnai (1980) shows that the existence of special configurations that are both statically and kinematically indeterminate is a general feature of trusses based on two interconnected regular polygons with n-sides (in Fig. 1.15 $n = 4$) and, in particular, configurations that admit finite amplitude inextensional mechanisms exist for all trusses with n even and ≥ 4. However, for n odd there no such special configurations.

Truss structures with a layout similar to Fig. 1.15(b) have been used for several applications, often in preference to the layout in Fig. 1.15(a), because their higher degree of symmetry leads to the expectation of a "more uniform" stiffness distribution. An example is the telescope structure shown in Fig. 1.18, here a truss structure with the same layout as that shown in Fig. 1.15(b) supports the secondary mirror. This is a standard design for Cassegrain configuration telescopes, but requires a massive ring beam at the top of the truss. As we have seen, a three- or five-sided truss would be mechanism-free and hence would not require such a massive ring beam.

In addition to the ring trusses described above, another family of trusses that show a strange pattern of static and kinematic indeterminacy are triangulated hyperbolic paraboloids.

Figure 1.19 shows a truss structure whose joints lie on a hyperbolic paraboloid, see Appendix A.3. Its edge joints lie on parabolas and are subject to vertical constraints, as shown in Fig. 1.19(a). Three additional constraints are applied to the whole surface, see Fig. 1.19(b), to prevent it from translating horizontally and rotating about a vertical axis.

Pellegrino (1988) has shown that this particular structure has $m = s = 4$ and, in general, a structure whose joints form a $n \times n$ grid has $m = s = n - 2$. Structures with different boundary shapes, e.g. straight, or with joints on a rectangular $n \times n'$ grid also

[5] This was pointed out by Kuznetsov (1991), who called such structures unprestressable.

