



DeepLearning.AI

# Probability and Statistics for Machine Learning and Data Science

---

## Week 3: Sampling and Point Estimates

# W3 Lesson 1



DeepLearning.AI

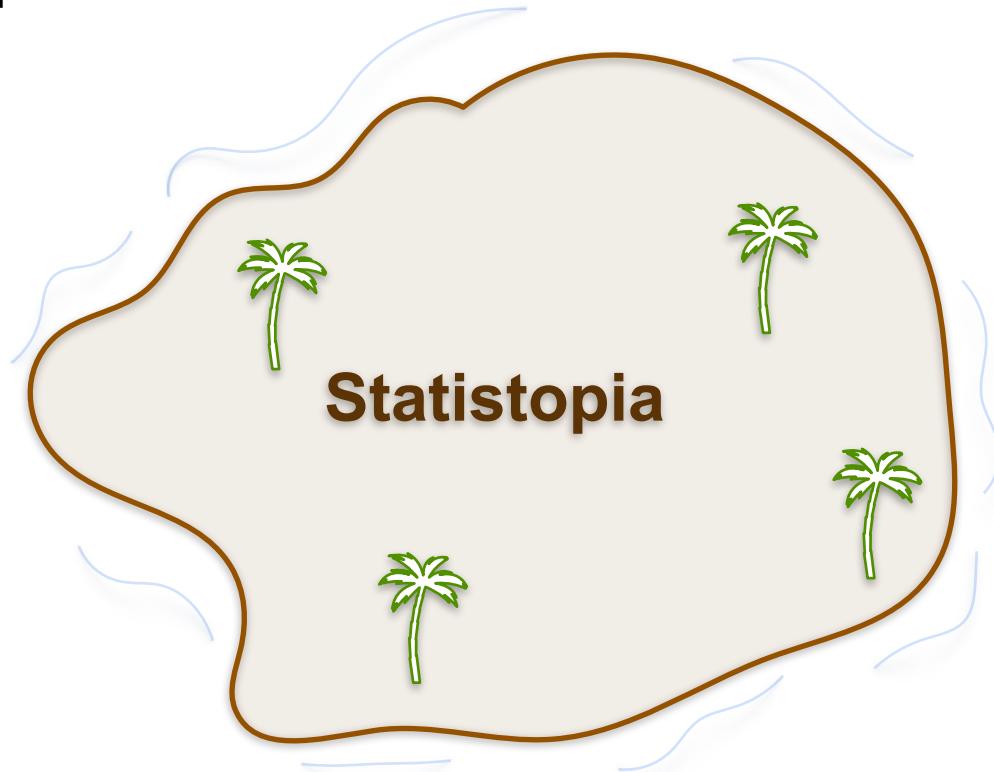
## Sample and Population

---

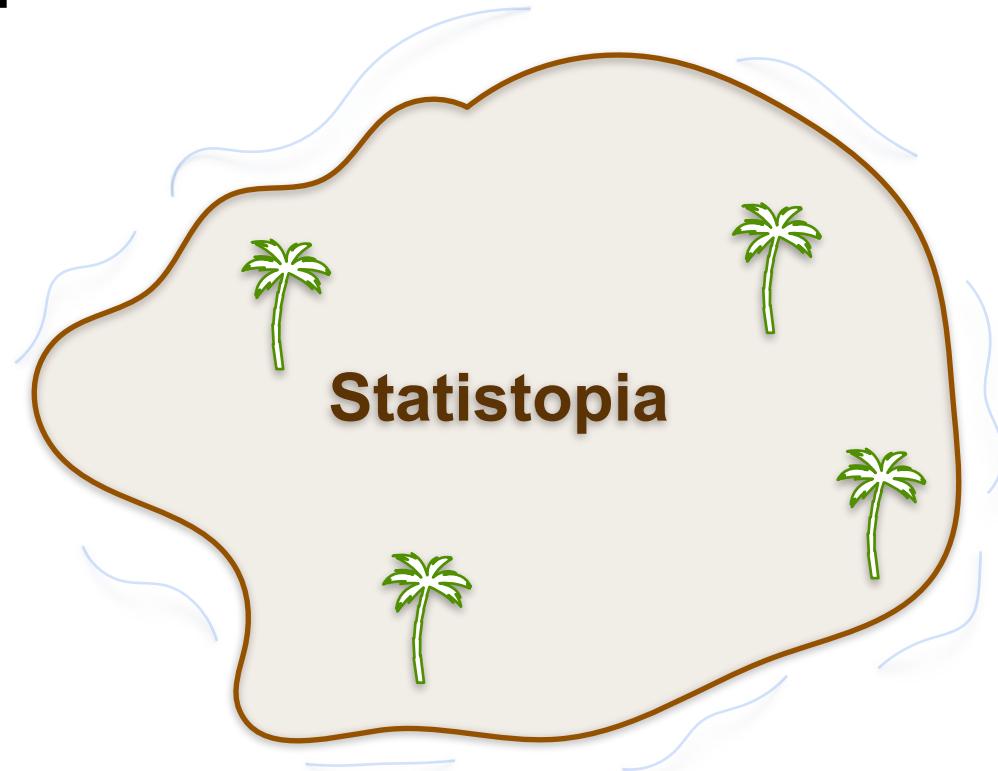
## Population and Sample

# Population and Sample

# Population and Sample



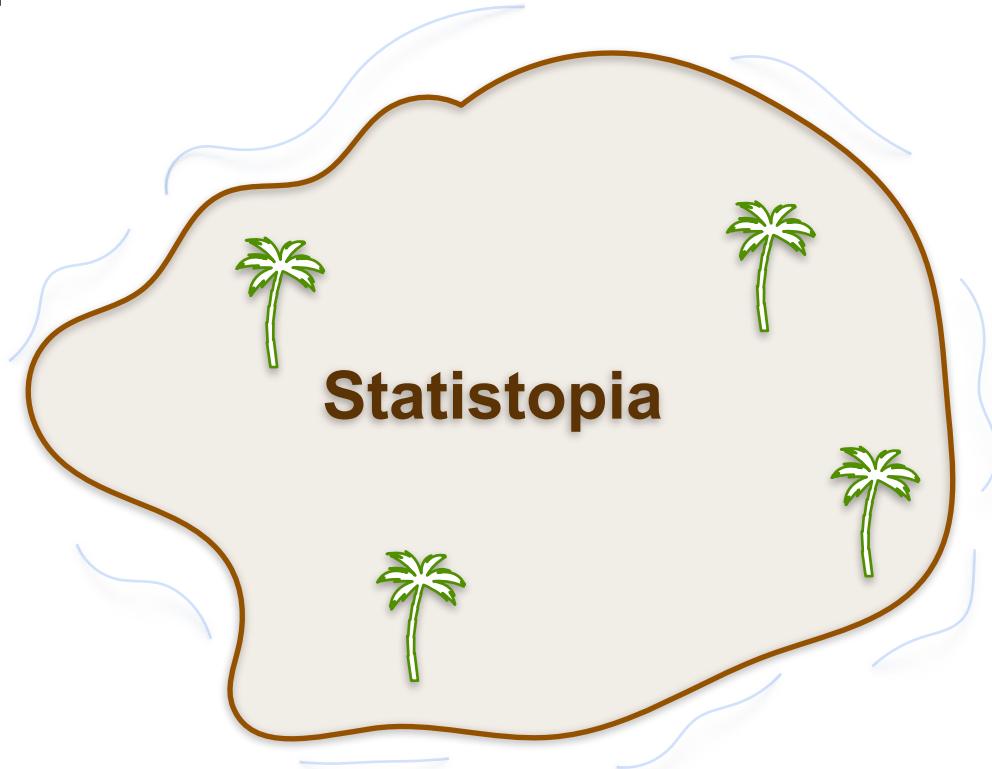
# Population and Sample



# Population and Sample



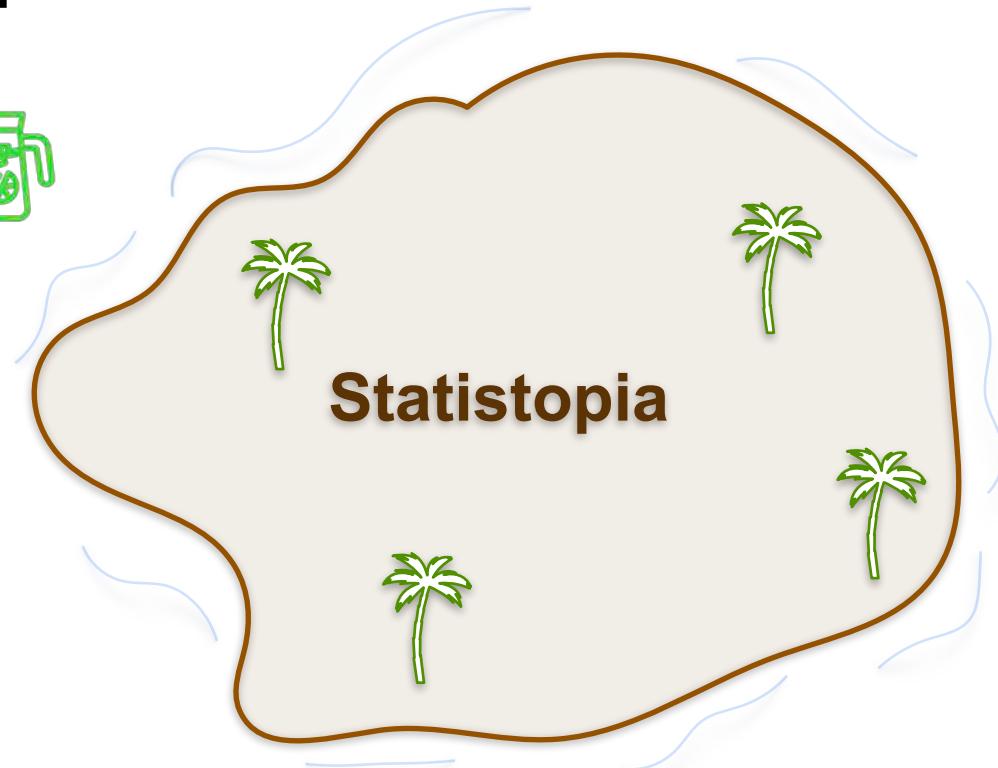
Find the **average height** of  
the people living on  
Statistopia



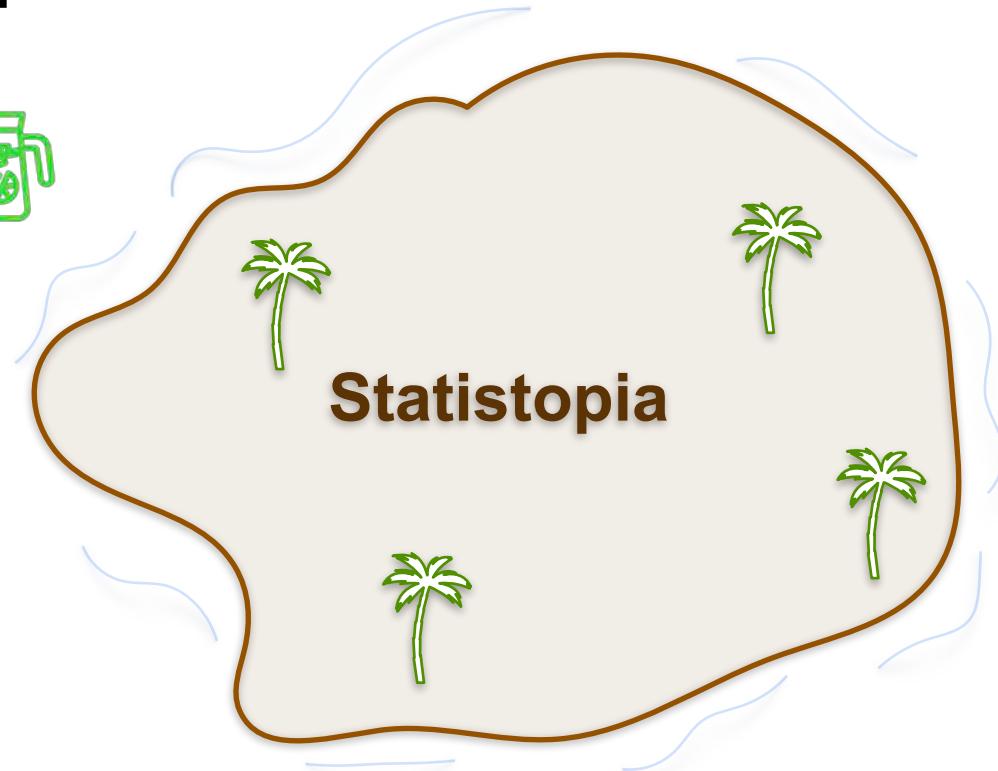
# Population and Sample



Find the **average height** of  
the people living on  
Statistopia



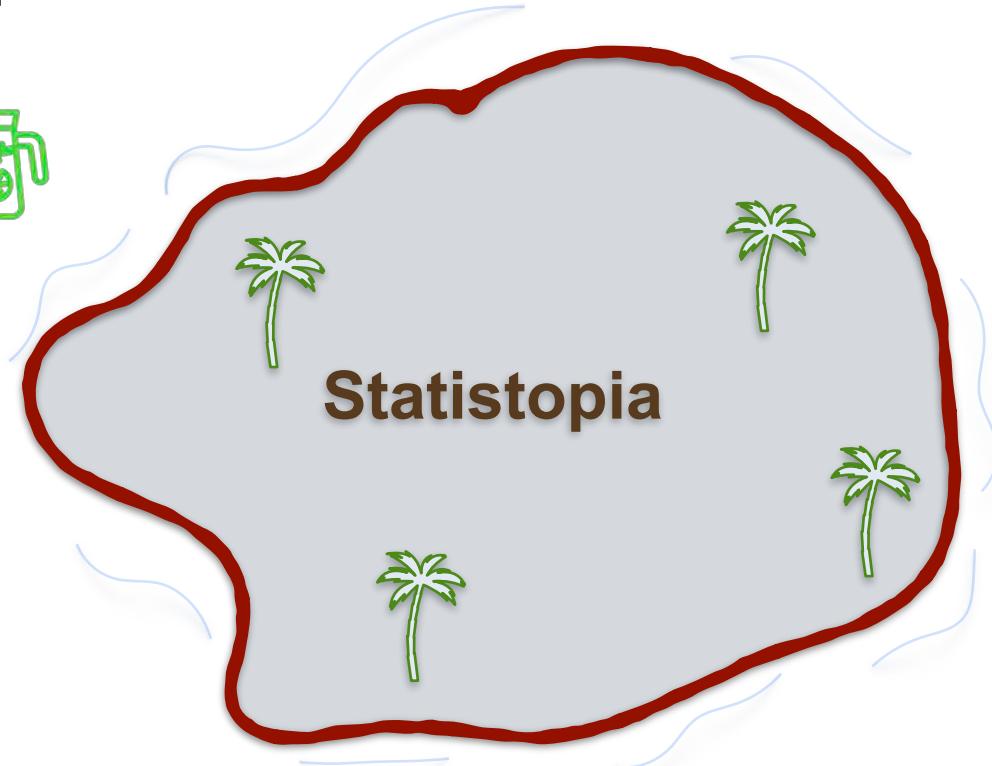
# Population and Sample



# Population and Sample



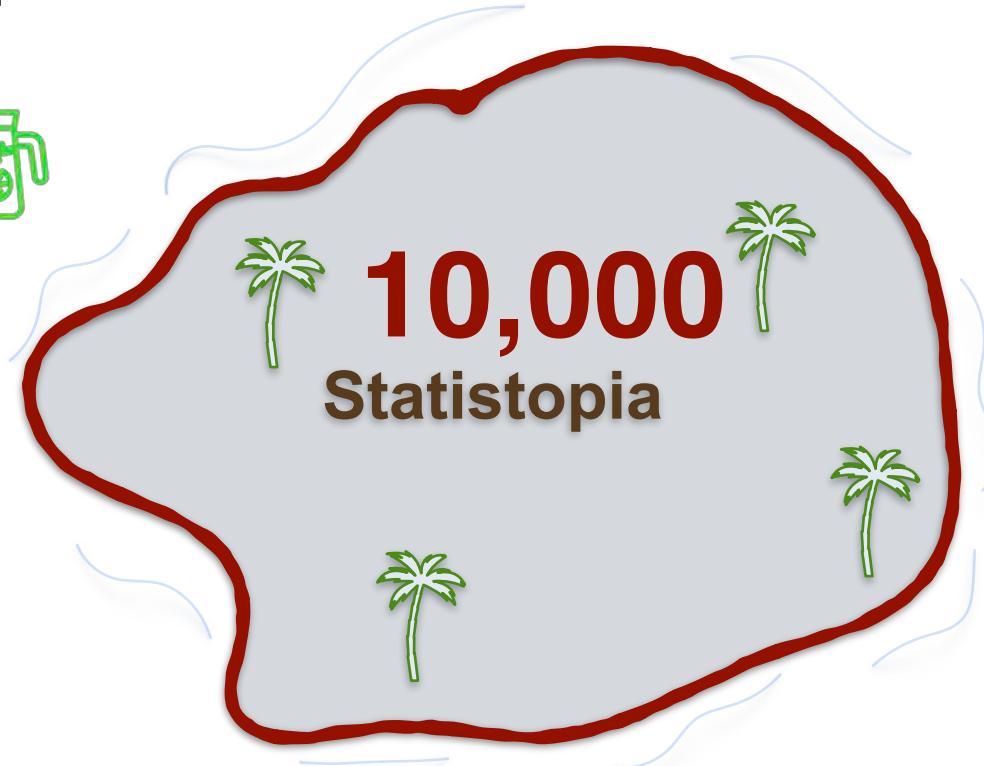
- Ask everyone on the island for their height.
- Divide by the total number



# Population and Sample



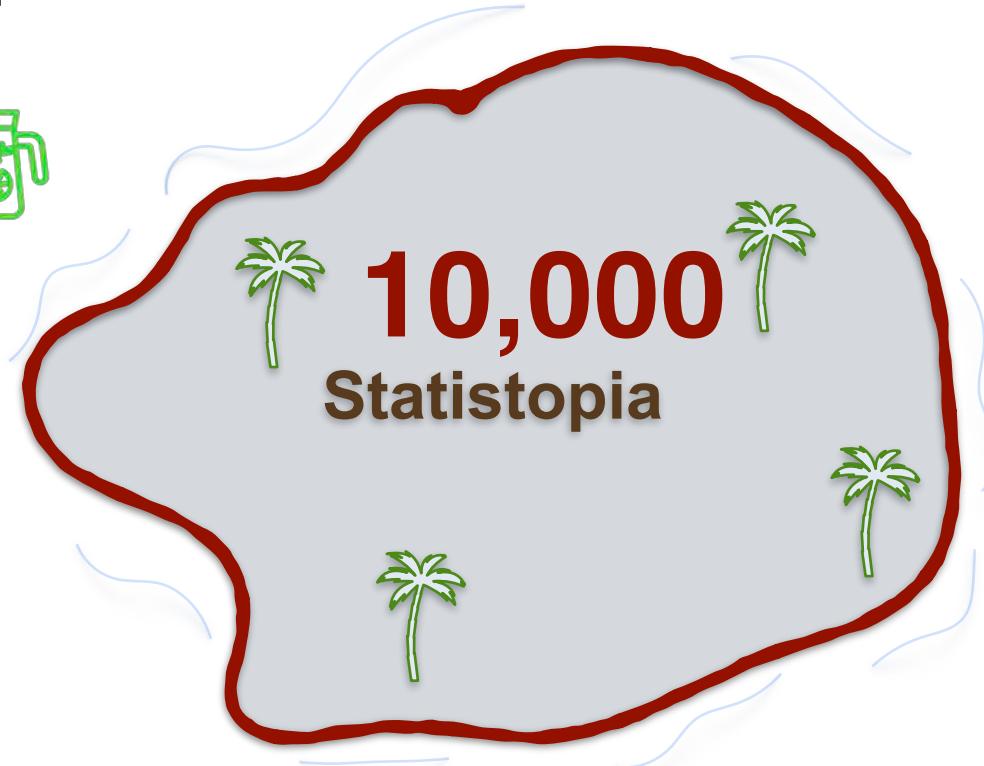
- Ask everyone on the island for their height.
- Divide by the total number



# Population and Sample



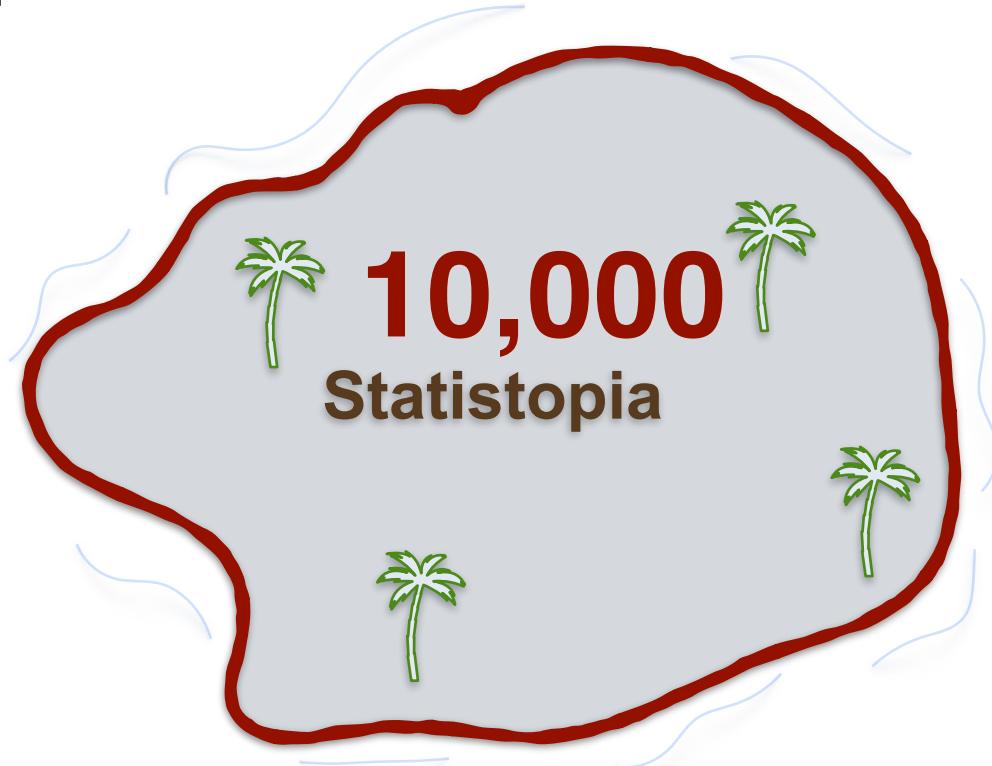
- Ask everyone on the island for their height.
- Divide by the total number



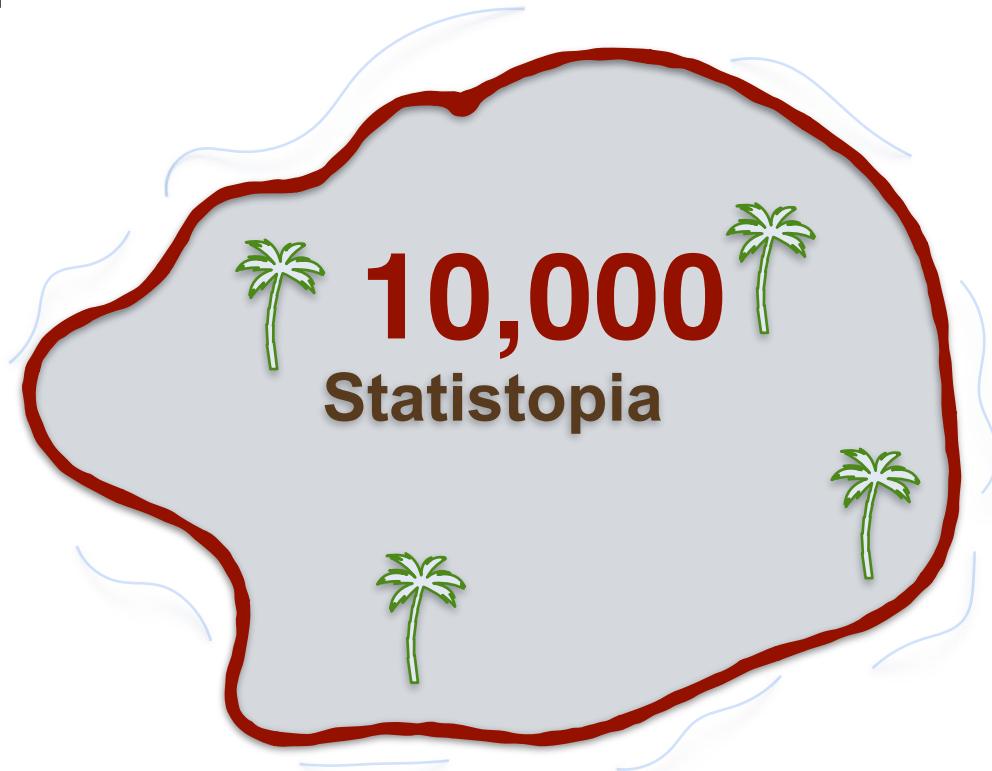
# Population and Sample



- Ask everyone on the island for their height.
- Divide by the total number



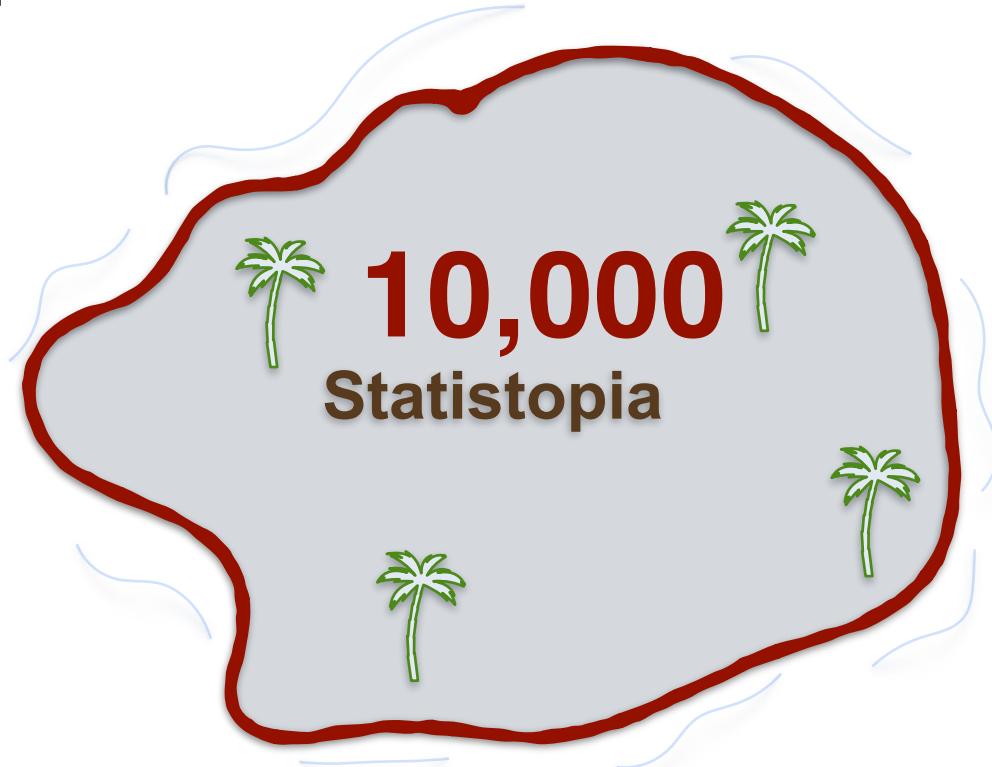
# Population and Sample



# Population and Sample



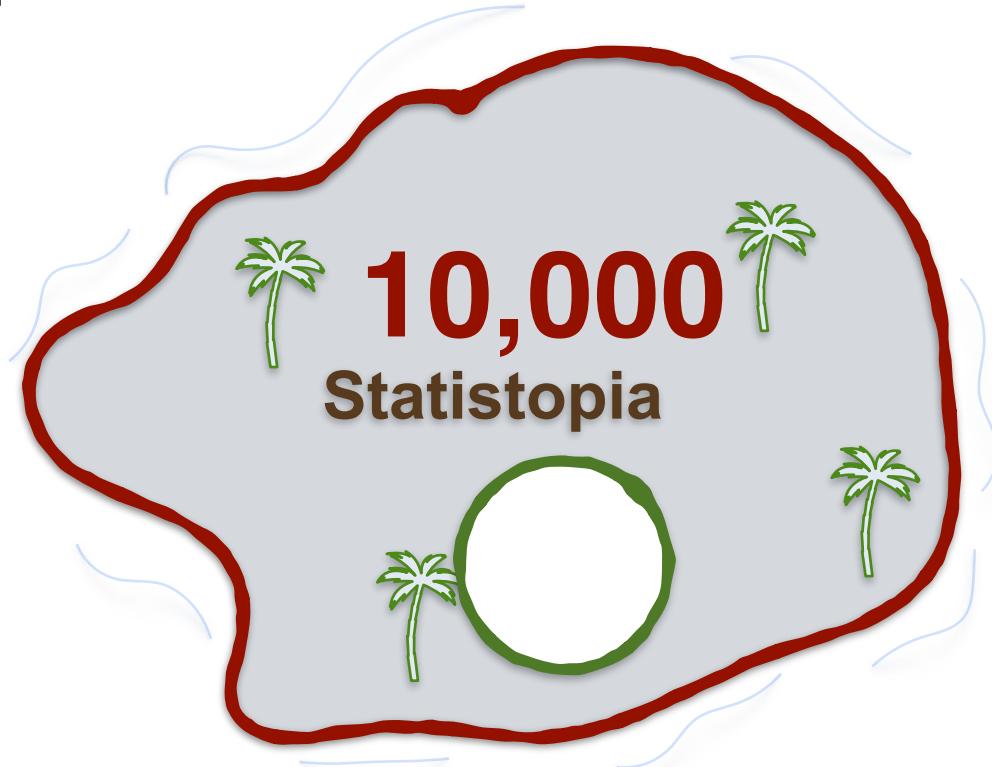
- Only ask a subset of the group to estimate the average height



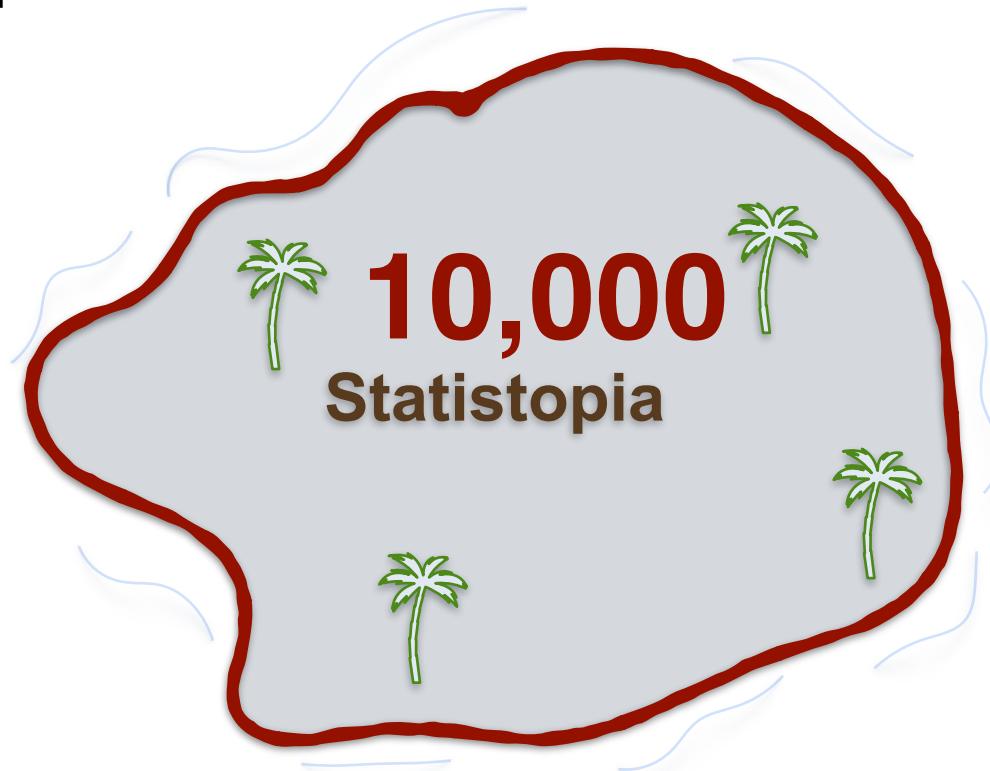
# Population and Sample



- Only ask a subset of the group to estimate the average height



# Population and Sample

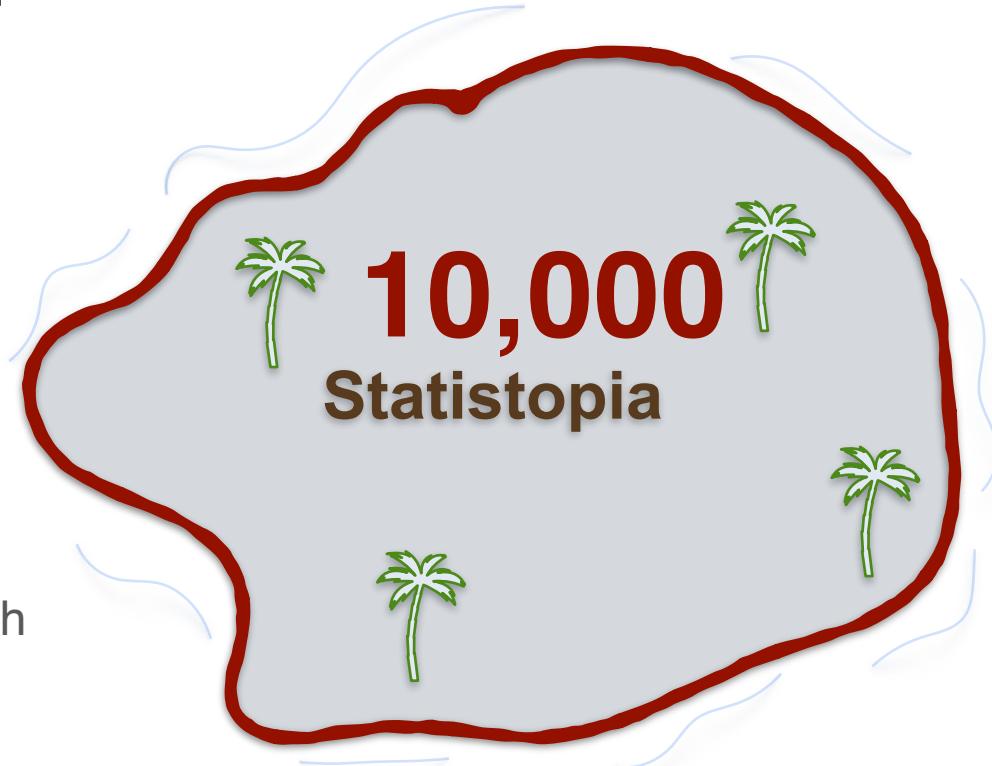


# Population and Sample



## Population:

the entire group of individuals or elements you want to study which share a common behaviour



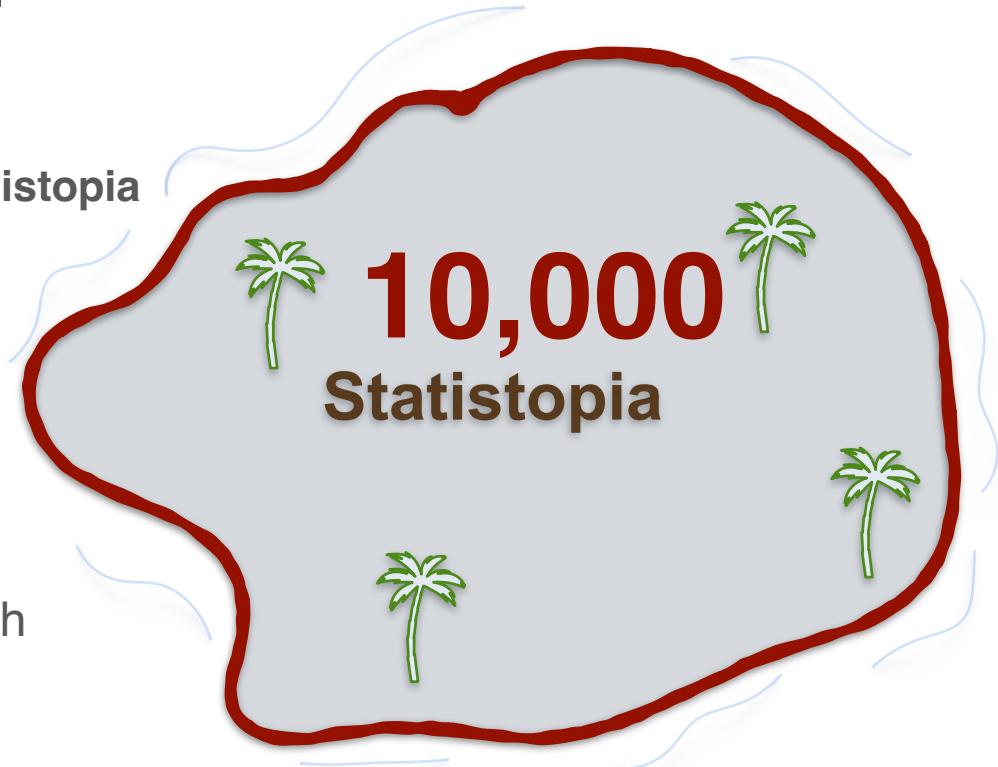
# Population and Sample



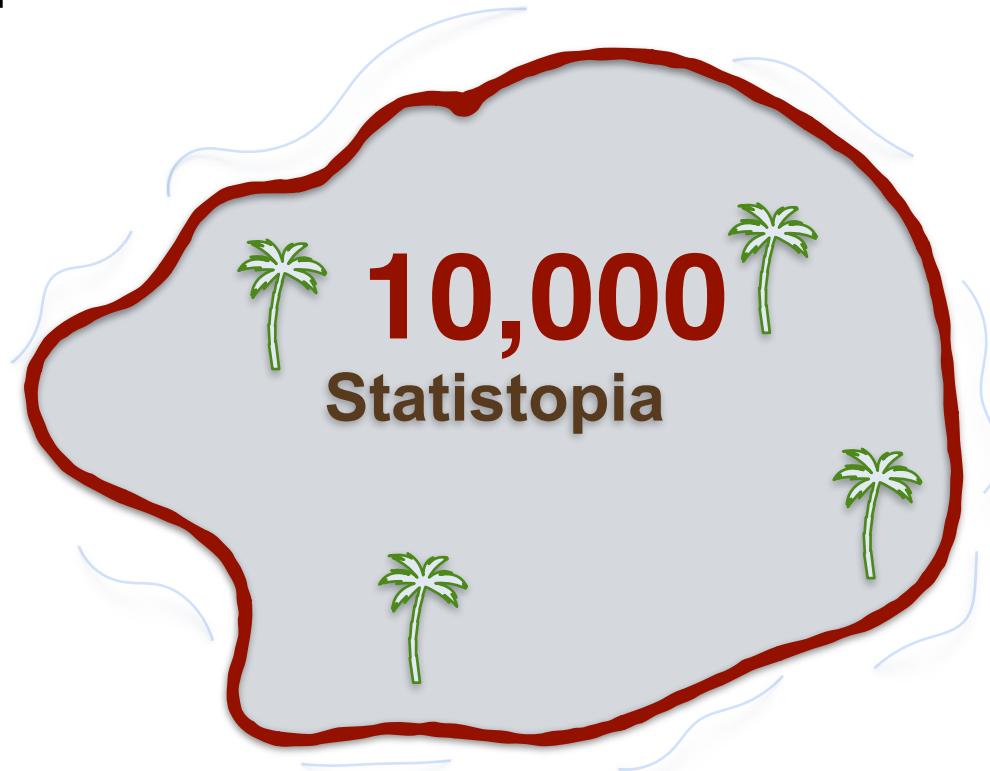
The people of statistopia

## Population:

the entire group of individuals or elements you want to study which share a common behaviour



# Population and Sample

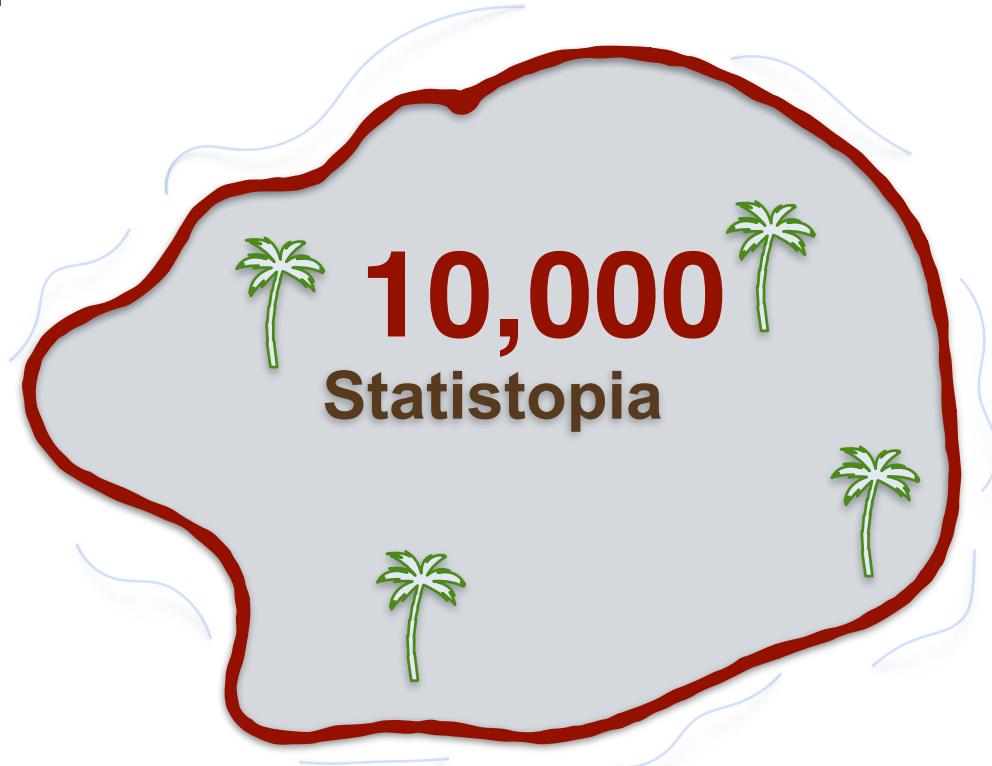


# Population and Sample



## Sample:

subset of the population you use  
to draw conclusions about the  
population as a whole



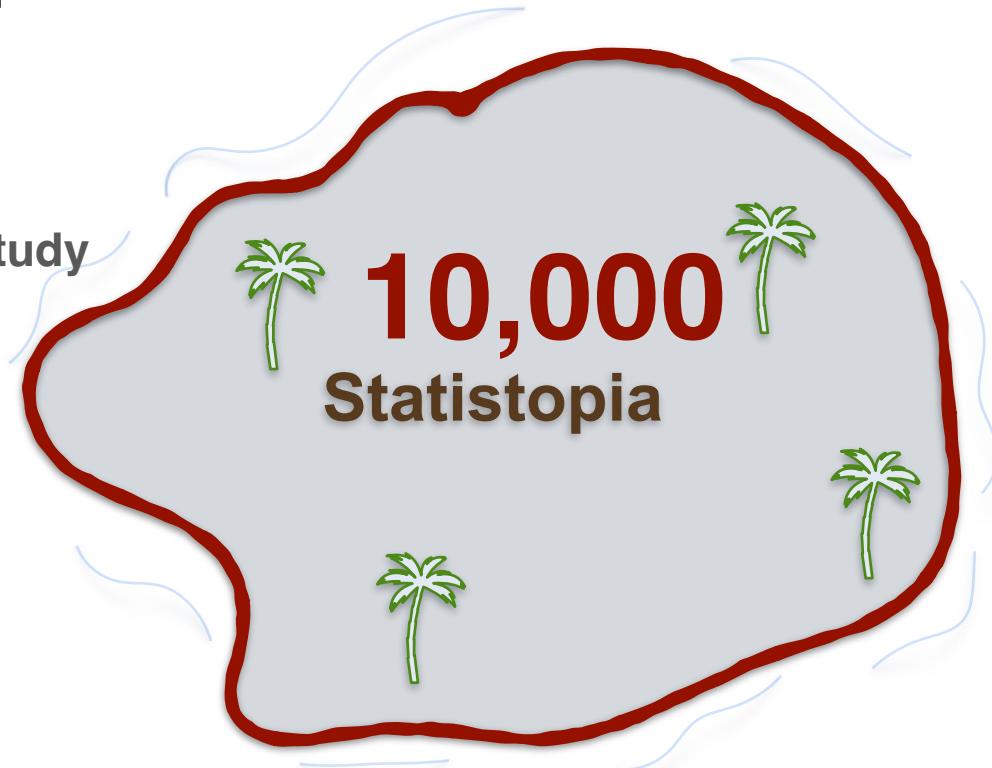
# Population and Sample



The people you  
select for your study

## Sample:

subset of the population you use  
to draw conclusions about the  
population as a whole



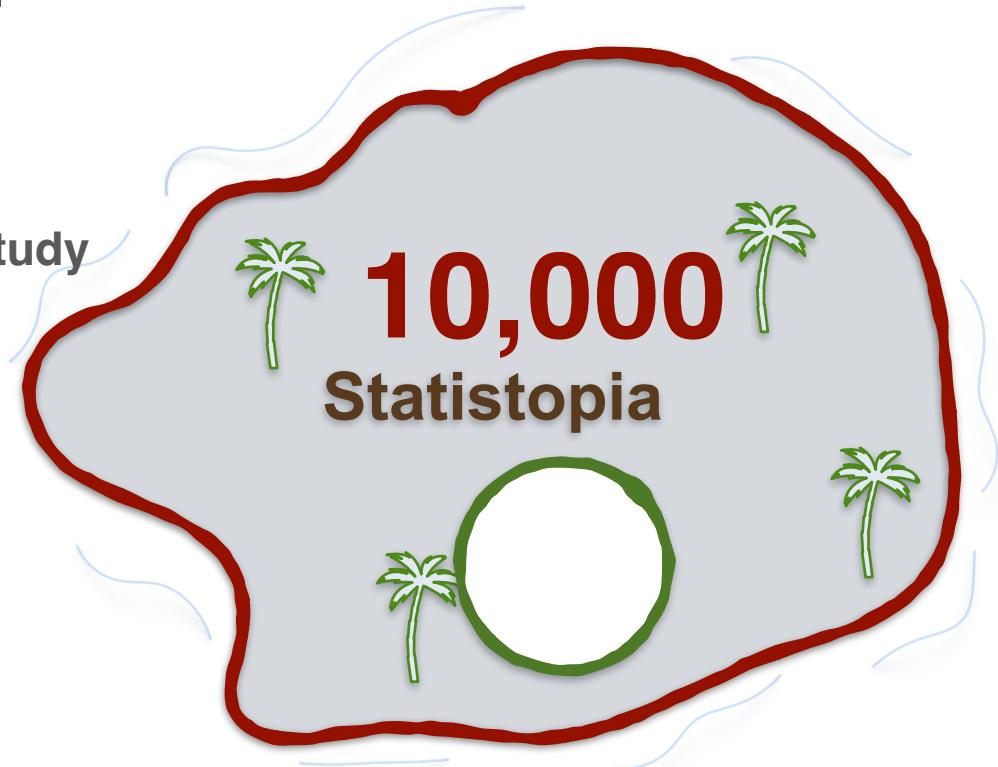
# Population and Sample



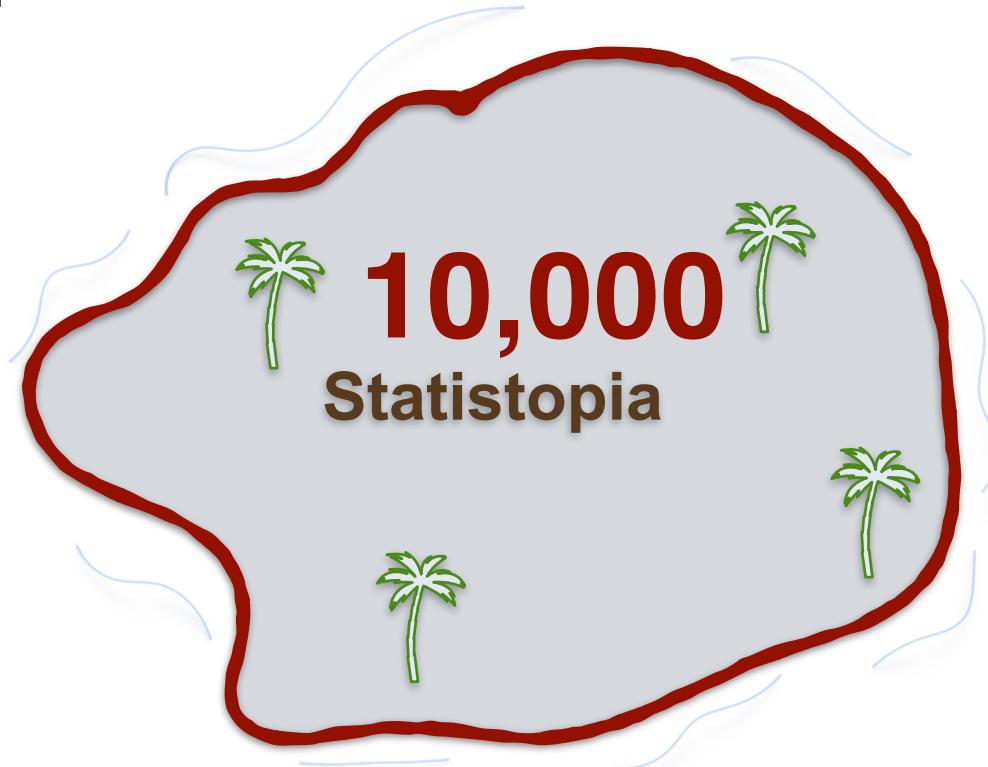
The people you  
select for your study

## Sample:

subset of the population you use  
to draw conclusions about the  
population as a whole



# Population and Sample

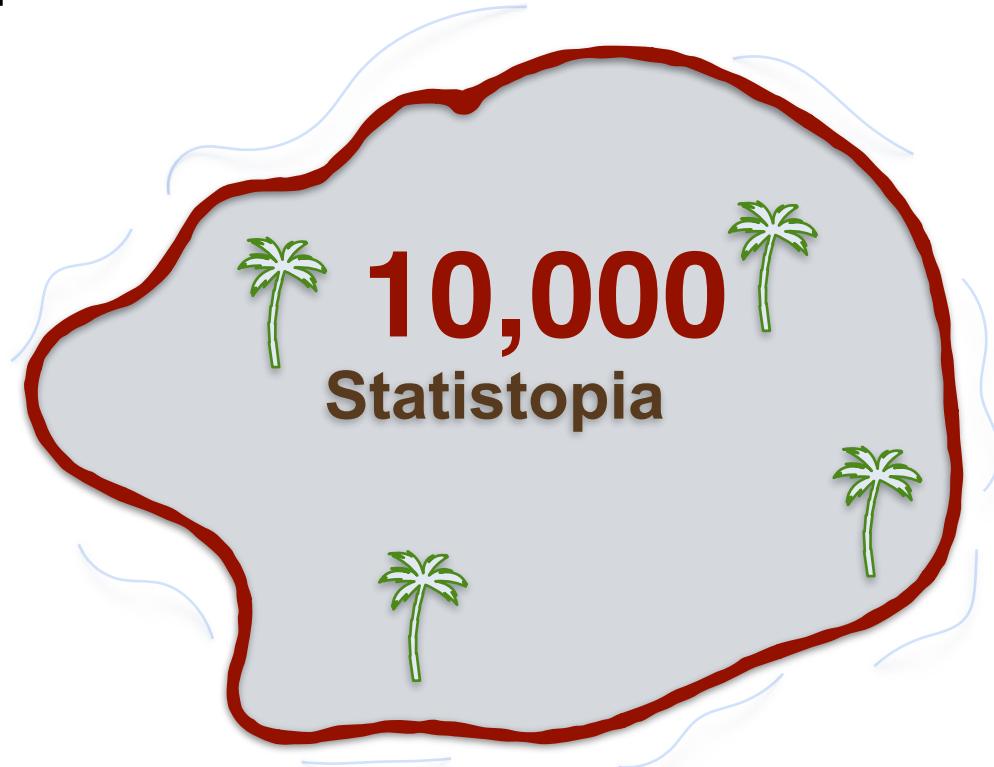


# Population and Sample



Population Size (N)

10,000

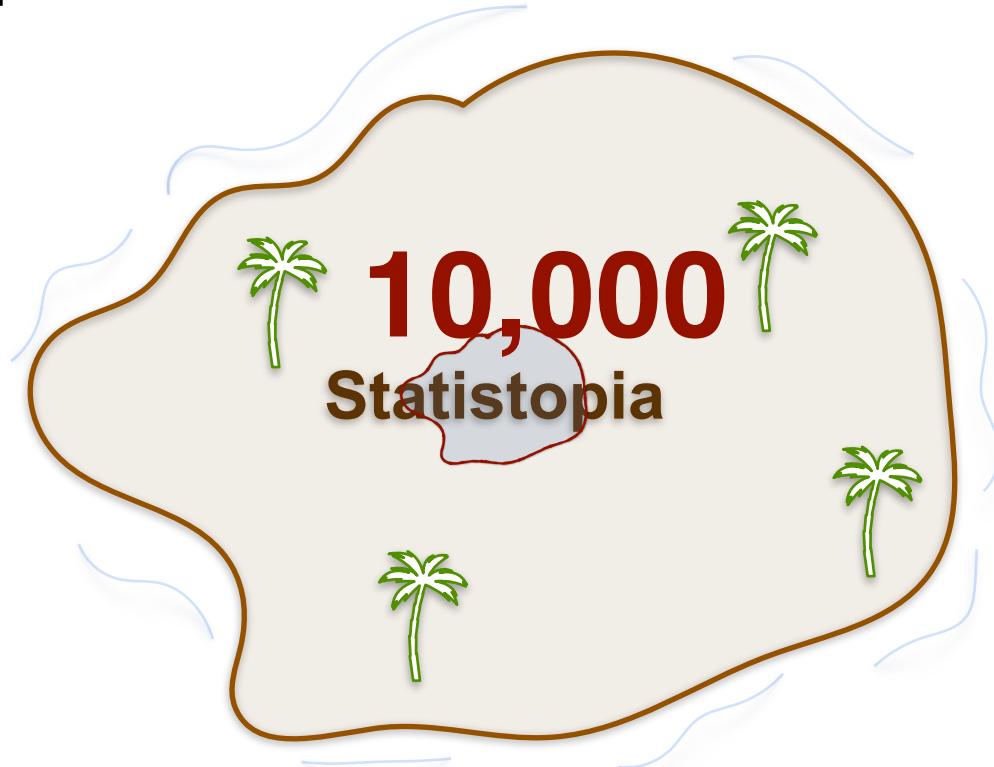


# Population and Sample



Population Size (N)

10,000



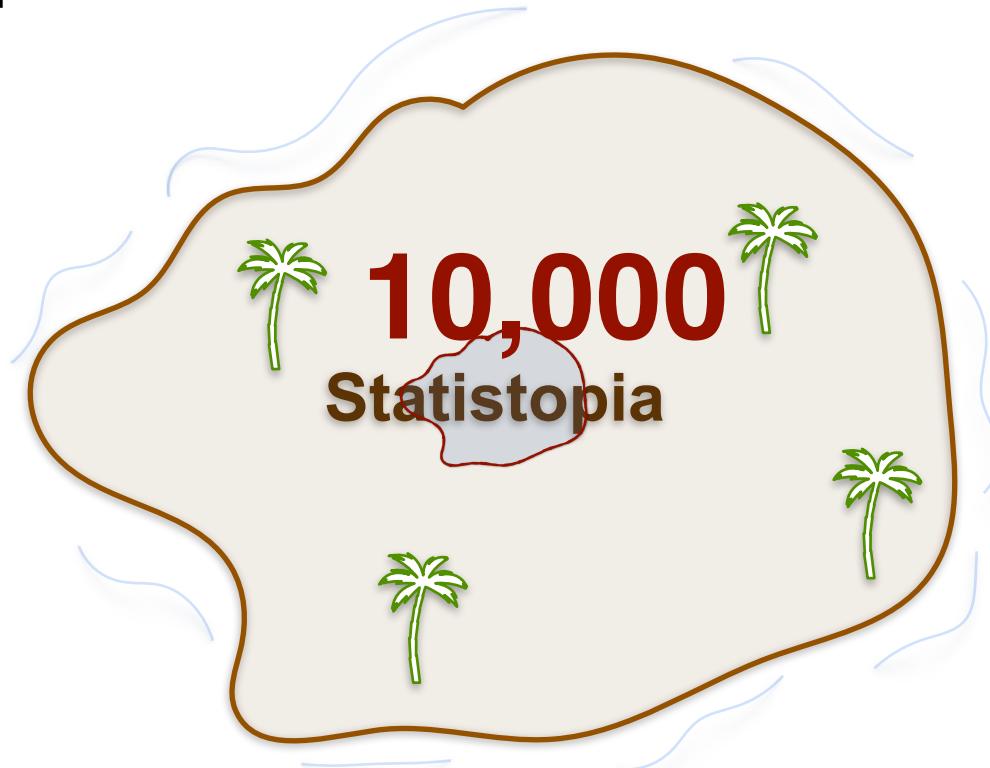
# Population and Sample



**Population Size (N)**

10,000

**Sample Size (n)**



# Population and Sample

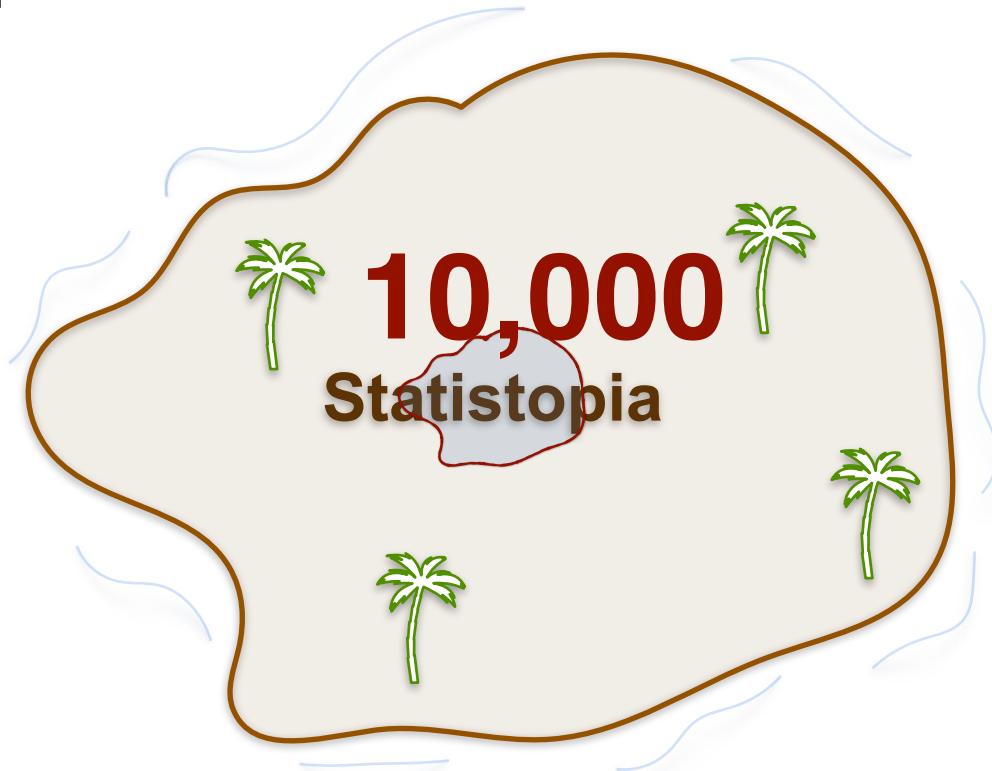


**Population Size (N)**

10,000

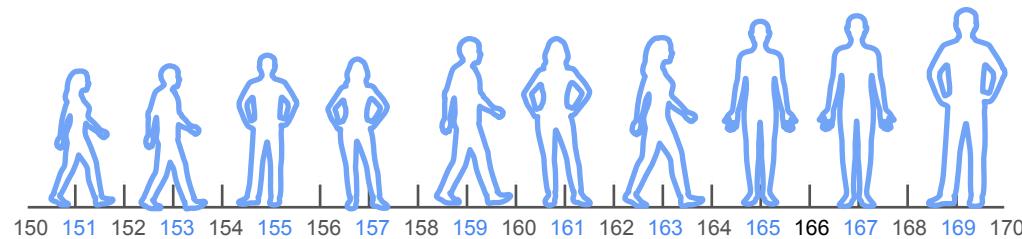
**Sample Size (n)**

1 - 9,999



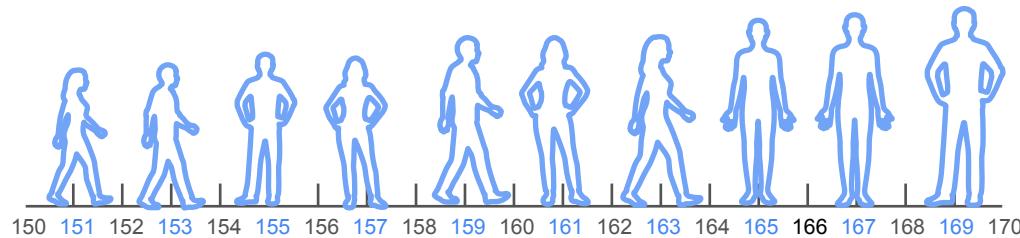
# Population and Sample

# Population and Sample

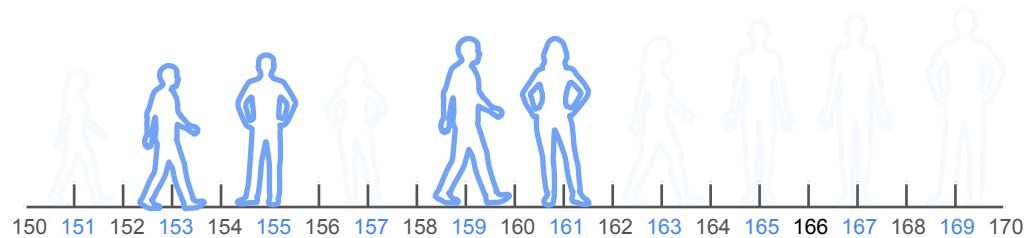


# Population and Sample

$N = 10$

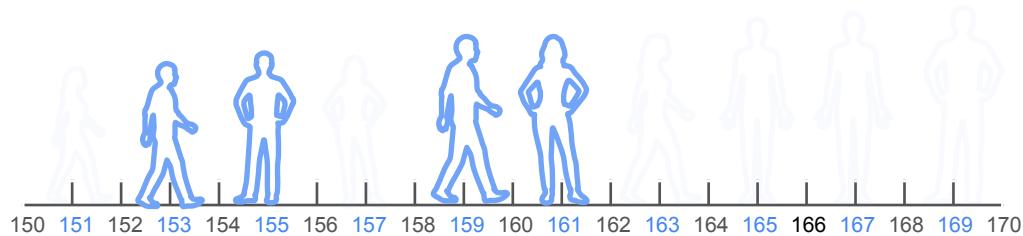


# Random Sampling



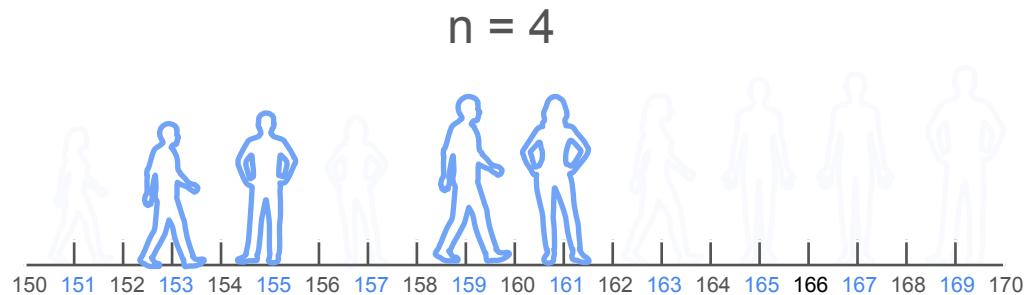
# Random Sampling

$n = 4$



# Random Sampling

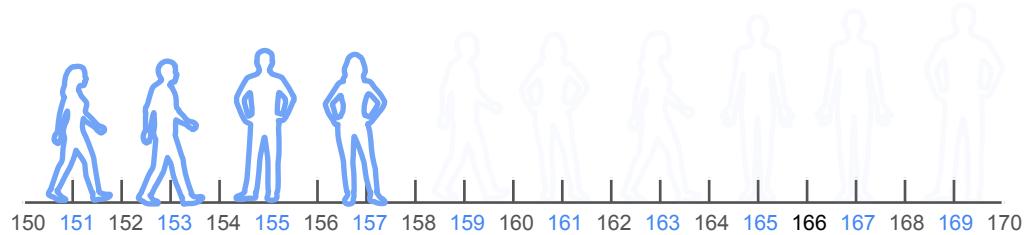
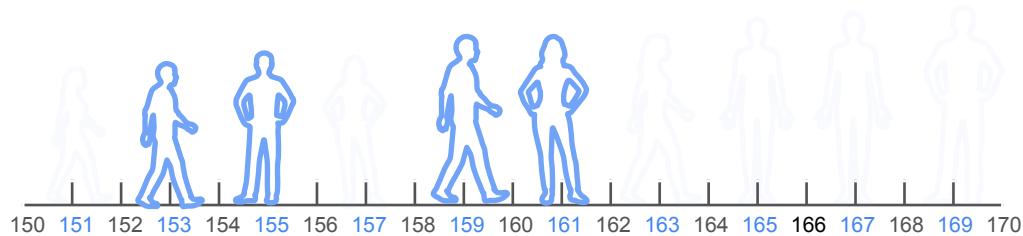
A



# Random Sampling

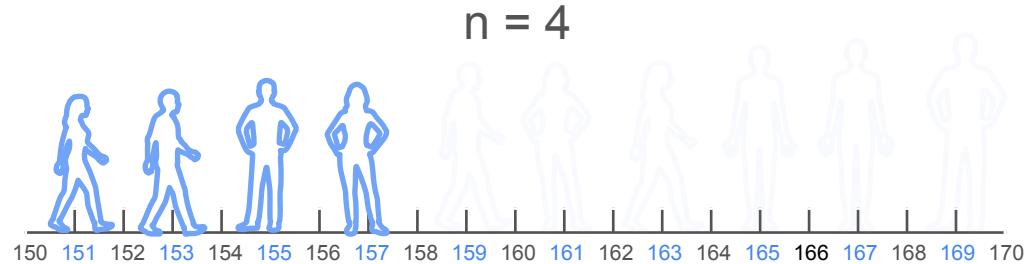
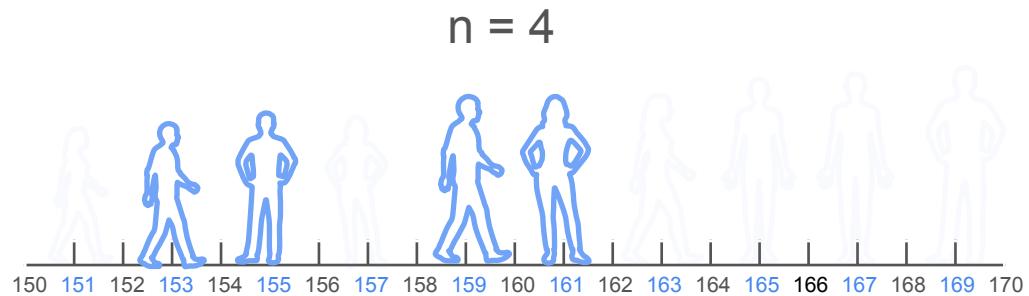
A

$n = 4$



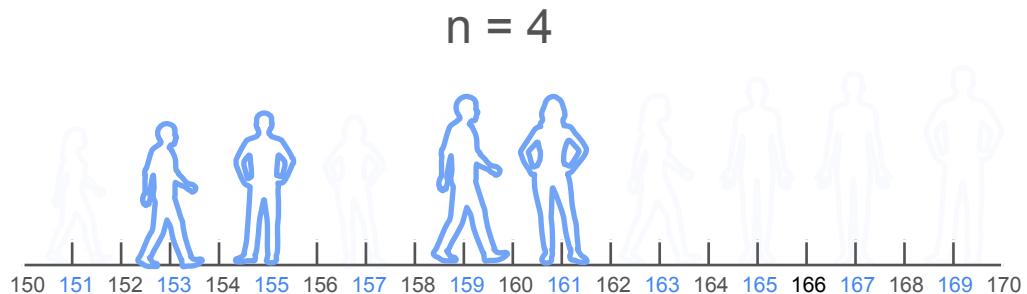
# Random Sampling

A

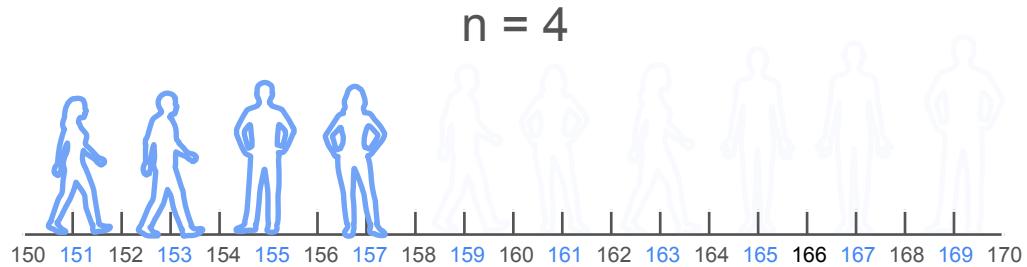


# Random Sampling

A



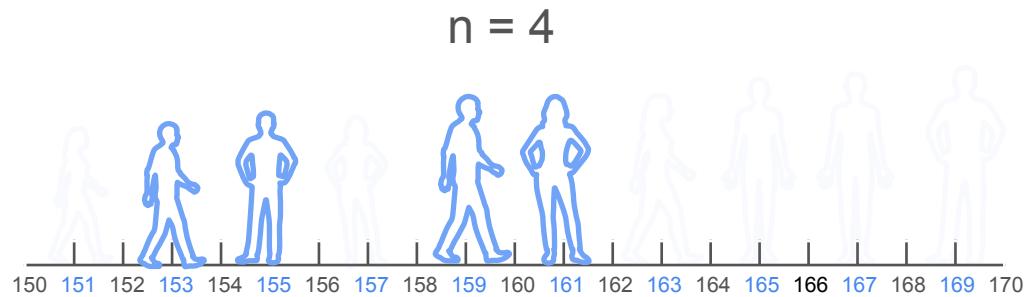
B



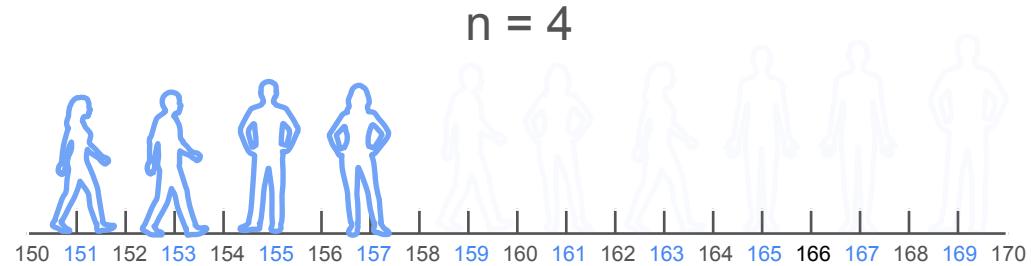
# Random Sampling

A

Which is the better sample  
to estimate the population  
mean height?

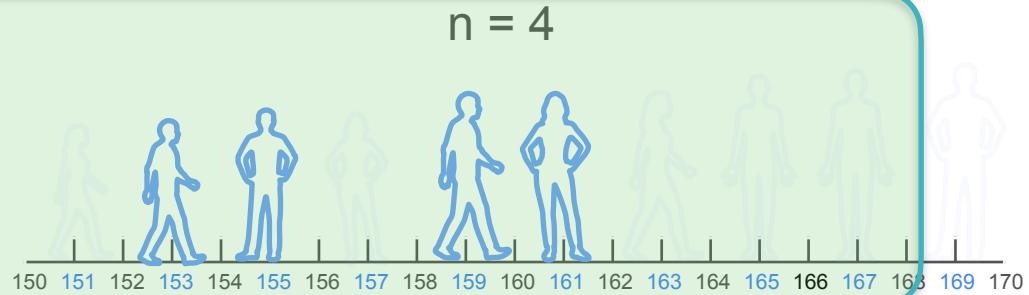


B



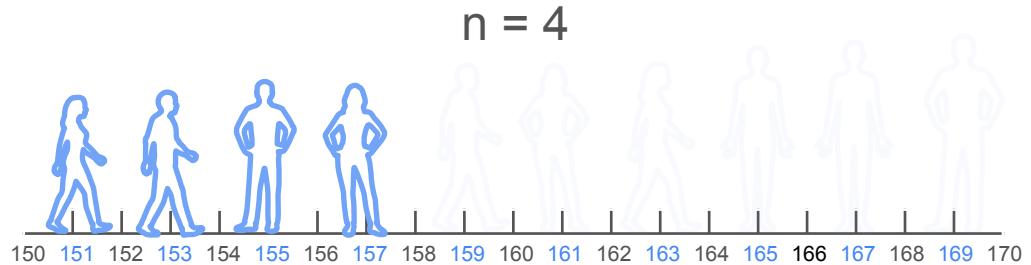
# Random Sampling

A



Which is the better sample  
to estimate the population  
mean height?

B



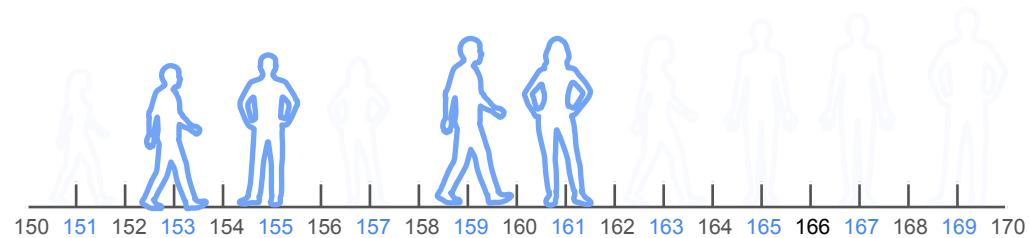
# Independent Sample

# Independent Sample

## Example 1

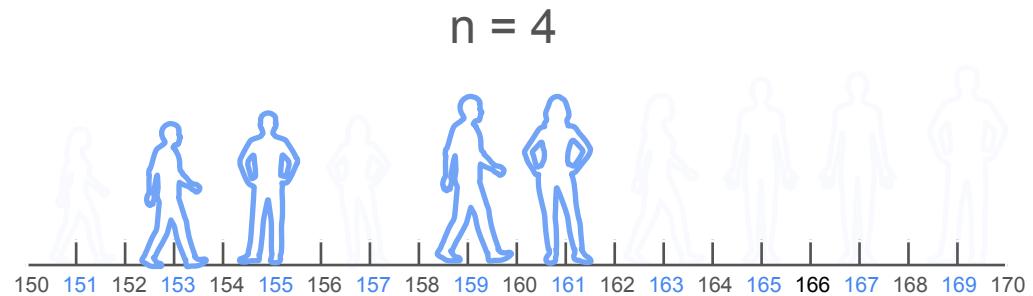
# Independent Sample

## Example 1



# Independent Sample

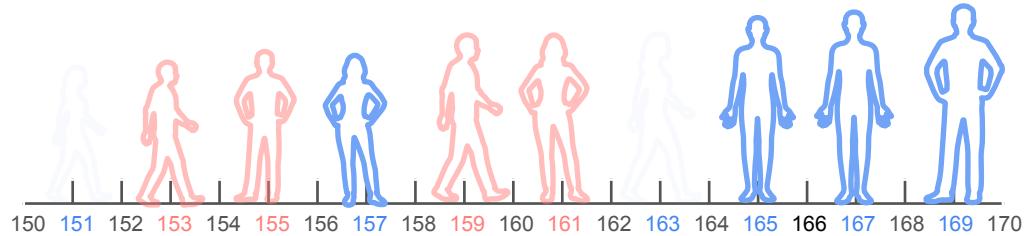
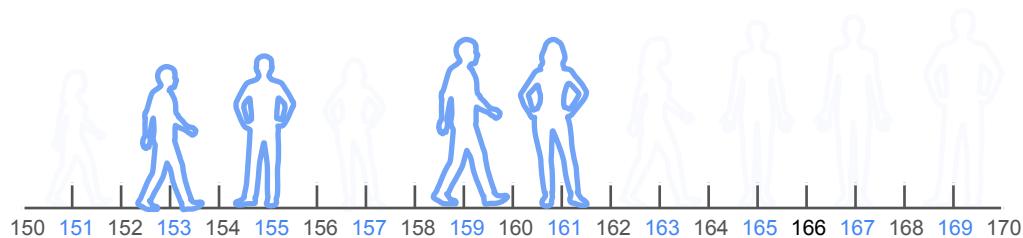
## Example 1



# Independent Sample

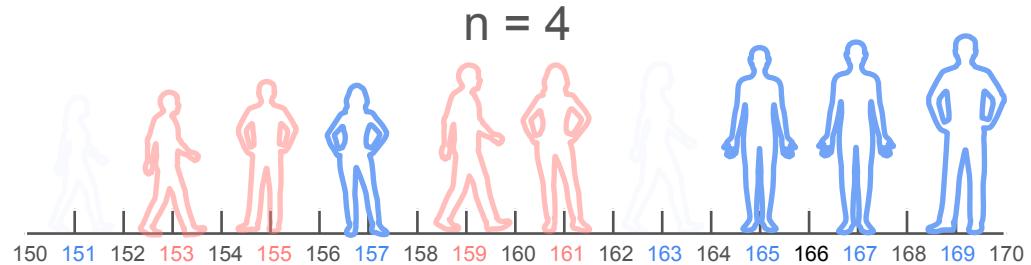
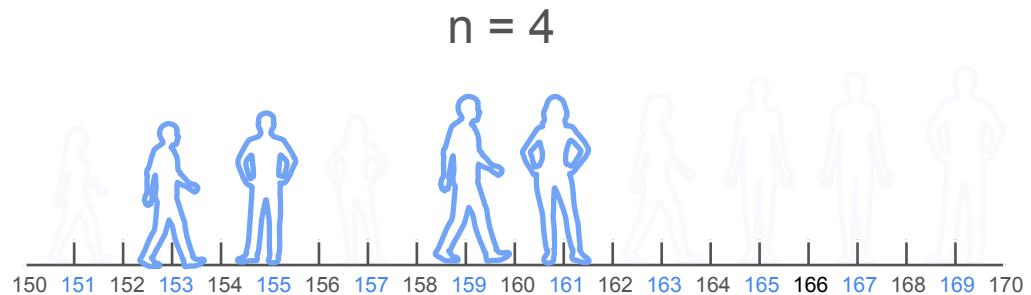
## Example 1

$n = 4$



# Independent Sample

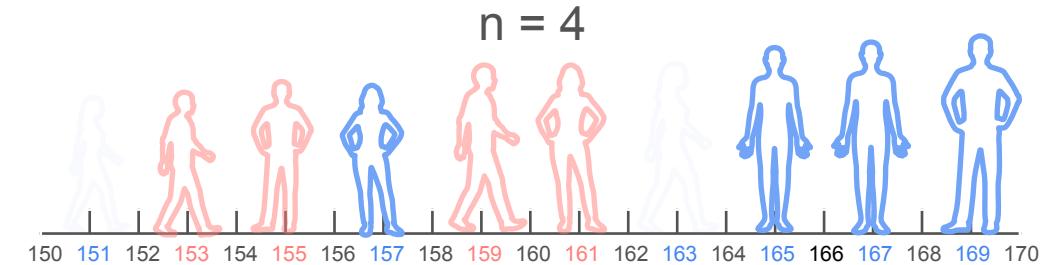
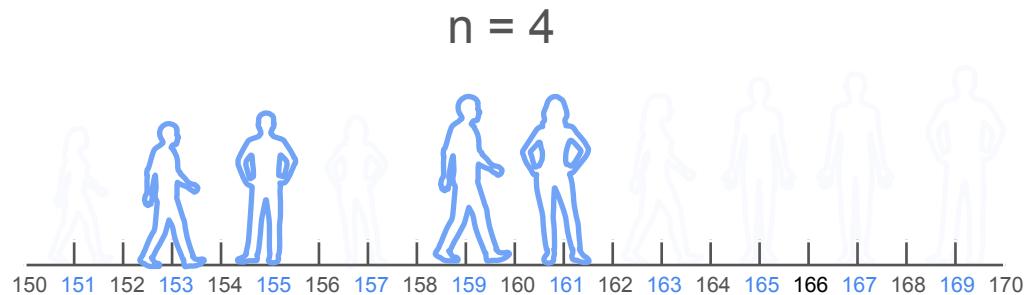
Example 1



# Independent Sample

Example 1

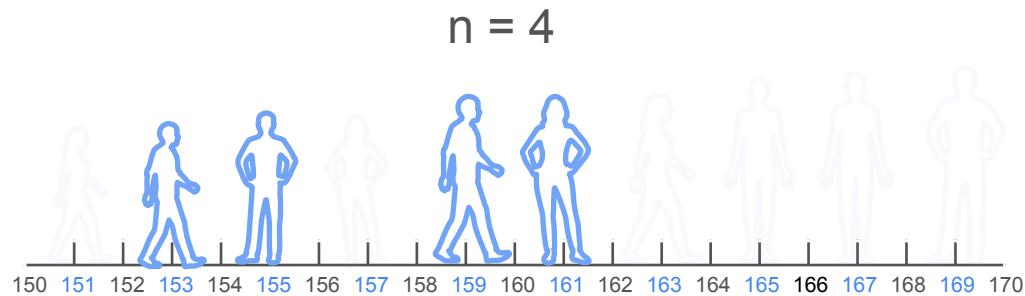
1st sample set



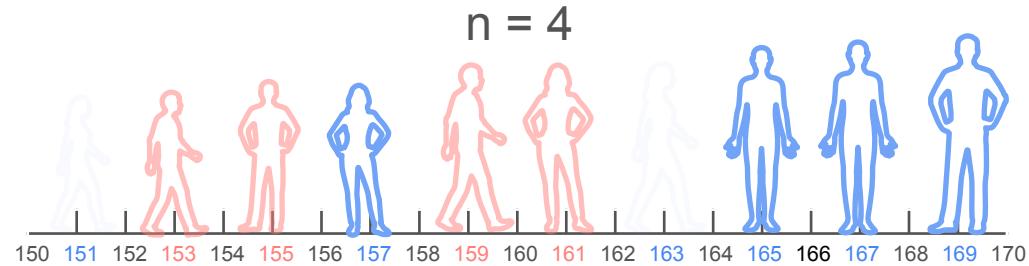
# Independent Sample

## Example 1

1st sample set



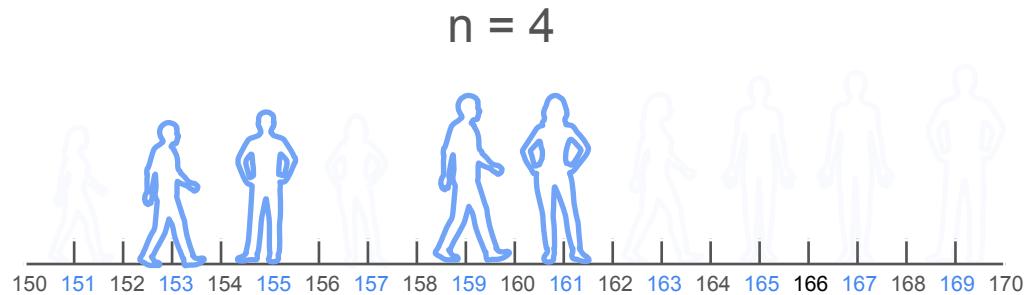
2nd sample set



# Independent Sample

Example 1

1st sample set



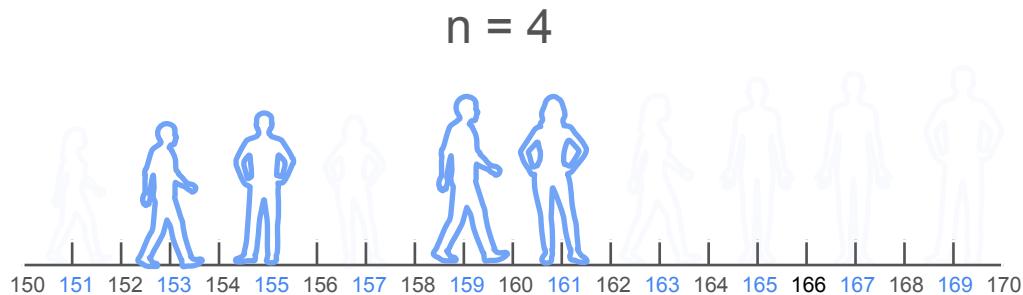
2nd sample set



# Independent Sample

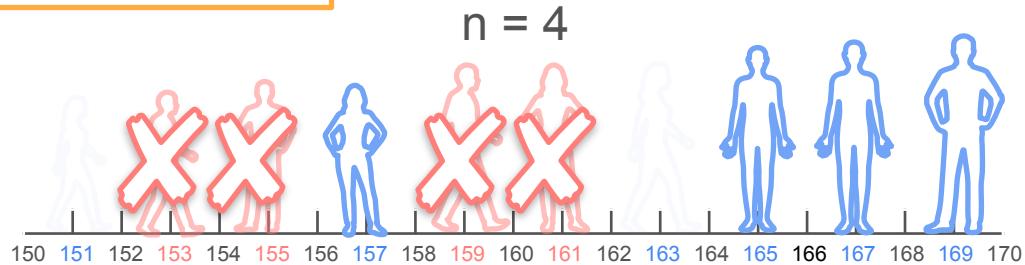
## Example 1

1st sample set



Why is sample set two not a good sample?

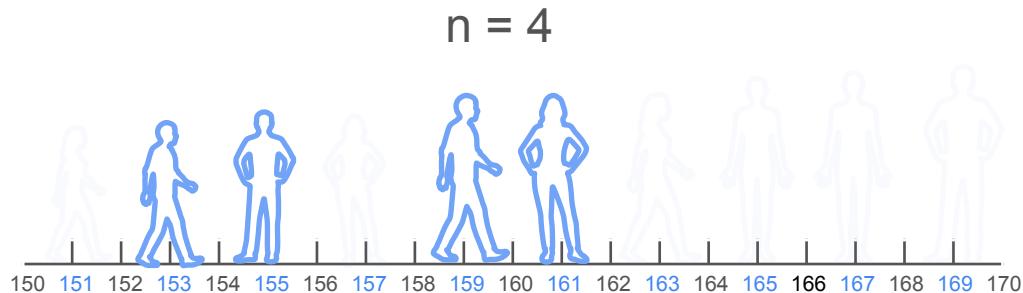
2nd sample set



# Independent Sample

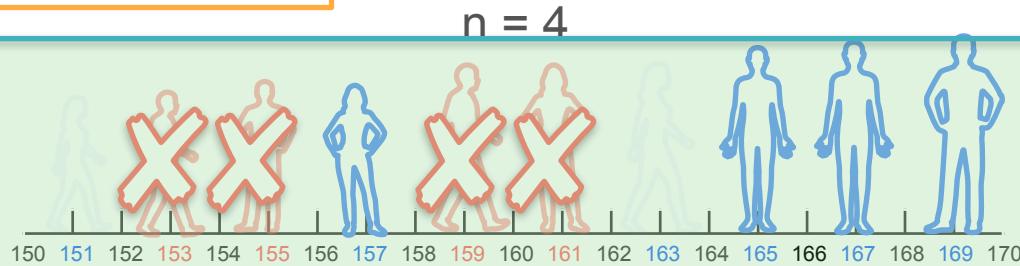
Example 1

1st sample set



Why is sample set two not a good sample?

