Consider a state variable, $\xi \in \mathbb{R}^d$ $\rightarrow$ define state of a robot system.

$\Rightarrow$ The temporal evolution of this state variable may be governed by either an autonomous (time-invariant)

varying with time $\rightarrow \dot{\xi}$

or non-Autonomous (time-varying) DS according to :

Autonomous $\rightarrow$ the system behaviour does not explicitly depend on time $\rightarrow$ $\dot{\xi} = f(\xi)$ — ①

non-Autonomous $\rightarrow$ the system behaviour may explicitly depend on time $\rightarrow$ $\dot{\xi} = f(t, \xi)$ — ②

To, explain this idea, let's say, we have a pendulum system and I run it with ~~same~~ same initial condition, ~~for~~ every day and the pendulum will follow the same trajectory and exhibit same behaviour regarless of which day we run the experiment.

Which means the system state evolve with time but does not depend ~~on the~~ explicitly ~~with~~ on time (getting same trajectory everyday)

OR

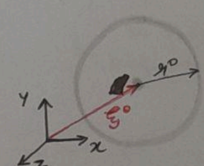
We can say that the governing Eq^n does not change based on when you start the Experiment.

But the state's will evolve with time and there Evolution can be computed using :

$$\xi_t = \xi_{t-1} + \dot{\xi} \cdot \Delta t \quad — ③$$

Now, we Introduce an obstacle any try to Induce a modulation on the trajectory of the system (state Evolution).

☆ Hyper-Sphere obstacle (d-dimensional sphere)



$\phi^s(\xi) : \mathbb{R}^d \rightarrow \mathbb{R}^d$

This obstacle will create modution throughout the state-space ~~through~~ based on a non-linear fun^c,
$$\phi^s(\xi) = \left[1 + \frac{(r^0)^2}{(\xi-\xi^0)^T(\xi-\xi^0)}\right](\xi-\xi^0)$$

Q) How $\phi^s(\xi)$ modulates the velocity of the robot?

Ans) compute Jacobian, $M(\xi) = \frac{d}{d\xi}\phi^s(\xi)$, ~~but~~ but first shift the coord-frame to object centre (ξ^0) and

Introduce a new coordinate, $\bar{\xi} = \xi - \xi^0$

$$\text{So, } M(\bar{\xi}) = \frac{d}{d\bar{\xi}}\phi^s(\bar{\xi}) = \frac{d}{d\bar{\xi}}\left[1 + \frac{r^2}{\bar{\xi}^T \cdot \bar{\xi}}\right] \cdot \bar{\xi} = \left[0 - \frac{(\bar{\xi} + \bar{\xi}^T)\, r^2}{(\bar{\xi}^T \cdot \bar{\xi})^2}\right]\bar{\xi} + \left[I + \frac{r^2}{\bar{\xi}^T \cdot \bar{\xi}}\right]$$

$$= I + \frac{r^2}{(\bar{\xi}^T \cdot \bar{\xi})} \times \frac{\bar{\xi}^T \cdot \bar{\xi}}{\bar{\xi}^T \cdot \bar{\xi}} - \frac{2\bar{\xi} \cdot \bar{\xi}^T r^2}{(\bar{\xi}^T \cdot \bar{\xi})^2} = I + \left(\frac{r}{\bar{\xi}^T \cdot \bar{\xi}}\right)^2 \left[\underbrace{\bar{\xi}^T \cdot \xi}_{\text{scalar}} \cdot I - 2\bar{\xi} \cdot \bar{\xi}^T\right]$$

final model for ~~local~~ real-time avoidance of ~~sp~~ Spherical object is obtained by applying the $M(\xi) \to$ Dynamic modulation matrix to the original DS given by:

$$\dot{\xi} = \underbrace{M(\bar{\xi}; r^o)}\cdot f(\cdot) \quad \text{— (4)}$$

locally deform the original dynamics $f(\,)$ such that robot does not hit the obstacle

Theorem: Consider a d-dimensional ~~sphere~~ static hyper-sphere obstacle in $\mathbb{R}^d$ with center ξ^o and radius r^o. The obstacle boundary consists of a hyper-surface $\mathcal{X}^b \subseteq \mathbb{R}^d = \{\xi \in \mathbb{R}^d : \|\xi - \xi^o\| = r\}$.