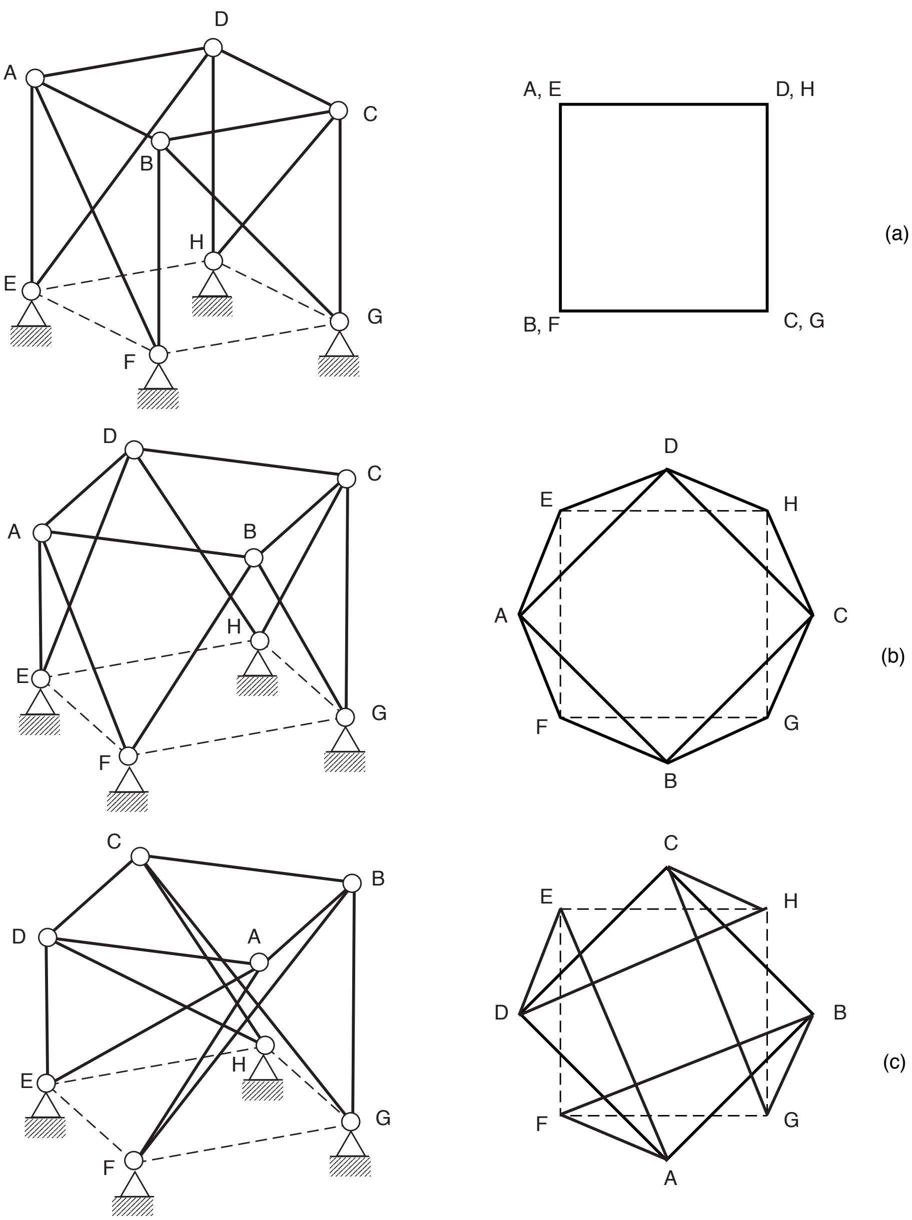


Figure 1.15: Perspective and top views of (a) cubic truss, and trusses obtained by rotating the upper square through (b) 45° and (c) 135°.

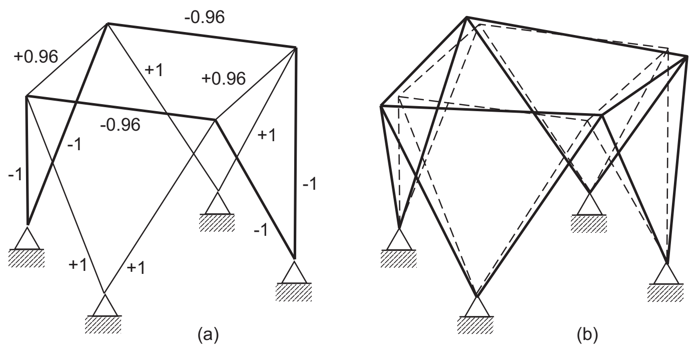


Figure 1.16: (a) State of self-stress and (b) inextensional mechanism of first special configuration of cubic truss.

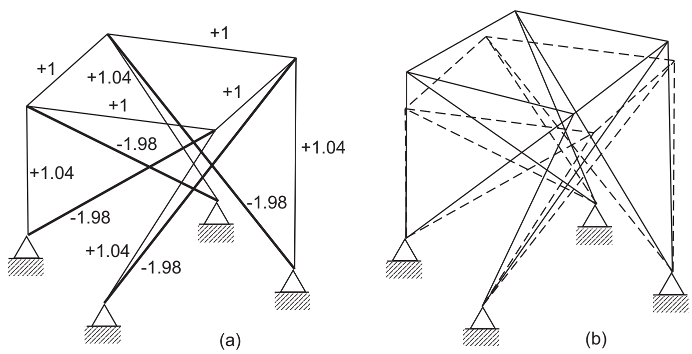


Figure 1.17: (a) State of self-stress and (b) inextensional mechanism of second special configuration of cubic truss.