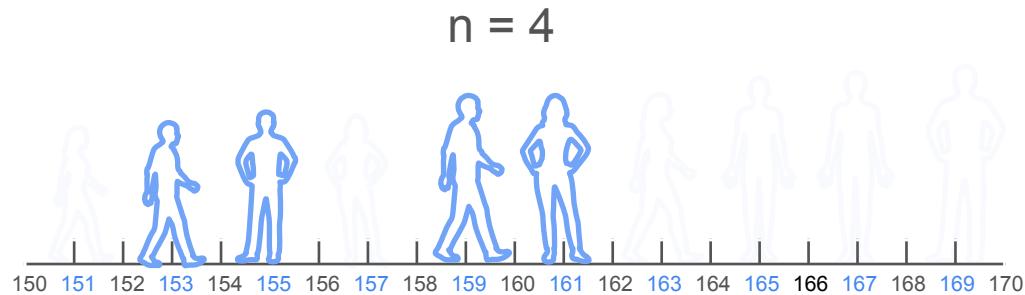
2nd sample set



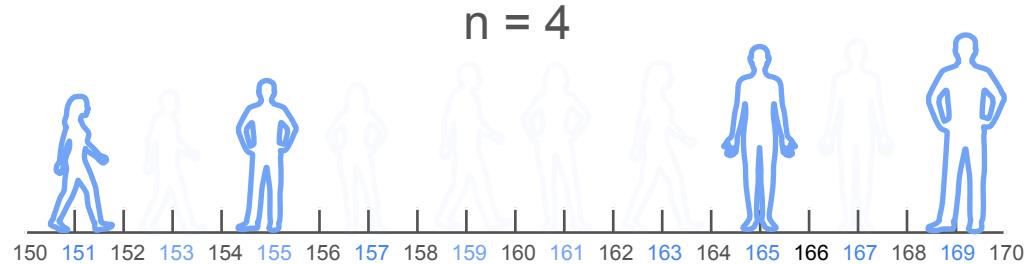
# Independent Sample

## Example 2

1st sample set



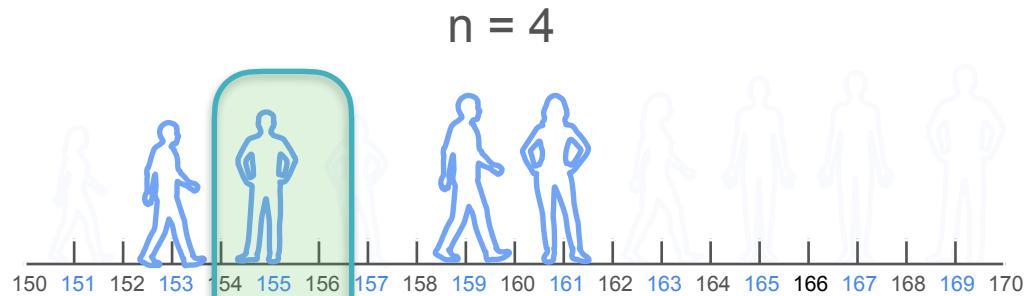
2nd sample set



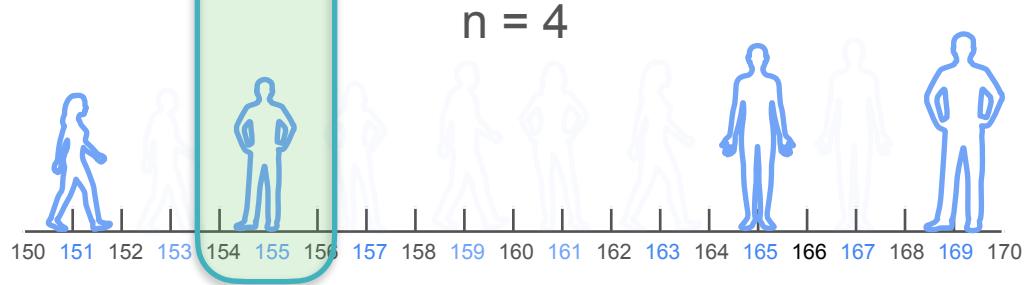
# Independent Sample

## Example 2

1st sample set



2nd sample set



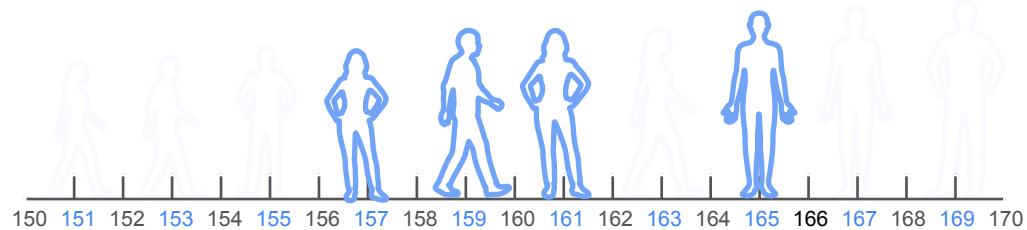
# Identically Distributed Samples

# Identically Distributed Samples

## Example 1

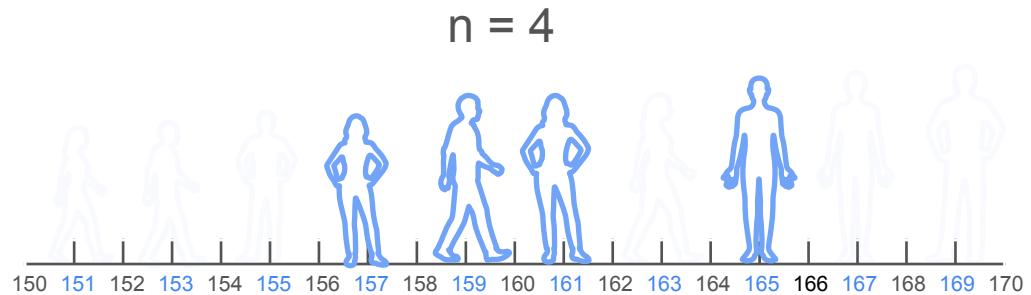
# Identically Distributed Samples

## Example 1



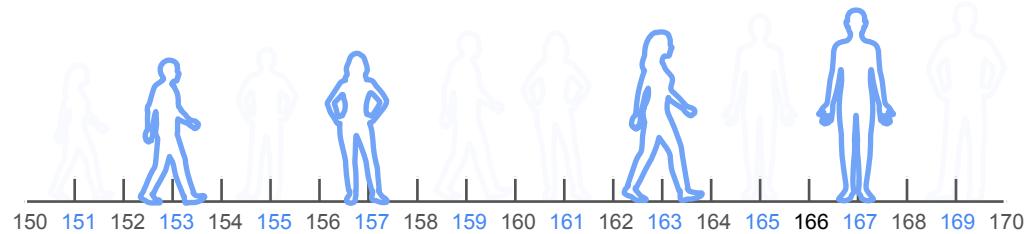
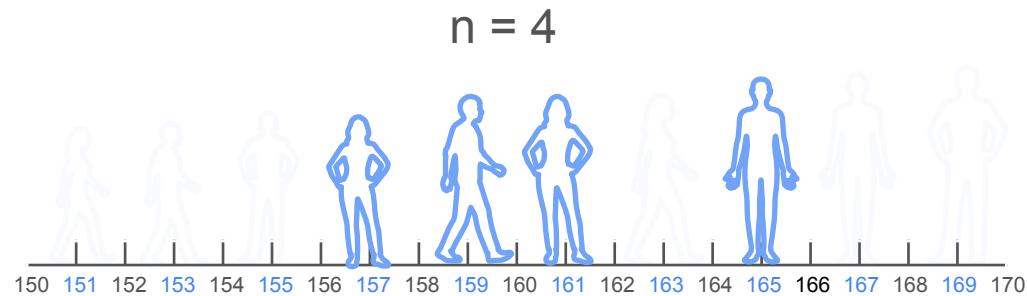
# Identically Distributed Samples

## Example 1



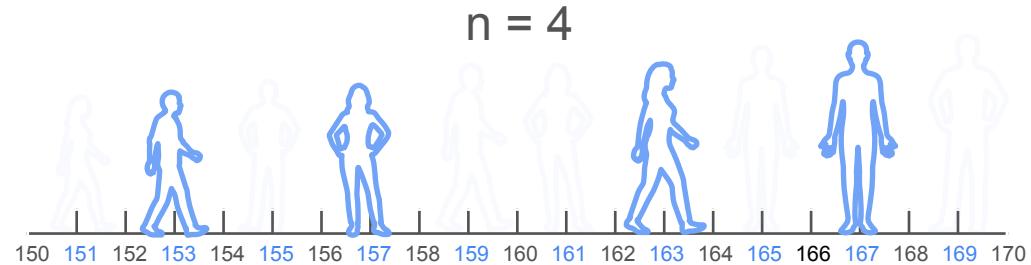
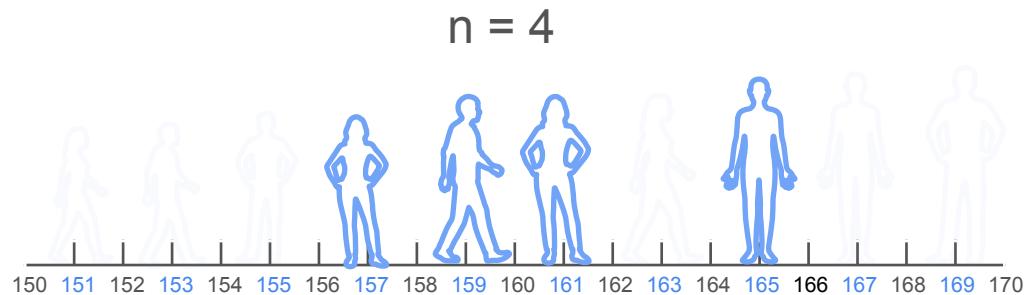
# Identically Distributed Samples

Example 1



# Identically Distributed Samples

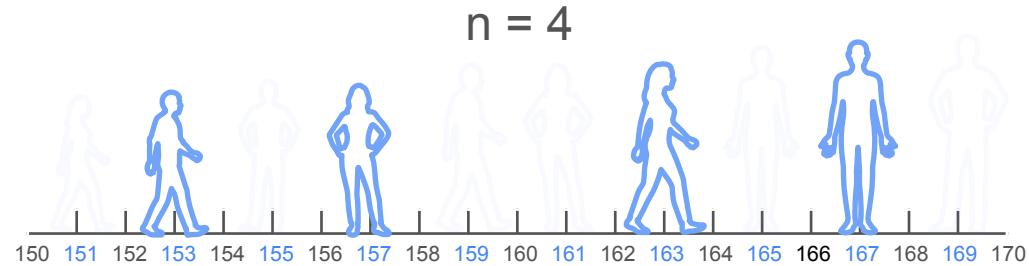
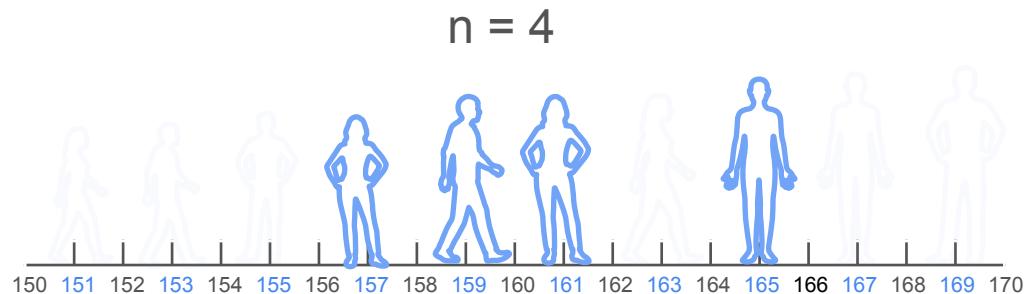
Example 1



# Identically Distributed Samples

Example 1

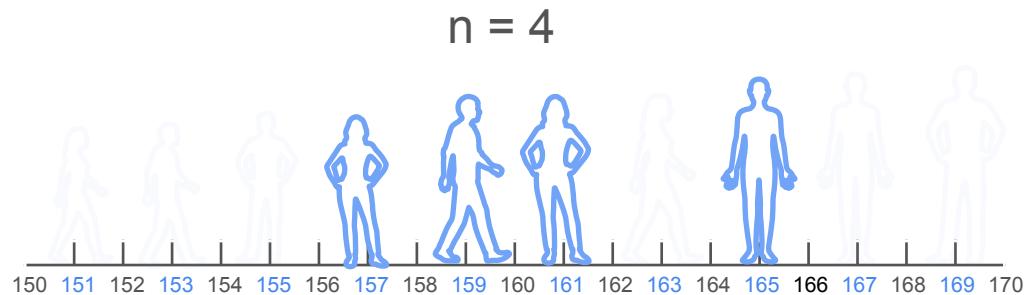
A



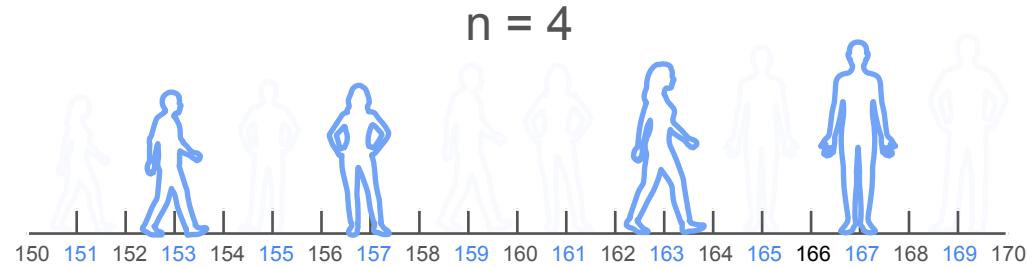
# Identically Distributed Samples

Example 1

A



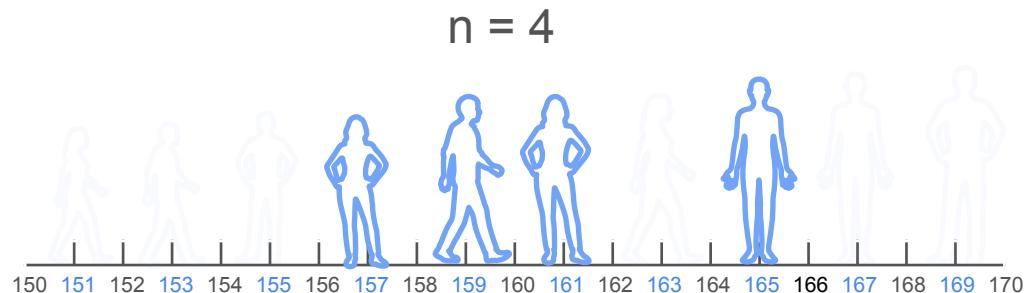
B



# Identically Distributed Samples

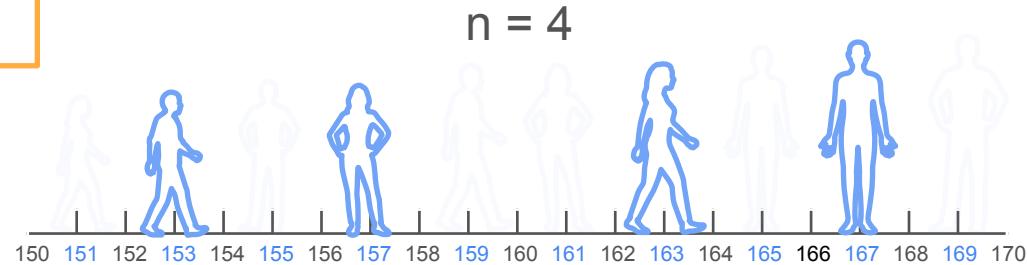
Example 1

A



Which of the following samples  
are identically distributed?

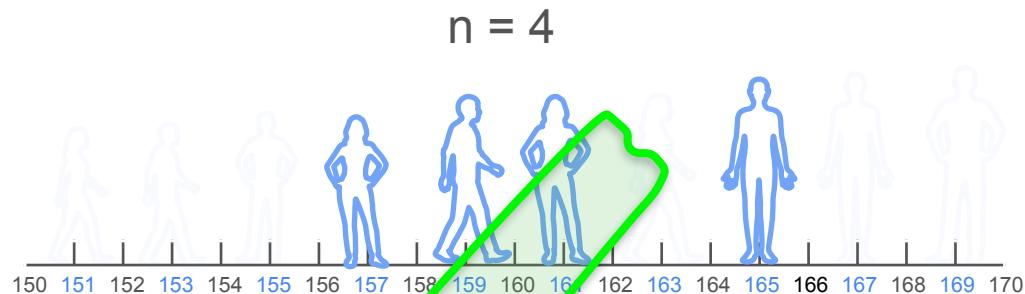
B



# Identically Distributed Samples

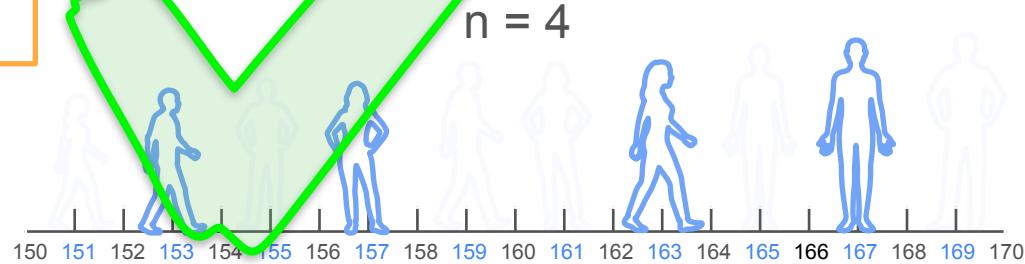
Example 1

A



Which of the following samples  
are identically distributed?

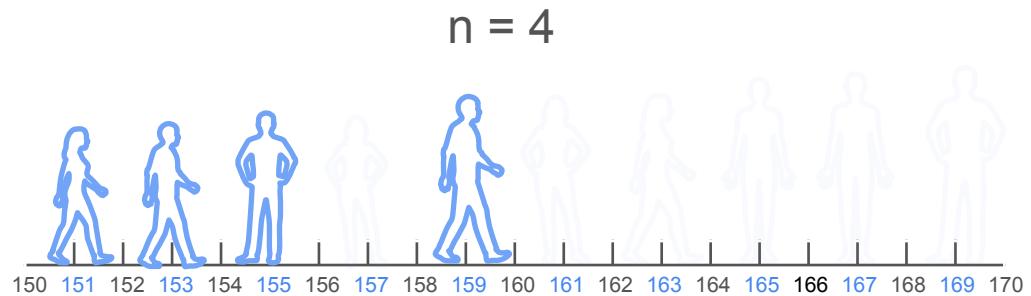
B



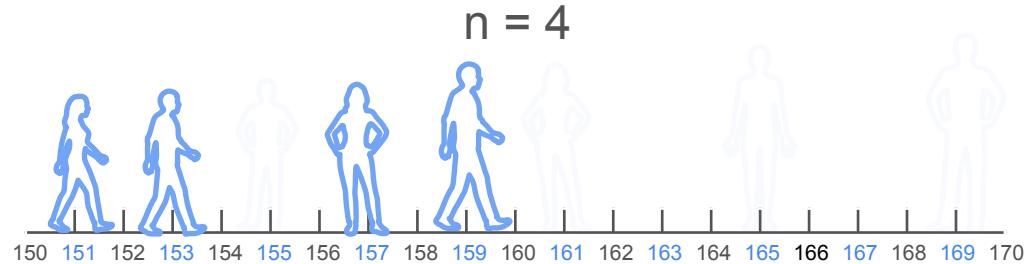
# Identically Distributed Samples

Example 2

A



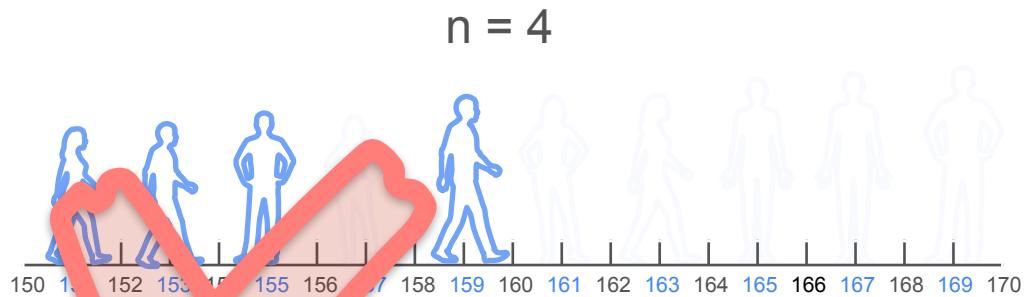
B



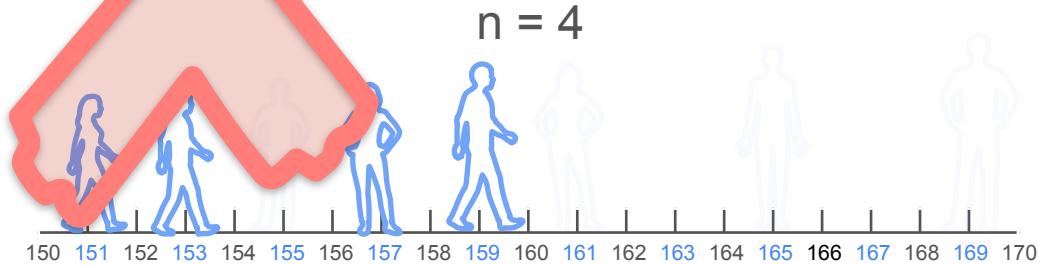
# Identically Distributed Samples

Example 2

A

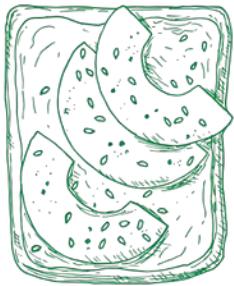


B



# The Avocado Toast Trend

# The Avocado Toast Trend



# The Avocado Toast Trend



Study the price of avocados  
in the United States



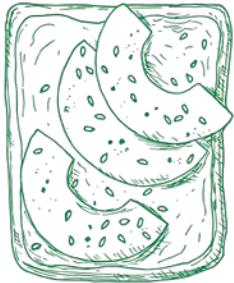
# The Avocado Toast Trend



Study the price of avocados  
in the United States



# The Avocado Toast Trend

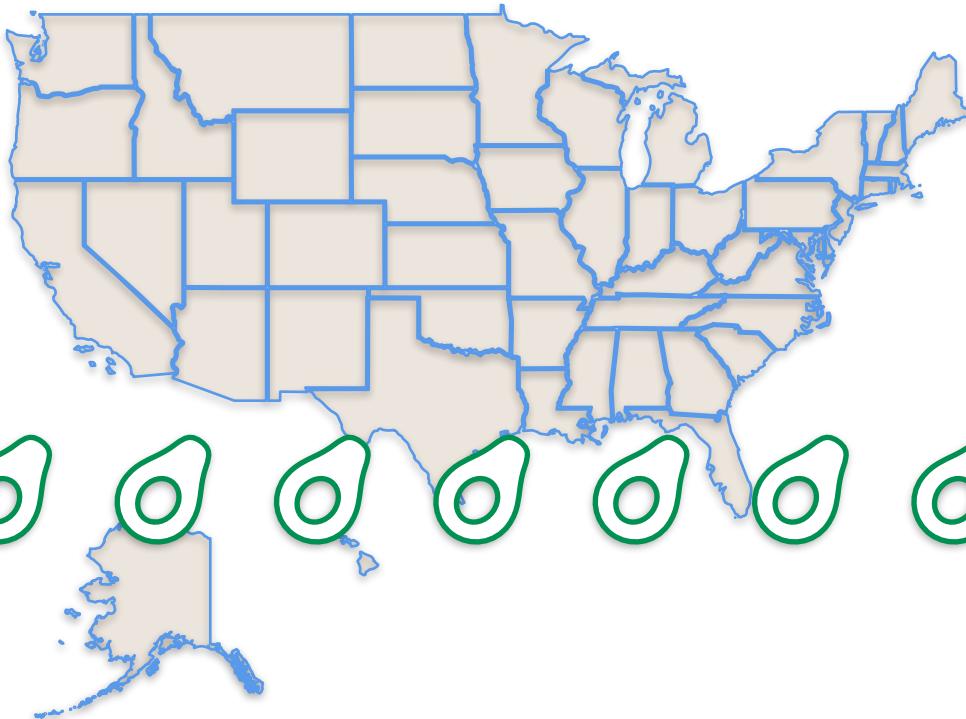


Study the price of avocados  
in the United States



What is the population of your study?

# The Avocado Toast Trend



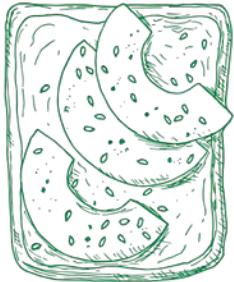
# The Avocado Toast Trend



Study the price of avocados  
in the United States



# The Avocado Toast Trend



Study the price of avocados  
in the United States



What is the sample of your study?

# The Avocado Toast Trend

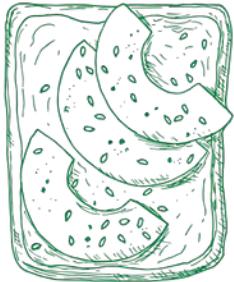


Study the price of avocados  
in the United States



What is the sample of your study?

# The Avocado Toast Trend



Study the price of avocados  
in the United States



What is the sample of your study?

**Avocados sold  
in the 4 stores  
you selected**

# Population and Sample in Machine Learning

# Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample  
**NOT** the population

Cats



Not cats

# Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample  
**NOT** the population

Cats



Not cats



# Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample  
**NOT** the population

Cats



Not cats



# Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample  
**NOT** the population

Cats



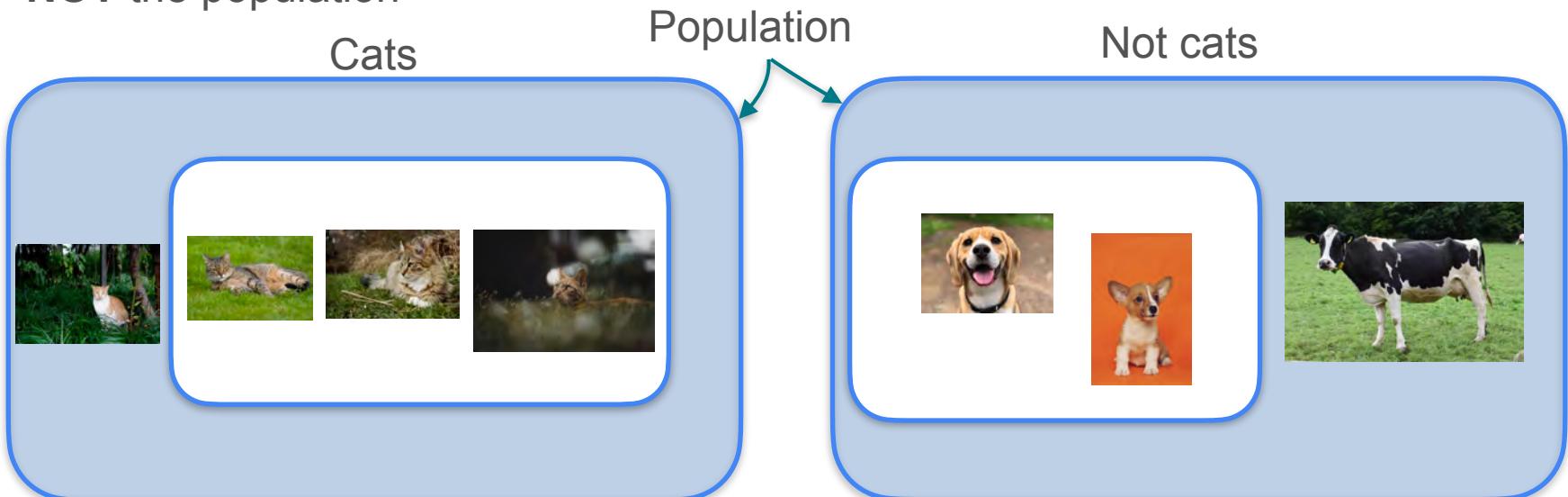
Not cats



# Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample

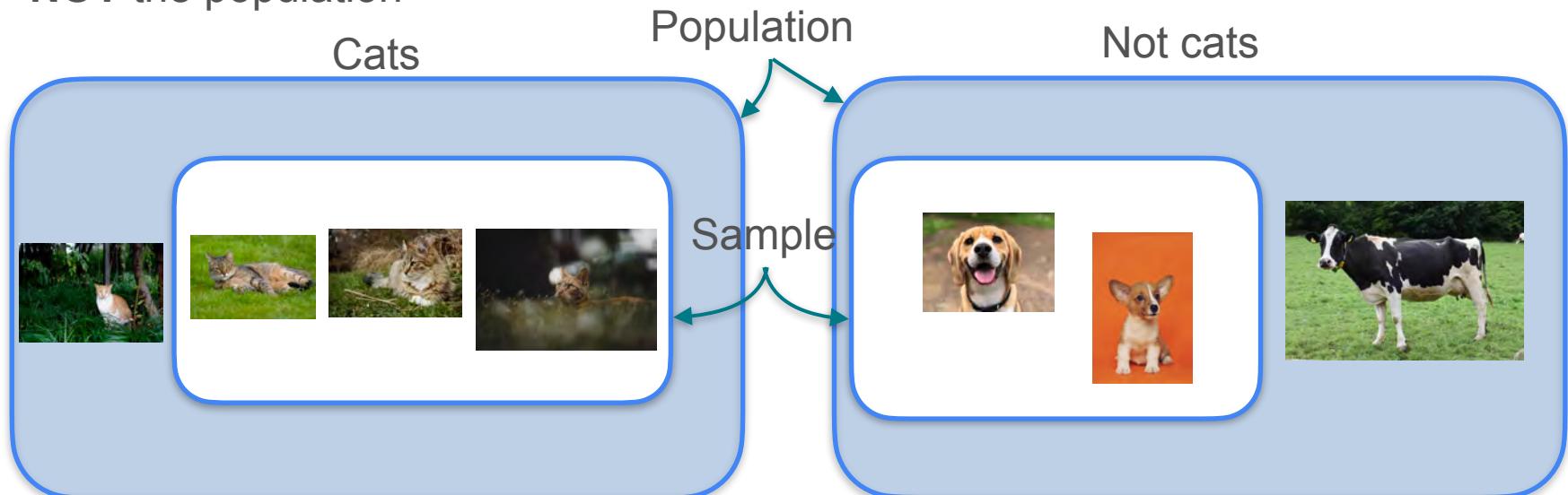
**NOT** the population



# Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample

**NOT** the population



# Recap

## Population

*the entire group of individuals or elements you want to study which share a common behaviour*



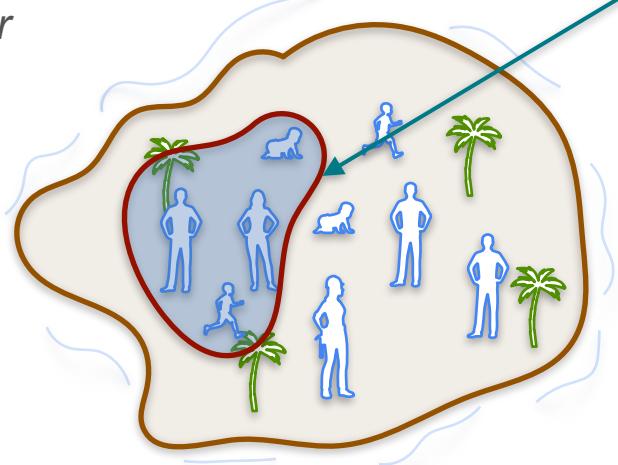
## Sample

*subset of the population you use to draw conclusions about the population as a whole*

# Recap

## Population

*the entire group of individuals or elements you want to study which share a common behaviour*



## Sample

*subset of the population you use to draw conclusions about the population as a whole*

Population Size:

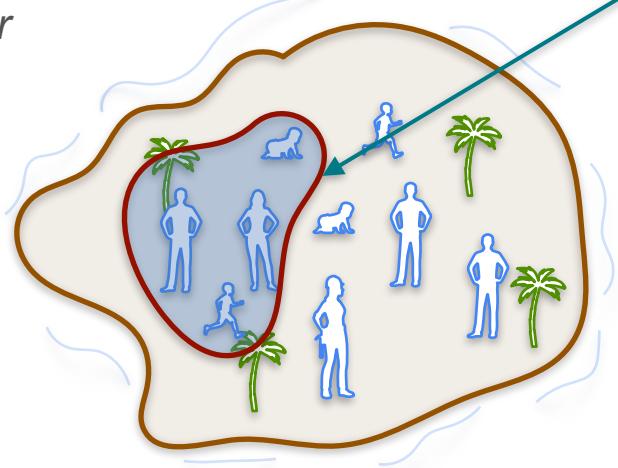
 $N$ 

Sample Size:

# Recap

## Population

*the entire group of individuals or elements you want to study which share a common behaviour*



## Sample

*subset of the population you use to draw conclusions about the population as a whole*

Population Size:  $N$

Sample Size:  $n$



DeepLearning.AI

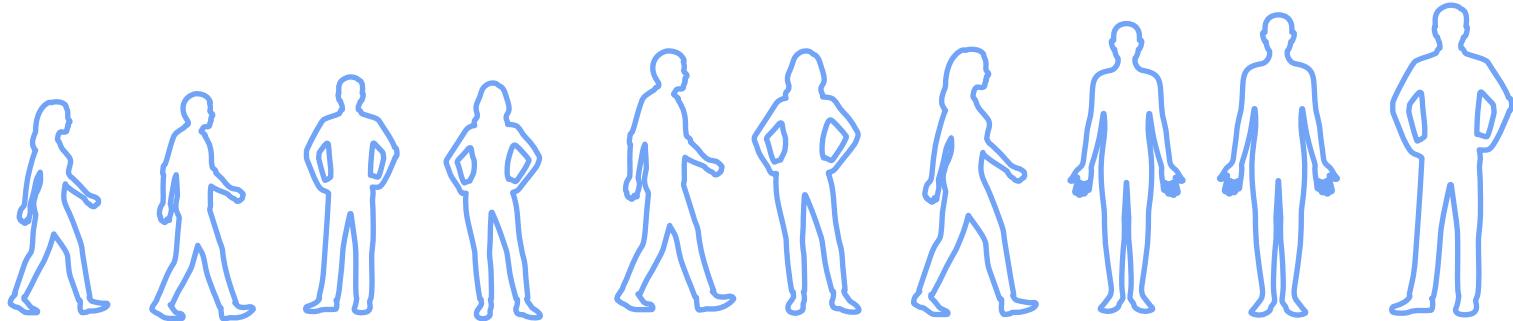
## Sample and Population

---

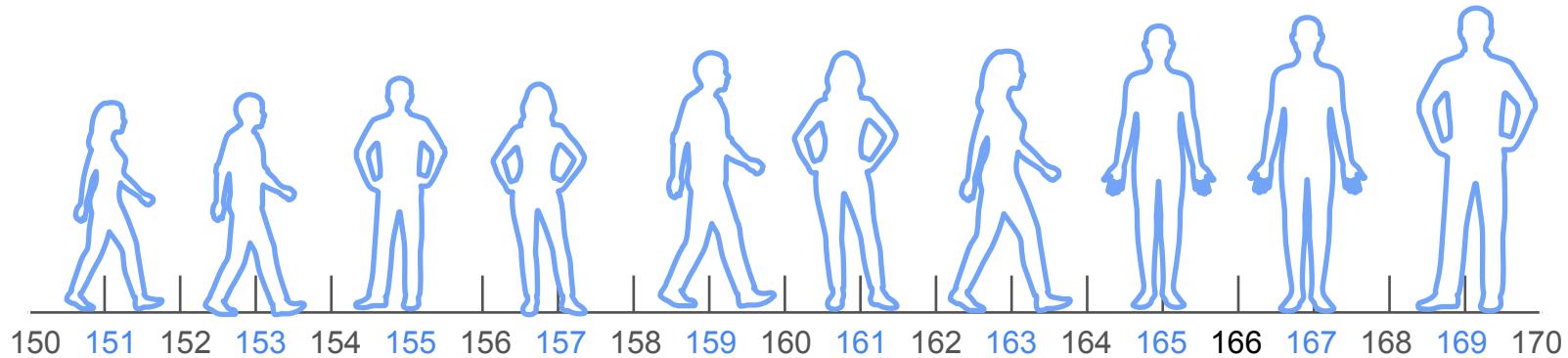
**Sample Mean, Proportion,  
and Variance**

# Population and Sample Mean

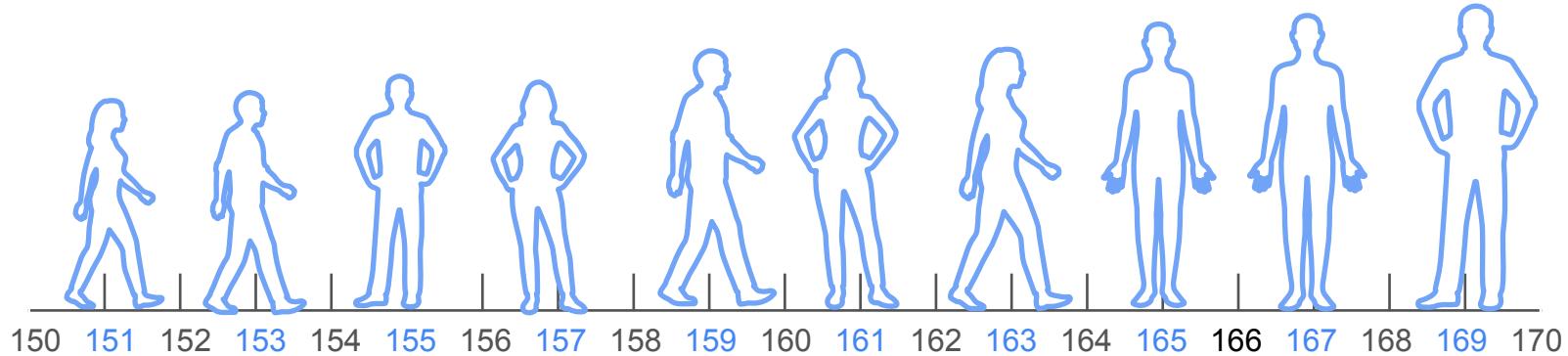
# Population and Sample Mean



# Population and Sample Mean

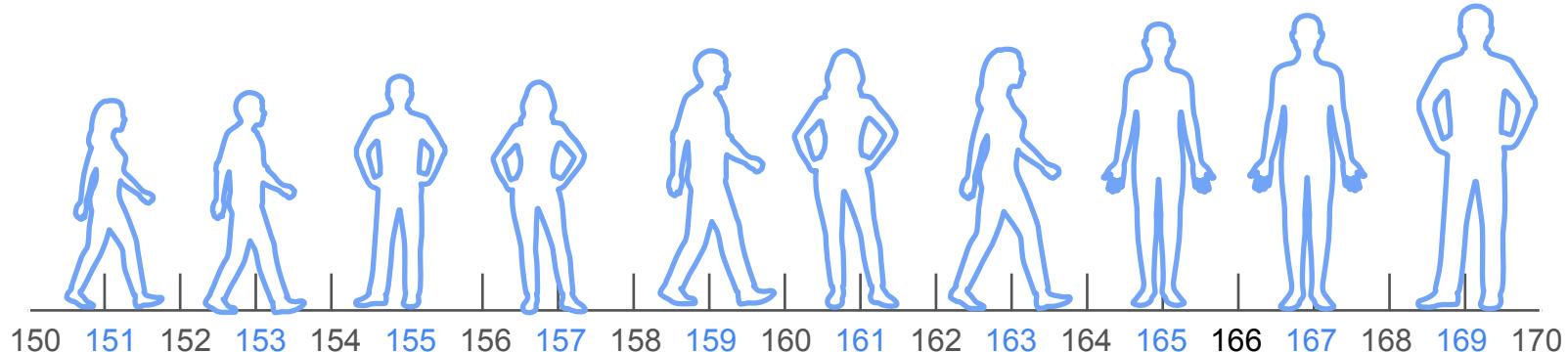


# Population and Sample Mean



What is the average  
height in statistopia?

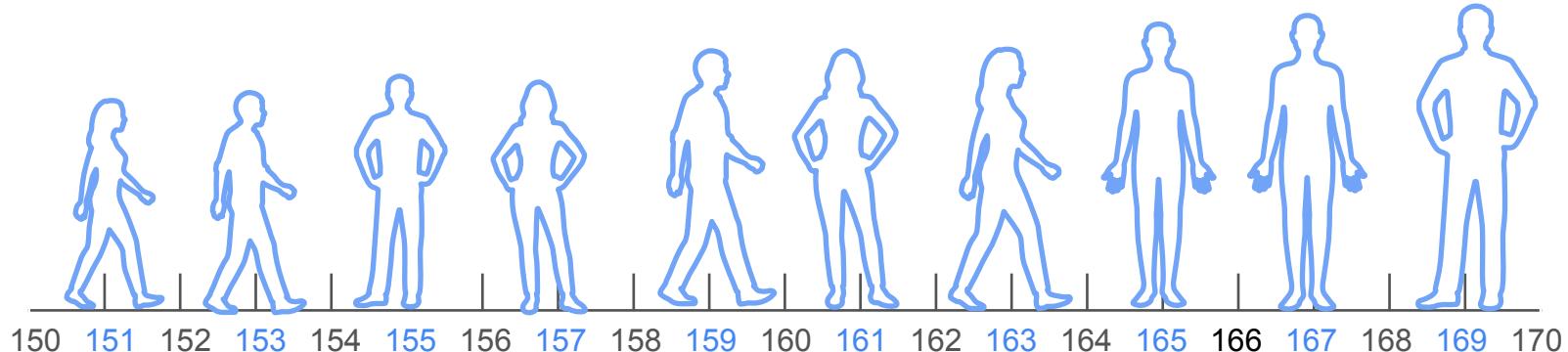
# Population and Sample Mean



What is the average  
height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161 + 163 + 165 + 167 + 169}{10}$$

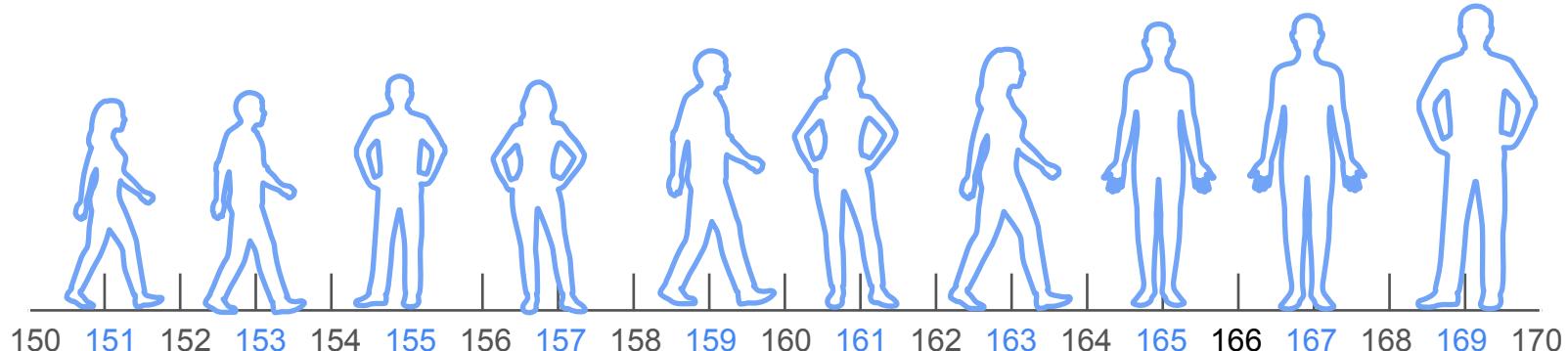
# Population and Sample Mean



What is the average height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161 + 163 + 165 + 167 + 169}{10} = \frac{1600}{10} = 160\text{cm}$$

# Population and Sample Mean



What is the average height in statistopia?

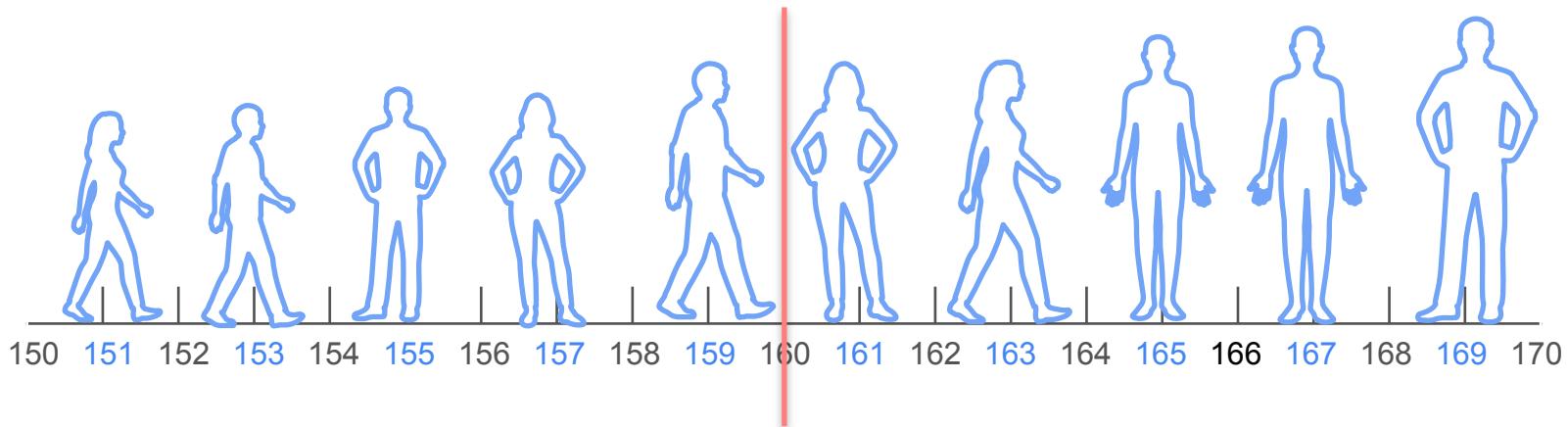
$$\frac{151 + 153 + 155 + 157 + 159 + 161 + 163 + 165 + 167 + 169}{10}$$

$$= \frac{1600}{10} = 160\text{cm}$$

Population mean

$\mu$

# Population and Sample Mean



What is the average height in statistopia?

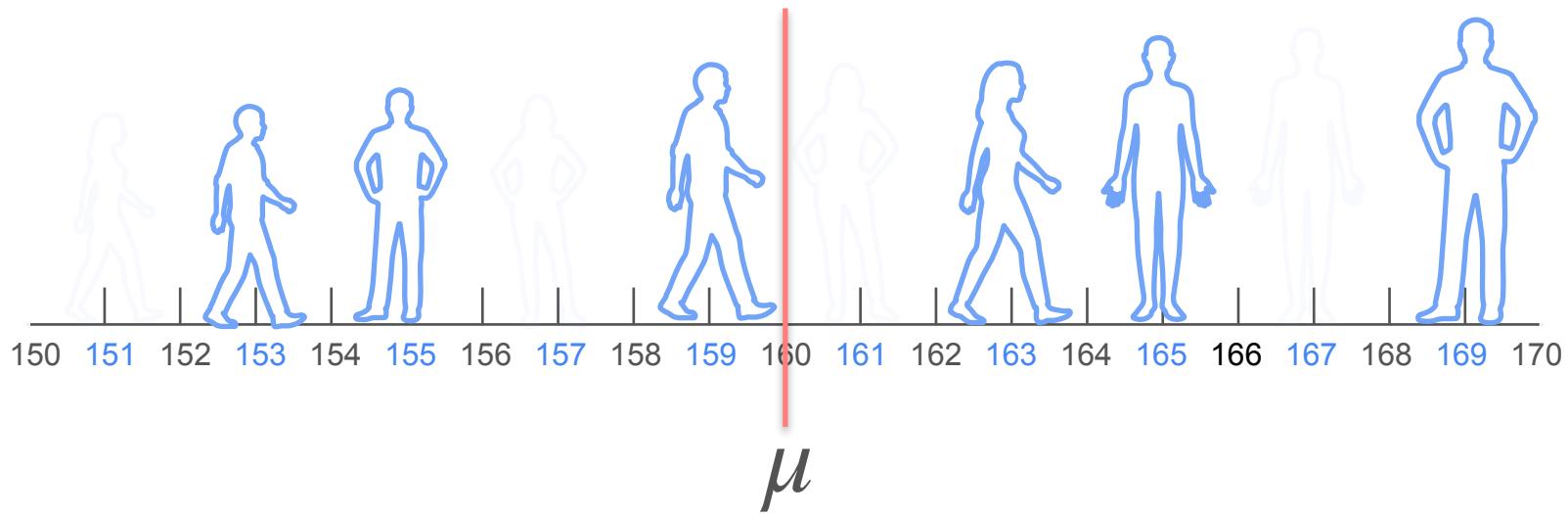
$$\frac{151 + 153 + 155 + 157 + 159 + 161 + 163 + 165 + 167 + 169}{10}$$

$$= \frac{1600}{10} = 160\text{cm}$$

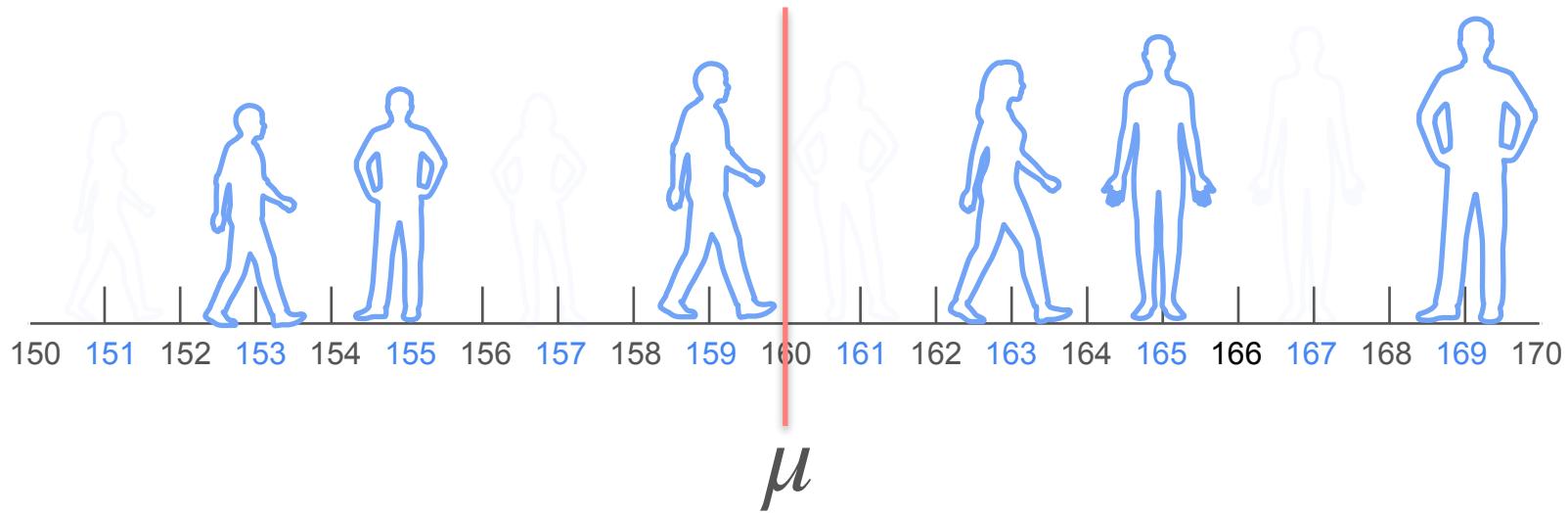
Population mean

$\mu$

# Population and Sample Mean

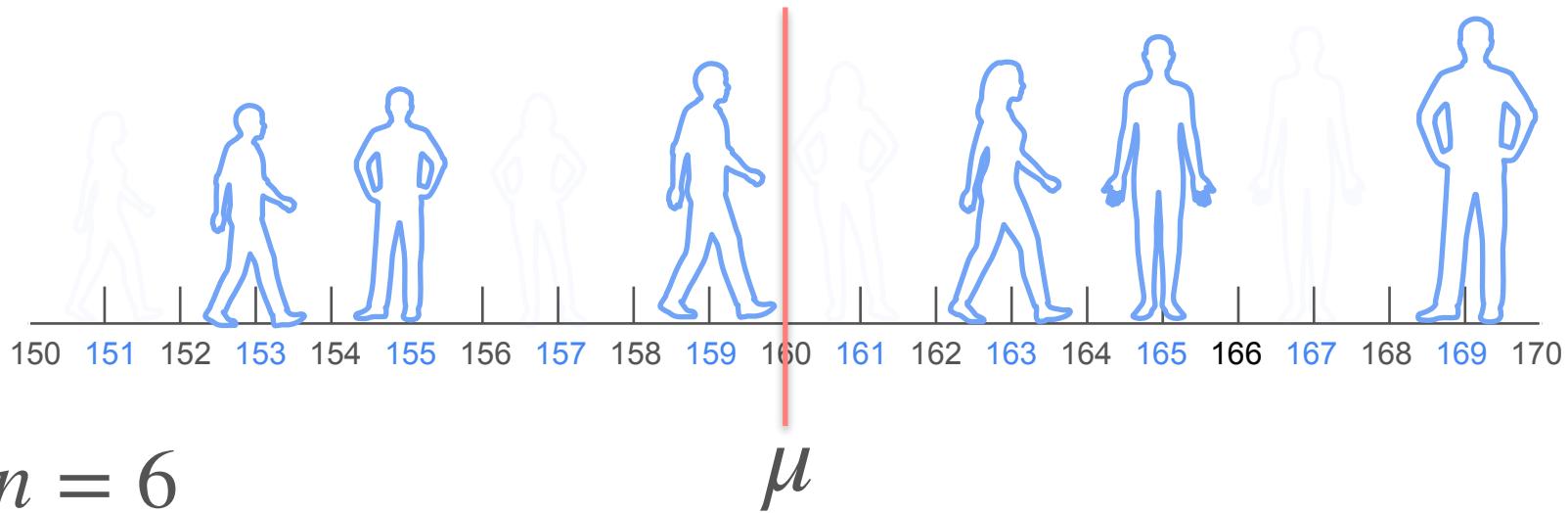


# Population and Sample Mean



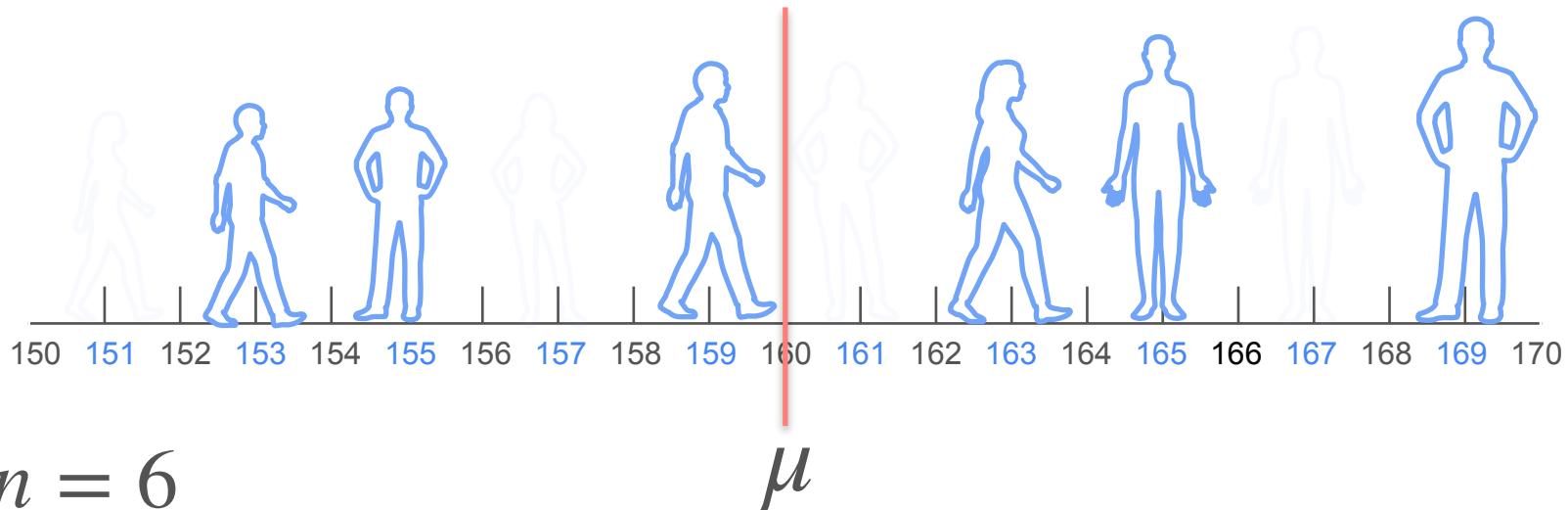
What is the average height in statistopia?

# Population and Sample Mean



What is the average  
height in statistopia?

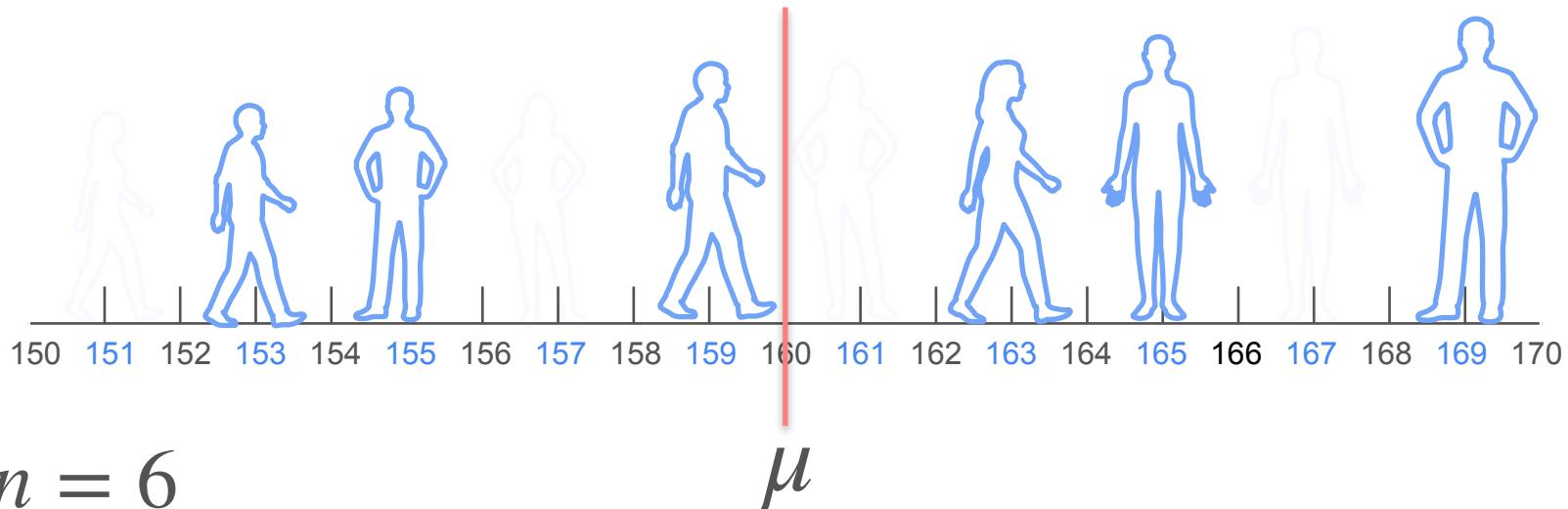
# Population and Sample Mean



What is the average height in statistopia?

$$\frac{153 + 155 + 159 + 163 + 165 + 169}{6}$$

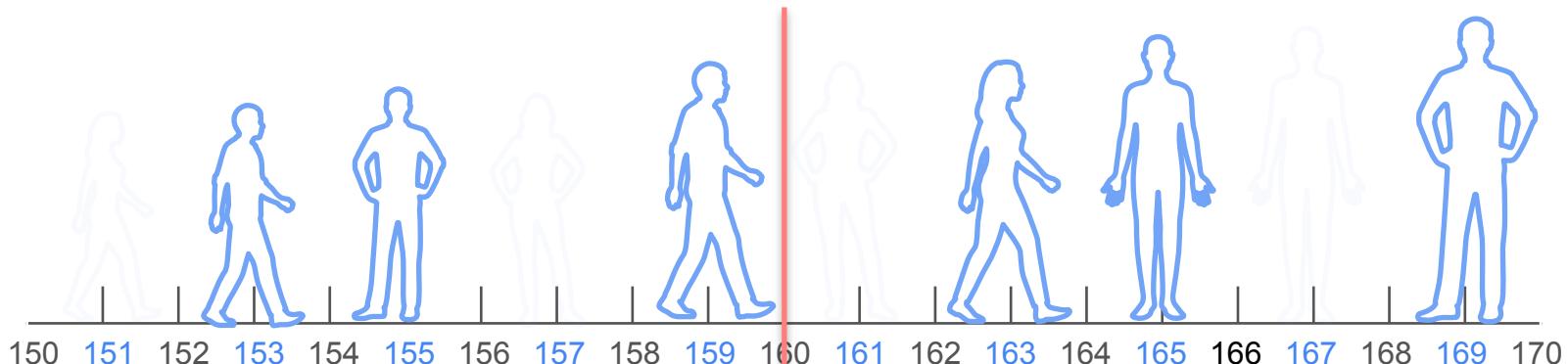
# Population and Sample Mean



What is the average height in statistopia?

$$\frac{153 + 155 + 159 + 163 + 165 + 169}{6} = \frac{964}{6} = 160.97$$

# Population and Sample Mean

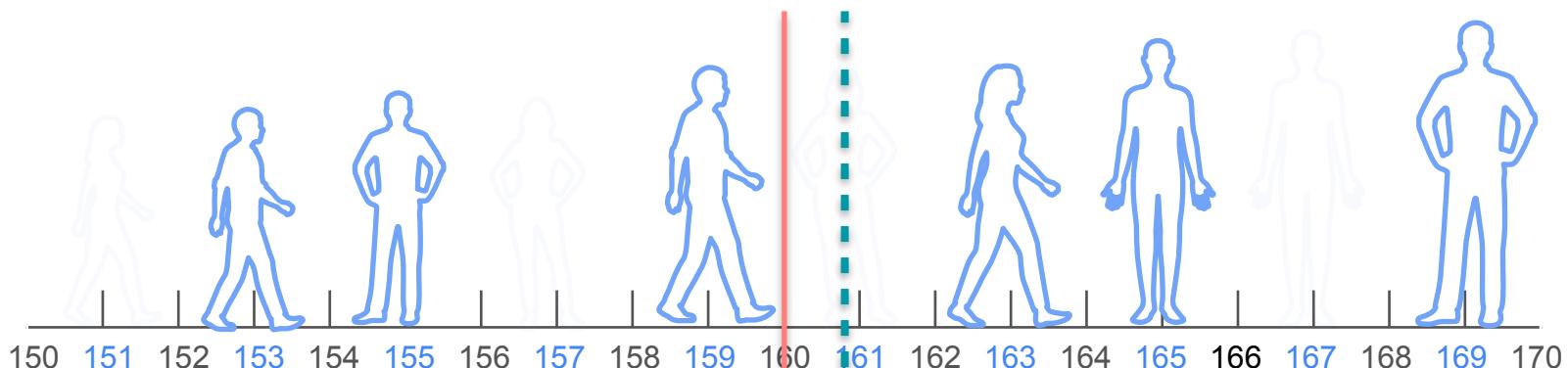


$$n = 6$$

What is the average height in statistopia?

$$\text{Sample mean} \quad \bar{x}_1$$
$$\frac{153 + 155 + 159 + 163 + 165 + 169}{6} = \frac{964}{6} = 160.97$$
$$\bar{x}$$

# Population and Sample Mean



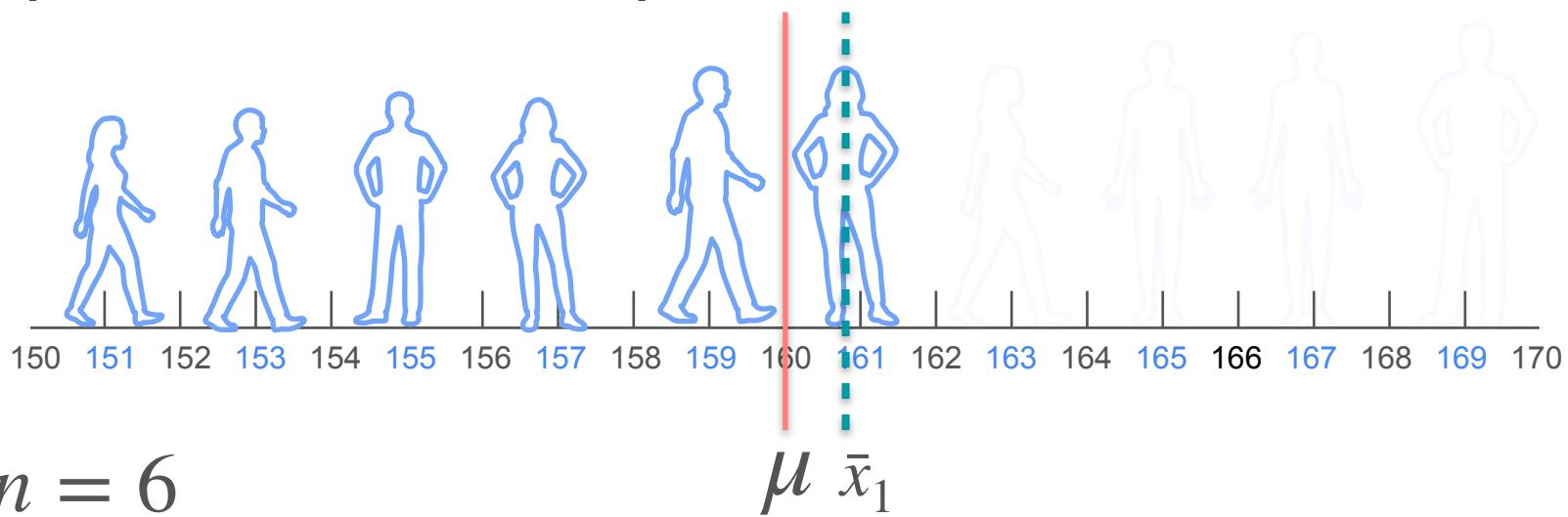
$$n = 6$$

What is the average height in statistopia?

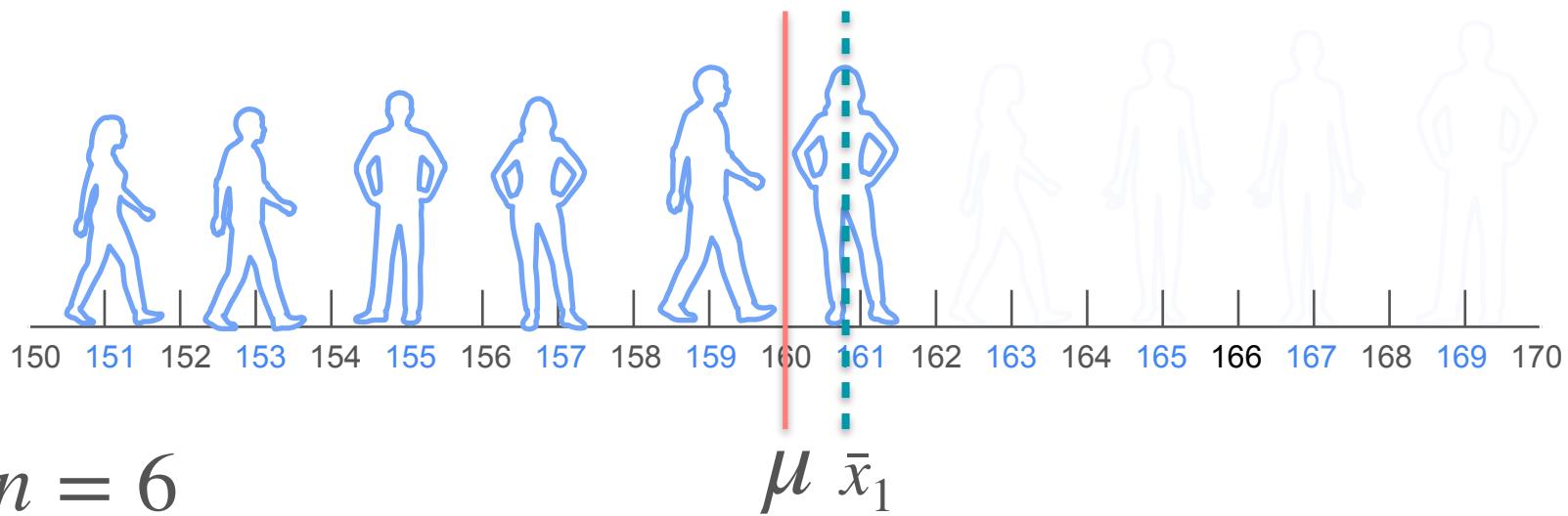
$$\frac{153 + 155 + 159 + 163 + 165 + 169}{6} = \frac{964}{6} = 160.97$$

$\bar{x}$

# Population and Sample Mean

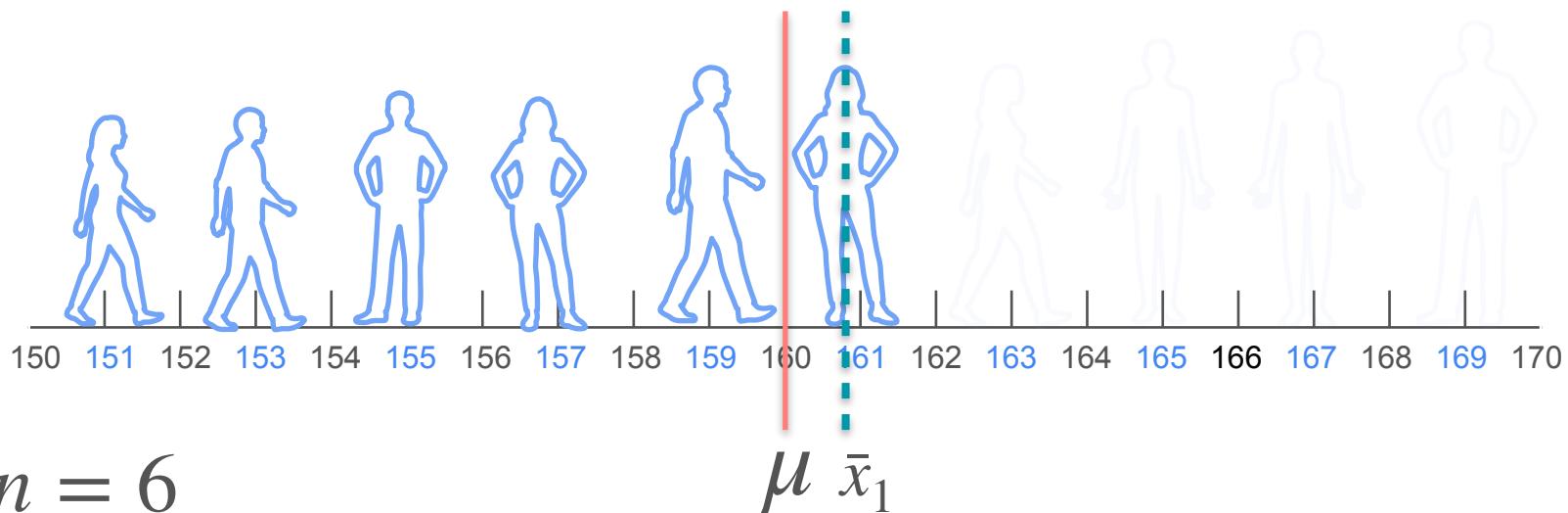


# Population and Sample Mean



What is the average  
height in statistopia?

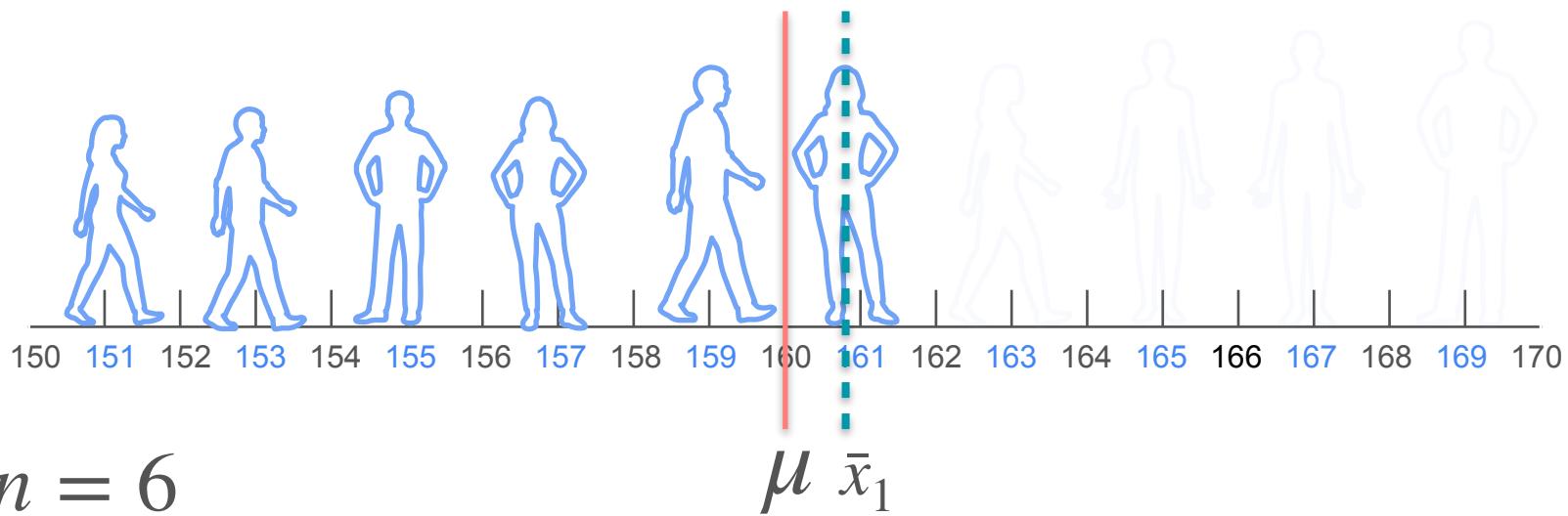
# Population and Sample Mean



What is the average height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161}{6}$$

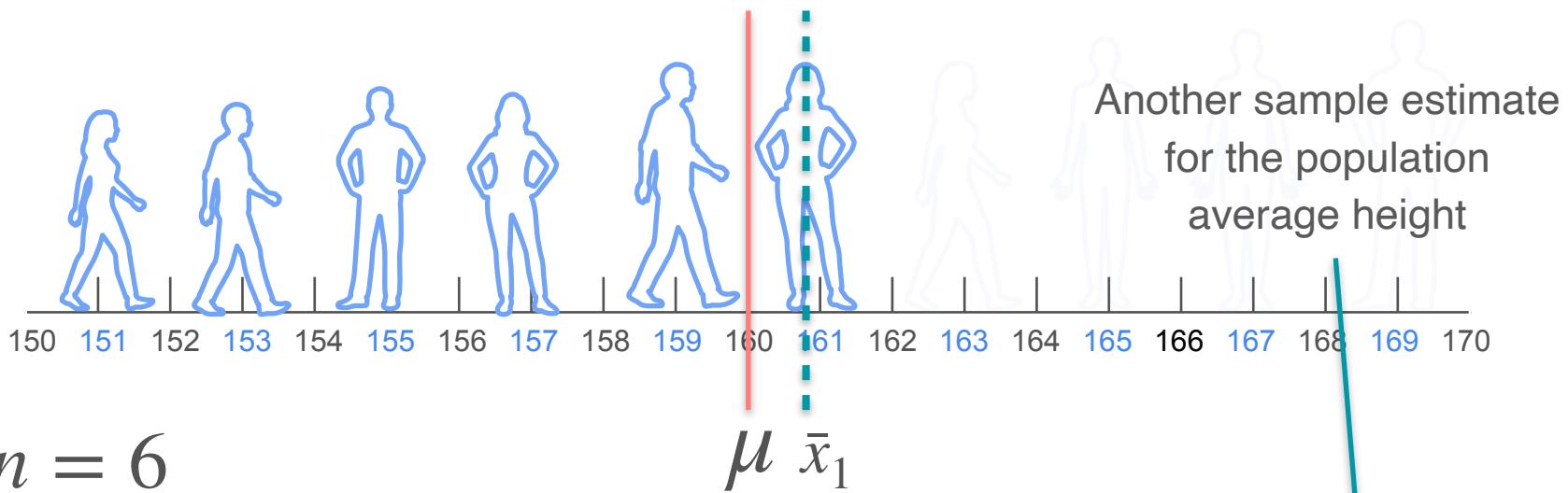
# Population and Sample Mean



What is the average height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161}{6} = \frac{936}{6} = 156\text{cm}$$

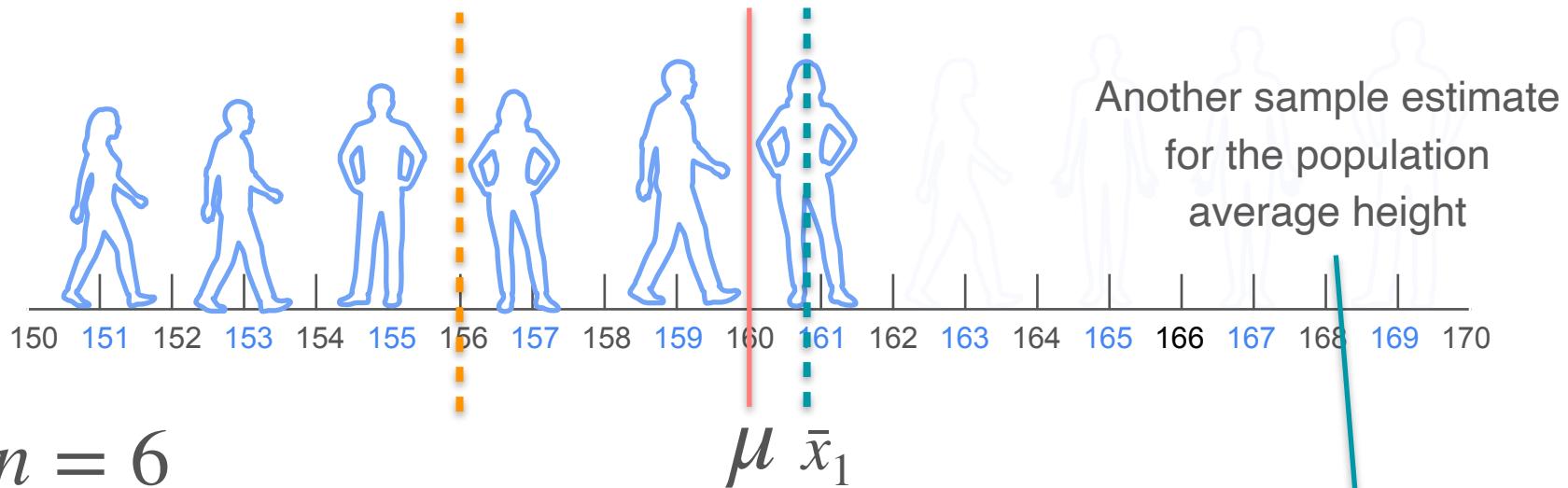
# Population and Sample Mean



What is the average height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161}{6} = \frac{936}{6} = 156\text{cm}$$

# Population and Sample Mean

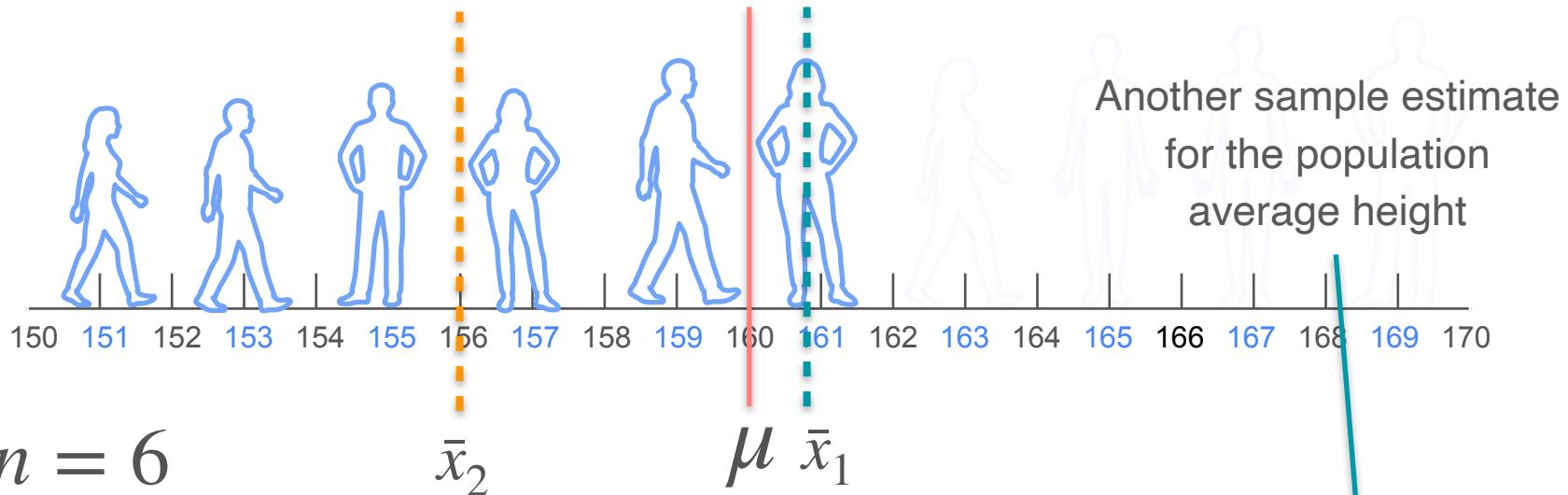


Another sample estimate  
for the population  
average height

What is the average  
height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161}{6} = \frac{936}{6} = 156\text{cm}$$

# Population and Sample Mean

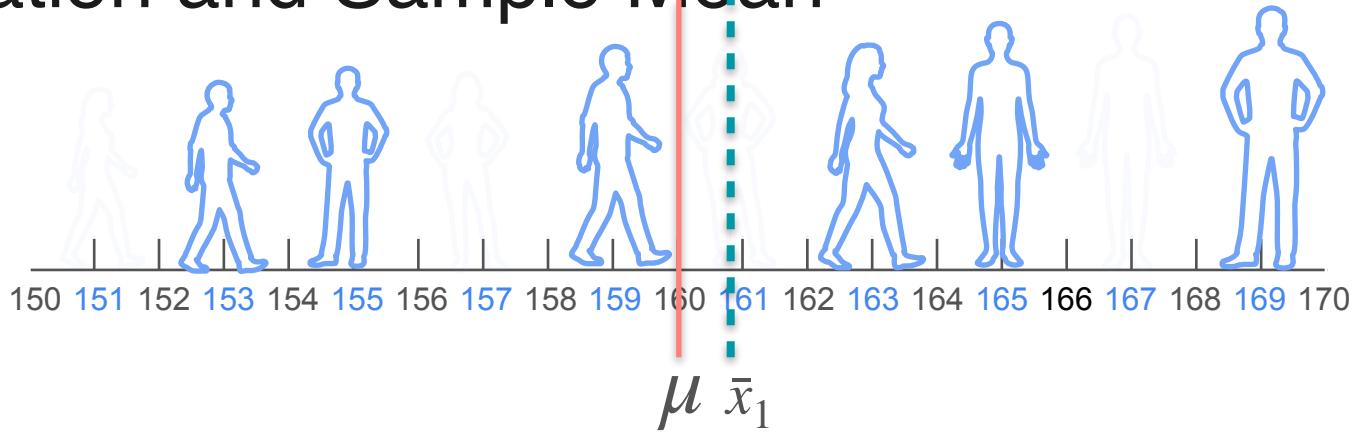


What is the average height in statistopia?

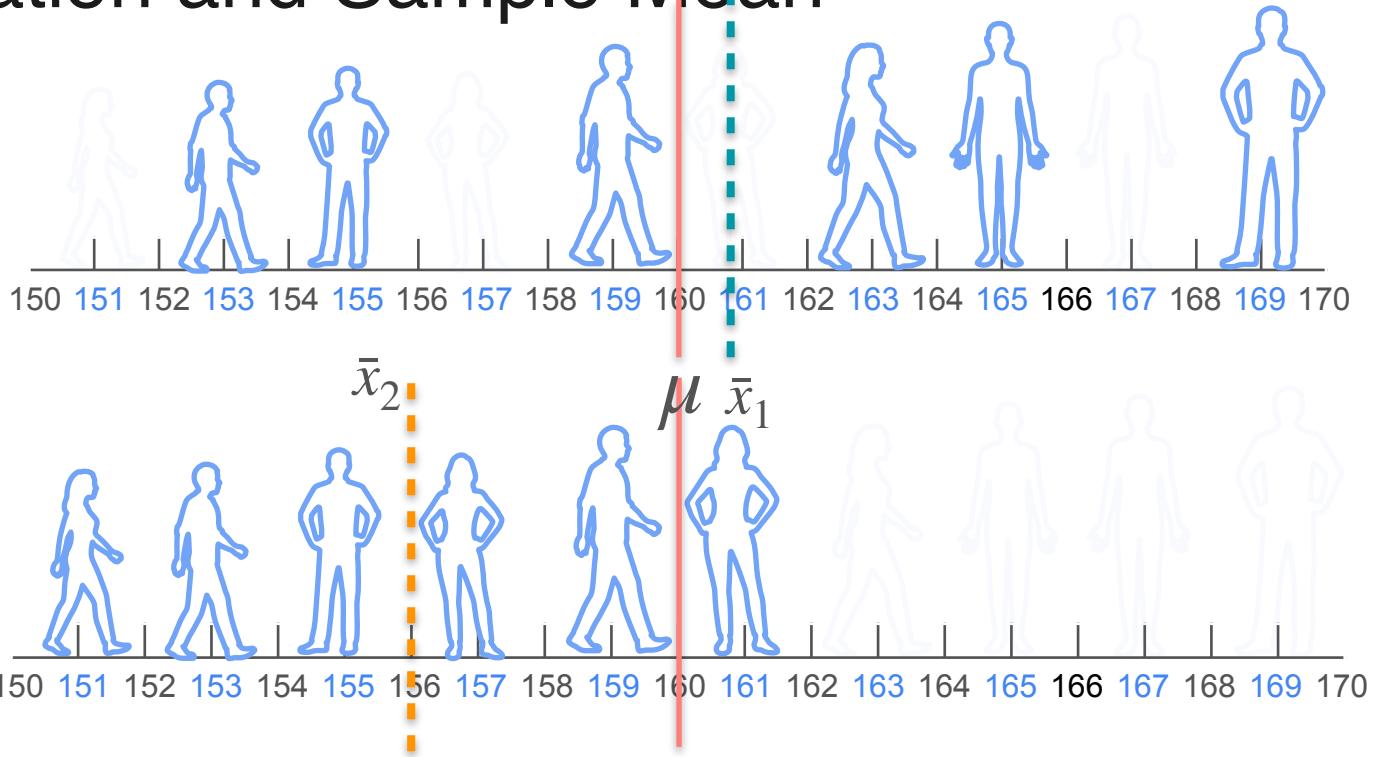
$$\frac{151 + 153 + 155 + 157 + 159 + 161}{6} = \frac{936}{6} = 156\text{cm}$$

# Population and Sample Mean

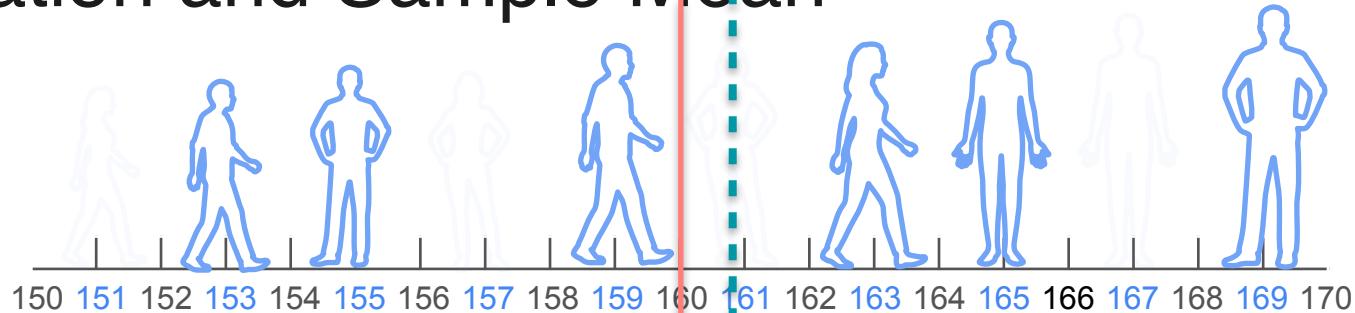
# Population and Sample Mean



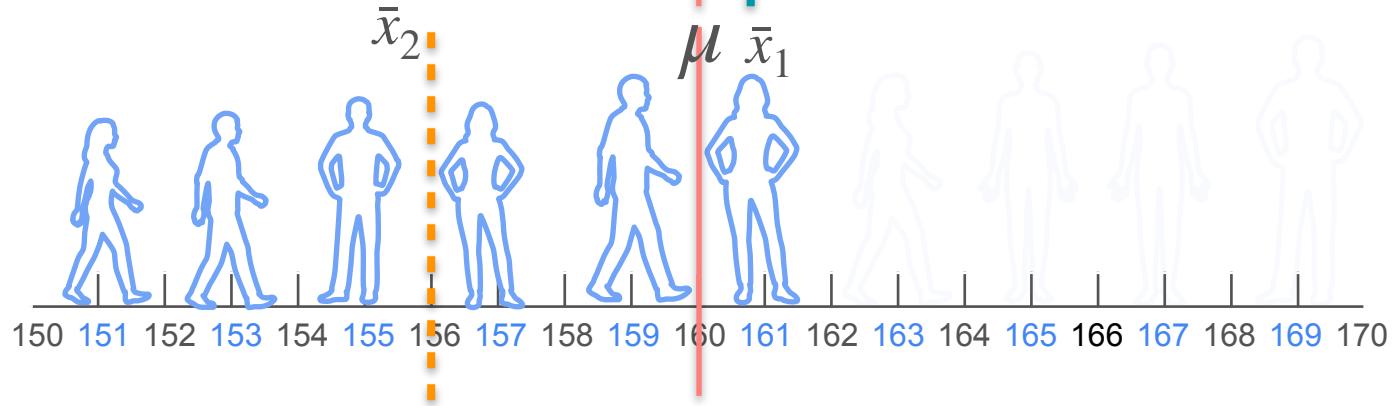
# Population and Sample Mean



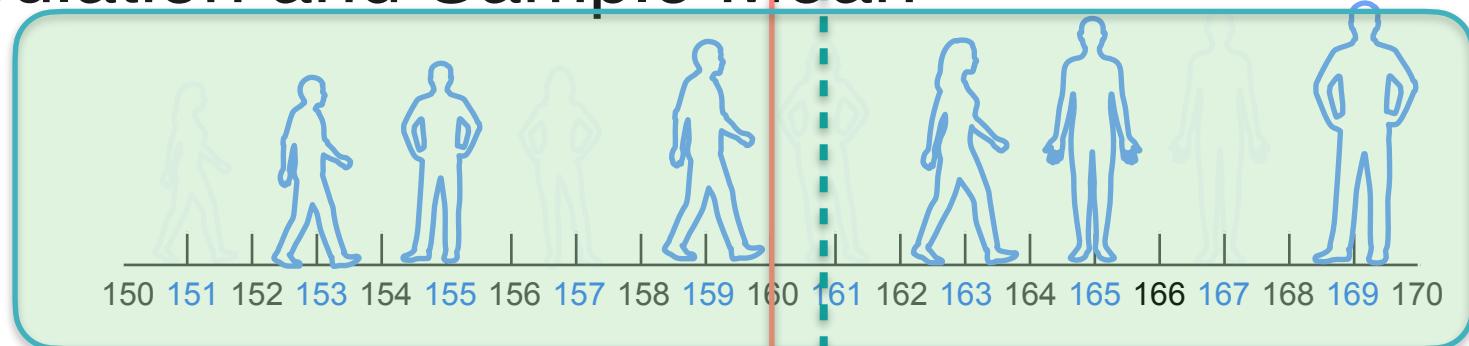
# Population and Sample Mean



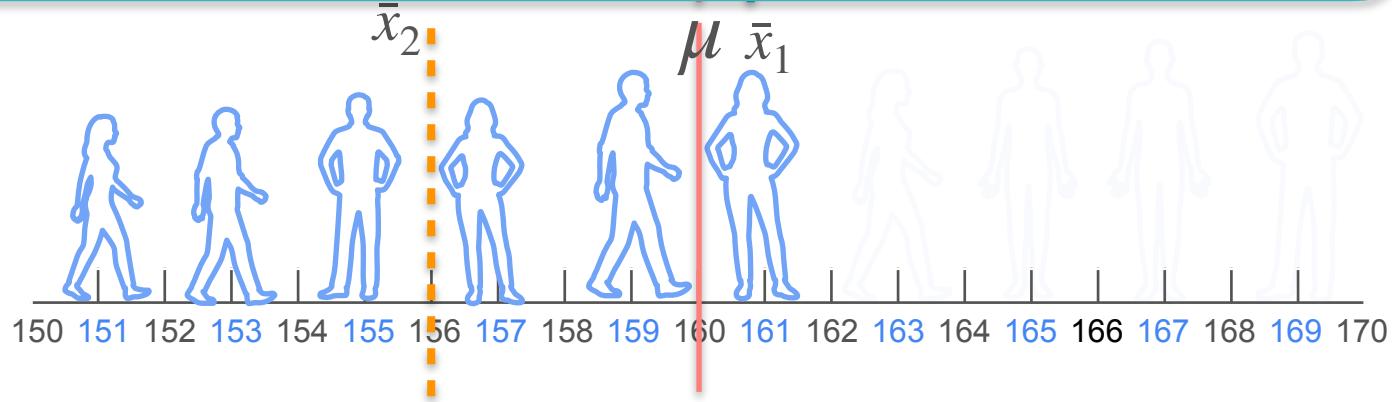
$$n = 6$$



# Population and Sample Mean



$$n = 6$$

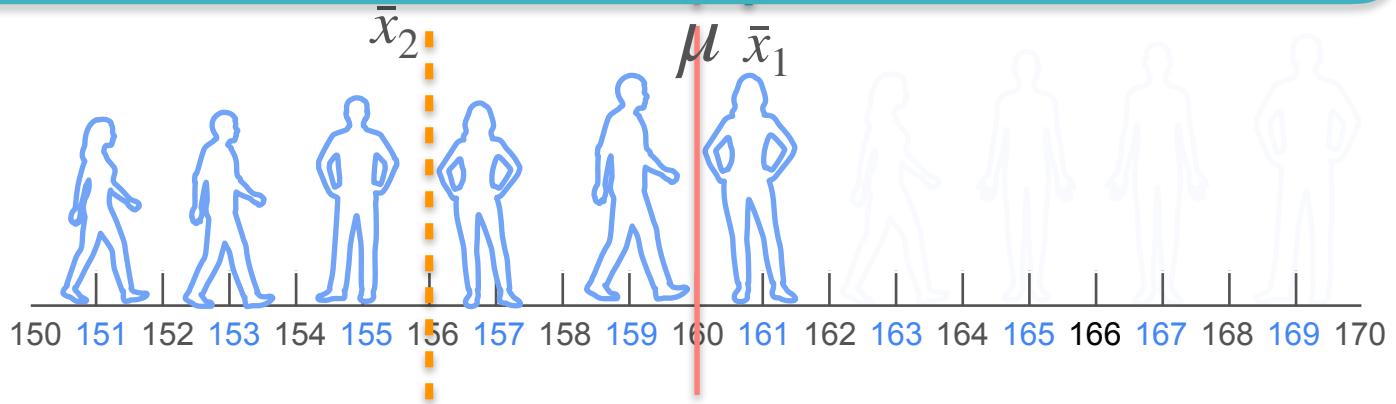


# Population and Sample Mean

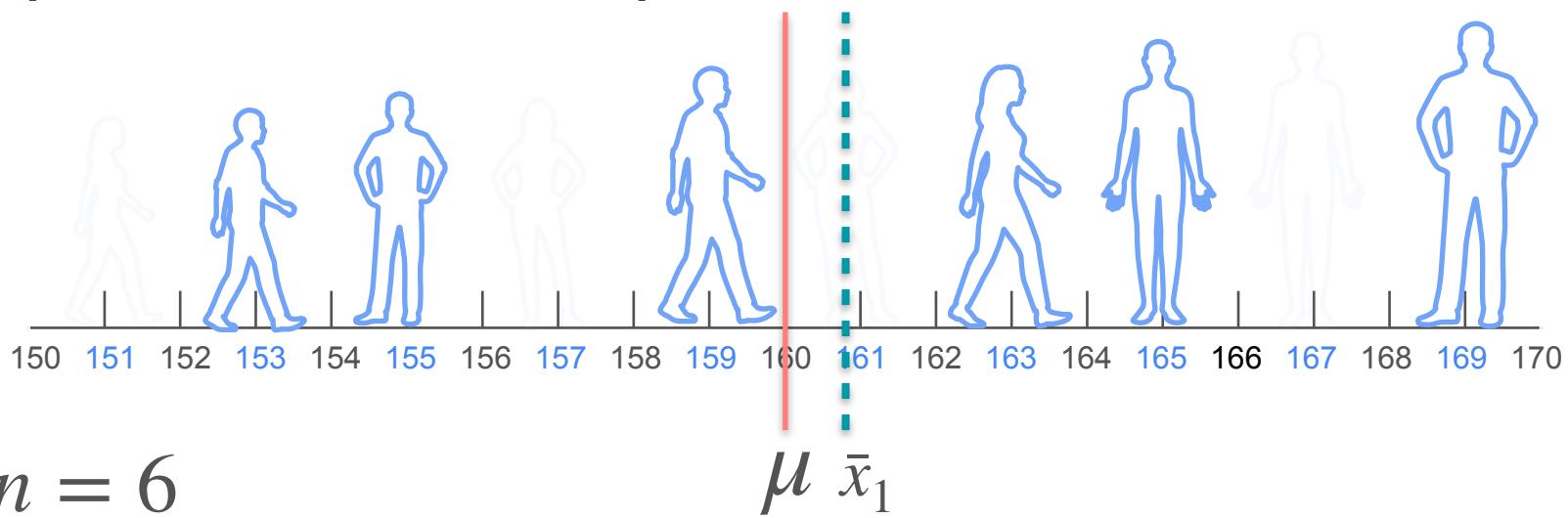
Better estimate of the population mean height

150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170

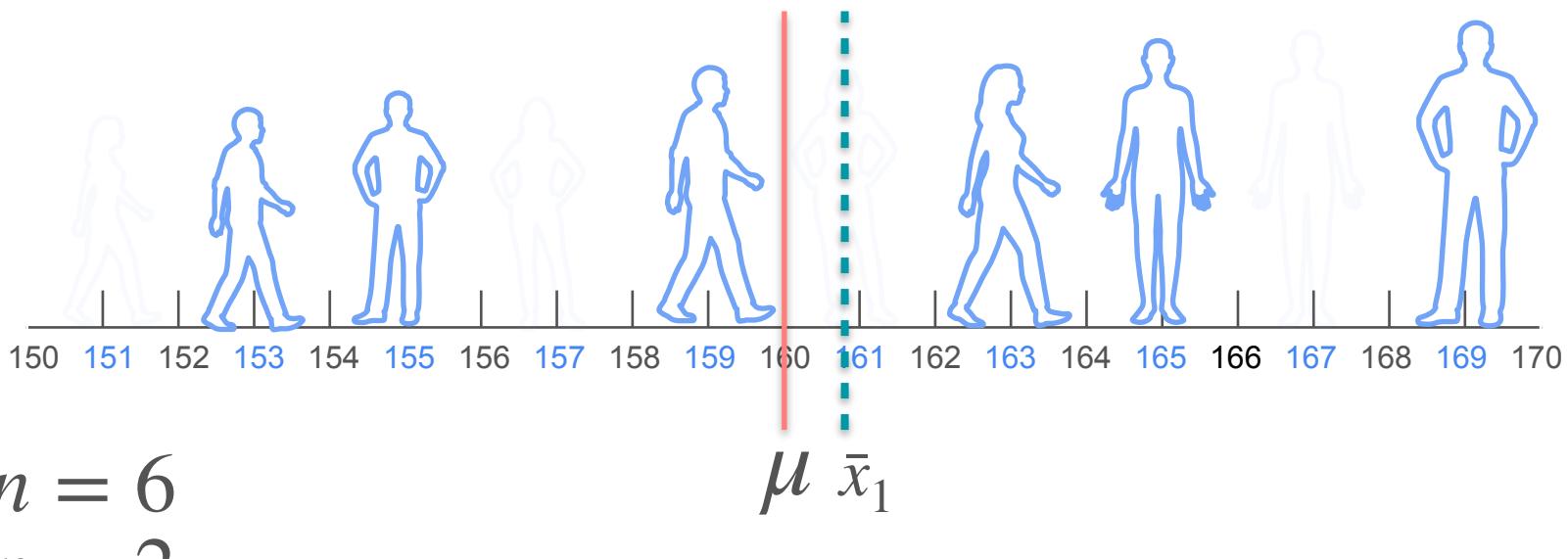
$$n = 6$$



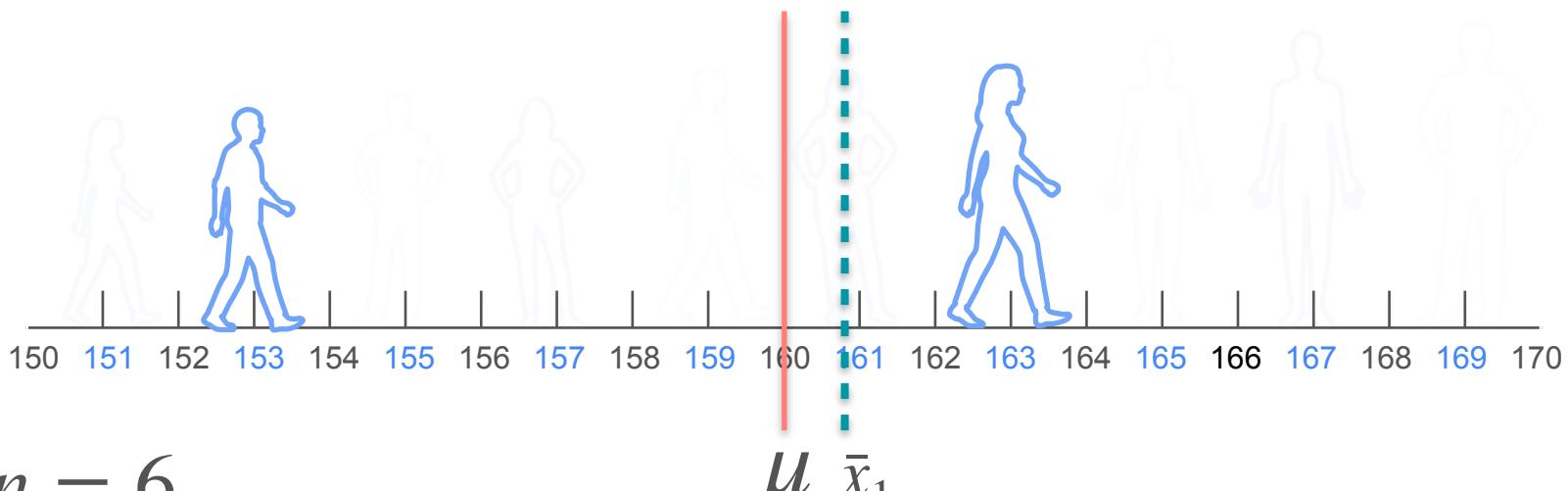
# Population and Sample Mean



# Population and Sample Mean

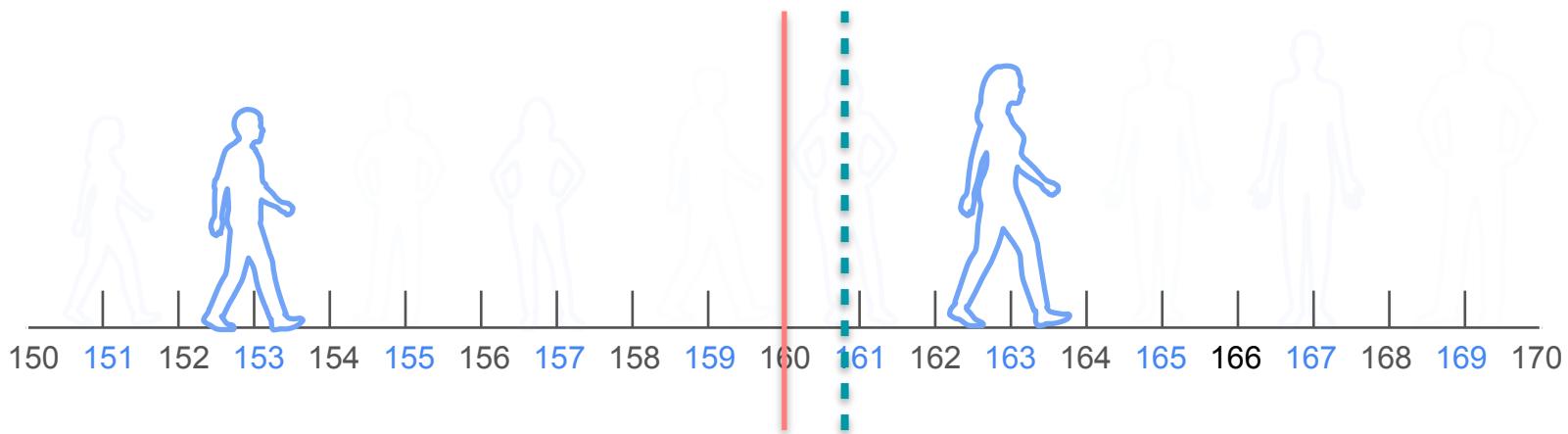


# Population and Sample Mean



$$n = 2$$

# Population and Sample Mean



$$n = 6$$

$$n = 2$$

What is the average height in statistopia?

