

A project report on

Design of Multiple Dependent State Variables-Based Acceptance Sampling Plans with Rectifying Inspection Based on Advanced Capability Index

A report submitted in partial fulfilment of the requirements for the degree of

Bachelor of Technology
in
Production and Industrial Engineering

Submitted by

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May, 2019**

CANDIDATE'S DECLARATION

I hereby declare that the work carried out in this project report entitled "**Design of Multiple Dependent State Variables-Based Acceptance Sampling Plans with Rectifying Inspection Based on Advanced Capability Index**" is submitted to the Department of mechanical and Industrial Engineering under partial fulfilment of the requirements for the award of the degree of Bachelor of Technology in Production and Industrial Engineering of the Indian Institute of Technology Roorkee, India. It is an authentic record of my own work carried out during the period from July, 2018 to April, 2019 under the supervision of **Dr. Akshay Dvivedi**, Associate Professor, Department of Mechanical & Industrial Engineering. The matter embodied in this report has not been submitted by me for the award of any other degree of this or any other Institute/University.

Date: May 2019

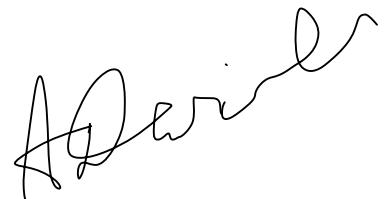
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CERTIFICATE

This is to certify that the report submitted by Ankit (15410005) on "**Design of Multiple Dependent State Variables-Based Acceptance Sampling Plans with Rectifying Inspection Based on Advanced Capability Index**" in partial fulfilment for the degree of Bachelor of Technology in Production and Industrial is an authentic record of his project work which he has satisfactorily completed under my supervision.



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A handwritten signature in black ink, appearing to read "Ankit".

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Abstract

Acceptance sampling uses statistical sampling procedures to determine whether to accept or reject a production lot of material. It has been a common quality control technique used in the industry. It can be performed at any stage during the process of production or while the products leave the factory. Acceptance sampling is an alternative to 100% inspection due to many reasons – 100% inspection is extremely expensive and time-consuming. Also, when the testing is destructive, sampling procedures are the only viable option available for deciding the fate of the lot.

Most often a producer supplies a consumer a number of items and a decision to accept or reject the items is made by determining the number of defective items in a sample from the lot. The lot is accepted if the number of defects falls below where the acceptance number or otherwise the lot is rejected. This type of cut-off criteria on the number of defective items is referred to as attributes based sampling. Another type of sampling in which the quality of the products is aggregated into a mean and standard deviation value or into a process capability index on the basis of a quality characteristic of the samples is known as variables based sampling. It has been well documented in the literature that variables based acceptance sampling plans provide better quality control for a smaller number of tested samples (Montgomery, 2009; Schilling and Neubauer, 2009). Similarly, the fate of the current lot can depend on whether the results for previous lots are considered or not; giving the terminology multiple-state plans to the former and single-state plans to the latter.

In this project, we develop variables-based multiple-dependent state (MDS) acceptance sampling plans based on the advanced process capability index, which was developed by combining the merits of the yield-based index and loss-based index. Both rectifying and non-rectifying inspection plans are developed. The operating characteristic function of the developed plans is derived based on the exact sampling distribution. The determination of plan parameters is formulated as an optimisation model with non-linear constraints, where the objective is to minimise the sample size required for inspection (for the non-rectifying inspection plans) and

average total inspection (for rectifying inspection plans) and the constraints are set by the vendor and the buyer to satisfy the desired quality levels and allowable risks. The performance of the developed plan is examined and compared with traditional sampling plans. A step-by-step procedure is provided, and the parameters of the plan under various conditions are tabulated for practical applications.

In addition, the economic aspects of considering the inspection costs, internal failure costs and external failure costs on the sampling plans are considered. The plans thus derived use total failure cost, which is the sum of inspection, internal and external failure costs as the objective function, with the constraints on producer's and consumer's risks as above. Finally, a sensitivity analysis of the plans w.r.t. to 3 types of costs is considered.

Keywords: *advanced capability index, variables-based sampling, economic sampling plan, multiple-dependent state sampling*

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Symbols and Abbreviations

- USL and LSL: the upper and lower specification limits, respectively. They are the targets set for the process/product by the customer.
- $d = (USL - LSL)/2$ is the half specification width
- T: the target value for the quality characteristic
- μ_X, σ_X : the mean and standard deviation of the random variable X, which represents the quality characteristic. Let X follows a normal distribution.
- process capability index, PCIs
 - Basic PCIs, $C_p = \frac{USL - LSL}{6\sigma}$, where $\sigma = E(X - \mu_X)^2$;
 - Yield-based PCIs, $C_{pk} = \min(C_{pu}, C_{pl})$, where $C_{pu} = \frac{USL - \mu}{3\sigma}$, and $C_{pl} = \frac{\mu - LSL}{3\sigma}$;
 - Loss-based PCIs, $C_{pm} = \frac{USL - LSL}{6\tau} = \frac{c_p}{\sqrt{1+\xi^2}}$ where $\tau^2 = E(X - T)^2$, $\xi^2 = [(\mu - T)/\sigma]^2$ and $T = (USL + LSL)/2$ is the target;
 - Yield-loss based PCIs, $C_{pkm} = \frac{c_{pk}}{\sqrt{1+\xi^2}}$ where ξ is defined as above
- C_{pkm}^{AQL} : Associated C_{pkm} value at Acceptable Quality Limit (AQL),
i.e, $P(\text{accept the lot} | C_{pkm} = C_{pkm}^{AQL}) \geq 1 - \alpha$
- C_{pkm}^{RQL} : Associated C_{pkm} value at Rejected Quality Limit (RQL) or Lot Tolerance Percent Defective (LTPD), i.e, $P(\text{reject the lot} | C_{pkm} = C_{pkm}^{RQL}) \leq \beta$
- N: Size of the lot
- n: Size of the sample randomly taken from the lot of size N
- p, the probability of defective item in the lot
- TQC: Total quality cost per lot (with rectifying inspection)
- Y: \hat{C}_{pkm} value calculated from the sample

- c^* : threshold of \hat{C}_{pkm} value, i.e., accept the lot if $Y \geq c^*$
- c_i : Cost of inspection per unit
- c_{if} : Cost of internal failure per unit
- c_{ef} : Cost of external failure per unit
- ATI: Average total inspection per lot
- D_d : Average no. of defects detected per lot
- D_n : Average no. of defects that go undetected per lot
- α : producer's risk, the probability of rejecting a good lot, the probability of Type I error under the testing H_0 : the lot is good, H_1 : the lot is poor
- β : consumer's risk, the probability of accepting a poor lot, the probability of Type II error under the testing H_0 : the lot is good, H_1 : the lot is poor
- \hat{C}_{pkm} = the estimated C_{pkm} which is a random variable
- $N(0,1)$: standard normal distribution

Chapter 1

Introduction and Literature Review

Acceptance sampling has been a cost and time-effective inspection tool adopted by the vendor and the buyer for product quality or reliability assurance (Montgomery 2009; Schilling and Neubauer 2009). In an acceptance sampling plan, a sample of items is randomly selected from a lot and inspected for defects. Based on the inspection results and a pre-determined criterion for acceptance/rejection, the disposition of the lot is determined. The acceptance/rejection decision can be based on attributes (e.g., percentage of defective items), referred to as attributes sampling or variables (e.g., using various process capability indices and in case of continuous quality characteristics), referred to as variables sampling.

With the ever-increasing demands of higher quality products by the consumers, it has become absolutely necessary for the industries to produce products of very high quality. Thus, there is a growing interest in the adoption of variables-based sampling plans by the industry as they provide better statistical control and require less sampling than attributes-based sampling plans. However, the precise measurements required by a variables-based plan would probably cost more than the simple classification of items required by an attributes-based plan, but the reduction in sample size, may more than offset this exact expense. Such saving may be especially marked if inspection is destructive and the item is expensive (Schilling, 1982; Duncan, 1986; Montgomery, 2001).

Constructing economic models of acceptance sampling plans has been a topic of interest to researchers for years (Wetherill and Chiu 1975). However, designing economically optimal acceptance sampling plans has not been widely addressed even though sampling remains a commonly used technique in many quality engineering systems (Ferrell and Chhoker, 2002). Hsu and Hsu (2012) developed an attributes-based economic model of acceptance sampling plans in a two-stage supply chain. Their model can be applied to any two-stage supply chain which includes

a vendor and a buyer, where a vendor delivers a lot of products to the buyer, and the buyer decides whether to accept or reject the entire lot based on the quality of the sample selected from the lot. However, they considered only single dependent state sampling plans, which although appropriate for industries with low-medium quality requirements, do not satisfy the needs of industries with very stringent quality requirements. At the same time, their plans are attributes-based which require a larger sample size as compared to variables-based plans; a caveat which we address in this research work.

Yen et al. (2015) proposed a variables-based acceptance sampling plan with rectifying inspection based on C_{pk} index, which is based on process yield. However, high-tech industrial processes have more strict requirements for product quality and reliability than other industries. Therefore, it would be more suitable for high-tech industry to use C_{pkm} rather than C_p , C_{pk} and C_{pm} which considers the process yield and quality loss simultaneously (Wu and Wang 2017). The plans proposed by Yen et al. (2015) are single state dependent plans which provide lesser statistical control and risk hedging for both the vendor and the consumer.