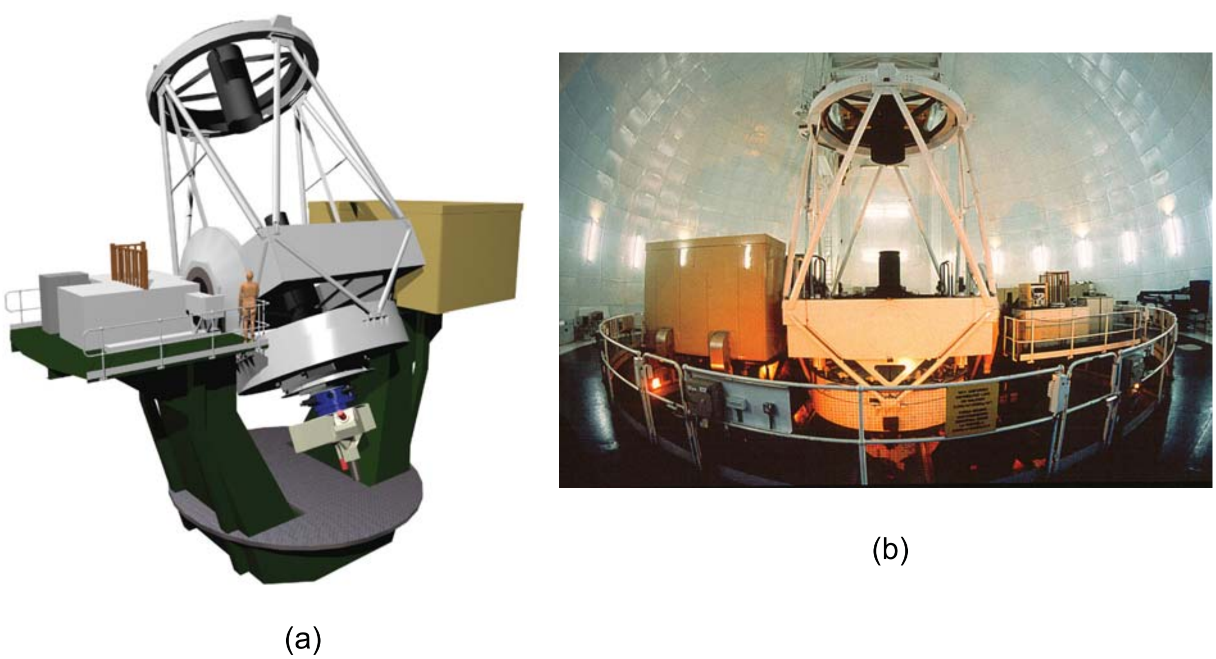


Figure 1.18: (a) 3D model of the William Herschel Telescope, La Palma, Canary Islands (2.54 m diameter) and (b) view of instrument inside its dome.

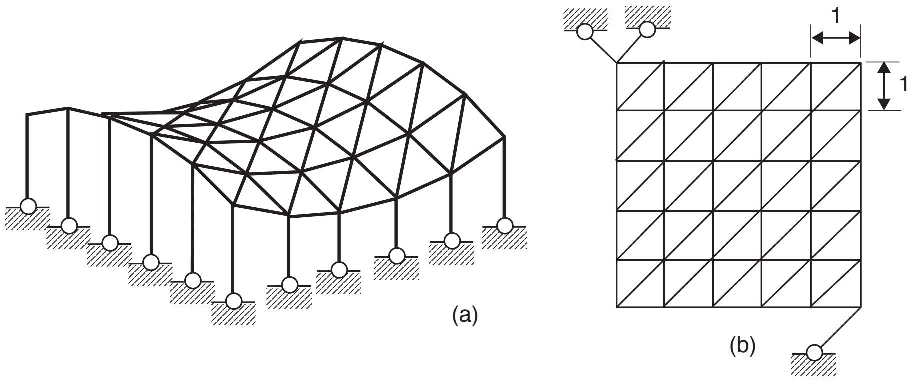


Figure 1.19: Triangulated hyperbolic paraboloid; (a) perspective view; (b) top view.

produce variable degrees of static and kinematic indeterminacy (Pellegrino 1988; Roth 1989).

1.5 Rigidity Computations

Consider a truss structure with j joints that are subject to k kinematic constraints, and b bars. Let d be the no. of dimensions of the space in which the structure is to be analysed ($d = 2$ or 3). By setting up a system of

$$n_r = dj - k \tag{1.11}$$

linear equilibrium equations that relate the

$$n_b = b \tag{1.12}$$

axial forces in the bars, arranged in the vector $\mathbf{t}$, to the external load components, arranged in the vector $\mathbf{p}$, we obtain the equilibrium matrix $\mathbf{A}$ for this structure

$$\mathbf{At} = \mathbf{p} \tag{1.13}$$

Usually $n_r \neq n_c$, and hence the equilibrium matrix is rectangular. Some details on a simple way of setting up this equilibrium matrix are given next.