# Population and Sample Mean



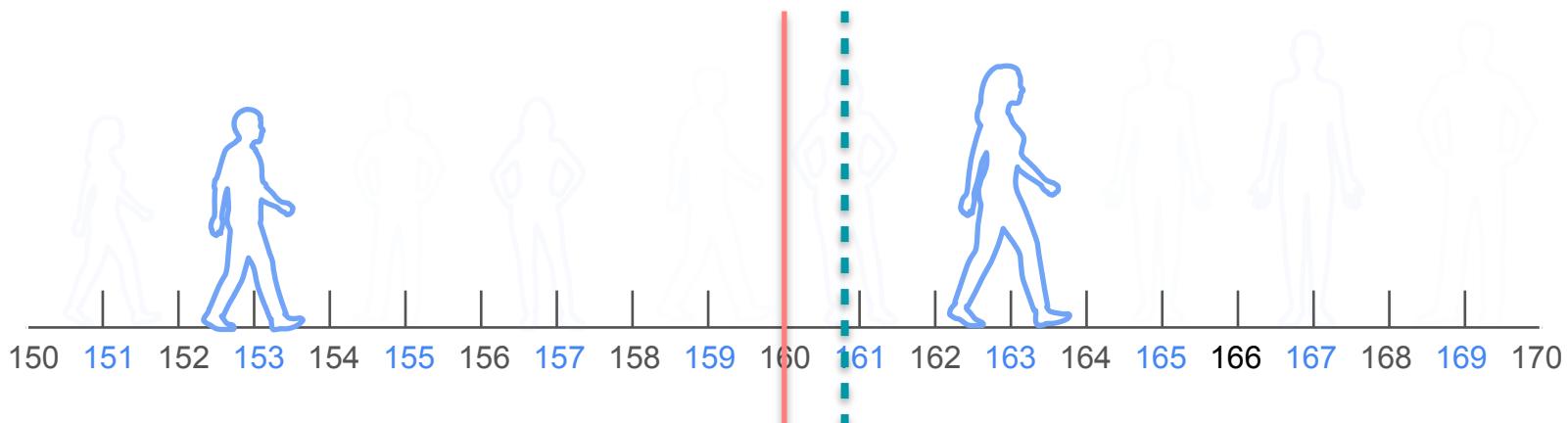
$$n = 6$$

$$n = 2$$

What is the average height in statistopia?

$$\frac{153 + 163}{2}$$

# Population and Sample Mean



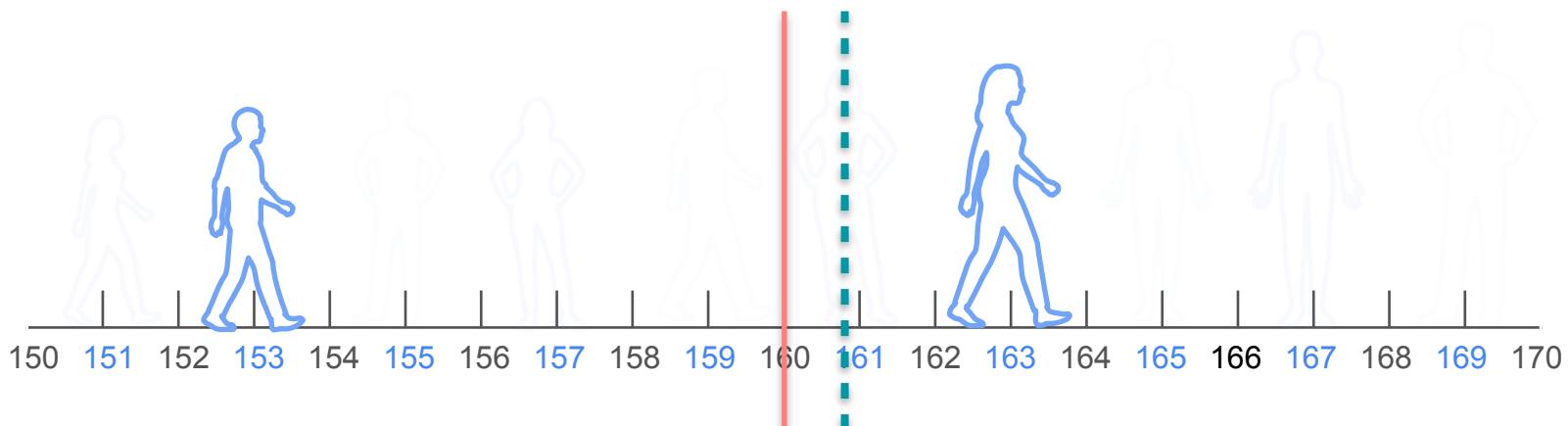
$$n = 6$$

$$n = 2$$

What is the average height in statistopia?

$$\frac{153 + 163}{2} = \frac{316}{2} = 158\text{cm}$$

# Population and Sample Mean



$$n = 6$$

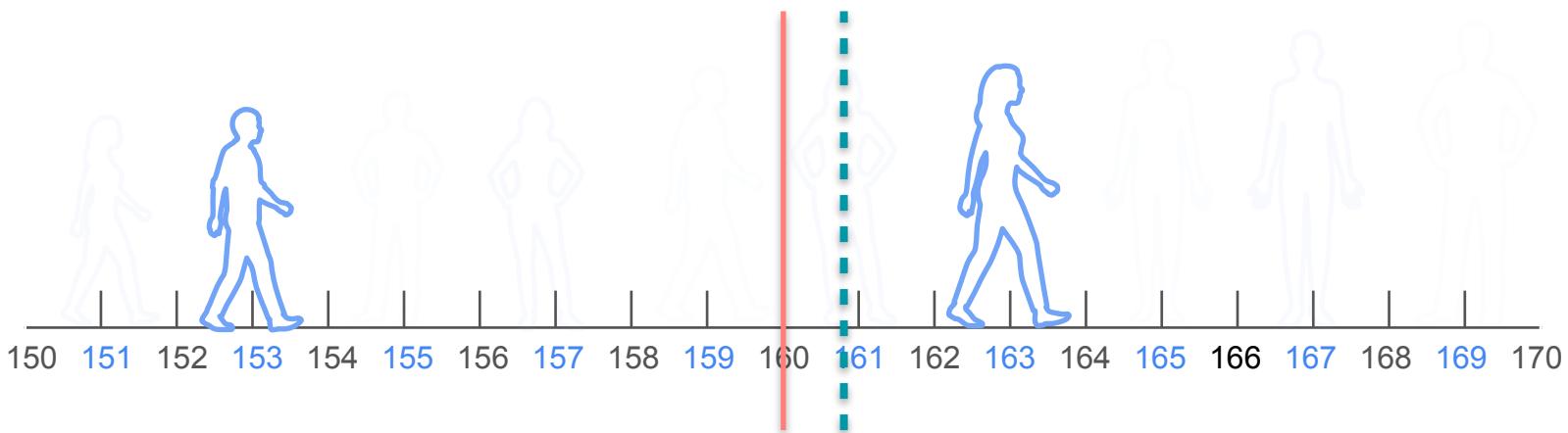
$$n = 2$$

What is the average height in statistopia?

$$\mu \bar{x}_1$$

$$\frac{153 + 163}{2} = \frac{316}{2} = 158\text{cm}$$

# Population and Sample Mean



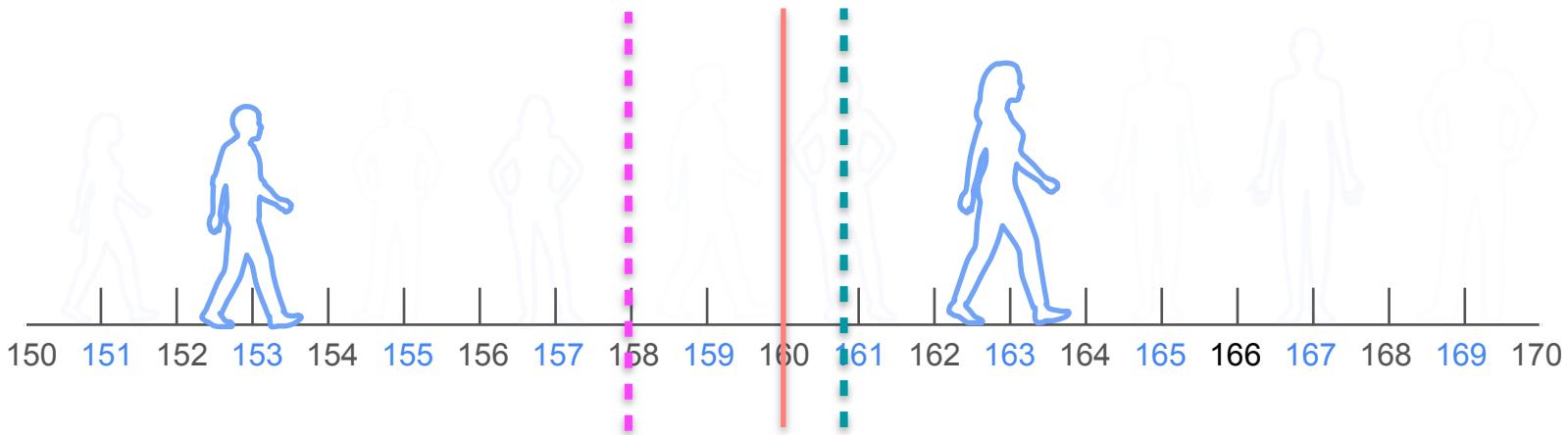
$$n = 6$$

$$n = 2$$

What is the average height in statistopia?

$$\frac{153 + 163}{2} = \frac{316}{2} = 158\text{cm}$$

# Population and Sample Mean



$$n = 6$$

$$n = 2$$

What is the average height in statistopia?

$$\frac{153 + 163}{2} = \frac{316}{2} = 158\text{cm}$$

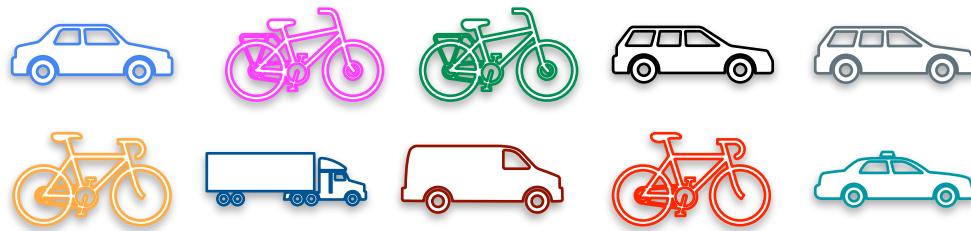
# Proportion

# Proportion

Population size: 10

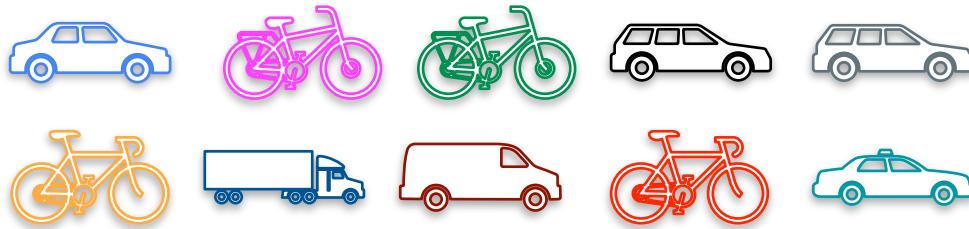
# Proportion

Population size: 10



# Proportion

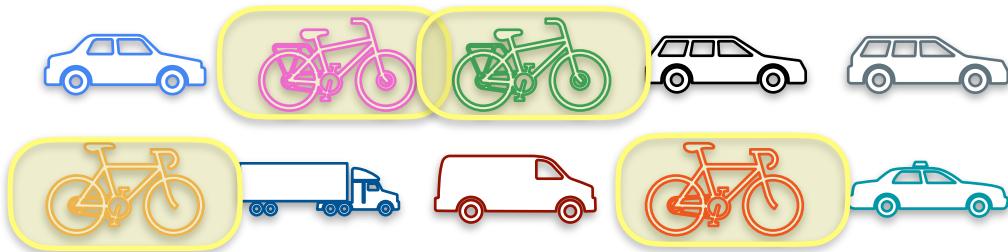
Population size: 10



What proportion of people own a bicycle?

# Proportion

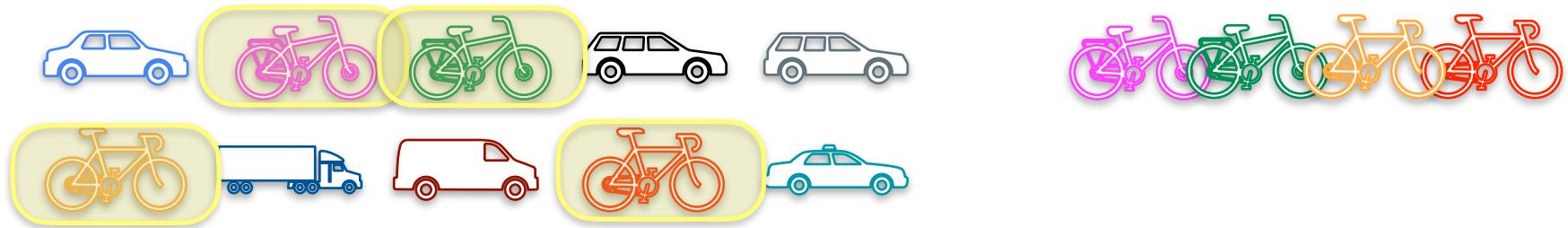
Population size: 10



What proportion of people own a bicycle?

# Proportion

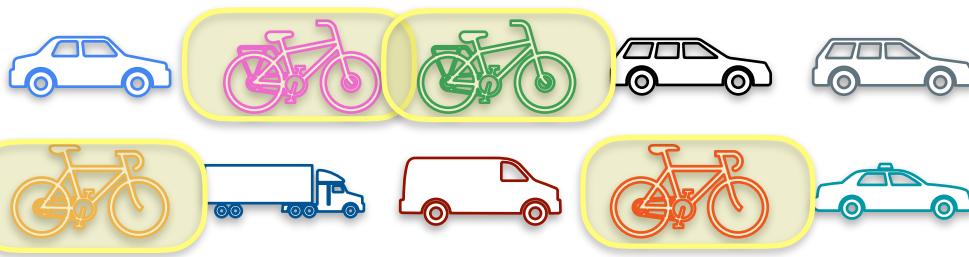
Population size: 10



What proportion of people own a bicycle?

# Proportion

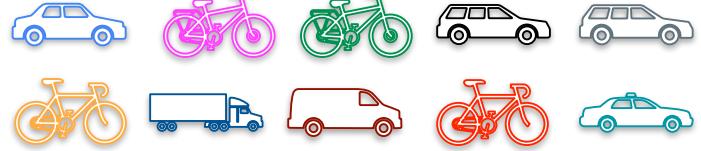
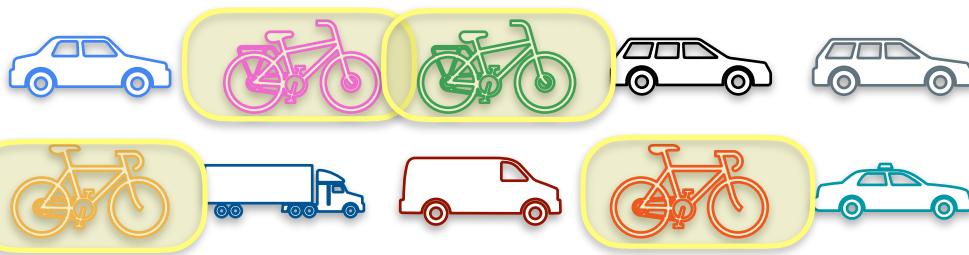
Population size: 10



What proportion of people own a bicycle?

# Proportion

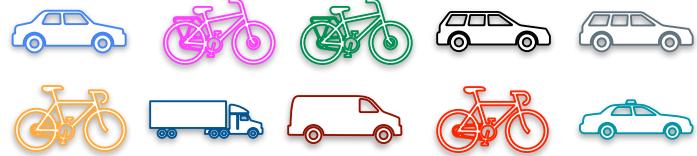
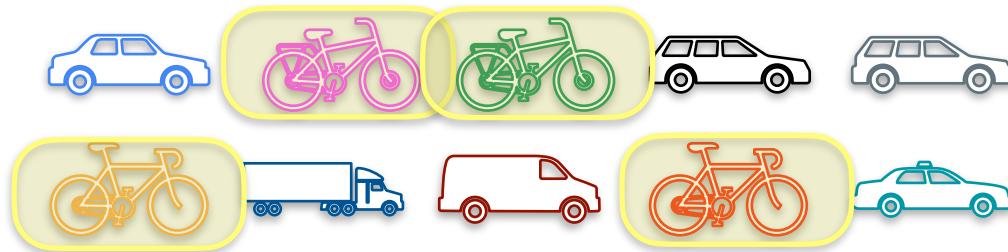
Population size: 10



What proportion of people own a bicycle?

# Proportion

Population size: 10

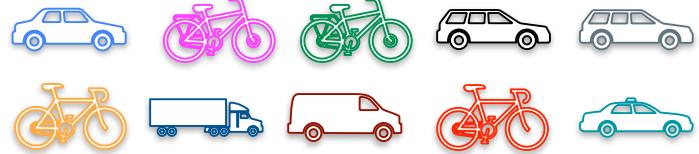
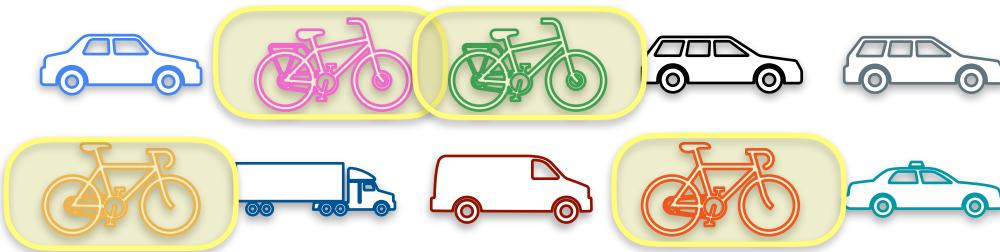


What proportion of people own a bicycle?

$$= \frac{4}{10} = 0.4 = 40\%$$

# Proportion

Population size: 10

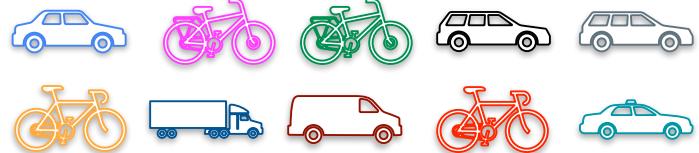
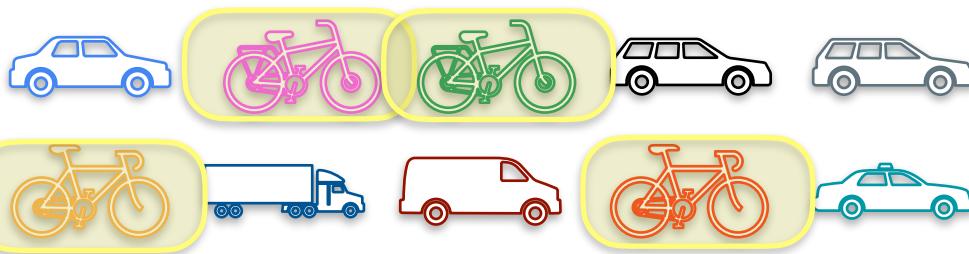


What proportion of people own a bicycle?

$$\text{population proportion} = \frac{4}{10} = 0.4 = 40\%$$

# Proportion

Population size: 10



What proportion of people own a bicycle?

*p*

$$\text{population proportion} = \frac{4}{10} = 0.4 = 40\%$$

# Proportion

$$P = \frac{\text{number of items with a given characteristic (}x\text{)}}{\text{population (}n\text{)}}$$

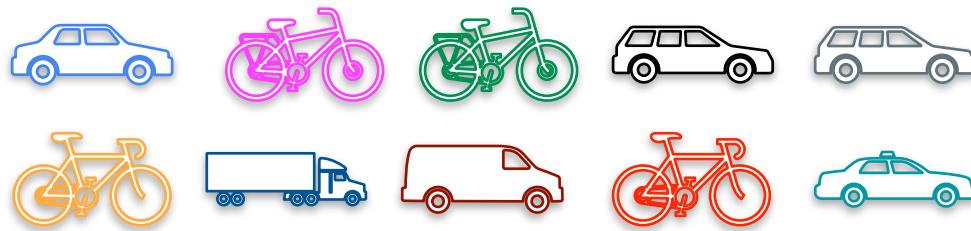
# Proportion

population proportion

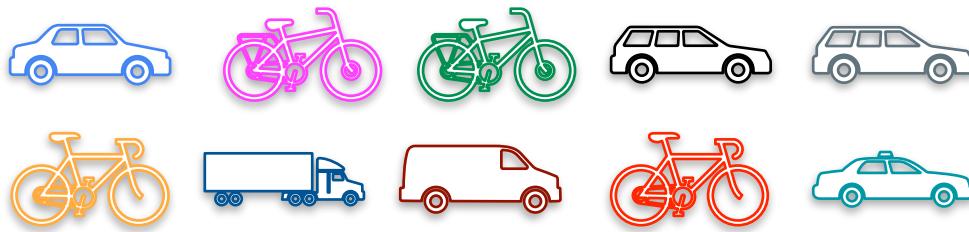
$$P = \frac{\text{number of items with a given characteristic } (x)}{\text{population } (n)}$$

# Proportion

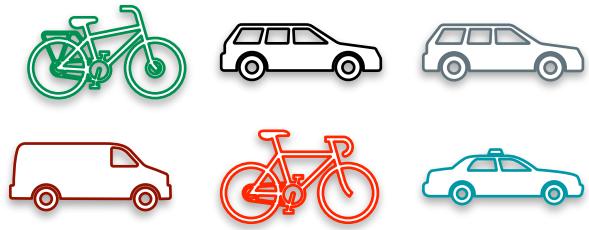
Population size: 10



# Proportion

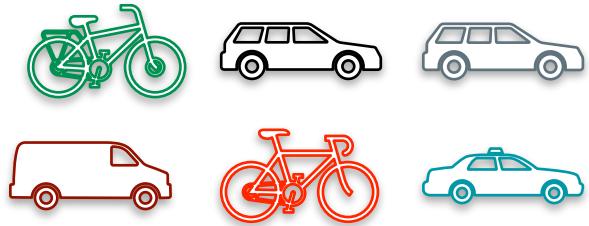


# Sample Proportion



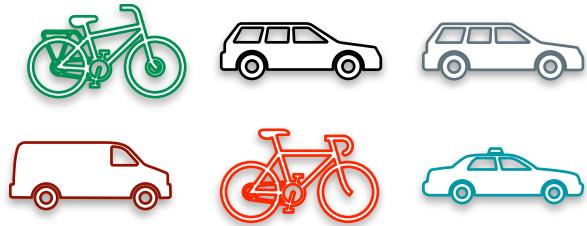
# Sample Proportion

Sample size: 6



# Sample Proportion

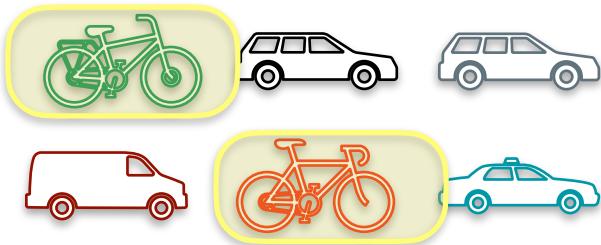
Sample size: 6



What proportion of people own a bicycle?

# Sample Proportion

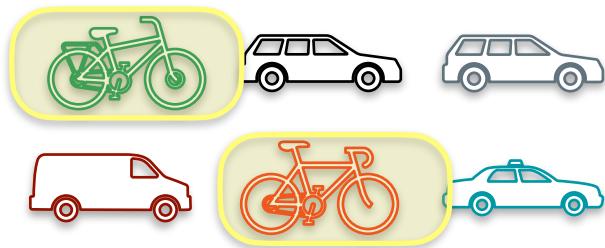
Sample size: 6



What proportion of people own a bicycle?

# Sample Proportion

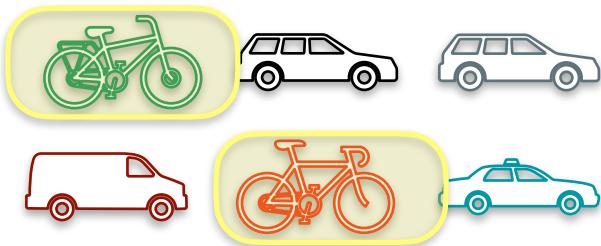
Sample size: 6



What proportion of people own a bicycle?

# Sample Proportion

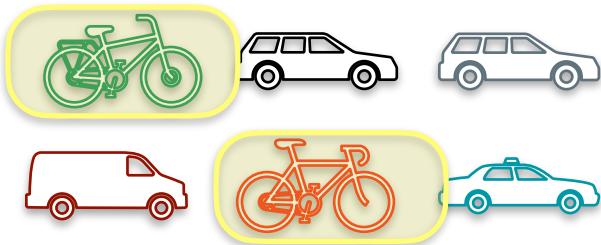
Sample size: 6



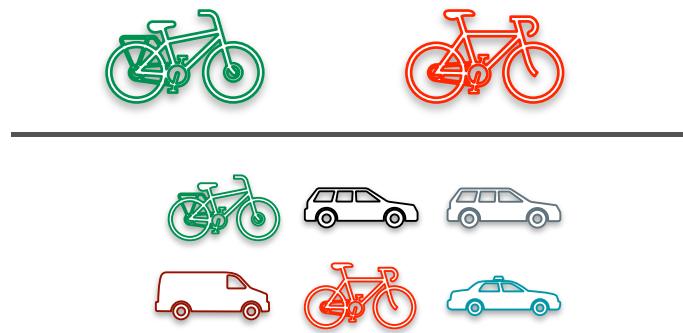
What proportion of people own a bicycle?

# Sample Proportion

Sample size: 6

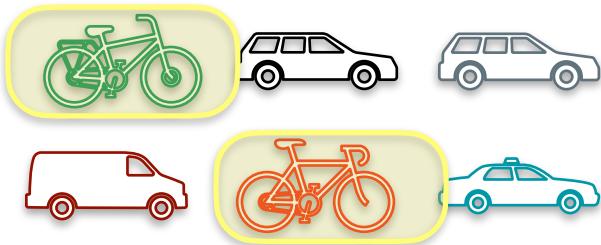


What proportion of people own a bicycle?

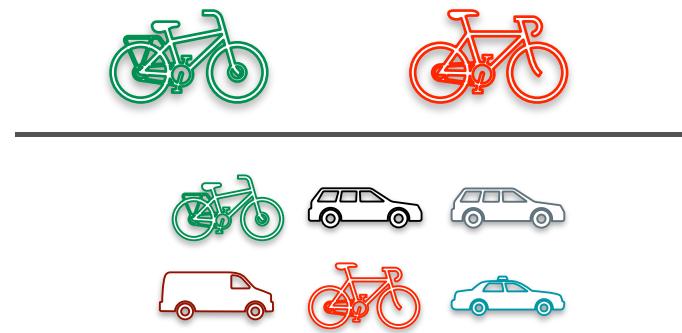


# Sample Proportion

Sample size: 6



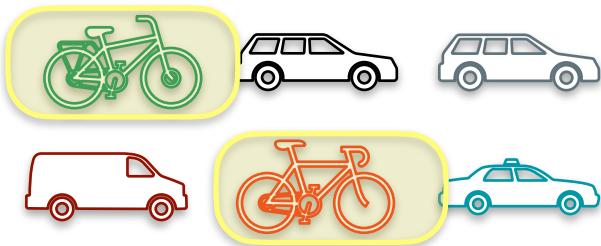
What proportion of people own a bicycle?



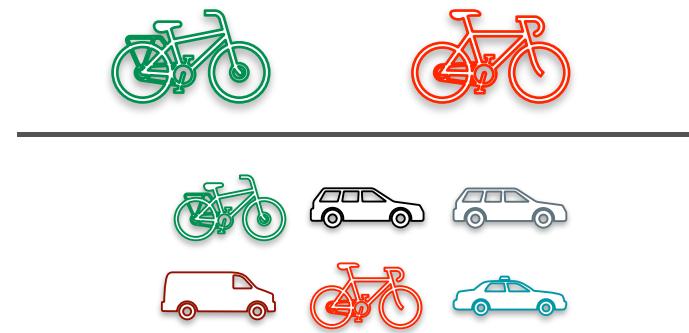
$$= \frac{2}{6} = 0.333 = 33.3\%$$

# Sample Proportion

Sample size: 6



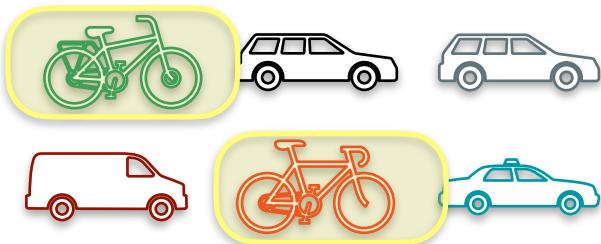
What proportion of people own a bicycle?



$$\text{sample proportion} = \frac{2}{6} = 0.333 = 33.3\%$$

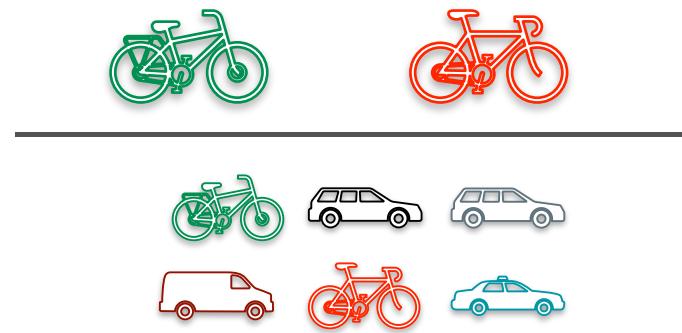
# Sample Proportion

Sample size: 6



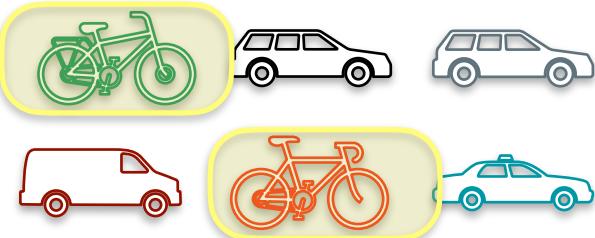
What proportion of people own a bicycle?

$$\hat{p} \text{ sample proportion} = \frac{2}{6} = 0.333 = 33.3\%$$

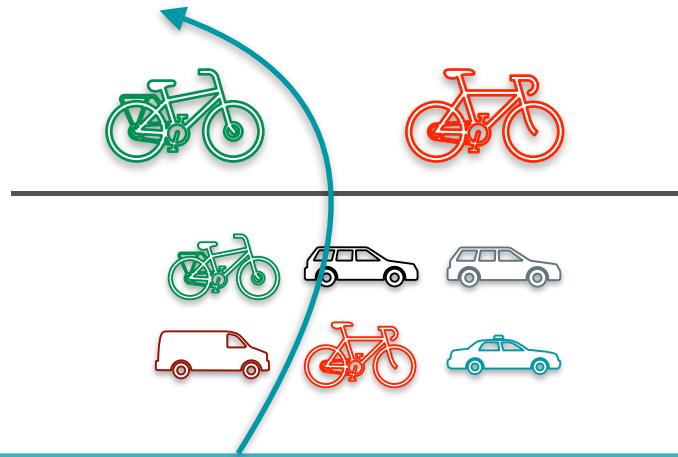


# Sample Proportion

Sample size: 6



What proportion of people own a bicycle?

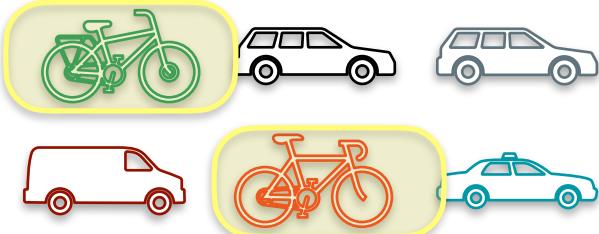


$$\hat{p} \text{ sample proportion} = \frac{2}{6} = 0.333 = 33.3\%$$

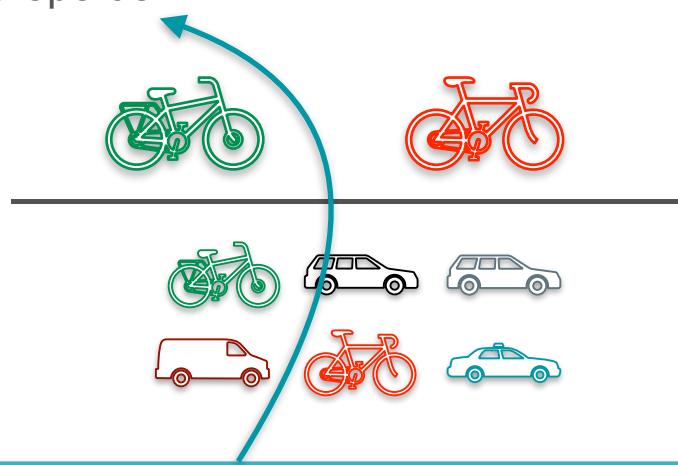
# Sample Proportion

Sample size: 6

estimate of the population proportion



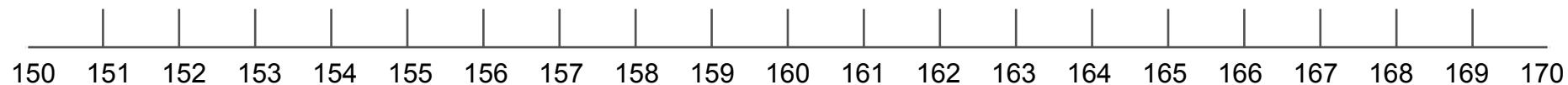
What proportion of people own a bicycle?



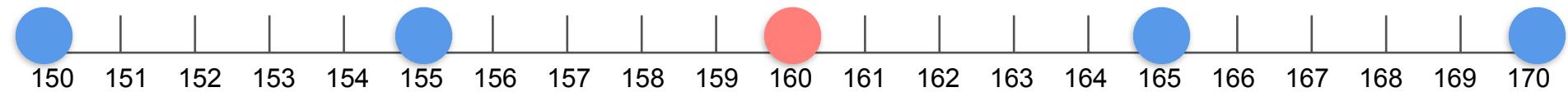
$$\hat{p} \text{ sample proportion } = \frac{2}{6} = 0.333 = 33.3\%$$

# Sample Variance

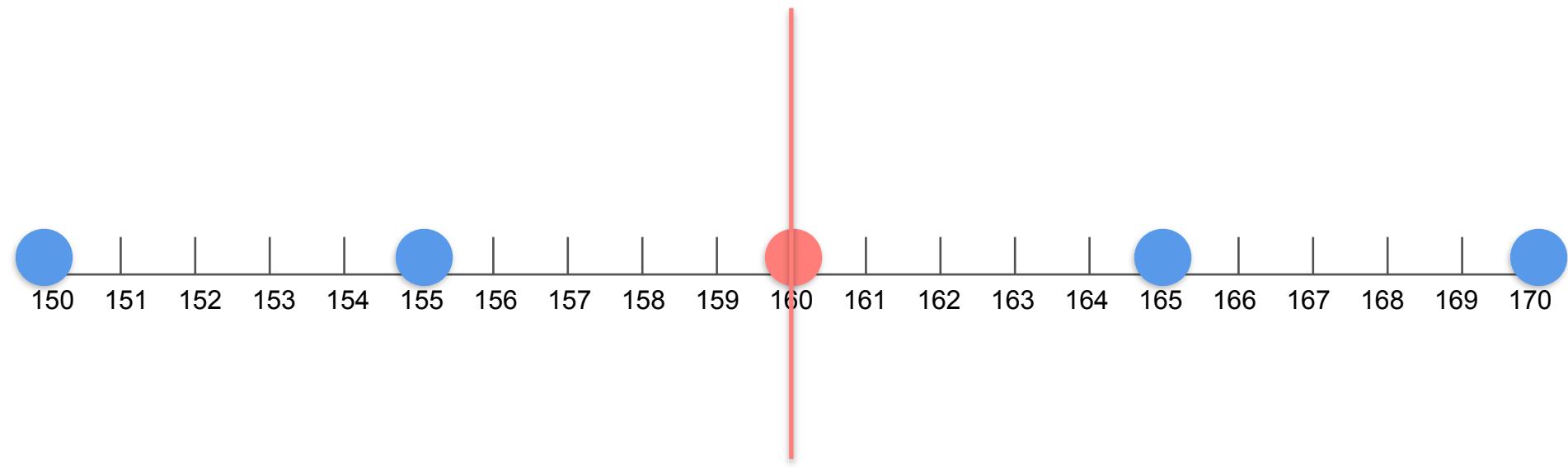
# Sample Variance



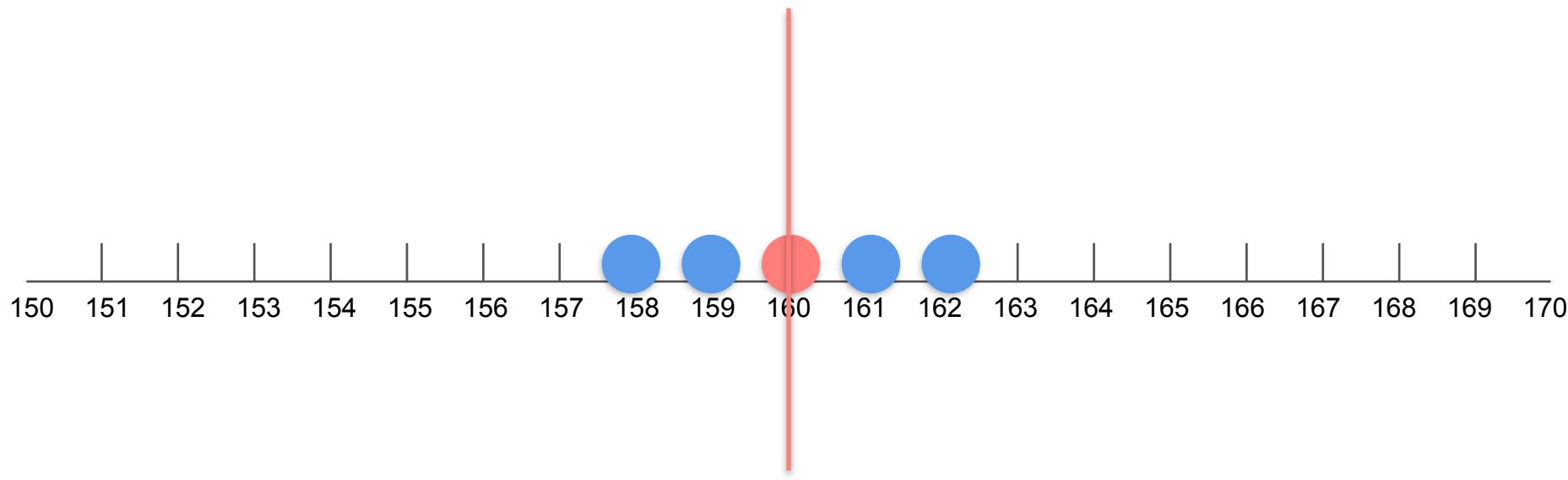
# Sample Variance



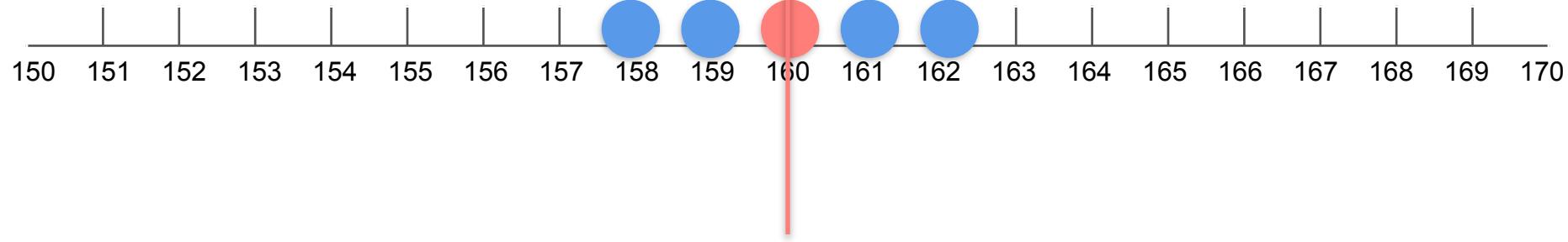
# Sample Variance



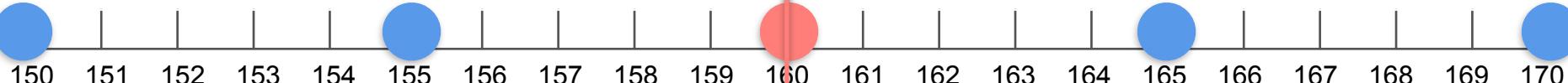
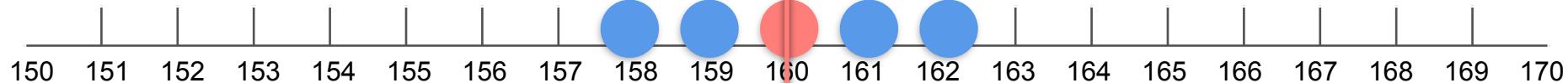
# Sample Variance



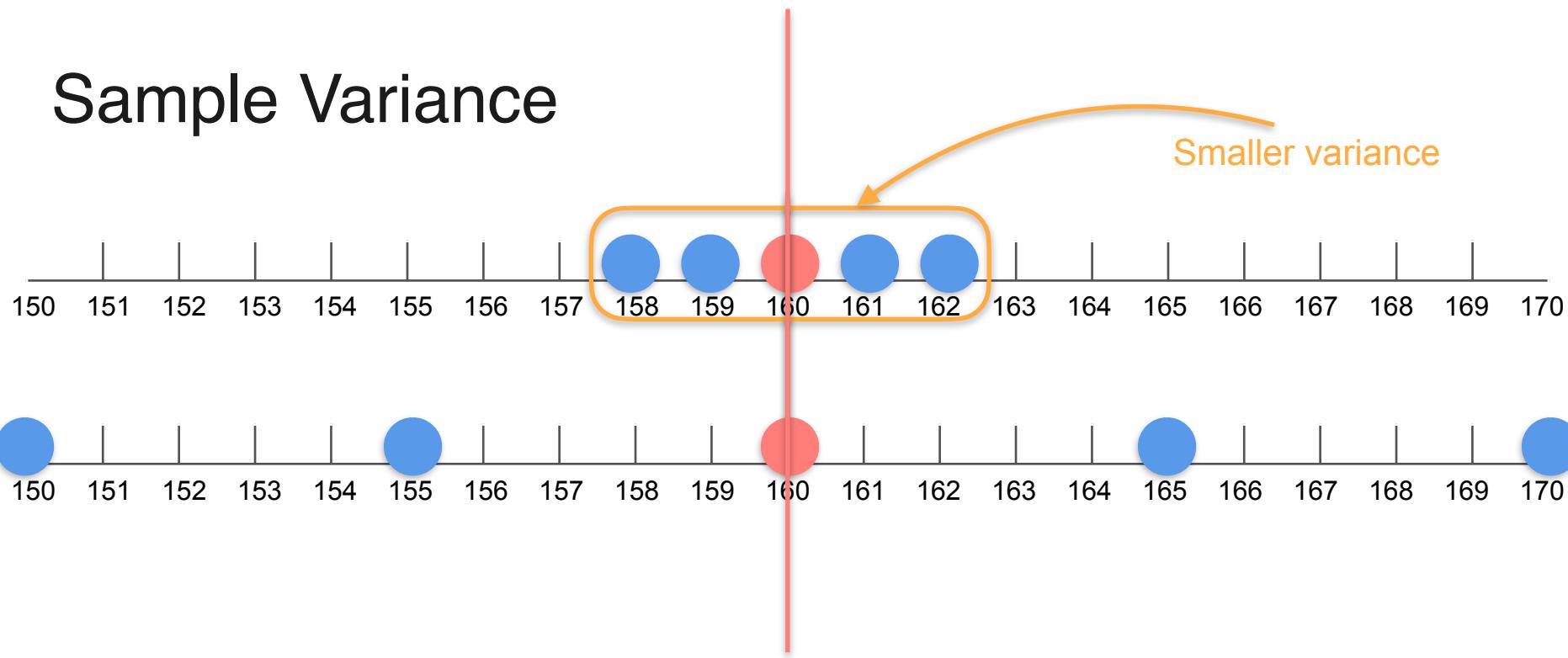
# Sample Variance



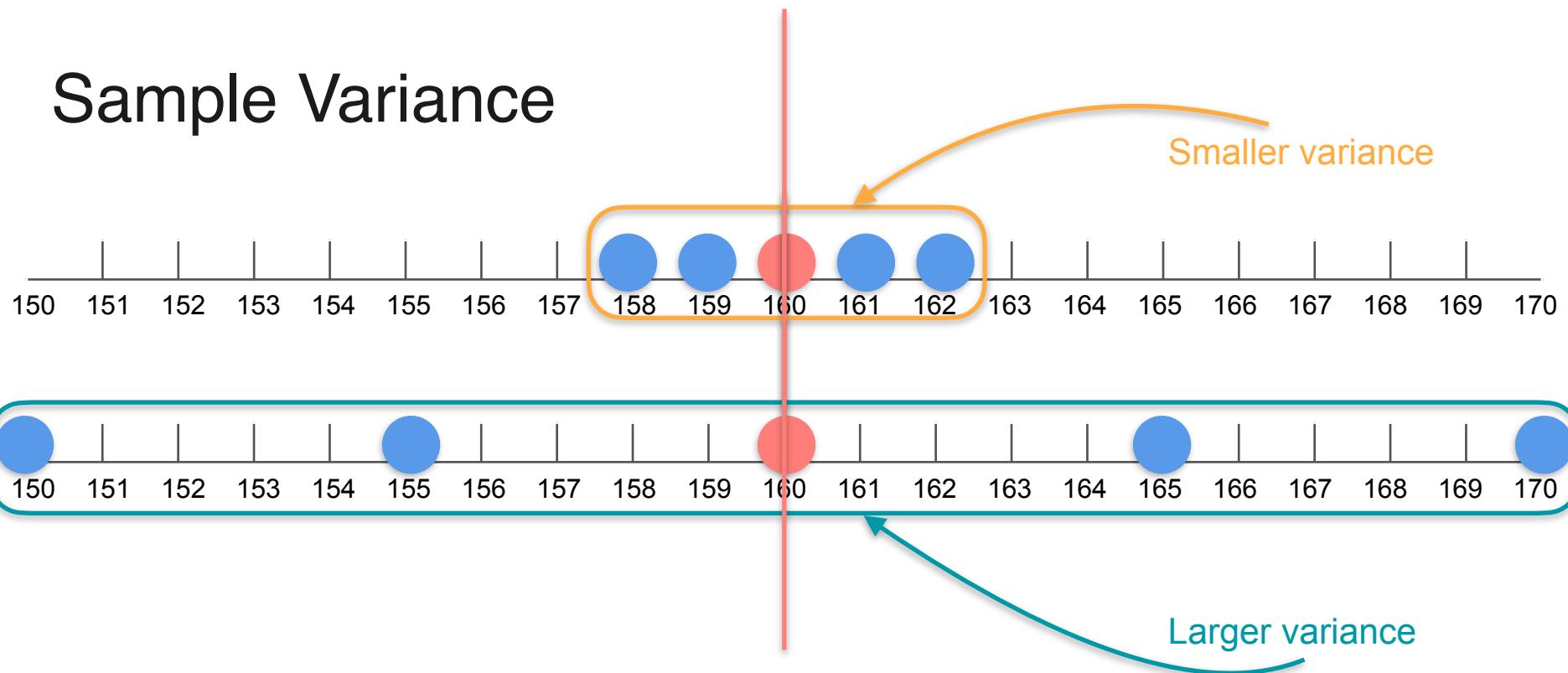
# Sample Variance



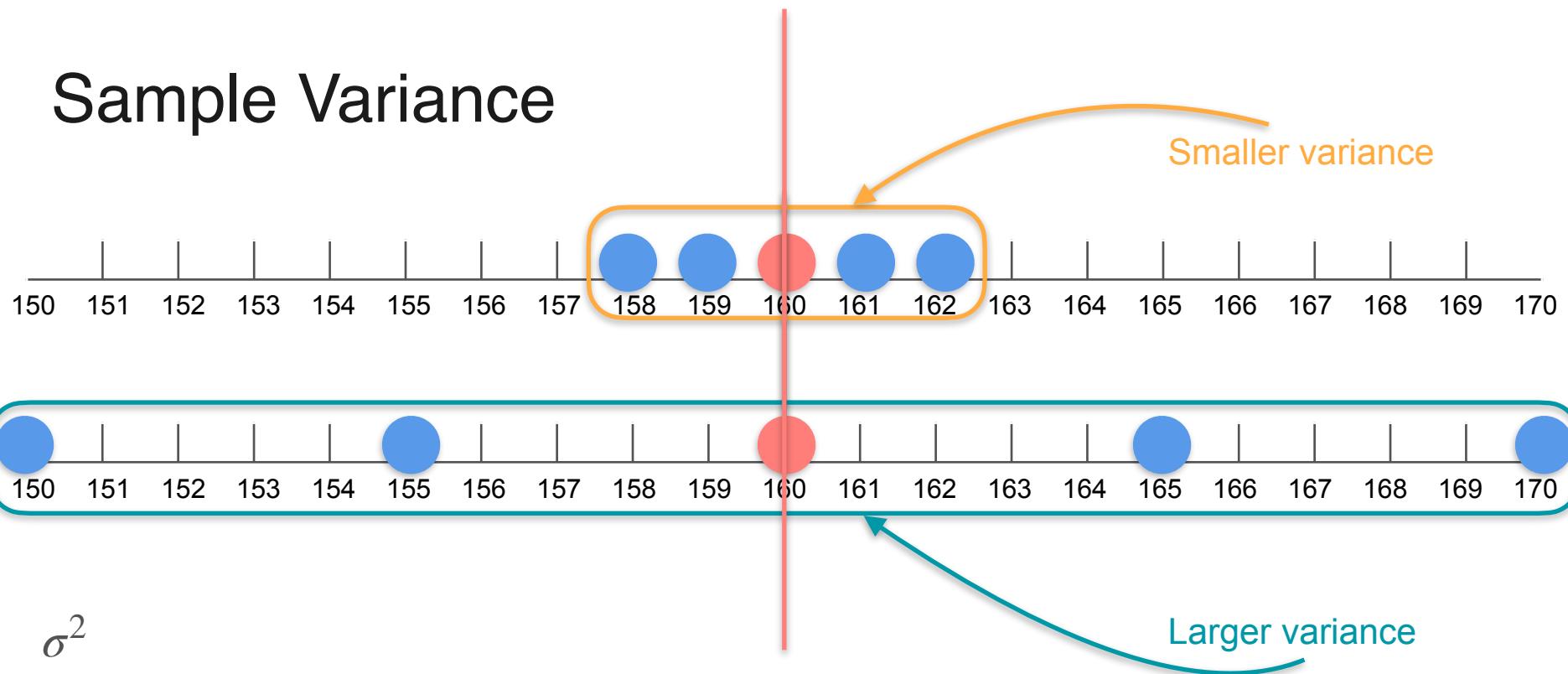
# Sample Variance



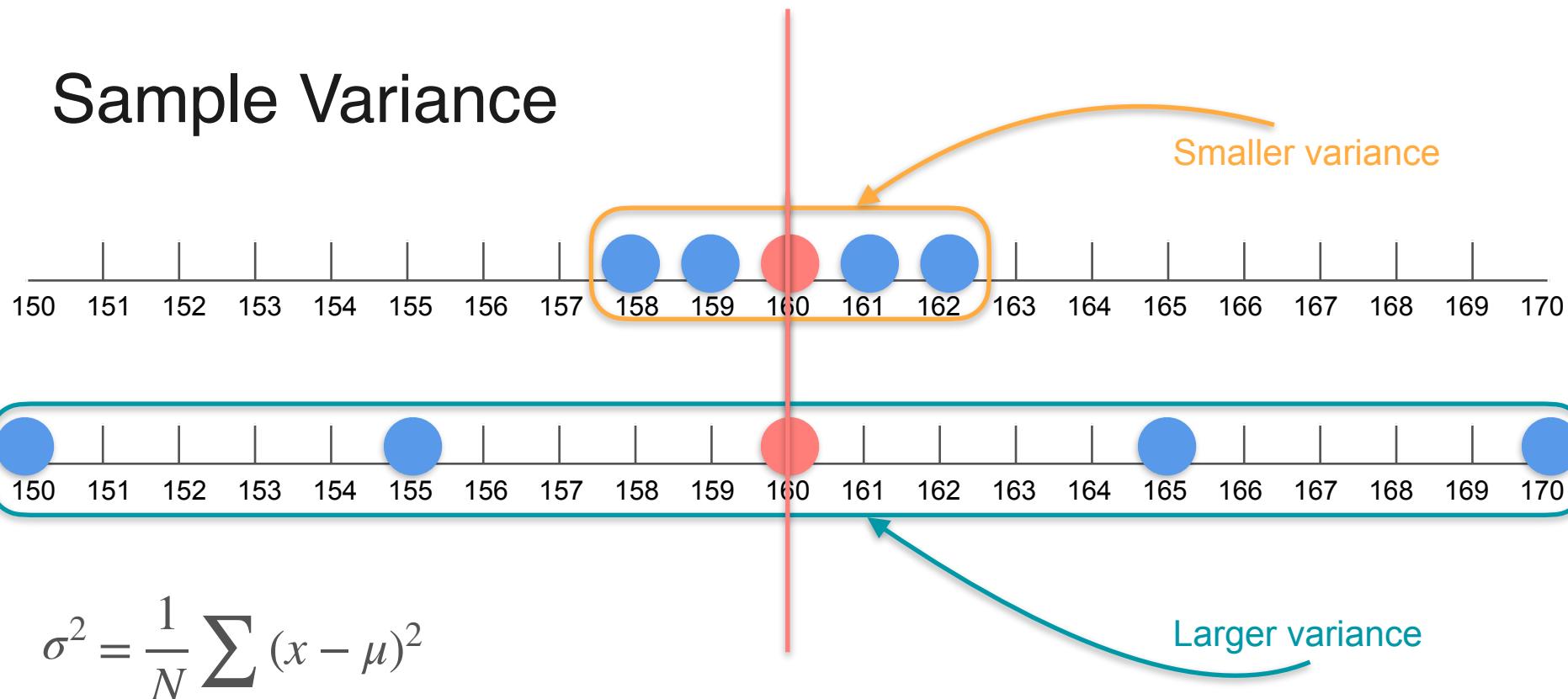
# Sample Variance



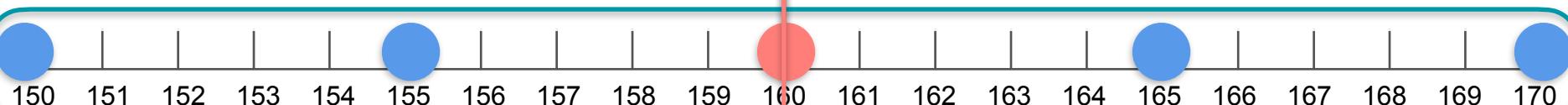
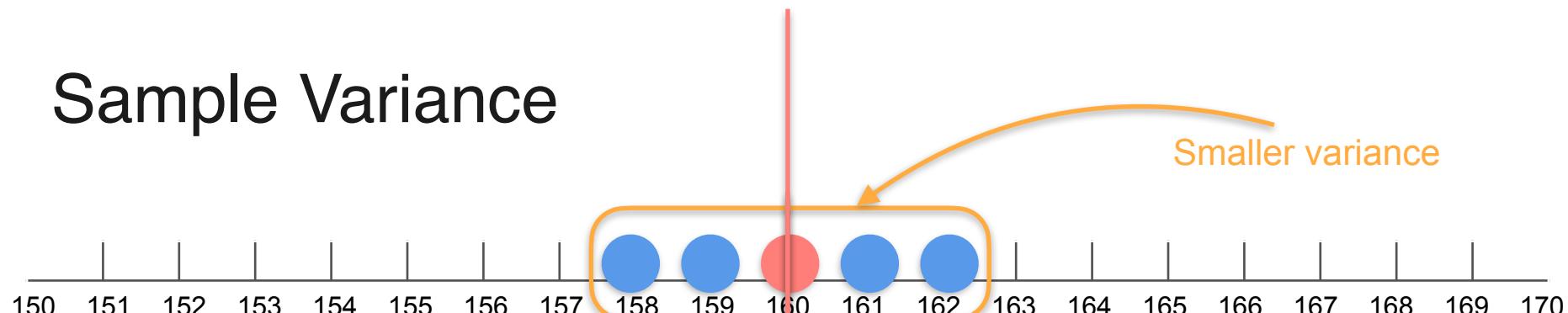
# Sample Variance



# Sample Variance



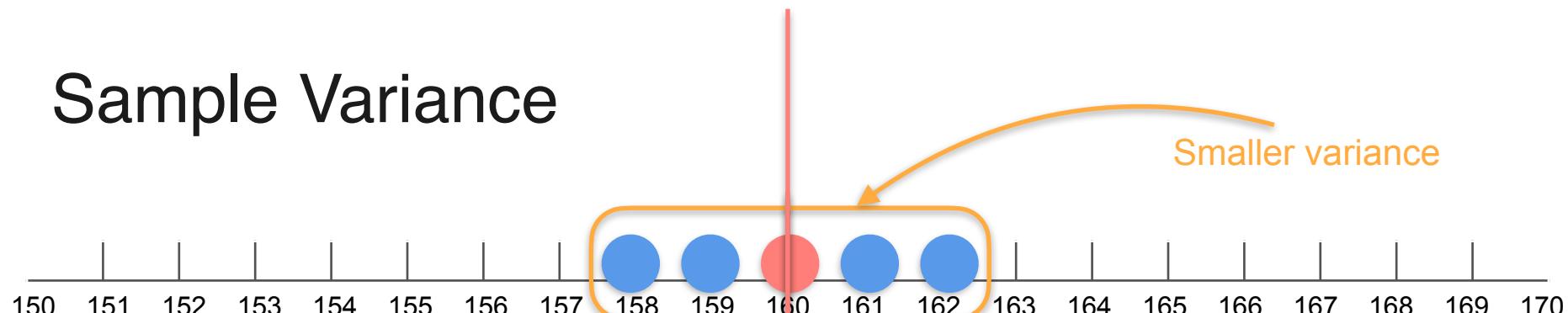
# Sample Variance



$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

population size

# Sample Variance



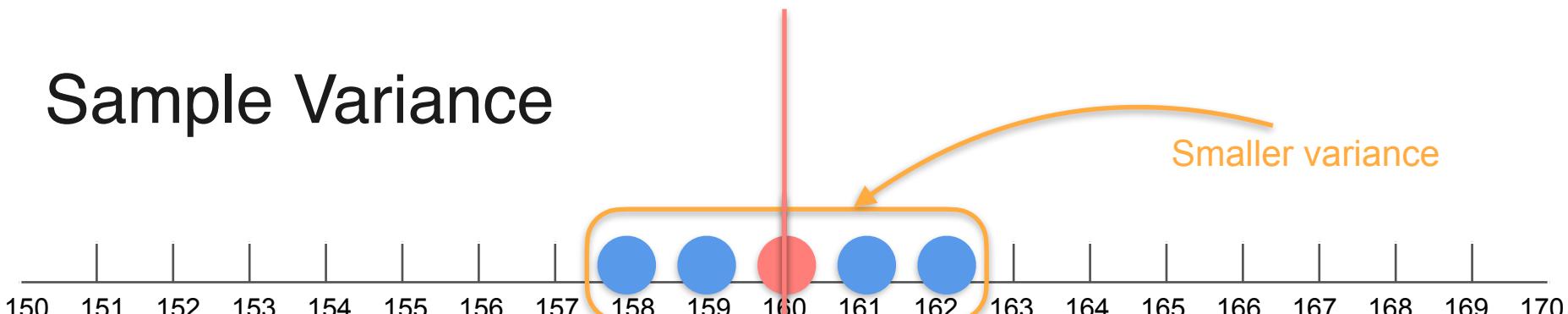
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

population mean

population size

Larger variance

# Sample Variance



$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

population size      population mean

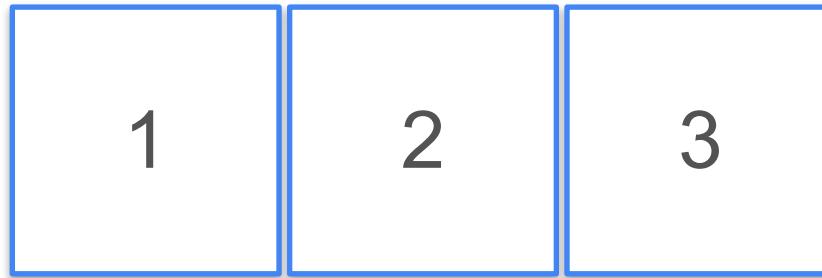
How to estimate population variance with the sample?

# Variance Estimation

# Variance Estimation

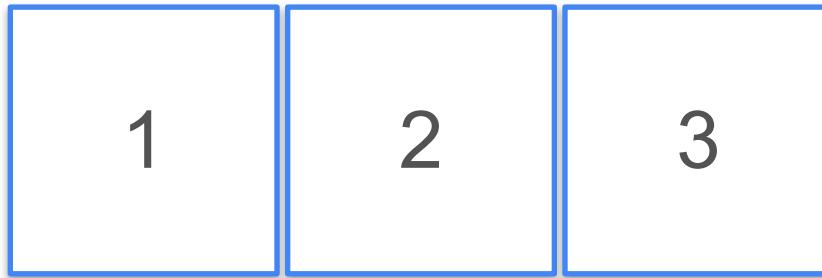


# Variance Estimation



$\mu$

# Variance Estimation



$$\mu = \frac{1 + 2 + 3}{3}$$

# Variance Estimation



$$\mu = \frac{1 + 2 + 3}{3} = \frac{6}{3}$$

# Variance Estimation



$$\mu = \frac{1 + 2 + 3}{3} = \frac{6}{3} = 2$$

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$x \quad x - \mu \quad (x - \mu)^2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

x
1
2
3

$$x - \mu \quad (x - \mu)^2$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$
1
2
3

$$x - 2$$

$$x - \mu$$

$$(x - \mu)^2$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1	1
2	0	0
3	1	1

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1	1
2	0	0
3	1	1

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1	1
2	0	0
3	1	1

$$\sum (x - \mu)^2$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1	1
2	0	0
3	1	1

$$\sum (x - \mu)^2 = 2$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1	1
2	0	0
3	1	1

$$\frac{\sum (x - \mu)^2}{N} = 2$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1	1
2	0	0
3	1	1

$$\frac{\sum (x - \mu)^2}{N} = \frac{2}{3}$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1	1
2	0	0
3	1	1

$$\frac{\sum (x - \mu)^2}{N} = \frac{2}{3}$$

$\sigma^2$   
Population variance

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

# Variance Estimation

1 2 3

$$n = 2$$

Samples

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

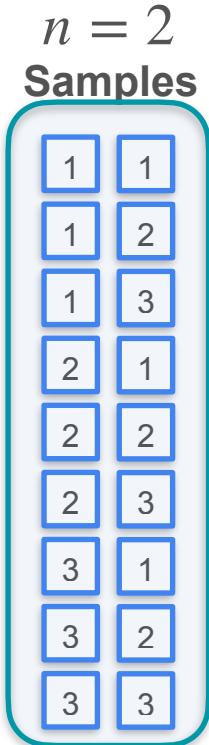
# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$	Samples
1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$$n = 2$$

Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$	Samples	$\bar{x}$
	1 1	
	1 2	
	1 3	
	2 1	
	2 2	
	2 3	
	3 1	
	3 2	
	3 3	

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$ Samples	$\bar{x}$
1 1	1
1 2	1,5
1 3	2
2 1	2
2 2	2,5
2 3	2
3 1	2,5
3 2	3
3 3	3

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$ Samples	$\bar{x}$
1 1	1
1 2	1,5
1 3	2
2 1	1,5
2 2	2
2 3	2,5
3 1	2
3 2	2,5
3 3	3

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$	Samples	$\bar{x}$
	1 1	1
	1 2	1,5
	1 3	2
	2 1	2
	2 2	2,5
	2 3	2
	3 1	2,5
	3 2	3
	3 3	3

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

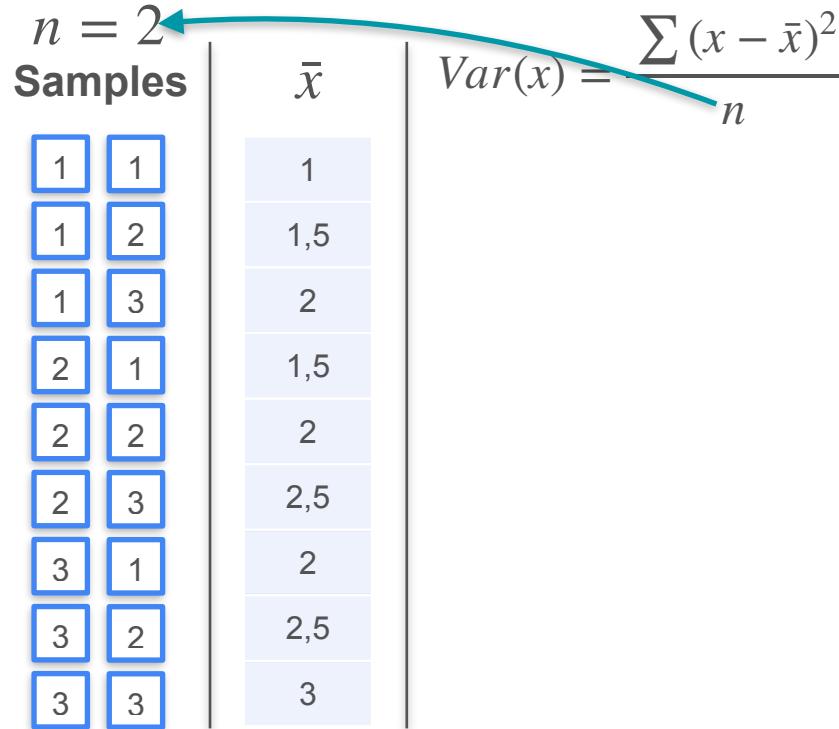
# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$ Samples	$\bar{x}$	$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$
1 1	1	0
1 2	1,5	0,25
1 3	2	1
2 1	1,5	0,25
2 2	2	0
2 3	2,5	0,25
3 1	2	1
3 2	2,5	0,25
3 3	3	0

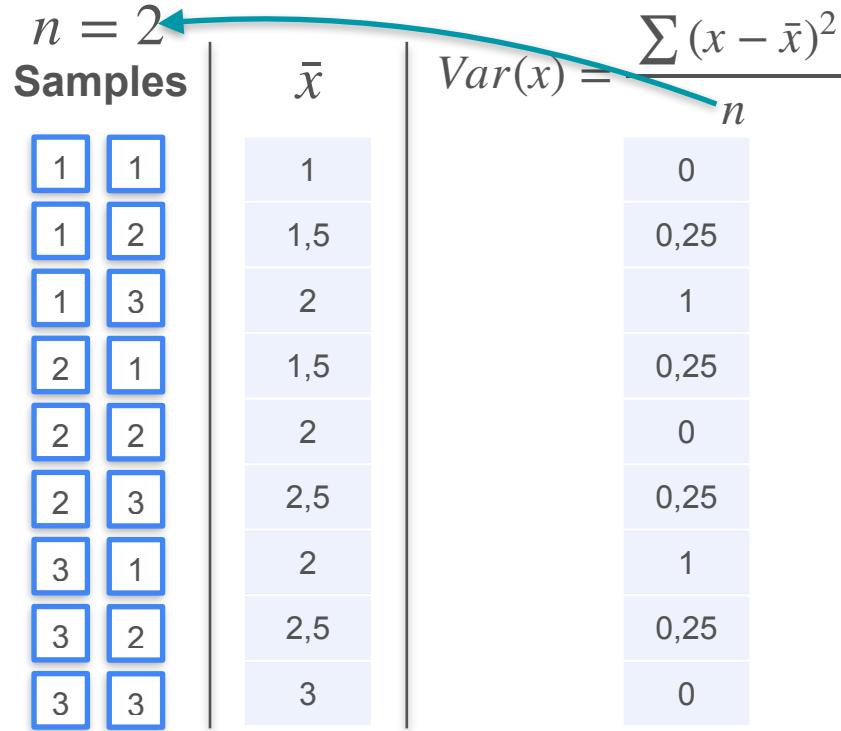
# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



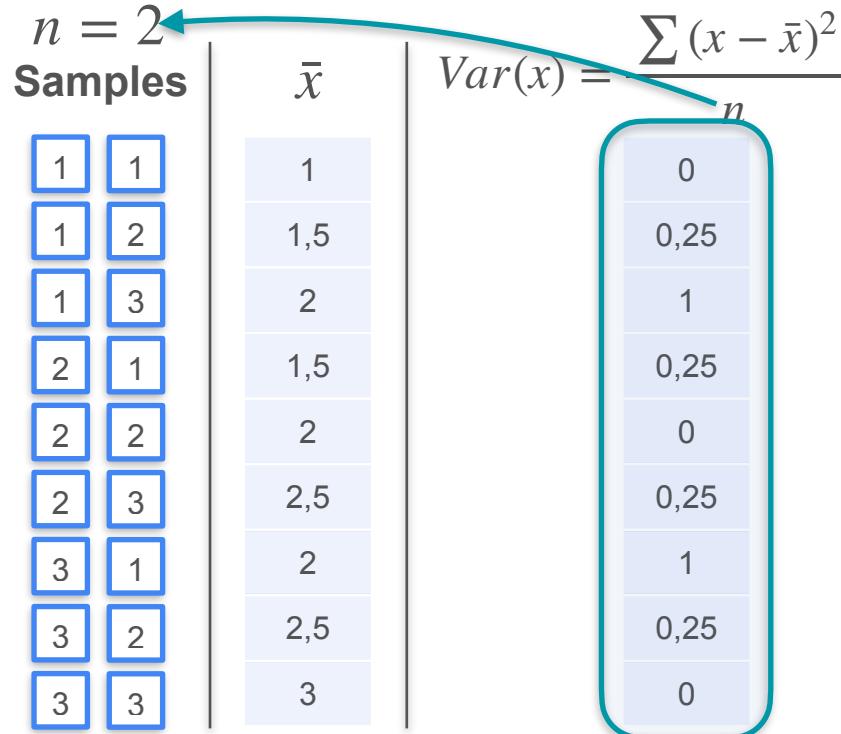
# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



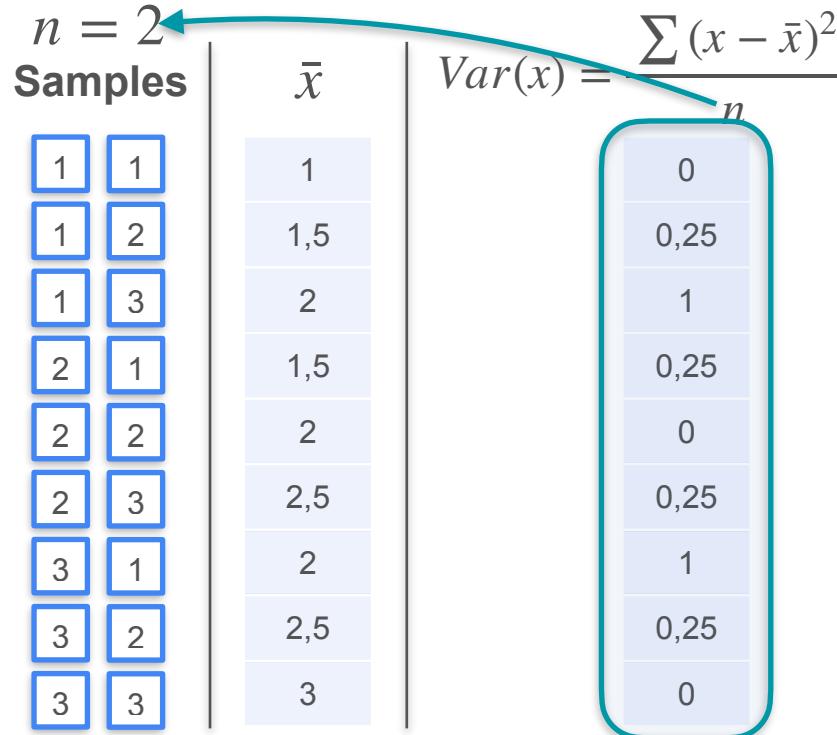
# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



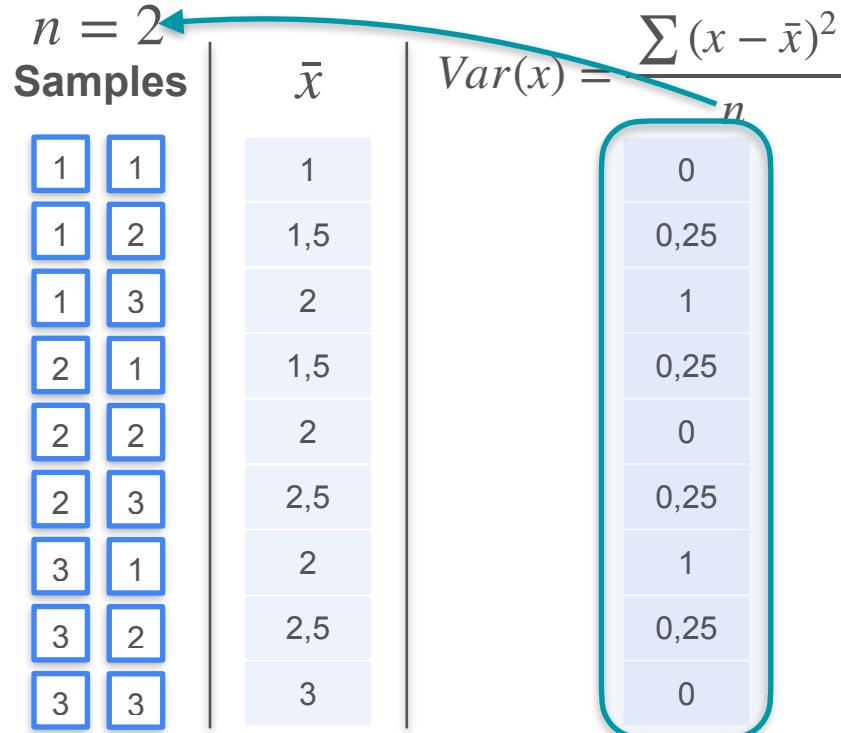
# Variance Estimation

$$\begin{matrix} 1 & 2 & 3 \end{matrix}$$

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



estimated variance

$$= 0.333$$

$$= \frac{1}{3}$$

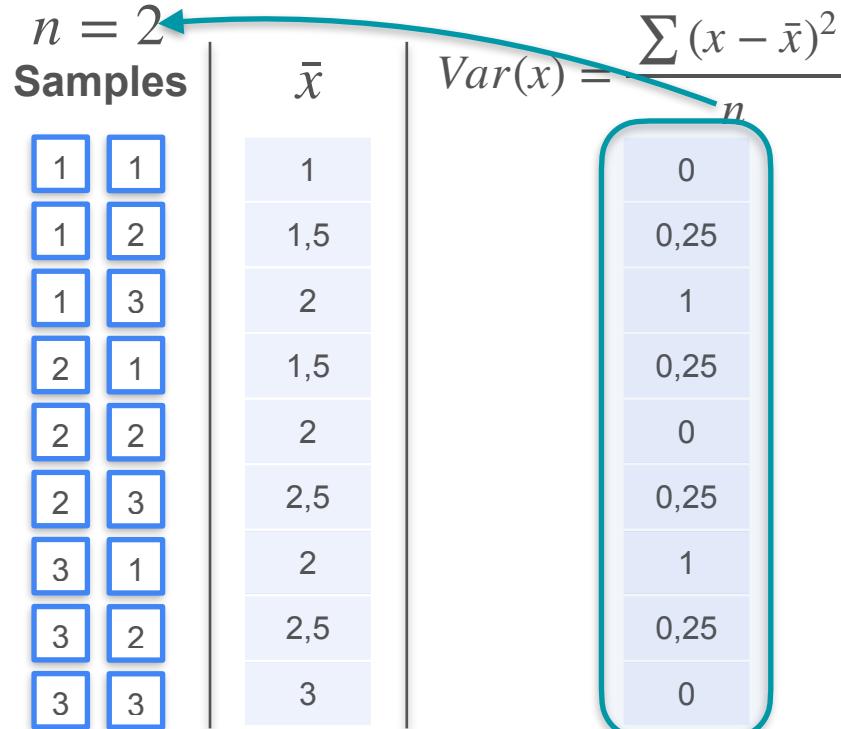
# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



estimated variance

$$= 0.333$$

$$= \frac{1}{3}$$

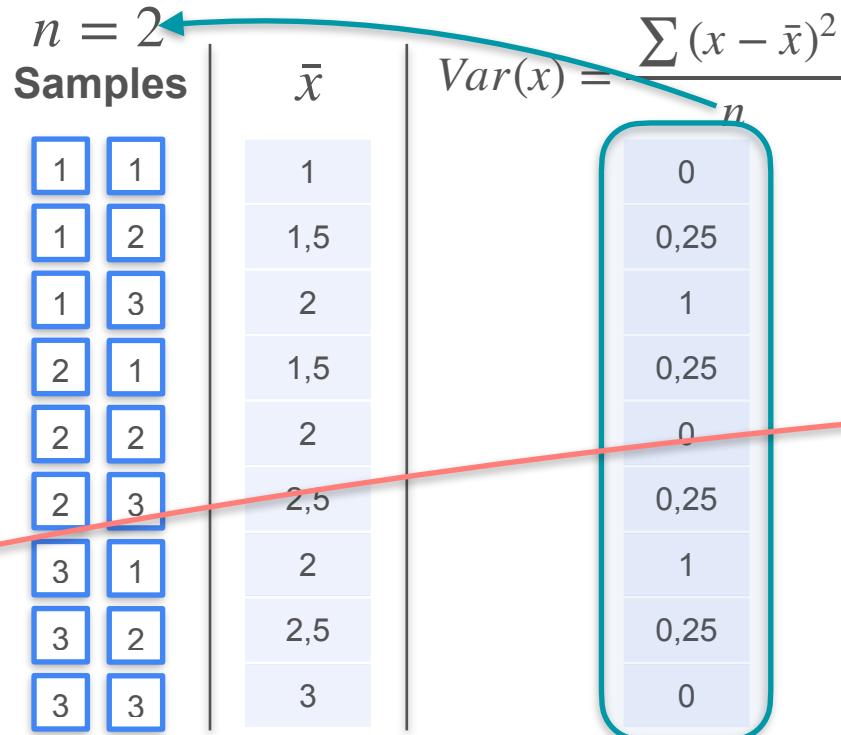
# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



estimated variance

$$= 0.333$$

$$= \frac{1}{3}$$

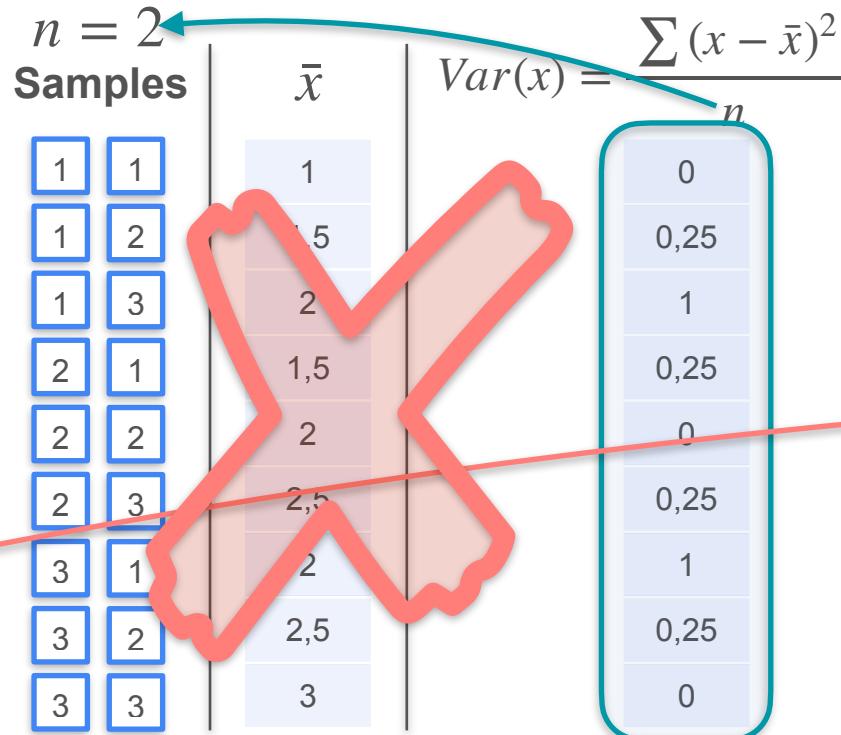
# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



estimated  
variance

$$= 0.333$$

$$= \frac{1}{3}$$

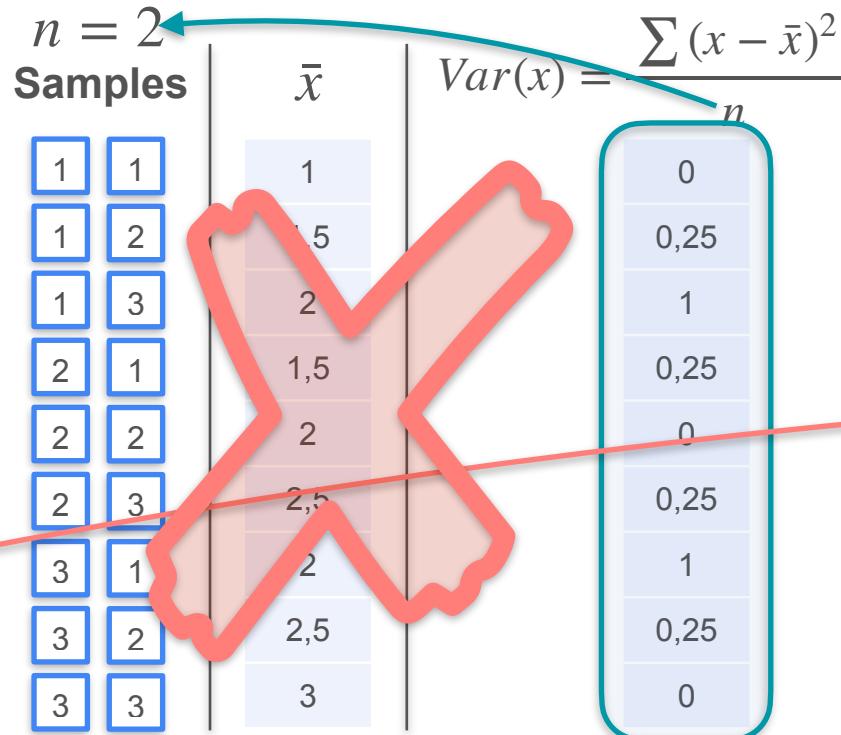
# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$



estimated  
variance

$$= 0.333$$

$$= \frac{1}{3}$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

# Variance Estimation

1	2	3
---	---	---

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

0
0,5
2
0,5
0
0,5
2
0,5
0

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

0
0,5
2
0,5
0
0,5
2
0,5
0

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

0
0,5
2
0,5
0
0,5
2
0,5
0

estimated  
variance

$$= 0.667$$

$$= \frac{2}{3}$$

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

0
0,5
2
0,5
0
0,5
2
0,5
0

estimated variance

$$= 0.667$$

$$= \frac{2}{3}$$

# Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$n = 2$   
Samples

1	1
1	2
1	3
2	1
2	2
2	3
3	1
3	2
3	3

$$\bar{x}$$

1
1,5
2
1,5
2
2,5
2
2,5
3

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

0
0,5
2
0,5
0
0,5
2
0,5
0

estimated variance

$$= 0.667$$

$$= \frac{2}{3}$$

# Variance Estimation

**Population Variance Formula**

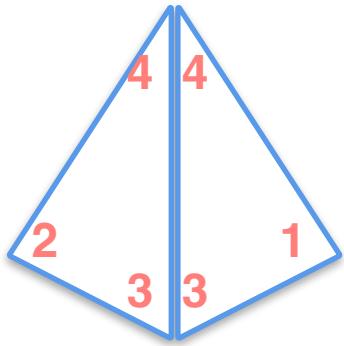
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