Subset of sphere in $\mathbb{R}^d$

Any motion, ξ_t; $t = 0, 1 \text{—} \infty$ that start outside the obstacle, i.e. $\|\xi_0 - \xi^o\| > r^o$ ($t=0$) and Evolve according to (4) never penetrate into obstacle, i.e, $\|\xi_t - \xi^o\| \geq r^o$

So, ~~we~~ to prove hyper-surface $\mathcal{X}^b \subset \mathbb{R}^d$ is Impentrable, the normal ~~vector~~ velocity at the boundary point $\xi^b \in \mathcal{X}^b$ should vanish: (Basically the velocity at the boundary and the normal vector to the boundary has to be $\perp$) $\longrightarrow$

$$\underbrace{n(\xi^b)^T}_{\text{unit normal vector at Boundary point } \xi^b} \cdot \underbrace{\dot{\xi}^b}_{\substack{\text{velocity} \\ \text{(state derivative at boundary)}}} = 0 \quad \forall \underbrace{\xi^b}_{\text{state at boundary}} \in \mathcal{X}^b \to \text{hyper~~sphere~~ surface}$$

$$n(\xi^b) = \frac{\xi^b - \xi^o}{\|\xi^b - \xi^o\|} \Rightarrow \frac{\bar{\xi}^b}{\|\bar{\xi}^b\|} = \frac{\bar{\xi}^b}{r^o} \quad \forall \xi^b \in \mathcal{X}^b \qquad (\bar{\xi}^b = \dot{\xi}^b - \xi^o)$$

Eigen-value decomposition of Square matrix $M(\bar{\xi}; r^o)$:

$$M(\bar{\xi}; r^o) = V(\bar{\xi}, r^o)\, D(\bar{\xi}, r^o)\, V(\bar{\xi}, r^o)^{-1}$$

$D \to \mathrm{Diag}(\lambda^1, \lambda^2, \lambda^3 \text{—} \lambda^d)$ where, $\lambda^1 = 1 - r^2/\bar{\xi}^T\bar{\xi}$

$\lambda^i = 1 + r^2/\bar{\xi}^T\bar{\xi}$, $\forall\, i \in 2 \text{--} d$

$V \to [v^1 \text{---} v^d]$, matrix of eigenvectors

Now, $n(\xi^b)\cdot\dot{\xi}^b = n(\xi^b)\cdot M(\bar{\xi}^b; r^o)\, f(\cdot)$

$$= \frac{(\bar{\xi}^b)^T}{r^o} \cdot V(\bar{\xi}^b; r^o)\cdot D(\bar{\xi}^b, r^o)\cdot V(\bar{\xi}^b, r^o)^{-1}\cdot f(\,)$$

Solve $\det(M(\bar{\xi}) - \lambda \hat{I}) = 0 \longrightarrow$ to find it's Eigen value

$$\det[M(\bar{\xi}) - \lambda I] = \det\left[I + \left(\frac{r}{\bar{\xi}^T \cdot \bar{\xi}}\right)^2 \left\{\bar{\xi}^T \cdot \bar{\xi} I - 2\bar{\xi}\,\bar{\xi}^T\right\} - \lambda I\right] = 0$$

let, $\alpha = \left(\frac{r}{\bar{\xi}^T \cdot \bar{\xi}}\right)^2$, $\beta = \frac{r}{\sqrt{\alpha}} = \bar{\xi}^T \cdot \bar{\xi}$

$$\Rightarrow \det\left[I + \alpha\left[\beta I - 2\bar{\xi} \cdot \bar{\xi}^T\right] - \lambda I\right] = 0$$

$$\det\left[(1 + \alpha\beta - \lambda) I - 2\alpha\,\bar{\xi} \cdot \bar{\xi}^T\right] = 0$$

★ sherman - morrison determinant formula :

$$\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)$$

let, $A = (1 + \alpha\beta - \lambda) I$, $u = -2\alpha\bar{\xi}$, $v = \bar{\xi}$

$$\det(A + uv^T) = \left(1 + \xi^T \left\{(1 + \alpha\beta - \lambda) I\right\}^{-1} (-2\alpha\bar{\xi})\right) \det\left[(1 + \alpha\beta - \lambda) I\right] = 0$$

$$\left(1 - \bar{\xi}^T \left(\frac{I}{(1 + \alpha\beta - \lambda)}\right) (+2\alpha)\,\bar{\xi}\right) \det \underbrace{\left[(1 + \alpha\beta - \lambda) I\right]}_{X} = 0$$

$$\left(1 - \left(\frac{2\alpha}{1 + \alpha\beta - \lambda}\right) \overbrace{\bar{\xi}^T I \bar{\xi}}^{\beta}\right) \det(X) = 0$$

$$\left(1 - \frac{2\alpha\beta}{1 + \alpha\beta - \lambda}\right) \cdot \det(X) = 0$$