Consider joint H of a two-dimensional pin-jointed structure, connected by bar HI to joint I and by bar HJ to joint J, as shown in Fig. 1.20(a). Let α_{HI} and α_{HJ} be the angles between these bars and the x-axis, respectively. These angles are defined to be positive if anti-clockwise. Figure 1.20(b) shows a free-body diagram for joint H, showing the x- and y-components of the bar forces acting on this joint. The bar forces (t_{HI} in bar HI, etc.) are assumed to be positive if tensile. The external force applied to joint H has components P_{Hx} and P_{Hy} in the x- and y-directions. For equilibrium in the x-direction, we must have

$$\cos\alpha_{HI}t_{HI} + \cos\alpha_{HJ}t_{HJ} + p_{Hx} = 0$$

or, in a form more suitable for Equation 1.13

$$-\cos\alpha_{HI}t_{HI} - \cos\alpha_{HJ}t_{HJ} = p_{Hx}$$

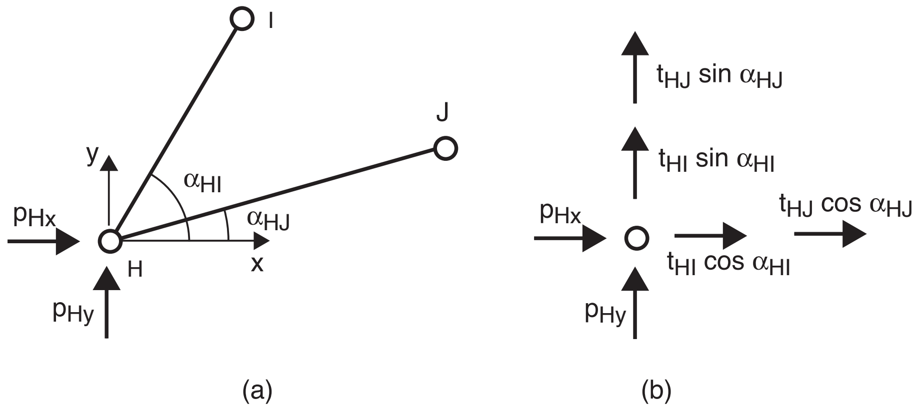


Figure 1.20: Equilibrium of a general pin-joint.

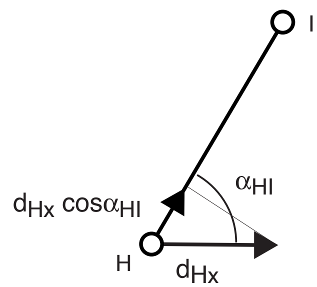


Figure 1.21: Relationship between change of length of a bar and one of the four components of joint displacement.

An analogous equilibrium equation in the y-direction can also be written. Hence, the equilibrium equations for joint H are

$$\begin{cases} -\cos\alpha_{HI}t_{HI} - \cos\alpha_{HJ}t_{HJ} = p_{Hx} \\ -\sin\alpha_{HI}t_{HI} - \sin\alpha_{HJ}t_{HJ} = p_{Hy} \end{cases} \tag{1.14}$$

A pair of equilibrium equations like this can be written for each unconstrained joint of a pin-jointed structure and, by grouping together these equations, we obtain a system whose coefficient matrix is the equilibrium matrix $\mathbf{A}$.

A dual set of equations to the equilibrium equations is the set of $n_b = b$ compatibility equations, that relate the extensions of the bars, arranged in the vector $\mathbf{e}$ and assumed to be small, to the displacement components of the joints, arranged in the vector $\mathbf{d}$ and also assumed to be small. These two vectors are related by the kinematic matrix $\mathbf{B}$:

$$\mathbf{Bd} = \mathbf{e} \tag{1.15}$$

The compatibility equation for a bar is obtained by considering the changes in bar length that are induced by each of the four displacement components of its joints, one of which is shown in Fig. 1.21. For a general bar HI, the compatibility equation is

$$-\cos\alpha_{HI}d_{Hx} - \sin\alpha_{HI}d_{Hy} + \cos\alpha_{HI}d_{Ix} + \sin\alpha_{HI}d_{Iy} = e_{HI} \tag{1.16}$$