**Sample Variance Formula**

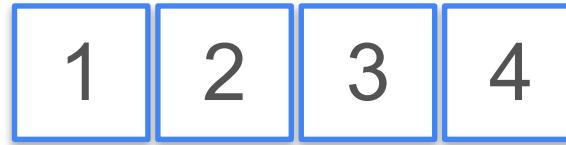
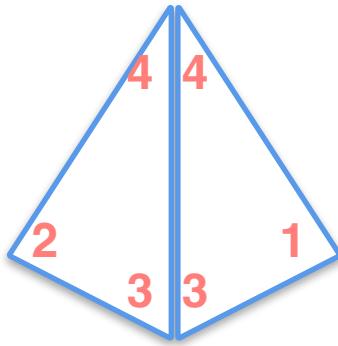
$$Var(x) = \frac{1}{n - 1} \sum (x - \bar{x})^2$$

# Variance Estimation

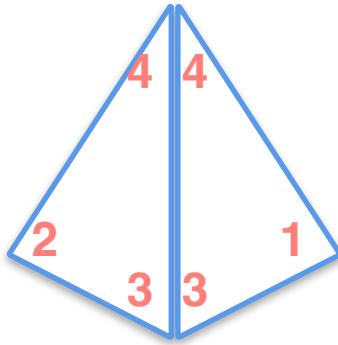
# Variance Estimation



# Variance Estimation

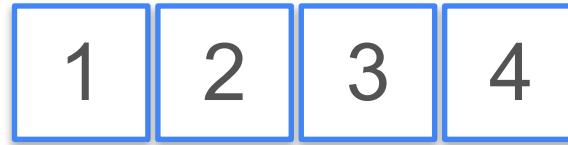
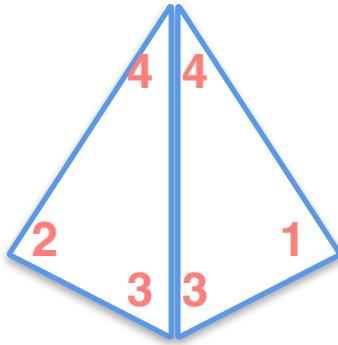


# Variance Estimation



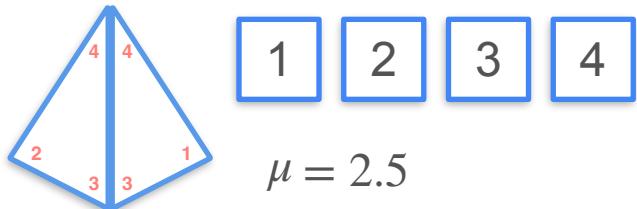
$$\mu = \frac{1 + 2 + 3 + 4}{4}$$

# Variance Estimation

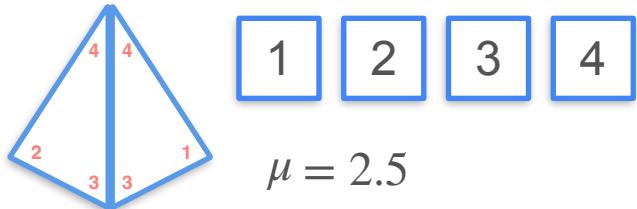


$$\mu = \frac{1 + 2 + 3 + 4}{4} = 2.5$$

# Variance Estimation

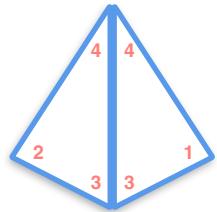


# Variance Estimation



$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

# Variance Estimation

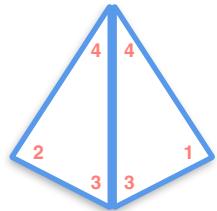


$$\mu = 2.5$$

$$x \quad x - \mu \quad (x - \mu)^2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

# Variance Estimation



1 2 3 4

$$\mu = 2.5$$

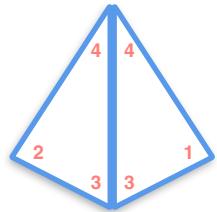
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$
1
2
3
4

$$x - \mu$$

$$(x - \mu)^2$$

# Variance Estimation



$$\mu = 2.5$$

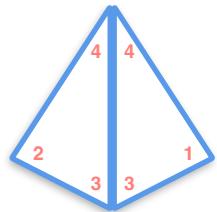
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$
1
2
3
4

$$x - 2.5$$

$$x - \mu \quad (x - \mu)^2$$

# Variance Estimation



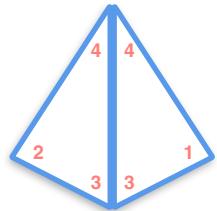
$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1,5	
2	-0,5	
3	0,5	
4	1,5	

$$x - 2.5$$

# Variance Estimation



1 2 3 4

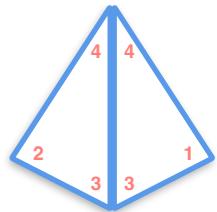
$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$x - 2.5$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1,5	2,25
2	-0,5	0,25
3	0,5	0,25
4	1,5	2,25

# Variance Estimation



1 2 3 4

$$\mu = 2.5$$

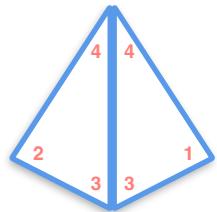
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1,5	2,25
2	-0,5	0,25
3	0,5	0,25
4	1,5	2,25

$$x - 2.5$$

$$\sum (x - \mu)^2$$

# Variance Estimation



$$\mu = 2.5$$

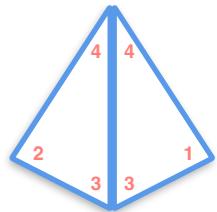
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1,5	2,25
2	-0,5	0,25
3	0,5	0,25
4	1,5	2,25

$$x - 2.5$$

$$\sum (x - \mu)^2 = 5$$

# Variance Estimation



$$\mu = 2.5$$

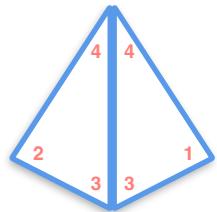
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1,5	2,25
2	-0,5	0,25
3	0,5	0,25
4	1,5	2,25

$$x - 2.5$$

$$\frac{\sum (x - \mu)^2}{N} = 5$$

# Variance Estimation



$$\mu = 2.5$$

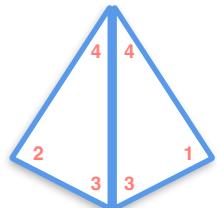
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$x$	$x - \mu$	$(x - \mu)^2$
1	-1,5	2,25
2	-0,5	0,25
3	0,5	0,25
4	1,5	2,25

$$x - 2.5$$

$$\frac{\sum (x - \mu)^2}{N} = \frac{5}{4}$$

# Variance Estimation



$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

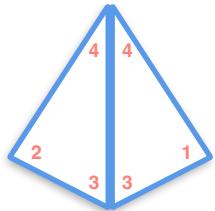
$x$	$x - \mu$	$(x - \mu)^2$
1	-1,5	2,25
2	-0,5	0,25
3	0,5	0,25
4	1,5	2,25

$$x - 2.5$$

$$\frac{\sum (x - \mu)^2}{N} = \frac{5}{4}$$

Population variance  
 $\sigma^2$

# Variance Estimation

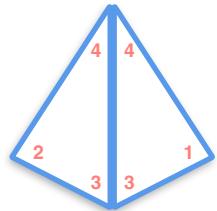


$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{5}{4}$$

# Variance Estimation



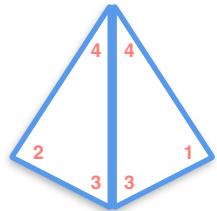
$$\mu = 2.5$$

$n = 2$   
**Samples**

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{5}{4}$$

# Variance Estimation



$$\mu = 2.5$$

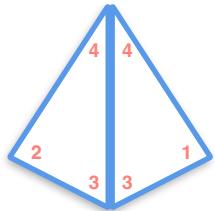
$n = 2$   
**Samples**

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{5}{4}$$

$x$

# Variance Estimation



$$\mu = 2.5$$

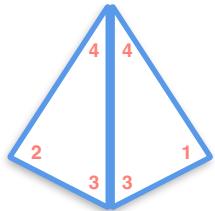
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{5}{4}$$

$n = 2$   
Samples

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----

# Variance Estimation



$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

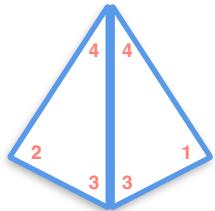
$$\sigma^2 = \frac{5}{4}$$

$n = 2$   
Samples

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----

$$\bar{x}$$

# Variance Estimation



1 2 3 4

$$\mu = 2.5$$

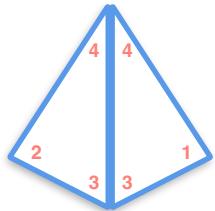
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{5}{4}$$

$n = 2$   
Samples

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3	2	2.5	3	3.5	2.5	3	3.5	4

# Variance Estimation



1 2 3 4

$$\mu = 2.5$$

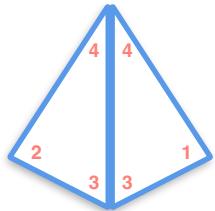
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{5}{4}$$

$n = 2$   
Samples

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3	2	2.5	3	3.5	2.5	3	3.5	4

# Variance Estimation



1    2    3    4

$$\mu = 2.5$$

$$n = 2 \\ \text{Samples}$$

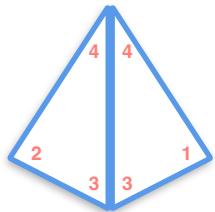
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{5}{4}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3	2	2.5	3	3.5	2.5	3	3.5	4

# Variance Estimation



$$\mu = 2.5$$

$$n = 2 \\ \text{Samples}$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

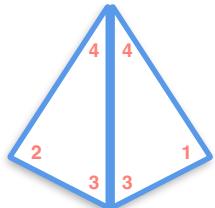
$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{5}{4}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3	2	2.5	3	3.5	2.5	3	3.5	4

$$var(x)$$

# Variance Estimation



1 2 3 4

$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

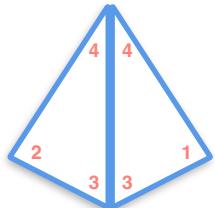
$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{5}{4}$$

$$n = 2 \\ \text{Samples}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4
$var(x)$	0	0,25	1	2,25	0,25	0	0,25	1	1	0,25	0	0,25	2,25	1	0,25	0

# Variance Estimation



1 2 3 4

$$\mu = 2.5$$

$$n = 2 \\ \text{Samples}$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

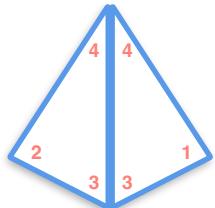
$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{5}{4}$$

$$var(x) = \frac{5}{8}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4
$var(x)$	0	0,25	1	2,25	0,25	0	0,25	1	1	0,25	0	0,25	2,25	1	0,25	0

# Variance Estimation



1 2 3 4

$$\mu = 2.5$$

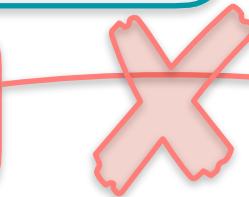
$$n = 2 \\ \text{Samples}$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

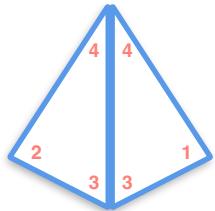
$$\sigma^2 = \frac{5}{4}$$

$$var(x) = \frac{5}{8}$$



$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4
$var(x)$	0	0,25	1	2,25	0,25	0	0,25	1	1	0,25	0	0,25	2,25	1	0,25	0

# Variance Estimation



$$\mu = 2.5$$

$n = 2$   
Samples

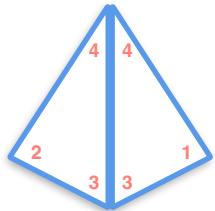
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{5}{4}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3	2	2.5	3	3.5	2.5	3	3.5	4

# Variance Estimation



$$\mu = 2.5$$

$n = 2$   
Samples

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

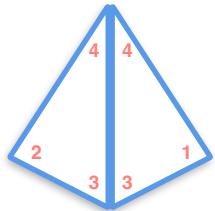
$$Var(x) = \frac{\sum (x - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{5}{4}$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4

# Variance Estimation



$$\mu = 2.5$$

$n = 2$   
Samples

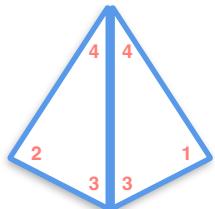
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

$$\sigma^2 = \frac{5}{4}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3	2	2.5	3	3.5	2.5	3	3.5	4

# Variance Estimation



$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

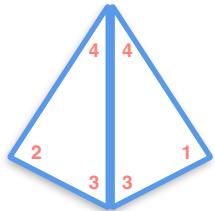
$$\sigma^2 = \frac{5}{4}$$

$n = 2$   
**Samples**

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4

$$var(x)$$

# Variance Estimation



$$\mu = 2.5$$

$$n = 2 \\ \text{Samples}$$

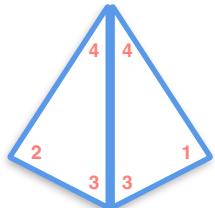
$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

$$\sigma^2 = \frac{5}{4}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4
$var(x)$	0	0,5	2	4,5	0,5	0	0,5	2	2	0,5	0	0,5	4,5	2	0,5	0

# Variance Estimation



$$\mu = 2.5$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

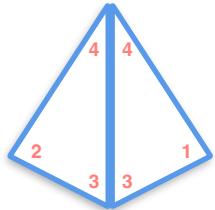
$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

$$\sigma^2 = \frac{5}{4}$$

$n = 2$   
Samples

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4
$var(x)$	0	0,5	2	4,5	0,5	0	0,5	2	2	0,5	0	0,5	4,5	2	0,5	0

# Variance Estimation



$$\mu = 2.5$$

$n = 2$   
Samples

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

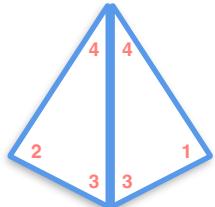
$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

$$\sigma^2 = \frac{5}{4}$$

$$Var(x) = \frac{5}{4}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4
$var(x)$	0	0,5	2	4,5	0,5	0	0,5	2	2	0,5	0	0,5	4,5	2	0,5	0

# Variance Estimation



$$\mu = 2.5$$

$n = 2$   
Samples

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

$$\sigma^2 = \frac{5}{4}$$

$$Var(x) = \frac{5}{4}$$

$x$	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
$\bar{x}$	1	1,5	2	2,5	1,5	2	2,5	3	2	2,5	3	3,5	2,5	3	3,5	4
$var(x)$	0	0,5	2	4,5	0,5	0	0,5	2	2	0,5	0	0,5	4,5	2	0,5	0

# Variance Estimation

**Population Variance Formula**

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

**Sample Variance Formula**

$$Var(x) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$



DeepLearning.AI

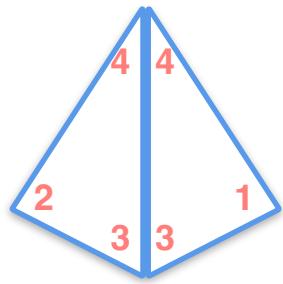
## Sample and Population

---

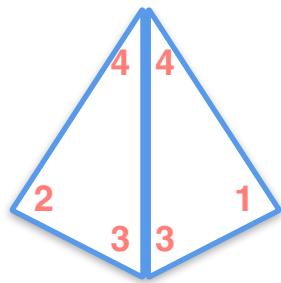
## Law of Large Numbers

# Law of Large Numbers

# Law of Large Numbers

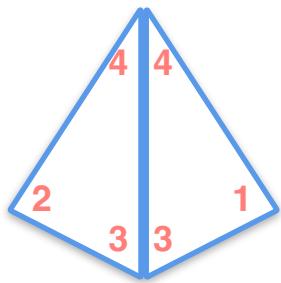


# Law of Large Numbers



1    2    3    4

# Law of Large Numbers



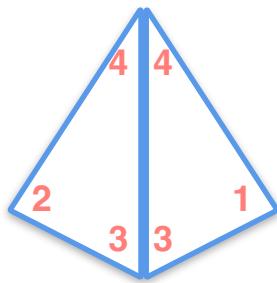
1    2    3    4

$$\mu = 2.5$$

## Experiment:

*Toss the 4-sided dice twice and record the average of your outcomes*

# Law of Large Numbers



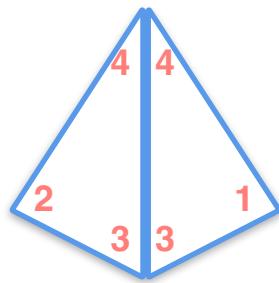
$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \mu = 2.5 \end{array}$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

## Experiment:

*Toss the 4-sided dice twice and record the average of your outcomes*

# Law of Large Numbers



1 2 3 4

$$\mu = 2.5$$

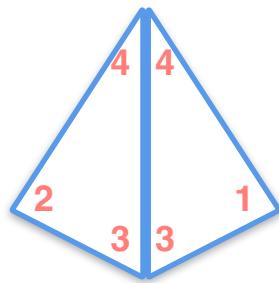
	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

	1	2	3	4
1	1	1.5	2	2.5
2	1.5	2	2.5	3
3	2	2.5	3	3.5
4	2.5	3	3.5	4

## Experiment:

*Toss the 4-sided dice twice and record the average of your outcomes*

# Law of Large Numbers



1 2 3 4

$$\mu = 2.5$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

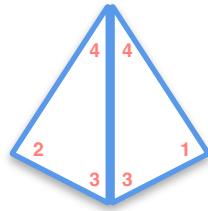
	1	2	3	4
1	1	1.5	2	2.5
2	1.5	2	2.5	3
3	2	2.5	3	3.5
4	2.5	3	3.5	4

	1	2	3	4
1				
2				
3				
4				

## Experiment:

*Toss the 4-sided dice twice and record the average of your outcomes*

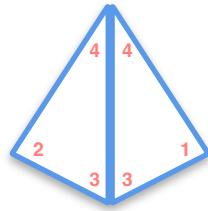
# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

# Law of Large Numbers

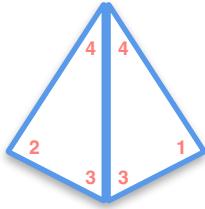


1 2 3 4  
 $\mu = 2.5$

1 trial

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

# Law of Large Numbers



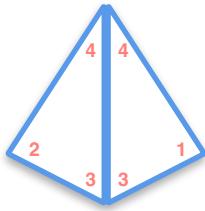
1 2 3 4  
 $\mu = 2.5$

1 trial

4,3

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

# Law of Large Numbers

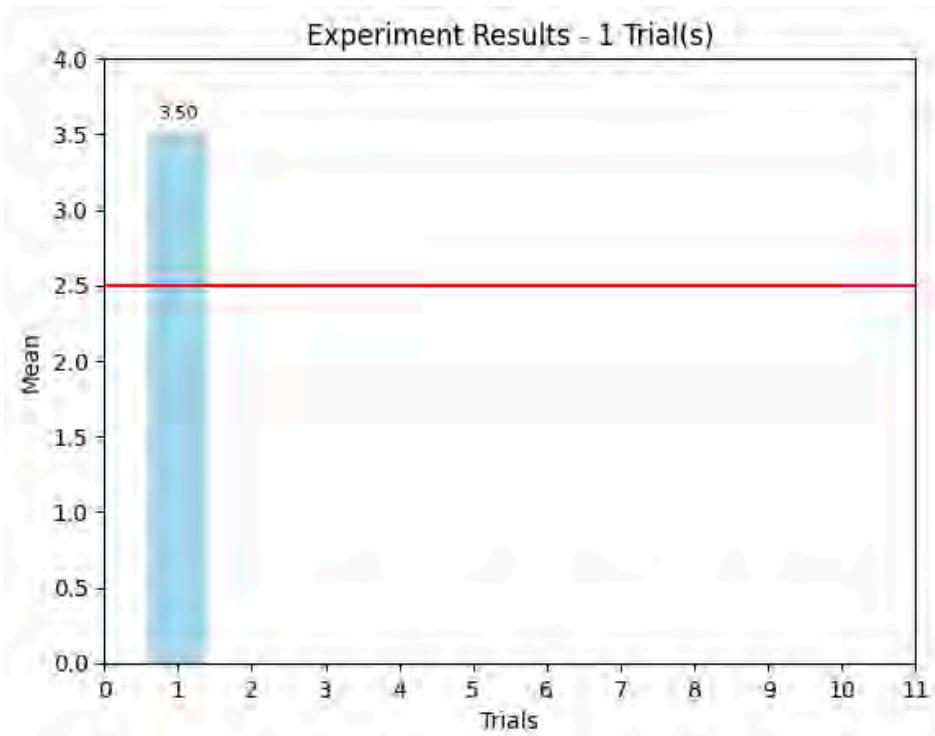


1 2 3 4  
 $\mu = 2.5$

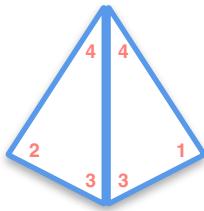
1 trial

4,3

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4



# Law of Large Numbers

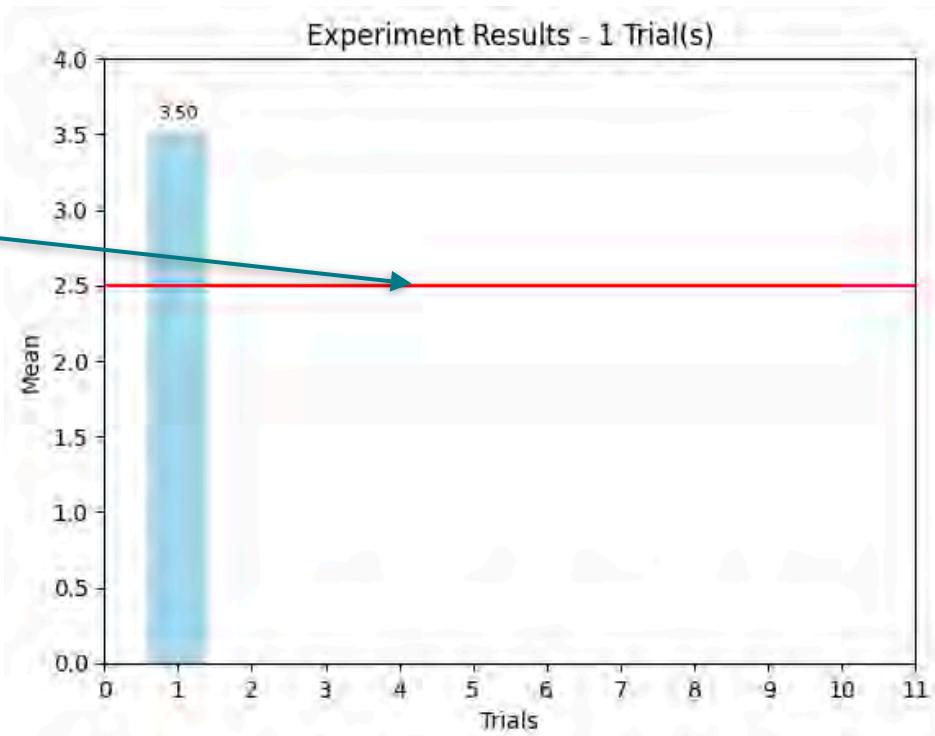


1 2 3 4  
 $\mu = 2.5$

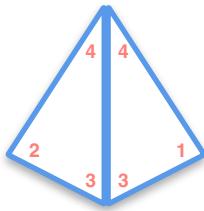
1 trial

4,3

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4



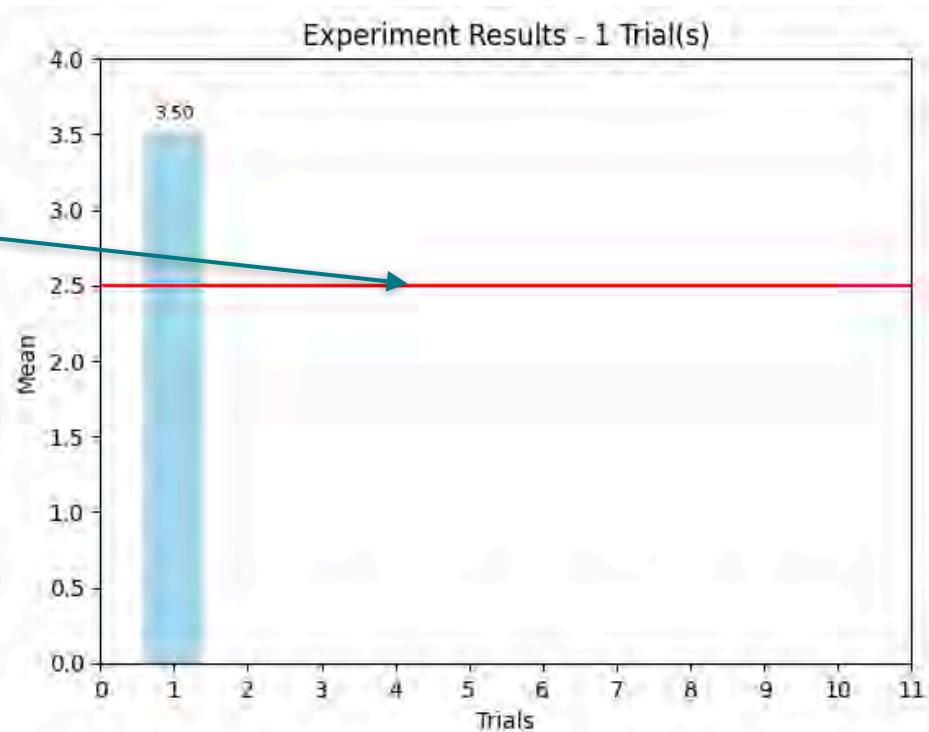
# Law of Large Numbers



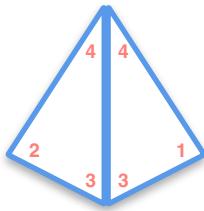
$$\begin{matrix} 1 & 2 & 3 & 4 \\ \mu = 2.5 \end{matrix}$$

1 trial  
4,3  $\bar{x}_1$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4



# Law of Large Numbers



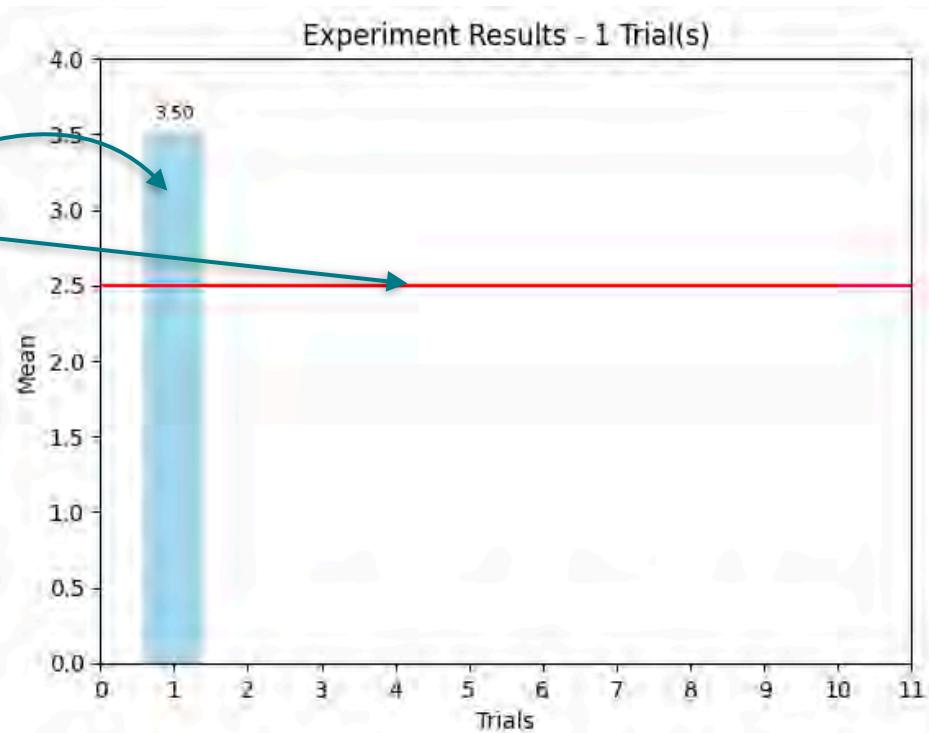
1 2 3 4  
 $\mu = 2.5$

1 trial

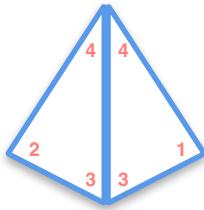
4,3

$\bar{x}_1$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4



# Law of Large Numbers



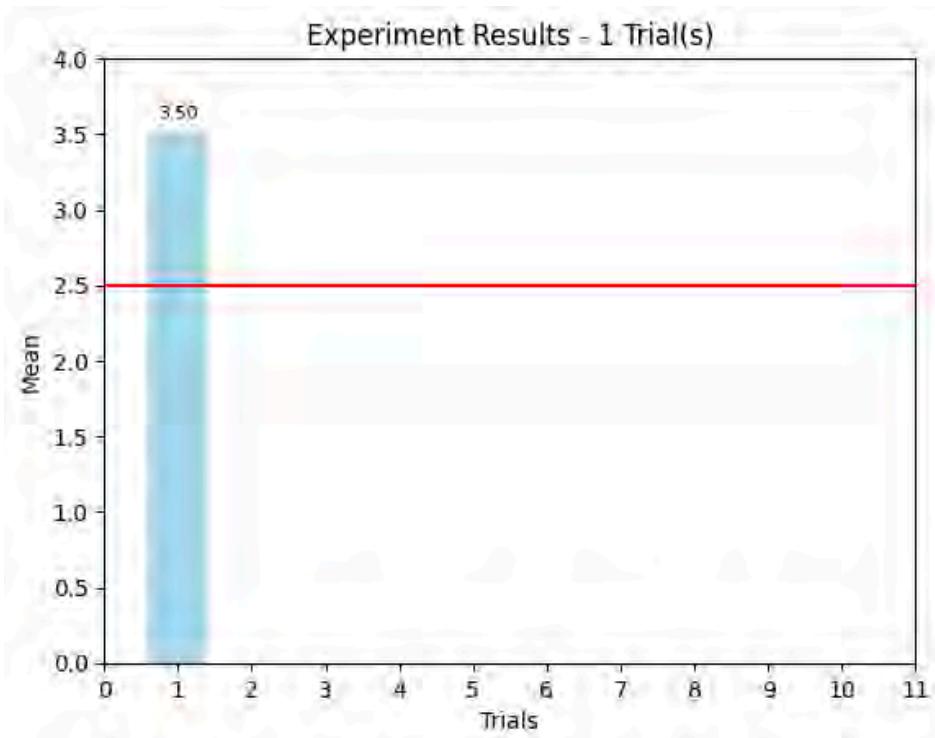
1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

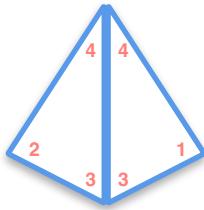
1 trial

4,3

2 trials



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

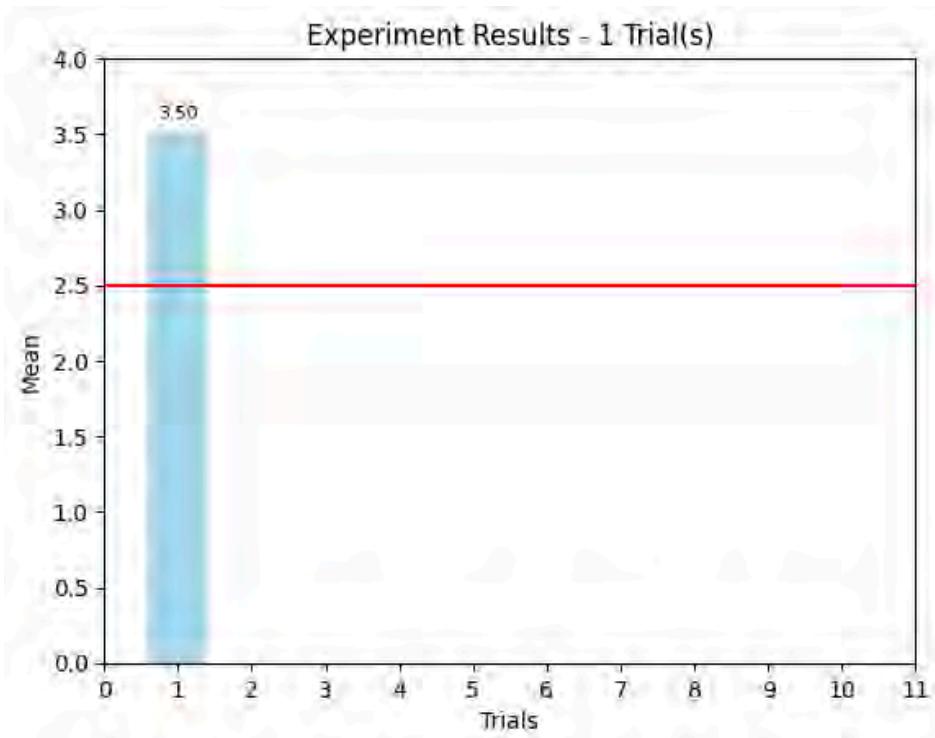
	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

1 trial

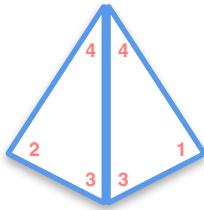
4,3

2 trials

3,4  
1,3



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

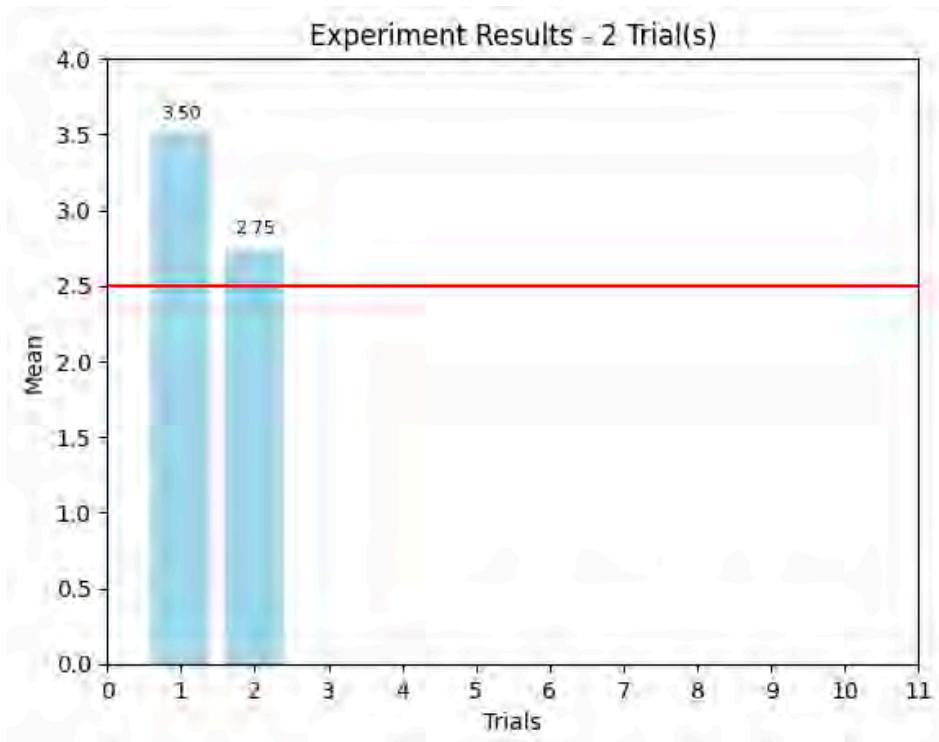
	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

1 trial

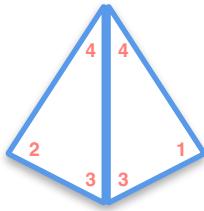
4,3

2 trials

3,4  
1,3



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

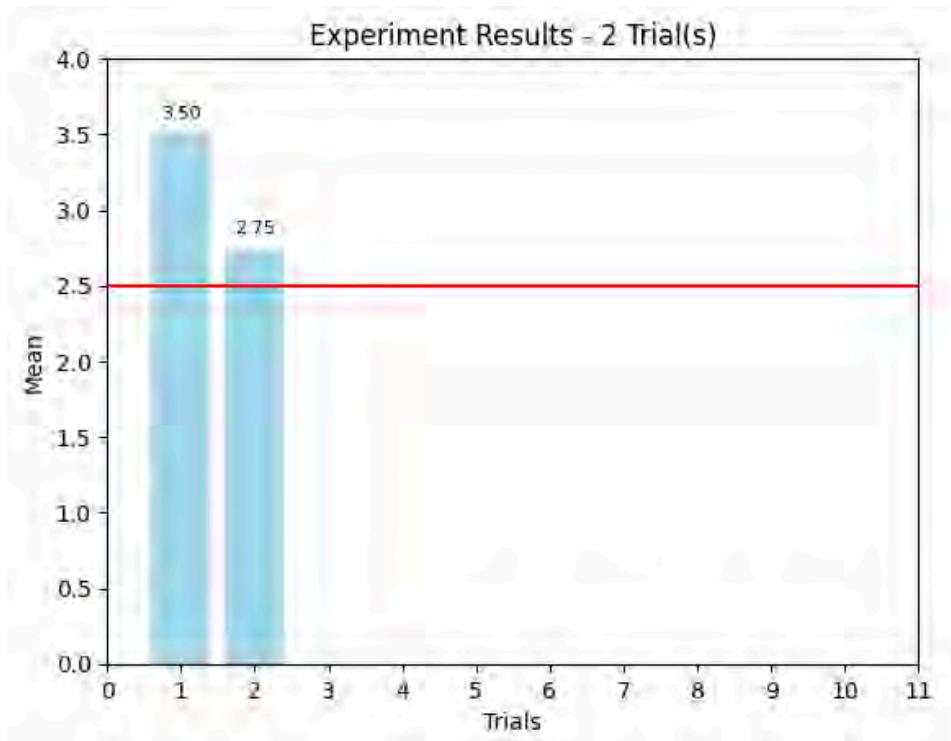
1 trial

4,3

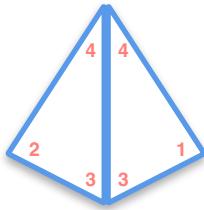
2 trials

3,4  
1,3

3 trials



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

1 trial

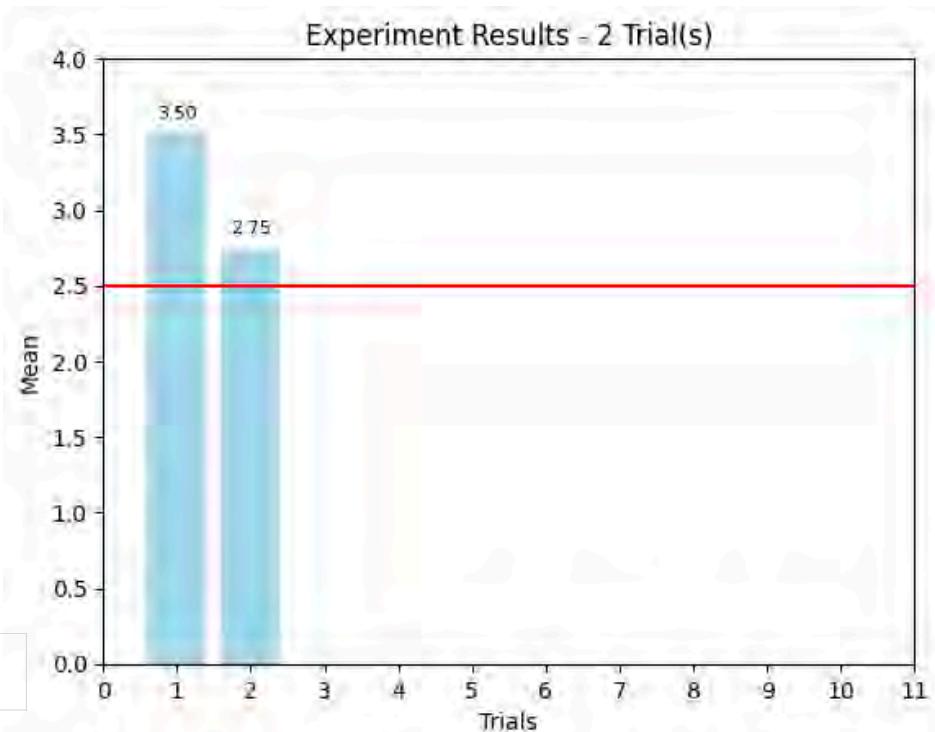
4,3

2 trials

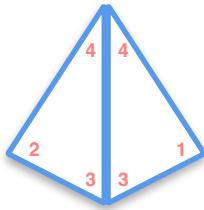
3,4  
1,3

3 trials

3,1    1,4    1,1



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

1 trial

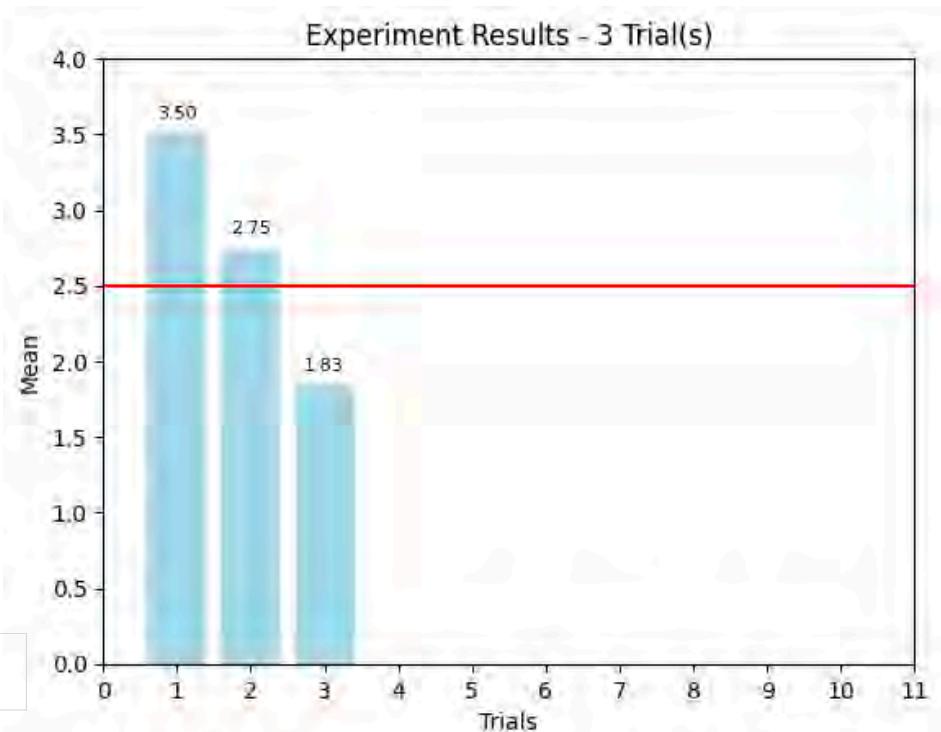
4,3

2 trials

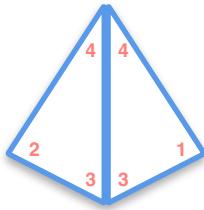
3,4  
1,3

3 trials

3,1    1,4    1,1



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

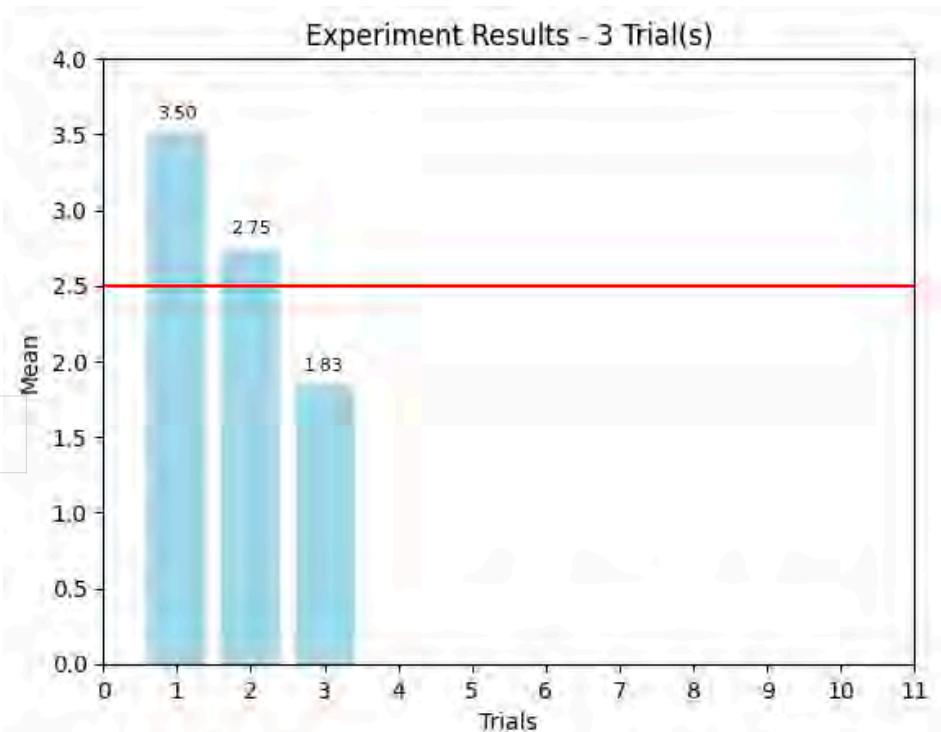
2 trials

3,4
1,3

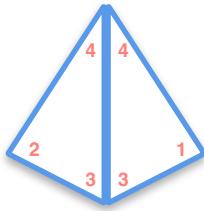
3 trials

3,1	1,4	1,1
-----	-----	-----

4 trials



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

2 trials

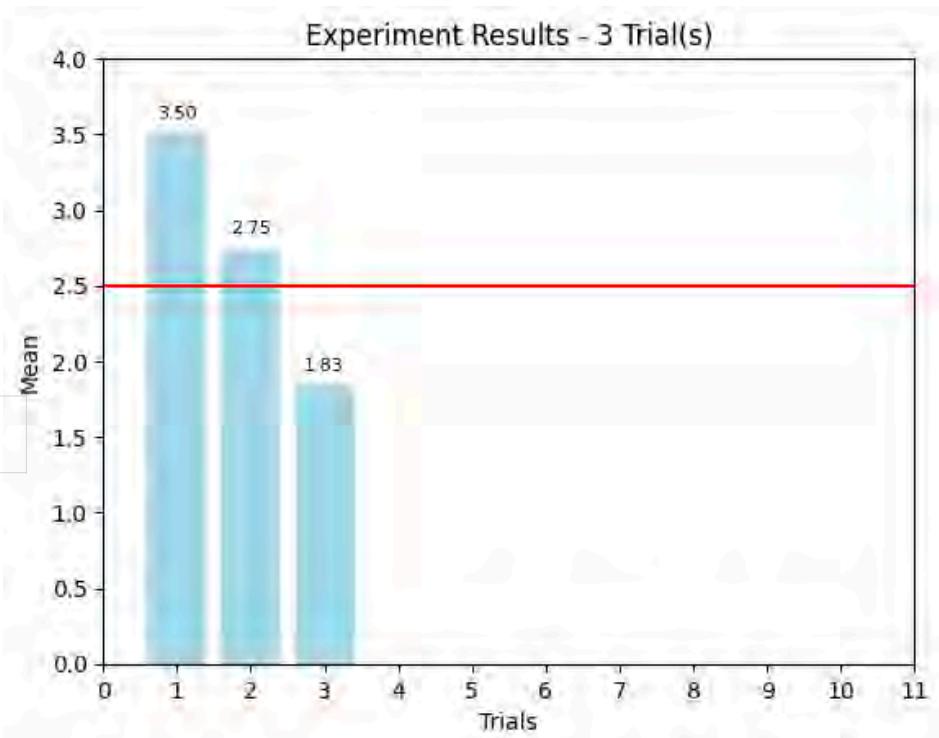
3,4
1,3

3 trials

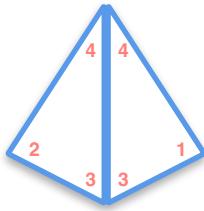
3,1	1,4	1,1
-----	-----	-----

4 trials

3,1	3,1
1,2	3,2



# Law of Large Numbers



1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

2 trials

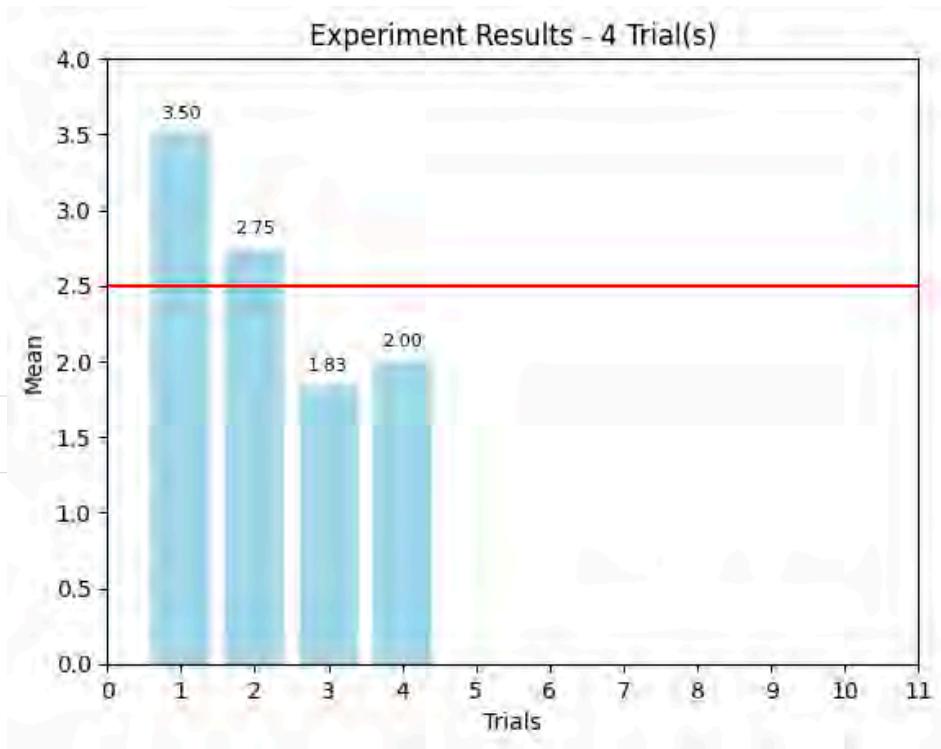
3,4
1,3

3 trials

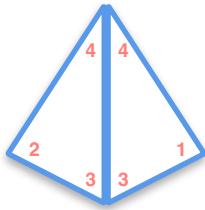
3,1	1,4	1,1
-----	-----	-----

4 trials

3,1	3,1
1,2	3,2

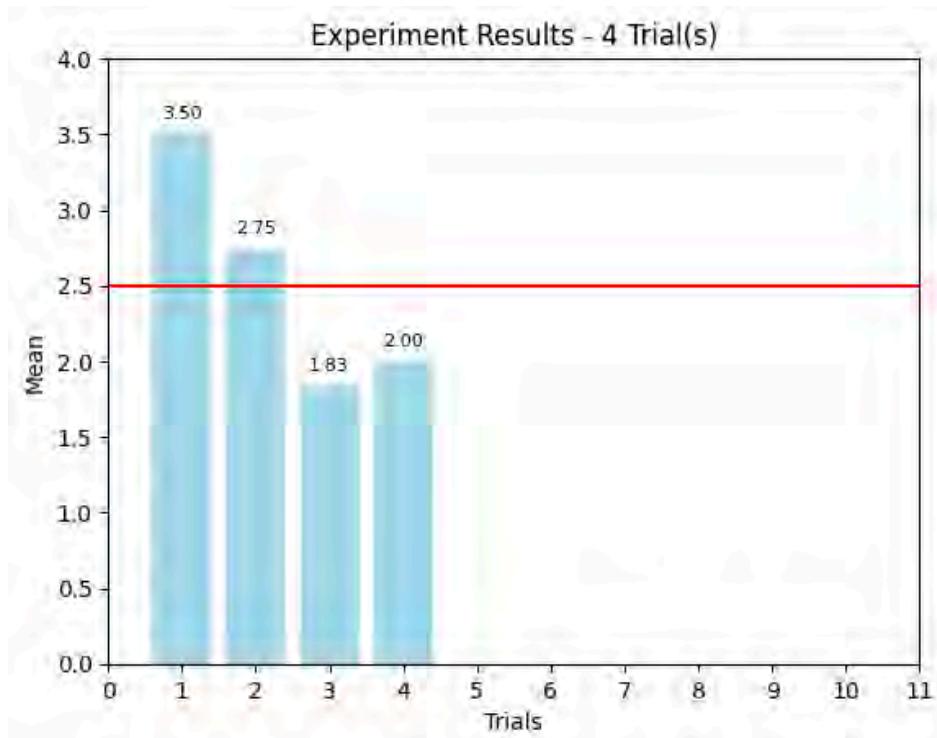


# Law of Large Numbers

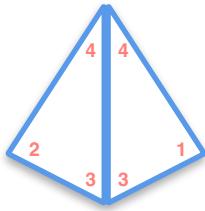


1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

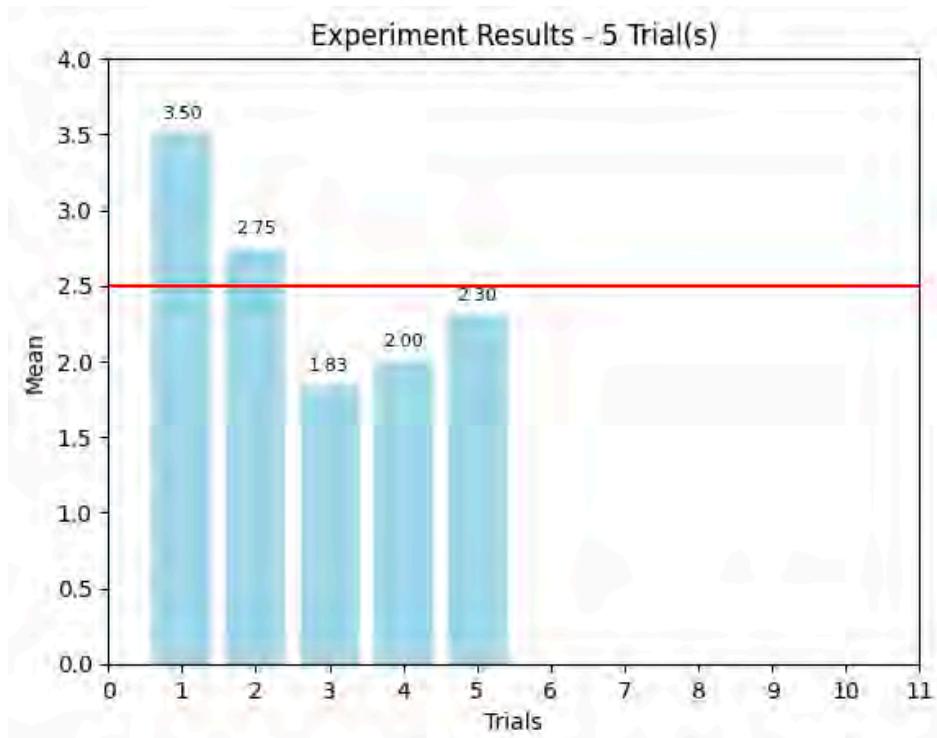


# Law of Large Numbers

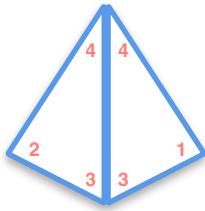


$$1 \ 2 \ 3 \ 4$$
$$\mu = 2.5$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

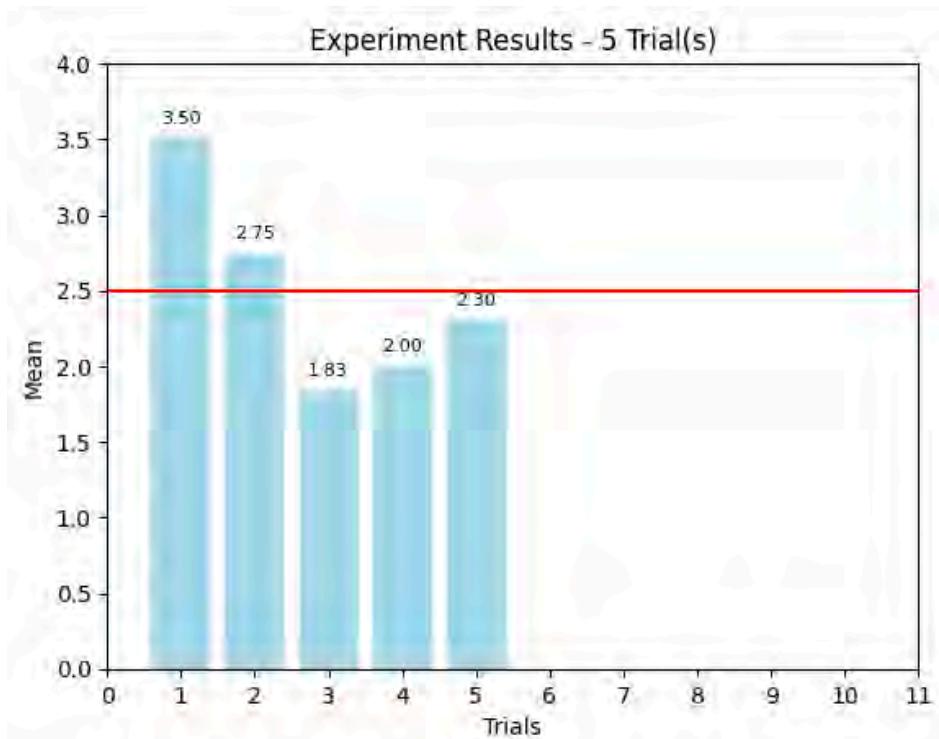


# Law of Large Numbers

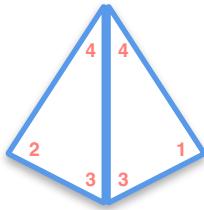


1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

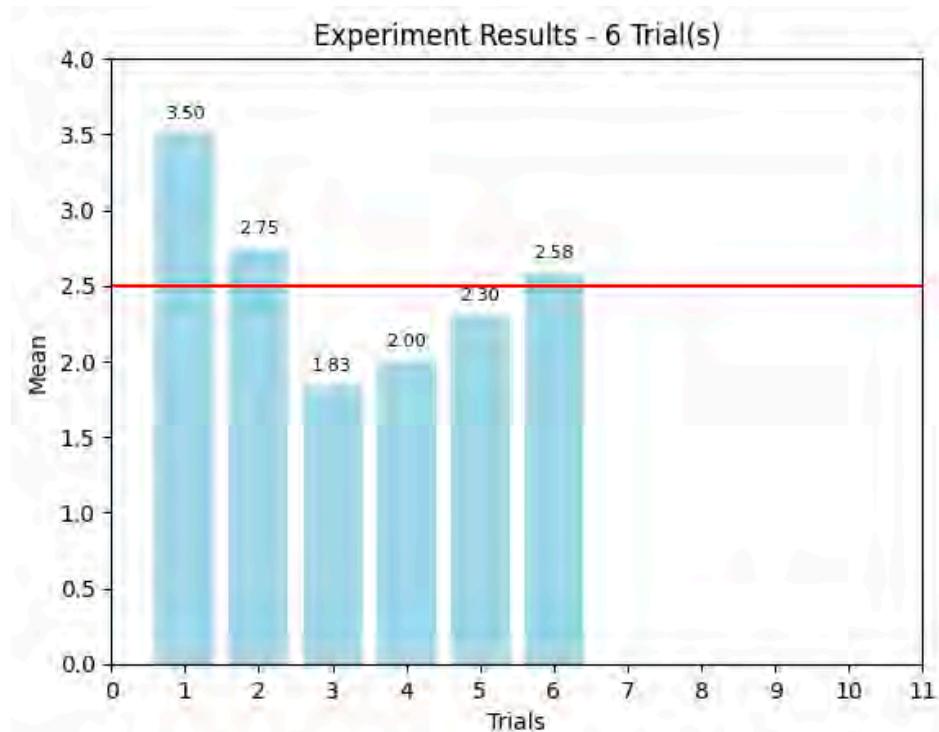


# Law of Large Numbers

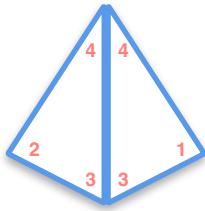


$$1 \ 2 \ 3 \ 4$$
$$\mu = 2.5$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

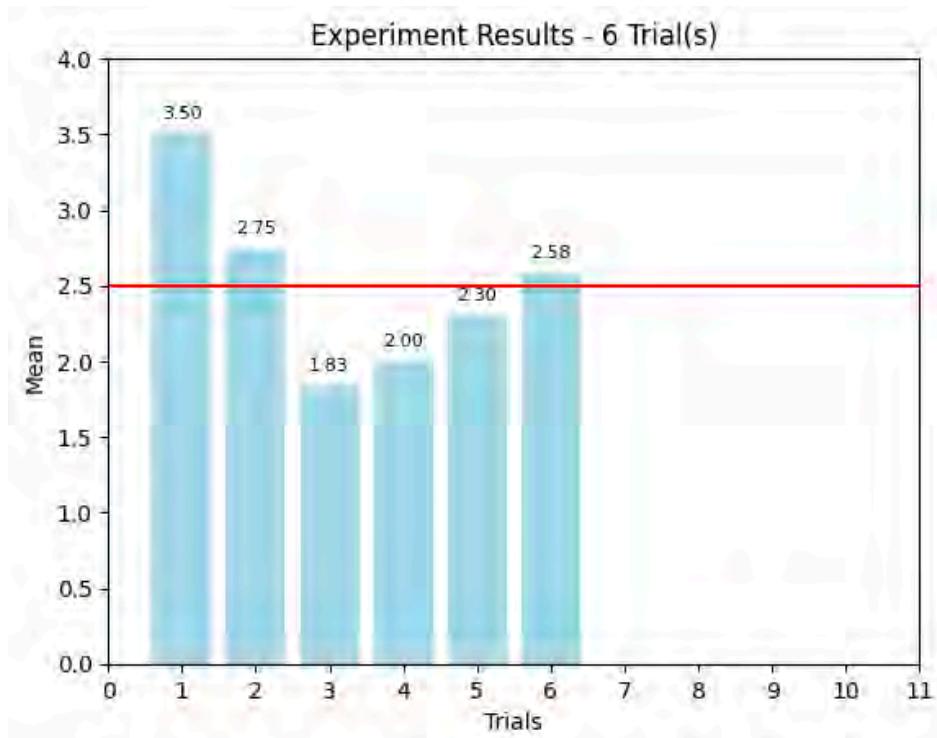


# Law of Large Numbers

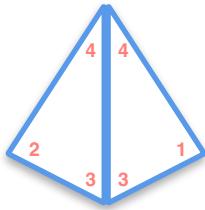


1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

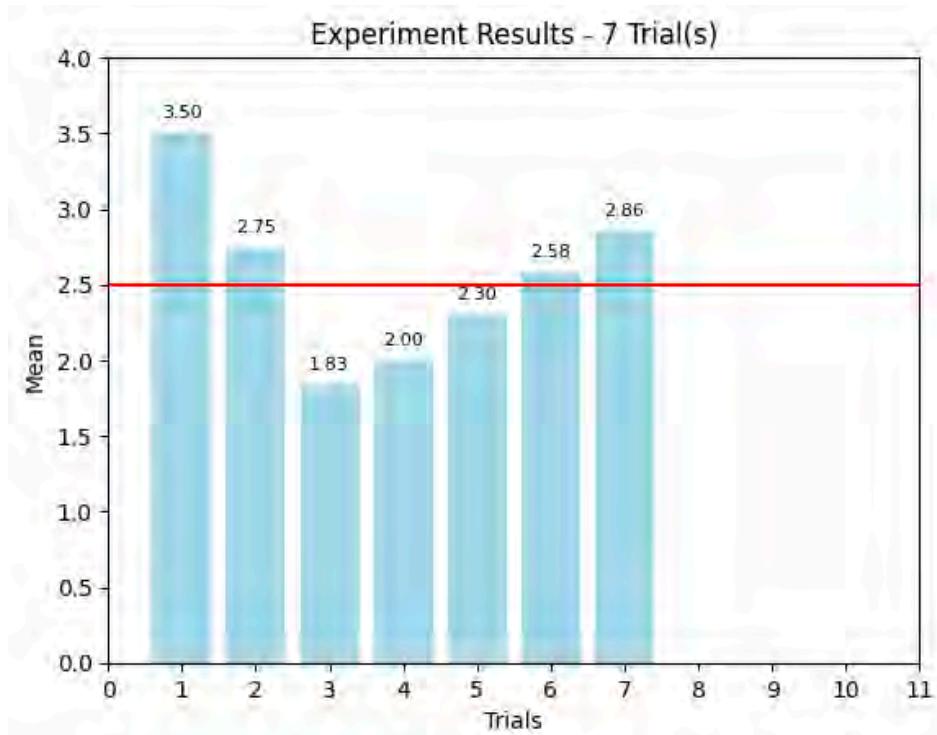


# Law of Large Numbers

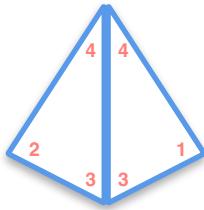


$$1 \ 2 \ 3 \ 4$$
$$\mu = 2.5$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

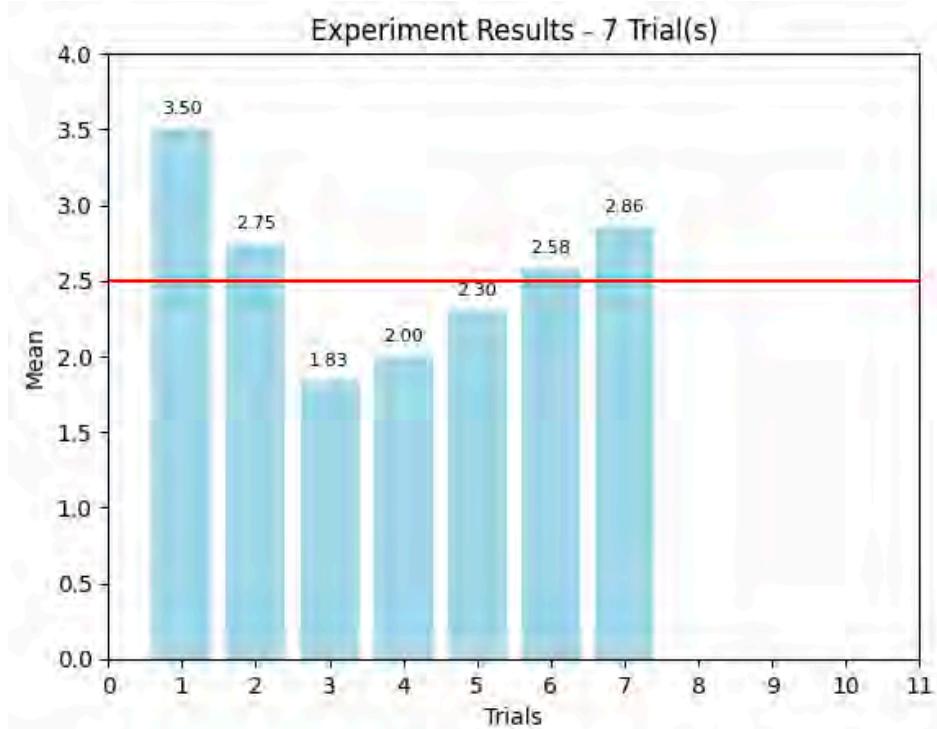


# Law of Large Numbers

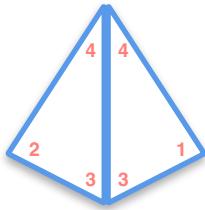


$$1 \ 2 \ 3 \ 4$$
$$\mu = 2.5$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

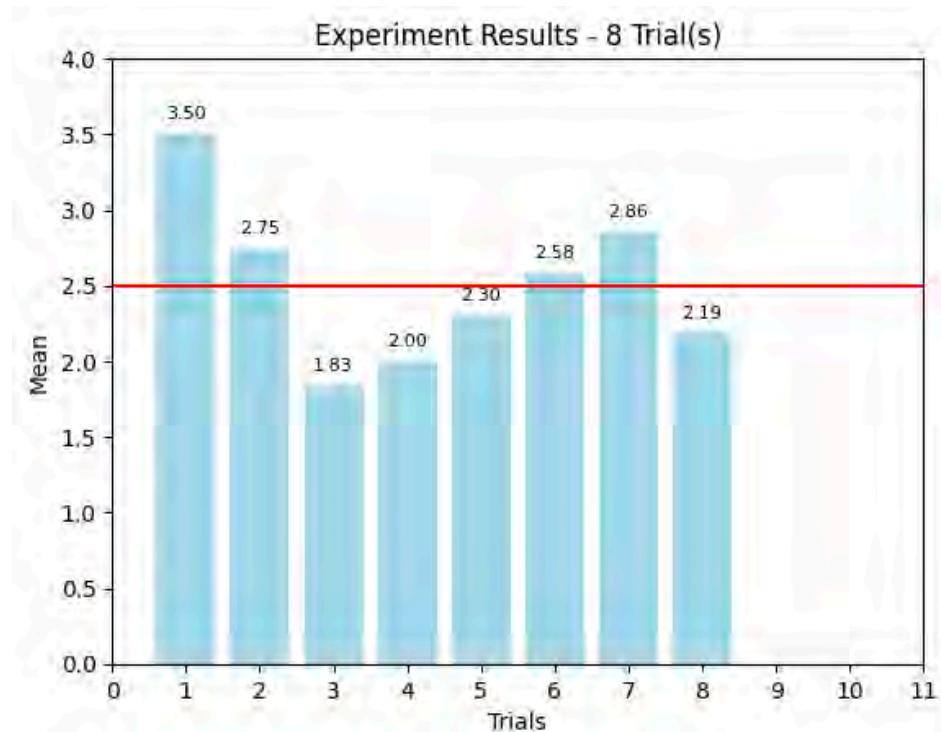


# Law of Large Numbers

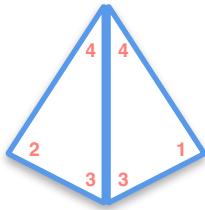


1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

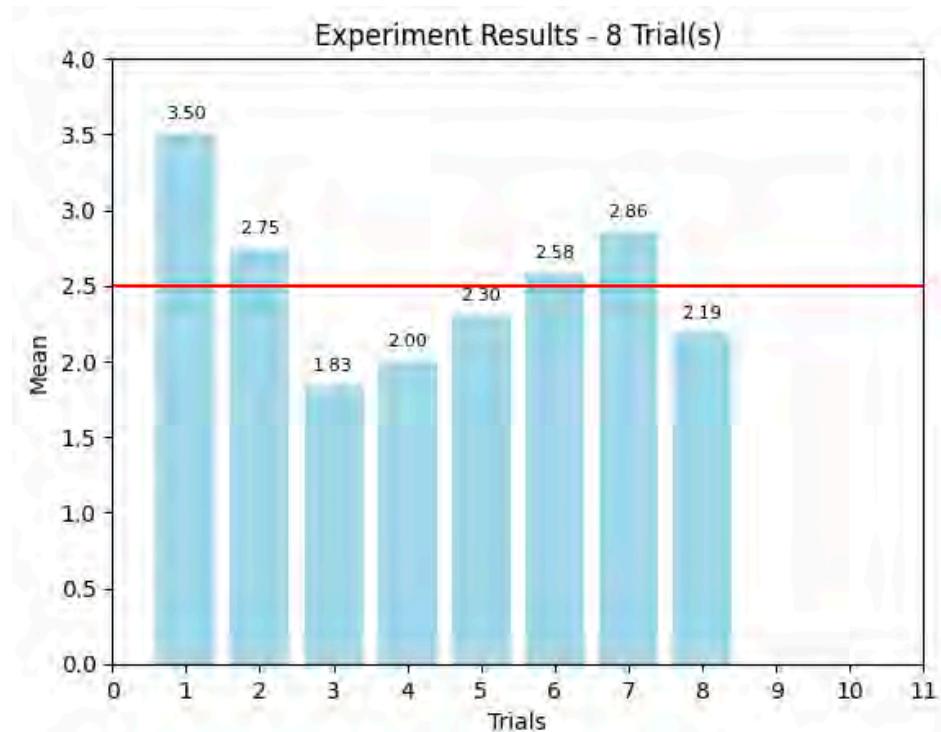


# Law of Large Numbers

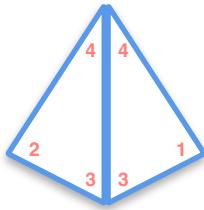


1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

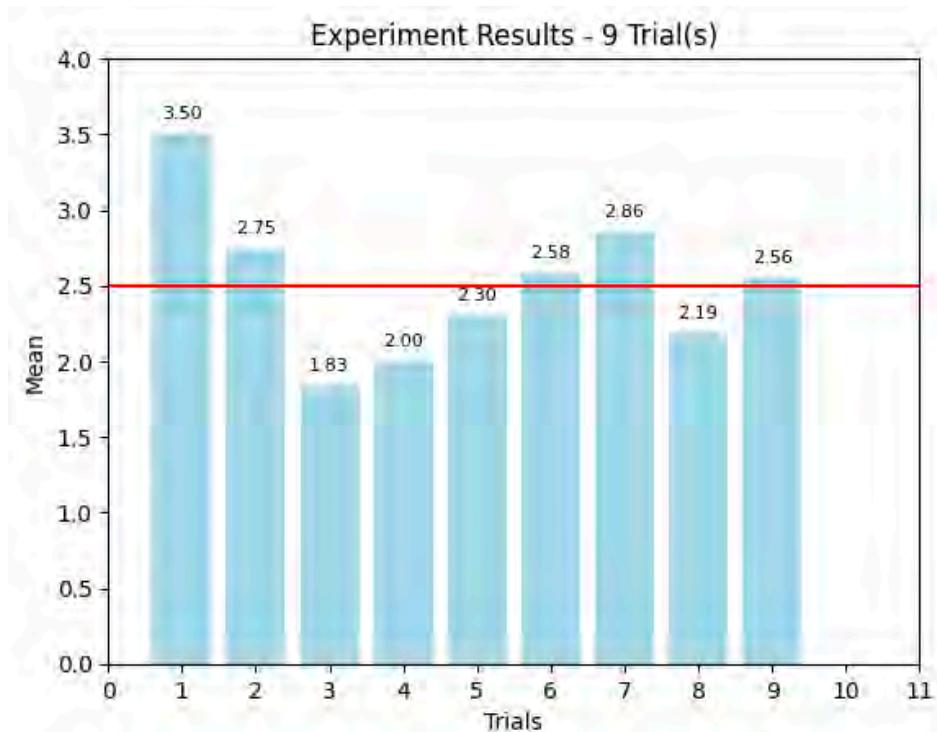


# Law of Large Numbers

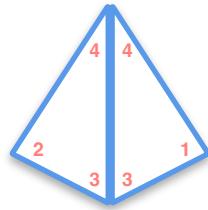


$$1 \ 2 \ 3 \ 4$$
$$\mu = 2.5$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

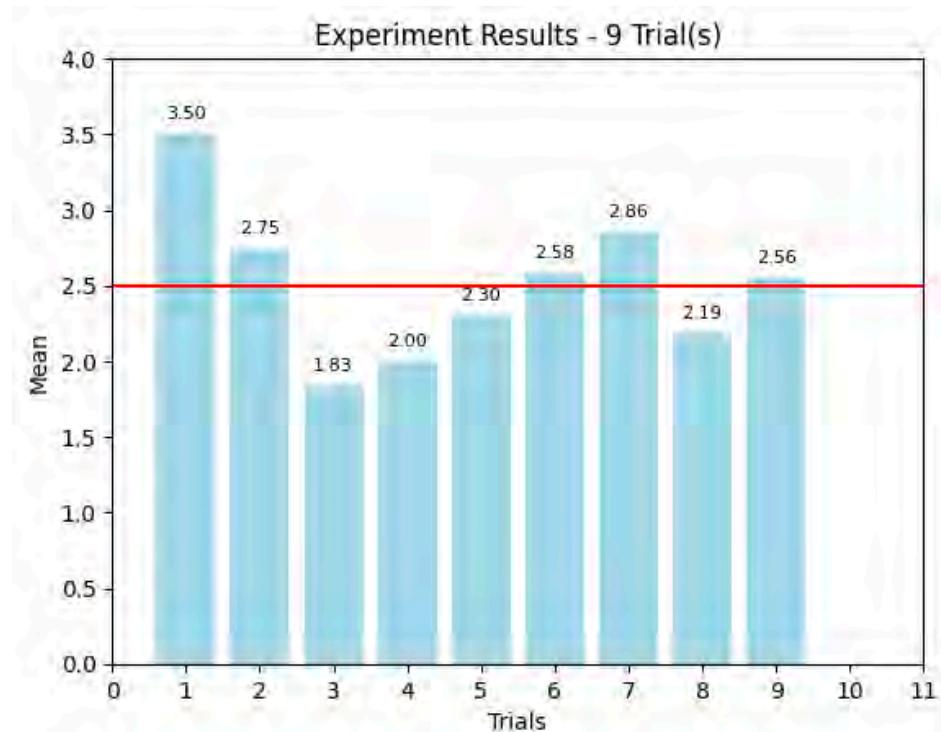


# Law of Large Numbers

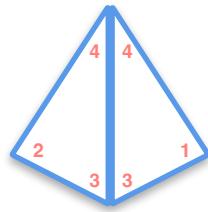


$$1 \ 2 \ 3 \ 4$$
$$\mu = 2.5$$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

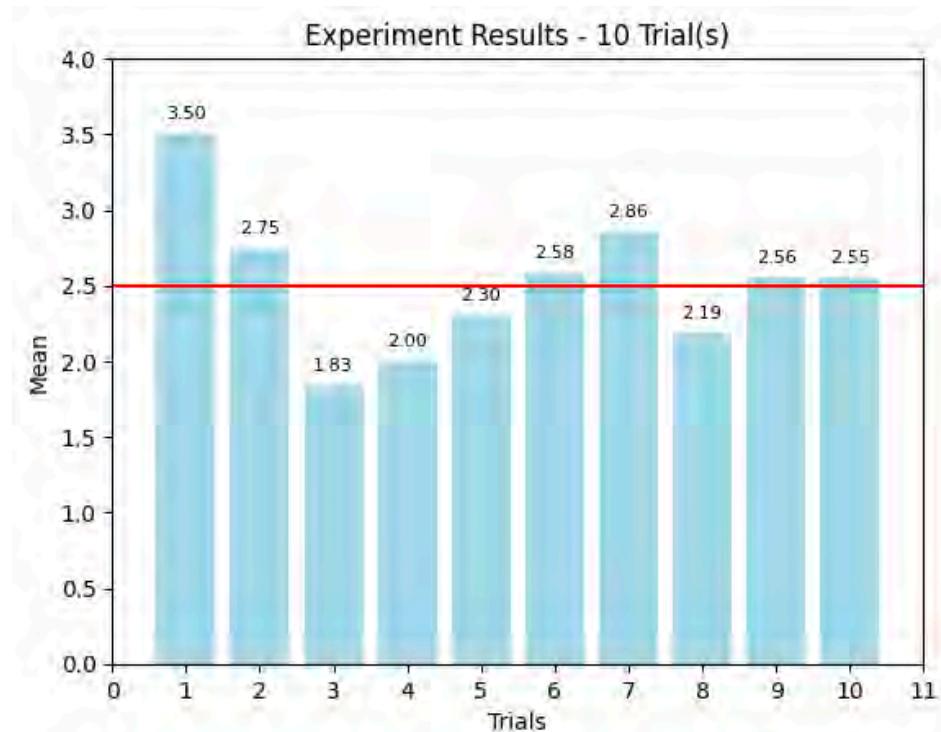


# Law of Large Numbers

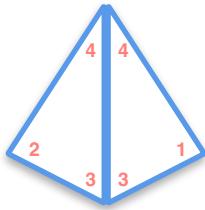


1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

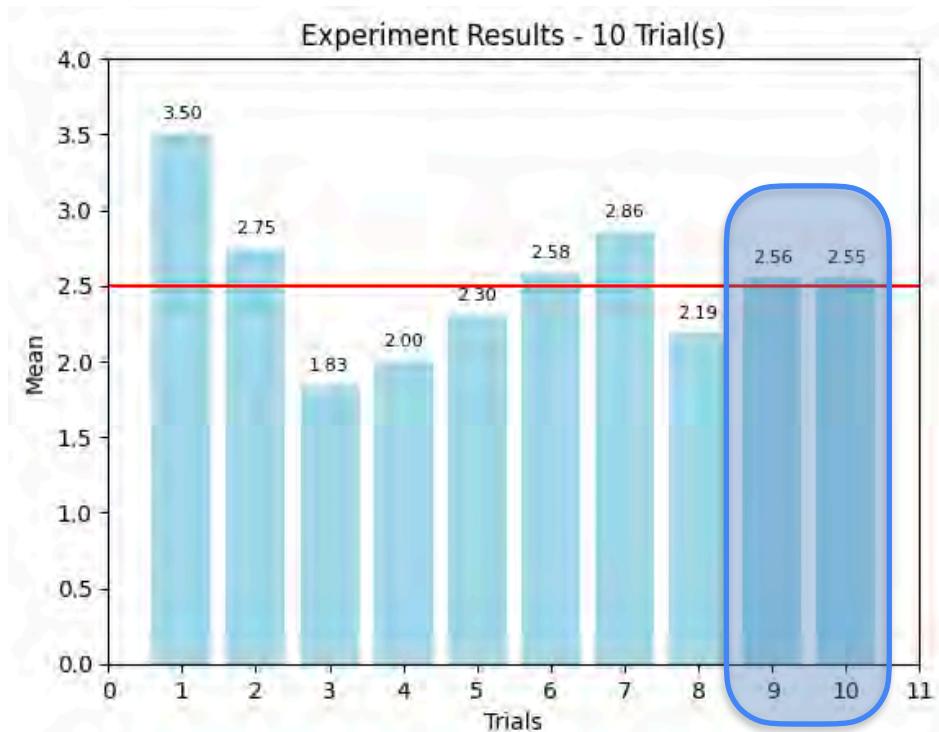


# Law of Large Numbers

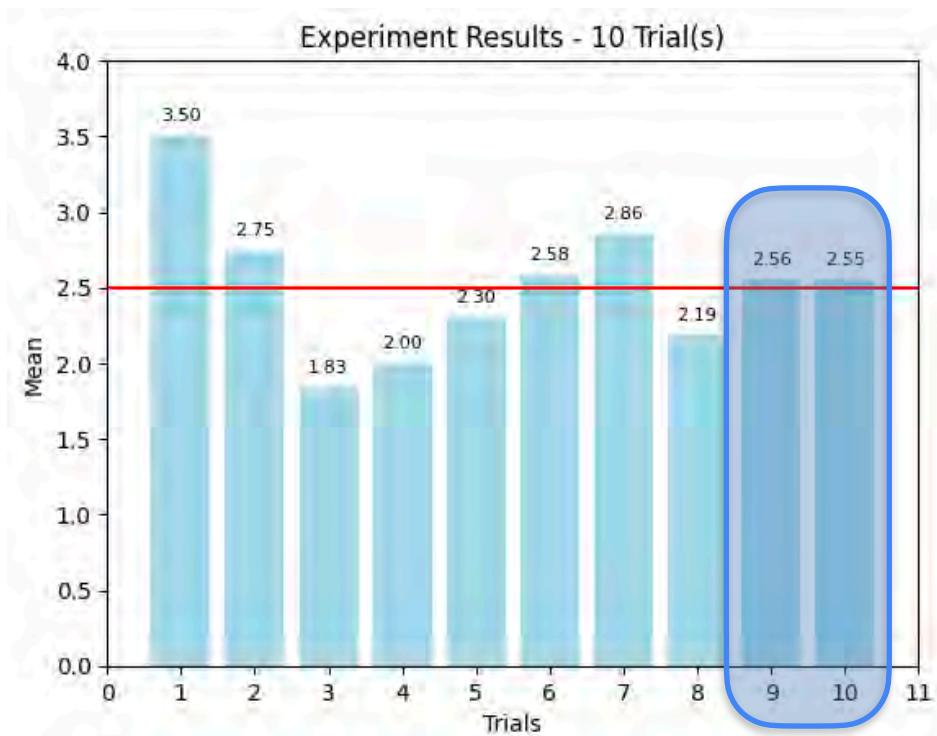


1 2 3 4  
 $\mu = 2.5$

	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4



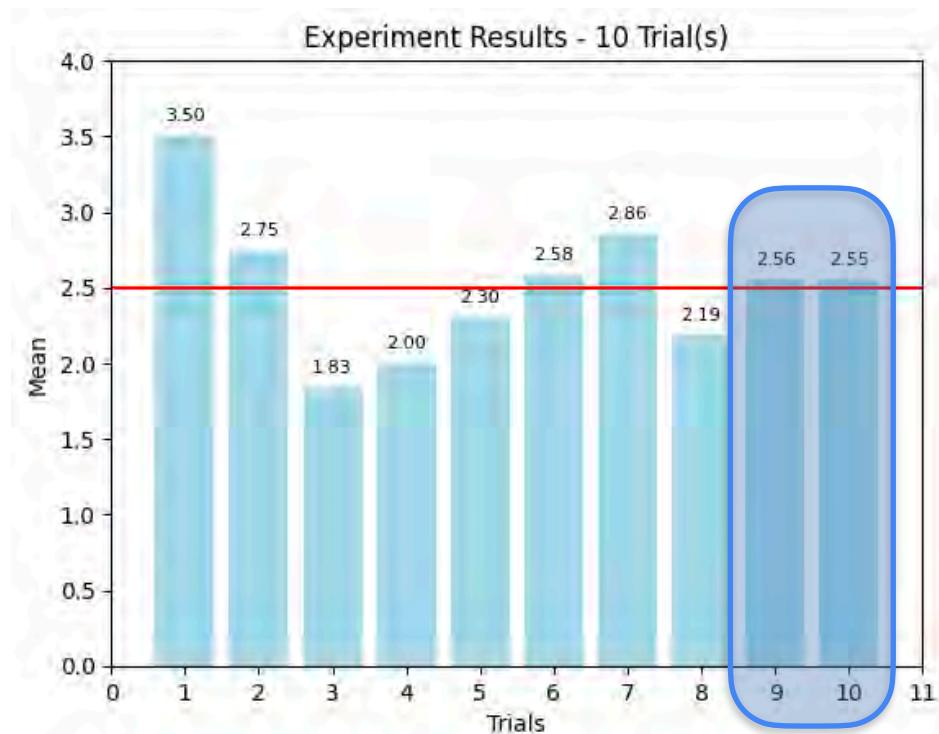
# Law of Large Numbers



# Law of Large Numbers

As the sample size increases, the average of the sample will tend to get closer to the average of the entire population.

## Law of Large Numbers



# Law of Large Numbers

Law of Large Numbers

# Law of Large Numbers

## Law of Large Numbers

$n$  : number of samples

# Law of Large Numbers

## Law of Large Numbers

$n$  : number of samples

$X_i$  : some estimate  $X$  for a sample size i

# Law of Large Numbers

## Law of Large Numbers

$n$  : number of samples

$X_i$  : some estimate  $X$  for a sample size  $i$

as  $n \rightarrow \infty$

# Law of Large Numbers

## Law of Large Numbers

$n$  : number of samples

$X_i$  : some estimate  $X$  for a sample size  $i$

as  $n \rightarrow \infty$

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mathbb{E}[X] = \mu_X$$

# Law of Large Numbers

as  $n \rightarrow \infty$

## Law of Large Numbers

$n$  : number of samples

$X_i$  : some estimate  $X$  for a sample size  $i$

$$\text{as } n \rightarrow \infty \quad \frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mathbb{E}[X] = \mu_X$$

# Law of Large Numbers

**Law of Large Numbers**

$n$  : number of samples

$X_i$  : some estimate  $X$  for a sample size  $i$

as  $n \rightarrow \infty$

$$\frac{X_1 + X_2 + X_3 + \dots + X_n}{n} \longrightarrow \mathbb{E}[X] = \mu_X$$

**UNDER CERTAIN CONDITIONS**

as  $n \rightarrow \infty$

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mathbb{E}[X] = \mu_X$$

# Law of Large Numbers

UNDER CERTAIN CONDITIONS

# Law of Large Numbers

## UNDER CERTAIN CONDITIONS

- Sample is randomly drawn.

# Law of Large Numbers

## UNDER CERTAIN CONDITIONS

- Sample is randomly drawn.

# Law of Large Numbers

## UNDER CERTAIN CONDITIONS

- Sample is randomly drawn.
- Sample size must be sufficiently large.

# Law of Large Numbers

## UNDER CERTAIN CONDITIONS

- Sample is randomly drawn.
- Sample size must be sufficiently large.

# Law of Large Numbers

## UNDER CERTAIN CONDITIONS

- Sample is randomly drawn.
- Sample size must be sufficiently large.
- Independent observations.