Although single sampling plans, in which the disposition of the current lot depends only upon the quality of current lot have been found to be effective, they do not take into account the sequential nature in which the lots are produced. Any manufacturing process is a continuous process and produces products in a sequential fashion. Due to this continuous nature, the quality of the lots which are produced subsequently can be deduced from the quality of the lots that have already been produced. This leads to the passing on of the information of quality characteristics from past lost to newer lots, thus making the acceptance sampling process more robust and less prone to errors. Such sampling plans in which the disposition of the current lot depends upon the quality of the current lot as well as on the quality of previous lots are known as multiple dependent state sampling plans, the state here refers to 0/1 nature of the sampling results, 0 for rejection and 1 for acceptance.

In this project, our aim is to develop multiple dependent state(MDS) sampling plans based on the advanced process capability index C_{pkm} which provides the pre-decided risk coverage to both the consumers and producers. As in an MDS plan, we only require the data of the previous

lots which was already recorded when that lot was sampled and tested, no extra cost is incurred in an MDS plan as compared to a single state sampling plan. Thus, MDS plans provide higher risk coverage and better-quality control as compared to single sampling plans at practically no additional cost.

The inspection in any acceptance sampling plan can be of 2 types: rectifying or non-rectifying. In rectifying inspection, the sample is tested and all the defective items are corrected for the defects and subsequently included in the lot. In a non-rectifying type of inspection, the defective items are not included in the lot and are discarded. Whether the sampling plans should be rectifying or non-rectifying in nature depends on the cost of rectification as compared to the cost of scrapping. When the discarded sample can be used elsewhere or sold as a refurbished item, rectification takes place. However, when the stakes of using the products are high, as in aerospace industry, or the inspection is destructive in nature, then non-rectifying inspection takes place. Our aim in this research is to first develop non-rectifying inspection plans and then extend the research to develop rectifying inspection plans, which lie at the pinnacle of quality control standards. Additionally, we perform the economic analysis of the sampling plans by considering the total quality cost (sum of inspection, internal failure and external failure costs) as the objective function.

1.1 A Review of Various Process Capability Indices (PCIs)

Process capability indices (PCIs) are useful management tools, particularly in the manufacturing industry. PCIs provide common quantitative measures on manufacturing capability and production quality. Process capability represents the performance of a process in a state of statistical control. It is determined by the total variability that exists because of all common causes present in the system. Process capability can also be viewed as the variation in the product quality characteristic that remains after all special causes have been removed. The product's performance is then predictable because the special causes are gone. This allows us to determine the ability of the product to meet customer expectations. Most supplier certification manuals describe the recommended procedure for computing a process capability index and use

them for making acceptance/rejection decisions (Wu and Pearn, 2007). Below we list four commonly used PCIs in the quality control literature:

$$(a) \text{ Basic PCI, } C_p = \frac{USL - LSL}{6\sigma}, \text{ where } \sigma = E(X - \mu_X)^2;$$

C_p relates the process spread (the difference between the natural tolerance limits, i.e., 6σ) to specification spread (difference between upper and lower specification limits), assuming two-sided specification limits. When σ is unknown, it is replaced by its estimate, $\hat{\sigma}$. The sample standard deviation s is one estimate of $\hat{\sigma}$. Above equation shows that C_p is the ratio of the allowable process spread to the actual process spread.

$$(b) \text{ Yield-based PCI, } C_{pk} = \min(C_{pu}, C_{pl}), \text{ where } C_{pu} = \frac{USL - \mu}{3\sigma}, \text{ and } C_{pl} = \frac{\mu - LSL}{3\sigma};$$

The indices C_{pu} and C_{pl} are useful in evaluating the process performance relative to the specification limits. They also aid in determining process parameter settings (such as the process mean μ) or process parameter requirements (such as the process standard deviation σ). However, both C_{pu} and C_{pl} fail to consider the deviation of the process mean from its target value, a caveat which is addressed by C_{pm} and C_{pkm} .

$$(c) \text{ Loss-based PCI, } C_{pm} = \frac{USL - LSL}{6\tau} = \frac{C_p}{\sqrt{1+\xi^2}} \text{ where } \tau^2 = E(X - T)^2, \xi^2 = [(\mu - T)/\sigma]^2 \text{ and } T = (USL + LSL)/2 \text{ is the target;}$$

C_{pm} is regarded as the second-generation capability index, developed from the original C_p index. It takes into consideration process location and the variability in the process but fails to consider the deviation of the process mean from the target value.

$$(d) \text{ Yield-loss based PCIs, } C_{pkm} = \frac{C_{pk}}{\sqrt{1+\xi^2}} \text{ where } \xi \text{ is defined as above;}$$

The yield-based PCI, C_{pkm} is known as the third-generation process capability index that incorporates the features of both C_{pk} and C_{pm} .

The first PCIs appearing in the engineering literature was presumably the simple precision index, C_p (Juran, 1974; Sullivan, 1984, 1985; Kane 1986). C_p is defined as the ratio of the process specification spread (also known as the allowable process spread) to actual process spread. C_p does not take into account the location of the process mean and hence turns out to be a suboptimal index. The index C_{pk} , on the other hand, takes both the magnitude of the process variation and the process departure from the midpoint into consideration (Wu et al. 2009). Both the C_p and C_{pk} indices are aimed at reducing the variability in a process but do not consider the loss incurred to the producer or the consumer.

To incorporate these losses into the process capability indices, Hsiang and Taguchi (1985) introduced a new and improved loss based process capability index, C_{pm} also known as the Taguchi index, which was also later proposed independently by Chan et al. (1988). However, C_{pm} is also unable to translate the effects of the deviation of process mean from its target value quantitatively.

To this end, Pearn et al. (1992) proposed the process capability index C_{pkm} , which combines the features of the yield based index C_{pk} and the loss based index C_{pm} . It alerts the user whenever the process variance increases and/or the process mean deviates from its target value (Wu et al. 2009).

C_{pkm} provides many advantages over C_p , C_{pk} , and C_{pm} :

- i) When the process mean departs from its target value, the amount of variation in C_p , C_{pk} , C_{pm} , and C_{pkm} is in the order: $C_{pkm} \geq C_{pm} \geq C_{pk} \geq C_p$ (Pearn and Kotz, 1995). Thus, C_{pkm} is most sensitive index among all 4 indices and responds quickly to any deviations in from the normality of the process.
- ii) For some positive value s , if $C_{pk} \geq s$, it does not imply that $C_{pm} \geq s$ and vice versa. But if $C_{pkm} \geq s$, then $C_{pk} \geq s$ and $C_{pm} \geq s$ (Wu and Pearn, 2008).
- iii) Assuming T , the target value of the process mean, to be the mid-point of specification limits (i.e, $T = (USL + LSL)/2$),

$|\mu - T| < d$, if $C_p = 1$,

$|\mu - T| < d/3$, if $C_{pk} = 1$,

$|\mu - T| < d/4$, if $C_{pkm} = 1$,

where $d = (USL - LSL)/2$, and μ is the process mean (Wu and Pearn 2008). Thus, C_{pkm} gives the smallest length of the confidence interval containing the process mean μ . That is, C_{pkm} leads to the greatest reduction in process-loss among all capability indices. Quantitatively, C_{pkm} decreases the length of the confidence interval of the mean by 75% as compared to C_p and 25% as compared to C_{pk} .

iv) As C_{pkm} is derived by combining C_{pk} and C_{pm} , it behaves more like C_{pk} when σ^2 is large and more like C_{pm} if σ^2 is small (Jessenberger and Weihs, 2000). Thus, C_{pkm} is more robust and adjust itself as per the needs of the quality requirements.

1.2 A Review of the Estimated C_{pkm} and its Sampling Distribution

Since its first introduction by Pearn et al. (1992), many researchers have described the statistical properties of the C_{pkm} index. As the process mean μ and process standard deviation σ are generally unknown, Pearn et al. (1992) suggested using the natural estimator of C_{pkm} denoted by \hat{C}_{pkm} :

$$\hat{C}_{pkm} = \min \left\{ \frac{USL - \bar{X}}{3\sqrt{S_n^2 + (\bar{X} - T)^2}}, \frac{\bar{X} - LSL}{3\sqrt{S_n^2 + (\bar{X} - T)^2}} \right\} = \frac{d - |\bar{X} - m|}{3\sqrt{S_n^2 + (\bar{X} - T)^2}}$$

where $\bar{X} = \sum_{i=1}^n (X_i / n)$ and $S_n^2 = \sum_{i=1}^n (X_i - \bar{X})^2 / (n - 1)$ are the maximum likelihood estimators (MLEs) of μ and σ^2 , respectively. Using these estimators, Chen and Hsu (1995) derived the asymptotic distribution of the C_{pkm} index. Wright (1998) proposed a rather complicated

expression for the probability density function of C_{pkm} . Under the normality assumption for the quality characteristic, Vännman (1997) derived the explicit expression for the cumulative distribution function (CDF) of the random variable \hat{C}_{pkm} (which we'll denote by Y from now on to simplify notation):

$$F_{\hat{C}_{pkm}}(x) = 1 - \int_0^{b\sqrt{n}/(1+3x)} G\left(\frac{(b\sqrt{n}-t)^2}{9x^2} - t^2\right) [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})] dt,$$

where:

- $b = 3*C_{pkm} * \sqrt{1 + \xi^2} + |\xi|$ where $\xi = (\mu - T)/\sigma$
- $G(\cdot)$ is the cumulative distribution function (CDF) of the χ^2_{n-1} distribution, where n = sample size is the number of degrees of freedom of the chi-square distribution
- $\phi(\cdot)$ is the probability distribution function of the standard normal distribution, $N(0,1)$

1.3 Producer's and Consumer's risk and Critical Acceptance value

Producer's risk is the probability that a good product will be rejected as a bad product by the consumer. As rejecting the lot of satisfactory quality is a risk faced by any producer, the probability of rejecting a lot under the sampling inspection plan is called Producer's risk. It calculates the probability of loss from:

- (1) rejecting a batch which, in fact, should have been accepted, or
- (2) accepting a batch that, in fact, will be rejected by the customer.