It can be shown, by inspection or by Virtual Work (Livesley, 1975), that the equilibrium matrix is equal to the transpose of the kinematic matrix, hence Eq. 1.15 can be written in the form:

$$\mathbf{A}^T\mathbf{d} = \mathbf{e} \tag{1.17}$$

The number of independent inextensional mechanisms, m, and the number of independent states of self-stress s can be calculated, after determining the rank r —recall that the rank of a matrix is equal to the number of linearly independent rows or columns (Strang, 1980)— of the equilibrium matrix, from the equations

$$m = n_r - r \tag{1.18}$$

$$s = n_c - r \tag{1.19}$$

Subtracting Eq. 1.19 to Eq. 1.19 and substituting Eqs 1.11 and 1.12 respectively for n_r and n_c yields the generalised Maxwell's equation in a form that coincides exactly with Eq. 1.2.

Table 1.2 introduces a general classification scheme for truss structures, which identifies four types of assemblies. Note that the actual values of m and s are not important: what matters is if they are zero or not: if $m = 0$ the truss is kinematically determinate (i.e. rigid if the bars are of fixed length), and if $s = 0$ the truss is statically determinate (i.e. the bar forces depend only the on the external forces).

Table 1.2: Classification of truss structures.

Type		**Static and kinematic properties**	**Equilibrium and kinematic equations**
I	$s = 0$ $m = 0$	Statically determinate and kinematically determinate	Both Eq. 1.13 and Eq. 1.15 have a unique solution for any $\mathbf{p}$ and $\mathbf{e}$.
II	$s = 0$ $m > 0$	Statically determinate and kinematically indeterminate.	Eq. 1.13 has one solution for some particular $\mathbf{p}$, otherwise no solution. Eq. 1.15 has infinitely many solutions for any $\mathbf{e}$.
III	$s > 0$ $m = 0$	Statically indeterminate and kinematically determinate.	Eq. 1.13 has infinitely many solutions for any $\mathbf{p}$. Eq. 1.15 has one solution for some particular $\mathbf{e}$, otherwise no solution.
IV	$s > 0$ $m > 0$	Statically indeterminate and kinematically indeterminate.	Eq. 1.13 has infinitely many solutions for some particular $\mathbf{p}$, otherwise no solution. Eq. 1.15 has infinitely many solutions for some particular $\mathbf{e}$, otherwise no solution.

1.5.1 SVD of equilibrium matrix

It is shown in Linear Algebra (e.g. Strang, 1988) that for any matrix $\mathbf{A}$ of dimensions $n_r \times n_c$ and rank r, there exist:

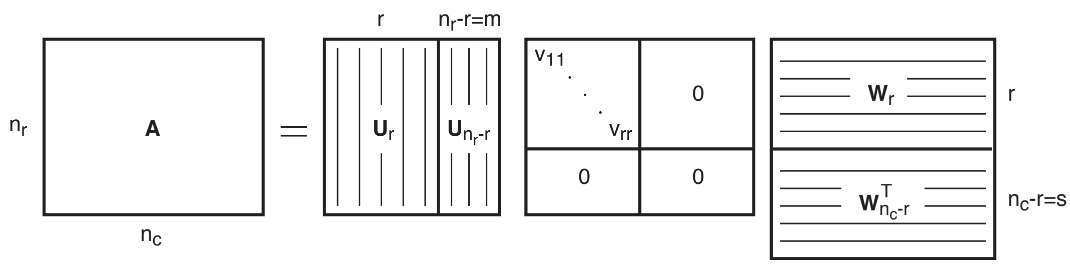


Figure 1.22: Graphical illustration of the SVD of the equilibrium matrix $\mathbf{A} = \mathbf{UVW}^T$.