DeepLearning.AI

## Sample and Population

---

## Central Limit Theorem

# Central Limit Theorem (CLT) - Example 1

If  $n = 1$

# Central Limit Theorem (CLT) - Example 1



If  $n = 1$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

If  $n = 1$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$

$$\mathbf{P}(T) = 0.5$$

Random  
variable  $\rightarrow X$

If  $n = 1$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$

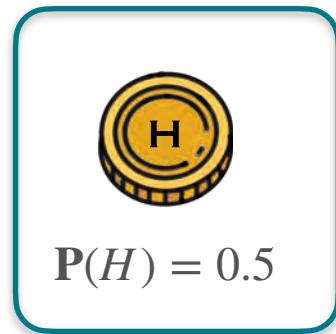


$$\mathbf{P}(T) = 0.5$$

Random variable  $\rightarrow X$  number of heads when a coin is flipped n times

If  $n = 1$

# Central Limit Theorem (CLT) - Example 1



Random variable  $\rightarrow X$  number of heads when a coin is flipped n times

If  $n = 1$        $X = 1$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

Random variable  $\rightarrow X$  number of heads when a coin is flipped n times

$$\text{If } n = 1$$

$$X = 1$$

$$X = 0$$

# Central Limit Theorem (CLT) - Example 1



Random variable  $\rightarrow X$  number of heads when a coin is flipped n times

If  $n = 1$

$X = 1$

$X = 0$

Discrete Random Variable

# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5 \quad P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

# Central Limit Theorem (CLT) - Example 1



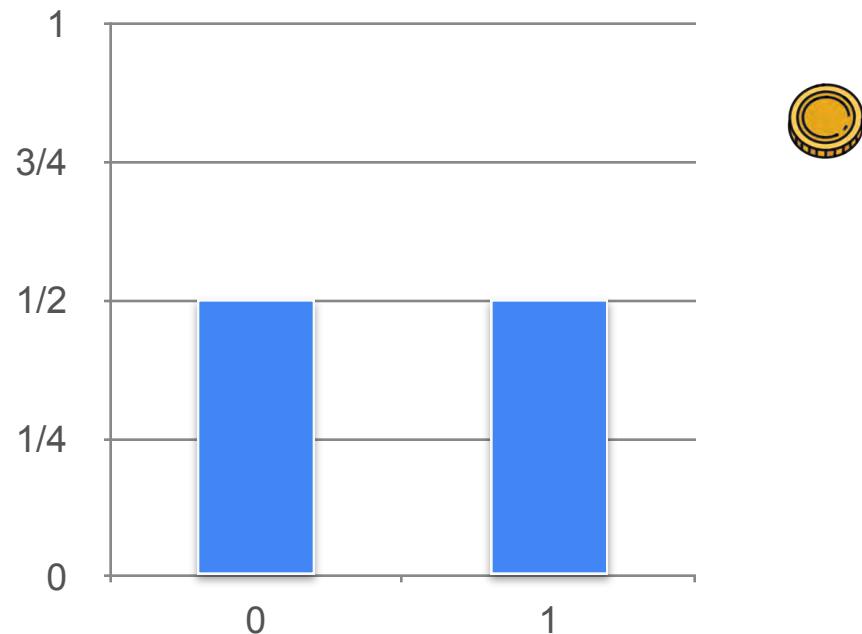
$$P(H) = 0.5$$



$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$



# Central Limit Theorem (CLT) - Example 1



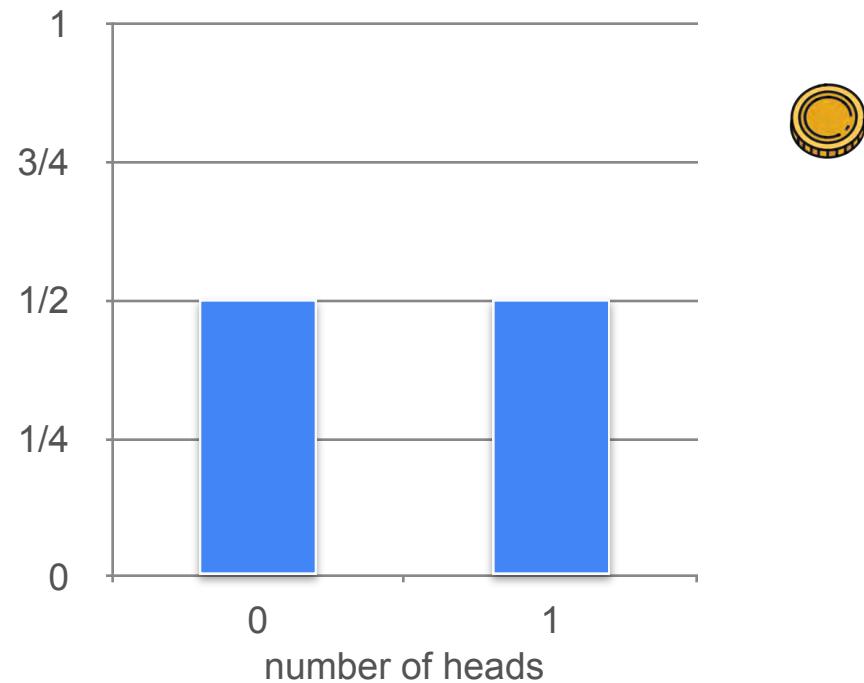
$$P(H) = 0.5$$



$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$



# Central Limit Theorem (CLT) - Example 1



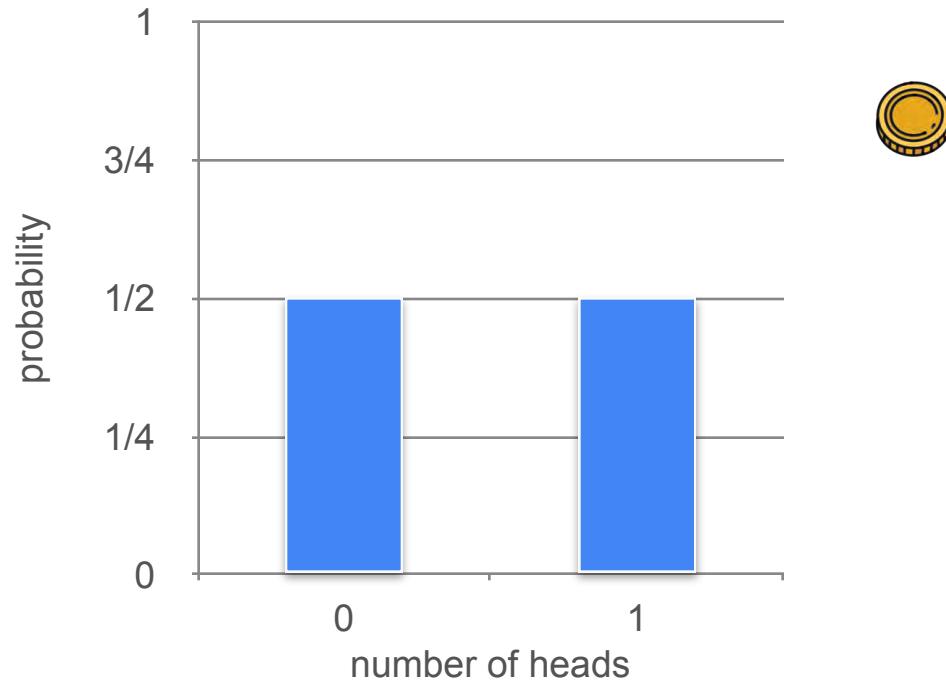
$$P(H) = 0.5$$



$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

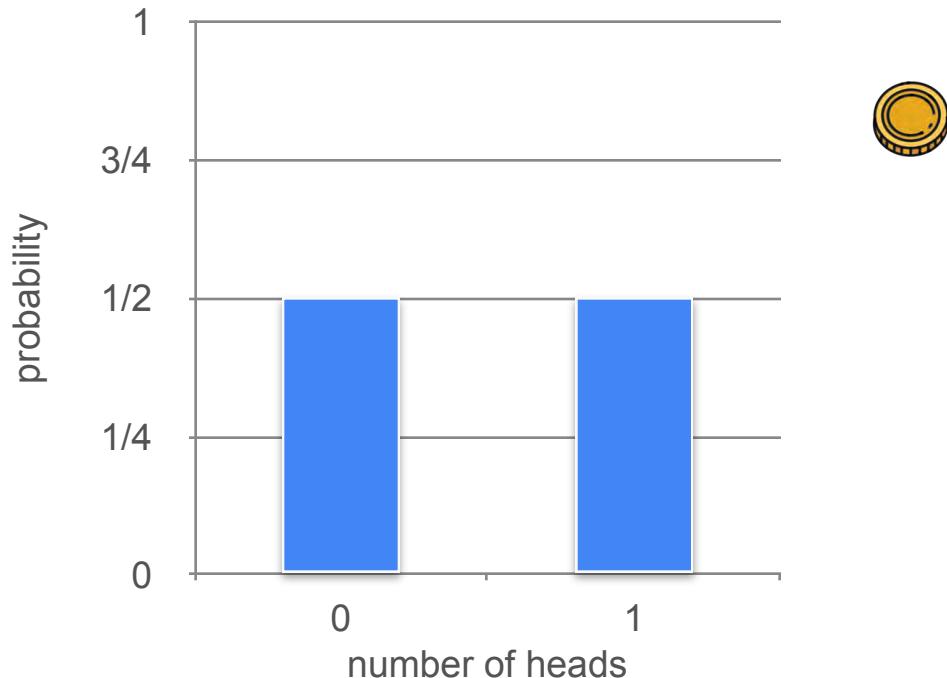


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

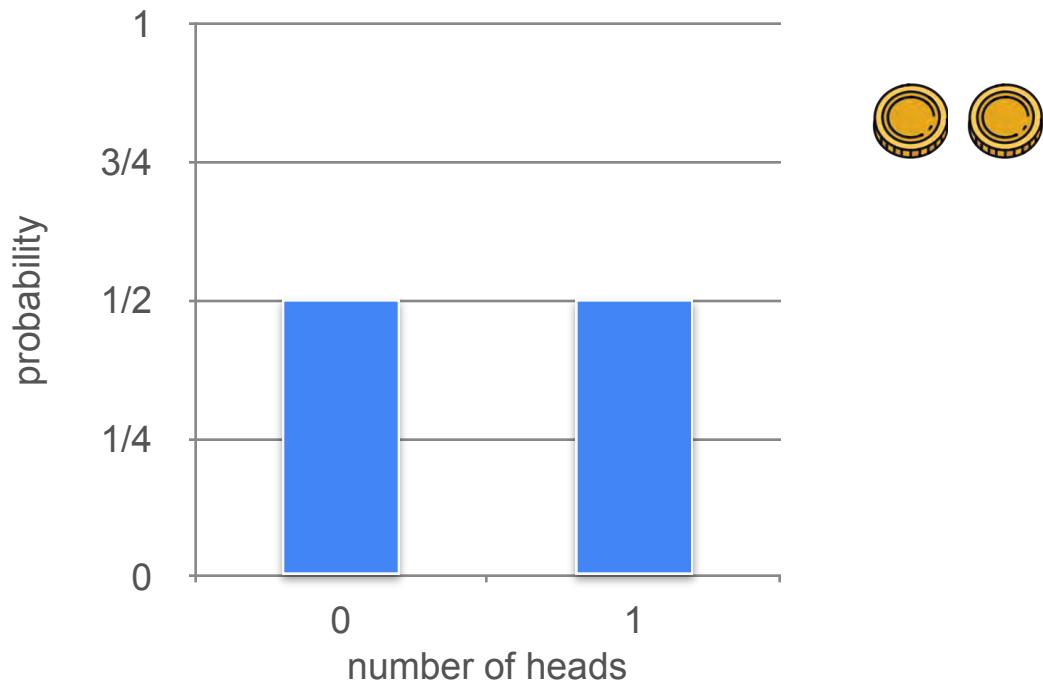


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

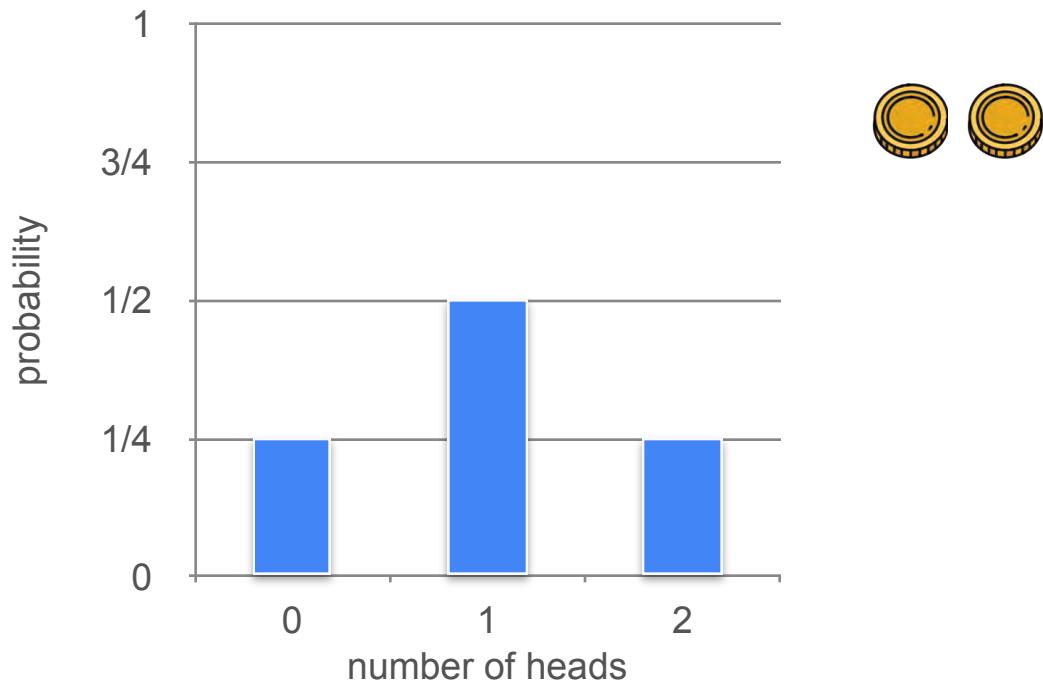


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

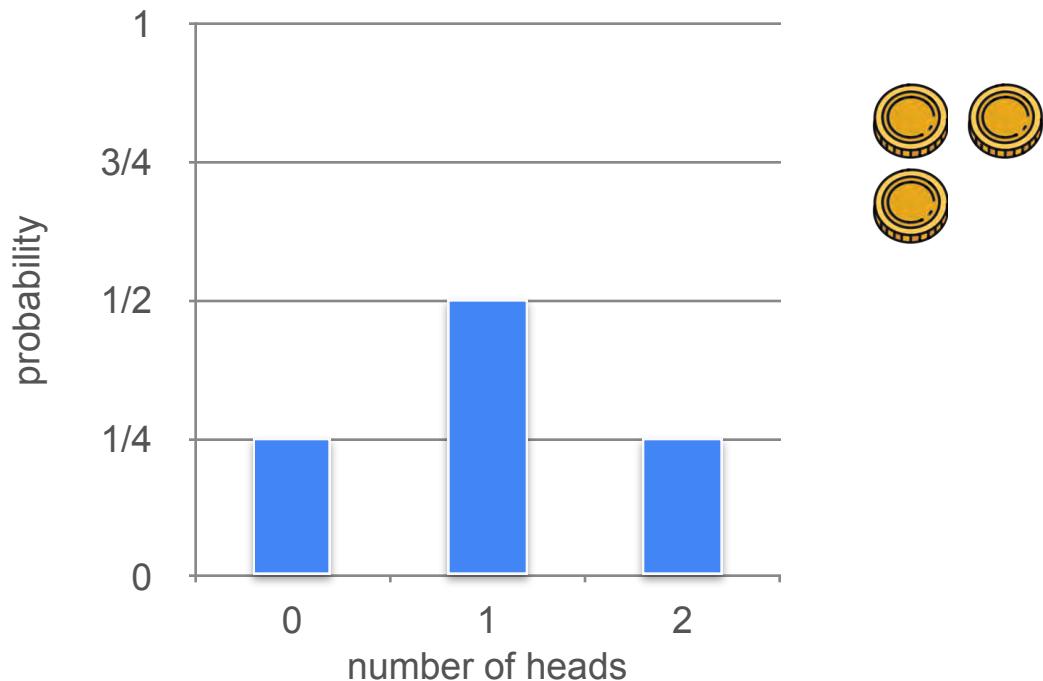


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

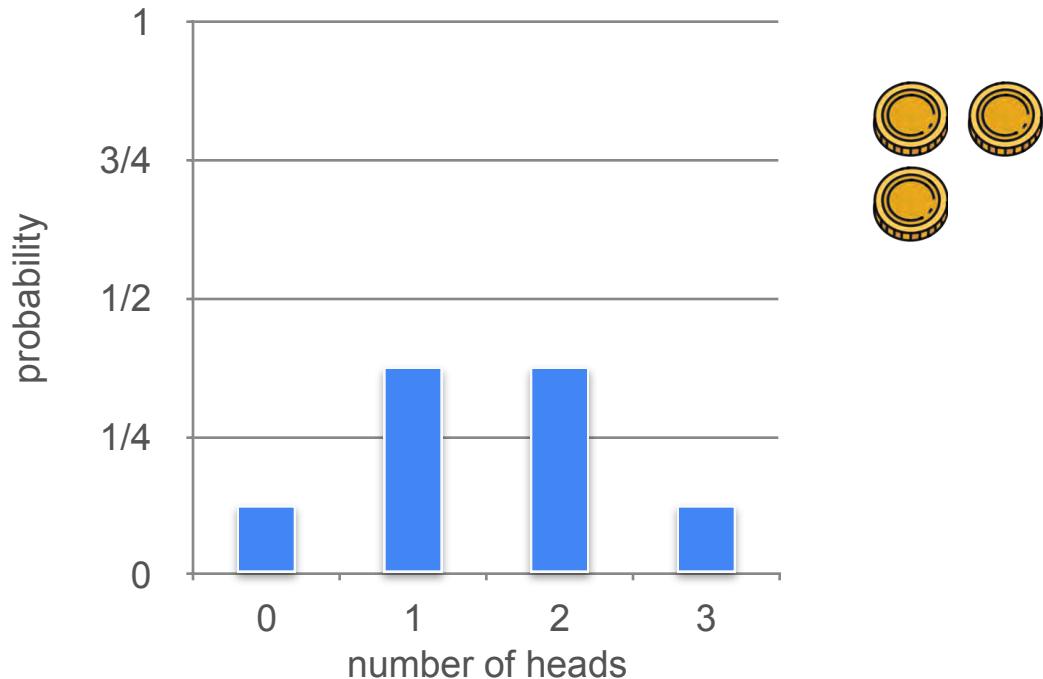


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

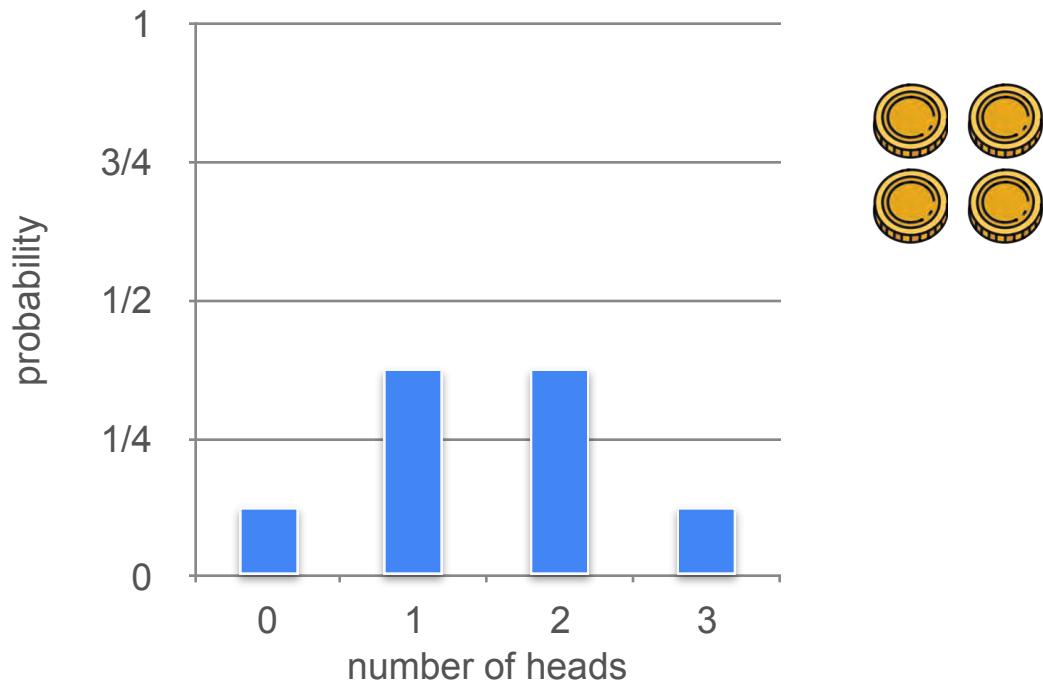


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

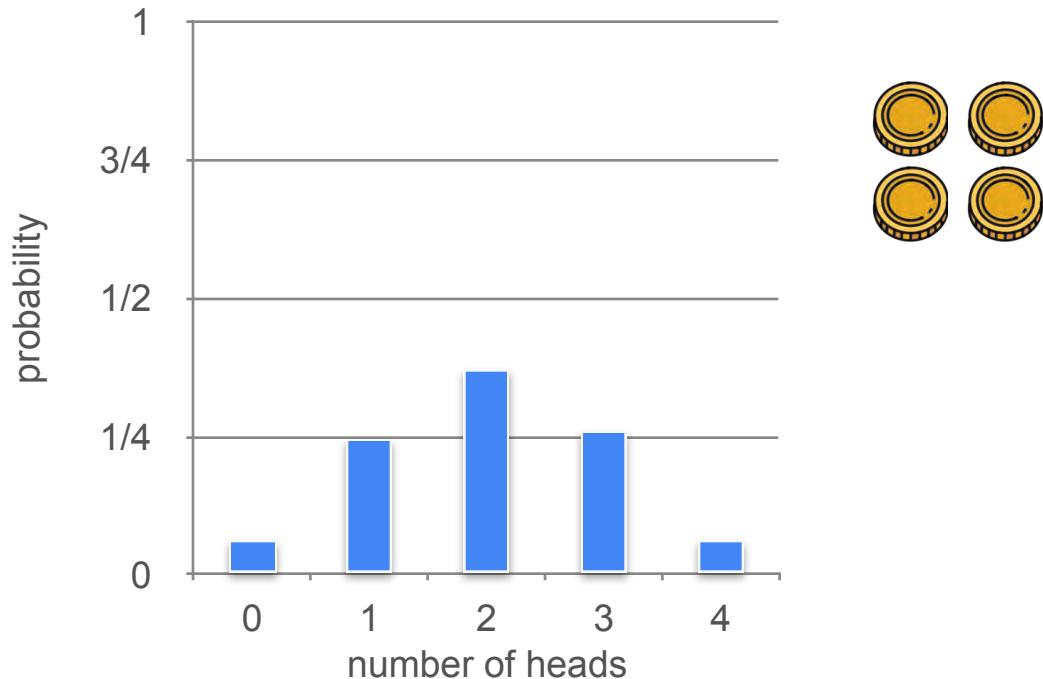


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

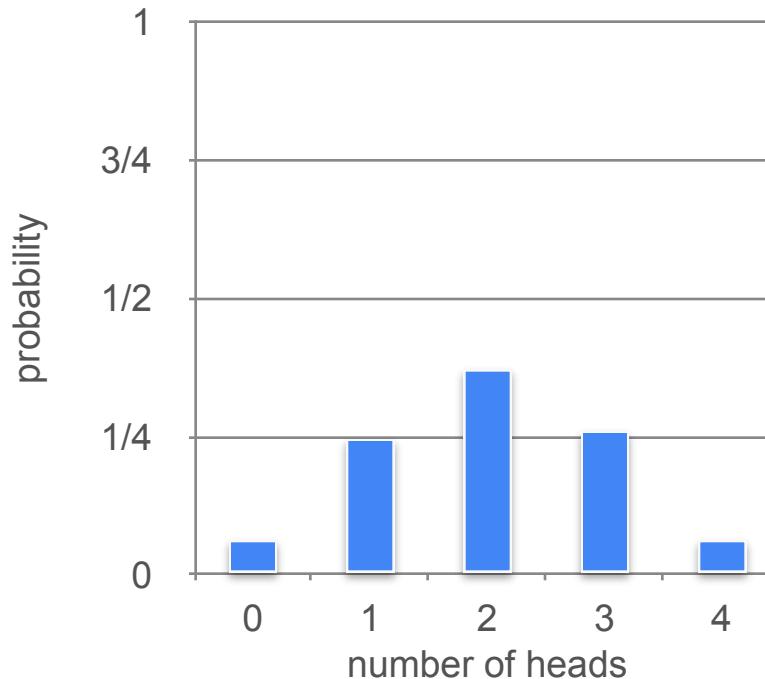


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the distribution when the number of coins we flip increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

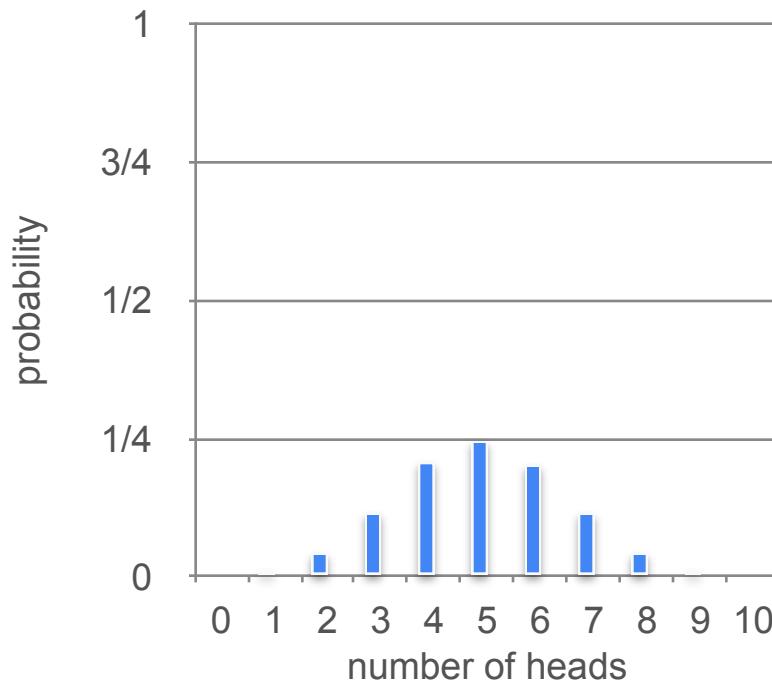


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the distribution when the number of coins we flip increases?



# Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

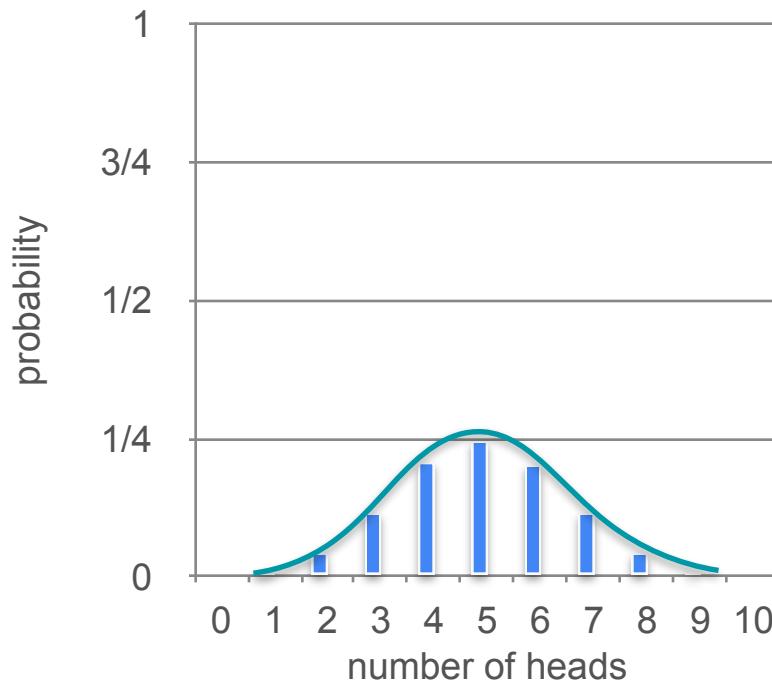


$$P(T) = 0.5$$

$$X = 1$$

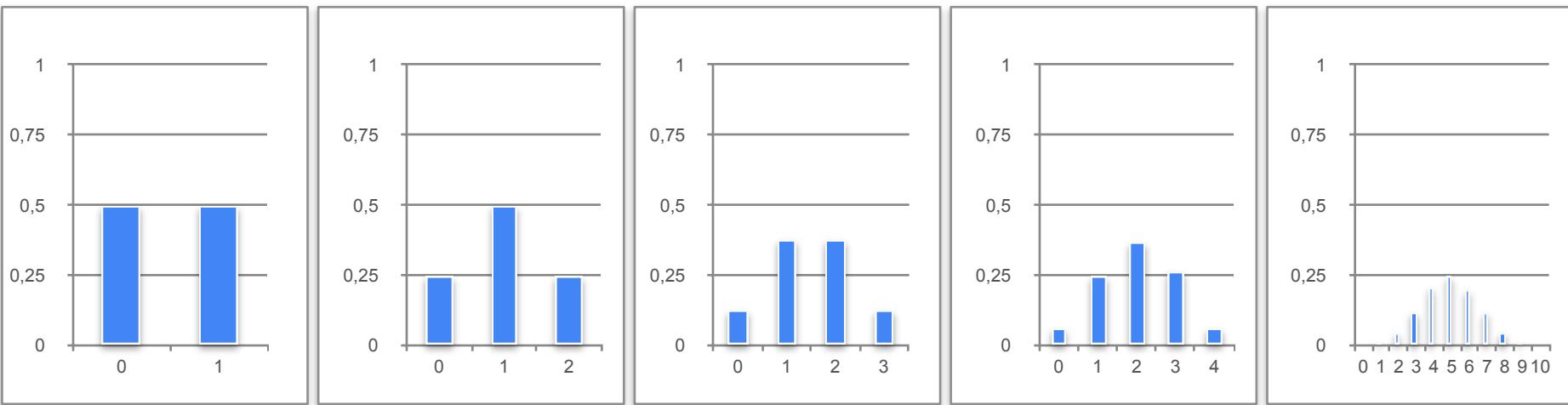
$$X = 0$$

What can we say about the distribution when the number of coins we flip increases?

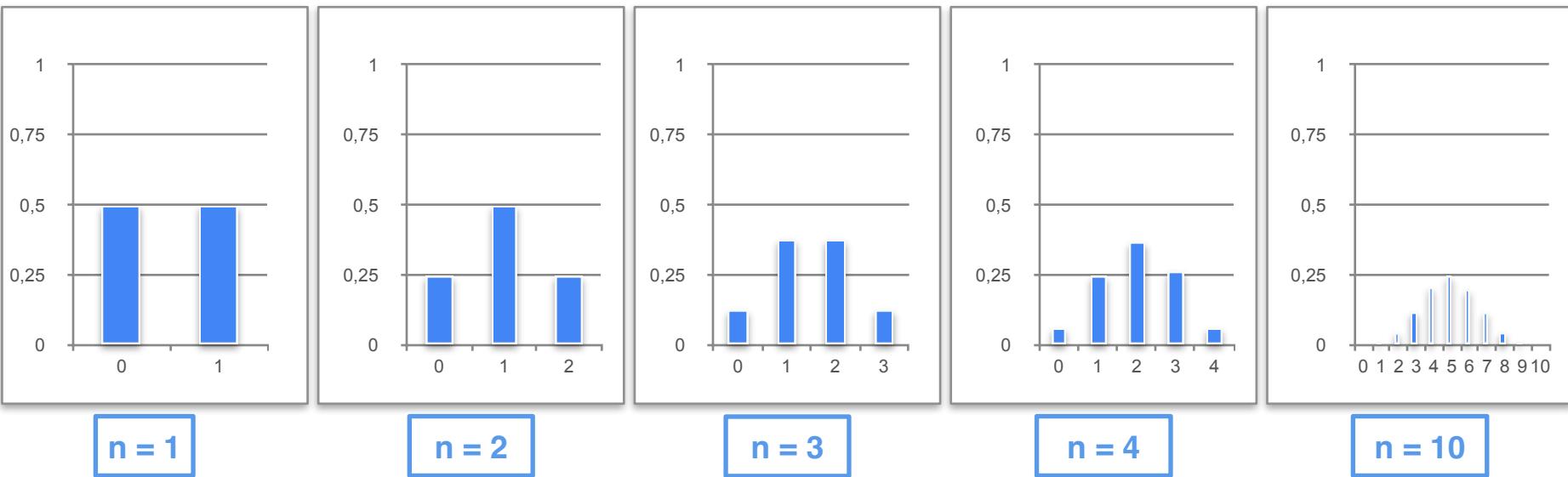


# Central Limit Theorem (CLT) - Example 1

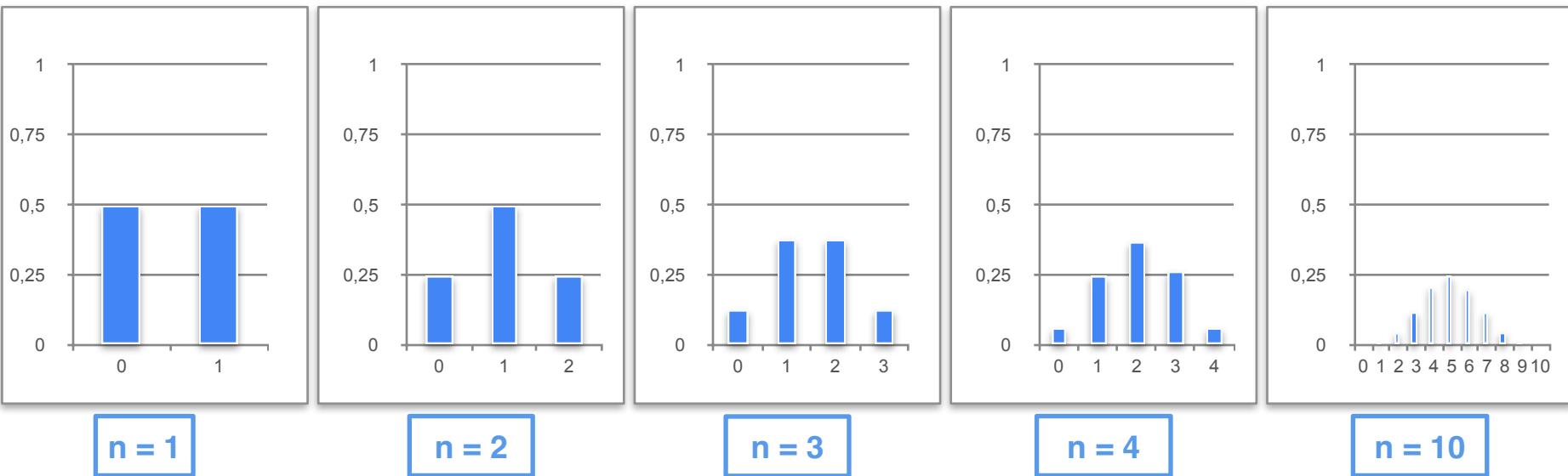
# Central Limit Theorem (CLT) - Example 1



# Central Limit Theorem (CLT) - Example 1



# Central Limit Theorem (CLT) - Example 1



As  $n$  increases, the probability distribution becomes closer to a Gaussian distribution

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

Random  
variable



$X$  number of heads when a coin is flipped n times

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

Random  
variable



**$X$  number of heads when a coin is flipped n times**

$$\mu = np$$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

Random variable



**X** number of heads when a coin is flipped n times

$$\mu = np = n\mathbf{P}(H)$$



# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

Random variable



**X** number of heads when a coin is flipped n times

$$\mu = np = n\mathbf{P}(H)$$



$$\sigma^2 = np(1 - p)$$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

Random variable



**$X$    number of heads when a coin is flipped n times**

$$\mu = np = n\mathbf{P}(H)$$



$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$



# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$


$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$


# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$


$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$


# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$


$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$


# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$


$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$




$n = 1$

$$\mu = np$$



# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$


$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$




$$n = 1$$

$$\mu = np$$



$$\mu = 1 \times 0.5$$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$


$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$




$n = 1$

$$\mu = np$$



$$\mu = 1 \times 0.5 = 0.5$$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$
Two yellow circular coins are shown. The top one has 'H' and the bottom one also has 'H'.

$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$
Two coins are shown side-by-side: a yellow one with 'H' and a teal one with 'T'.



$$n = 1$$

$$\mu = np$$



$$\mu = 1 \times 0.5 = 0.5$$

$$\sigma^2 = np(1 - p)$$



# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$
Two yellow circular coins, one with 'H' and one with 'T', positioned below the equation.

$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$
Two coins, one yellow with 'H' and one teal with 'T', side-by-side below the equation.



$$n = 1$$

$$\mu = np$$



$$\mu = 1 \times 0.5 = 0.5$$

$$\sigma^2 = np(1 - p)$$



$$\sigma^2 = (1 \times 0.5)(0.5)$$

# Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$
Two yellow circular coins, one with 'H' and one with 'T', positioned below the equation.

$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$
Two coins, one yellow with 'H' and one teal with 'T', side-by-side below the equation.



$$n = 1$$

$$\mu = np$$

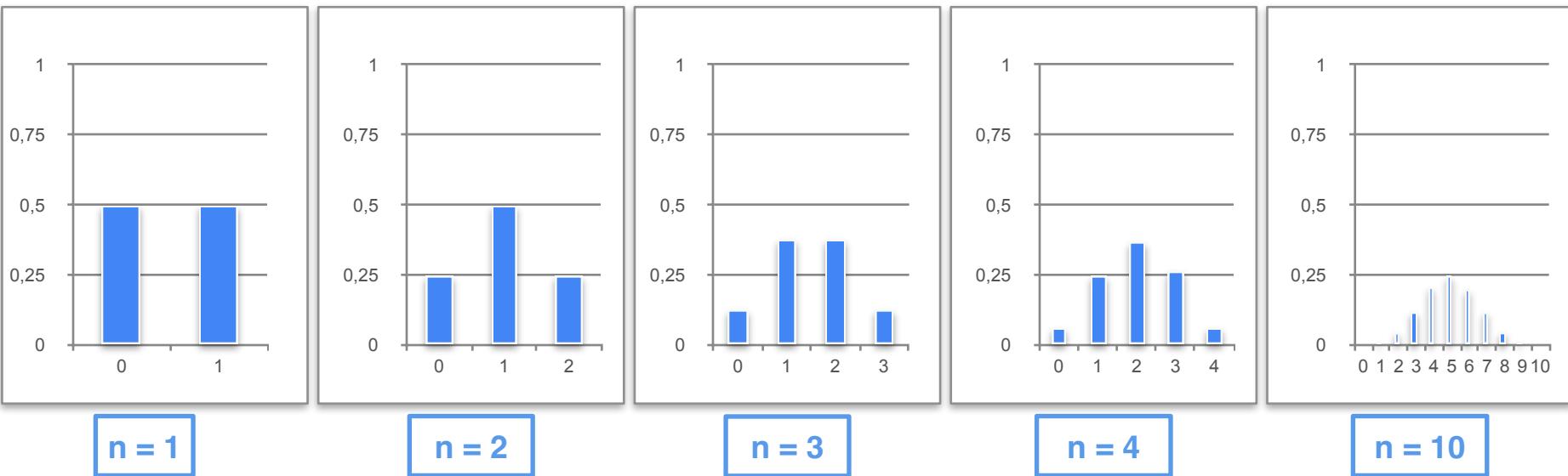


$$\mu = 1 \times 0.5 = 0.5$$

$$\sigma^2 = np(1 - p)$$
Two coins, one yellow with 'H' and one teal with 'T', side-by-side below the equation.

$$\sigma^2 = (1 \times 0.5)(0.5) = 0.25$$

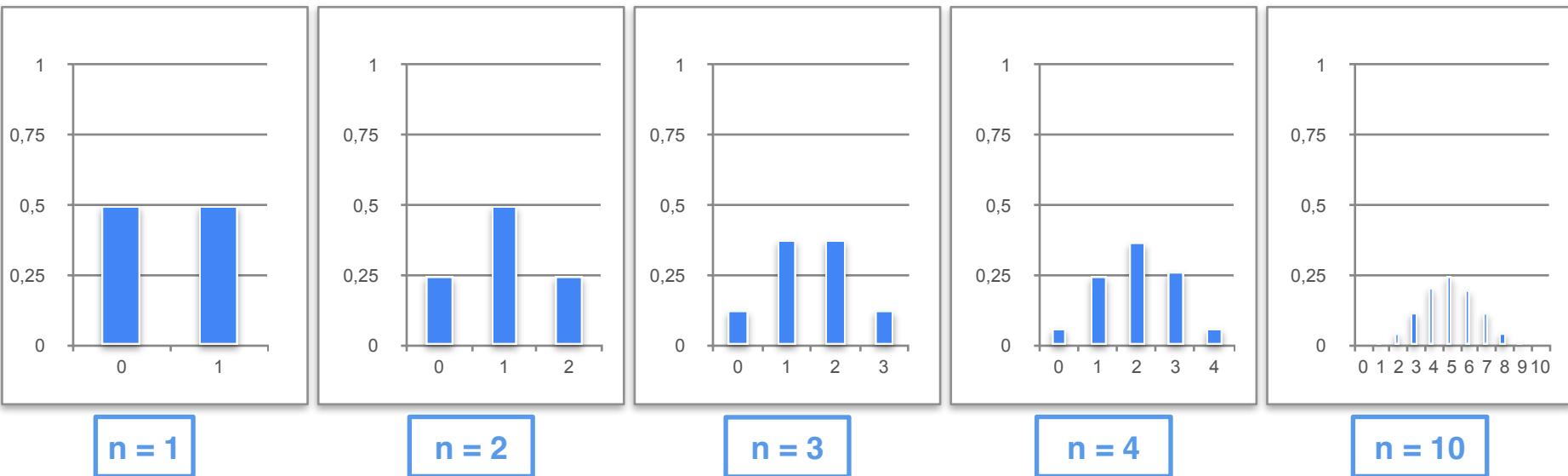
# Central Limit Theorem (CLT) - Example 1



As  $n$  increases, the probability distribution becomes closer to a gaussian distribution

# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

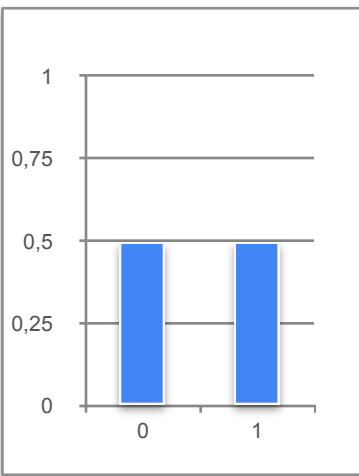


As n increases, the probability distribution becomes closer to a gaussian distribution

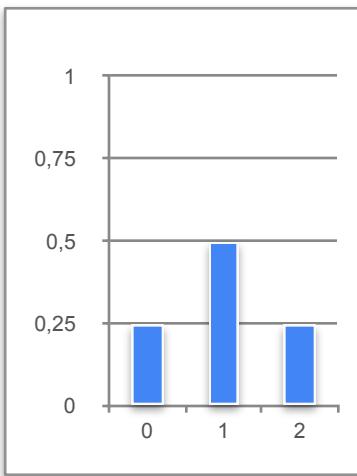
# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

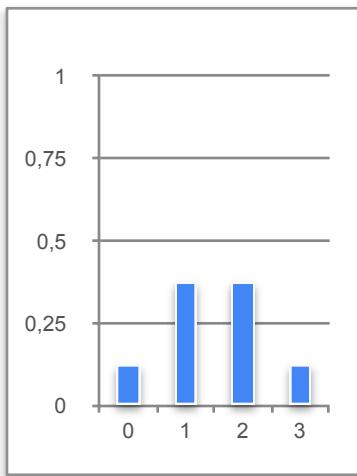
$$\sigma^2 = np(1 - p)$$



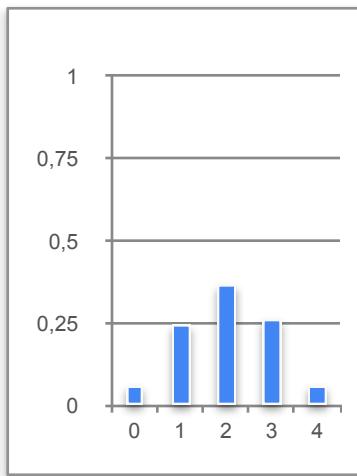
n = 1



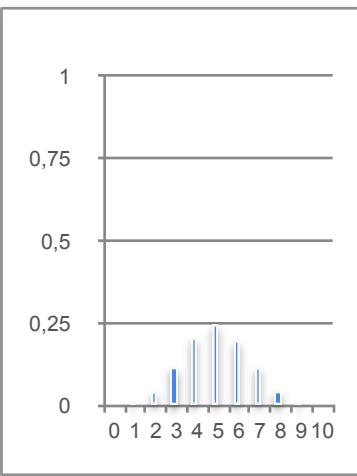
n = 2



n = 3



n = 4



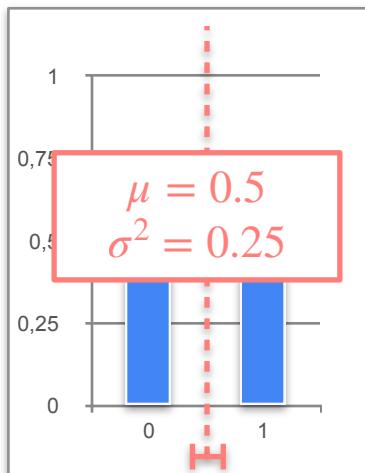
n = 10

As n increases, the probability distribution becomes closer to a gaussian distribution

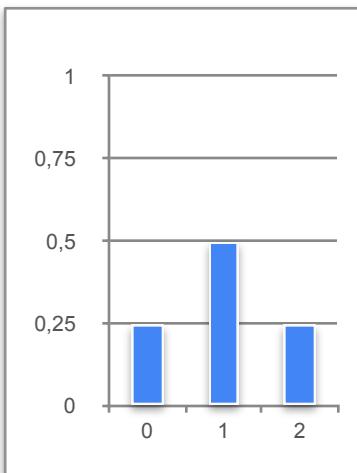
# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

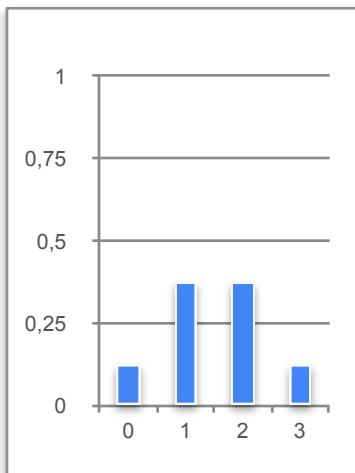
$$\sigma^2 = np(1 - p)$$



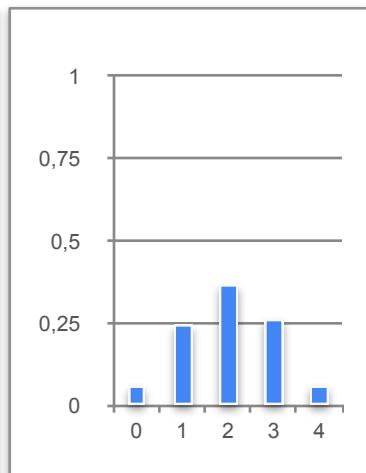
**$n = 1$**



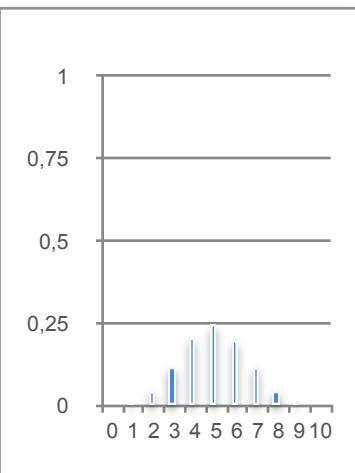
**$n = 2$**



**$n = 3$**



**$n = 4$**



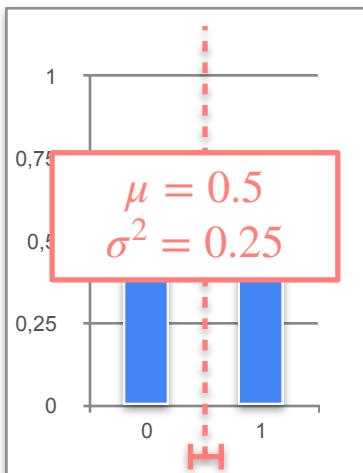
**$n = 10$**

As  $n$  increases, the probability distribution becomes closer to a gaussian distribution

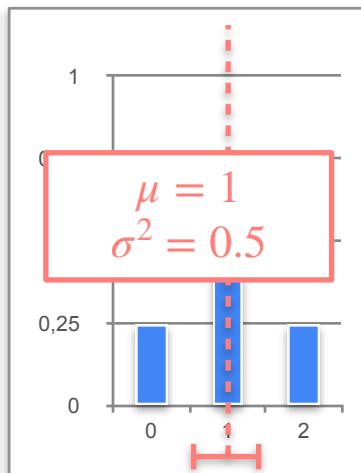
# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

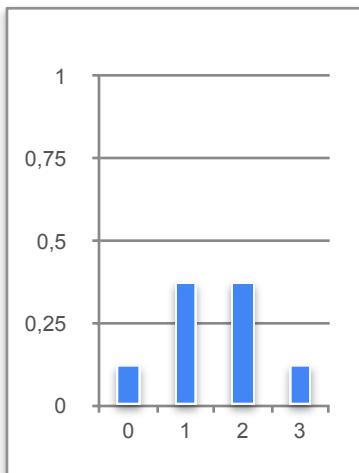
$$\sigma^2 = np(1 - p)$$



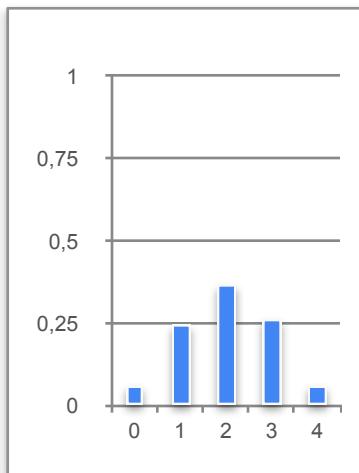
$n = 1$



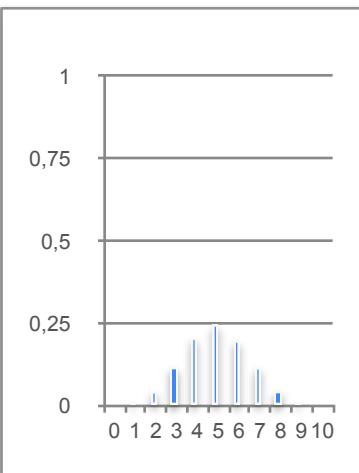
$n = 2$



$n = 3$



$n = 4$



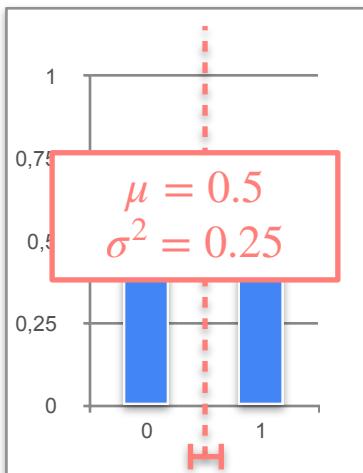
$n = 10$

As  $n$  increases, the probability distribution becomes closer to a gaussian distribution

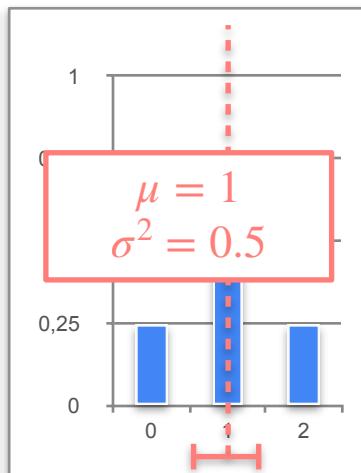
# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

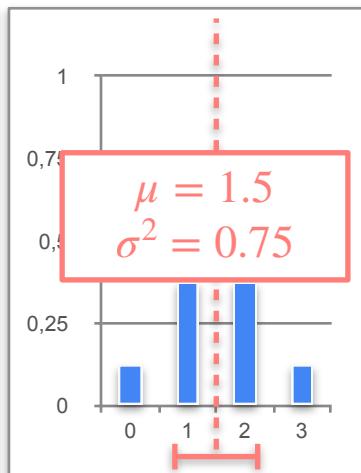
$$\sigma^2 = np(1 - p)$$



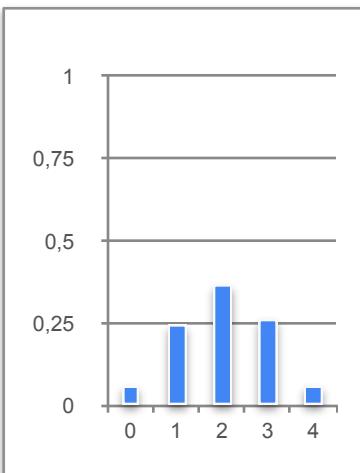
$n = 1$



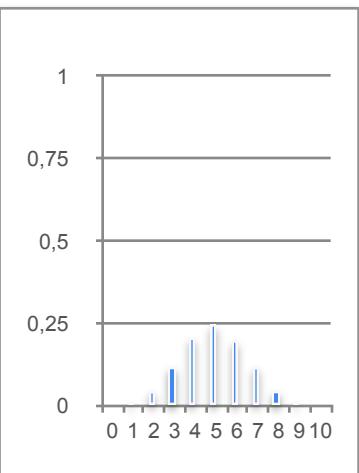
$n = 2$



$n = 3$



$n = 4$



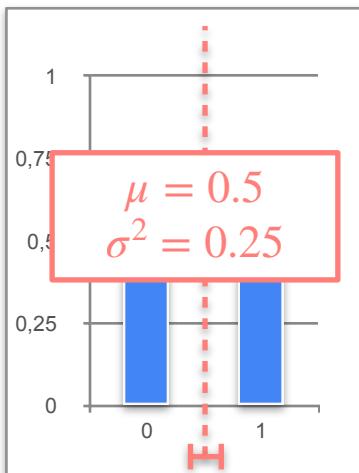
$n = 10$

As  $n$  increases, the probability distribution becomes closer to a gaussian distribution

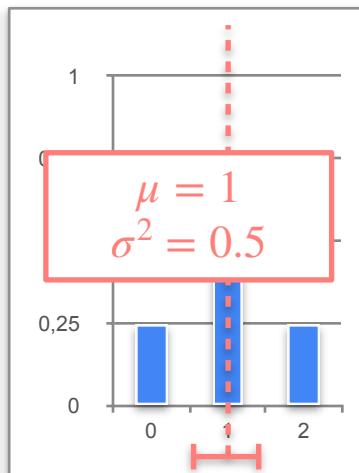
# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

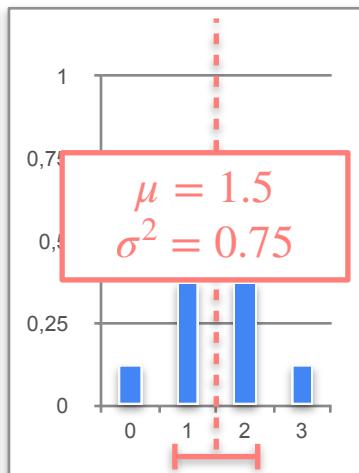
$$\sigma^2 = np(1 - p)$$



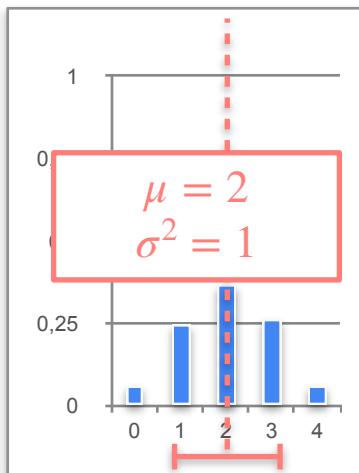
$n = 1$



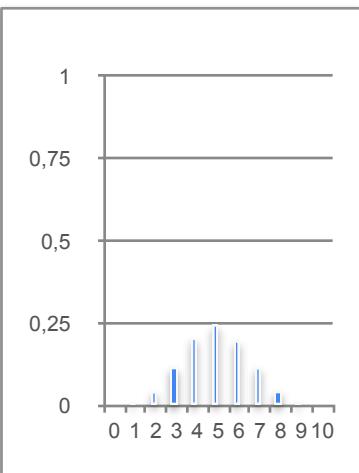
$n = 2$



$n = 3$



$n = 4$



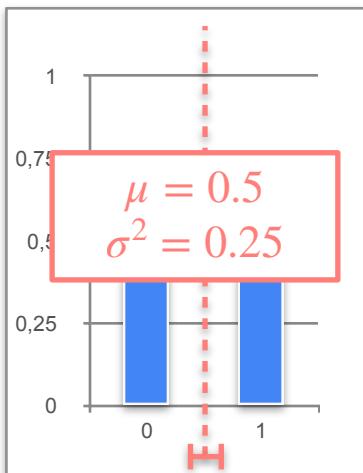
$n = 10$

As  $n$  increases, the probability distribution becomes closer to a gaussian distribution

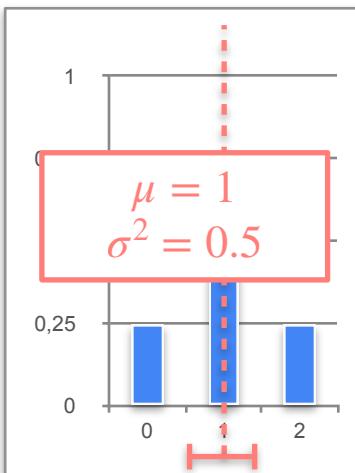
# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

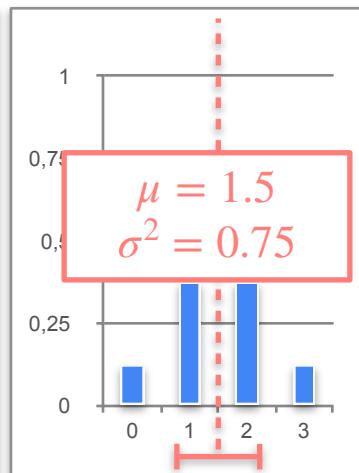
$$\sigma^2 = np(1 - p)$$



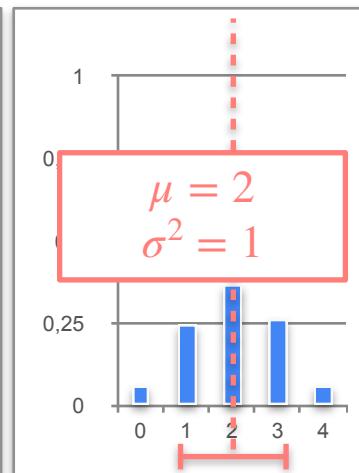
$n = 1$



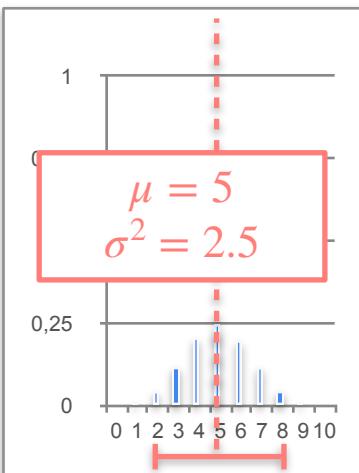
$n = 2$



$n = 3$



$n = 4$



$n = 10$

As  $n$  increases, the probability distribution becomes closer to a gaussian distribution

# Central Limit Theorem (CLT) - Example 1

$$\begin{aligned}\mu &= 0.5 \\ \sigma^2 &= 0.25\end{aligned}$$

$$\begin{aligned}\mu &= 1 \\ \sigma^2 &= 0.5\end{aligned}$$

$$\begin{aligned}\mu &= 1.5 \\ \sigma^2 &= 0.75\end{aligned}$$

$$\begin{aligned}\mu &= 2 \\ \sigma^2 &= 1\end{aligned}$$

$$\begin{aligned}\mu &= 5 \\ \sigma^2 &= 2.5\end{aligned}$$

# Central Limit Theorem (CLT) - Example 1

$$\begin{aligned}\mu &= 0.5 \\ \sigma^2 &= 0.25\end{aligned}$$

$$\begin{aligned}\mu &= 1 \\ \sigma^2 &= 0.5\end{aligned}$$

$$\begin{aligned}\mu &= 1.5 \\ \sigma^2 &= 0.75\end{aligned}$$

$$\begin{aligned}\mu &= 2 \\ \sigma^2 &= 1\end{aligned}$$

$$\begin{aligned}\mu &= 5 \\ \sigma^2 &= 2.5\end{aligned}$$

as  $n$  become sufficiently large we will get a normal distribution with parameters

# Central Limit Theorem (CLT) - Example 1

$$\begin{aligned}\mu &= 0.5 \\ \sigma^2 &= 0.25\end{aligned}$$

$$\begin{aligned}\mu &= 1 \\ \sigma^2 &= 0.5\end{aligned}$$

$$\begin{aligned}\mu &= 1.5 \\ \sigma^2 &= 0.75\end{aligned}$$

$$\begin{aligned}\mu &= 2 \\ \sigma^2 &= 1\end{aligned}$$

$$\begin{aligned}\mu &= 5 \\ \sigma^2 &= 2.5\end{aligned}$$

as  $n$  become sufficiently large we will get a normal distribution with parameters

$$\mu = np$$

# Central Limit Theorem (CLT) - Example 1

$$\begin{aligned}\mu &= 0.5 \\ \sigma^2 &= 0.25\end{aligned}$$

$$\begin{aligned}\mu &= 1 \\ \sigma^2 &= 0.5\end{aligned}$$

$$\begin{aligned}\mu &= 1.5 \\ \sigma^2 &= 0.75\end{aligned}$$

$$\begin{aligned}\mu &= 2 \\ \sigma^2 &= 1\end{aligned}$$

$$\begin{aligned}\mu &= 5 \\ \sigma^2 &= 2.5\end{aligned}$$

as  $n$  become sufficiently large we will get a normal distribution with parameters

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

$$\begin{aligned}\mu &= 0.5 \\ \sigma^2 &= 0.25\end{aligned}$$

$$\begin{aligned}\mu &= 1 \\ \sigma^2 &= 0.5\end{aligned}$$

$$\begin{aligned}\mu &= 1.5 \\ \sigma^2 &= 0.75\end{aligned}$$

$$\begin{aligned}\mu &= 2 \\ \sigma^2 &= 1\end{aligned}$$

$$\begin{aligned}\mu &= 5 \\ \sigma^2 &= 2.5\end{aligned}$$

as  $n$  become sufficiently large we will get a normal distribution with parameters

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

# Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

$$\begin{aligned}\mu &= 0.5 \\ \sigma^2 &= 0.25\end{aligned}$$

$$\begin{aligned}\mu &= 1 \\ \sigma^2 &= 0.5\end{aligned}$$

$$\begin{aligned}\mu &= 1.5 \\ \sigma^2 &= 0.75\end{aligned}$$

$$\begin{aligned}\mu &= 2 \\ \sigma^2 &= 1\end{aligned}$$

$$\begin{aligned}\mu &= 5 \\ \sigma^2 &= 2.5\end{aligned}$$

as  $n$  become sufficiently large we will get a normal distribution with parameters

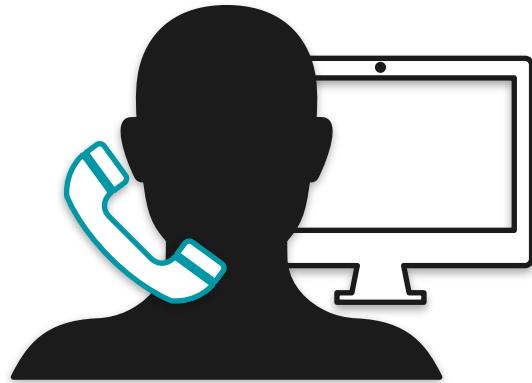
$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

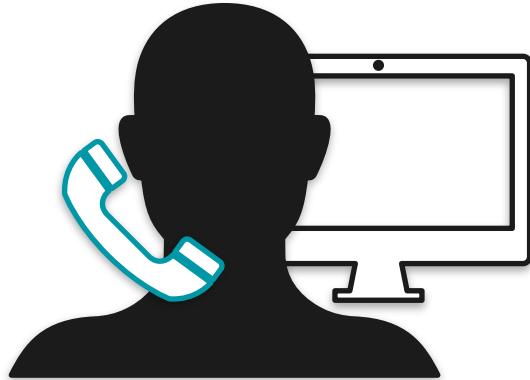
# Video 4b: Central Limit Theorem

- Example 2: Continuous Random Variable (Sample mean of a sampling distribution)

# Uniform Distribution: Motivation

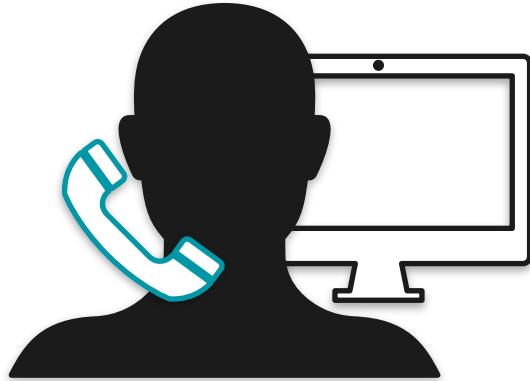


# Uniform Distribution: Motivation



You're calling a tech support line. They can answer any time between zero and 15 minutes and if they don't answer in this time, the line is disconnected.

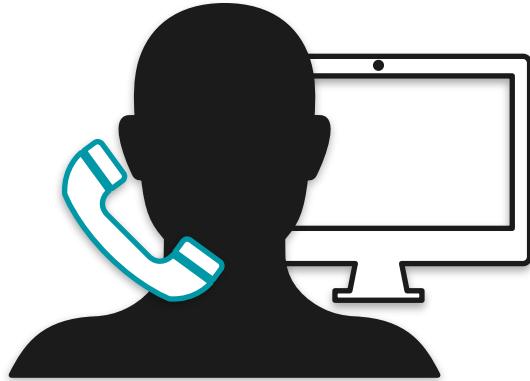
# Uniform Distribution: Motivation



You're calling a tech support line. They can answer any time between zero and 15 minutes and if they don't answer in this time, the line is disconnected.

$X$  = "Wait time for a called to be answered "

# Uniform Distribution: Motivation



You're calling a tech support line. They can answer any time between zero and 15 minutes and if they don't answer in this time, the line is disconnected.

$X$  = "Wait time for a called to be answered "

$$X \sim \mathcal{U}(0,15)$$

# Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

# Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

$$n = 2 \quad Y_2 = \frac{X_1 + X_2}{2}$$

# Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

$$n = 2 \quad Y_2 = \frac{X_1 + X_2}{2}$$

$$n = 3 \quad Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

⋮

⋮

# Central Limit Theorem (CLT) - Example 2

$$n = 1$$

$$Y_1 = \frac{X_1}{1}$$

Record the average of all  $n$  experiments

$$n = 2$$

$$Y_2 = \frac{X_1 + X_2}{2}$$

$$n = 3$$

$$Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

⋮

⋮

# Central Limit Theorem (CLT) - Example 2

$$n = 1$$

$$Y_1 = \frac{X_1}{1}$$

Record the average of all  $n$  experiments

$$n = 2$$

$$Y_2 = \frac{X_1 + X_2}{2}$$

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$

$$n = 3$$

$$Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

⋮

⋮

# Central Limit Theorem (CLT) - Example 2

$$n = 1$$

$$Y_1 = \frac{X_1}{1}$$

Record the average of all  $n$  experiments

$$n = 2$$

$$Y_2 = \frac{X_1 + X_2}{2}$$

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$

$$n = 3$$

$$Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

⋮

⋮

Can we say anything about the distribution of this average?

# Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

# Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

What happens to the distribution of these averages as  $n$  increases?

# Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

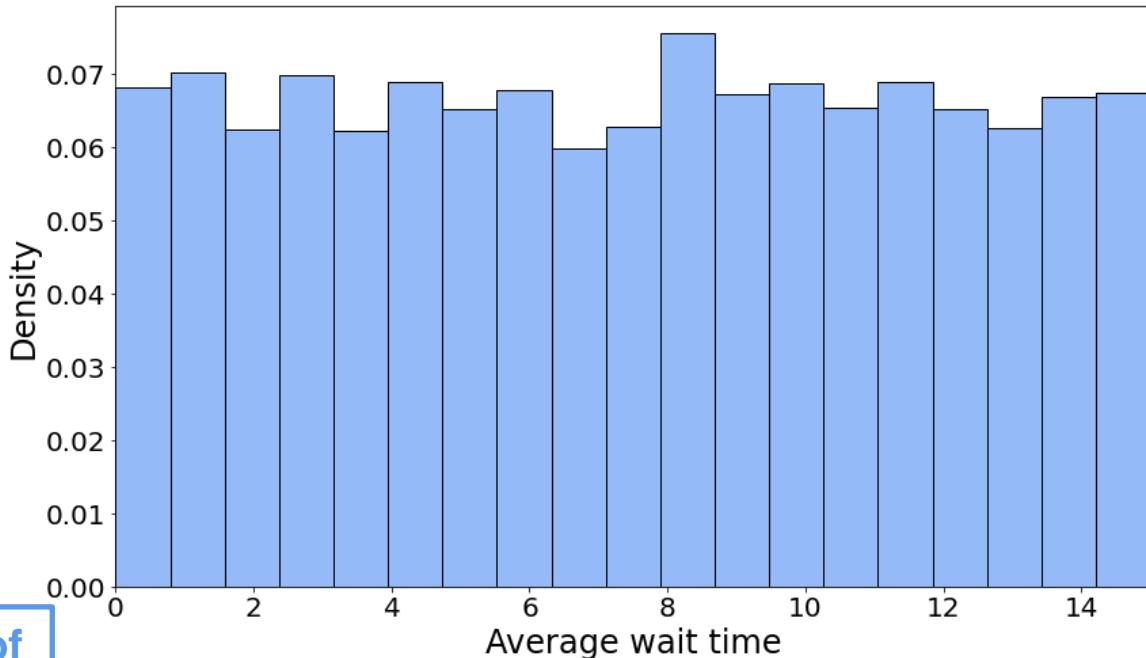
Create many samples of  $Y_1$  so you can get a pretty histogram

What happens to the distribution of these averages as  $n$  increases?

# Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

Create many samples of  $Y_1$  so you can get a pretty histogram

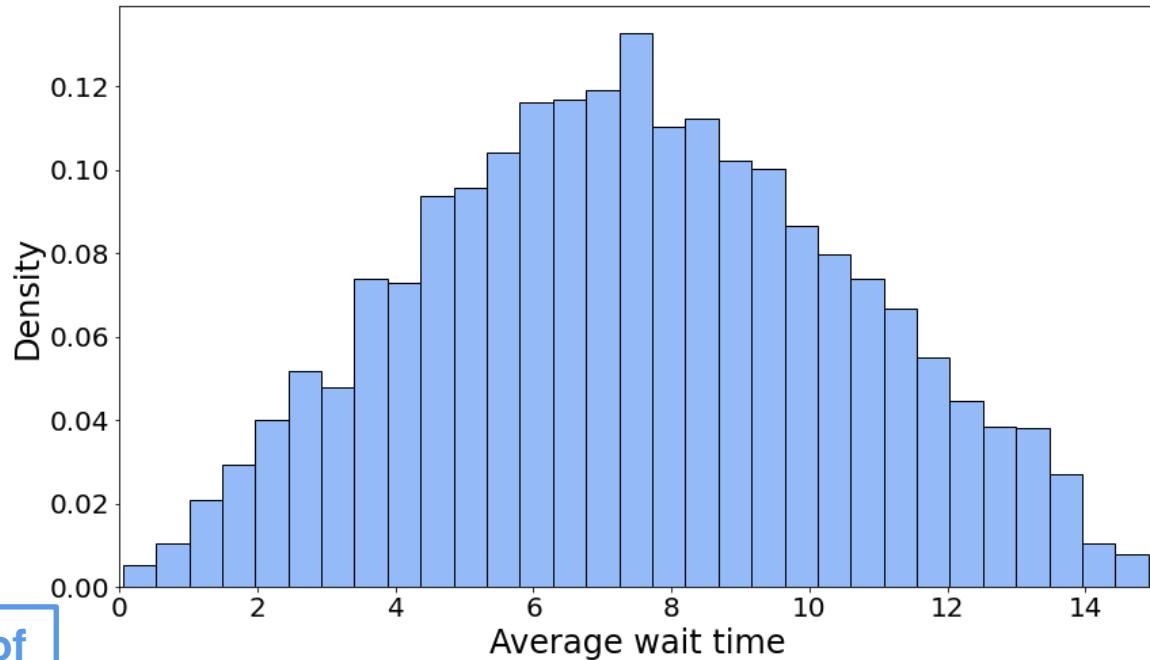


What happens to the distribution of these averages as  $n$  increases?

# Central Limit Theorem (CLT) - Example 2

$$n = 2 \quad Y_2 = \frac{X_1 + X_2}{2}$$

Create many samples of  $Y_2$  so you can get a pretty histogram

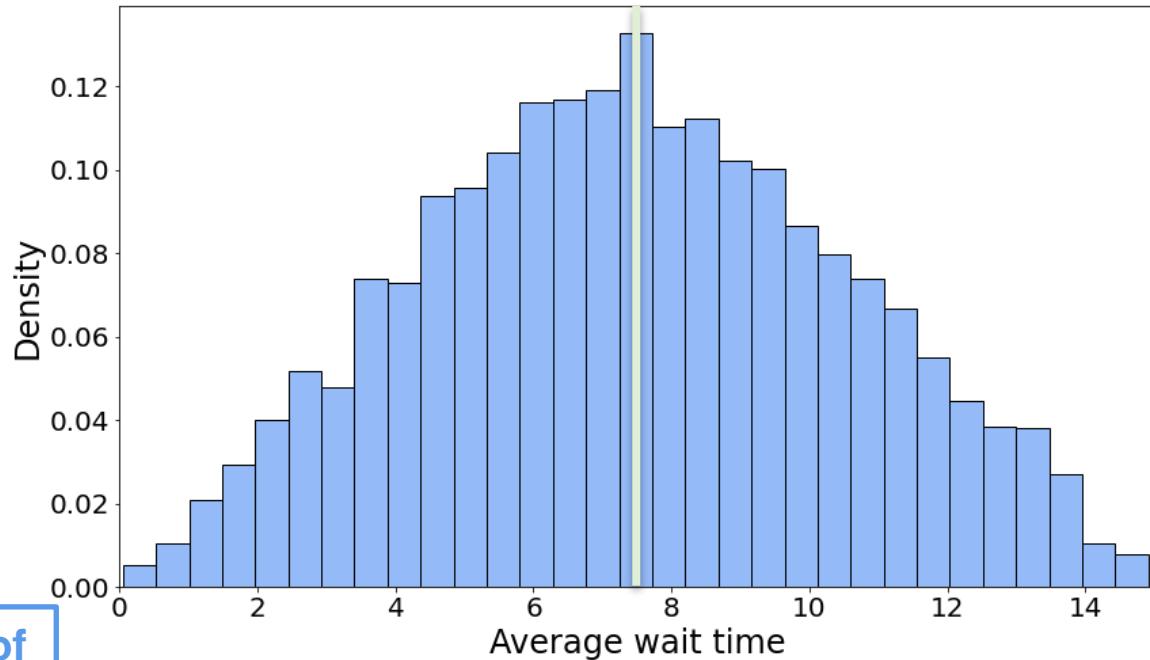


What happens to the distribution of these averages as  $n$  increases

# Central Limit Theorem (CLT) - Example 2

$$n = 2 \quad Y_2 = \frac{X_1 + X_2}{2}$$

Create many samples of  $Y_2$  so you can get a pretty histogram



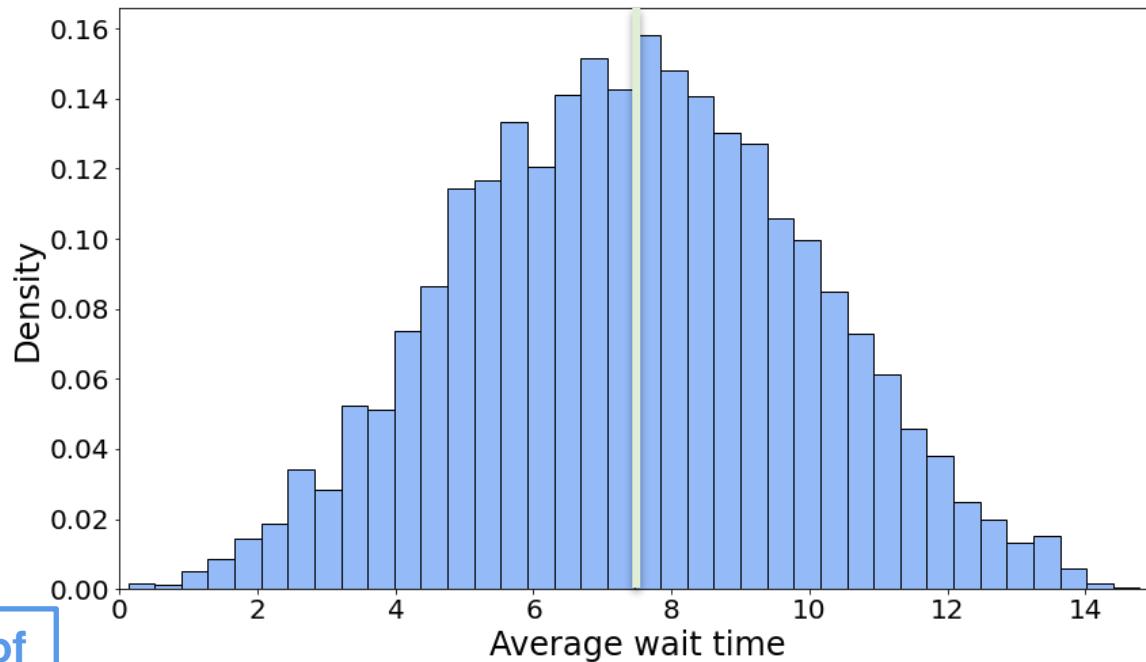
What happens to the distribution of these averages as  $n$  increases

# Central Limit Theorem (CLT) - Example 2

$$n = 3 \quad Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

Create many samples of  $Y_3$  so you can get a pretty histogram

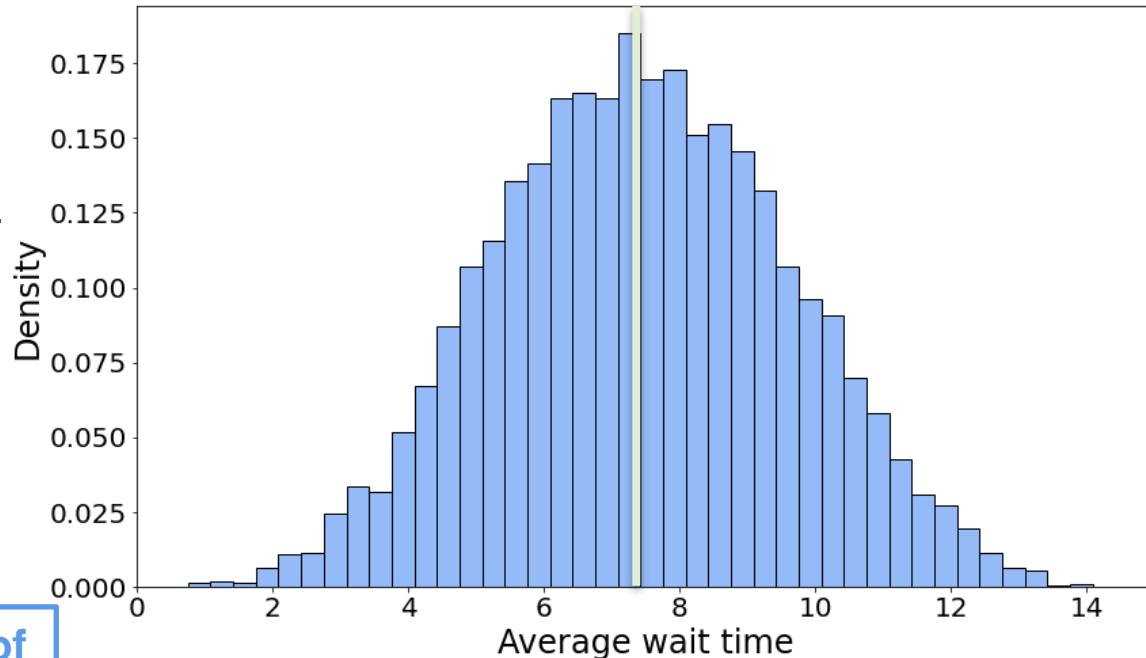
What happens to the distribution of these averages as  $n$  increases



# Central Limit Theorem (CLT) - Example 2

$$n = 4 \quad Y_4 = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

Create many samples of  $Y_4$  so you can get a pretty histogram

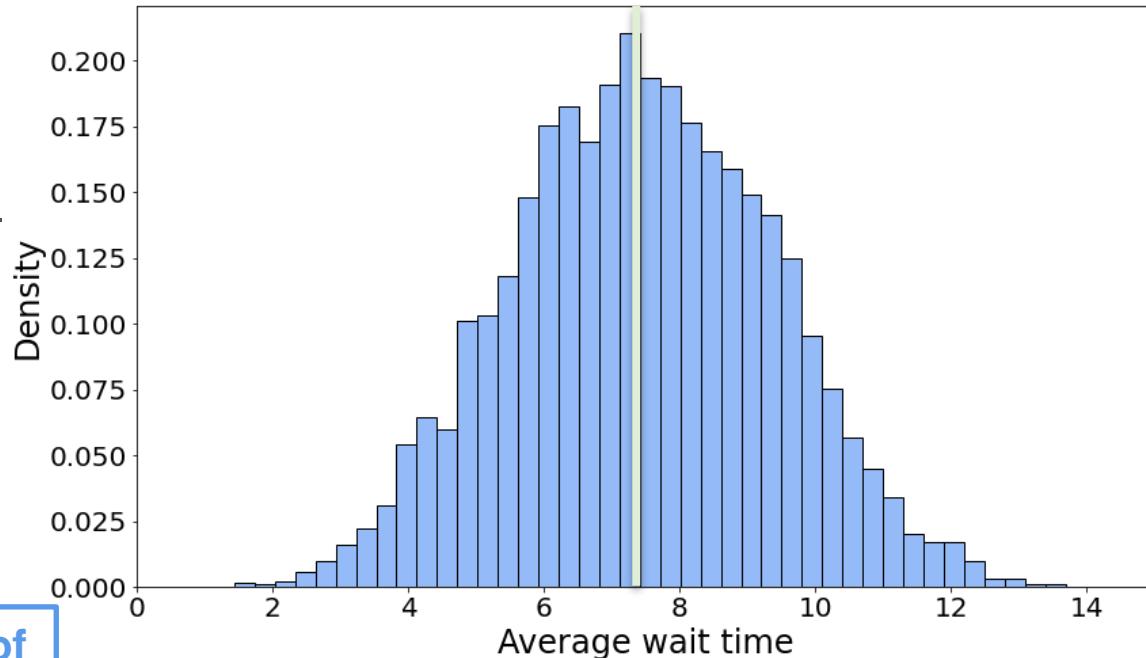


What happens to the distribution of these averages as  $n$  increases

# Central Limit Theorem (CLT) - Example 2

$$n = 5 \quad Y_5 = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{5}$$

Create many samples of  $Y_5$  so you can get a pretty histogram

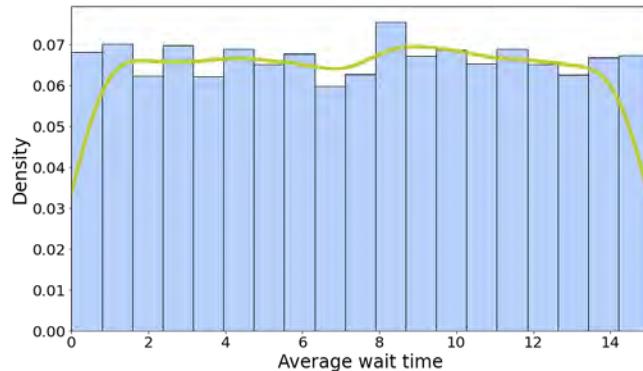


What happens to the distribution of these averages as  $n$  increases

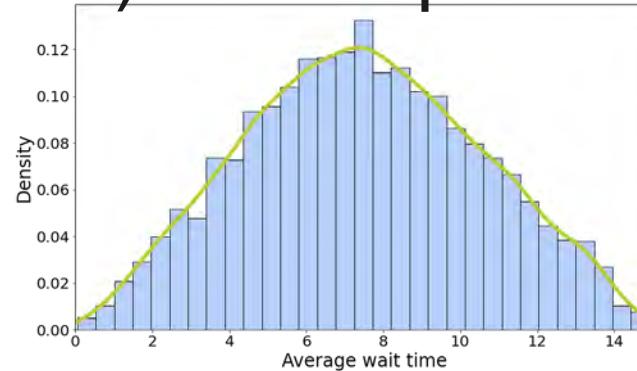
# Central Limit Theorem (CLT) - Example 2

# Central Limit Theorem (CLT) - Example 2

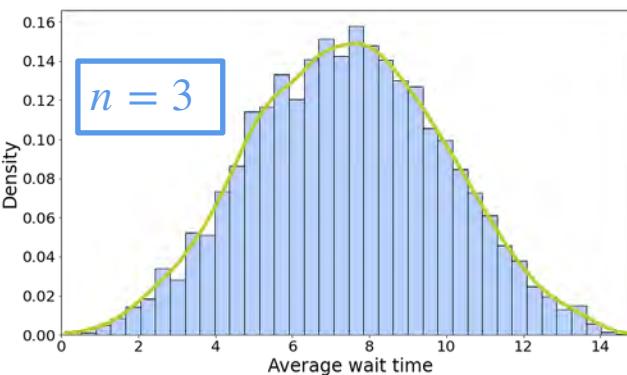
$n = 1$



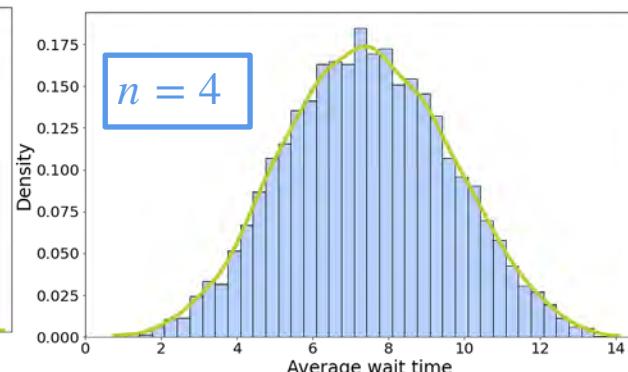
$n = 2$



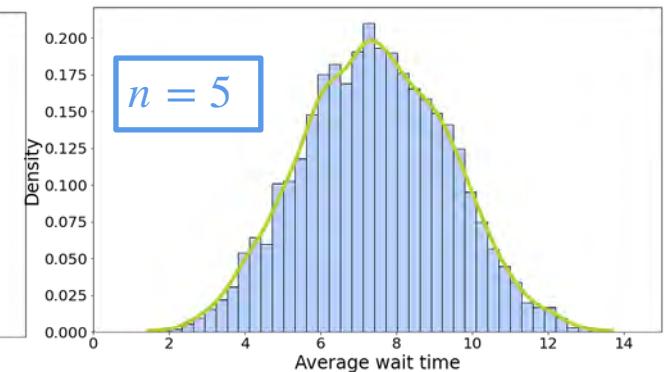
$n = 3$



$n = 4$



$n = 5$



# Central Limit Theorem (CLT) - Example 2

$$= \frac{1}{n^2}$$

# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right]$$

$$= \frac{1}{n^2}$$

# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i]$$

$$= \frac{1}{n^2}$$

# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n \mathbb{E}[X] = \mathbb{E}[X]$$

$$= \frac{1}{n^2}$$

# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n \mathbb{E}[X] = \mathbb{E}[X] = 7.5$$

$$= \frac{1}{n^2}$$

# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n \mathbb{E}[X] = \mathbb{E}[X] = 7.5$$

$$Var[Y_n] = Var \left( \frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n^2}$$

# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n \mathbb{E}[X] = \mathbb{E}[X] = 7.5$$

$$Var[Y_n] = Var \left( \frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i)$$

# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n \mathbb{E}[X] = \mathbb{E}[X] = 7.5$$

$$\begin{aligned} Var[Y_n] &= Var \left( \frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i) \\ &= \frac{1}{n^2} n Var(X) = \frac{Var(X)}{n} \end{aligned}$$

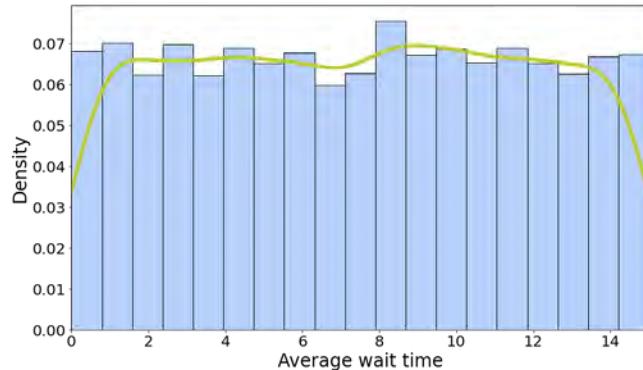
# Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n \mathbb{E}[X] = \mathbb{E}[X] = 7.5$$

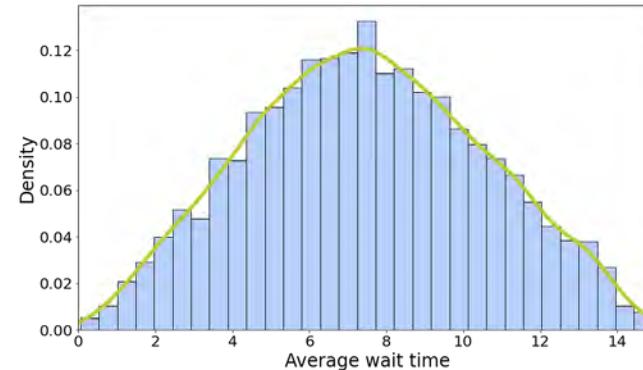
$$\begin{aligned} Var[Y_n] &= Var \left( \frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i) \\ &= \frac{1}{n^2} n Var(X) = \frac{Var(X)}{n} = \frac{18.75}{n} \end{aligned}$$