In terms of Hypothesis testing, producer's risk is equivalent to rejecting a null hypothesis when it is true. It is also called Type I error or α error. Acceptable quality level (AQL) is the terminology used to define this level of quality. The AQL is a percent defective that is the base line requirement for the quality of the producer's product. The acceptable quality limit (AQL) is the worst tolerable process average (mean) in percentage or ratio that is still considered

acceptable; that is, it is at an acceptable quality level. In a quality control procedure, a process is said to be at an acceptable quality level if the appropriate statistic used to construct a control chart does not fall outside the bounds of the acceptable quality limits.

Consumer's risk is a potential risk found in all consumer-oriented products which is the risk that a product not meeting quality standards will pass undetected through the manufacturer's quality control system and enter the consumer marketplace. Accepting the lot of unsatisfactory quality is a risk for any consumer. Limiting quality level (LQL) or rejectable quality level (RQL) is the terminology used to define this level of unacceptable quality. An alternative terminology, when the quality level is expressed in terms of percentage non-conformance, is lot tolerance percent defective (LTPD). The rejectable quality level (RQL) is the highest defective rate or defect rate that the consumer is willing to tolerate in an individual lot. Thus, the RQL describes what the sampling plan will reject, and the AQL describes what the sampling plan will accept.

The critical acceptance value, c^* , is the minimum value of \hat{C}_{pkm} (as calculated from the sample) for which the lot will be accepted. If $\hat{C}_{pkm} \geq c^*$, the lot is accepted, but if $\hat{C}_{pkm} < c^*$, the lot undergoes 100% inspection and is then accepted.

For a given C_{pkm} for the process, the operating characteristic equation, i.e., the probability of acceptance of the lot as a function of C_{pkm} is given by:

$$P_a(C_{pkm}) = P(Y \geq c^*) = 1 - P(Y \leq c^*) = 1 - F_Y(c^*)$$

More explicitly, we can write:

$$P_a(C_{pkm}) = \int_0^{b\sqrt{n}/(1+3c^*)} G\left(\frac{(b\sqrt{n}-t)^2}{9c^{*2}} - t^2\right) [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})] dt$$

Chapter 2

Problem Definition and Mathematical Formulations

For the sake of completeness, we start with a general structure of a sampling plan and develop the proposed models of a single variables-based non-rectifying acceptance sampling plan based on C_{pkm} . The plan is further extended to include dependence on multiple states, rectifying inspection and total quality cost (sum of inspection, internal failure and external failure costs) considerations.

2.1 Mathematical formulation of a general acceptance sampling plan

A general single sampling plan broadly consists of two steps:

- (1) Finding the criterion for acceptance or rejection of lots.
 - (i) Determining the cost function (or objective function) of the sampling plan to be minimized, e.g. sample size, average total inspection, total quality cost, etc.
 - (ii) Determining the sample statistic to be used for making acceptance and rejection decisions, e.g. number of poor items in the sample or PCIs value for the sample.
 - (iii) Defining constraint equations for pre-specified producer's risk α and consumer's risk β using the sample statistic.
 - (iv) Minimizing the cost function as per the constraints to obtain the optimum sample size and critical value of sample statistic.

- (2) Comparing the sample statistic calculated by picking a random sample of optimum size from a lot with the critical value of the statistic to make the acceptance/rejection decision for that lot.

We now consider one attributes-based and one variables-based model which use different sample statistic. The cost function may or may not be the same.

Model 1: Attributes-based acceptance sampling plan

An attributes-based plan (traditional approach) with the critical acceptance number c^* as the decision-making criterion.

Minimize Cost

such that: $P(\hat{T} \leq c^* | p = p^{AQL}) \geq 1-\alpha$

$$P(\hat{T} \leq c^* | p = p^{RQL}) \leq \beta$$

Here, \hat{T} is the number of defective items in the sample of size n , and p denotes the probability of defective items in the lot.

Model 2: Variables-based acceptance sampling plan

A variables-based plan (contemporary) approach with the critical acceptance value c^* (in terms of process capability indices, say C_{pkm}) as the decision-making criterion.

Minimize Cost

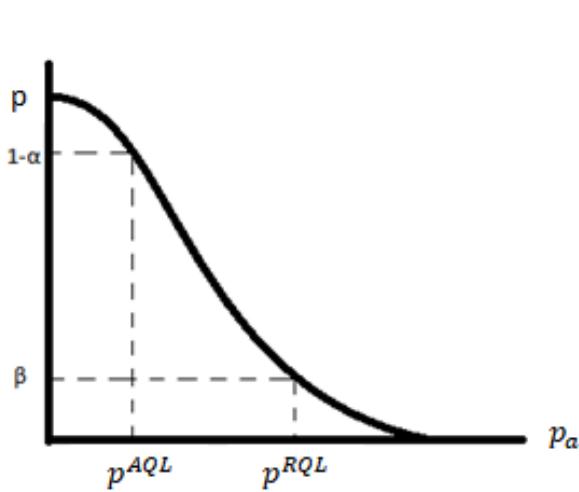
such that: $P(\hat{T} \geq c^* | C_{pkm} = C_{pkm}^{AQL}) \geq 1-\alpha$

$$P(\hat{T} \geq c^* | C_{pkm} = C_{pkm}^{RQL}) \leq \beta$$

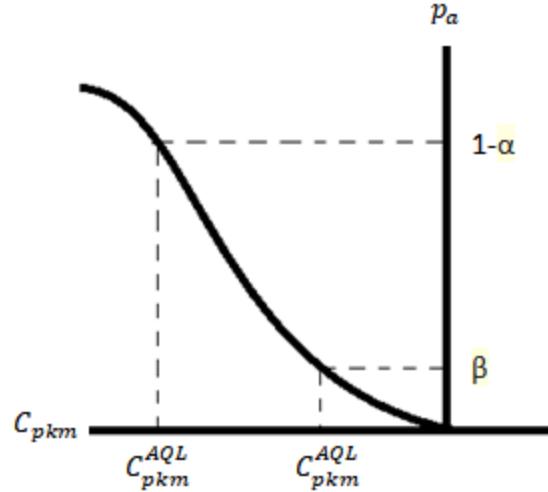
Here, T is the \hat{C}_{pkm} value of the sample.

Figure 1 illustrates the operating characteristic curve of an attributes-based sampling plan (i.e., the probability of acceptance of the lot as a function of the proportion of defective items in

the lot) and a variables-based sampling plan (i.e., the probability of acceptance of the lot as a



(a) A Traditional Sampling Plan



(b) The Proposed Sampling Plan

Figure 1: Operating characteristic curves of (a) traditional sampling plan (b) proposed sampling plan

function of the C_{pkm} value of the process).

It has been well established in the literature of acceptance sampling that the area under the curve of any process capability index gives the probability of acceptance of the lot (Wu and Pearn, 2017). Thus, we need the integral of C_{pkm} which will give us the probability of acceptance of the lot w.r.t. its C_{pkm} value. The pioneering work by Vännman and Kotz (1995) derived the genuine C_{pkm} sampling distribution under normal distribution conditions. Using a variable transformation and integration technique similar to Vännman (1997), the explicit form of the cumulative distribution function (CDF) of C_{pkm} is given as follows:

$$F_{\hat{C}_{pkm}}(x) = 1 - \int_0^{b\sqrt{n}/(1+3x)} G\left(\frac{(b\sqrt{n} - t)^2}{9x^2} - t^2\right) [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})] dt$$

where:

- $x > 0$
- $b = 3*C_{pkm} * \sqrt{(1 + \xi^2) + |\xi|}$

- $\zeta = (\mu - T)/\sigma$
- $G(\cdot)$ is the cumulative distribution function of the chi-square distribution χ^2_{n-1} , where $n = \text{sample size}$ is the number of degrees of freedom of the chi-square distribution
- $\phi(\cdot)$ is the probability distribution function of the standard normal distribution $N(0,1)$

Thus, the probability of acceptance of a lot as a function of C_{pkm} , denoted by $\pi_{A,m}$ can be represented as:

$$\begin{aligned}\pi_{A,m}(C_{pkm}) &= P(\hat{C}_{pkm} \geq k_a | C_{pkm}) + P(k_r < \hat{C}_{pkm} < k_a | C_{pkm}) \times P(\hat{C}_{pkm} \geq k_a | C_{pkm})^m \\ &= 1 - F_{\hat{C}_{pkm}}(k_a | C_{pkm}) + [F_{\hat{C}_{pkm}}(k_a | C_{pkm}) - F_{\hat{C}_{pkm}}(k_r | C_{pkm})] \times [1 - F_{\hat{C}_{pkm}}(k_a | C_{pkm})]^m.\end{aligned}$$

The first term in this equation, $P(\hat{C}_{pkm} \geq k_a | C_{pkm})$ is equal to the probability of acceptance of the lot given the \hat{C}_{pkm} value calculated using the samples lies above the critical acceptance value. However, if the \hat{C}_{pkm} value lies between the critical acceptance and rejection limits, then we check whether previous m lots (m signifies the number of dependent states for the sampling plan) were all accepted or not. Thus, the second term is multiplied m times with the probability of acceptance of each previous lot.