So, $\left(1 - \frac{2\alpha\beta}{1 + \alpha\beta - \lambda}\right) = 0 \Rightarrow 2\alpha\beta = 1 + \alpha\beta - \lambda \Rightarrow \lambda = 1 - \alpha\beta = 1 - \alpha\frac{r}{\sqrt{\alpha}}$

$$\lambda = 1 - (\sqrt{\alpha}) r = 1 - (r)\left(\frac{r}{\bar{\xi}^T \cdot \bar{\xi}}\right) = 1 - \left(\frac{r^2}{\bar{\xi}^T \cdot \bar{\xi}}\right)$$

Eigen vector matrix, $V(\bar{\xi}, r^0) = [\nu^1, \nu^2 — \nu^d]$

⇒ $\nu^1 = \bar{\xi}$ (vector from obstacle centre to current position), first vector points in the radial direction from the obstacle centre ξ^0.

⇒ other Eigen vector $\nu^2 - \nu^d$ form an ~~orthogonal~~ orthonormal (orthogonal & $\|\nu^i\| = 1$) basis vector for the hyperplane tangent to the obstacle ~~centre~~ surface (∂C^b).

So, $V(\bar{\xi}^b, r^0) = [\bar{\xi}^b, \nu^2, \nu^3 - \nu^d]$ ★ $\bar{\xi}^b \perp \nu^2 \perp \nu^3 \perp - - \perp \nu^d$ → basis vector

now, $n(\bar{\xi}^b) \cdot \dot{\bar{\xi}}^b = n(\bar{\xi}^b) \cdot M(\bar{\xi}^b; r^0) f(\cdot)$

$$= \frac{(\bar{\xi}^b)^T}{r^0} \cdot \underbrace{V(\bar{\xi}^b, r^0)}_{d \times d} \cdot \underbrace{D(\bar{\xi}^b, r^0)}_{d \times d} \cdot \underbrace{V(\bar{\xi}^b, r^0)^{-1}}_{d \times d} \cdot f()$$

$$\Rightarrow \frac{\overbrace{(\bar{\xi}^b)^T}^{1 \times d}}{r^0} \cdot [\bar{\xi}^b, \nu^2, \nu^3 - \nu^d] \cdot D(\bar{\xi}^b, r^0) \cdot V(\bar{\xi}^b, r^0)^{-1} \cdot f()$$

★ $(\xi^b)^T \cdot [\bar{\xi}^b, \nu^2, \nu^3 - \nu^d] = [1, 0, 0 — 0]_{1 \times d}$

★ $[1, 0, 0, — 0] \cdot D(\bar{\xi}^b, r^0) = [1, 0, 0 — 0] \cdot \begin{bmatrix} \lambda_1 & 0 & 0 & — & 0 \\ 0 & \lambda_2 & 0 & — & 0 \\ 0 & 0 & \lambda_3 & & | \\ 0 & 0 & 0 & \ddots & \lambda_d \end{bmatrix} = 0$

at the boundary x^b, $\lambda_1 = 1 - \dfrac{r^2}{\bar{\xi}^{bT} \cdot \bar{\xi}^b} = 1 - \dfrac{r^2}{r^2} = 0$

⇒ $\underbrace{0^T}_{1 \times d} \cdot \underbrace{V(\bar{\xi}^b, r^0)^{-1}}_{d \times d} \cdot f() = 0$

So, $n(\bar{\xi}^b) \cdot \dot{\bar{\xi}}^b = 0$

★ Convex obstacles ⇒ This is a general approach can be used for various shapes and the modulation matrix $M(\bar{\xi}) = E(\bar{\xi}) \cdot D(\bar{\xi}) \cdot E(\bar{\xi})^{-1}$.

⇒ $\Gamma(\bar{\xi})$ is a continuous func that project $\mathbb{R}^d$ to $\mathbb{R}$

⇒ This func $\Gamma(\bar{\xi})$ is used to define different region ~~[illegible]~~ of the obstacle.

So, at the boundary $\Gamma(\bar{\xi}) = 1$ and inside the boundary, $\Gamma(\bar{\xi}) < 1$

for ex: for d-dimensional ellipsoid with axis length a_i.

$$\Gamma(\bar{\xi}) = \sum_{i=1}^{d} \left(\frac{\bar{\xi}_i}{a_i}\right)^2 = 1 \rightarrow \text{implicitly define a hypersurface}$$

at, point on boundary, $\xi^b \in \mathcal{X}^b$, we can compute a tangential hyper-plane define by a normal vector:

$$n(\bar{\xi}^b) = \left[\frac{d\Gamma(\bar{\xi}^b)}{d\bar{\xi}_1^b}, \frac{d\Gamma(\bar{\xi}^b)}{d\bar{\xi}_2^b}, \dots, \frac{d\Gamma(\bar{\xi}^b)}{d\bar{\xi}_d^b}\right]^T$$

★ The normal vector to a hyperplane is defined as the gradient of the func that describes the hyperplane.