# Central Limit Theorem (CLT) - Example 2

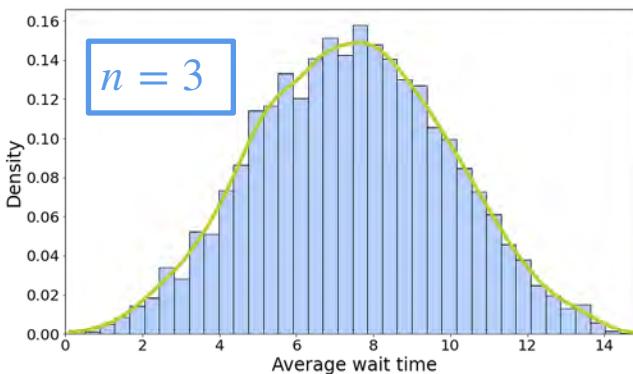
$n = 1$



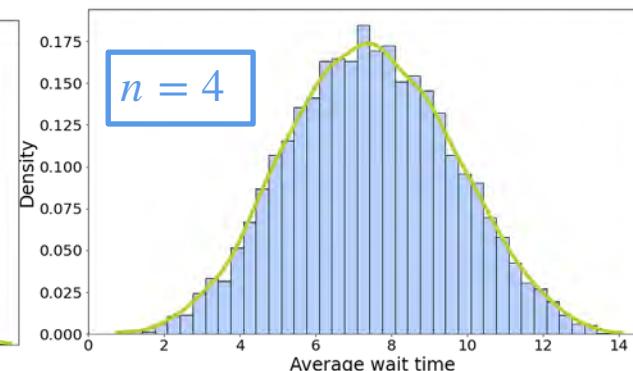
$n = 2$



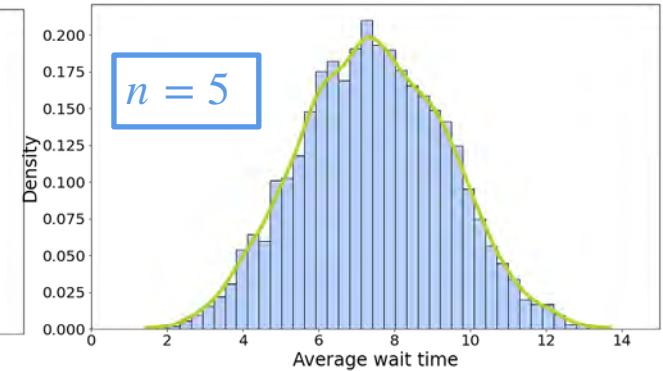
$n = 3$



$n = 4$

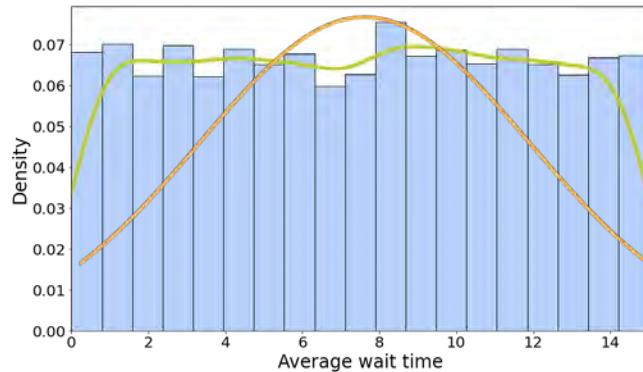


$n = 5$

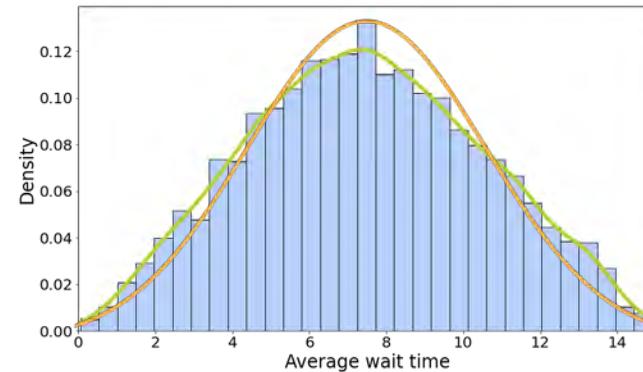


# Central Limit Theorem (CLT) - Example 2

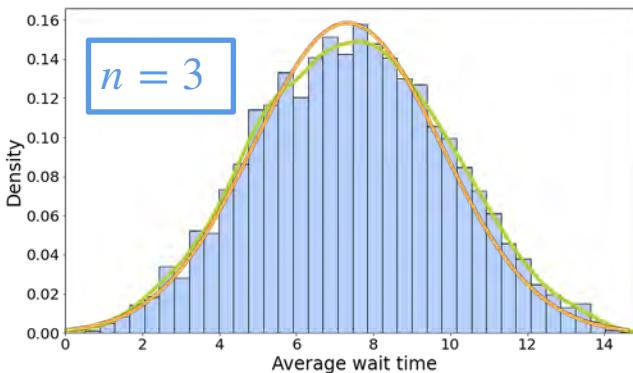
$n = 1$



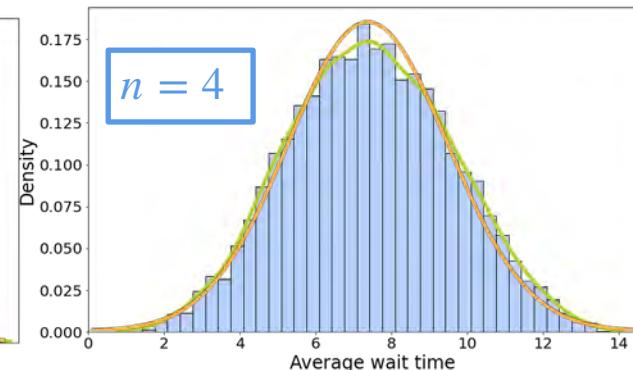
$n = 2$



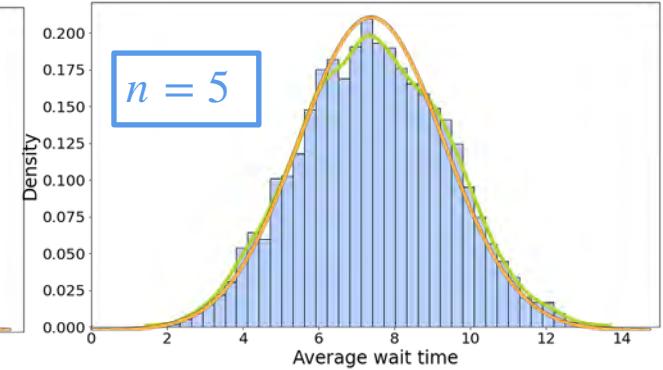
$n = 3$



$n = 4$

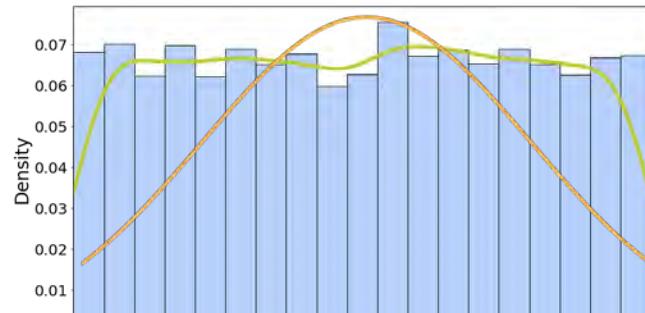


$n = 5$

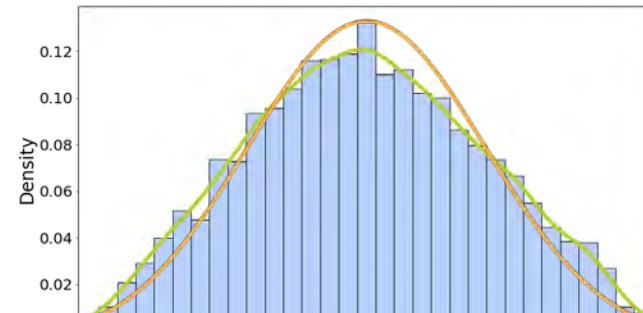


# Central Limit Theorem (CLT) - Example 2

$n = 1$

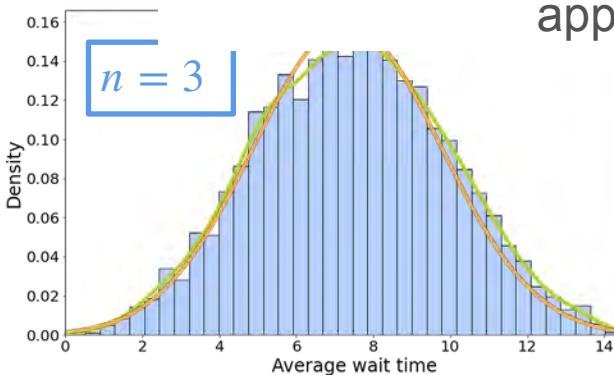


$n = 2$

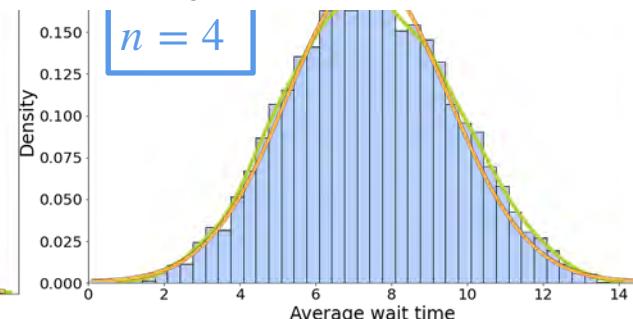


When you average a large enough number of variables, the distribution will approximately follow a normal distribution

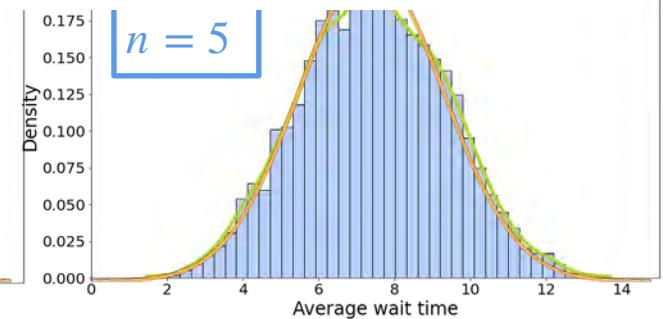
$n = 3$



$n = 4$

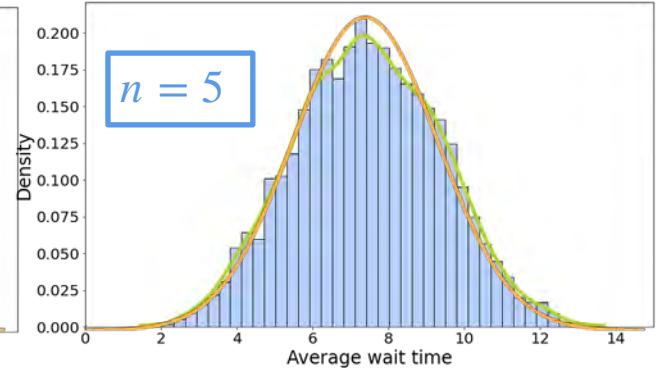
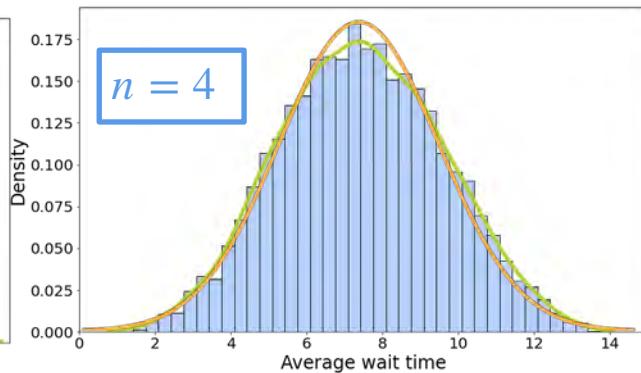
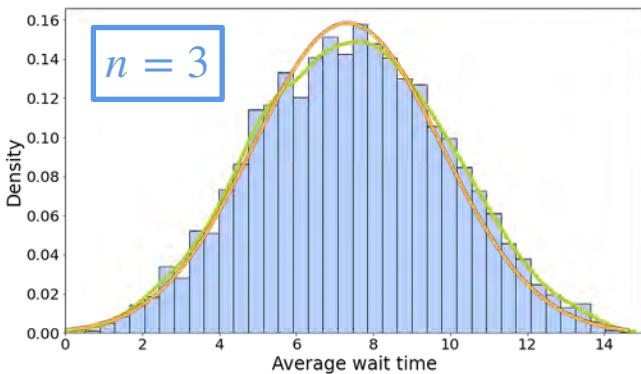


$n = 5$



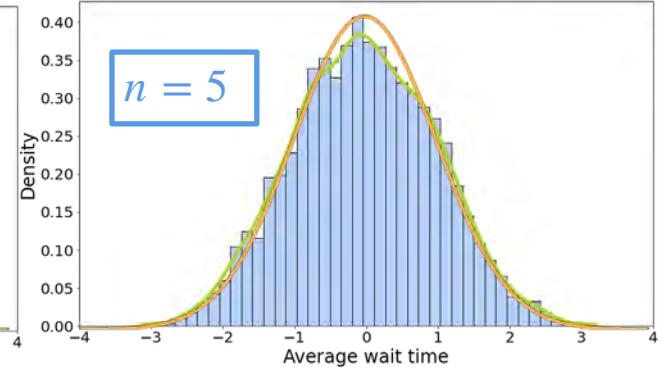
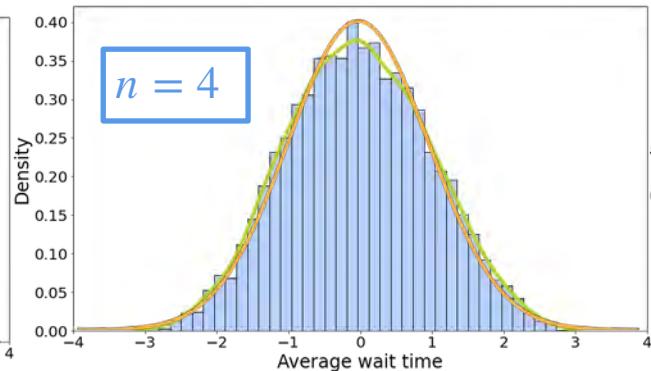
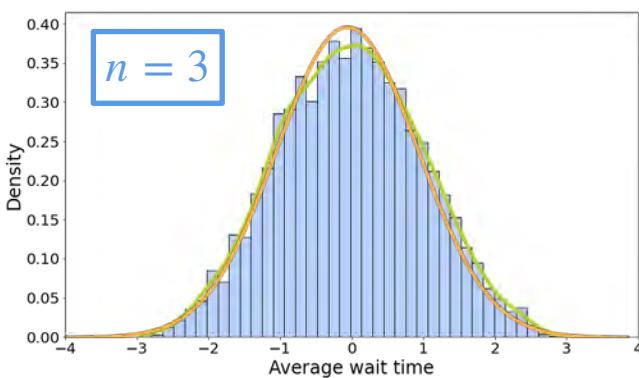
# Central Limit Theorem (CLT) - Example 2

$$\frac{Y_n - 7.5}{\sqrt{18.75/n}}$$



# Central Limit Theorem (CLT) - Example 2

$$\frac{Y_n - 7.5}{\sqrt{18.75/n}} \xrightarrow{n \uparrow} \mathcal{N}(0,1)$$



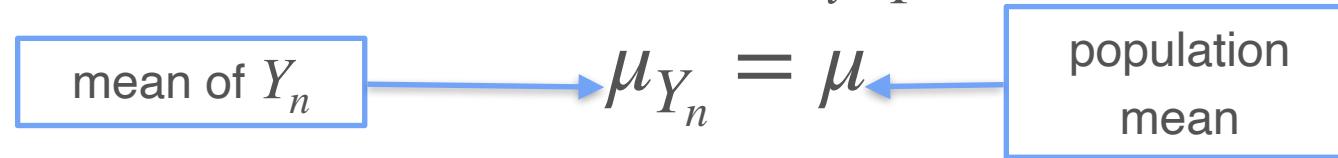
# Central Limit Theorem (CLT) - Example 2

# Central Limit Theorem (CLT) - Example 2

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$

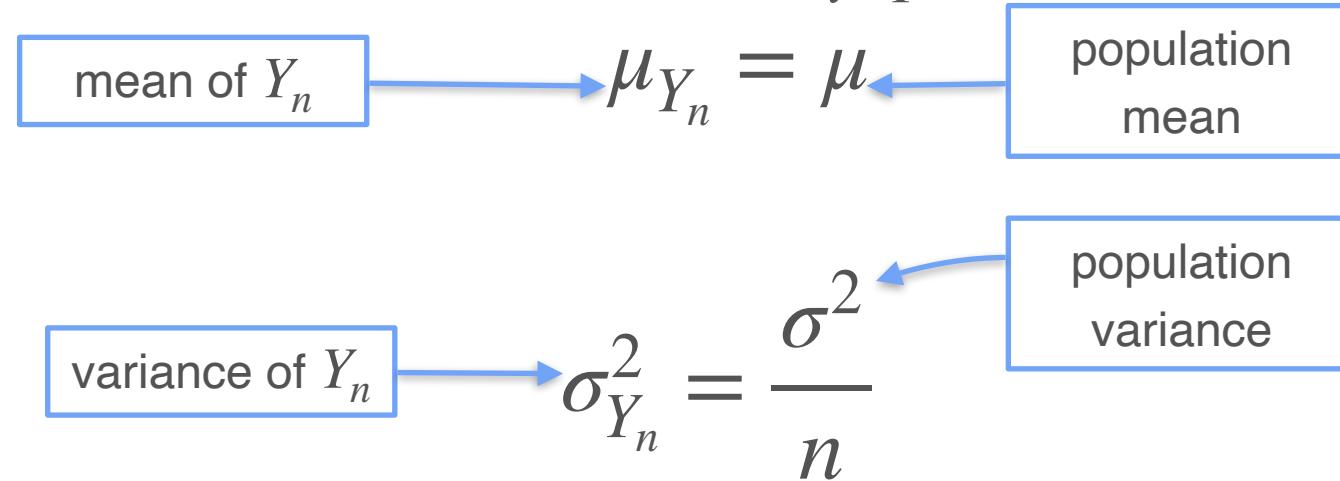
# Central Limit Theorem (CLT) - Example 2

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$



# Central Limit Theorem (CLT) - Example 2

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$



# Central Limit Theorem (CLT) - Formal Definition

# Central Limit Theorem (CLT) - Formal Definition

As  $n \rightarrow \infty$

$$\frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

# Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \rightarrow \infty \quad \frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

$$\text{As } n \rightarrow \infty \quad \frac{1}{n} \left( \frac{\sum_{i=1}^n X_i - \frac{1}{n} n \mathbb{E}[X]}{\sigma_X} \right) \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

# Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \rightarrow \infty \quad \frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

$$\text{As } n \rightarrow \infty \quad \frac{1}{n} \left( \frac{\sum_{i=1}^n X_i - \cancel{\frac{1}{n} n \mathbb{E}[X]}}{\sigma_X} \right) \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

# Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \rightarrow \infty \quad \frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

$$\text{As } n \rightarrow \infty \quad \frac{1}{\cancel{\sqrt{n}}} \left( \frac{\sum_{i=1}^n X_i - \cancel{\frac{1}{n} n \mathbb{E}[X]}}{\sigma_X} \right) \cancel{\sqrt{n}} \sim \mathcal{N}(0, 1^2)$$

# Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \rightarrow \infty \quad \frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

As  $n \rightarrow \infty$

# Central Limit Theorem (CLT) - Formal Definition

As  $n \rightarrow \infty$

$$\frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

As  $n \rightarrow \infty$

$$\frac{\sum_{i=1}^n X_i - n\mathbb{E}[X]}{\sqrt{n}\sigma_X} \sim \mathcal{N}(0, 1^2)$$

# W3 Lesson 2



DeepLearning.AI

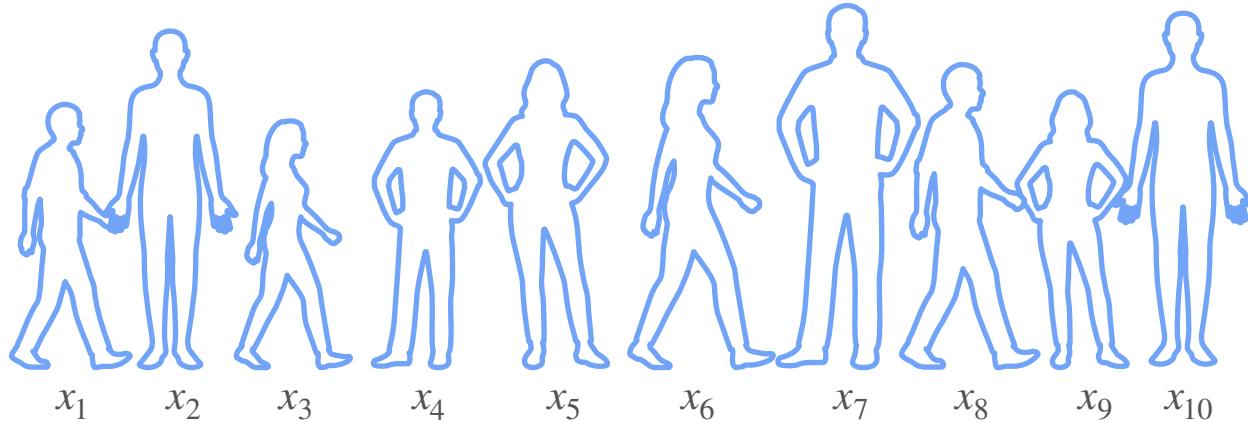
## Point Estimation

---

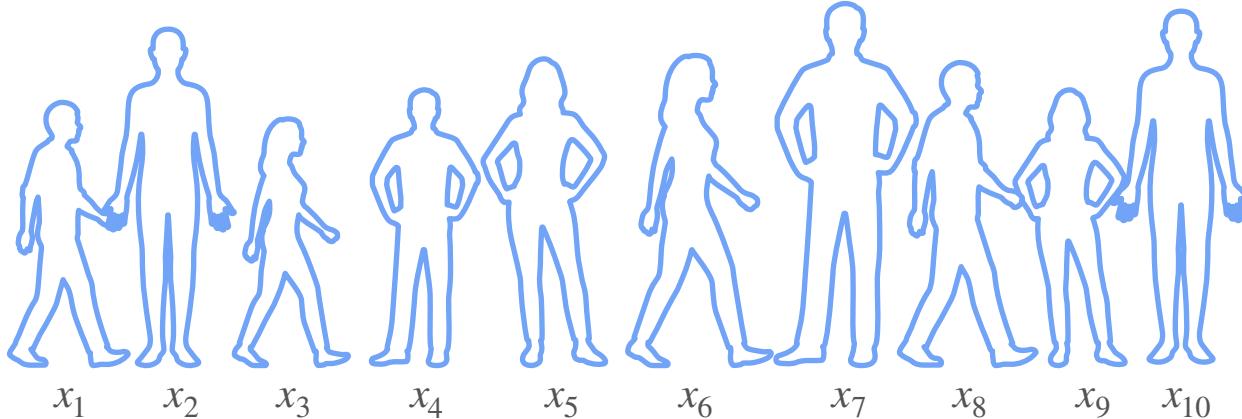
### What is Point Estimation?

# Point Estimates

# Point Estimates



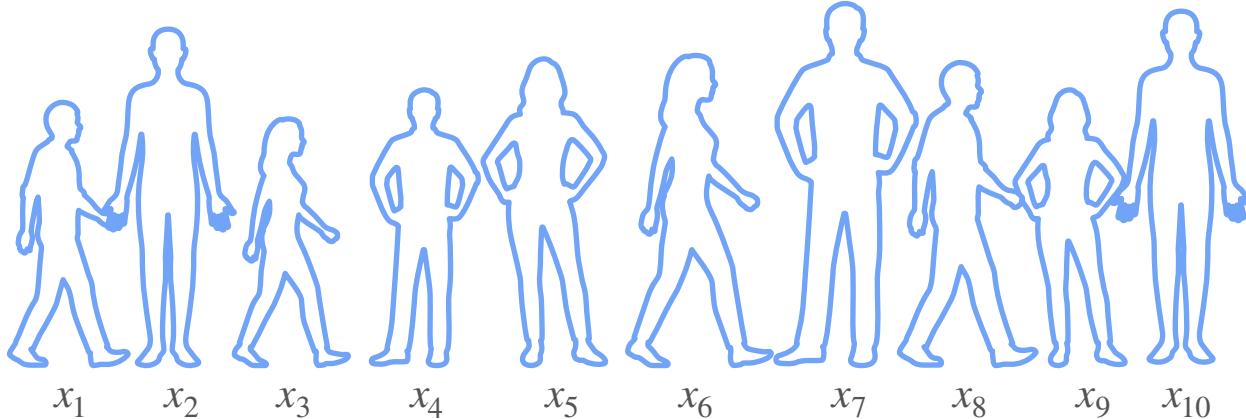
# Point Estimates



Mean of the population?

$$\mu \approx \frac{1}{10} \sum_{i=1}^{10} x_i = \bar{x}$$

# Point Estimates



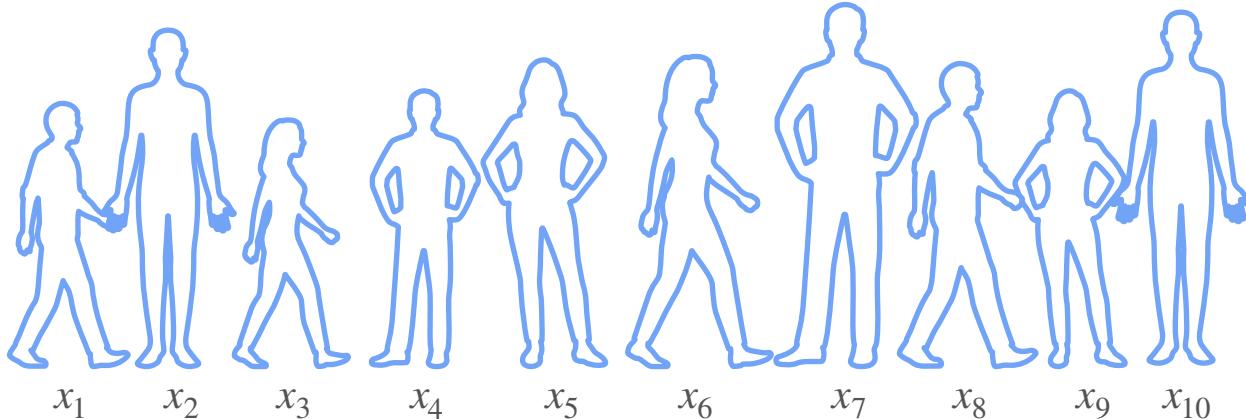
Mean of the population?

$$\mu \approx \frac{1}{10} \sum_{i=1}^{10} x_i = \bar{x}$$

Variance of the population?

$$\sigma^2 \approx \frac{1}{10 - 1} \sum_{i=1}^{10} (x_i - \bar{x})^2 = s^2$$

# Point Estimates



Mean of the population?

$$\mu \approx \frac{1}{10} \sum_{i=1}^{10} x_i = \bar{x}$$

Variance of the population?

$$\sigma^2 \approx \frac{1}{10 - 1} \sum_{i=1}^{10} (x_i - \bar{x})^2 = s^2$$

$\bar{x}$  and  $s^2$  are point estimates

# Point Estimates

# Point Estimates

A **point estimate** is a **single numerical value** based on **sample data** that is used to **approximate an unknown parameter** of a population or model parameter.

# Point Estimates

# Point Estimates



$\mathbf{P}(H) ?$

# Point Estimates



$P(H)$ ?



# Point Estimates



$P(H)$ ?

10 throws  
7 heads  
3 tails



# Point Estimates



$P(H)$ ?

10 throws  
7 heads  
3 tails

$$P(H) = \frac{7}{10}$$

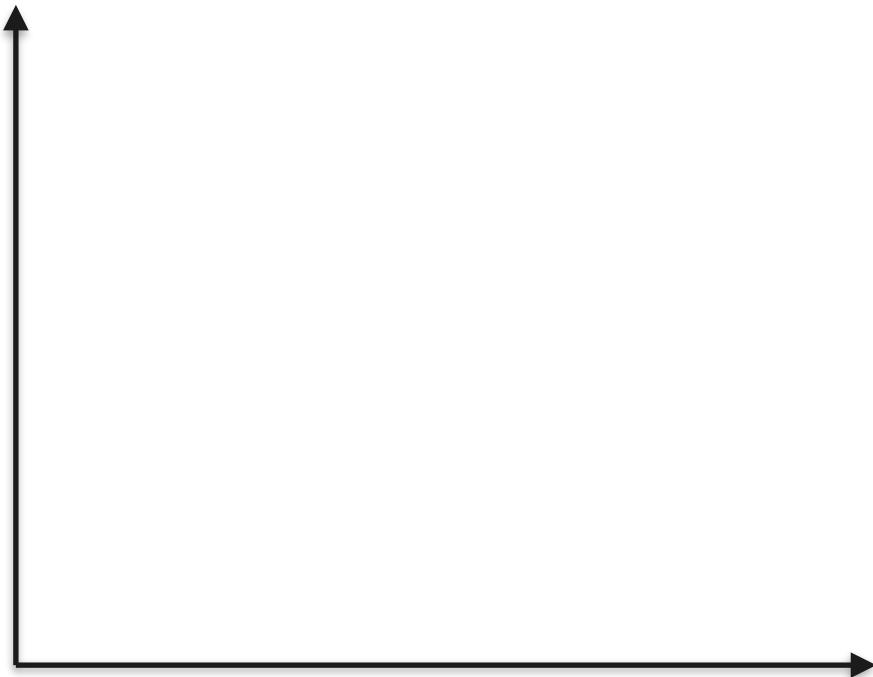


# Point Estimates

	$x$	$y$
0		
1		
2		
$\vdots$		
50		

# Point Estimates

	$x$	$y$
0		
1		
2		
$\vdots$		
50		

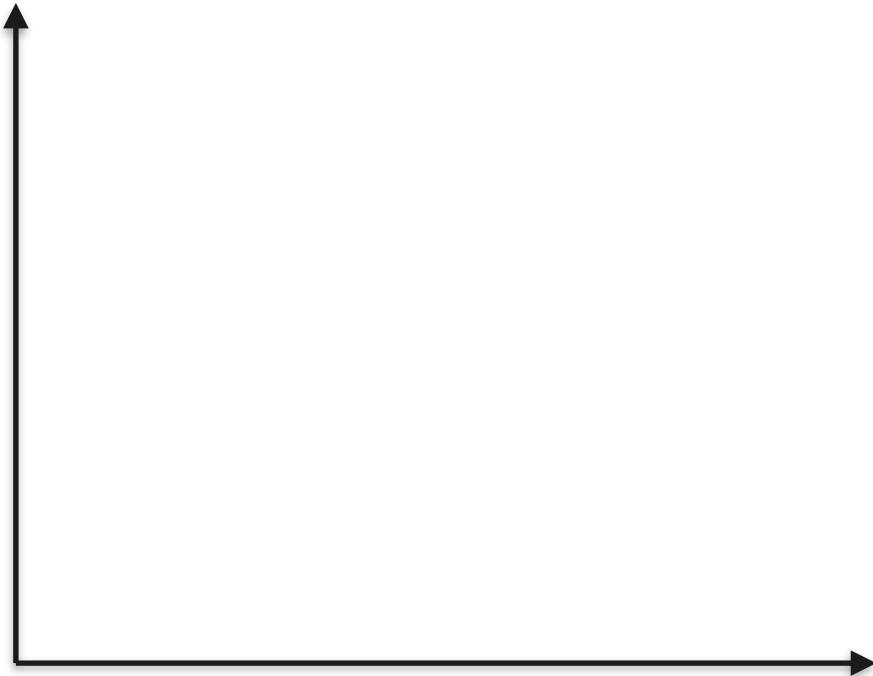


# Point Estimates

	$x$	$y$
0		
1		
2		
$\vdots$		
50		

$$y = \beta_0 + \beta_1 x$$

$$\beta_0, \beta_1 = ??$$

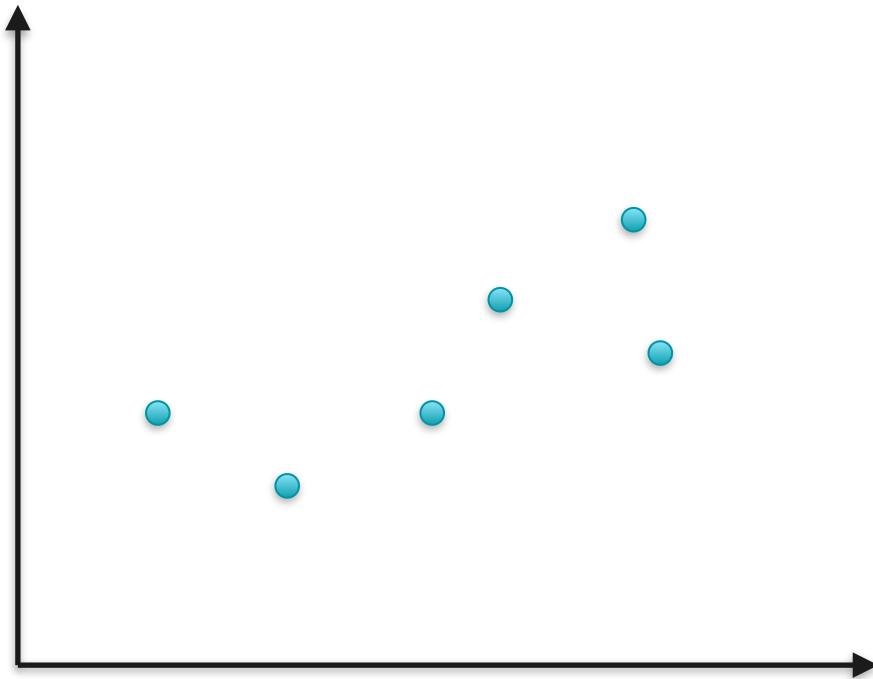


# Point Estimates

	$x$	$y$
0		
1		
2		
$\vdots$		
50		

$$y = \beta_0 + \beta_1 x$$

$$\beta_0, \beta_1 = ??$$

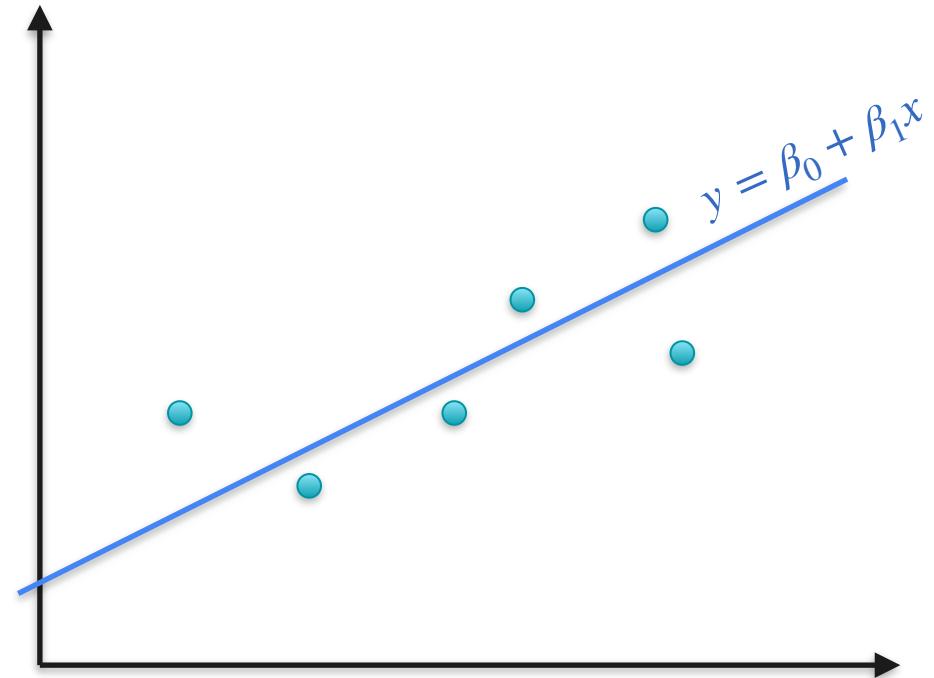


# Point Estimates

	$x$	$y$
0		
1		
2		
$\vdots$		
50		

$$y = \beta_0 + \beta_1 x$$

$$\beta_0, \beta_1 = ??$$





DeepLearning.AI

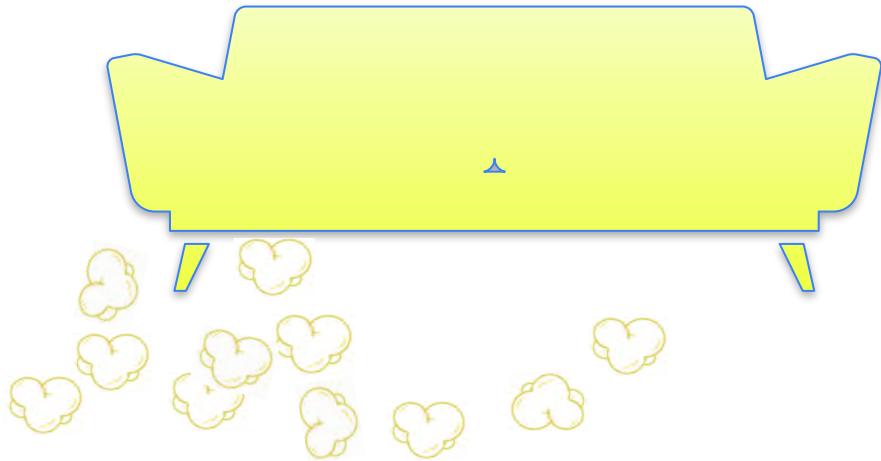
## Point Estimation

---

**Maximum Likelihood  
Estimation: Motivation**

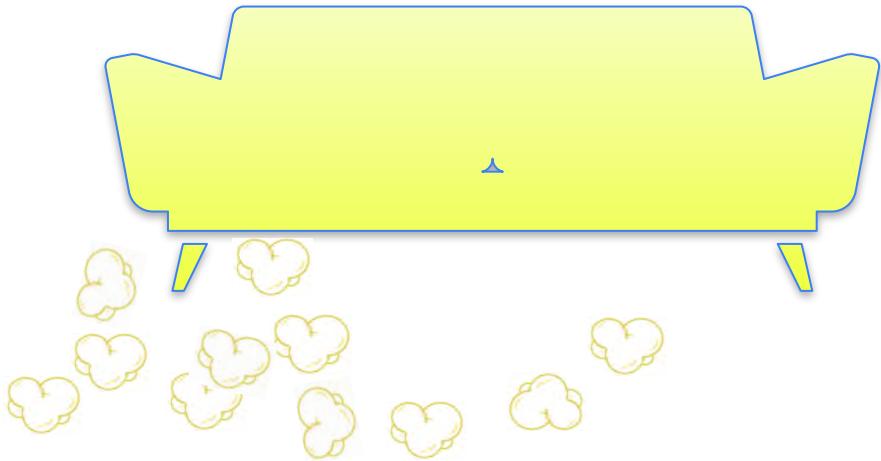
# There's Popcorn on the Floor. What Happened?

# There's Popcorn on the Floor. What Happened?

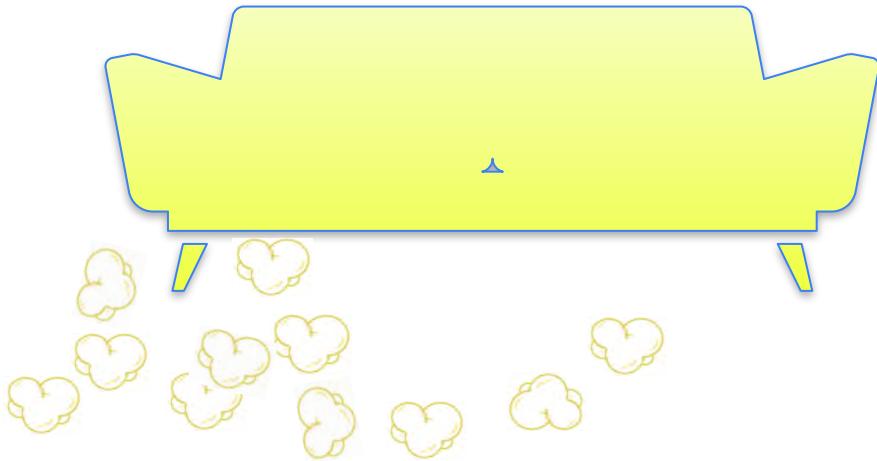


# There's Popcorn on the Floor. What Happened?

Movies



# There's Popcorn on the Floor. What Happened?



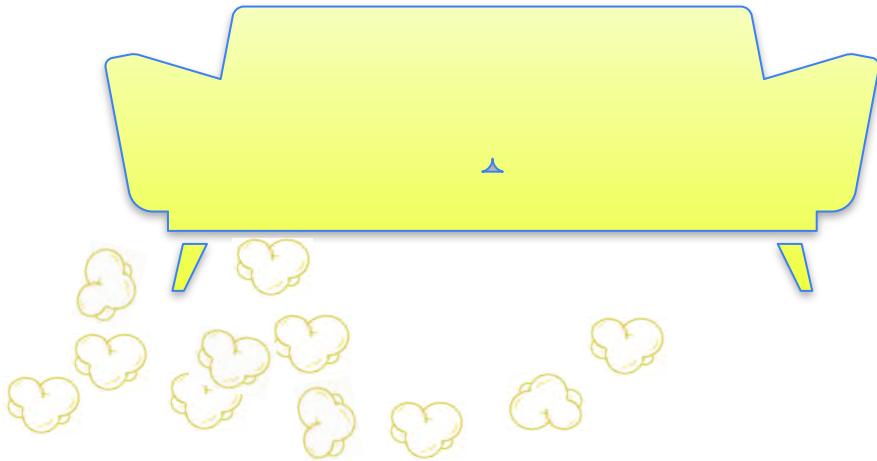
Movies



Board Games



# There's Popcorn on the Floor. What Happened?



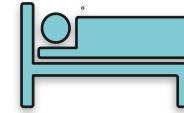
Movies



Board Games



Nap



# Quiz

What do you think happened?

- A. People were watching a movie
- B. People were playing boardgames
- C. People were taking a nap

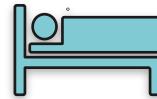
# There's Popcorn on the Floor. What Happened?



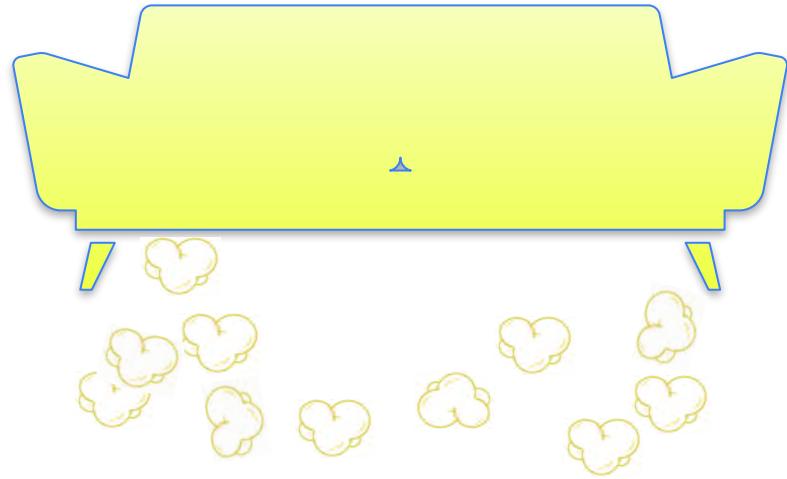
Movies



Board  
Games



Nap



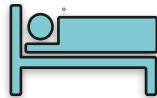
# There's Popcorn on the Floor. What Happened?



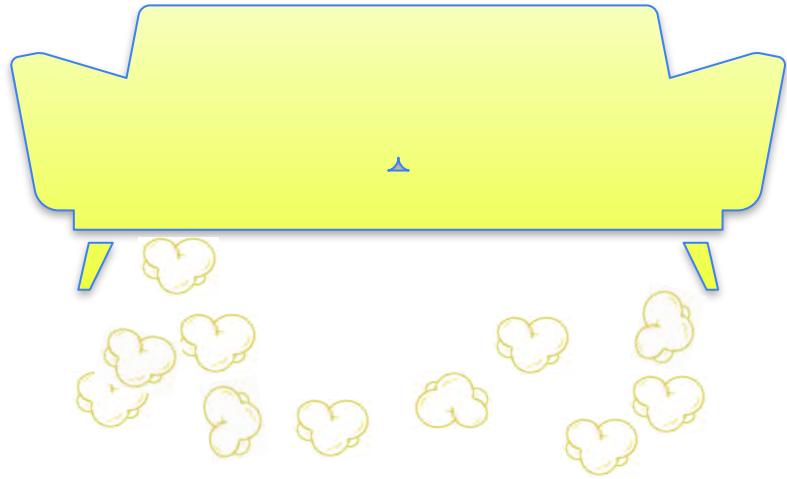
Movies *High*



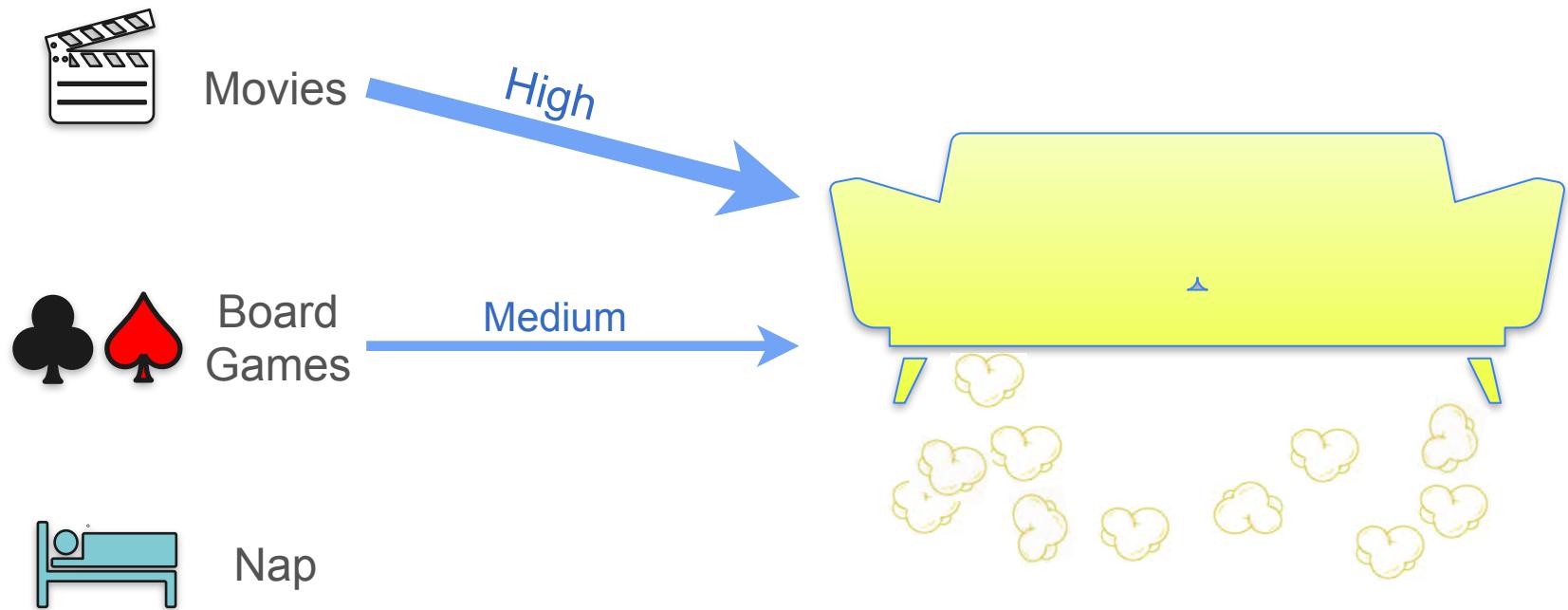
Board Games



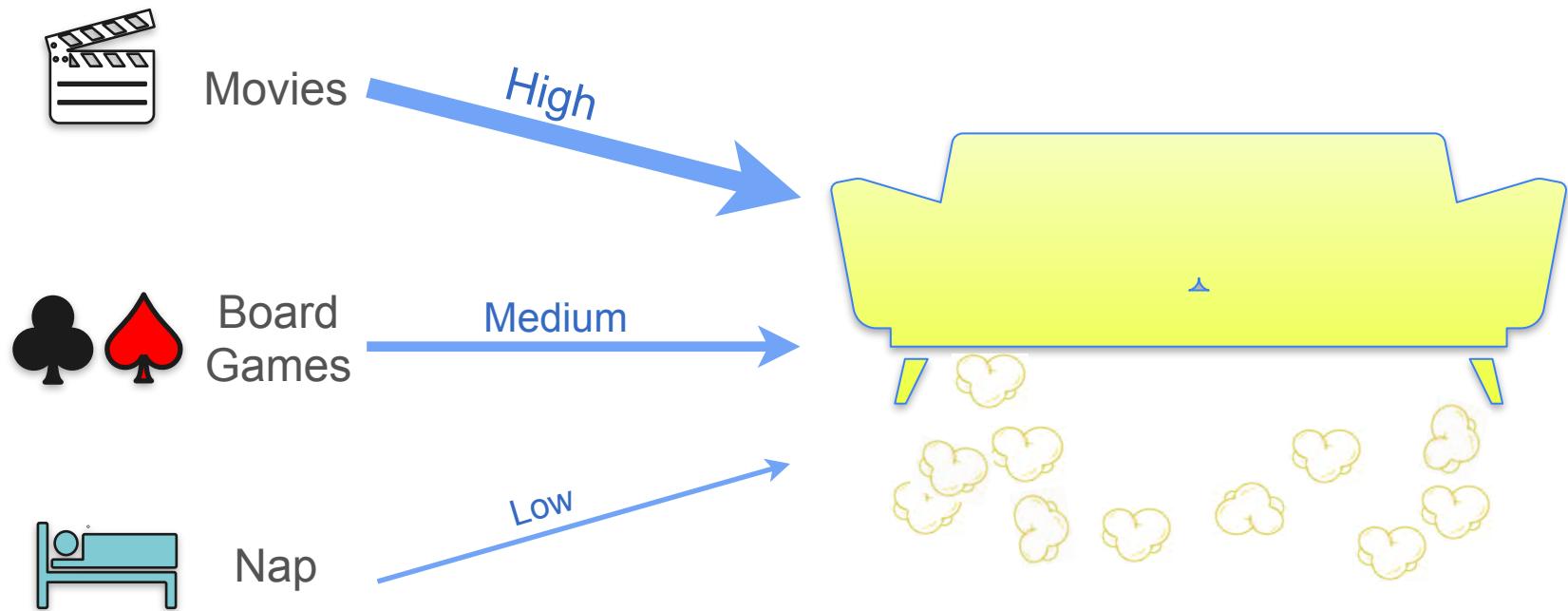
Nap



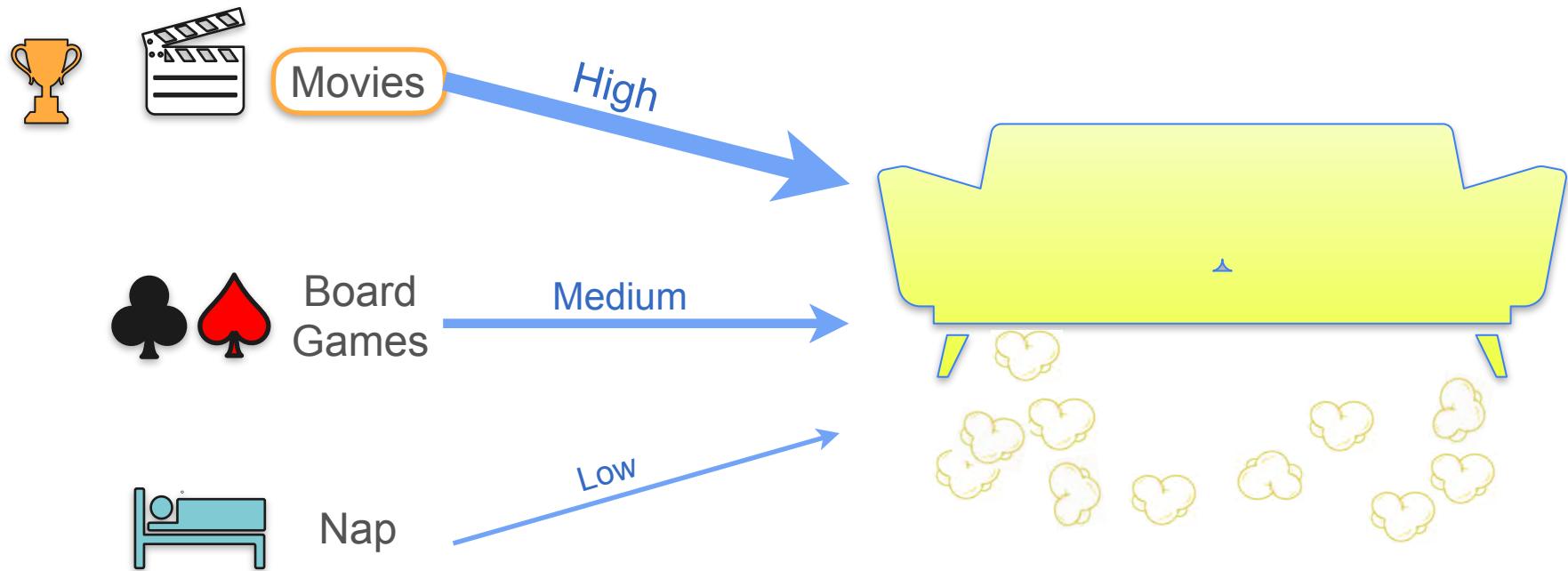
# There's Popcorn on the Floor. What Happened?



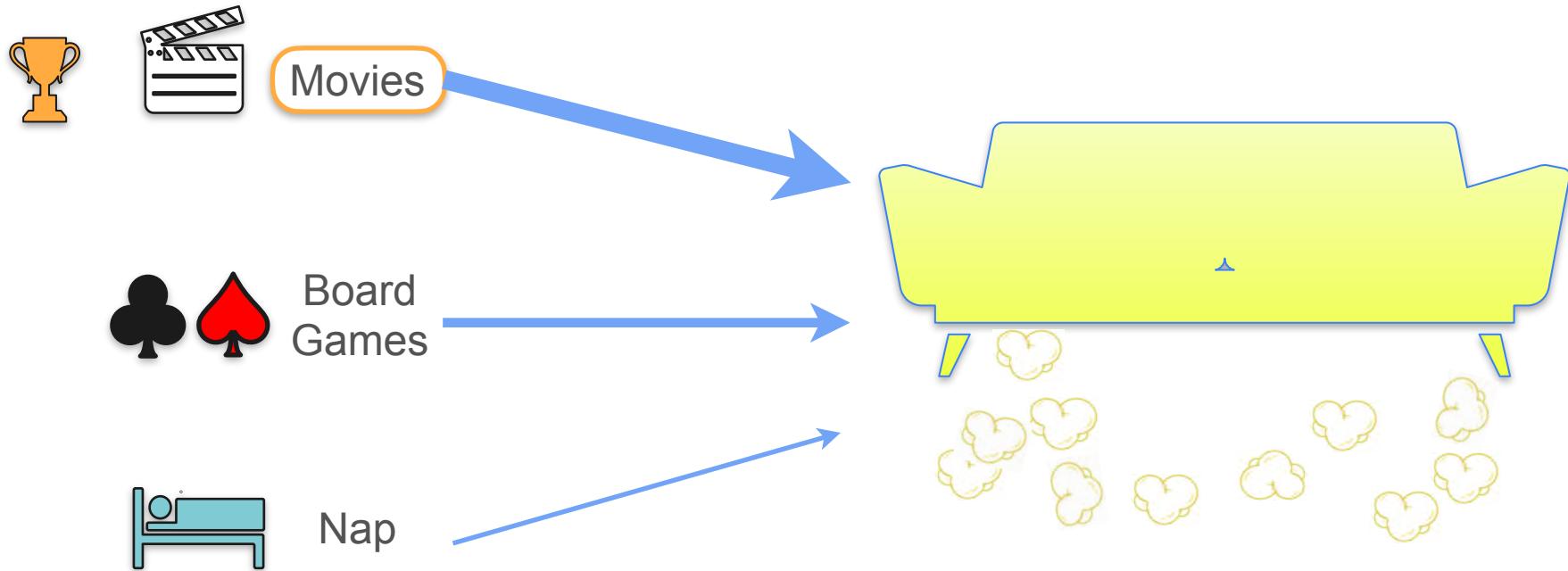
# There's Popcorn on the Floor. What Happened?



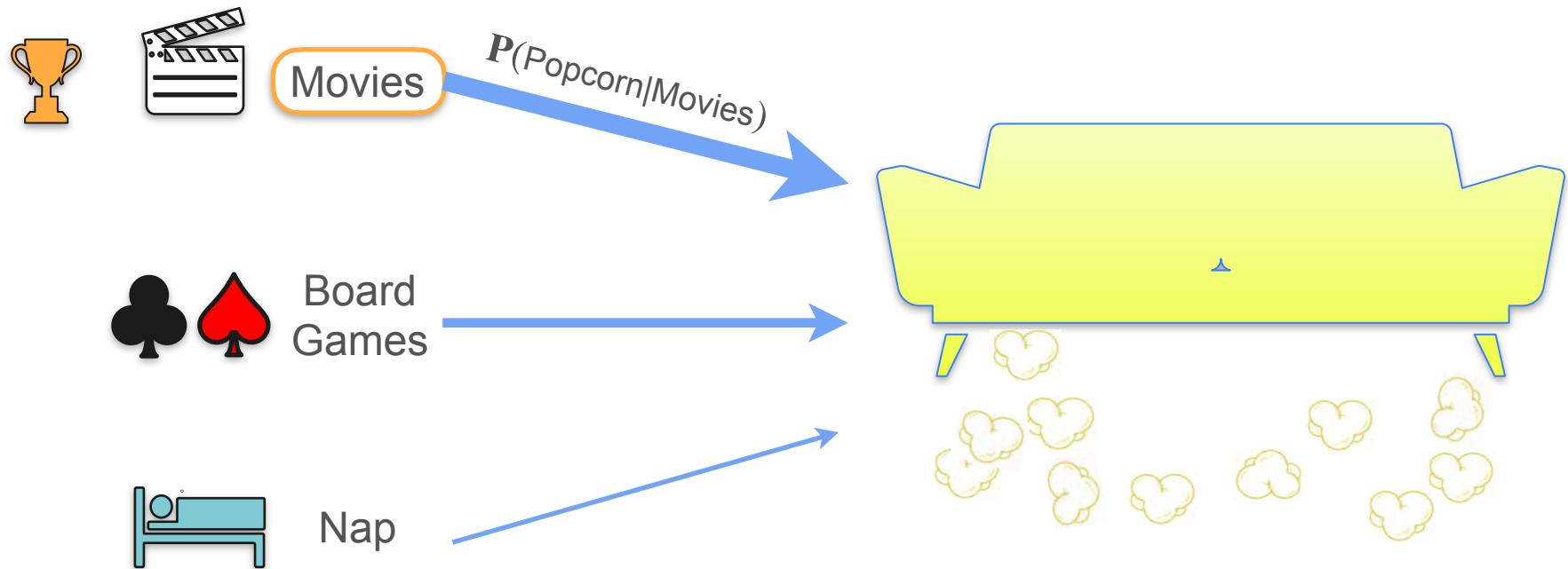
# There's Popcorn on the Floor. What Happened?



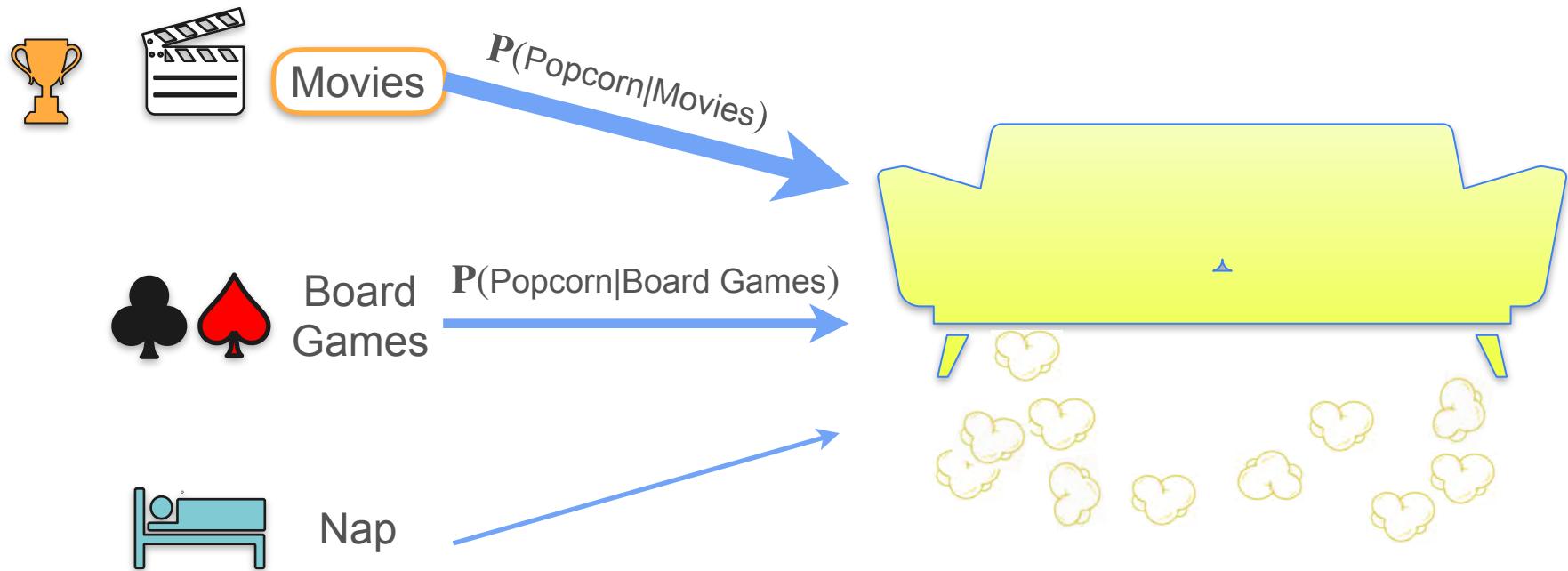
# There's Popcorn on the Floor. What Happened?



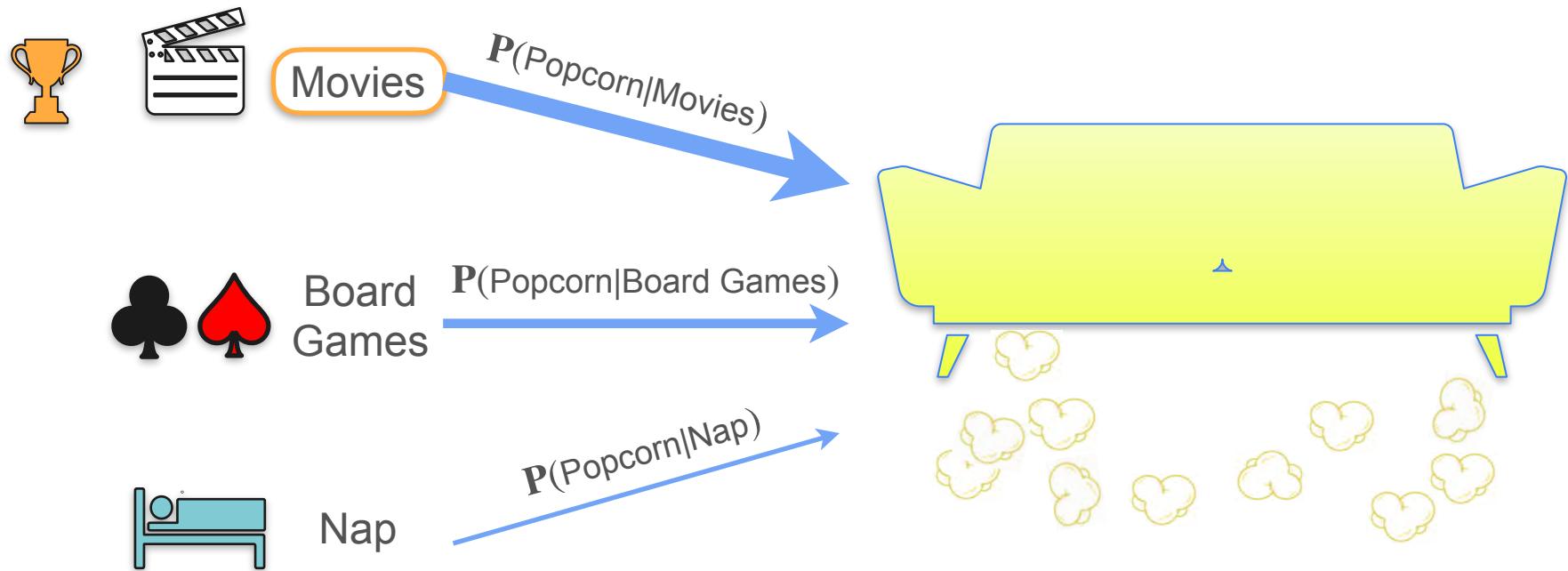
# There's Popcorn on the Floor. What Happened?



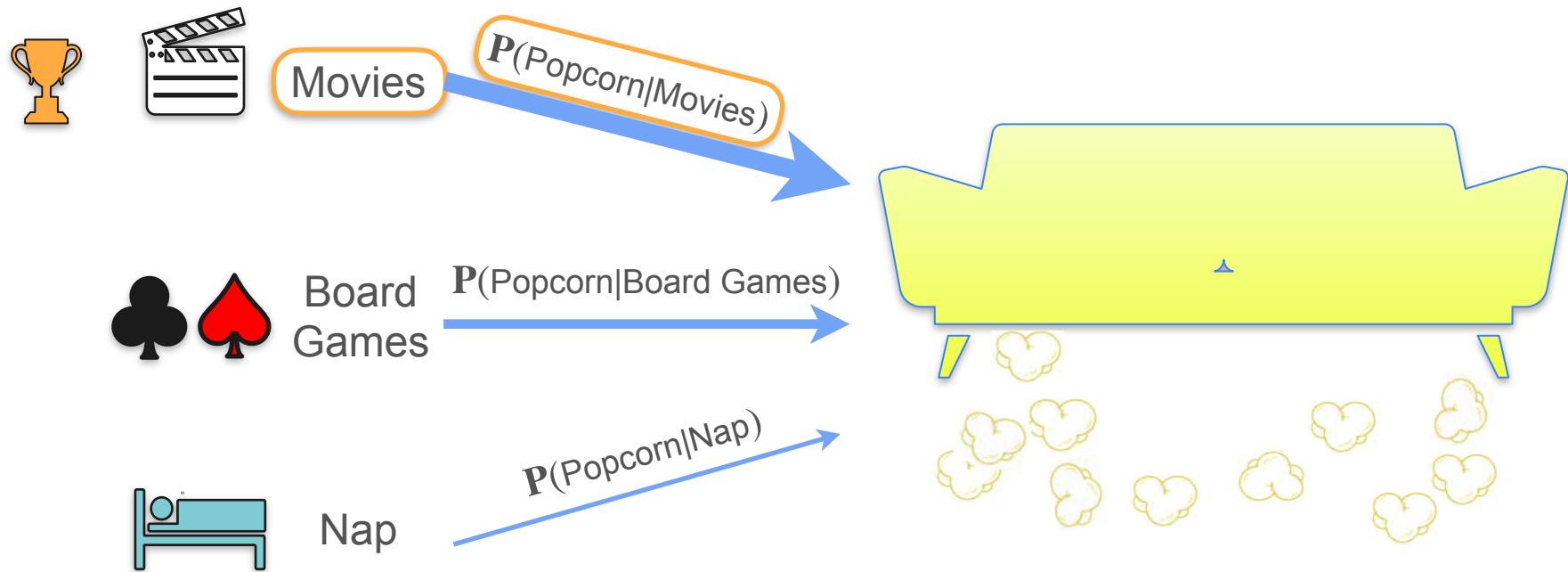
# There's Popcorn on the Floor. What Happened?



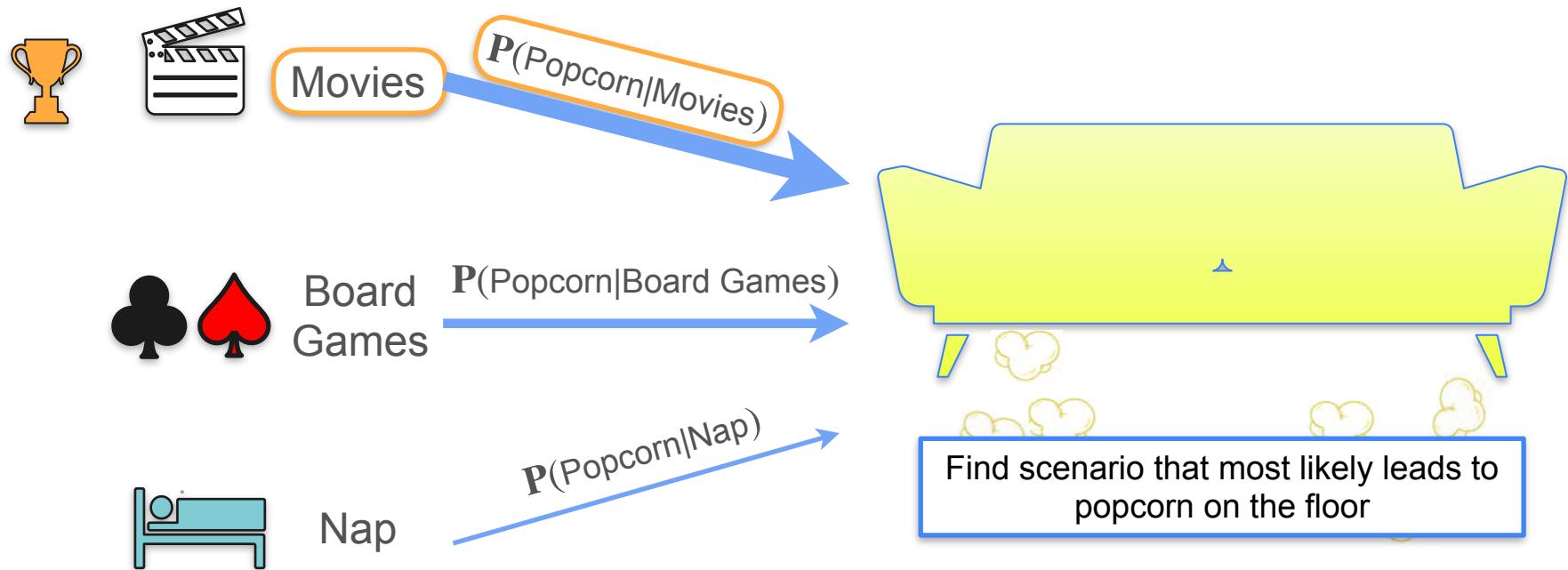
# There's Popcorn on the Floor. What Happened?



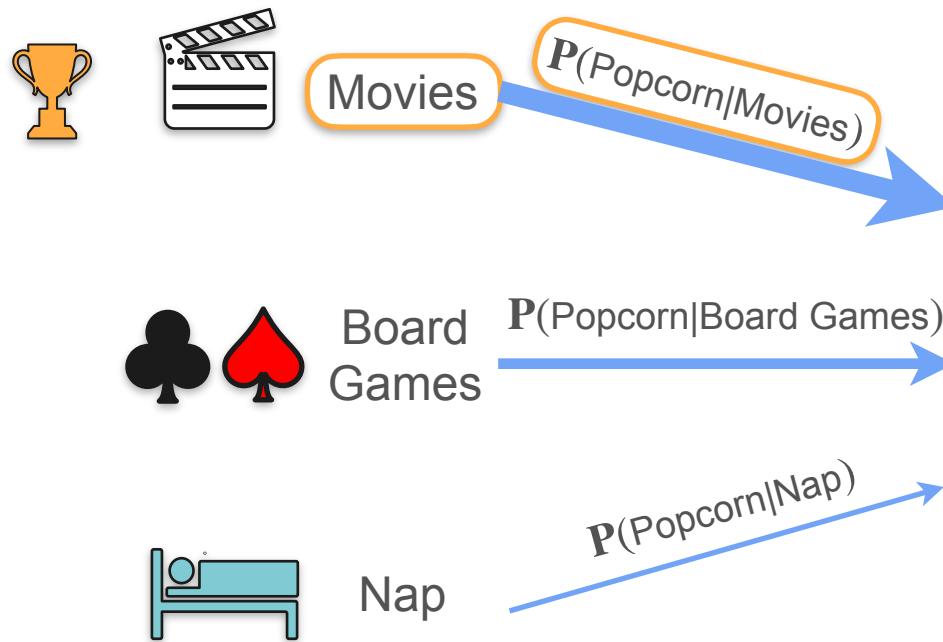
# There's Popcorn on the Floor. What Happened?



# There's Popcorn on the Floor. What Happened?



# There's Popcorn on the Floor. What Happened?



Find scenario that most likely leads to popcorn on the floor

Maximum Likelihood

# Maximum Likelihood

# Maximum Likelihood



Data

# Maximum Likelihood



Model 1



Data

# Maximum Likelihood



Model 1



Model 2



Data

# Maximum Likelihood



Model 1



Model 2

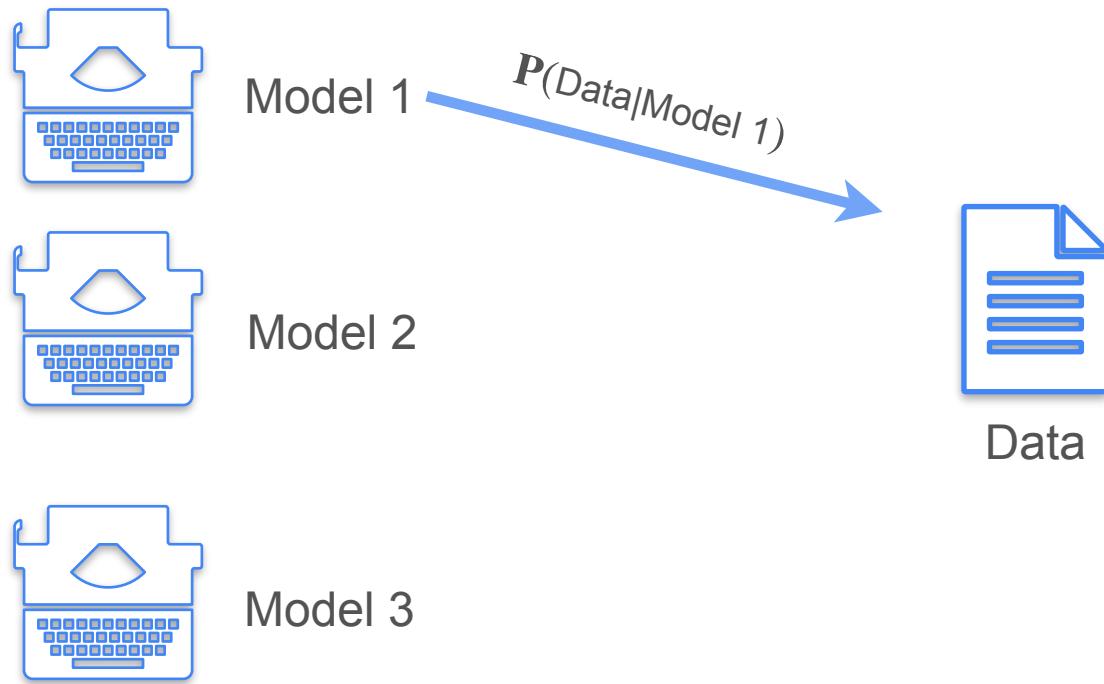


Model 3

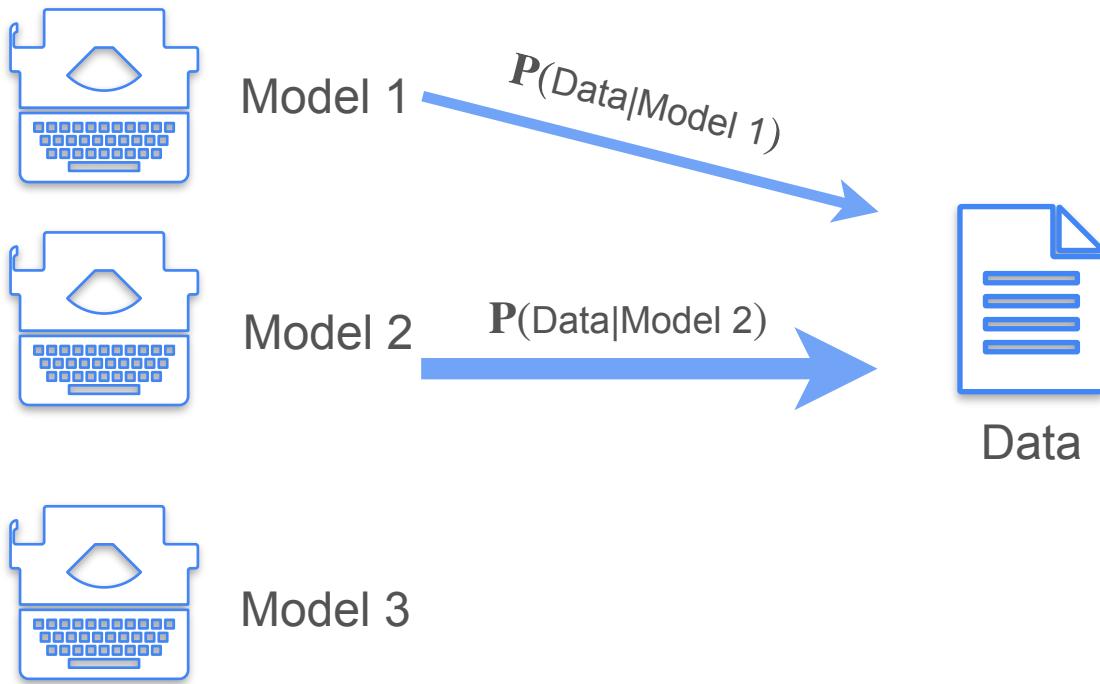


Data

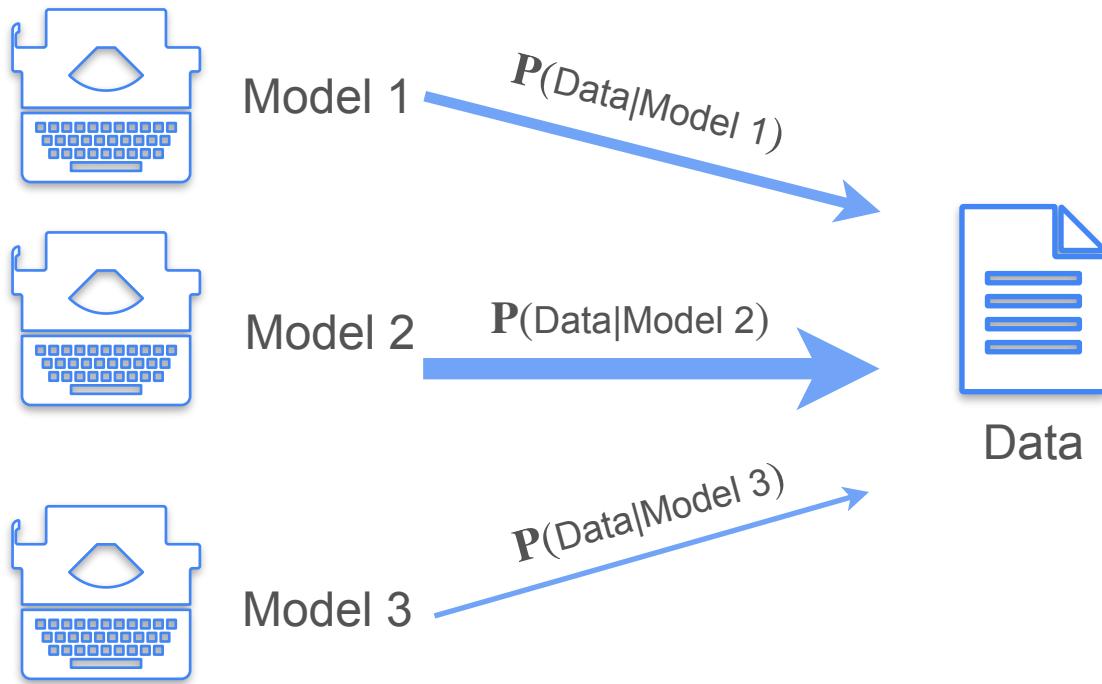
# Maximum Likelihood



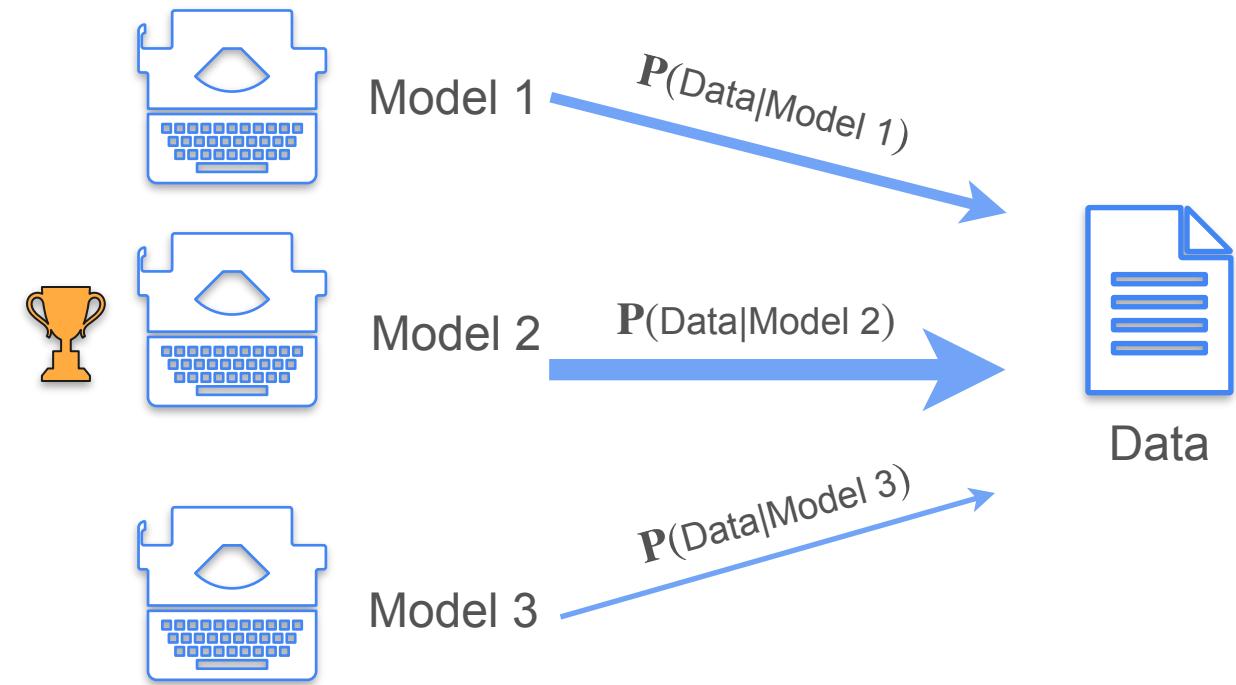
# Maximum Likelihood



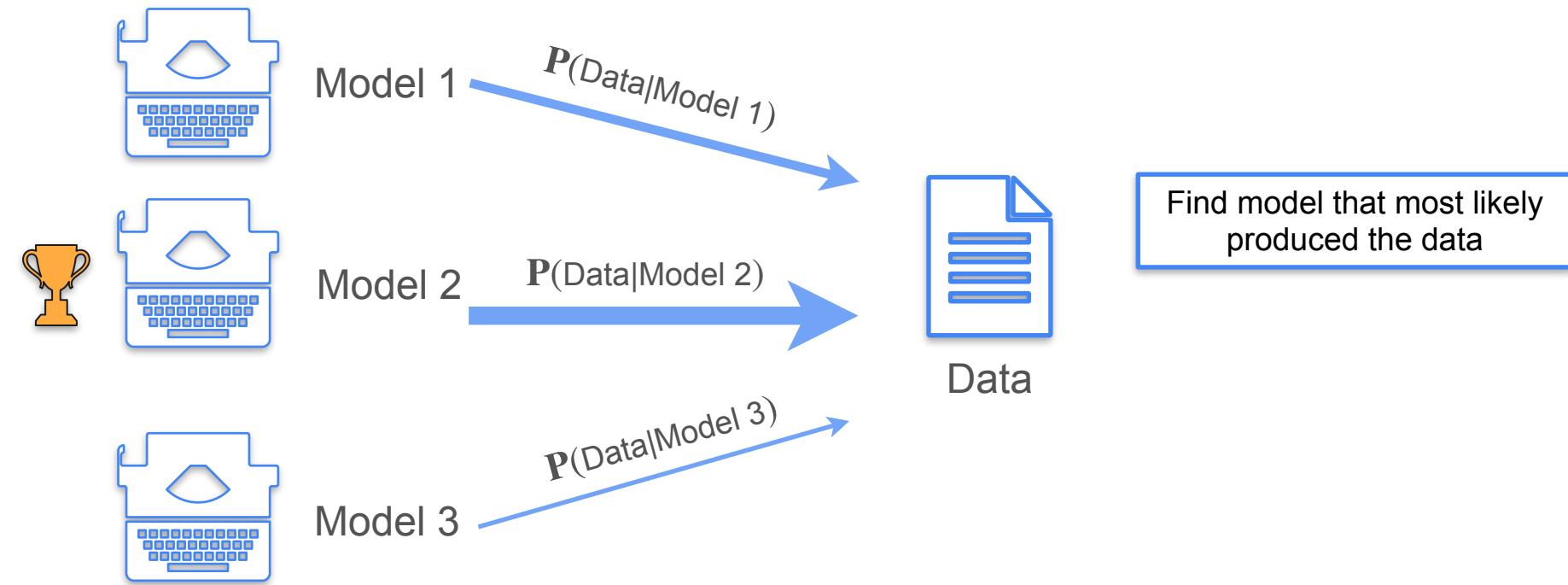
# Maximum Likelihood



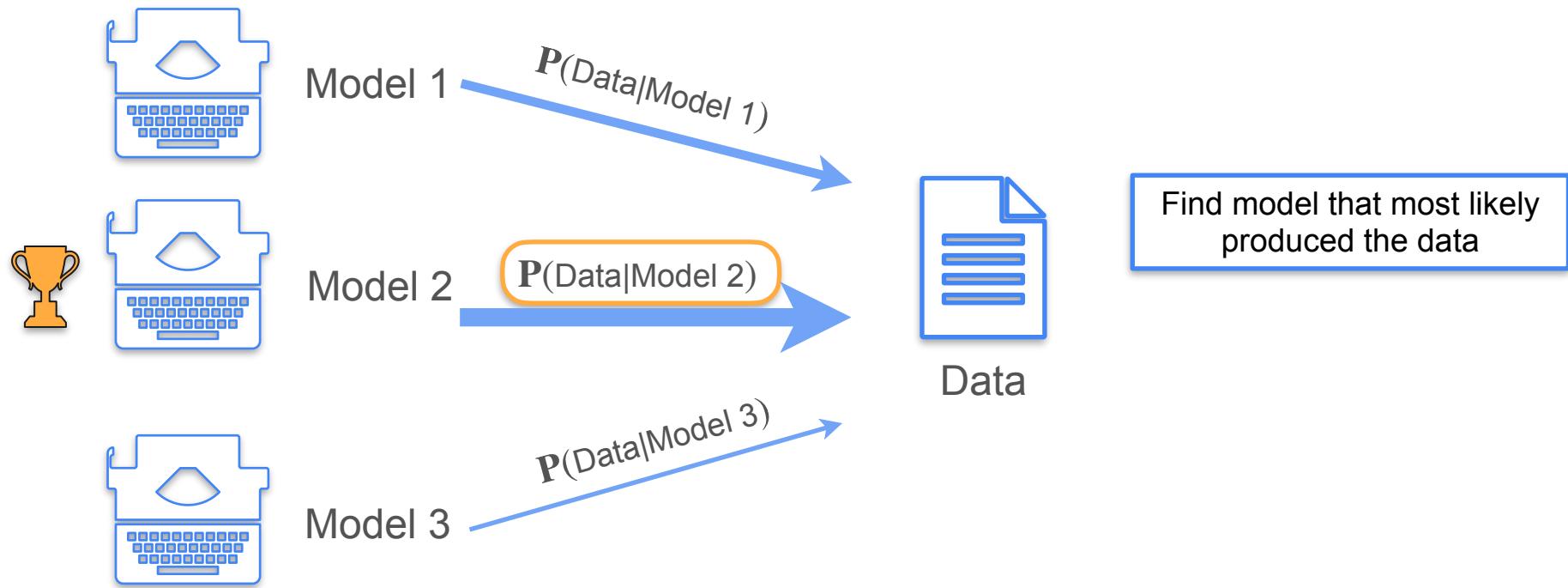
# Maximum Likelihood



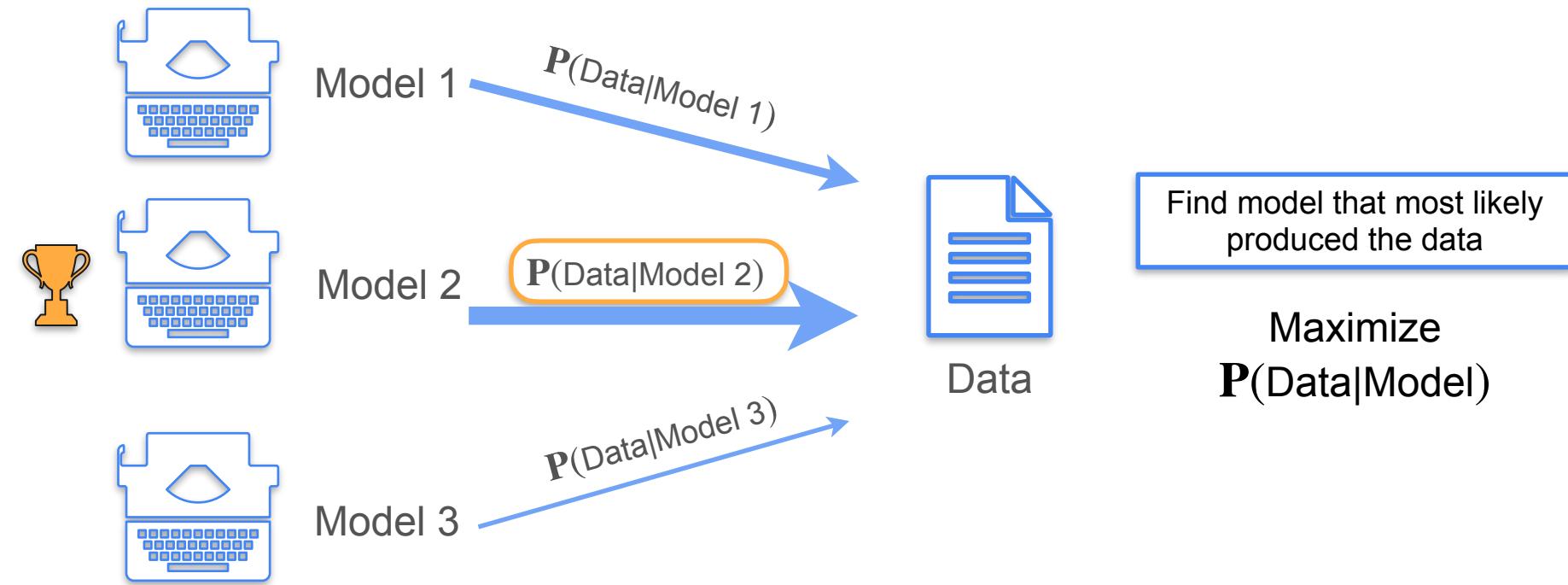
# Maximum Likelihood



# Maximum Likelihood

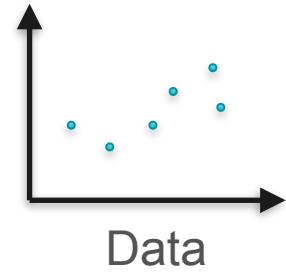


# Maximum Likelihood

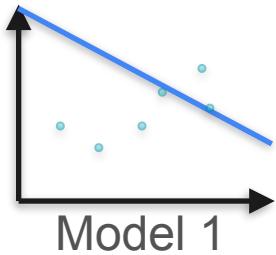


# Example: Linear Regression

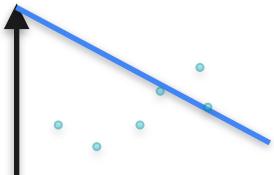
# Example: Linear Regression



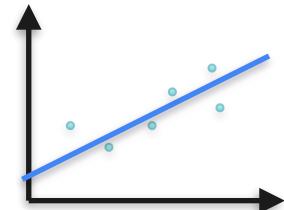
# Example: Linear Regression



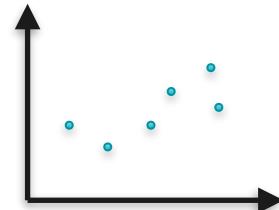
# Example: Linear Regression



Model 1

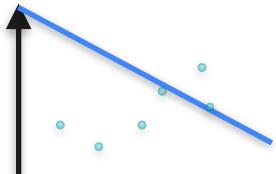


Model 2

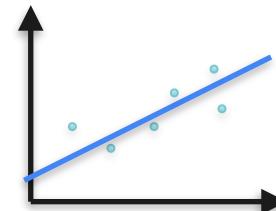


Data

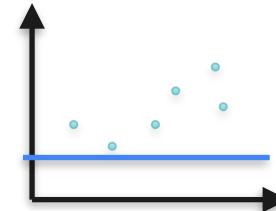
# Example: Linear Regression



Model 1



Model 2

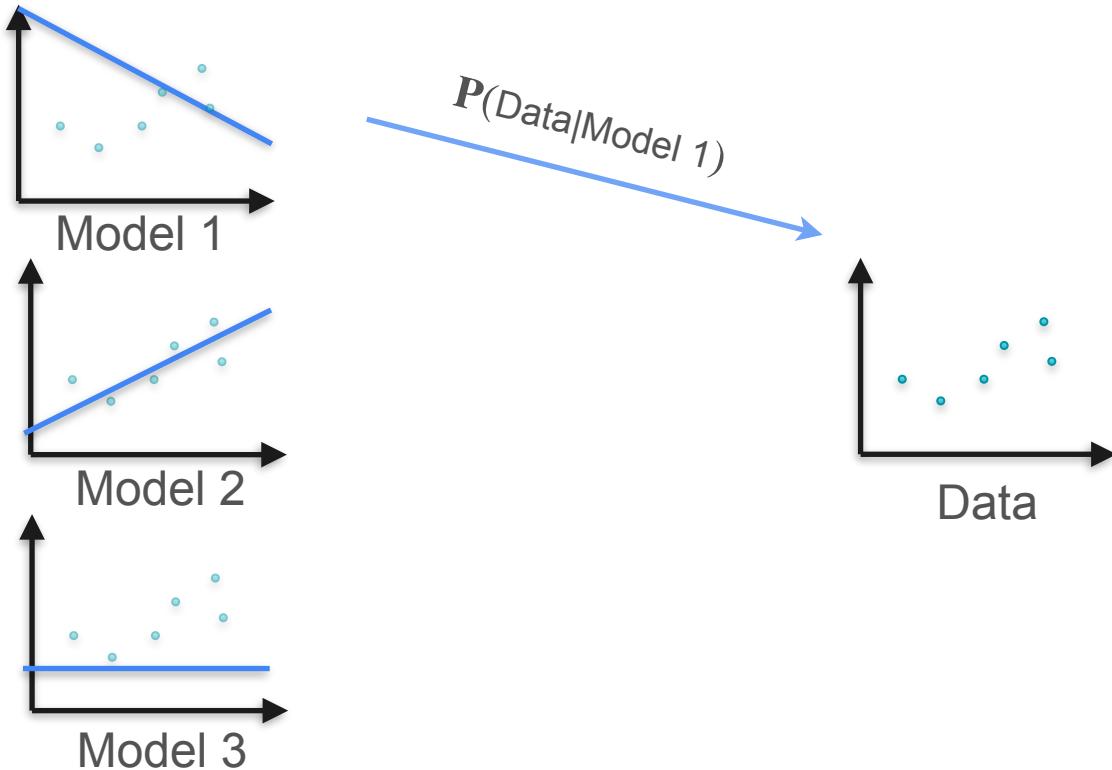


Model 3

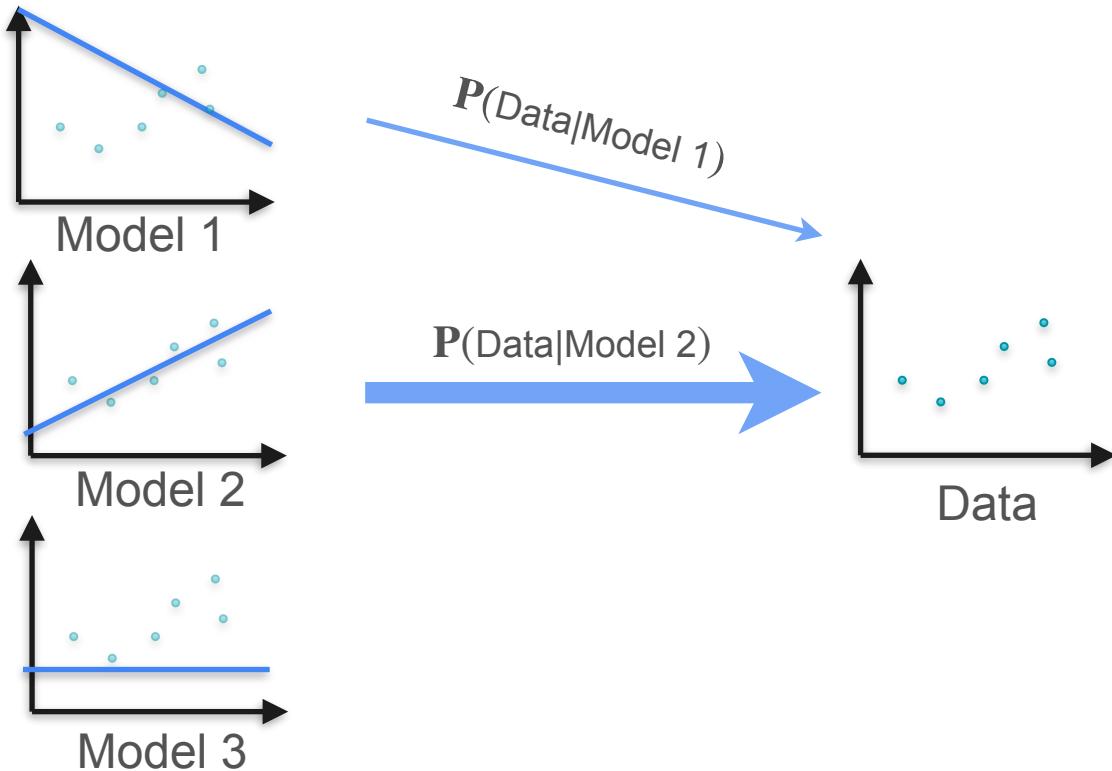


Data

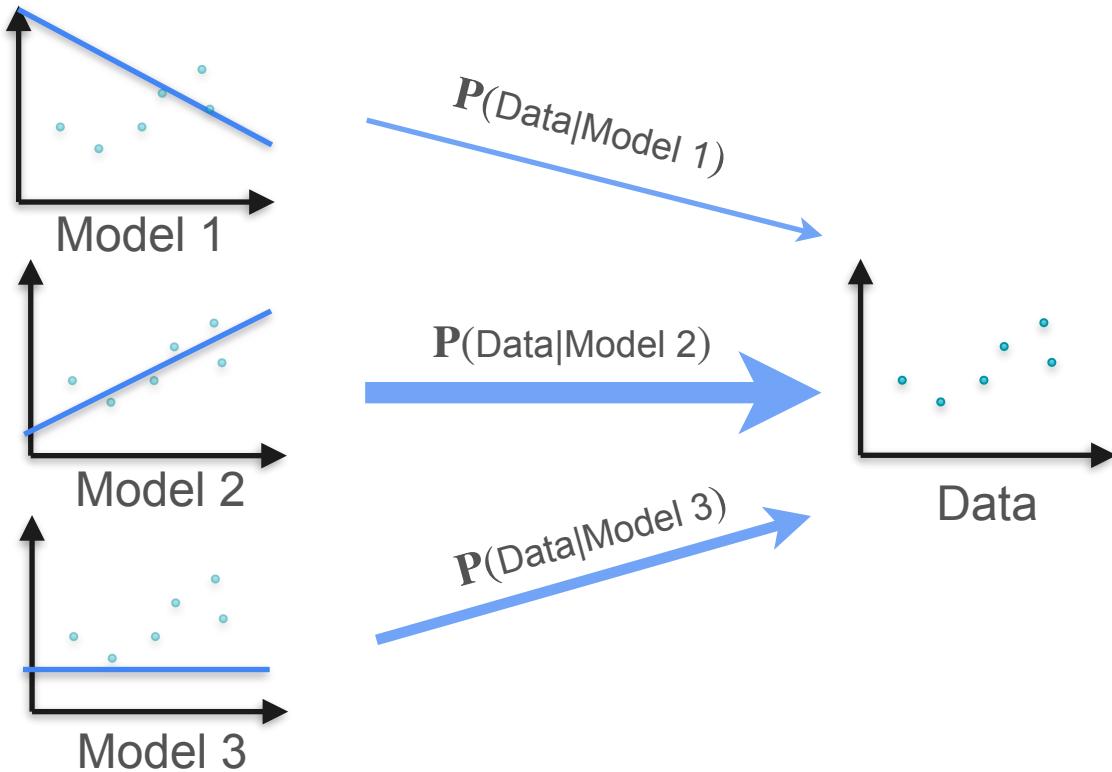
# Example: Linear Regression



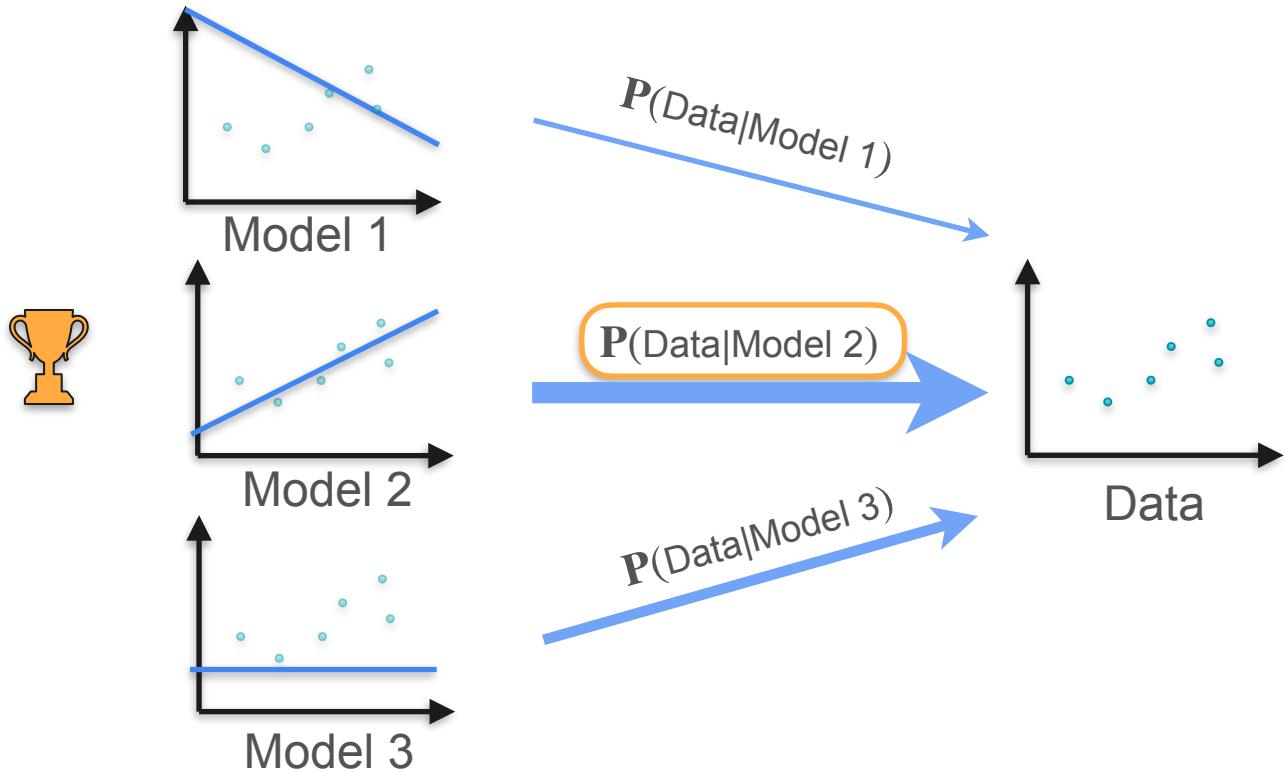
# Example: Linear Regression



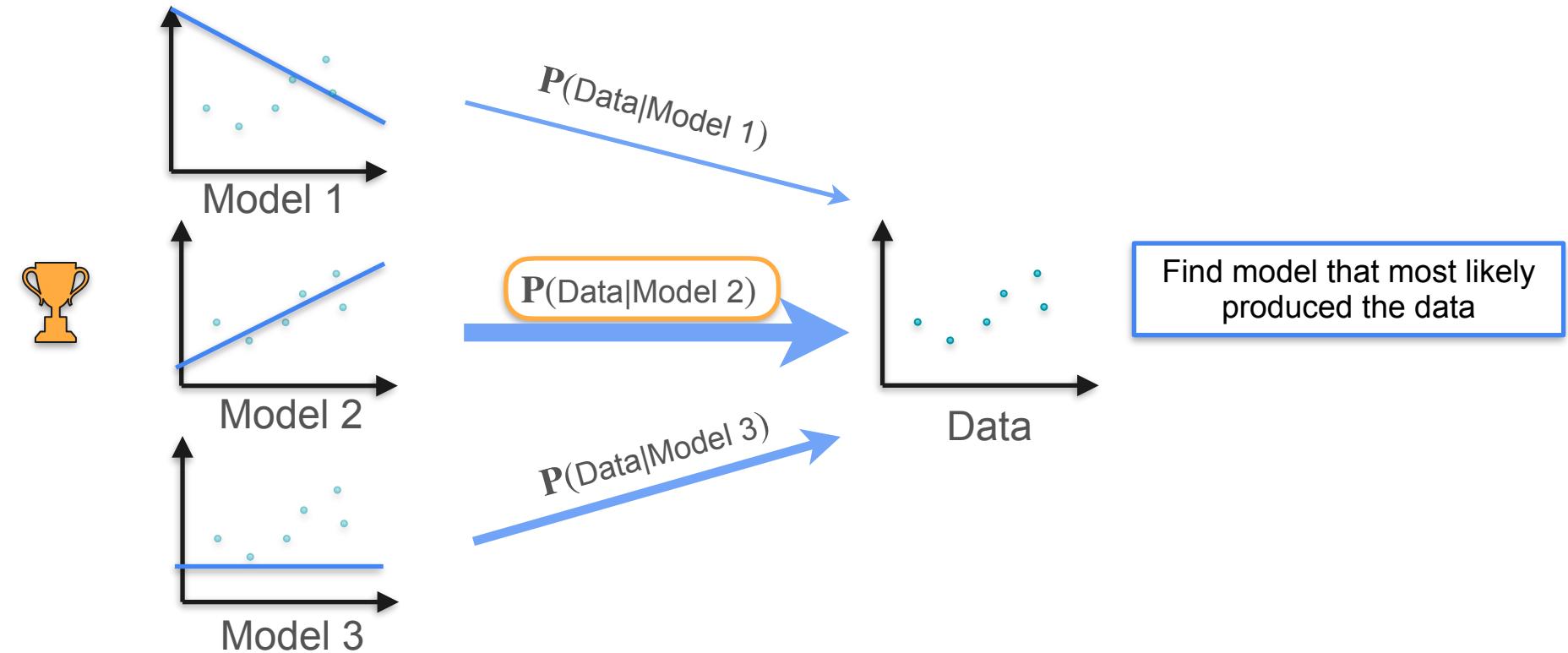
# Example: Linear Regression



# Example: Linear Regression



# Example: Linear Regression





DeepLearning.AI

## Point Estimation

---

### MLE: Bernoulli Example

# Maximum Likelihood: Bernoulli Example

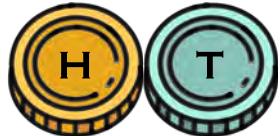
# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



Coin 1



$$P(H) = 0.7$$

$$P(T) = 0.3$$

Coin 2



$$P(H) = 0.5$$

$$P(T) = 0.5$$

Coin 3



$$P(H) = 0.3$$

$$P(T) = 0.7$$

# Quiz

Which of the coins is more likely?