The producer's and consumer's risk constitute the constraints for our model and are given as follows:

$$P(\text{accepting the lot} | C_{pkm} = C_{pkm}^{AQL}) \geq 1 - \alpha$$

$$P(\text{accepting the lot} | C_{pkm} = C_{pkm}^{RQL}) \leq \beta$$

Using the above definition of the probability of acceptance of a lot, these equations can be further written down as:

$$\pi_{A,m}(c_{AQL}) = 1 - F_{\hat{C}_{pkm}}(k_a | c_{AQL}) + [F_{\hat{C}_{pkm}}(k_a | c_{AQL}) - F_{\hat{C}_{pkm}}(k_r | c_{AQL})] \times [1 - F_{\hat{C}_{pkm}}(k_a | c_{AQL})]^m \geq 1 - \alpha,$$

$$\pi_{A,m}(c_{RQL}) = 1 - F_{\hat{C}_{pkm}}(k_a | c_{RQL}) + [F_{\hat{C}_{pkm}}(k_a | c_{RQL}) - F_{\hat{C}_{pkm}}(k_r | c_{RQL})] \times [1 - F_{\hat{C}_{pkm}}(k_a | c_{RQL})]^m \leq \beta.$$

2.2 Complete mathematical models

2.2.1 Optimisation equations for the non-rectifying multiple state acceptance sampling plan

Minimize n

such that:

$$\pi_{A,m}(c_{AQL}) = 1 - F_{\hat{C}_{pmk}}(k_a | c_{AQL}) + \left[F_{\hat{C}_{pmk}}(k_a | c_{AQL}) - F_{\hat{C}_{pmk}}(k_r | c_{AQL}) \right] \times \left[1 - F_{\hat{C}_{pmk}}(k_a | c_{AQL}) \right]^m \geq 1 - \alpha,$$

$$\pi_{A,m}(c_{RQL}) = 1 - F_{\hat{C}_{pmk}}(k_a | c_{RQL}) + \left[F_{\hat{C}_{pmk}}(k_a | c_{RQL}) - F_{\hat{C}_{pmk}}(k_r | c_{RQL}) \right] \times \left[1 - F_{\hat{C}_{pmk}}(k_a | c_{RQL}) \right]^m \leq \beta.$$

As it is a multiple dependent state sampling plan, the dependence on the disposition of the previous lots is decided by the value of m . To derive single sampling plan from these equations, the second summation term in the constraints equations has to be made zero. Thus, while solving these equations, $m = 1, 2, 3, \dots$ for multiple state sampling and $m \rightarrow \infty$ for single sampling. In practise, m is set to a large number such as 1000.

2.2.2 Optimisation equations for the rectifying multiple state acceptance sampling plan

For deriving the equations for the rectifying multiple sampling plans, the objective function is set to average total inspection. The average total inspection is given by:

$$ATI = n + (1 - \pi_A)^*(N - n)$$

Thus, the final mathematical to be optimised becomes:

Minimize: $ATI = n + (1 - \pi_A)^*(N - n)$

such that:

$$\pi_{A,m}(c_{AQL}) = 1 - F_{\hat{C}_{pmk}}(k_a | c_{AQL}) + \left[F_{\hat{C}_{pmk}}(k_a | c_{AQL}) - F_{\hat{C}_{pmk}}(k_r | c_{AQL}) \right] \times \left[1 - F_{\hat{C}_{pmk}}(k_a | c_{AQL}) \right]^m \geq 1 - \alpha,$$

$$\pi_{A,m}(c_{RQL}) = 1 - F_{\hat{C}_{pmk}}(k_a | c_{RQL}) + \left[F_{\hat{C}_{pmk}}(k_a | c_{RQL}) - F_{\hat{C}_{pmk}}(k_r | c_{RQL}) \right] \times \left[1 - F_{\hat{C}_{pmk}}(k_a | c_{RQL}) \right]^m \leq \beta.$$

2.2.3 Optimisation equations for the economic rectifying single state acceptance sampling plan

For deriving the optimisation model for the rectifying economic acceptance sampling plan, the total quality cost has to minimized. The total quality cost is a linear function of 3 types of quality costs:

- i) inspection cost, which is the fixed cost incurred when inspecting one item from the sample.

Let the inspection cost per item be fixed and equal to c_i . Then the total inspection cost will be given by:

$$\text{Total inspection cost} = c_i * ATI$$

where, ATI is the average total inspection per lot.

- ii) internal failure cost, such as scrap and rework costs for the materials, labour and overhead costs associated with production. Internal failure cost would be incurred in those items which are defective and which have been identified during the inspection process. Let the internal failure cost per item be given by c_{if} . Then, the total internal failure cost for the lot will be given by:

$$\text{Total internal failure cost} = c_{if} * D_d$$

where D_d is the number of defects detected during the sampling and is given by:

$$D_d = (1 - \Phi(3 * C_{pk} * \sqrt{1 + \xi^2})) * (n + (1 - \pi_A) * (N - n))$$

- iii) external failure costs, comprising primarily of costs due to customer complaints, which includes the cost of investigation and adjustments, and those associated with receipt, handling, repair and replacement of non-conforming items. External failure cost would be incurred in those items which are defective and which have not been identified during the

inspection process. Let the external failure cost per item be given by c_{ef} . Then, the total external failure cost for the lot will be given by:

$$\text{Total external failure cost} = c_{ef} * D_n$$

where D_n is the number of defects not detected during the sampling and is given by:

$$D_n = (\pi_A) * (1 - \Phi(3 * C_{pk} * \sqrt{(1 + \xi^2)})) * (N-n)$$

The total quality cost(TQC), then becomes:

$$\text{Total Quality Cost} = c_i * ATI + c_{if} * D_d + c_{ef} * D_n$$

Thus, the final optimisation model becomes:

$$\text{Minimize } c_i * ATI + c_{if} * D_d + c_{ef} * D_n$$

such that:

$$\pi_{A,m}(c_{AQL}) = 1 - F_{\hat{C}_{pmk}}(k_a | c_{AQL}) + [F_{\hat{C}_{pmk}}(k_a | c_{AQL}) - F_{\hat{C}_{pmk}}(k_r | c_{AQL})] \times [1 - F_{\hat{C}_{pmk}}(k_a | c_{AQL})]^m \geq 1 - \alpha,$$

$$\pi_{A,m}(c_{RQL}) = 1 - F_{\hat{C}_{pmk}}(k_a | c_{RQL}) + [F_{\hat{C}_{pmk}}(k_a | c_{RQL}) - F_{\hat{C}_{pmk}}(k_r | c_{RQL})] \times [1 - F_{\hat{C}_{pmk}}(k_a | c_{RQL})]^m \leq \beta.$$

2.3 Solution methodology for the models

In all the three optimization schemes developed above, the constraint equations are non-linear in nature. We used an algorithm known as sequential quadratic programming (SQP) to solve for the plan parameters. SQP is an iterative method for solving constrained non-linear optimisation problems which is a generalisation of the Newton's method for unconstrained optimisation. It replaces the objective function with a quadratic approximation and the non-linear constraints with linear approximations. It has been implemented in almost all popular scientific computing languages. For our purpose, we utilised “*fmincon*” function in **MATLAB®** by setting

the Algorithm option to “*sqp*”. All the **MATLAB®** codes, along with their modes of usage, are available in the Appendix.

Chapter 3

Results and Conclusions

We are now well-versed to solve the above described optimisation schemes. Here, we first present a representative scheme of solving for the plan parameters in general and then move on to concrete solutions by considering exemplar values for the input plan parameters.

3.1 A general optimisation scheme for the economic sampling plan

i) Given the pre-determined values:

- process capability requirements ($C_{pkm}^{AQL}, 1 - \alpha$) and (C_{pkm}^{RQL}, β)
- C_{pkm} of the process
- cost values: c_i , c_{if} and c_{ef}
- lot size N

ii) Find the optimum sample size n and critical acceptance value c^* using the model described in

iii) Randomly pick a sample of size n from the lot of size N and calculate the associated value of \hat{C}_{pkm} .

iv) The decision criterion are given as follows:

- If $\hat{C}_{pkm} \geq c^*$, accept the lot and replace the defective items in the sample with good ones.
- If $\hat{C}_{pkm} < c^*$, perform 100% inspection and replace all the defective items with good ones, and then accept the lot.

Both the other optimisation schemes can also be solved in a similar manner.