- a $n_r \times n_r$ orthogonal matrix $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_{n_r}]$ (by orthogonal matrix we mean that $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, i.e. that the vectors $\mathbf{u}_i$ are orthonormal);
- a $n_c \times n_c$ orthogonal matrix $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_{n_c}]$; and
- a $n_r \times n_c$ matrix $\mathbf{V}$ with r positive elements v_{ii} $(i = 1, ...r)$ on the leading diagonal: all other elements are zero;

such that

$$\mathbf{A} = \mathbf{UVW}^T \tag{1.20}$$

See Fig. 1.22 for a pictorial representation of this decomposition.

The coefficients v_{ii} are the singular values of $\mathbf{A}$ and the vectors $\mathbf{u}_i$ and $\mathbf{w}_i$ are, respectively, the i-th left singular vector and the i-th right singular vector. This is called the Singular Value Decomposition (SVD) of the matrix $\mathbf{A}$, and generalises the spectral decomposition of a square matrix. The SVD is described in many textbooks on matrix computations, such as Golub and Van Loan (1983) where further details, proofs, etc. can be found.

For our purposes, given $\mathbf{A}$, all that is needed is a way of computing its SVD; the easiest way of doing it is to type `svd(A)` in Matlab (Mathworks, 1997). Once the SVD of the equilibrium matrix has been computed, the value of its rank has to be decided. The problem is that, although the definition of the SVD at the beginning of this section states that $\mathbf{V}$ contains only r non-zero elements, in practice one often finds that the leading diagonal of $\mathbf{V}$ contains a number of singular values of decreasing magnitude, none of which is actually equal to zero although some may be quite small. For accurate and stable computations a threshold value ϵ has to be set, whose value depends on an acceptable round-off error and on the accuracy to which the nodal coordinates are defined: all singular values smaller than ϵ are treated as zero, and the rank of $\mathbf{A}$ is defined accordingly.

The value of ϵ can be set after the full set of singular values has been calculated and in fact, in particularly sensitive calculations, its choice can be based upon information provided by the singular values themselves. See Pellegrino (1993).

1.5.2 Physical interpretation of the decomposition

Because of the correspondence between statics and kinematics described above, the singular value decomposition of the equilibrium matrix can be interpreted in two different, but corresponding ways.

Figure 1.23(a) shows that the first r left singular vectors are the load systems in equilibrium with the sets of bar forces in the corresponding right singular vectors, multiplied by the corresponding singular values. The remaining $s = n_c - r$ right singular vectors are *independent states of self-stress.*

Similarly, Fig. 1.23(b) shows that the first r left singular vectors are the displacement systems compatible with the sets of bar extensions in the corresponding right singular vectors, divided by the corresponding singular values. The remaining $m = n_r - r$ left singular vectors are *independent inextensional displacement modes, or mechanisms.*

Furthermore, it can be shown that the last m left singular vectors are *load conditions which the truss cannot equilibrate* in its current configuration, and also that the last s right singular vectors are orthogonal sets of *incompatible strains.*

Note that the SVD is a way of finding bases[6] for the four fundamental subspaces of a matrix —the rowspace, the nullspace, the column space, and the left-nullspace (Strang, 1980)— and, although computationally more expensive than alternative techniques based on row operations (Pellegrino and Calladine, 1986), two advantages of the approach presented here are that (i) ill-conditioning of the equilibrium matrix poses no problems, and (ii) the one-to-one static and kinematic correspondence between the left singular vectors in $\mathbf{U}_r$ and the corresponding right singular vectors in $\mathbf{W}_r$, explained above, is very useful for structural computations (Pellegrino, 1993).

1.5.3 Examples

We will now apply the above computational method to the analysis of some of the three-dimensional truss structures discussed in Section 1.2.

Consider the truss shown in Fig. 1.24(a). The vector of bar forces contains the axial forces T_i in the four bars, and the load vector contains the load components in the x-, y-, and z-directions at joint A. Equation 1.13 takes the form:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} t_{AB} \\ t_{AC} \\ t_{AD} \\ t_{AE} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

The equilibrium matrix $\mathbf{A}$ has $n_r = 3$ rows and $n_c = 4$ columns. The vectors of kinematic variables $\mathbf{e}$ and $\mathbf{d}$ contain the four bar extensions e_i and the three displacement components of joint A, respectively.