- A. Coin 1 ( $P(H) = 0.7$ )
- B. Coin 2 ( $P(H) = 0.5$ )
- C. Coin 3 ( $P(H) = 0.3$ )

# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



Coin 1    0.7    0.7    0.7    0.7    0.7    0.7    0.7    0.7    0.3    0.3 = 0.0051

# Maximum Likelihood: Bernoulli Example



**Coin 1** 0.7 0.7 0.7 0.7 0.7 0.7 0.7 0.7 0.7 0.3 = 0.0051

**Coin 2** 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 = 0.0010

# Maximum Likelihood: Bernoulli Example



**Coin 1** 0.7 0.7 0.7 0.7 0.7 0.7 0.7 0.7 0.7 0.3 = 0.0051

**Coin 2** 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 = 0.0010

**Coin 3** 0.3 0.3 0.3 0.3 0.3 0.3 0.3 0.30 0.7 0.7 = 0.00003

# Maximum Likelihood: Bernoulli Example



<b>Coin 1</b>	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.3	0.3 = 0.0051
---------------	-----	-----	-----	-----	-----	-----	-----	-----	-----	--------------

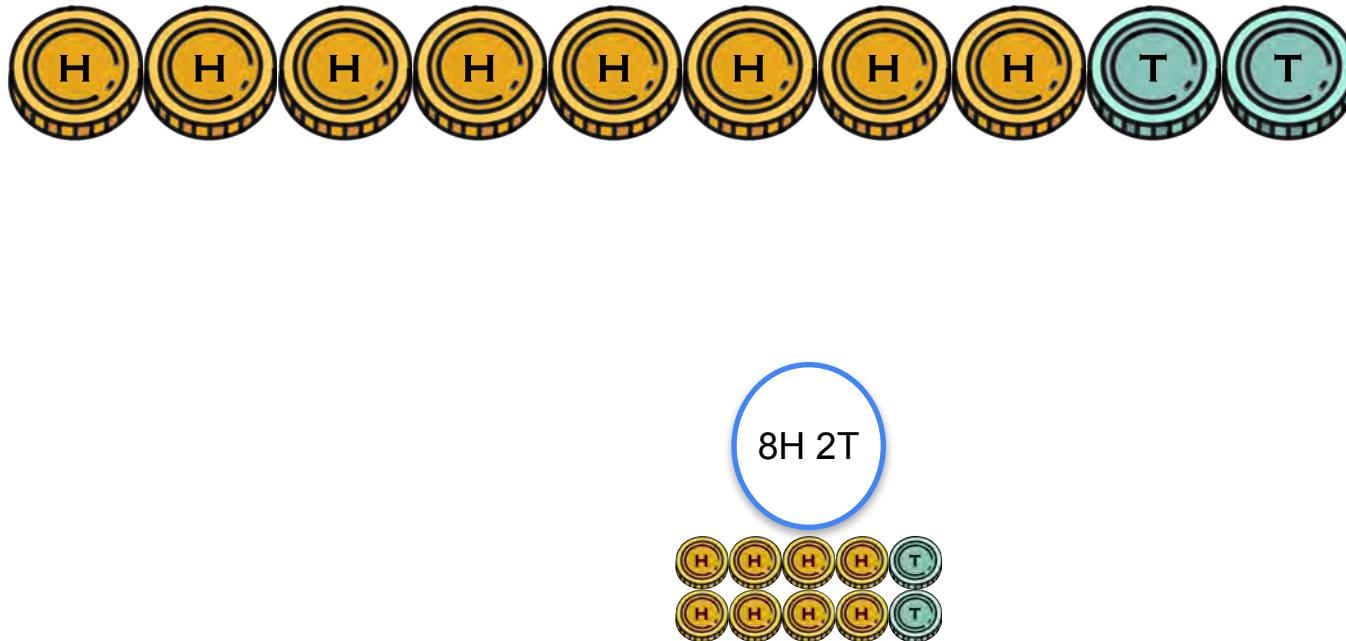
<b>Coin 2</b>	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5 = 0.0010
---------------	-----	-----	-----	-----	-----	-----	-----	-----	-----	--------------

<b>Coin 3</b>	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.30	0.7	0.7 = 0.00003
---------------	-----	-----	-----	-----	-----	-----	-----	------	-----	---------------

# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



**Coin 1**

$$P(H) = 0.7$$



**Coin 2**

$$P(H) = 0.5$$

8H 2T



# Maximum Likelihood: Bernoulli Example



**Coin 1**

$$P(H) = 0.7$$



**Coin 2**

$$P(H) = 0.5$$



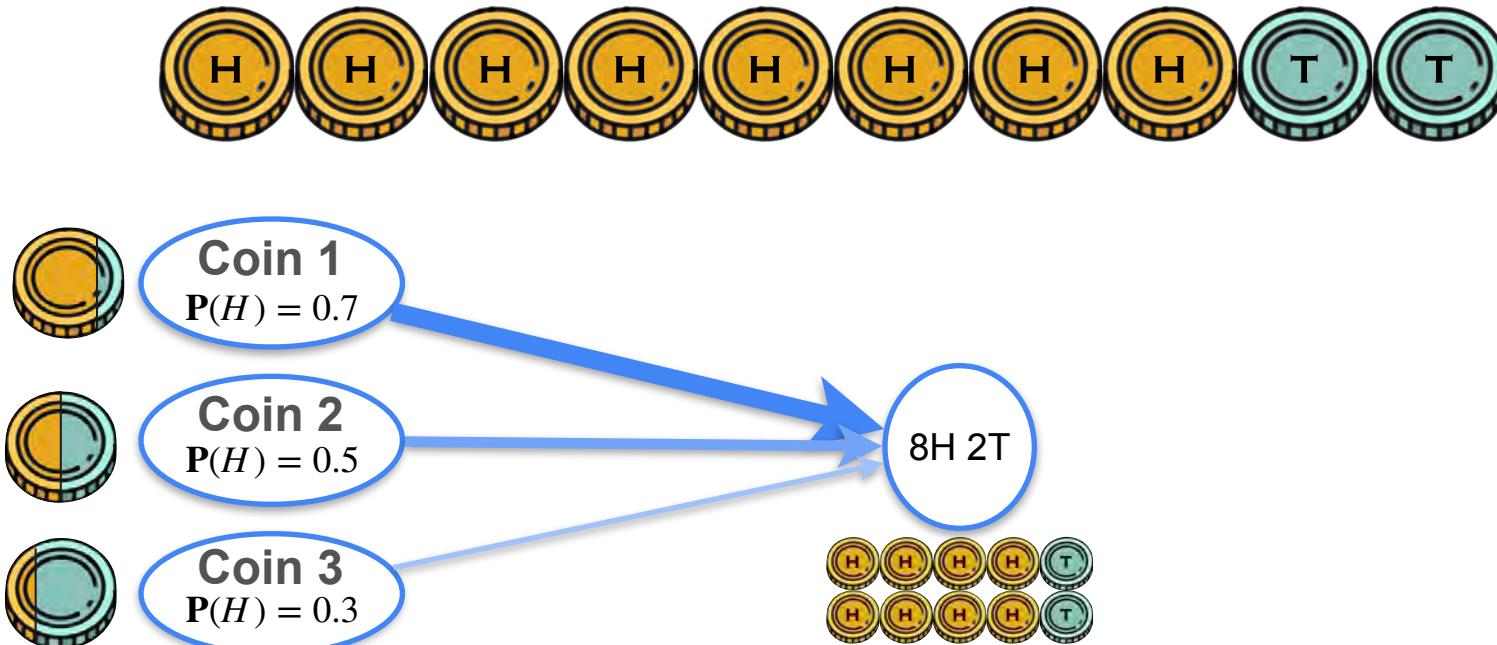
**Coin 3**

$$P(H) = 0.3$$

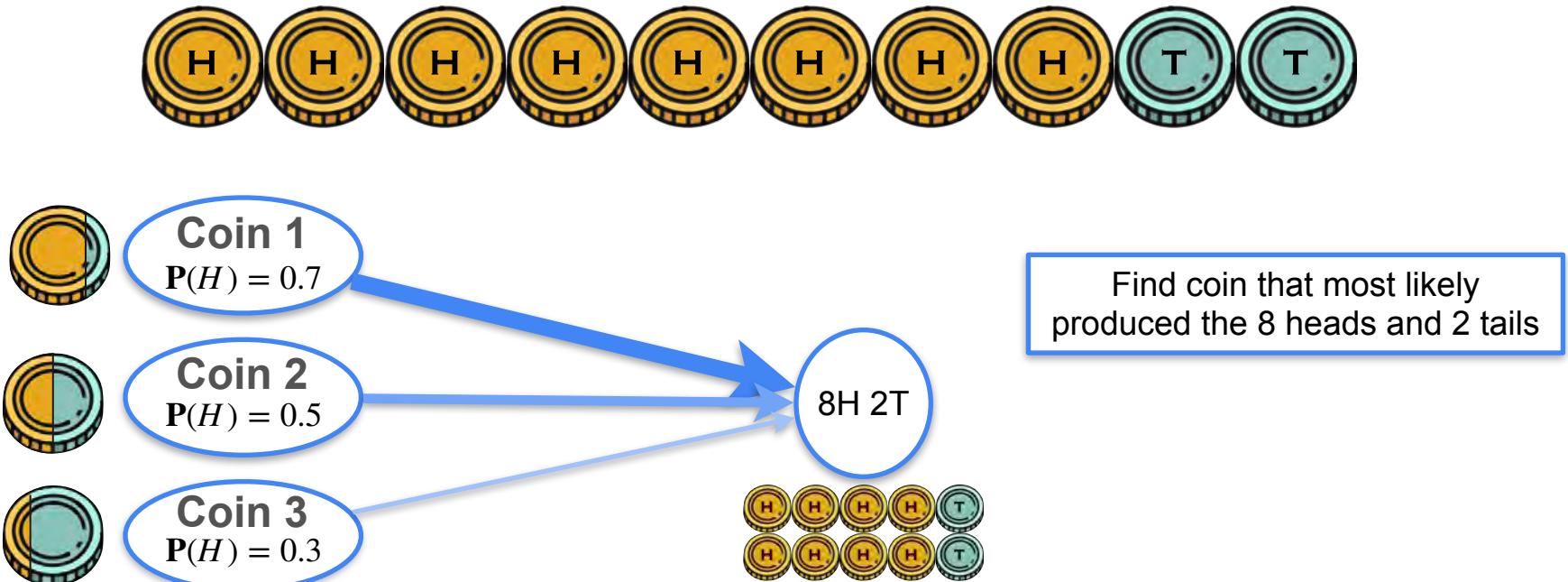
8H 2T



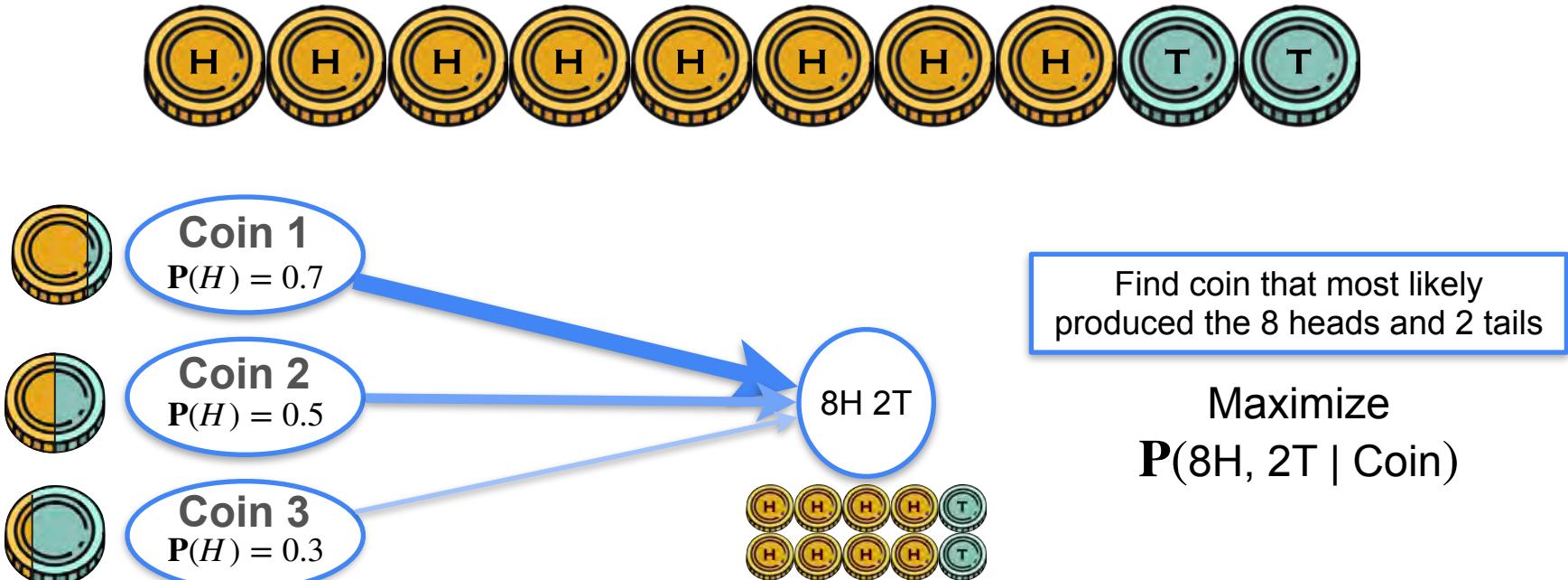
# Maximum Likelihood: Bernoulli Example



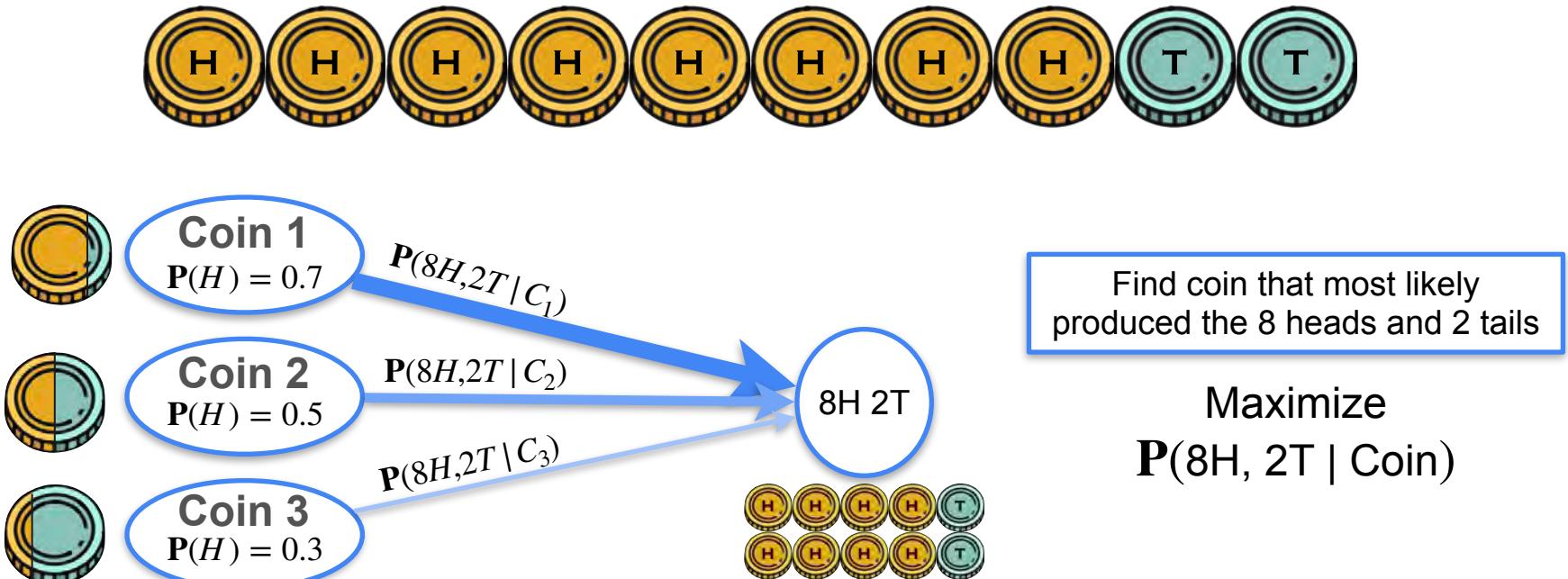
# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



# Maximum Likelihood: Bernoulli Example



**Coin 1**  
 $P(H) = 0.7$

$$P(8H, 2T | C_1) = 0.0051$$



**Coin 2**  
 $P(H) = 0.5$

$$P(8H, 2T | C_2) = 0.0010$$



**Coin 3**  
 $P(H) = 0.3$

$$P(8H, 2T | C_3) = 0.00003$$

8H 2T



Find coin that most likely produced the 8 heads and 2 tails

Maximize  
 $P(8H, 2T | \text{Coin})$

# Maximum Likelihood: Bernoulli Example



**Coin 1**  
 $P(H) = 0.7$

$$P(8H, 2T | C_1) = 0.0051$$



**Coin 2**  
 $P(H) = 0.5$

$$P(8H, 2T | C_2) = 0.0010$$



**Coin 3**  
 $P(H) = 0.3$

$$P(8H, 2T | C_3) = 0.00003$$



Find coin that most likely produced the 8 heads and 2 tails

Maximize  
 $P(8H, 2T | \text{Coin})$

# Maximum Likelihood: Bernoulli Example

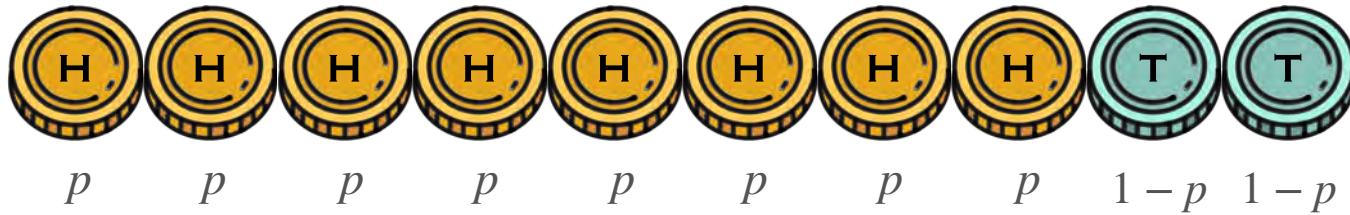


# Maximum Likelihood: Bernoulli Example



Can you do any better?

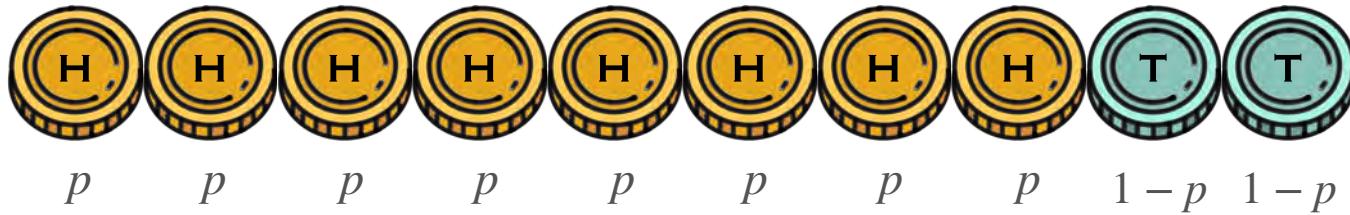
# Maximum Likelihood: Bernoulli Example



Can you do any better?

$$p = \mathbf{P}(H)$$

# Maximum Likelihood: Bernoulli Example

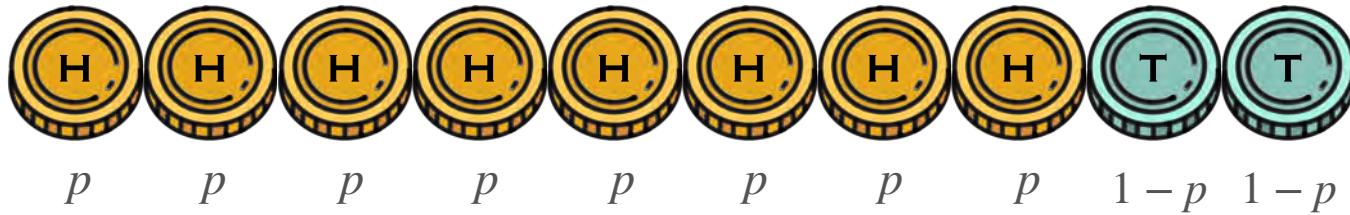


Can you do any better?

$$p = \mathbf{P}(H)$$

$$p^8(1 - p)^2$$

# Maximum Likelihood: Bernoulli Example



Can you do any better?

$$p = \mathbf{P}(H)$$

$$p^8(1 - p)^2$$

You want  $p$  that maximizes the chances of seeing 8H

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H)$$

$$p^8(1 - p)^2$$

You want  $p$  that maximizes the chances of seeing 8H

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1 - p)^2$$

You want  $p$  that maximizes the chances of seeing 8H

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1 - p)^2 \quad \text{Function of } p$$

You want  $p$  that maximizes the chances of seeing 8H

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1-p)^2 \quad \text{Function of } p$$

You want  $p$  that maximizes the chances of seeing 8H

$$\log((p^8(1-p)^2))$$

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1-p)^2 \quad \text{Function of } p$$

You want  $p$  that maximizes the chances of seeing 8H

$$\log((p^8(1-p)^2)) = 8\log(p) + 2\log(1-p)$$

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1-p)^2 \quad \text{Function of } p$$

You want  $p$  that maximizes the chances of seeing 8H

$$\text{Log-likelihood} \quad \ell(p; 8H) = \log((p^8(1-p)^2)) = 8\log(p) + 2\log(1-p)$$

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1-p)^2 \quad \text{Function of } p$$

You want  $p$  that maximizes the chances of seeing 8H

$$\text{Log-likelihood} \quad \ell(p; 8H) = \log((p^8(1-p)^2)) = 8\log(p) + 2\log(1-p)$$

$$\frac{d}{dp} (8\log(p) + 2\log(1-p))$$

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1-p)^2 \quad \text{Function of } p$$

You want  $p$  that maximizes the chances of seeing 8H

$$\text{Log-likelihood} \quad \ell(p; 8H) = \log((p^8(1-p)^2)) = 8\log(p) + 2\log(1-p)$$

$$\frac{d}{dp} (8\log(p) + 2\log(1-p)) = \frac{8}{p} + \frac{2}{1-p}(-1)$$

# Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1-p)^2 \quad \text{Function of } p$$

You want  $p$  that maximizes the chances of seeing 8H

$$\text{Log-likelihood} \quad \ell(p; 8H) = \log((p^8(1-p)^2)) = 8\log(p) + 2\log(1-p)$$

$$\frac{d}{dp} (8\log(p) + 2\log(1-p)) = \frac{8}{p} + \frac{2}{1-p}(-1) = 0 \rightarrow \hat{p} = \frac{8}{10}$$

# The General Case

# Maximum Likelihood: Bernoulli Example

$n$  coins

$k$  heads

# Maximum Likelihood: Bernoulli Example

$n$  coins

$k$  heads

$X_1$

$X_2$

$X_3$

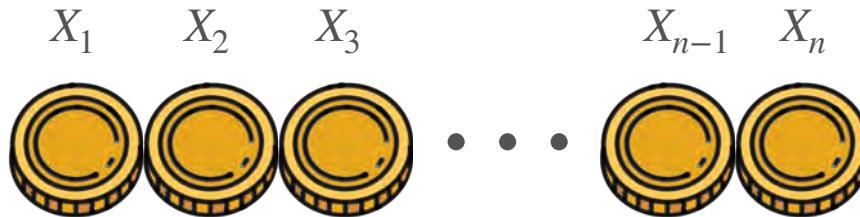
$X_{n-1}$

$X_n$



# Maximum Likelihood: Bernoulli Example

$n$  coins  
 $k$  heads



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

# Maximum Likelihood: Bernoulli Example

$n$  coins  
 $k$  heads



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i)$$

# Maximum Likelihood: Bernoulli Example

$n$  coins  
 $k$  heads



$$\mathbf{X} = (X_1, \dots, X_n)$$

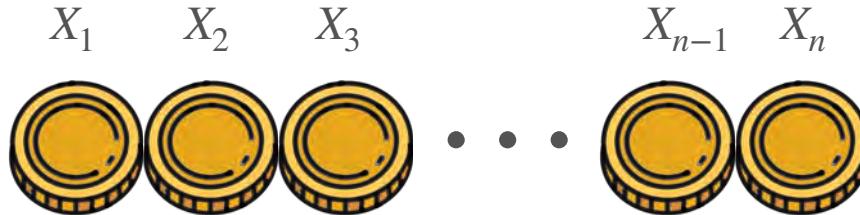
$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}$$

# Maximum Likelihood: Bernoulli Example

$n$  coins  
 $k$  heads



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

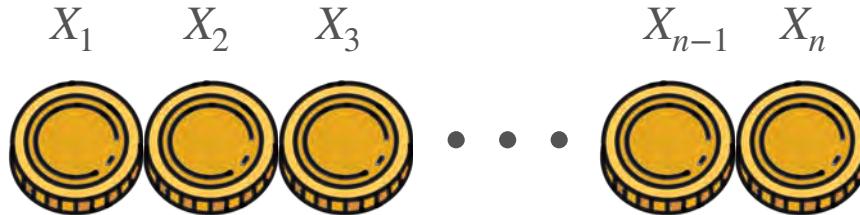
Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}$$

If  $x_i = 1$ ,  $p^{[x_i]}(1-p)^{[1-x_i]} = p$   
If  $x_i = 0$ ,  $p^{[x_i]}(1-p)^{[1-x_i]} = (1-p)$

# Maximum Likelihood: Bernoulli Example

$n$  coins  
 $k$  heads



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}$$

If  $x_i = 1$ ,  $p^{[x_i]}(1-p)^{[1-x_i]} = p$   
If  $x_i = 0$ ,  $p^{[x_i]}(1-p)^{[1-x_i]} = (1-p)$

$$\sum_{i=1}^n x_i = \# \text{ heads}$$

$$n - \sum_{i=1}^n x_i = \# \text{ tails}$$

# Maximum Likelihood: Bernoulli Example



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}$$

# Maximum Likelihood: Bernoulli Example



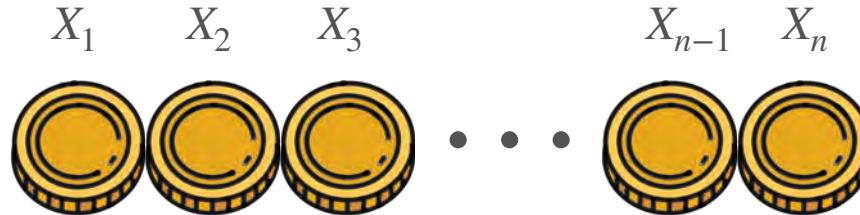
$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} = p^{\left(\sum_{i=1}^n x_i\right)} (1-p)^{\left(n - \sum_{i=1}^n x_i\right)}$$

# Maximum Likelihood: Bernoulli Example



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

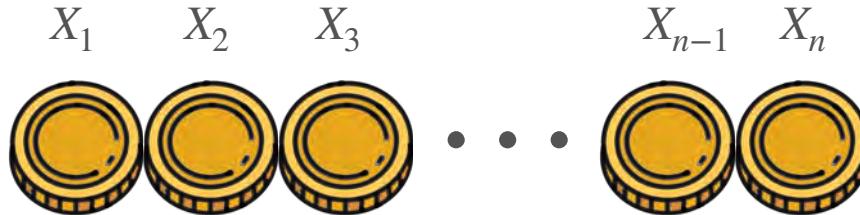
Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} = p^{\left(\sum_{i=1}^n x_i\right)}(1-p)^{\left(n - \sum_{i=1}^n x_i\right)}$$

Log-likelihood

$$\ell(p; \mathbf{x}) = \log \left( (p^{\sum_{i=1}^n x_i})(1-p)^{n - \sum_{i=1}^n x_i} \right)$$

# Maximum Likelihood: Bernoulli Example



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

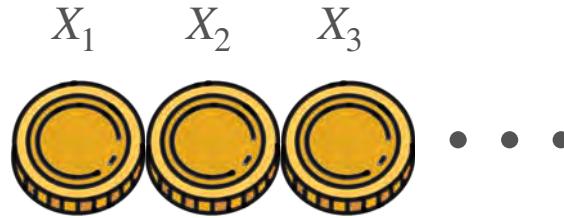
Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} = p^{\left(\sum_{i=1}^n x_i\right)} (1-p)^{\left(n - \sum_{i=1}^n x_i\right)}$$

Log-likelihood

$$\ell(p; \mathbf{x}) = \log \left( (p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}) \right) = \left( \sum_{i=1}^n x_i \right) \log(p) + \left( n - \sum_{i=1}^n x_i \right) \log(1-p)$$

# Maximum Likelihood: Bernoulli Example



X<sub>1</sub>    X<sub>2</sub>    X<sub>3</sub>

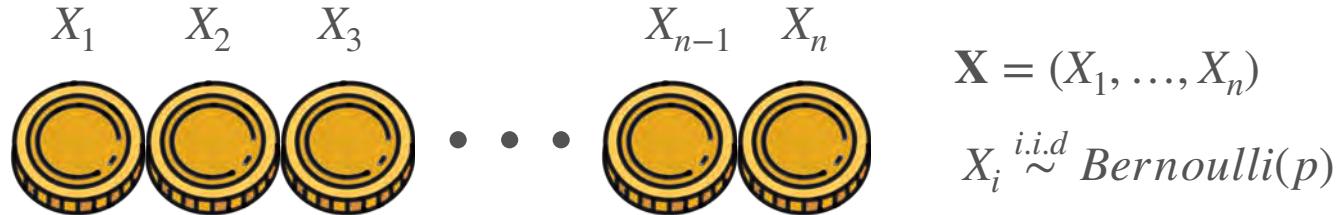


$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

$$\ell(p; \mathbf{x}) = \log \left( (p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}) \right) = \left( \sum_{i=1}^n x_i \right) \log(p) + \left( n - \sum_{i=1}^n x_i \right) \log(1-p)$$

# Maximum Likelihood: Bernoulli Example



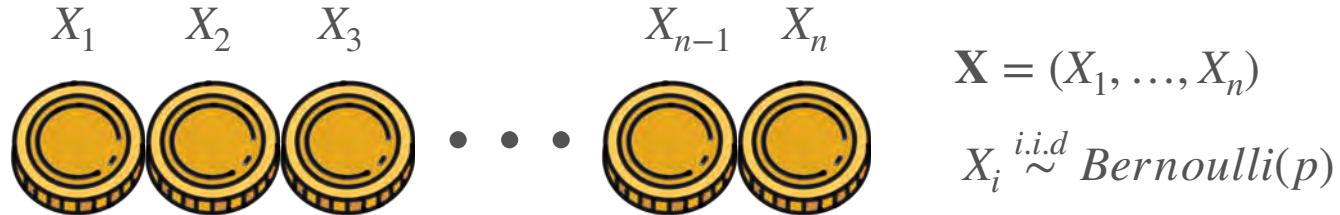
$$\ell(p; \mathbf{x}) = \log \left( (p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}) \right) = \left( \sum_{i=1}^n x_i \right) \log(p) + \left( n - \sum_{i=1}^n x_i \right) \log(1-p)$$

Find the maximum!

$$\frac{d}{dp} \ell(p; \mathbf{x}) = \frac{d}{dp} \left( \left( \sum_{i=1}^n x_i \right) \log(p) + \left( n - \sum_{i=1}^n x_i \right) \log(1-p) \right)$$

$$= \frac{\sum_{i=1}^n x_i}{p} + \frac{n - \sum_{i=1}^n x_i}{1-p} (-1) = 0$$

# Maximum Likelihood: Bernoulli Example



$$\ell(p; \mathbf{x}) = \log \left( (p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}) \right) = \left( \sum_{i=1}^n x_i \right) \log(p) + \left( n - \sum_{i=1}^n x_i \right) \log(1-p)$$

Find the maximum!

$$\begin{aligned} \frac{d}{dp} \ell(p; \mathbf{x}) &= \frac{d}{dp} \left( \left( \sum_{i=1}^n x_i \right) \log(p) + \left( n - \sum_{i=1}^n x_i \right) \log(1-p) \right) \\ &= \frac{\sum_{i=1}^n x_i}{p} + \frac{n - \sum_{i=1}^n x_i}{1-p} (-1) = 0 \end{aligned} \quad \rightarrow \quad \hat{p} = \frac{\sum_{i=1}^n x_i}{n} = \bar{x}$$



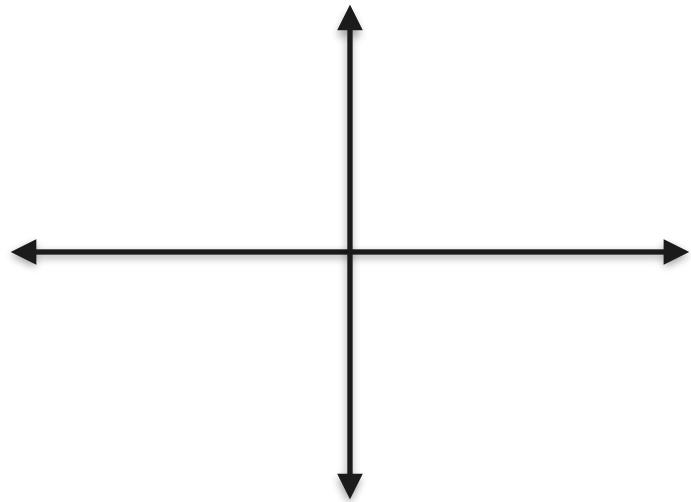
DeepLearning.AI

## Point Estimation

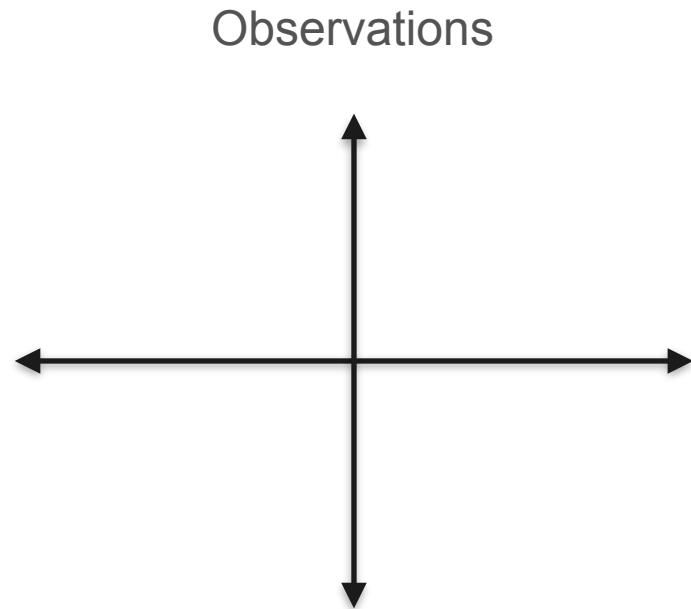
---

### MLE: Gaussian Example

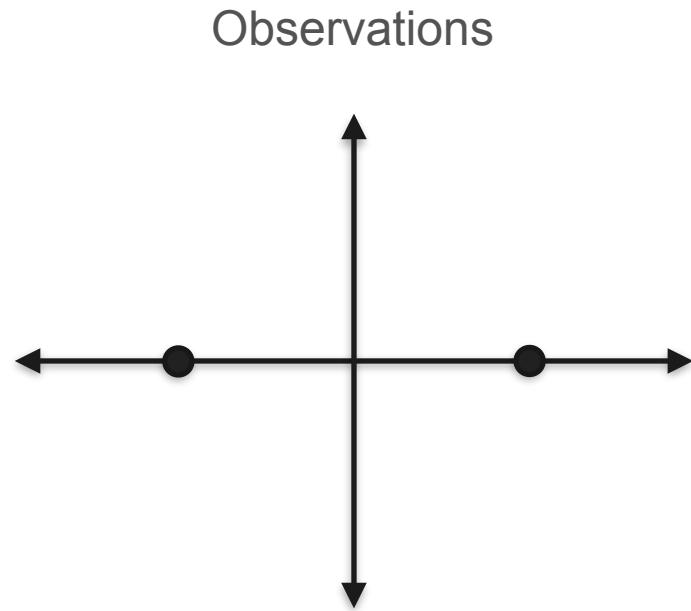
# Maximum Likelihood: Gaussian Example



# Maximum Likelihood: Gaussian Example



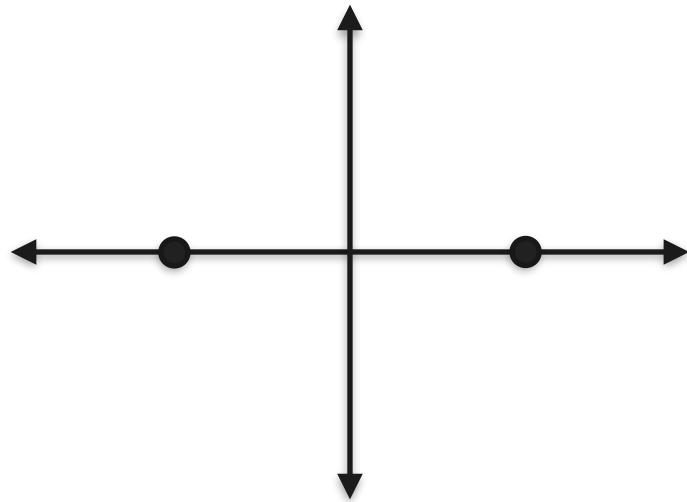
# Maximum Likelihood: Gaussian Example



# Maximum Likelihood: Gaussian Example

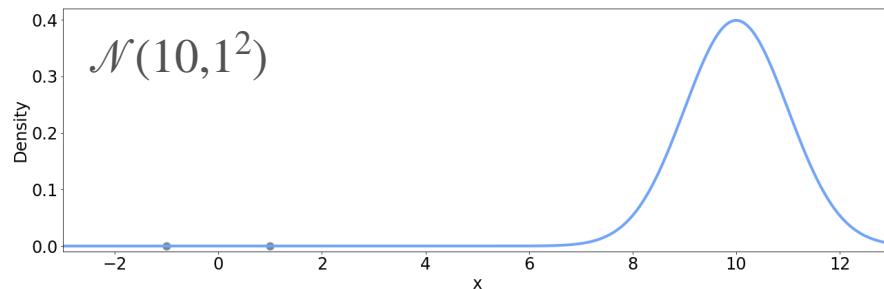
Candidates

Observations

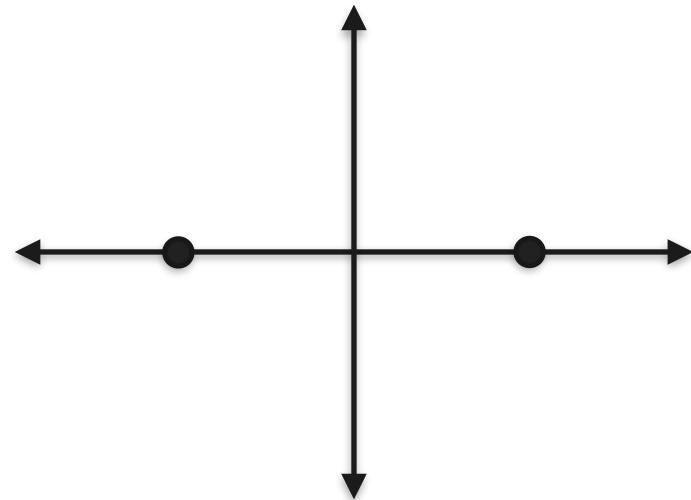


# Maximum Likelihood: Gaussian Example

Candidates

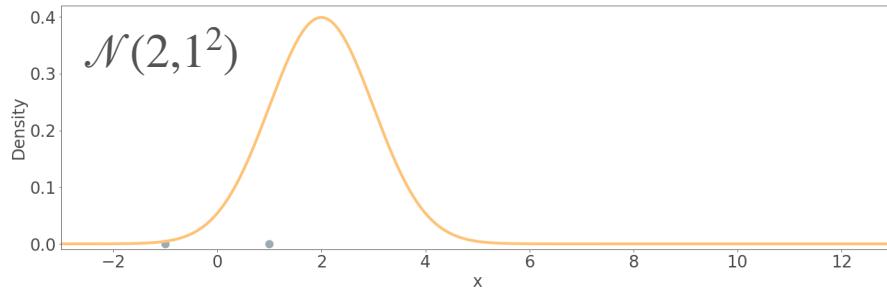
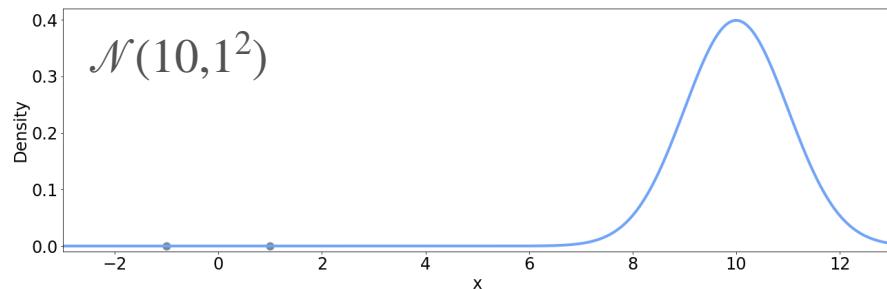


Observations

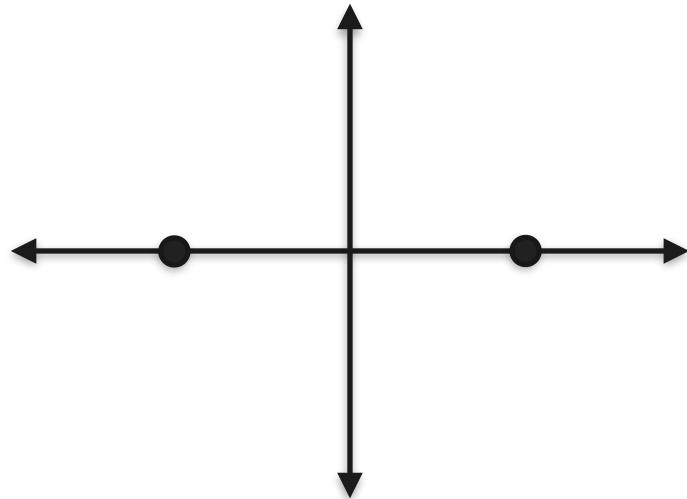


# Maximum Likelihood: Gaussian Example

Candidates

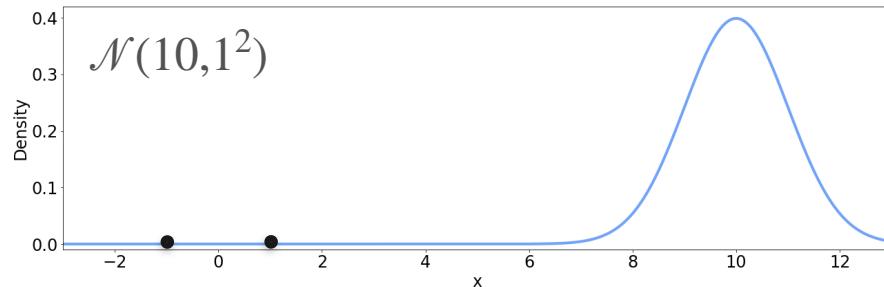


Observations

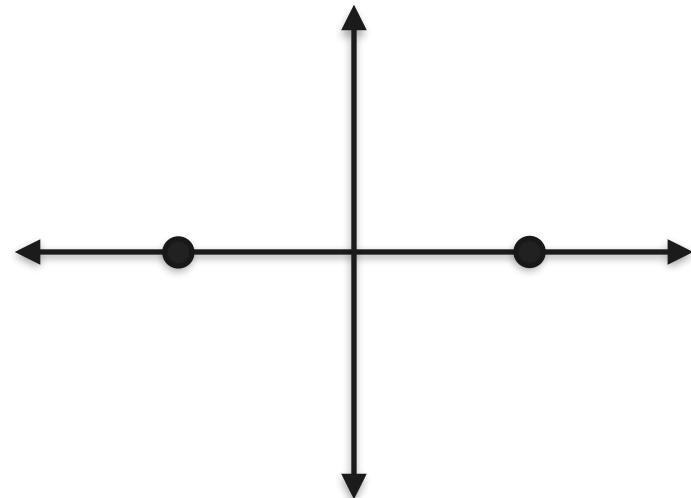


# Maximum Likelihood: Gaussian Example

Candidates

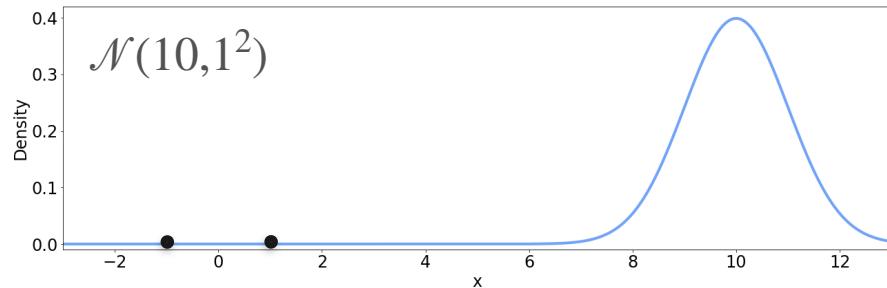


Observations

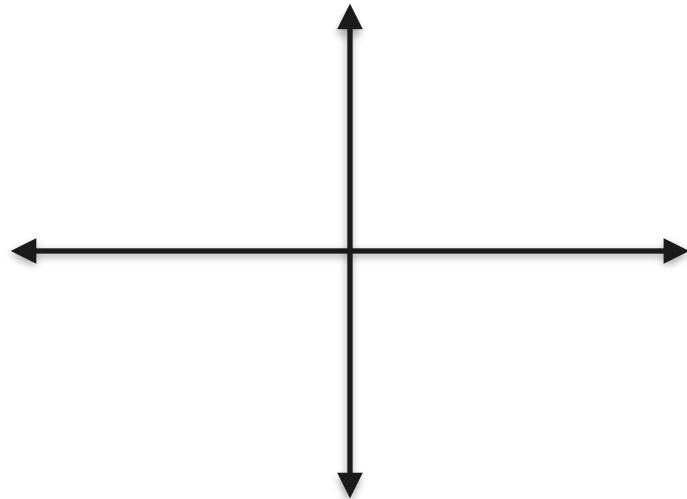


# Maximum Likelihood: Gaussian Example

Candidates

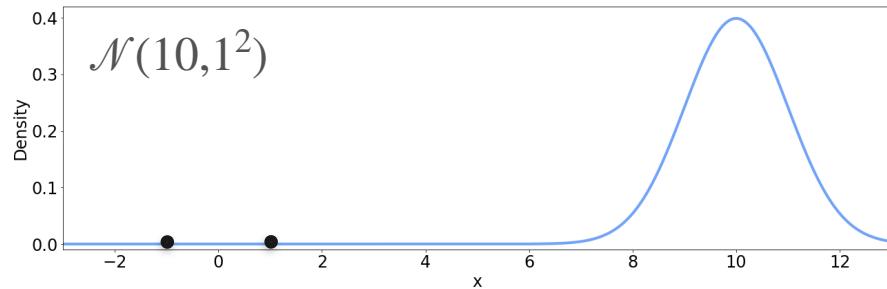


Observations

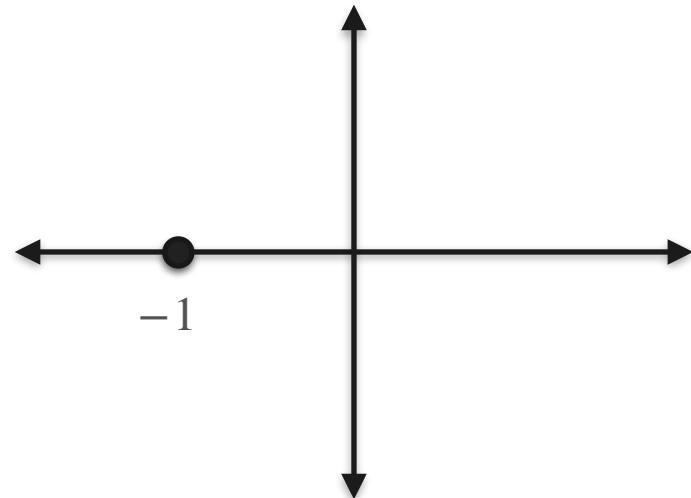


# Maximum Likelihood: Gaussian Example

Candidates

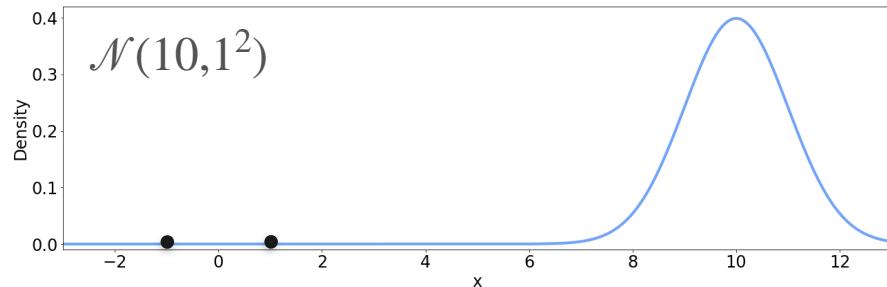


Observations

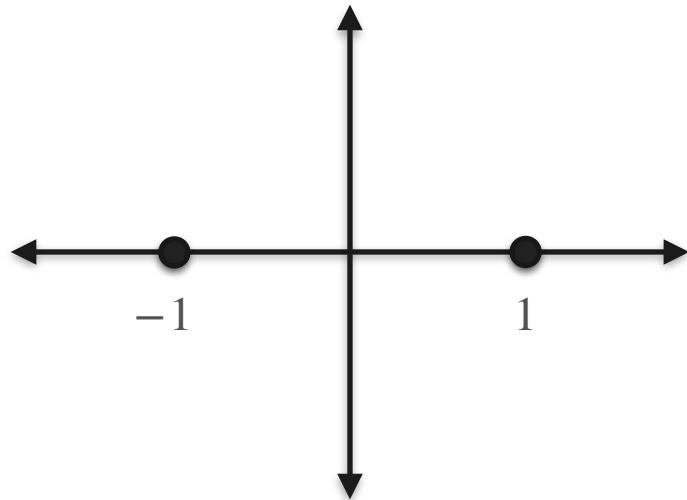


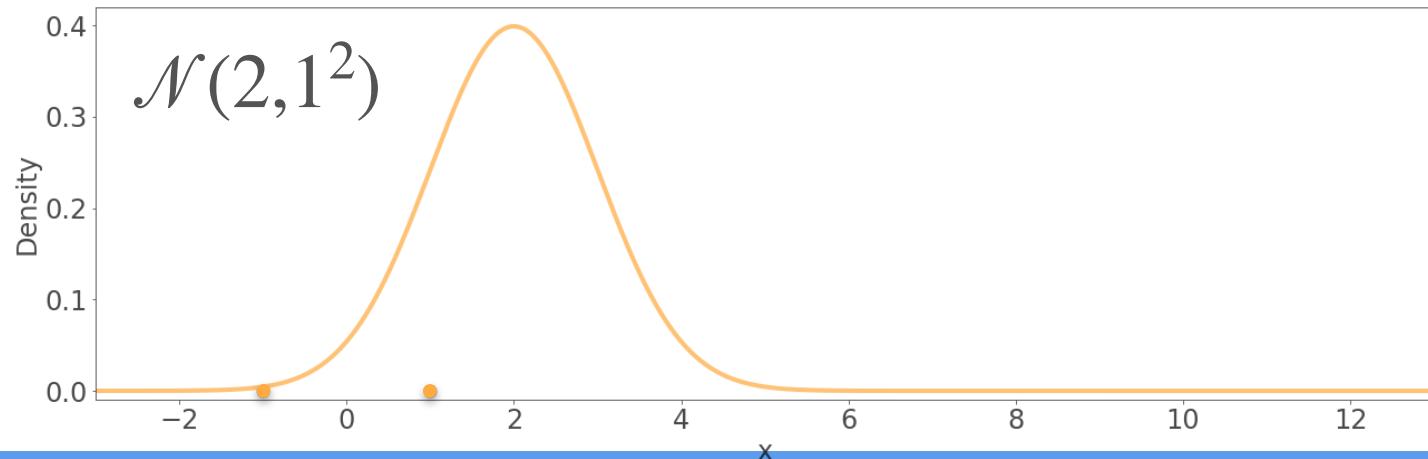
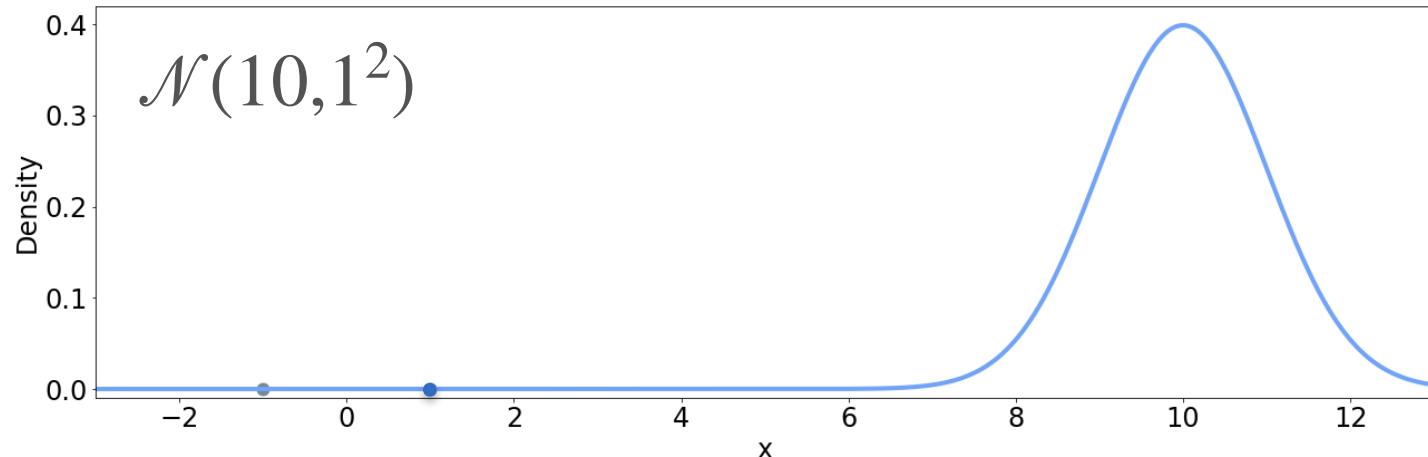
# Maximum Likelihood: Gaussian Example

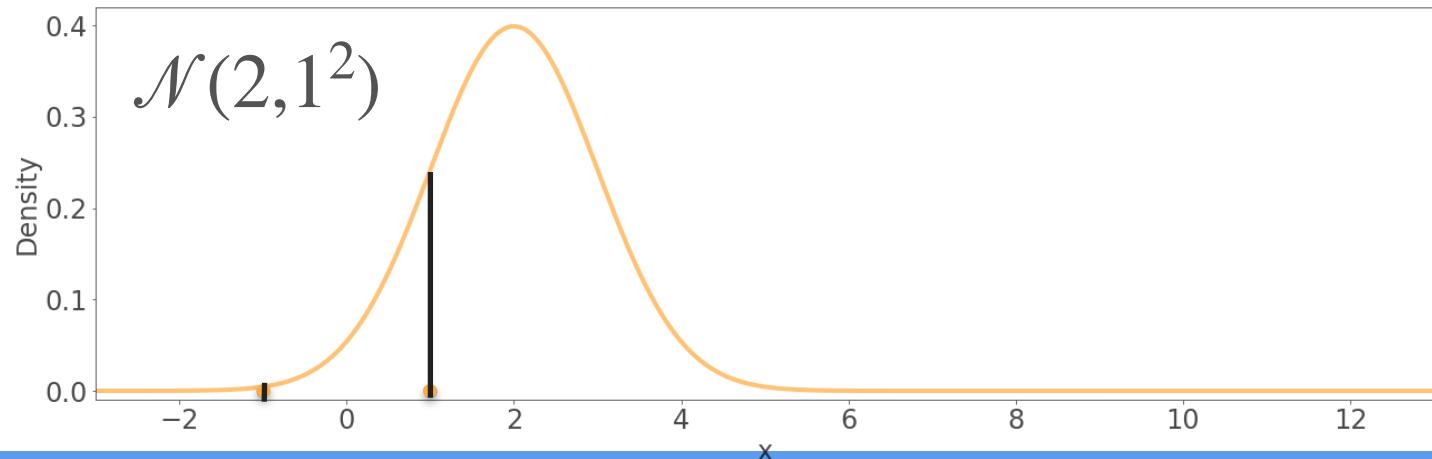
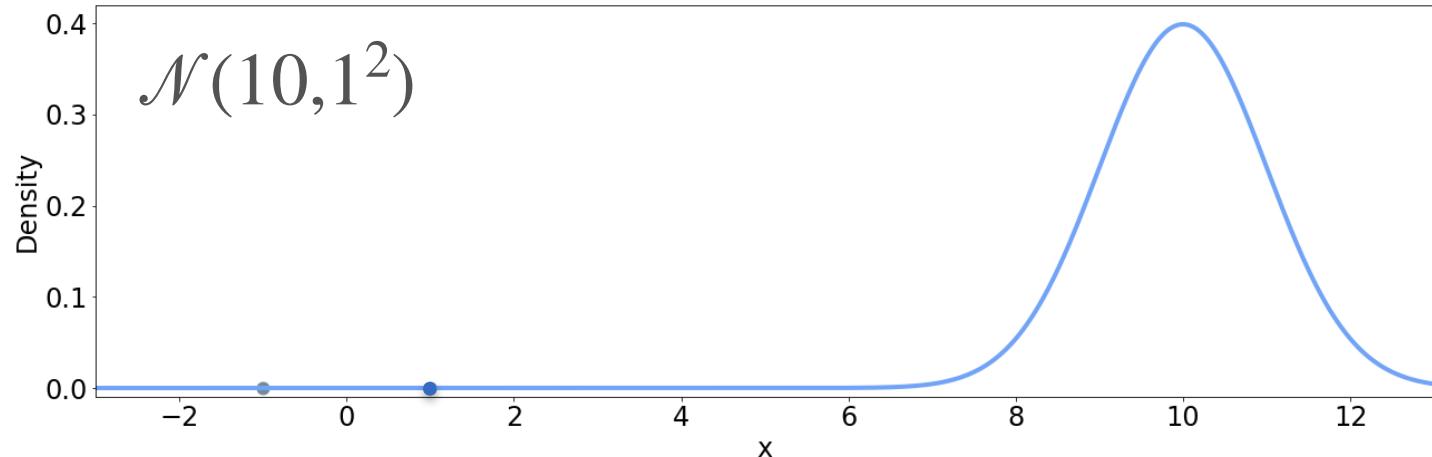
Candidates

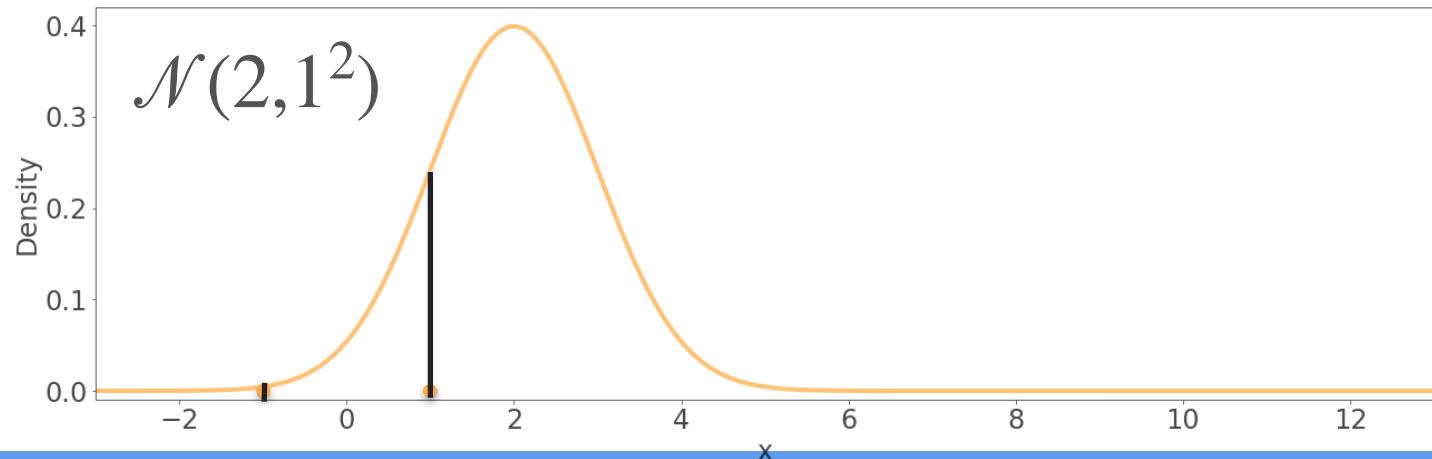
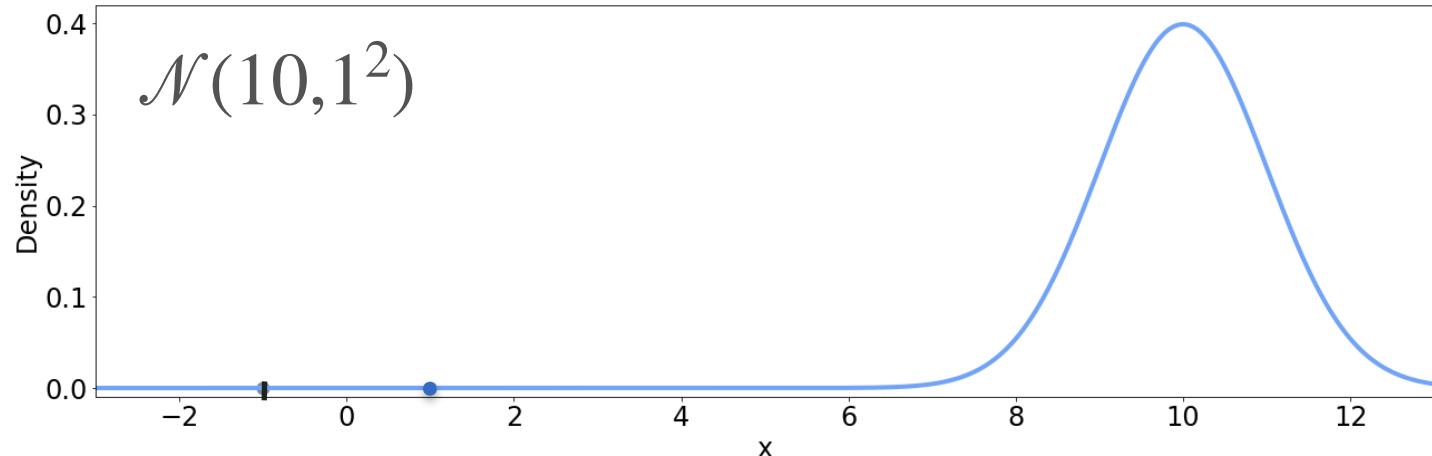


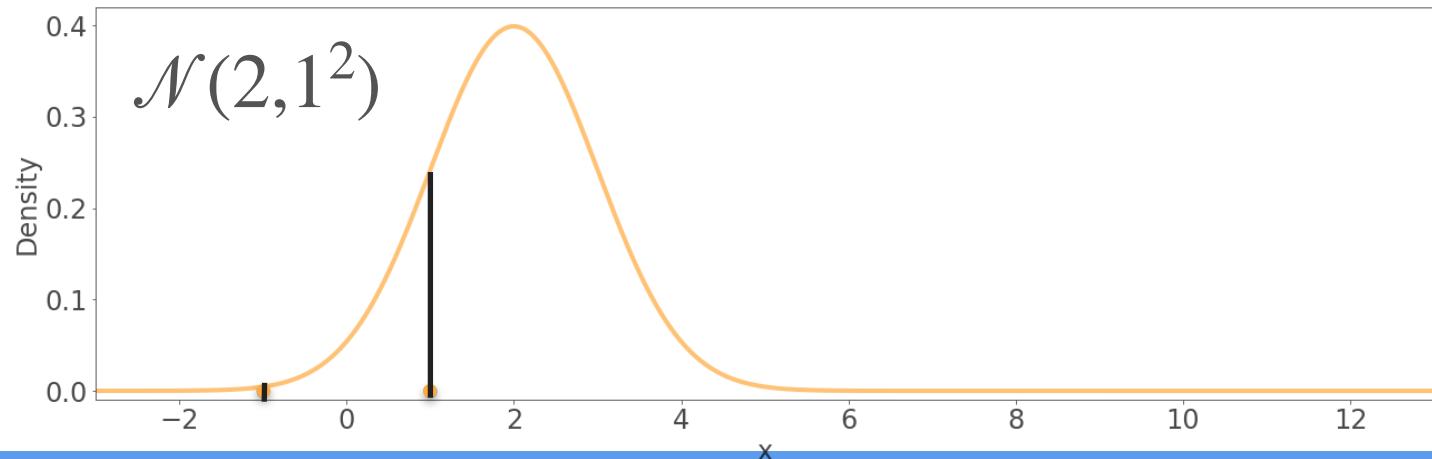
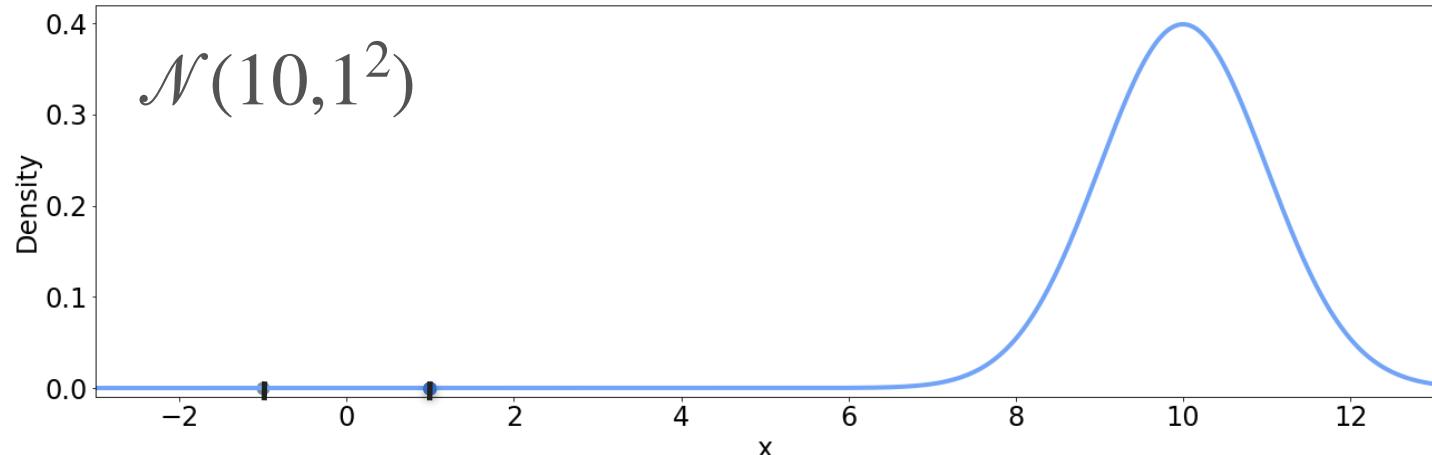
Observations

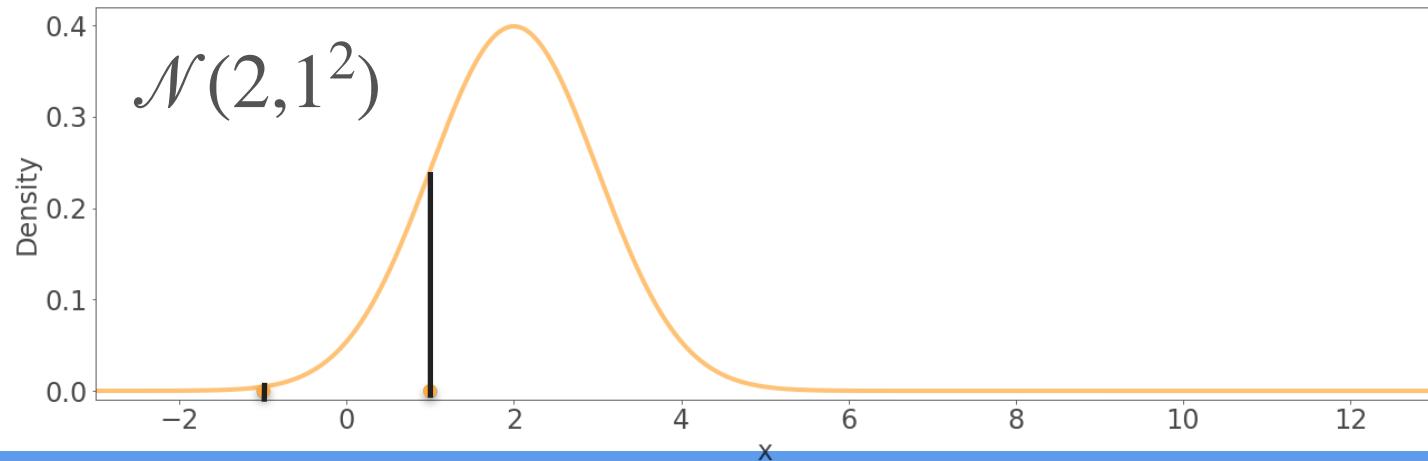
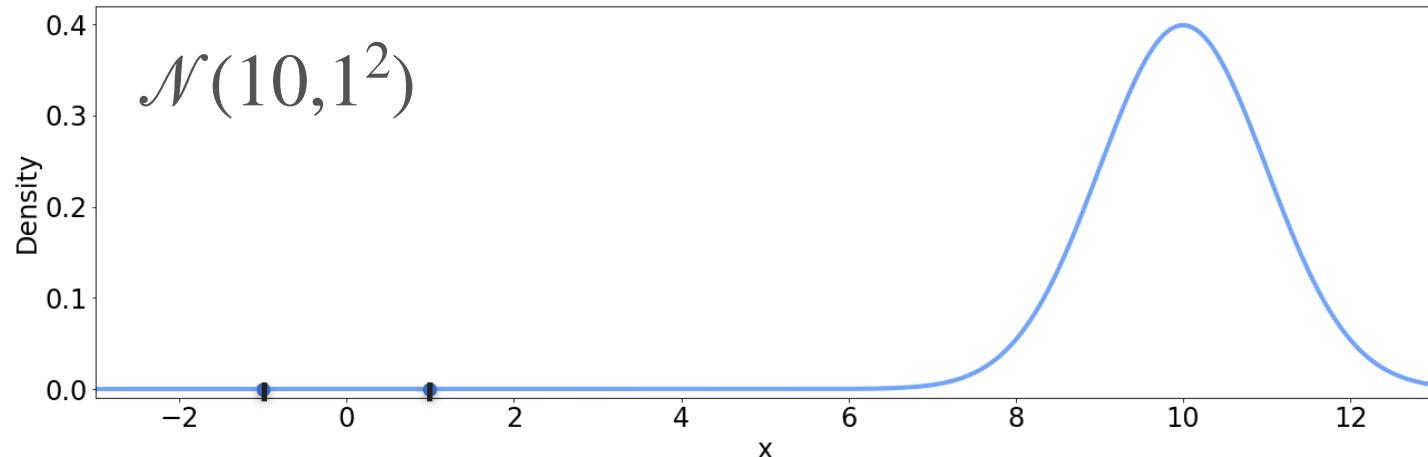


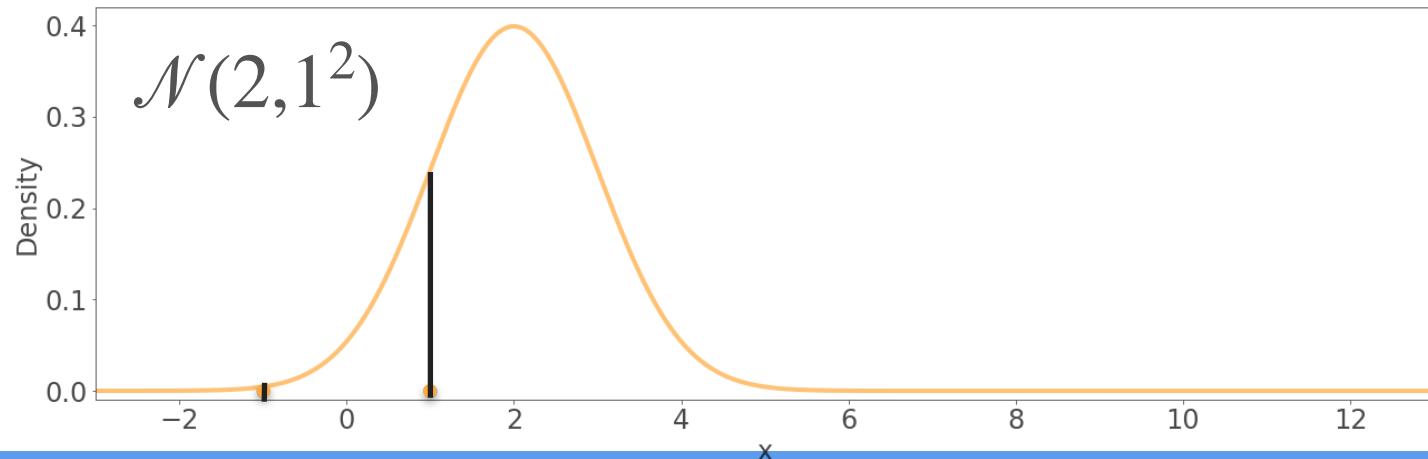
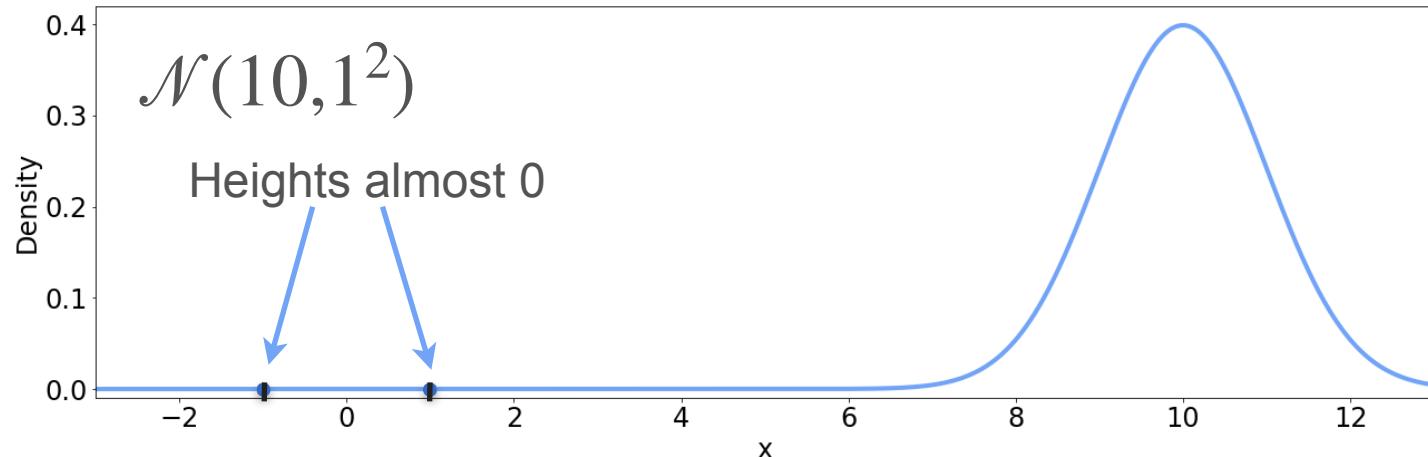


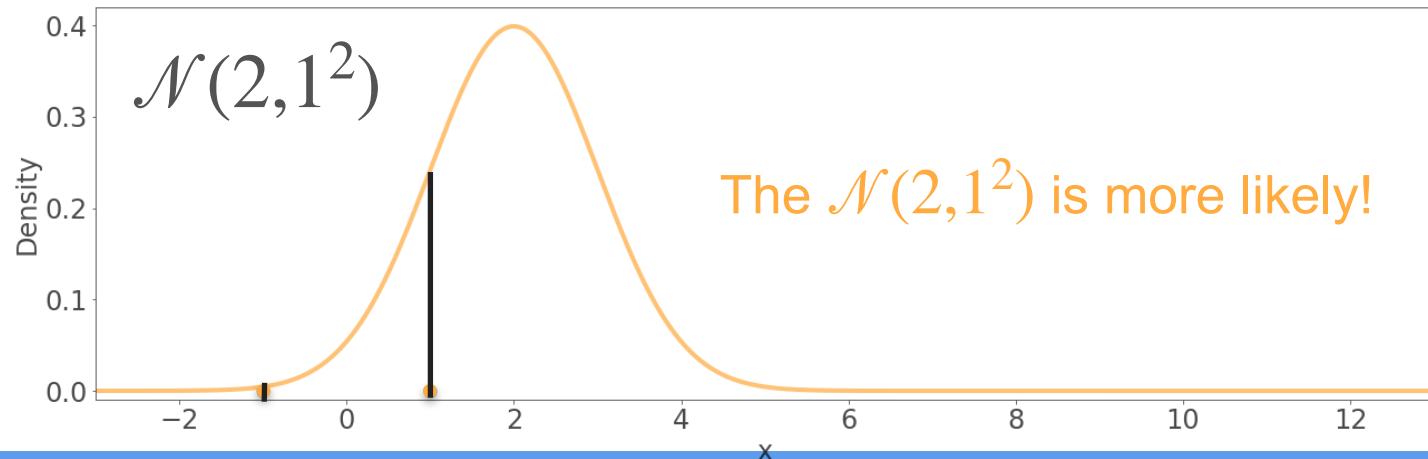
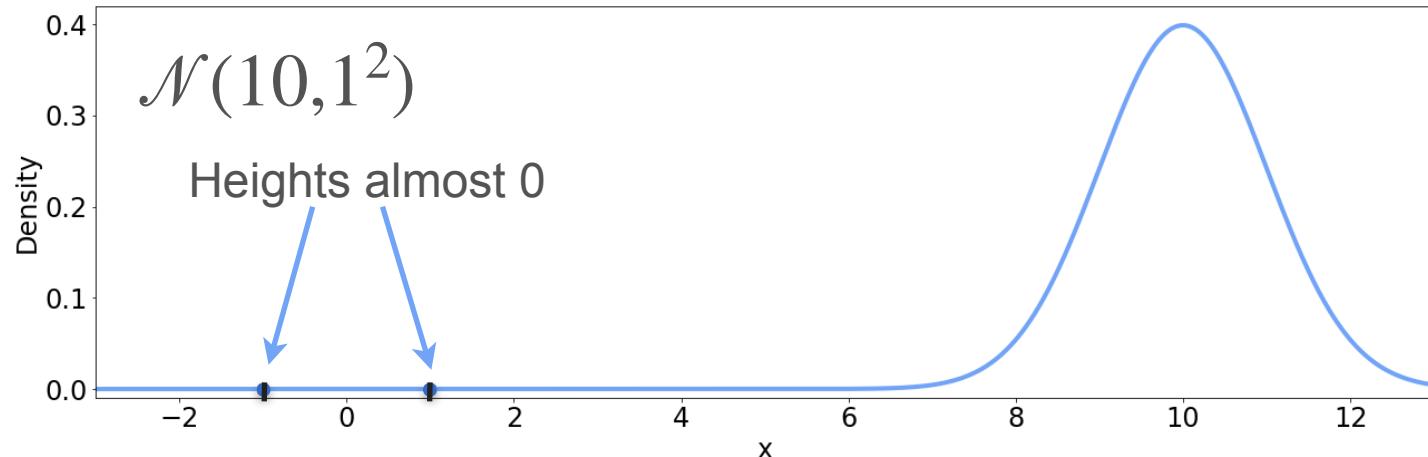