3.2 Single and multiple dependent state non-rectifying inspection plans

For illustration, we now use a numerical example to demonstrate the results of our sampling plan. First, we provide solutions to optimisation scheme 1: Optimisation equations for the non-rectifying multiple state acceptance sampling plan.

The values for various parameters are as follows:

- $N = 1000$
- $C_{AQL} = 1.33$ and $C_{RQL} = 1.00$
- $\alpha = \{0.01, 0.025, 0.05, 0.075, 0.1\}$
- $\beta = \{0.01, 0.025, 0.05, 0.075, 0.1\}$

Table 1. The plan parameters of the variables MDS non-rectifying sampling plan with $m = 1$.

1. Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{RQL} = 1.00$

α	β	n, sample size	k_a , critical acceptance value	k_r , critical rejection value
0.01	0.01	133	1.21	1.07
0.01	0.025	109	1.20	1.05
0.01	0.05	92	1.19	1.03
0.01	0.075	81	1.18	1.02
0.01	0.1	74	1.17	1.01
0.025	0.01	115	1.23	1.07
0.025	0.025	94	1.22	1.06
0.025	0.05	77	1.21	1.04
0.025	0.075	68	1.20	1.03
0.025	0.1	61	1.19	1.01
0.05	0.01	101	1.25	1.08
0.05	0.025	81	1.24	1.06
0.05	0.05	66	1.23	1.04
0.05	0.075	57	1.22	1.03
0.05	0.1	51	1.22	1.02
0.075	0.01	92	1.26	1.08
0.075	0.025	73	1.25	1.07
0.075	0.05	59	1.24	1.05
0.075	0.075	50	1.24	1.03
0.075	0.1	44	1.23	1.02
0.1	0.01	86	1.27	1.09

0.1	0.025	67	1.27	1.07
0.1	0.05	54	1.26	1.05
0.1	0.075	46	1.25	1.03
0.1	0.1	40	1.25	1.02

Table 2. The plan parameters of the variables MDS non-rectifying sampling plan with $m = 2$. Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{RQL} = 1.00$

α	β	n, sample size	k_a , critical acceptance value	k_r , critical rejection value
0.01	0.01	138	1.20	0.99
0.01	0.025	112	1.19	0.98
0.01	0.05	92	1.18	0.97
0.01	0.075	80	1.17	0.96
0.01	0.1	72	1.16	0.95
0.025	0.01	121	1.22	0.99
0.025	0.025	97	1.21	0.98
0.025	0.05	78	1.19	0.97
0.025	0.075	67	1.18	0.95
0.025	0.1	60	1.18	0.95
0.05	0.01	107	1.23	0.99
0.05	0.025	84	1.22	0.97
0.05	0.05	67	1.21	0.96
0.05	0.075	57	1.20	0.95
0.05	0.1	50	1.19	0.94
0.075	0.01	99	1.24	0.99
0.075	0.025	77	1.23	0.97
0.075	0.05	61	1.22	0.96
0.075	0.075	51	1.22	0.95
0.075	0.1	45	1.21	0.93
0.1	0.01	92	1.25	0.98
0.1	0.025	71	1.24	0.97
0.1	0.05	56	1.23	0.95
0.1	0.075	47	1.23	0.94
0.1	0.1	40	1.22	0.93

Table 3. The plan parameters of the variables MDS non-rectifying sampling plan with $m = 3$. Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{RQL} = 1.00$

α	β	n, sample size	k_a , critical acceptance value	k_r , critical rejection value
0.01	0.01	144	1.20	0.94
0.01	0.025	117	1.18	0.93
0.01	0.05	96	1.17	0.92
0.01	0.075	84	1.16	0.91
0.01	0.1	75	1.15	0.90
0.025	0.01	128	1.21	0.93
0.025	0.025	102	1.20	0.92
0.025	0.05	83	1.18	0.91
0.025	0.075	71	1.17	0.90
0.025	0.1	63	1.17	0.90
0.05	0.01	114	1.22	0.92
0.05	0.025	90	1.21	0.91
0.05	0.05	72	1.20	0.90
0.05	0.075	61	1.19	0.90
0.05	0.1	53	1.18	0.90
0.075	0.01	105	1.23	0.91
0.075	0.025	82	1.22	0.90
0.075	0.05	65	1.21	0.90
0.075	0.075	55	1.20	0.90
0.075	0.1	48	1.19	0.90
0.1	0.01	99	1.24	0.91
0.1	0.025	77	1.23	0.90
0.1	0.05	60	1.22	0.90
0.1	0.075	50	1.21	0.90
0.1	0.1	43	1.20	0.90

Tables 1, 2 and 3 represent the results obtained from the proposed model when the number of previous samples inspected is $m = 1, 2$ and 3 respectively.

Table 4. The plan parameters of the variables MDS non-rectifying plan with $m = 1\text{--}10$. Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{RQL} = 1.00$

	m=1	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9	m=10
n	92	92	97	101	103	106	108	109	111	112
k_a	1.19	1.18	1.17	1.17	1.16	1.16	1.16	1.16	1.16	1.16
k_r	1.04	0.98	1.00	1.01	0.98	1.02	1.02	0.99	1.02	1.01

Table 5. The plan parameters of the variables non-rectifying single state sampling plan.
Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{RQL} = 1.00$

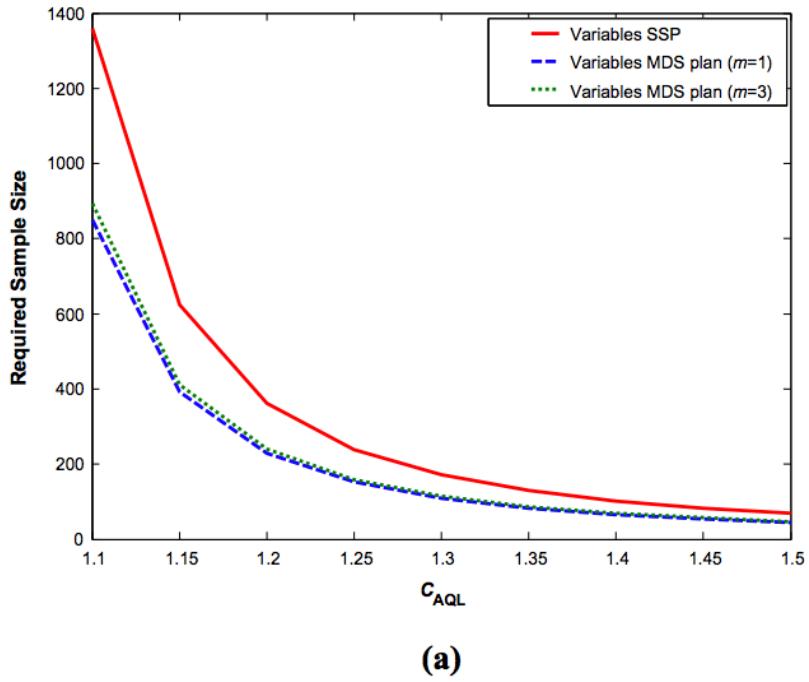
α	β	n, sample size	k_a , critical acceptance value	k_r , critical rejection value
0.01	0.01	202	1.16	1.01
0.01	0.025	169	1.15	0.99
0.01	0.05	144	1.14	0.97
0.01	0.075	128	1.13	0.95
0.01	0.1	117	1.12	0.95
0.025	0.01	173	1.18	1.20
0.025	0.025	143	1.16	1.20
0.025	0.05	120	1.15	1.20
0.025	0.075	106	1.14	1.20
0.025	0.1	95	1.13	1.20
0.05	0.01	150	1.19	1.20
0.05	0.025	123	1.18	1.20
0.05	0.05	101	1.16	1.20
0.05	0.075	88	1.15	1.20
0.05	0.1	79	1.15	1.20
0.075	0.01	136	1.20	1.20
0.075	0.025	110	1.19	1.20
0.075	0.05	90	1.18	1.20
0.075	0.075	78	1.17	1.20
0.075	0.1	69	1.16	1.20
0.1	0.01	126	1.21	1.20
0.1	0.025	101	1.20	1.20
0.1	0.05	81	1.19	1.20
0.1	0.075	70	1.18	1.20
0.1	0.1	62	1.17	1.20

Table 6. The required sample size of the variables single sampling plan and the proposed variables multiple dependent state plan under various values of C_{pmk}^{AQL} , C_{pmk}^{RQL} , α and β .