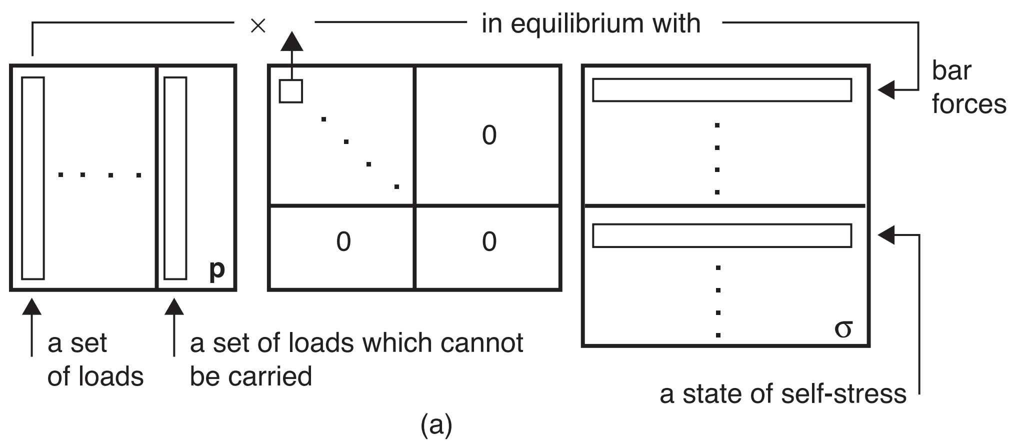
The SVD of $\mathbf{A}$, obtained from Matlab (Mathworks, 1997), is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -0.000 & -0.000 & 1.000 \\ 0.000 & 1.000 & 0.000 \\ 1.000 & -0.000 & 0.000 \end{bmatrix}\begin{bmatrix} 1.414 & 0 & 0 & 0 \\ 0 & 1.000 & 0 & 0 \\ 0 & 0 & 1.000 & 0 \end{bmatrix}\begin{bmatrix} 0.500 & 0.500 & 0.500 & 0.500 \\ 0.000 & 0.707 & -0.000 & -0.707 \\ -0.707 & 0.000 & 0.707 & 0.000 \\ 0.500 & -0.500 & 0.500 & -0.500 \end{bmatrix}$$

It can be verified by inspection that this decomposition satisfies the conditions stated at the beginning of Section 1.5.1.

[6] A basis of a vector space is a set of linearly independent vectors whose linear combinations span the space (Strang, 1980).



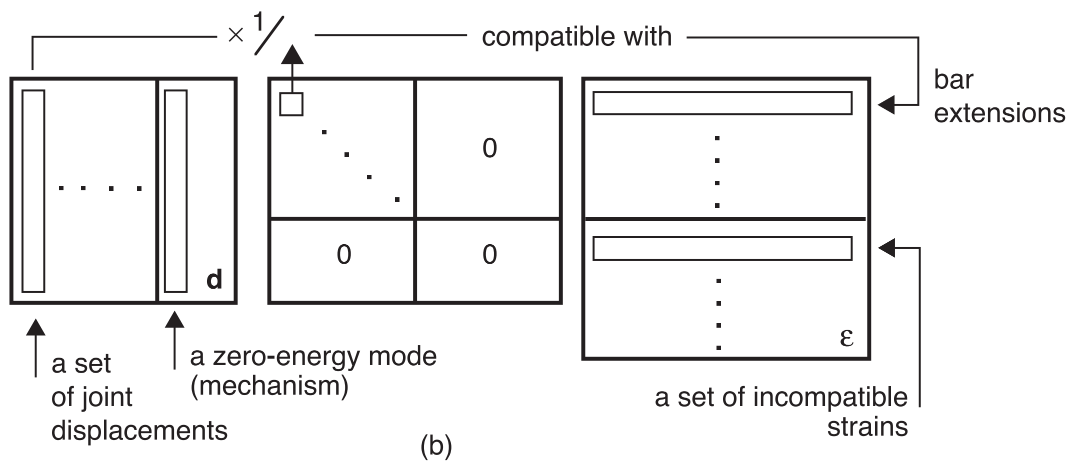


Figure 1.23: The first r left- and right-singular vectors are in one-to-one correspondence both statically and kinematically.