So, $E(\bar{\xi}) = [n(\bar{\xi}), e^1(\bar{\xi}), \dots, e^{d-1}(\bar{\xi})]$

$$D(\bar{\xi}) = \begin{bmatrix} \lambda^1(\bar{\xi}) & 0 & \cdots \\ 0 & \lambda^2(\bar{\xi}) & \cdots \\ \vdots & \vdots & \ddots & \lambda^d(\bar{\xi}) \end{bmatrix}, \quad \lambda^1(\bar{\xi}) = 1 - \frac{1}{|\Gamma(\bar{\xi})|}$$

So, $\boxed{M(\bar{\xi}) = E(\bar{\xi})\, D(\bar{\xi})\, E(\bar{\xi})^{-1}}$

★ Since, $\Gamma(\bar{\xi})$ is monotonically increasing with $\|\bar{\xi}\|$. The matrix $D(\bar{\xi})$ of eigen values, converges to identity matrix as the distance to the obstacle increases.

So, the effect of Dynamic modulation matrix is maximum at the boundary of the obstacles and vanishes for points far from it.

Reactivity

The magnitude of the modulation created by the obstacle can be tuned by modifying the Eigen values of dynamic modulation matrix:

$$\lambda^1(\xi) = 1 - \frac{1}{|\Gamma(\xi)|^{1/\rho}}$$

$$\lambda^i(\xi) = 1 - \frac{1}{|\Gamma(\xi)|^{1/\rho}} , \quad 2 \leq i \leq d$$

★ larger the reactivity (ρ), → larger the amplitude of deflection

Stability of Dynamical System

⇒ let the target point be ξ^* (outside the obstacle boundary)

⇒ Suppose a d-dimensional DS which is globally asymptotically stable is defined by

$$\dot{\xi} = f(\xi) \quad \text{or} \quad \dot{\xi} = f(t, \xi)$$

for the DS to be globally stable, the velocity should vanish solely at the target point ξ^*

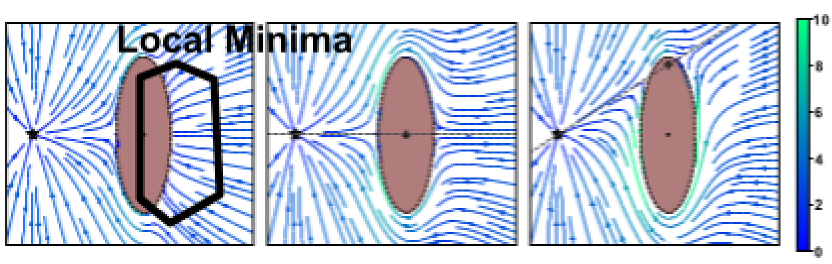
$$\text{i.e,} \quad f(\xi^*) = 0 \quad \text{or} \quad \lim_{t \to \infty} f(t, \xi^*) = 0$$

→ After $f(\cdot)$ being modulated by dynamic modulation matrix $M(\xi)$, ξ^* will remain an equilibrium point, if the velocity still vanishes at the target ξ^*.

$$\text{i.e,} \quad M(\xi^* - \xi^0) f(\xi^*) = 0 \;/\; \lim_{t \to 0} M(\xi^* - \xi^0) f(t, \xi^*) = 0$$

But, in the presence of obstacle, the target ξ^* may not remain the unique Equilibrium point of the system and other Equilib point may be created due to $M(\xi)$

★ on the boundary $\mathcal{X}^b$ of the obstacle, i.e $\xi^b \in \mathcal{X}^b$, $M(\xi)$ loses one rank, yielding a no of Equilibrium point. and we can see these Equilib point $\xi^s \in \mathcal{X}^b$, where there is a collinearity b/w velocity and normal vector at the boundary point.



(a) No reference **(b)** Reference center **(c)** Reference top

Fig. 2. (a) Using an orthonormal basis matrix $\mathbf{E}(\xi)$ as in [17], a local minimum might occur on the surface of the obstacle. The placement of the reference point $\xi^r \in \mathcal{X}^i$ marked as '+' in (b,c) guides the modulated DS around the obstacle.