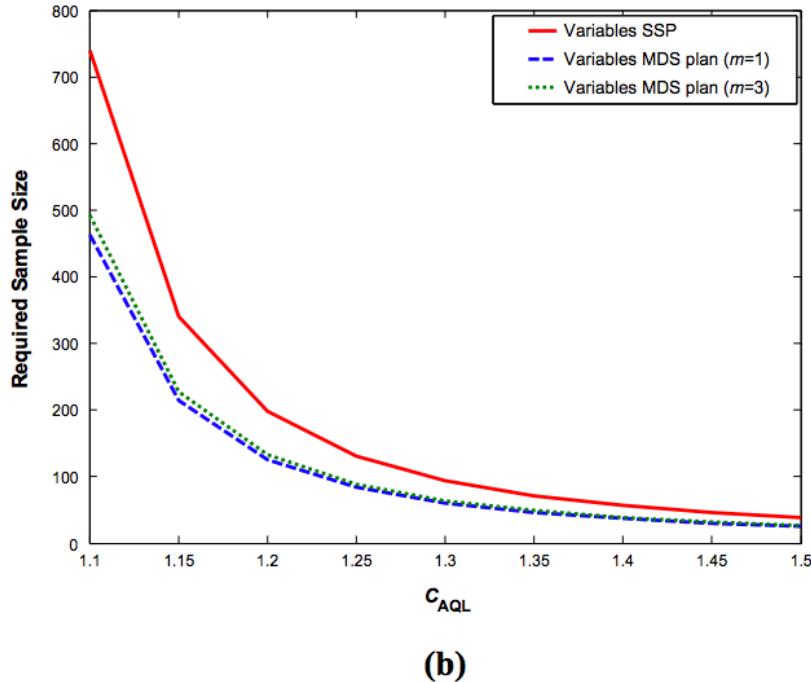
		$c_{AQL} = 1.33$ $c_{RQL} = 1.00$			$c_{AQL} = 1.50$ $c_{RQL} = 1.33$			$c_{AQL} = 1.67$ $c_{RQL} = 1.33$			$c_{AQL} = 2.00$ $c_{RQL} = 1.67$						
α	β	SSP	MDS Plan			MDS Plan			MDS Plan			MDS Plan					
			$m = 1$	$m = 2$	$m = 3$	SSP	$m = 1$	$m = 2$	$m = 3$	SSP	$m = 1$	$m = 2$	$m = 3$	SSP	$m = 1$	$m = 2$	$m = 3$
0.010	0.010	202	133	138	145	1039	673	699	734	286	187	195	204	426	278	289	303
	0.025	170	110	112	118	877	558	570	599	240	155	158	166	358	230	235	247
	0.050	144	92	92	97	749	470	470	494	204	130	130	137	305	193	193	203
	0.075	129	82	81	84	671	418	412	432	182	115	114	119	273	172	169	177
	0.100	117	74	72	75	614	381	371	387	166	105	102	106	249	156	152	158
0.025	0.010	174	116	122	128	886	580	610	644	245	162	171	180	365	241	253	267
	0.025	144	94	97	103	738	474	489	519	203	132	137	145	303	196	202	214
	0.050	120	78	79	83	621	393	397	422	170	109	111	117	254	162	164	174
	0.075	106	68	68	72	550	345	344	364	150	96	96	101	224	142	142	150
	0.100	96	61	60	63	499	311	306	323	136	86	85	89	203	128	126	133
0.050	0.010	151	101	108	114	765	505	537	572	213	142	151	161	316	210	224	238
	0.025	123	81	85	91	627	406	424	454	173	114	119	127	258	169	176	188
	0.050	102	66	68	72	520	332	339	364	143	93	95	102	213	138	141	151
	0.075	89	57	58	62	455	288	290	310	125	80	81	87	187	119	120	128
	0.100	79	51	51	54	409	257	256	272	112	72	71	76	167	106	112	112
0.075	0.010	137	93	99	106	691	459	492	527	193	130	139	149	286	192	206	220
	0.025	111	74	77	83	560	365	385	414	156	103	109	116	231	152	160	172
	0.050	90	59	61	66	459	295	304	328	127	83	86	92	189	123	127	136
	0.075	78	51	52	55	399	254	258	277	110	71	72	78	164	105	107	115
	0.100	69	45	45	48	355	224	225	242	98	63	63	68	146	93	93	100
0.100	0.010	127	86	93	100	636	424	459	493	178	121	130	139	264	178	192	206
	0.025	101	68	72	77	511	335	355	384	143	95	101	108	211	140	148	160
	0.050	82	54	56	61	415	267	278	301	115	76	79	85	171	112	116	125
	0.075	70	46	47	51	357	228	234	253	99	64	66	71	147	95	97	105
	0.100	62	40	41	44	316	201	203	219	87	57	57	62	130	84	84	91

As is clear from Table 6, the sample size required by a MDS sampling plan is always smaller than that required by a single sampling plan for same values of plan parameters. For example, for $C_{pmk}^{AQL} = 1.33$, $C_{pmk}^{RQL} = 1.00$, $\alpha = 0.075$ and $\beta = 0.01$, the single sampling plan requires 137 samples to be inspected whereas the multiple sampling plan with $m = 1$ requires 93, $m = 2$ requires 99 and $m = 106$ samples respectively. Thus, on an average, the required sample size is reduced by at least 20%.

Figure 2 depicts the required sample size vs C_{pmk}^{AQL} values for the single state and multiple dependent state sampling plans with $m = 1, 2$ and 3 . As is clear from the figure, the single sampling plan always requires more number of samples as compared to the multiple state plans.



(a)



(b)

Figure 2: Required sample size of the variables SSP and the proposed MDS plan with different values of C_{pmk}^{AQL} under $C_{pmk}^{RQL} = 1$ (a) with $(\alpha, \beta) = (0.01, 0.05)$ and (b) with $(\alpha, \beta) = (0.05, 0.10)$.

The operating characteristic (OC) curve depicts the discriminatory power of an acceptance sampling plan. The OC curve plots the probabilities of accepting a lot versus the fraction defective or process capability index. Higher the slope of the OC curve, the higher the discriminatory power of the sampling plan. Figure 3 shows the OC curve of the variables-based single sampling plan and the proposed multiple-dependent state sampling plan for $m = 1, 2$ and 3 , while keeping all other parameters the same.

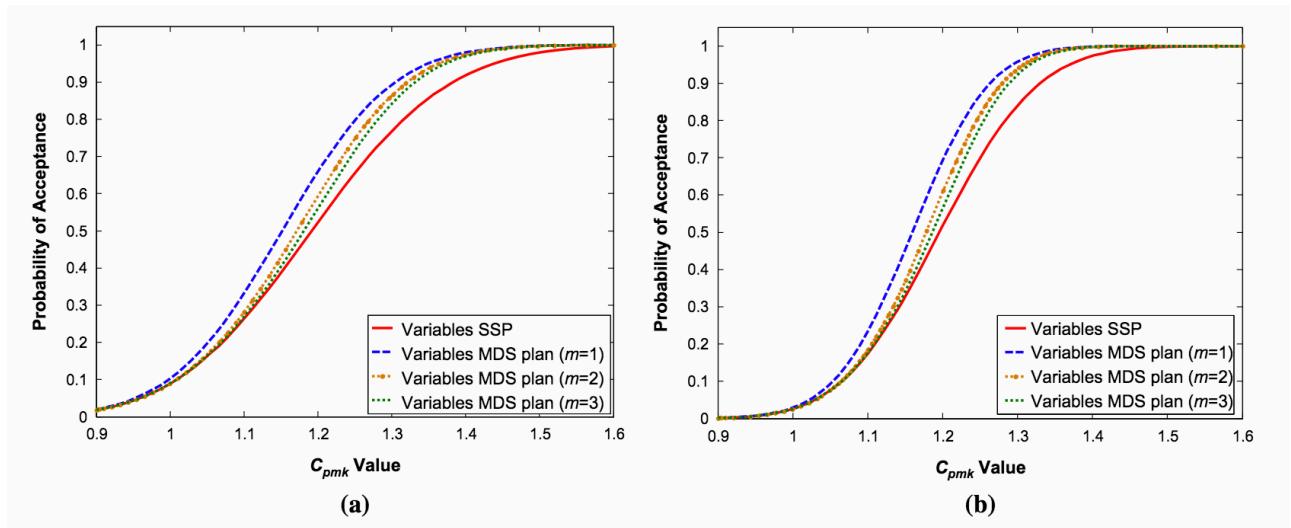


Figure 3: Operating characteristic(OC) curve of the variables-based single sampling plan and the proposed multiple dependent state sampling plan for $m = 1, 2$ and 3

As is clear from figure 3, the discriminatory power of the MDS plan is higher as compared to SSP for the same values of the plan parameters.

3.3 Multiple dependent state rectifying inspection plans

We now provide solutions to optimisation scheme 2: Optimisation equations for the rectifying multiple state acceptance sampling plan.