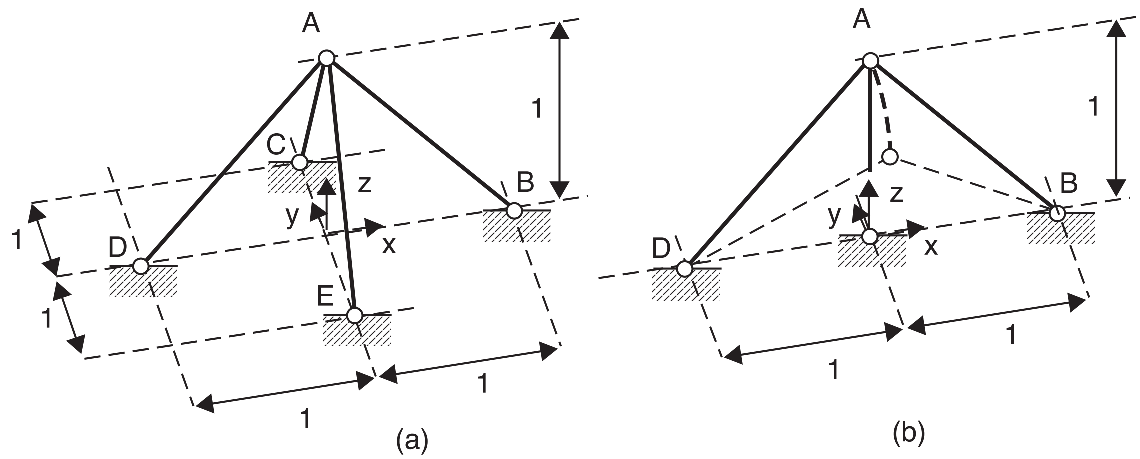


Figure 1.24: Truss structures discussed in Section 1.2; (a) consists of four bars connected to one joint and four foundation joints, while (b) consists of three bars, with the three foundation joints collinear.

There are three non-zero singular values, hence $r = 3$, $m = n_r - r = 0$ and $s = n_c - r = 1$. Therefore, the pin-jointed assembly of Fig. 1.24(a) is of Type III. The structure has a single state of self-stress [set of incompatible extensions] involving axial forces [extensions] of equal magnitude but opposite signs in AB, AD and AC, AD, see the bottom row of the $\mathbf{W}^T$ matrix.

The static and kinematic correspondence between the first three left and right singular vectors can be easily verified. For example, the load condition $\mathbf{u}_1$ which consists of a unit z-load at joint A, amplified by $\sqrt{2}$ —which is the singular value v_{11}—, is in equilibrium with axial forces of +0.5 in the four bars, forming the stress state w_1. From a kinematic viewpoint, the displacement mode u_1 which involves a unit z-displacement of joint A, divided by the singular value $v_{11} = \sqrt{2}$, is compatible with extensions of +0.5 of the four bars, forming the set of extensions w_1.

Next, consider the truss shown in Fig. 1.24(b). The singular value of its equilibrium matrix (which is obtained by deleting the second and fourth columns from the previous equilibrium matrix) is

$$\begin{bmatrix} -0.707 & 0.707 \\ 0 & 0 \\ 0.707 & 0.707 \end{bmatrix} = \begin{bmatrix} -0.707 & -0.707 & 0.000 \\ 0.000 & 0.000 & -1.000 \\ 0.707 & -0.707 & 0.000 \end{bmatrix} \begin{bmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \\ 0.000 & 0.000 \end{bmatrix} \begin{bmatrix} 1.000 & 0.000 \\ 0.000 & -1.000 \end{bmatrix}$$

There are two non-zero singular values, hence $r = 2$, $m = n_r - r = 1$ and $s = n_c - r = 0$. Therefore, the pin-jointed assembly of Fig. 1.24(b) is of Type II. The single mechanism [and corresponding set of loads which cannot be equilibrated] involves the translation of joint A in the y-direction, see the last left singular vector.