$\cancel{\Delta x}$ $\quad \Delta X = J \cdot \Delta\theta$ — (1)

★ Instant k represents the no of steps taken by the manipulator to reach the position and it is not specifying a fixed time step.

$$X(k+1) - X(k) = J \cdot \Delta\theta$$

$$X(k+1) = X(k) + J \cdot \Delta\theta(k) \quad \text{— (2)} \Longrightarrow$$

Eq (2) represent positioning task as a discrete-time input affine form $\left[x_{k+1} = f(x_k) + g(x_k)\,u_k\right]$

The closed loop dynamics which move the EE from current position X_c to desired position X_d are derived as,

$$\tilde{X}(k+1) = \tilde{X}(k) - J\,\Delta\theta(k) = A\,\tilde{X}(k) + B\,U(k) \quad \text{— (3)}$$

where, $\tilde{X} = X_d - X$, $A = I$, $B = -J$, $U = \Delta\theta$

It is assumed that $X_d(k+1) = X_d(k)$

The error dynamics (3) represents the closed-loop kinematic control of the manipulator as a discrete-time input affine system.

$$\underbrace{\tilde{X}(k+1)}_{\begin{bmatrix}\tilde{x}(k+1)\\ \tilde{y}(k+1)\\ \tilde{z}(k+1)\end{bmatrix}_{3\times1}} = \underbrace{\tilde{X}(k)}_{\begin{bmatrix}\tilde{x}(k)\\ \tilde{y}(k)\\ \tilde{z}(k)\end{bmatrix}_{3\times1}} - J\,\Delta\theta(k) = \tilde{X}(k) + \bar{B}\cdot\underbrace{U(k)}_{\begin{bmatrix}\theta_1(k)\\ \theta_2(k)\\ \theta_3(k)\\ \vdots\\ \theta_n(k)\end{bmatrix}_{n\times1}}$$

$n \rightarrow$ no of actuated joint

convergent of X to X_d means convergence of $\tilde{X}$ to the set $O = \{X \mid (\tilde{x}, \tilde{y}, \tilde{z}) = (0,0,0)\}$

tracking of a reference trajectory being possible with linear MPC.

$$\left.\begin{aligned}
\tilde{X}[1|k] &= A\cdot\tilde{X}[0|k] + B\cdot U[0|k]\\
\tilde{X}[2|k] &= A\cdot\tilde{X}[1|k] + B\cdot U[1|k]\\
\tilde{X}[3|k] &= A\cdot\tilde{X}[2|k] + B\cdot U[2|k]
\end{aligned}\right\} \quad \tilde{X}[i+1|k] = A\cdot\tilde{X}[i|k] + B\cdot U[i|k]$$

where $i = 0, 1 \ldots \underbrace{N-1}_{}$ $\rightarrow$ prediction horizon

further expand,

$$\tilde{X}[1|k] = A\cdot\tilde{X}[0|k] + B\cdot U[0|k]$$

$$\tilde{X}[2|k] = A\left[A\tilde{X}[0|k] + B\cdot U[0|k]\right] + B\cdot U[1|k] = A^2\cdot\tilde{X}[0|k] + AB\cdot U[0|k] + B\cdot U[1|k]$$

$$\tilde{X}[3|k] = A\left[A^2\cdot\tilde{X}[0|k] + AB\cdot U[0|k] + B\cdot U[1|k]\right] + B\cdot U[2|k]$$

$$= A^3\cdot\tilde{X}[0|k] + A^2B\cdot U[0|k] + AB\cdot U[1|k] + B\cdot U[2|k]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\begin{bmatrix} \tilde{X}[1|k] \\ \tilde{X}[2|k] \\ \tilde{X}[3|k] \\ \tilde{X}[4|k] \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ A^3 \\ A^4 \end{bmatrix} \tilde{X}[0|k] + \begin{bmatrix} B & 0 & 0 & 0 \\ AB & B & 0 & 0 \\ A^2B & AB & B & 0 \\ A^3B & A^2B & AB & B \end{bmatrix} \begin{bmatrix} U[0|k] \\ U[1|k] \\ U[2|k] \\ U[3|k] \end{bmatrix} \begin{matrix} \to 4\times1 \\ \to 4\times1 \\ \to 4\times1 \\ \\ (4N\times1) \end{matrix}$$

$$\underbrace{\underbrace{\bar{\mathcal{X}}[k_t+1|k]}_{} = \bar{A}\cdot \underbrace{\bar{\mathcal{X}}[0|k]}_{}}_{\text{Since all terms are Error in state, we can construct cost fun}^{c}} + \bar{B}\cdot\bar{U}[k_t|k]$$

$$J(\bar{U}) = \bar{\mathcal{X}}^T Q \bar{\mathcal{X}} + \bar{U}^T R \bar{U}$$

$$J(\bar{U}) = \left\{\bar{\mathcal{X}}^T[k_t+1|k]\, Q\, \bar{\mathcal{X}}[k_t+1|k]\right\} + \bar{U}^T R \bar{U}$$

$$J(\bar{U}) = \left\{\left(\bar{A}\cdot\bar{\mathcal{X}}[0|k] + \bar{B}\cdot\bar{U}[k_t|k]\right)^T Q \left(\bar{A}\cdot\bar{\mathcal{X}}[0|k] + \bar{B}\cdot\bar{U}[k_t|k]\right)\right\} + \bar{U}^T R \bar{U}$$

$$\frac{dJ(\bar{U})}{d\bar{U}} = 2\left\{\left(\bar{A}\cdot\bar{\mathcal{X}}[0|k] + \bar{B}\cdot\bar{U}[k_t|k]\right)^T\cdot Q\cdot\bar{B}\right\} + 2\bar{U}^T R = 0$$

$$\left(\bar{A}\cdot\bar{\mathcal{X}}[0|k]\right)^T Q\,\bar{B} + \left(\bar{B}\cdot\bar{U}\right)^T Q\,\bar{B} + \bar{U}^T R = 0$$

$$\bar{U}^T\cdot\bar{B}^T\cdot Q\cdot\bar{B} + \bar{U}^T\cdot R = -\bar{\mathcal{X}}^T[0|k]\cdot\bar{A}^T\cdot Q\cdot\bar{B}$$

$$\bar{U}^T\cdot\left[\bar{B}^T\cdot Q\cdot\bar{B} + R\right] = -\bar{\mathcal{X}}^T[0|k]\cdot\bar{A}^T\cdot Q\cdot\bar{B}$$

$$\left[\bar{B}^T\cdot Q\cdot\bar{B} + R\right]^T\cdot\bar{U} = -\left[\bar{\mathcal{X}}_0^T\cdot\bar{A}^T\cdot Q\cdot\bar{B}\right]^T = \cancel{-\left(\bar{\mathcal{X}}_0^T\cdot\bar{A}^T\right)}$$

$$\left[\bar{B}^T\cdot Q^T\cdot\bar{B} + R^T\right]\cdot\bar{U} = -\left(Q\cdot\bar{B}\right)^T\cdot\left(\bar{\mathcal{X}}_0^T\cdot\bar{A}^T\right)^T = -\bar{B}^T Q^T\cdot\bar{A}\cdot\bar{\mathcal{X}}_0$$

$$\boxed{\bar{U}^* = -\left[\bar{B}^T Q^T \bar{B} + R^T\right]^{-1}\left(\bar{B}^T\cdot Q^T\cdot\bar{A}\cdot\bar{\mathcal{X}}_0\right)}$$

If, $Q = I$, $R = I$ then, $Q^T = Q$, $R^T = R$