The values for various parameters are as follows:

- $N = 1000$
- $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{RQL} = 1.00$

- $\alpha = \{0.01, 0.025, 0.05, 0.075, 0.1\}$
- $\beta = \{0.01, 0.025, 0.05, 0.075, 0.1\}$

Table 7. The plan parameters of the variables rectifying state sampling plan for m=1. Plan parameters: N = 1000, $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{RQL} = 1.00$

α	β	n, sample size	k_a	k_r
0.01	0.01	380	1.20	0.90
0.01	0.025	271	1.18	0.90
0.01	0.05	204	1.16	0.90
0.01	0.01	380	1.20	0.90
0.01	0.025	271	1.18	0.90
0.01	0.05	204	1.16	0.90
0.01	0.075	173	1.15	0.90
0.01	0.1	153	1.14	0.90
0.025	0.01	379	1.22	0.90
0.025	0.025	267	1.20	0.90
0.025	0.05	193	1.18	0.90
0.025	0.075	158	1.17	0.90
0.025	0.1	137	1.16	0.90
0.05	0.01	379	1.24	0.90
0.05	0.025	266	1.22	0.90
0.05	0.05	187	1.20	0.90
0.05	0.075	149	1.19	0.90
0.05	0.1	126	1.18	0.90
0.075	0.01	379	1.25	0.90
0.075	0.025	265	1.23	0.90
0.075	0.05	185	1.22	0.90
0.075	0.075	144	1.20	0.90
0.075	0.1	120	1.19	0.90
0.1	0.01	379	1.25	0.90
0.1	0.025	265	1.24	0.90
0.1	0.05	183	1.23	0.90
0.1	0.075	141	1.21	0.90
0.1	0.1	115	1.20	0.90

Table 8. The plan parameters of the variables rectifying state sampling plan. Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 1.50$ and $C_{pmk}^{RQL} = 1.33$

α	β	n, sample size	k_a	k_r
0.01	0.01	834	1.42	1.38
0.01	0.025	561	1.43	1.36
0.01	0.05	474	1.42	1.35
0.01	0.075	423	1.42	1.34
0.01	0.1	385	1.41	1.34
0.025	0.01	582	1.44	1.37
0.025	0.025	477	1.44	1.36
0.025	0.05	397	1.43	1.35
0.025	0.075	350	1.43	1.35
0.025	0.1	316	1.42	1.34
0.05	0.01	507	1.45	1.37
0.05	0.025	409	1.45	1.37
0.05	0.05	336	1.44	1.36
0.05	0.075	293	1.43	1.35
0.05	0.1	262	1.43	1.35
0.075	0.01	461	1.46	1.38
0.075	0.025	368	1.45	1.37
0.075	0.05	299	1.45	1.36
0.075	0.075	258	1.44	1.36
0.075	0.1	230	1.44	1.35
0.1	0.01	427	1.46	1.38
0.1	0.025	338	1.46	1.37
0.1	0.05	298	1.45	1.35
0.1	0.075	233	1.45	1.36
0.1	0.1	206	1.44	1.36

Table 9. The plan parameters of the variables rectifying state sampling plan. Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 1.67$ and $C_{pmk}^{RQL} = 1.33$

α	β	n, sample size	k_a	k_r
0.01	0.01	189	1.54	1.42
0.01	0.025	157	1.53	1.41
0.01	0.05	133	1.51	1.39
0.01	0.075	119	1.50	1.38
0.01	0.1	109	1.49	1.38
0.025	0.01	164	1.56	1.43
0.025	0.025	134	1.55	1.42
0.025	0.05	112	1.53	1.41
0.025	0.075	99	1.52	1.40
0.025	0.1	90	1.51	1.39
0.05	0.01	172	1.56	1.42
0.05	0.025	150	1.55	1.38
0.05	0.05	96	1.55	1.42
0.05	0.075	84	1.54	1.41
0.05	0.1	76	1.53	1.41
0.075	0.01	131	1.59	1.45
0.075	0.025	105	1.58	1.44
0.075	0.05	86	1.56	1.44
0.075	0.075	75	1.55	1.42
0.075	0.1	67	1.54	1.42
0.1	0.01	122	1.60	1.45
0.1	0.025	97	1.59	1.45
0.1	0.05	79	1.58	1.44
0.1	0.075	68	1.56	1.43
0.1	0.1	61	1.55	1.43

Table 10. The plan parameters of the variables rectifying state sampling plan. Plan parameters: $N = 1000$, $C_{pmk}^{AQL} = 2.00$ and $C_{pmk}^{RQL} = 1.67$

α	β	n, sample size	k_a	k_r
0.01	0.01	285	1.88	1.73
0.01	0.025	242	1.86	1.73
0.01	0.05	198	1.86	1.70
0.01	0.075	180	1.84	1.70
0.01	0.1	320	1.85	1.65
0.025	0.01	247	1.89	1.76
0.025	0.025	213	1.89	1.73
0.025	0.05	283	1.89	1.805
0.025	0.075	394	1.86	1.81
0.025	0.1	184	1.86	1.69
0.05	0.01	221	1.91	1.76
0.05	0.025	182	1.90	1.73
0.05	0.05	167	1.88	1.73
0.05	0.075	424	1.87	1.74
0.05	0.1	545	1.86	1.80
0.075	0.01	238	1.92	1.74
0.075	0.025	189	1.91	1.78
0.075	0.05	201	1.90	1.73
0.075	0.075	548	1.85	1.82
0.075	0.1	543	1.86	1.81
0.1	0.01	191	1.93	1.79
0.1	0.025	146	1.93	1.74
0.1	0.05	183	1.92	1.67
0.1	0.075	542	1.86	1.81
0.1	0.1	84	1.91	1.71

3.4 Economic analysis of the rectifying single state acceptance sampling plan

We now provide solutions to optimisation scheme 3: Optimisation equations for economic analysis of the rectifying single state acceptance sampling plan.

The values for various parameters are as follows:

- $N = 1000$
- $C_{pmk}^{AQL} = 1.33$ and $C_{pmk}^{LQL} = 1.00$
- $c_i = 10, c_{if} = 20$ and $c_{ef} = 50$
- $\alpha = \{0.01, 0.025, 0.05, 0.075, 0.1\}$
- $\beta = \{0.01, 0.025, 0.05, 0.075, 0.1\}$

The optimal combination of n and c^* for both the plans are summarized in Table 11. The values in columns 3, 4 and 5 are the n, c^* and TQC values, respectively, for our plan. Similarly, the values in columns 6, 7 and 8 are the n, c^* and TQC values, respectively, for Yen et al. plan (Yen et al., 2015).

Table 11 also presents the comparative analysis between plan proposed by Yen et. al (2015) and our plan.

Table 11: Comparison of sample size and total quality cost values between Yen et. al., 2015 plan and our proposed single sampling plan

		Our Plan		Yen et. al. 2015		Reduction	
α	β	n	TQC	n	TQC	% ↓ in n	% ↓ in TQC
0.01	0.01	233	2909	250	3170	6.80	8.23
0.01	0.025	197	2493	218	2712	9.63	8.08
0.01	0.05	169	2159	183	2343	7.65	7.85
0.01	0.075	151	1955	165	2119	8.48	7.74
0.01	0.1	139	1804	149	1954	6.71	7.68
0.025	0.01	233	2909	250	3170	6.80	8.23
0.025	0.025	197	2493	218	2712	9.63	8.08
0.025	0.05	169	2159	183	2343	7.65	7.85
0.025	0.075	151	1955	165	2119	8.48	7.74
0.025	0.1	139	1804	149	1954	6.71	7.68
0.05	0.01	233	2909	250	3170	6.80	8.23
0.05	0.025	197	2493	218	2712	9.63	8.08
0.05	0.05	169	2159	183	2343	7.65	7.85
0.05	0.075	151	1955	165	2119	8.48	7.74
0.05	0.1	139	1804	149	1954	6.71	7.68
0.075	0.01	233	2909	250	3170	6.80	8.23
0.075	0.025	197	2493	218	2712	9.63	8.08
0.075	0.05	169	2159	183	2343	7.65	7.85
0.075	0.075	151	1955	165	2119	8.48	7.74
0.075	0.1	139	1804	149	1954	6.71	7.68
0.1	0.01	233	2909	250	3170	6.80	8.23
0.1	0.025	197	2493	218	2712	9.63	8.08
0.1	0.05	169	2159	183	2343	7.65	7.85
0.1	0.075	151	1955	165	2119	8.48	7.74
0.1	0.1	139	1804	149	1954	6.71	7.68

Observations and inferences from Table 11:

- The required sample size decreases as the consumer's risk (β) increases but remains constant with an increase in producer's risk (α).
- The total quality cost decreases as consumer's risk (β) increases but remains constant with

an increase in producer's risk (α).

- When C_{pkm} is used instead of C_{pk} as the PCI, then on an average both n and TQC decrease approximately by 8%. This implies that our sampling plan not only achieves a lower total quality cost for same producer's and consumer's risk but also achieves it with a smaller sample size. Thus, we are able to attain a win-win situation for both the producer and the consumer.

3.4.1 Sensitivity analysis of n , c^* and TQC

The values for various parameters are as follows:

- $N = 1000$
- $C_{pkm}^{AQL} = 1.33$ and $C_{pkm}^{RQL} = 1.00$
- $c_i = 10$, $c_{if} = 20$ and $c_{ef} = 50$
- $\alpha = 0.05$
- $\beta = 0.05$

Table 12 shows how TQC varies as c_i varies from 10 to 100, while $c_{if} = 20$ and $c_{ef} = 50$. For Yen et al. plan, $n = 183$ while for our plan, $n = 169$ which remain constant for all values of c_i .

Table 12: Comparison of TQC: c_i varies from 10 to 100, while $c_{if} = 20$ and $c_{ef} = 50$

c_i	10	20	30	40	50	60	70	80	90	100
TQC: Our plan	2159	4311	6463	8616	10768	12920	15072	17224	19376	21529
TQC: Yen et. al.	2343	4679	7015	9351	11687	14024	16360	18696	21032	23368
% \downarrow in TQC	7.85	7.86	7.87	7.86	7.87	7.87	7.87	7.87	7.87	7.87

Table 13 shows how TQC varies as c_{if} varies from 10 to 100, while $c_i = 10$ and $c_{ef} = 50$. For Yen et al. plan, $n = 183$ and for our plan, $n = 169$ which remain constant for all values of c_{if} .

Table 13: Comparison of TQC: c_{if} varies from 10 to 100, while $c_i = 20$ and $c_{ef} = 50$

c_{if}	10	20	30	40	50	60	70	80	90	100
TQC: Our plan	2159	2159	2159	2159	2160	2160	2160	2161	2161	2161
TQC: Yen et. al.	2343	2343	2343	2344	2344	2344	2345	2345	2345	2345
% ↓ in TQC	7.85	7.85	7.85	7.89	7.85	7.85	7.89	7.85	7.85	7.85

Table 14 shows how TQC varies as c_{ef} varies from 10 to 100, while $c_i = 10$ and $c_{if} = 20$. For Yen et al. plan, $n = 183$ and for our plan, $n = 169$ which remain constant for all values of c_{if} .

Table 14: Comparison of TQC: c_{ef} varies from 10 to 100, while $c_i = 10$ and $c_{if} = 20$

c_{ef}	10	20	30	40	50	60	70	80	90	100
TQC: Our plan	2338	2339	2341	2342	2343	2344	2345	2347	2348	2349
TQC: Yen et. al.	2154	2155	2156	2158	2159	2161	2168	2163	2164	2165
% ↓ in TQC	7.87	7.87	7.90	7.86	7.85	7.81	7.80	7.84	7.84	7.83

From Tables 12, 13 and 14, we conclude that the total quality cost is primarily dependent on the inspection cost and almost independent of the internal and external failure costs. Thus, industries should focus on minimising inspection costs to lower their total quality costs. Moreover, we observe that the optimum sample size is independent of all cost values

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Appendix

Appendix B.1

MATLAB Code for defining the optimization function

```
% This file contains the definitions of all the functions that will be used for the optimisation.
```

```
format shortG
```

```
c=1.2;
```

```
syms t
```

```
% c_aql = C_pmk value at acceptable quality level (AQL)
```

```
c_aql = 1.33;
```

```
% c_rql = C_pmk value at lot tolerance percent defective (LTPD)
```

```
c_rql = 1.00;
```

```
% N = size of the lot (Important: N is the size of the whole lot from which
```

```
% samples are taken, it is not the size of the sample)
```

```
N=1000;
```

```
% zeta = (mu-T)/sigma, mu = mean of the population, T = target value of the
```

```
% statistic, sigma = population variance
```

```
zeta=0.50;
```

```
tqc2 = @(x) x(1)
```

Appendix B.2

MATLAB code to define the constraints (producers and consumers risks)

```
function [c,ceq] = constraints_btp(x, alpha, beta)
```

```
m=1000;
```

```
zeta = 0.50;
```

```
c_aql = 1.33;
```

```
b1 = 3*c_aql*sqrt(1+zeta^2) + zeta;
```

```
c_rql = 1.00;
```

```
b2 = 3*c_rql*sqrt(1+zeta^2) + zeta;
```

```
ll = 0;
```

```
ul_b1_x2 = b1*sqrt(x(1))/(1+3*x(2));
```

```
ul_b1_x3 = b1*sqrt(x(1))/(1+3*x(3));
```

```
ul_b2_x2 = b2*sqrt(x(1))/(1+3*x(2));
```

```
ul_b2_x3 = b2*sqrt(x(1))/(1+3*x(3));
```

```
syms t
```

```
c(1) = 1 - alpha - ( integral( @(t) chi2cdf( (b1.*sqrt(x(1))-t).^2/(9*x(2).^2) - t.^2, x(1)-1).*...  
    (normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1)))), ll,ul_b1_x2 ) + ...  
    ( integral( @(t) chi2cdf( (b1.*sqrt(x(1))-t).^2/(9*x(3).^2) - t.^2, x(1)-1).*...
```

```

(normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1))), ll,ul_b1_x3) -...
integral( @(t) chi2cdf( (b1.*sqrt(x(1))-t).^2/(9*x(2).^2) - t.^2, x(1)-1).*...
(normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1))), ll,ul_b1_x2) ).*...
(integral( @(t) chi2cdf( (b1.*sqrt(x(1))-t).^2/(9*x(2).^2) - t.^2, x(1)-1).*...
(normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1))), ll,ul_b1_x2))^m );

c(2) = integral( @(t) chi2cdf( (b2.*sqrt(x(1))-t).^2/(9*x(2).^2) - t.^2, x(1)-1).*...
(normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1))), ll,ul_b2_x2) + ...
( integral( @(t) chi2cdf( (b2.*sqrt(x(1))-t).^2/(9*x(3).^2) - t.^2, x(1)-1).*...
(normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1))), ll,ul_b2_x3) -...
integral( @(t) chi2cdf( (b2.*sqrt(x(1))-t).^2/(9*x(2).^2) - t.^2, x(1)-1).*...
(normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1))), ll,ul_b2_x2) ).*...
(integral( @(t) chi2cdf( (b2.*sqrt(x(1))-t).^2/(9*x(2).^2) - t.^2, x(1)-1).*...
(normpdf(t+zeta*sqrt(x(1))) + normpdf(t-zeta*sqrt(x(1))), ll,ul_b2_x2))^m - beta;

```

ceq=[];

Appendix B.3

MATLAB Code for running the optimization using sample size(tqc2) as objective function and producers and consumers risk as constraints

```
% This code runs the optimisation for finding optimum values of the  
% decision variables n (sample size) and c (critical acceptance value of  
% Cpmk). In our code, n and c are represented by x(1) and x(2)  
% respectively.
```

```
format shortG
```

```
m=1000;
```

```
% For the optimisation to start, we need to supply an initial value for n  
% and c. x0 is that initial value.
```

```
x0 = [100,1.0,1.0];
```

```
% a and b are the LHS and RHS of linear inequality constraints a*x <=b.
```

```
%As we do not have linear inequality constraints, they are left blank.
```

```
a=[];
```

```
b=[];
```

```
% aeq and beq are the LHS and RHS of linear equality constraints aeq*x = beq.
```

```
% As we do not have linear inequality constraints, they are left blank.
```

```
aeq=[];
```

```
beq=[];
```

```
% lb and ub are the lower and upper bounds of decision variables n and c
```

```
% lb = [0, 0] means that n>=0 and c>=0
```

```
% ub = [1000, 2.0] means that n<=1000 and c<=2.0
```

```

lb=[20,0.9,0.9];
ub=[1000,1.5,1.5];

% alpha and beta are the sets of values of producer's and consumer's risk
% We will carry out the optimisation for each pair of alpha and beta values.
% So, in total we will have 25 optimisation runs.

```

```

alpha = [0.01,0.025,0.05,0.075,0.1];
beta = [0.01,0.025,0.05,0.075,0.1];

```

```

% After the optimisation process, n will contain the optimum sample size,
% c_critical will contain the critical Cpmk values and q_cost will contain
% the optimum sample size for each (alpha, beta) pair

```

```

n = zeros(5,5);
c_critical = zeros(5,5);
q_cost = zeros(5,5);

```

```
% code for the optimisation
```

```

for i=1:5
    alpha_ = alpha(i);

```

```

    for j =1:5
        beta_ = beta(j);

```

```
[x,f_eval] = fmincon(tqc2,x0,a,b,aeq,beq,lb,ub, @(x) constraints_btp(x,alpha_,beta_));
```

```

n(i,j) = round(x(1));
c_critical1(i,j) = x(2);
c_critical2(i,j) = x(3);
q_cost(i,j) = round(f_eval);

```

% ati = average total inspection (the function for ati is defined in the file named
functiondef.m)

% dd = no. of defects detected (the function for dd is defined in the file named
functiondef.m)

% dn = no. of defects not detected (the function for dn is defined in the file named
functiondef.m)

% pa = probability of acceptance of the lot based on n and c values

```
avg_total_ins(i,j) = ati(x);  
d_detected(i,j) = dd(x);  
d_not_detected(i,j) = dn(x);  
p_acceptance(i,j) = pa(x);  
end  
end
```

```
alpha = meshgrid(alpha,alpha);  
alpha_table = alpha(:);
```

```
beta = meshgrid(beta,beta);  
temp = beta';  
beta_table = temp(:);
```

```
temp = n';  
n_table = temp(:);
```

```
temp=c_critical1';  
c_table1 = temp(:);
```

```
temp=c_critical2';  
c_table2 = temp(:);
```

```

temp=q_cost';
q_cost_table = temp(:);

%temp=avg_total_ins';
%avg_total_ins_table = temp(:);

%temp = d_detected';
%d_detected_table = temp(:);

%temp = d_not_detected';
%d_not_detected_table = temp(:);

table = [alpha_table, beta_table, n_table, c_table, avg_total_ins_table, d_detected_table,
d_not_detected_table, q_cost_table];

table = [alpha_table, beta_table, n_table, c_table1,c_table2, q_cost_table];

%filename = 'external_failure_cost.xlsx'
%xlswrite(filename,table,'cef=100')

row_names = {};
column_names =
{'alpa','beta','sample_size','c_critical','ATI','Defects_detected','Defects_not_detected','Total_qualit
y_cost'};

column_names = {'alpa','beta','sample_size','c_critical1','c_critical2','Total_quality_cost'};

sTable = array2table(table,'RowNames',row_names,'VariableNames',column_names)

```