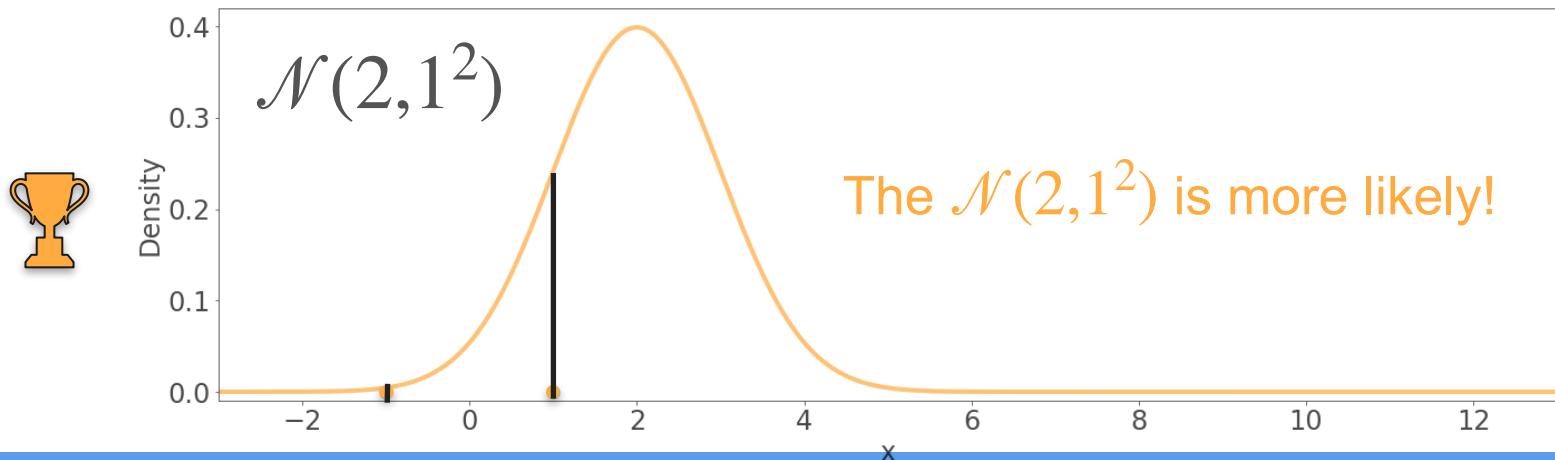
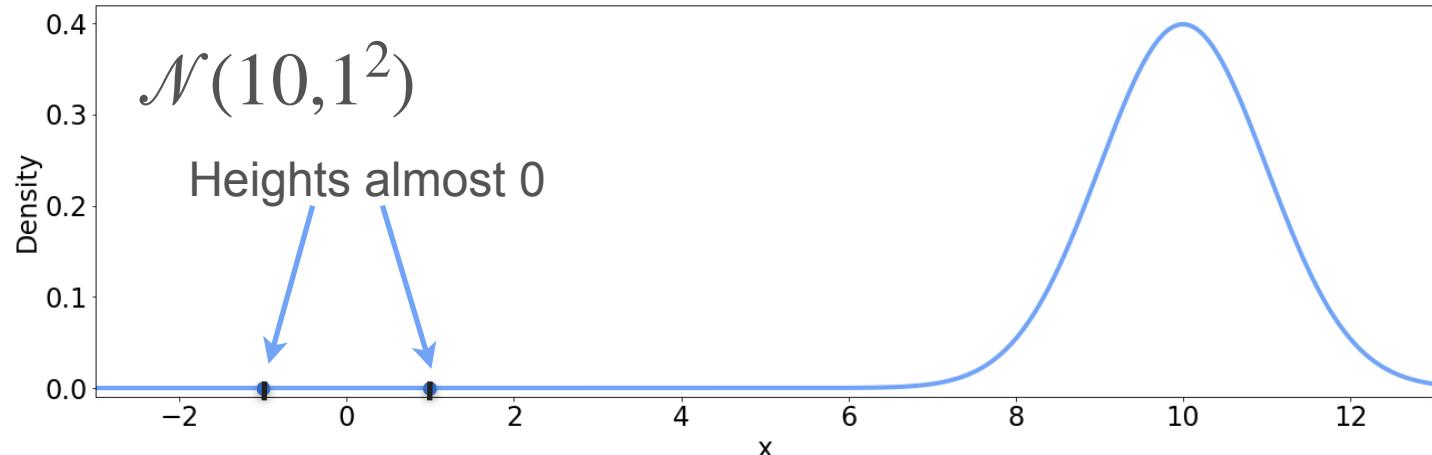






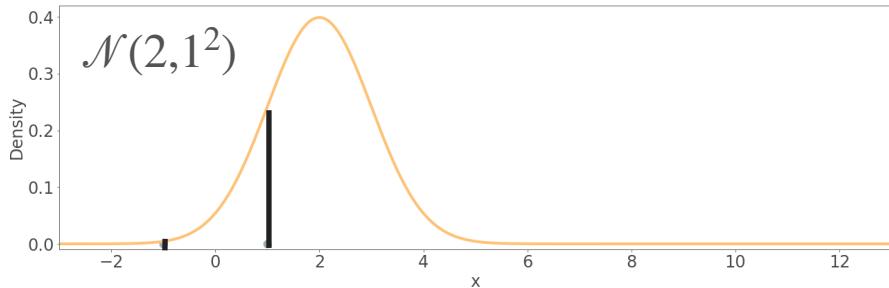
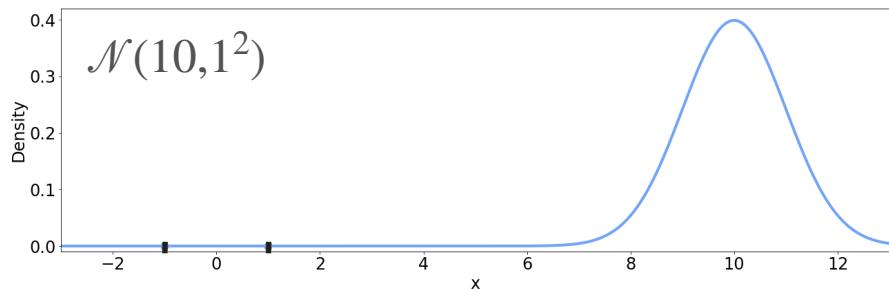




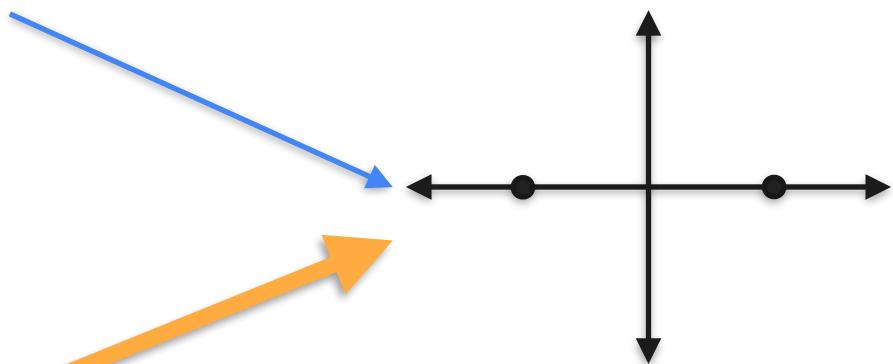


# Maximum Likelihood: Gaussian Example

Candidates

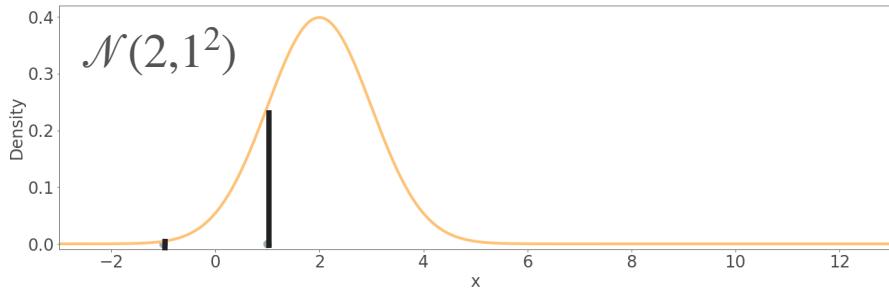
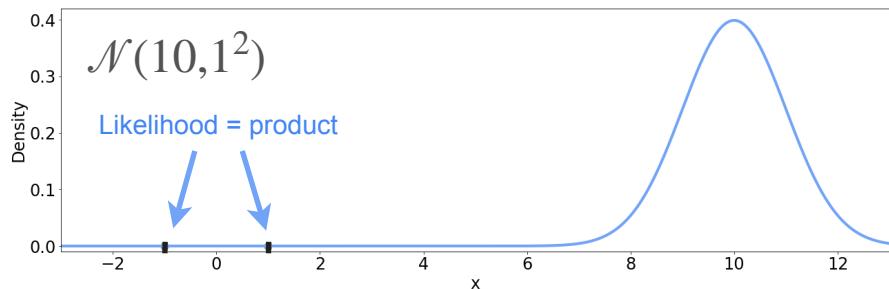


Observations

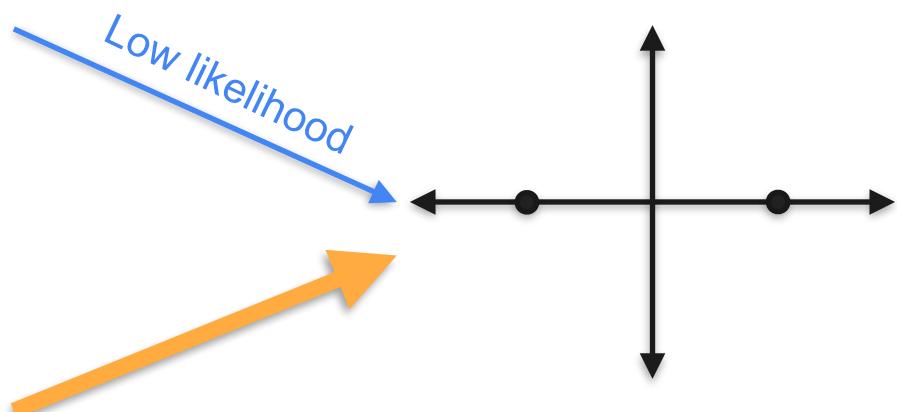


# Maximum Likelihood: Gaussian Example

Candidates

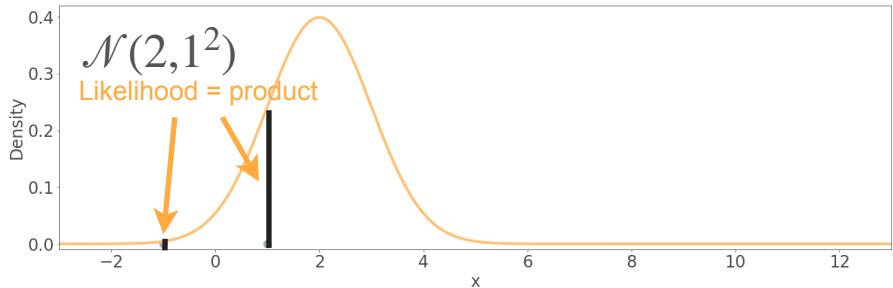
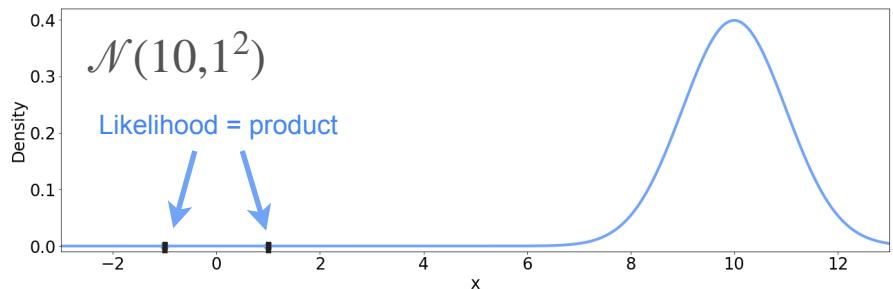


Observations

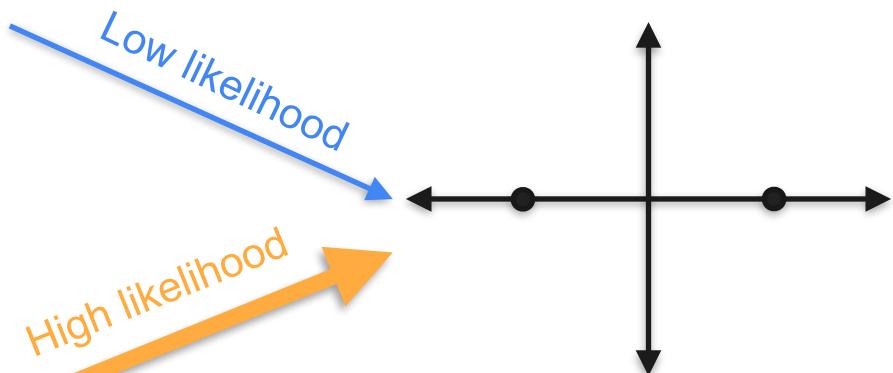


# Maximum Likelihood: Gaussian Example

Candidates



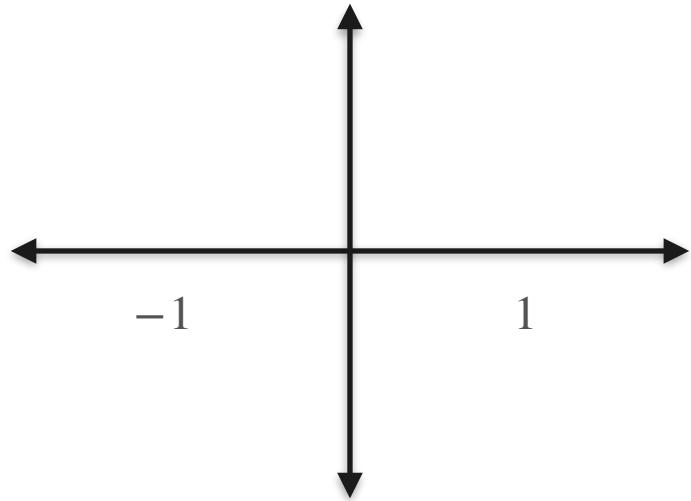
Observations



# Gaussians With Three Different Means

Candidates

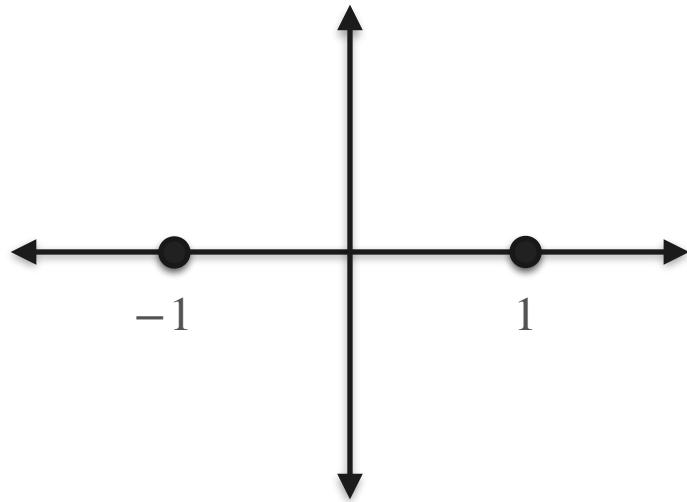
Observations



# Gaussians With Three Different Means

Candidates

Observations

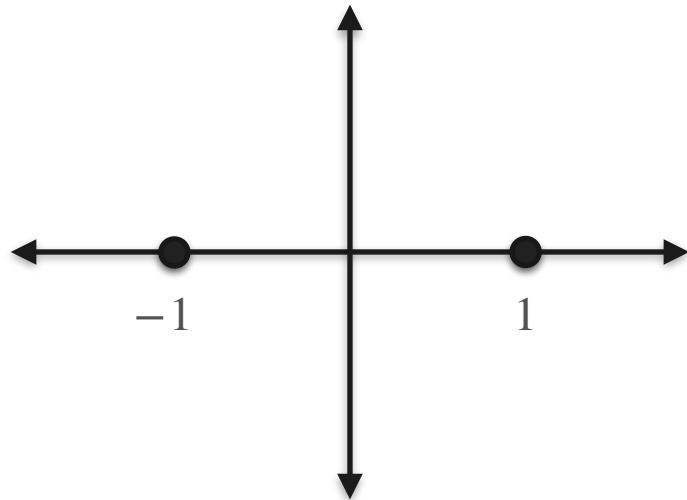


# Gaussians With Three Different Means

Candidates

$$\mathcal{N}(-1, 1^2)$$

Observations



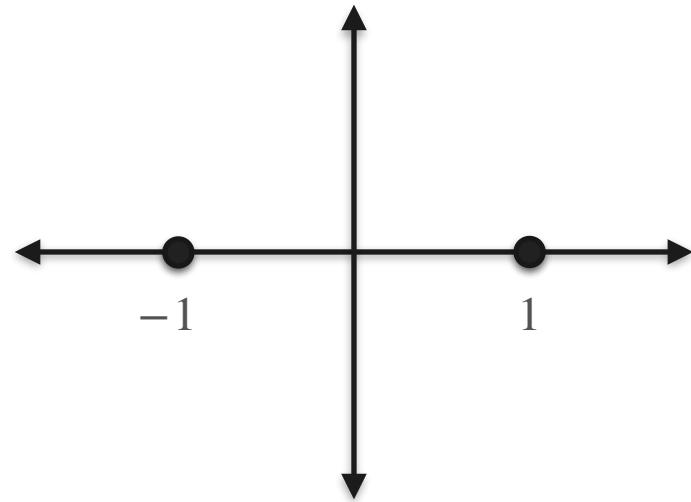
# Gaussians With Three Different Means

Candidates

$$\mathcal{N}(-1, 1^2)$$

$$\mathcal{N}(0, 1^2)$$

Observations



# Gaussians With Three Different Means

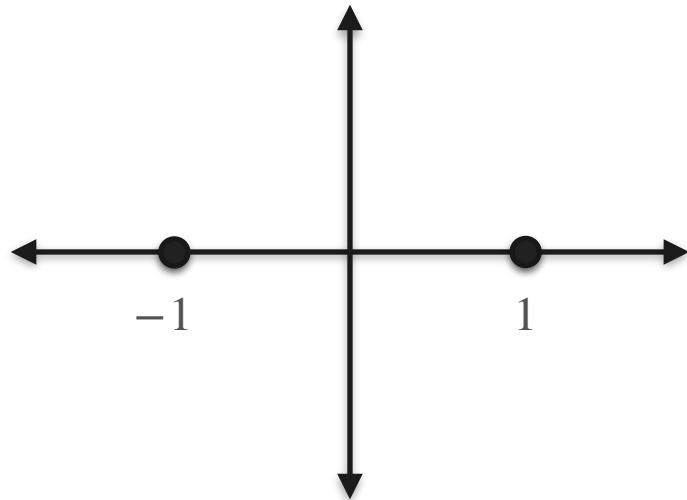
Candidates

$$\mathcal{N}(-1, 1^2)$$

$$\mathcal{N}(0, 1^2)$$

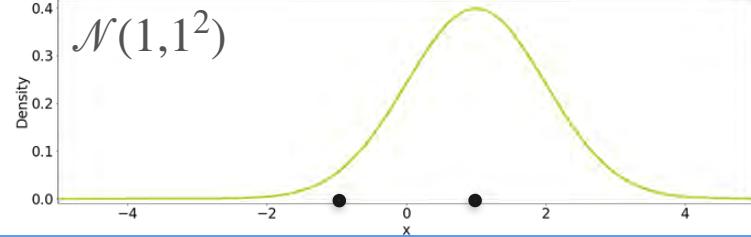
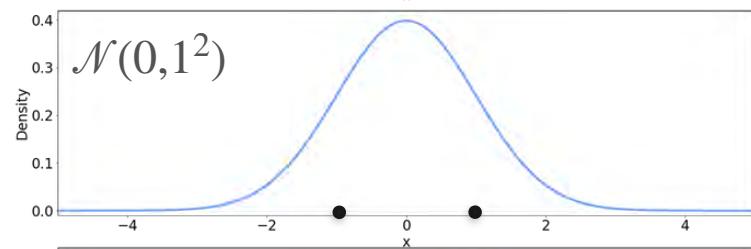
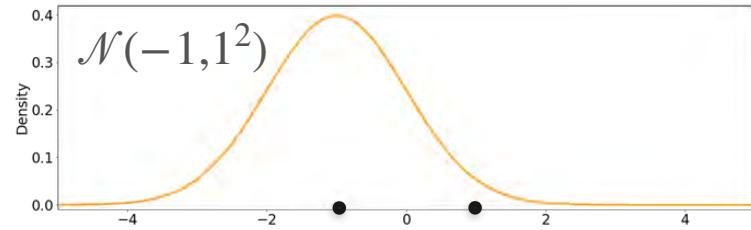
$$\mathcal{N}(1, 1^2)$$

Observations

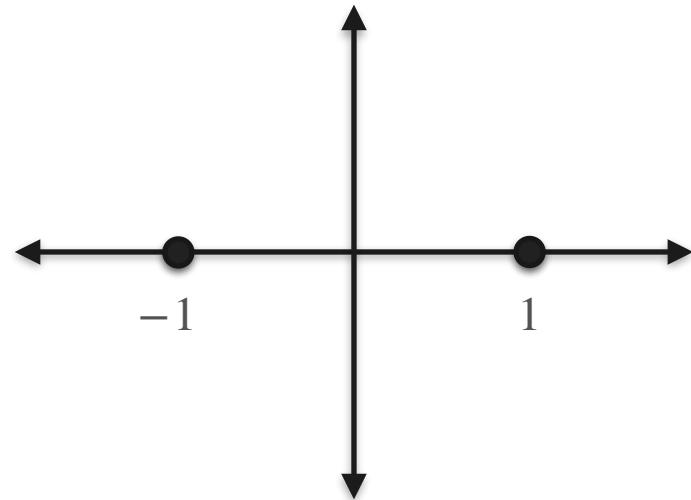


# Gaussians With Three Different Means

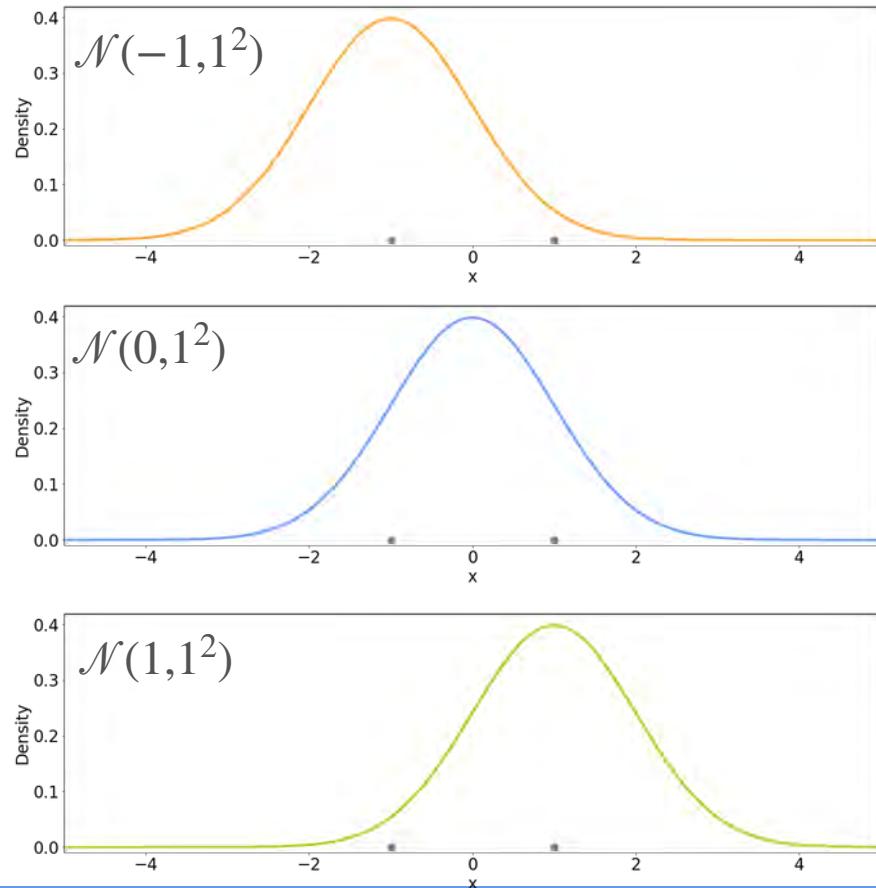
Candidates



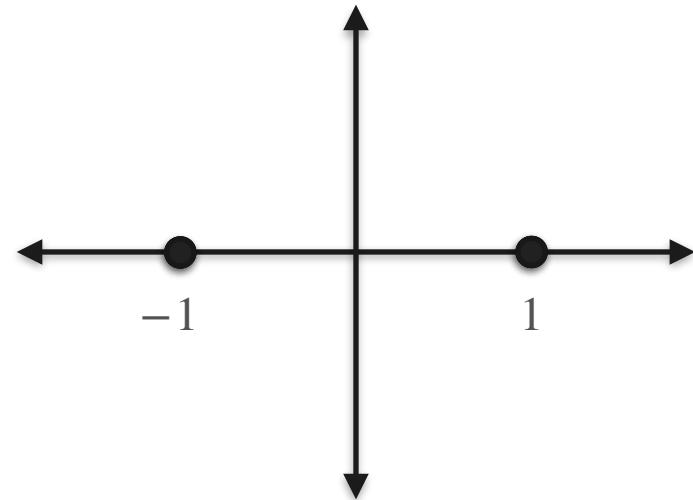
Observations



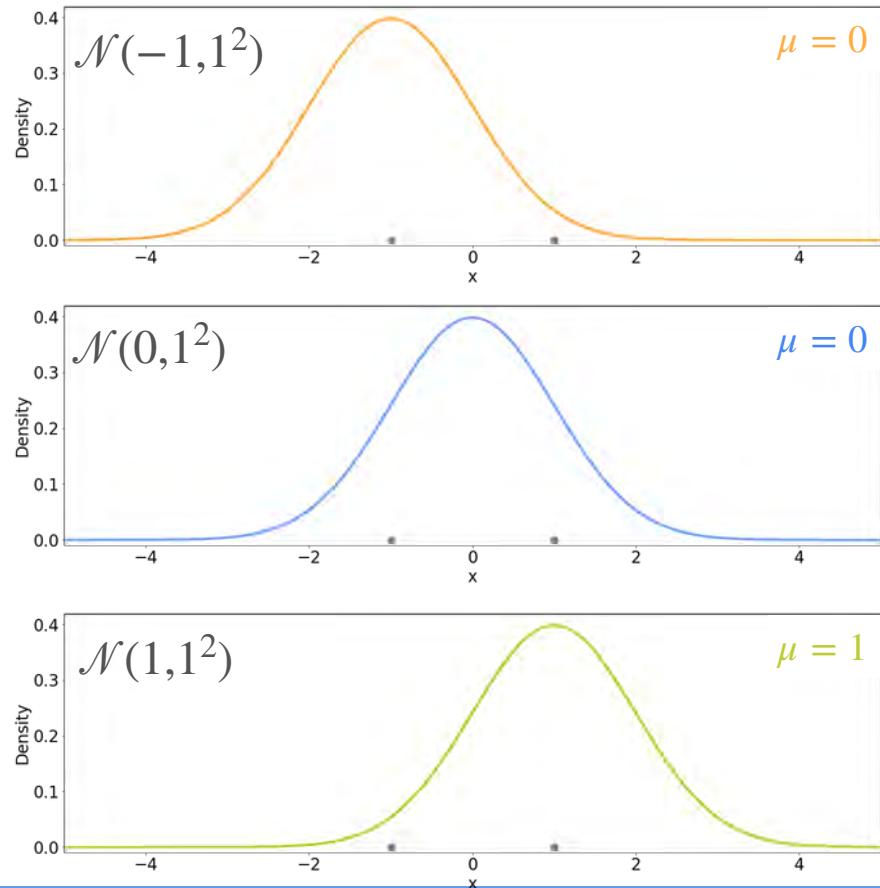
## Candidates



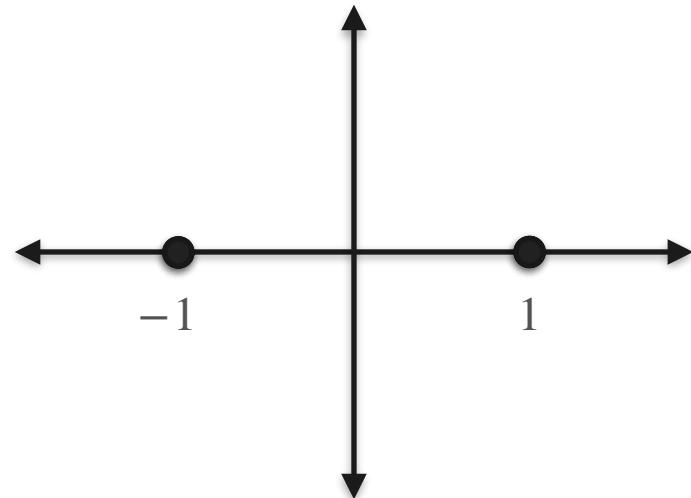
## Observations



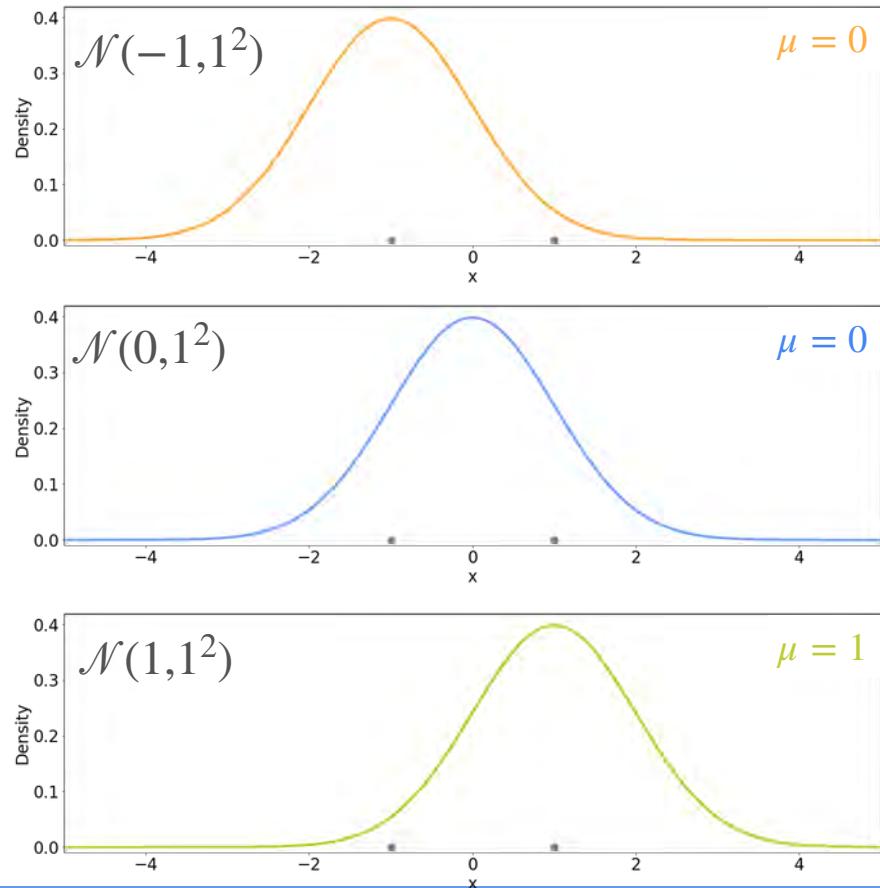
## Candidates



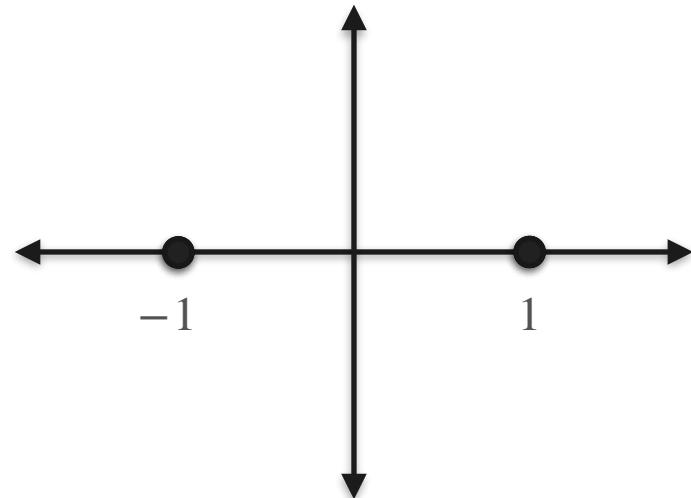
## Observations



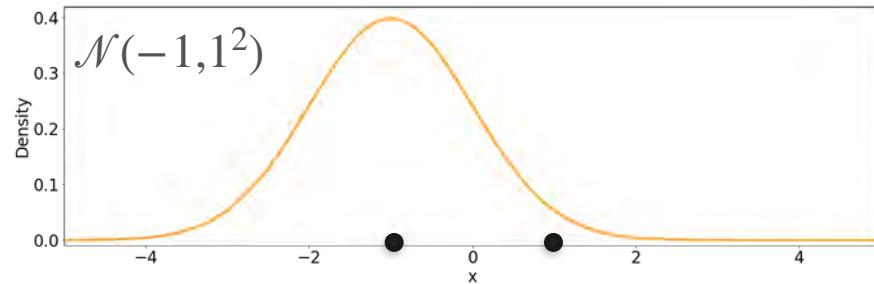
## Candidates



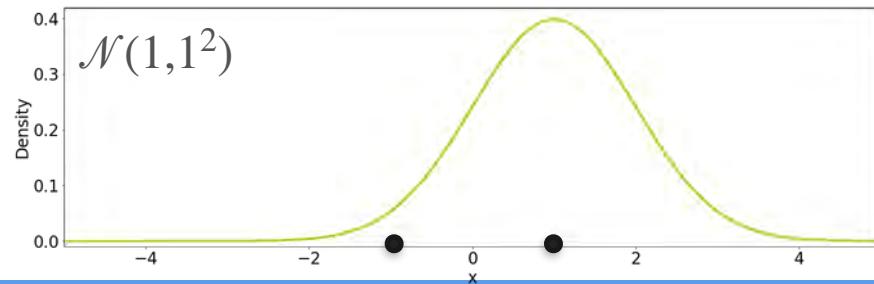
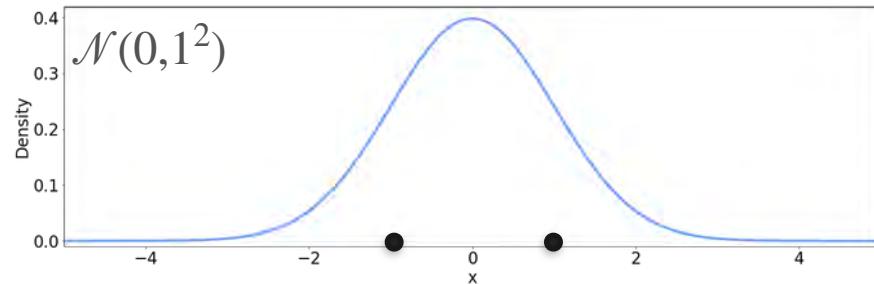
## Observations



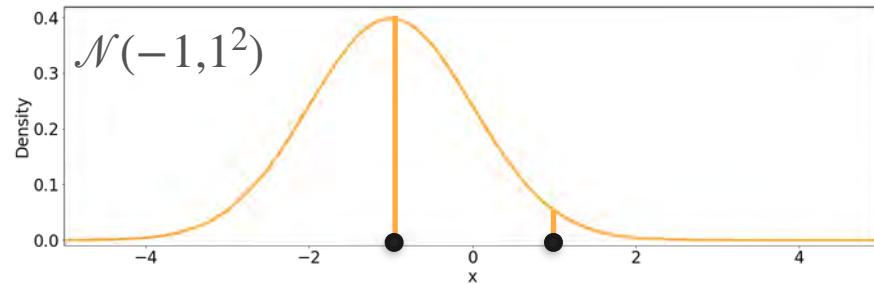
## Candidates



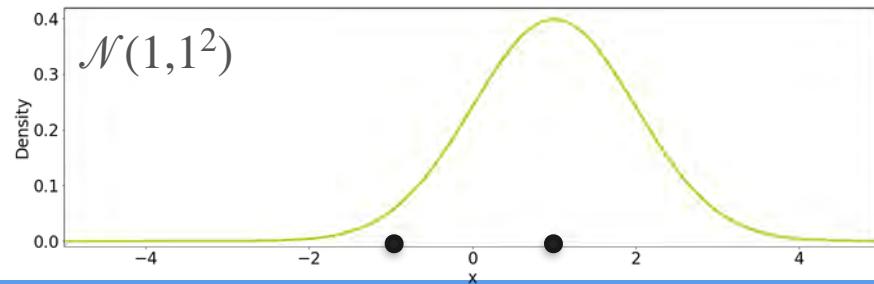
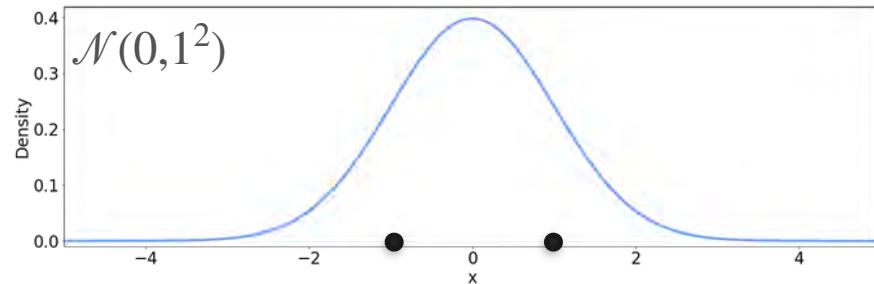
## Observations



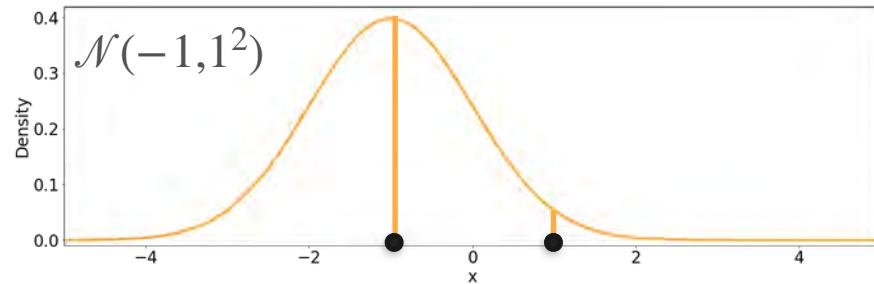
## Candidates



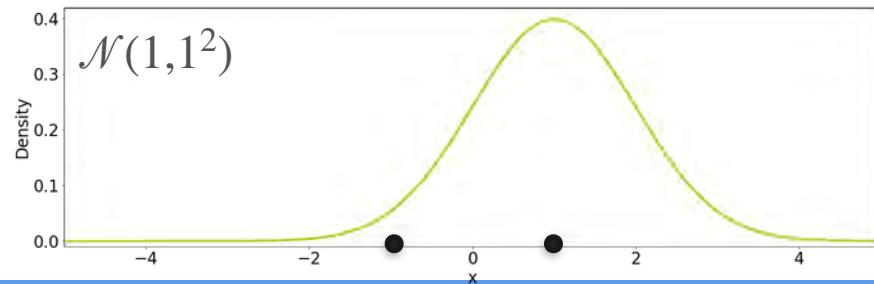
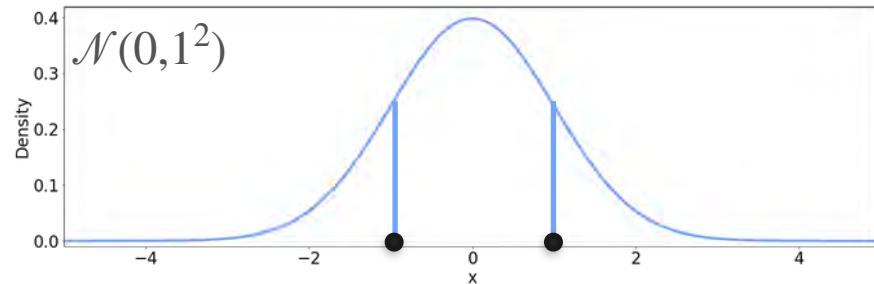
## Observations



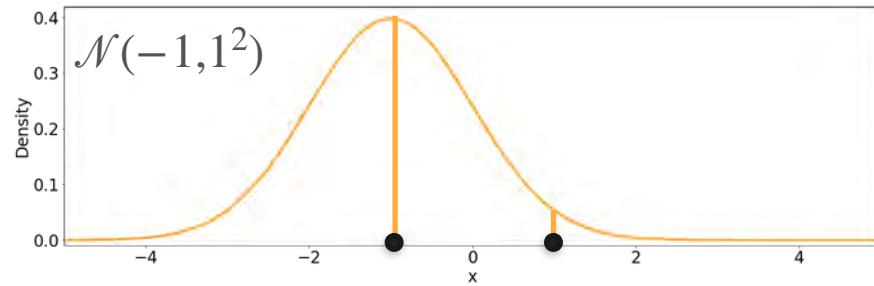
## Candidates



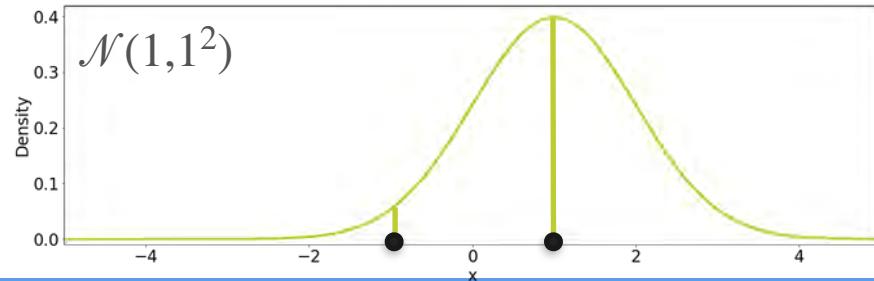
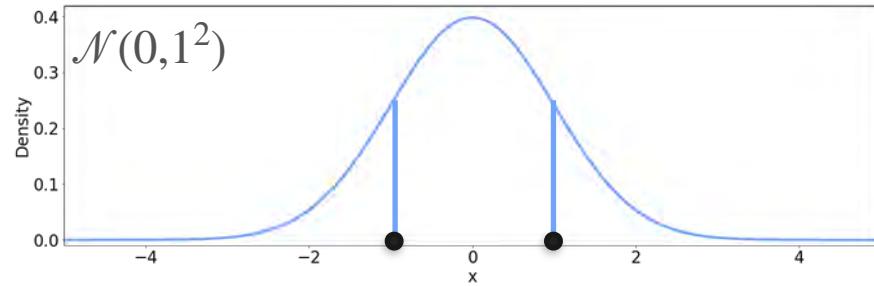
## Observations



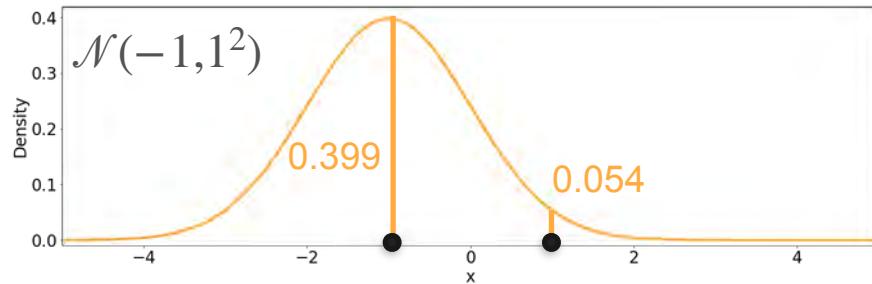
## Candidates



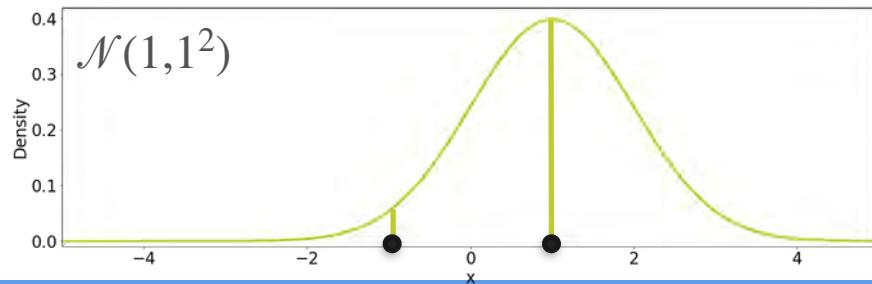
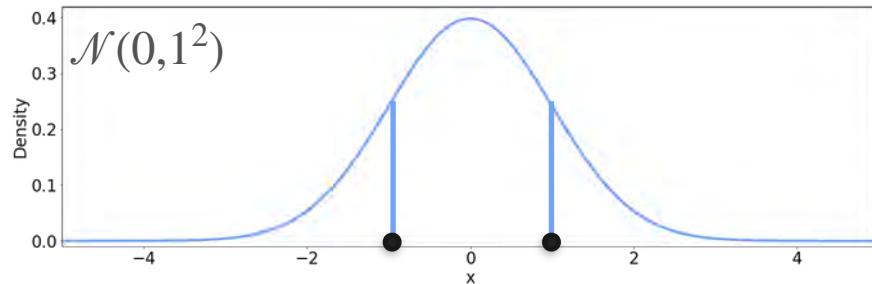
## Observations



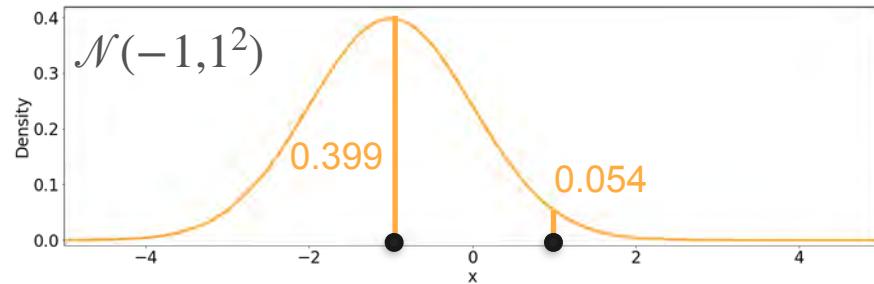
## Candidates



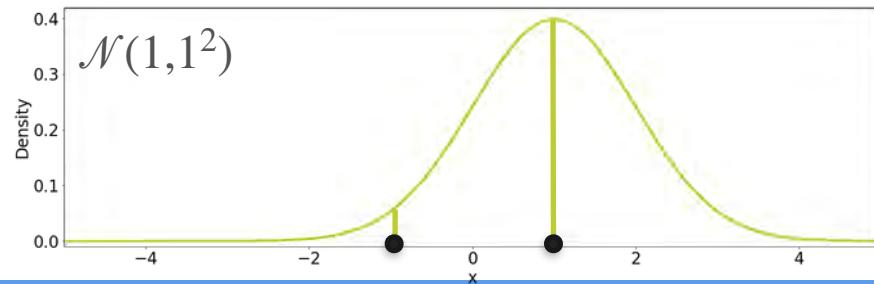
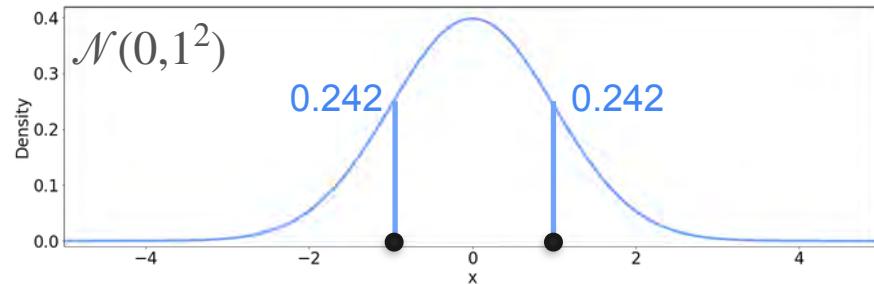
## Observations



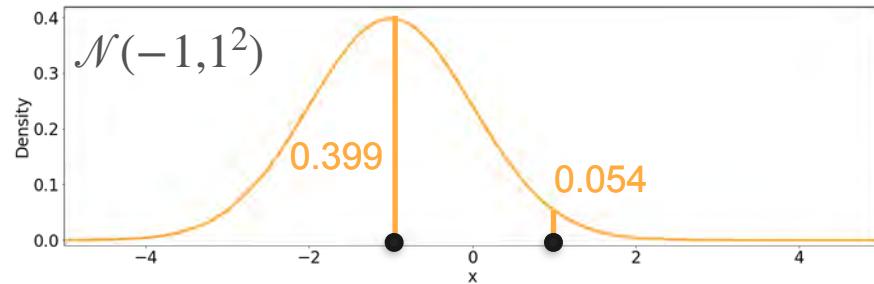
## Candidates



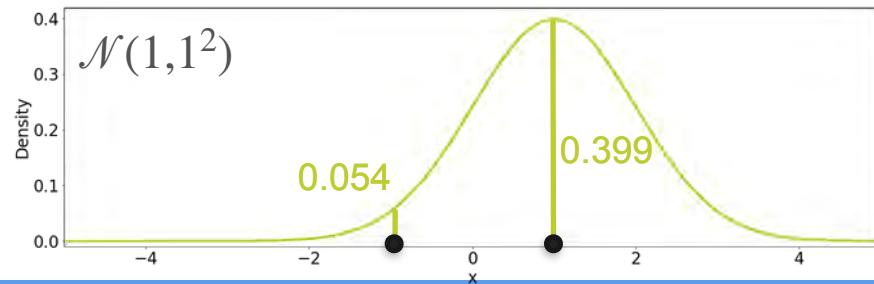
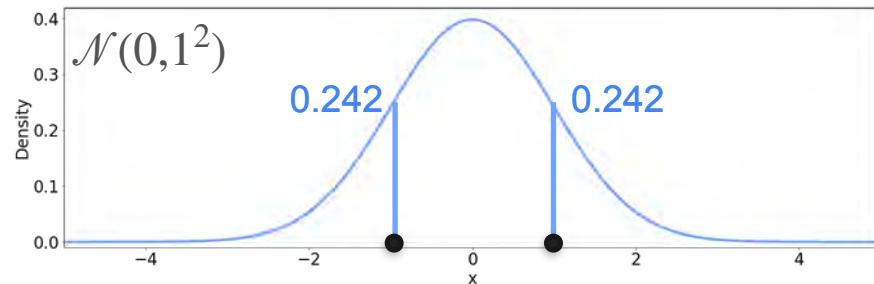
## Observations



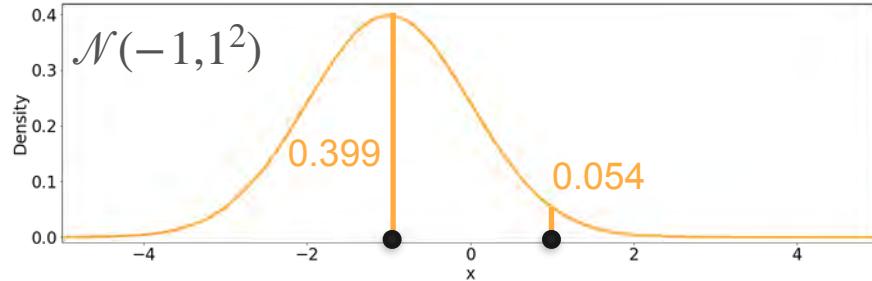
## Candidates



## Observations

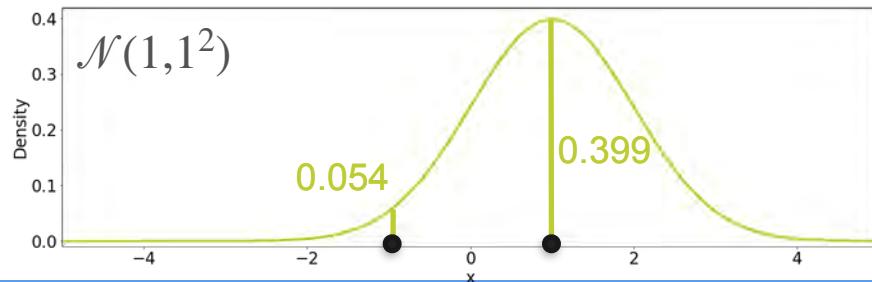
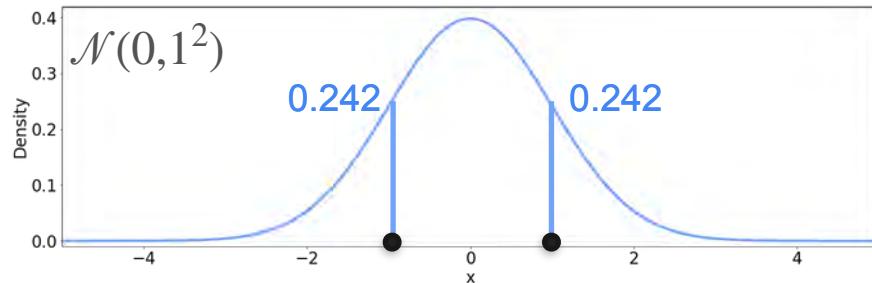


## Candidates

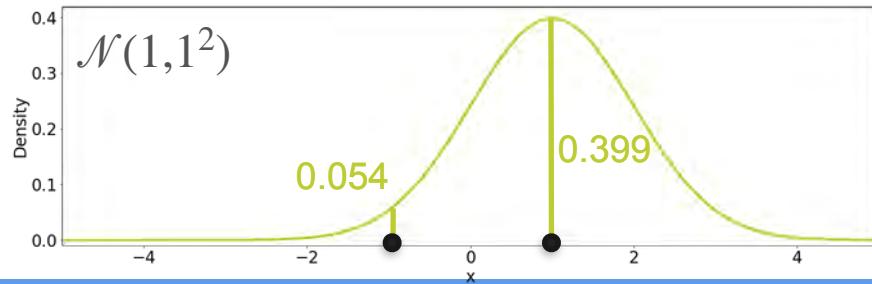
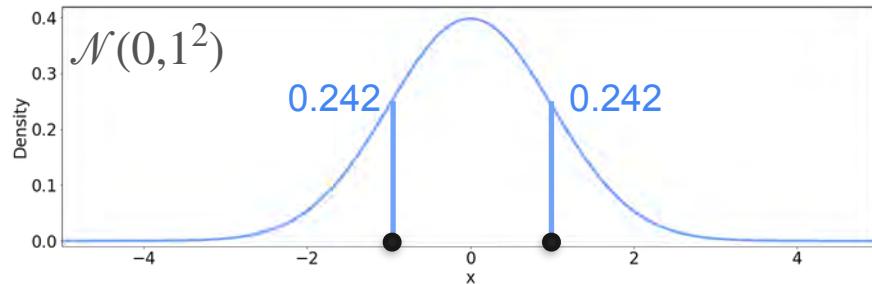
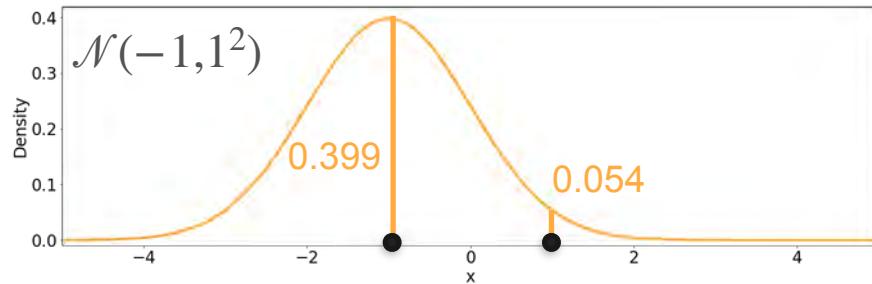


## Observations

$$0.399 \cdot 0.054 = 0.022$$



## Candidates

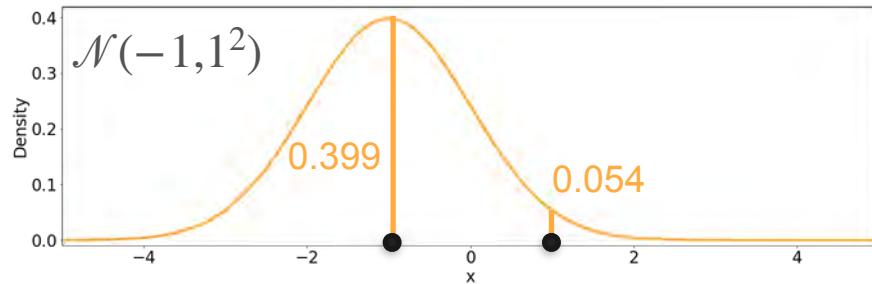


## Observations

$$0.399 \cdot 0.054 = 0.022$$

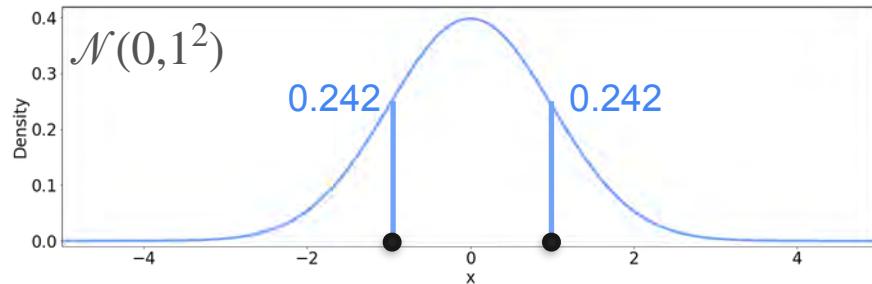
$$0.242 \cdot 0.242 = 0.059$$

## Candidates

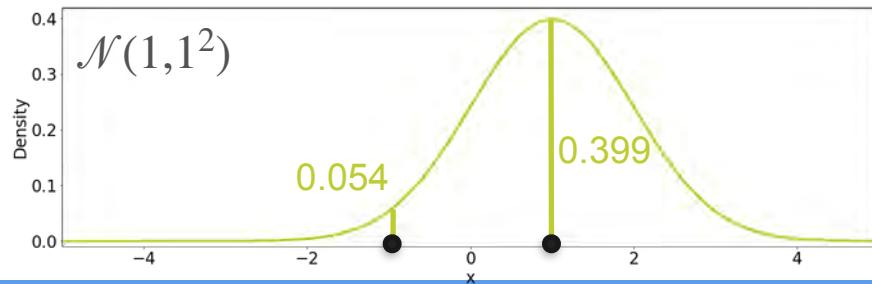


## Observations

$$0.399 \cdot 0.054 = 0.022$$

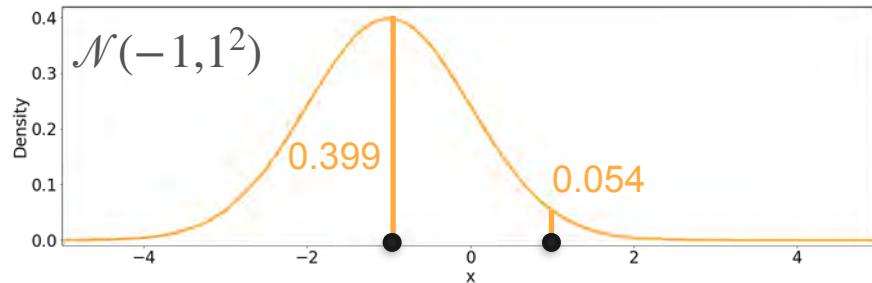


$$0.242 \cdot 0.242 = 0.059$$



$$0.054 \cdot 0.399 = 0.022$$

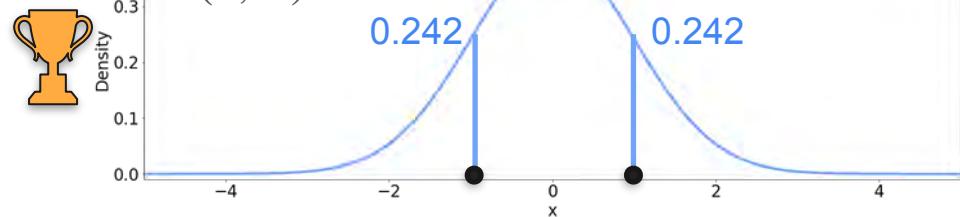
## Candidates



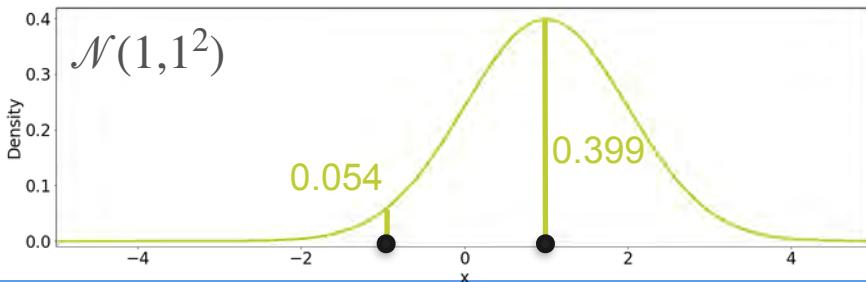
## Observations

$$0.399 \cdot 0.054 = 0.022$$

The  $\mathcal{N}(0, 1^2)$  is more likely!

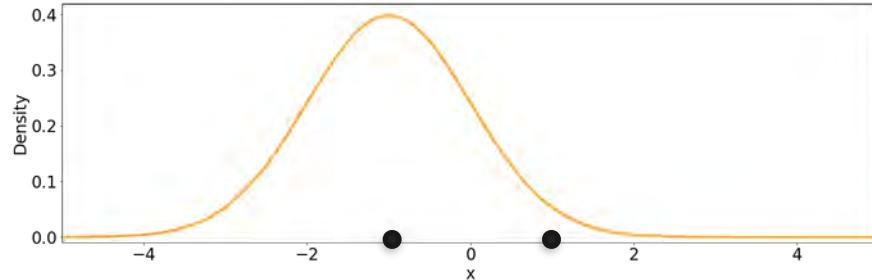


$$0.242 \cdot 0.242 = 0.059$$

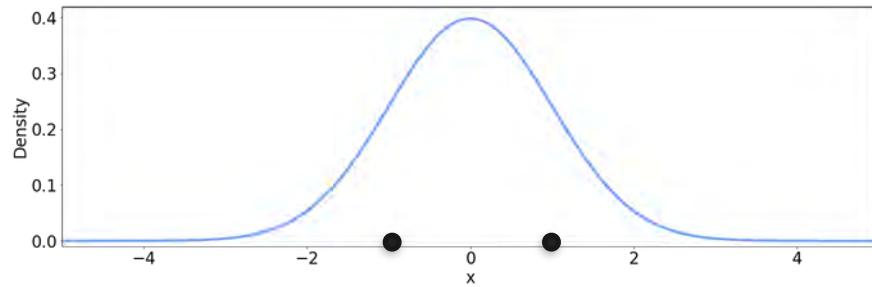


$$0.054 \cdot 0.399 = 0.022$$

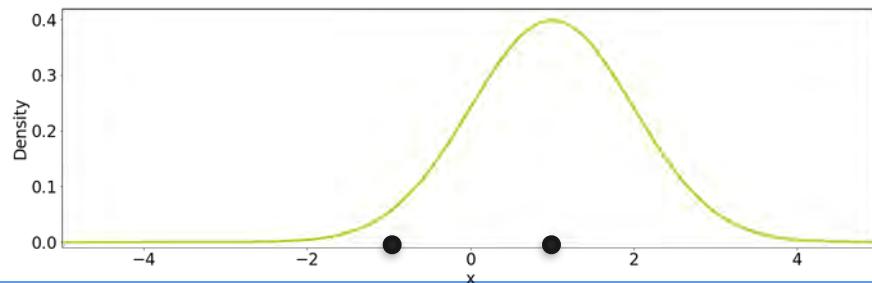
# Candidates



Likelihood = 0.022

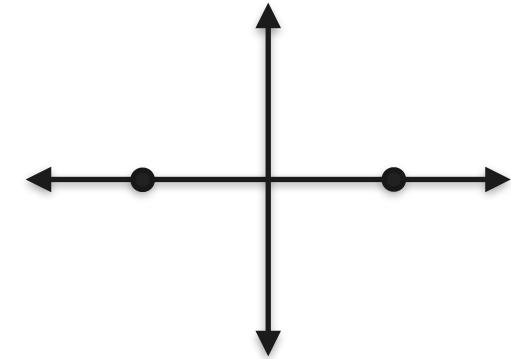


Likelihood = 0.059

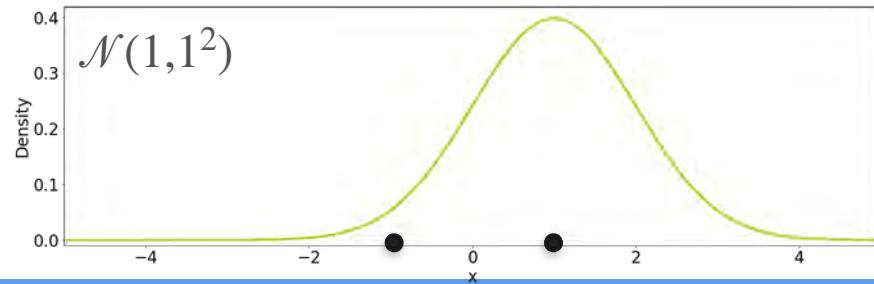
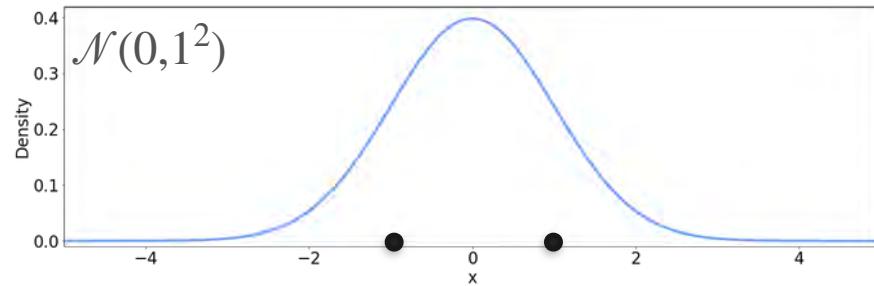
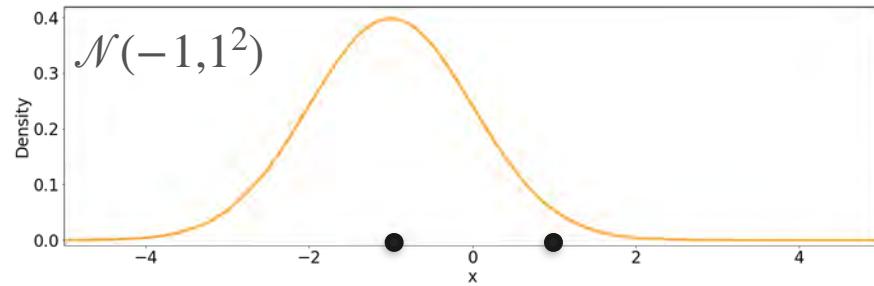


Likelihood = 0.022

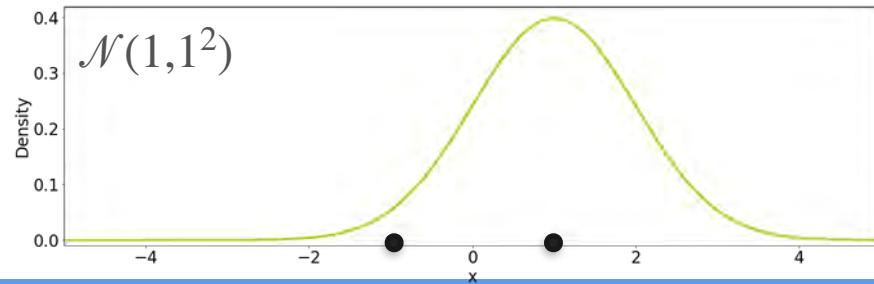
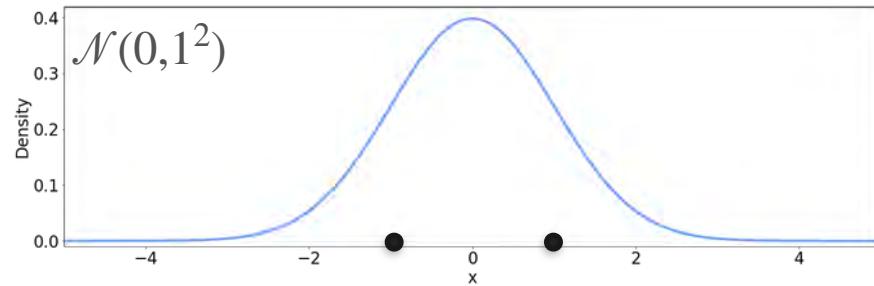
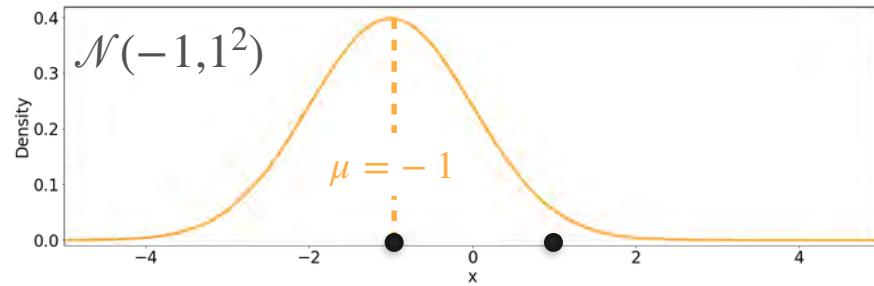
# Observations



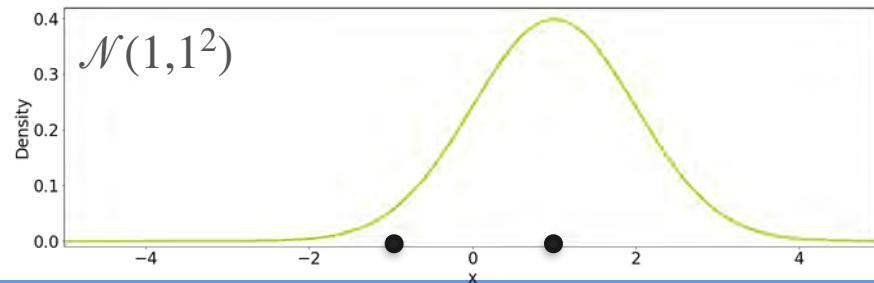
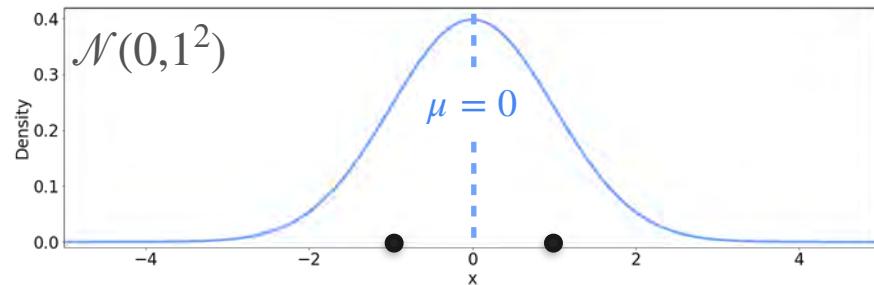
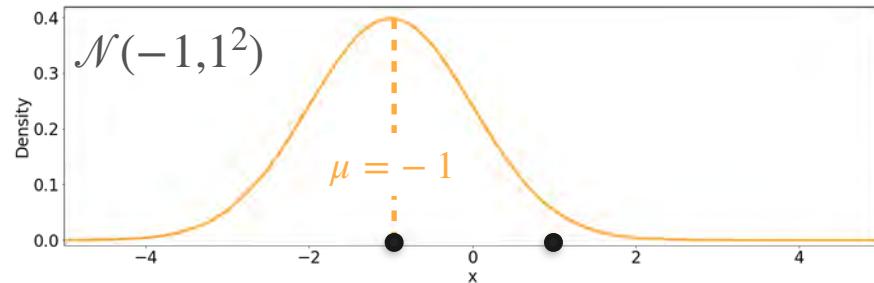
# Candidates



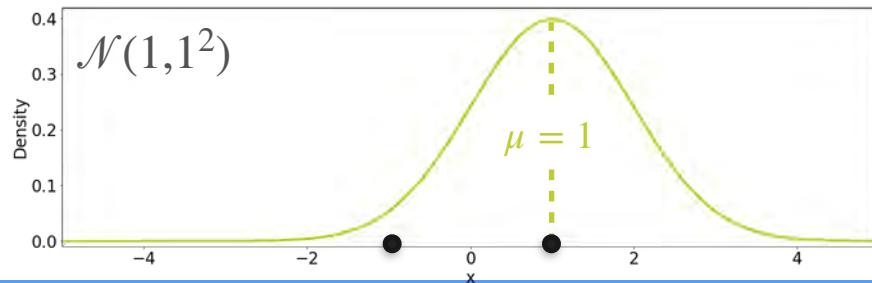
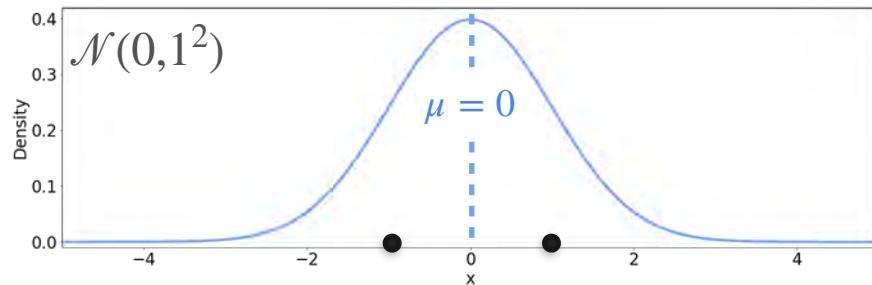
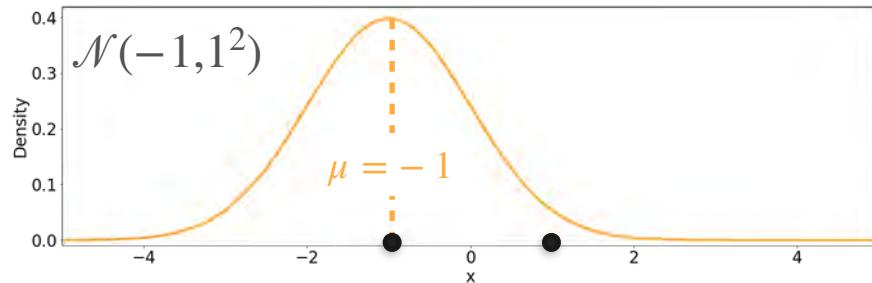
# Candidates



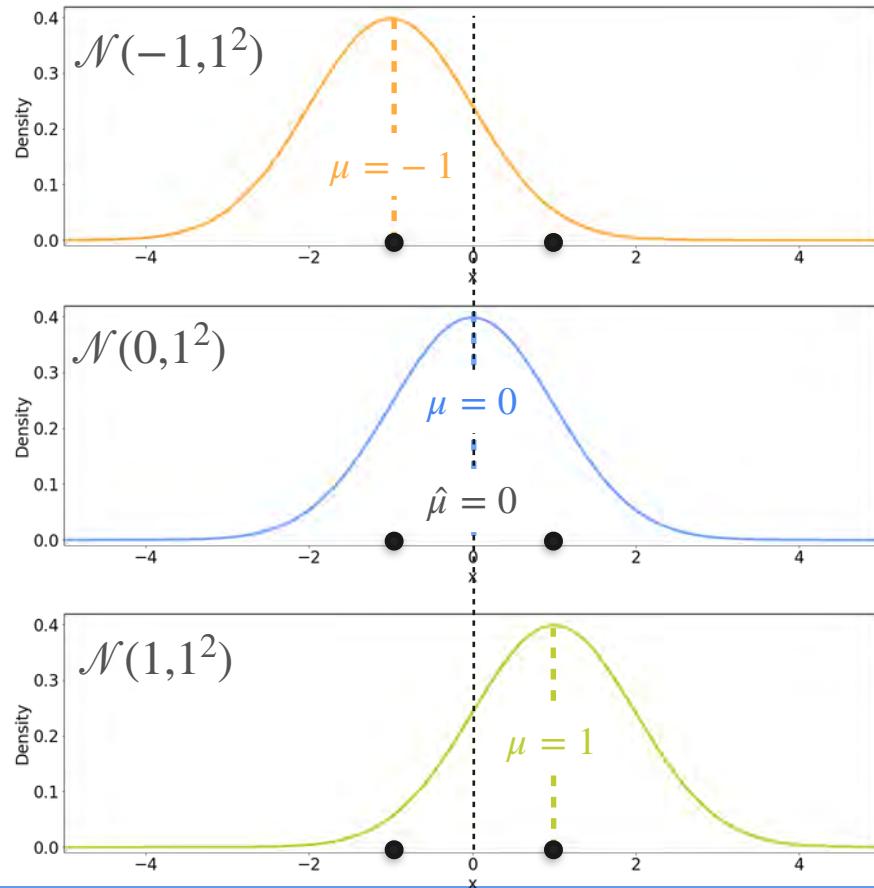
# Candidates



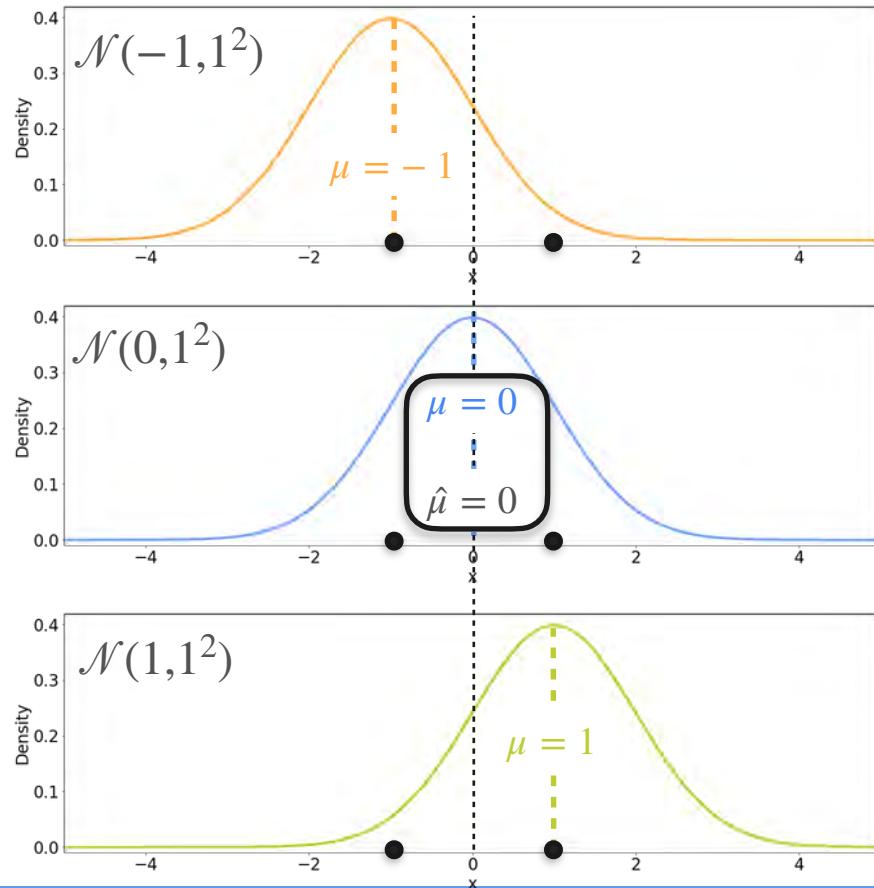
# Candidates



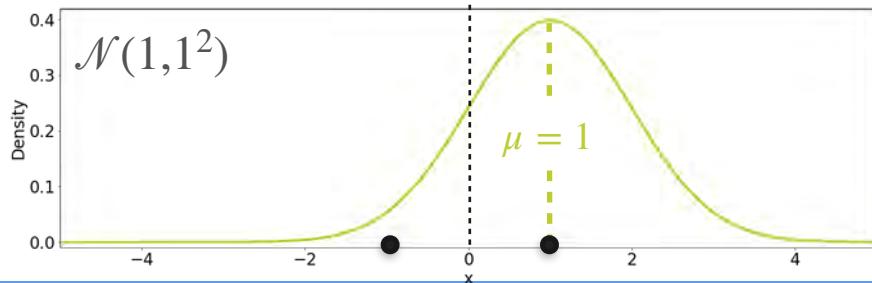
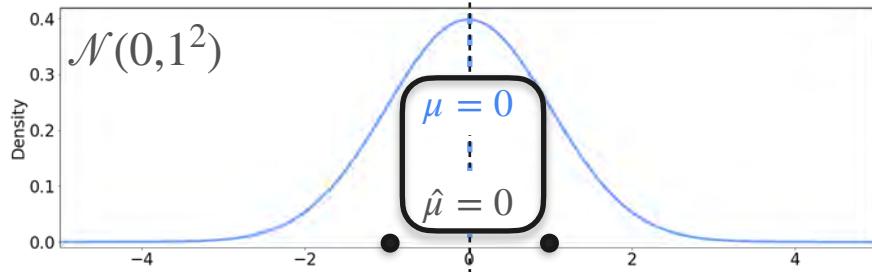
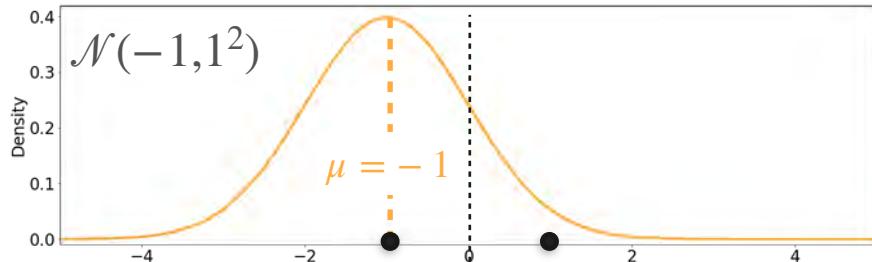
# Candidates



# Candidates



## Candidates

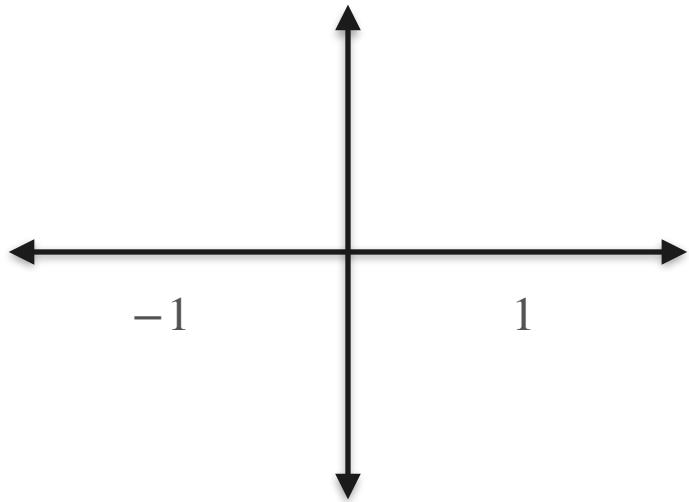


The best distribution is the one where  
the **mean** of the distribution is the  
**mean** of the sample

# Gaussians With Three Different Variance

Candidates

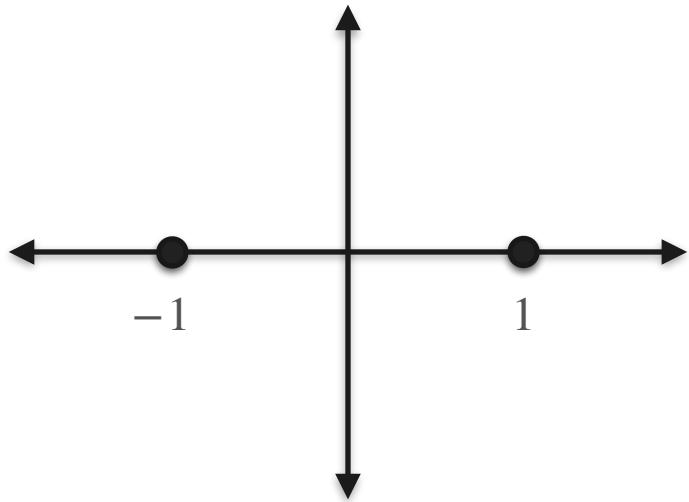
Observations



# Gaussians With Three Different Variance

Candidates

Observations

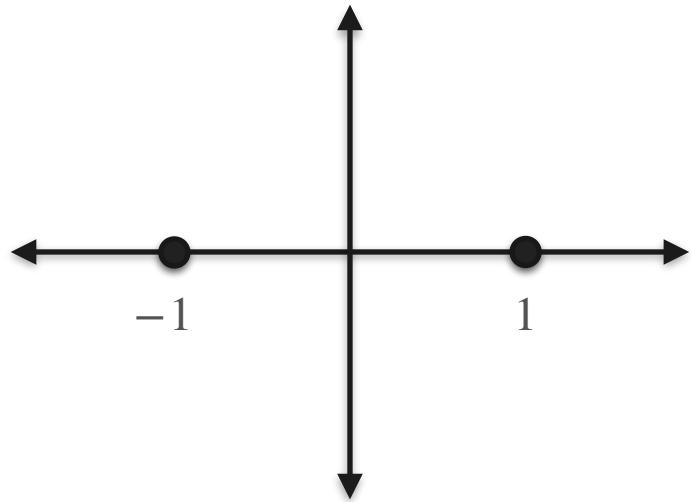


# Gaussians With Three Different Variance

Candidates

$$\mathcal{N}(0, 0.5^2)$$

Observations



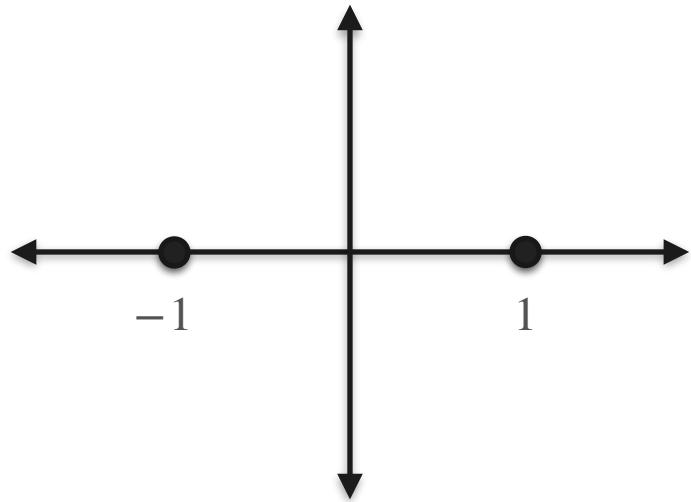
# Gaussians With Three Different Variance

Candidates

$$\mathcal{N}(0, 0.5^2)$$

$$\mathcal{N}(0, 1^2)$$

Observations



# Gaussians With Three Different Variance

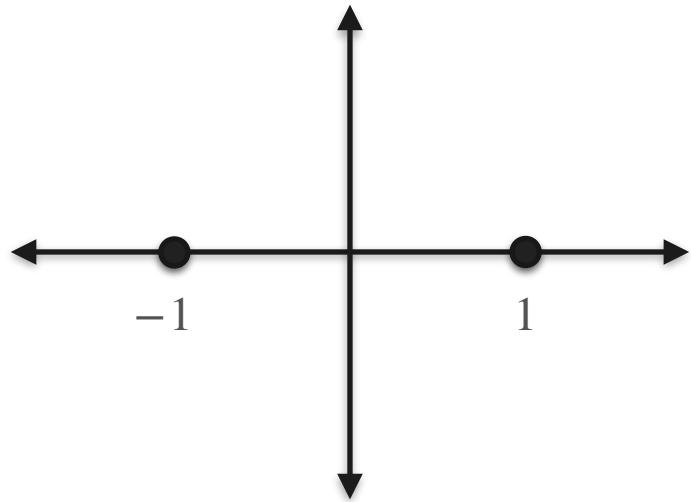
Candidates

$$\mathcal{N}(0,0.5^2)$$

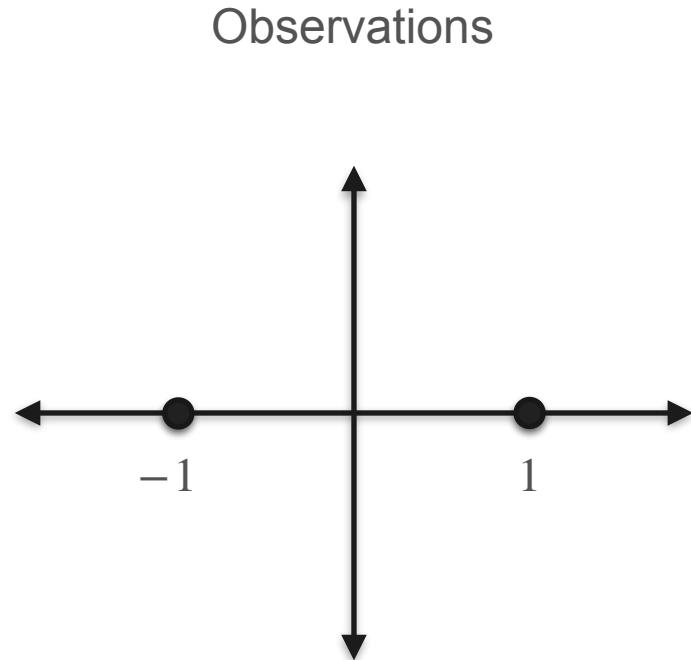
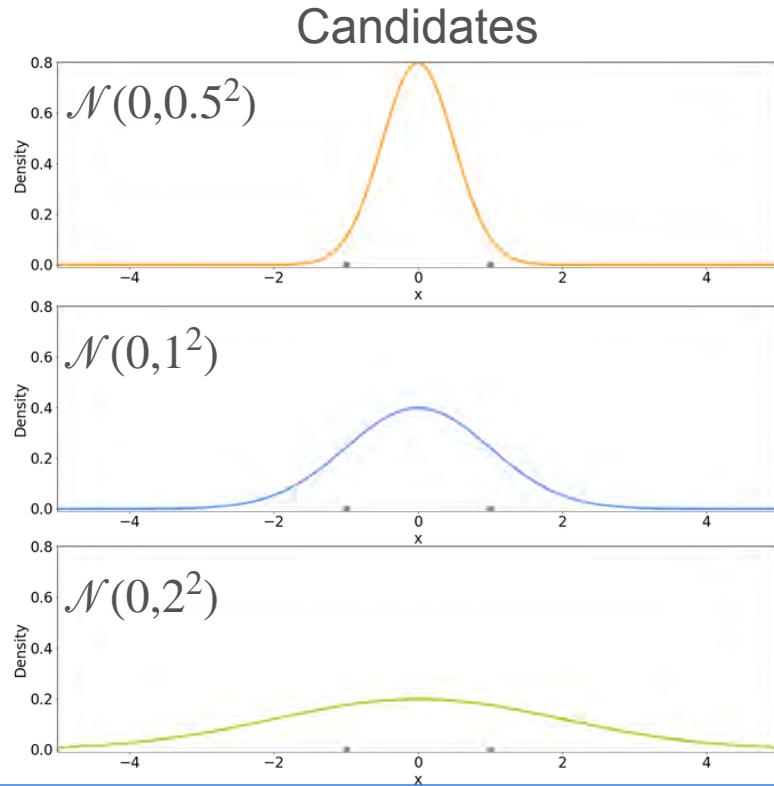
$$\mathcal{N}(0,1^2)$$

$$\mathcal{N}(0,2^2)$$

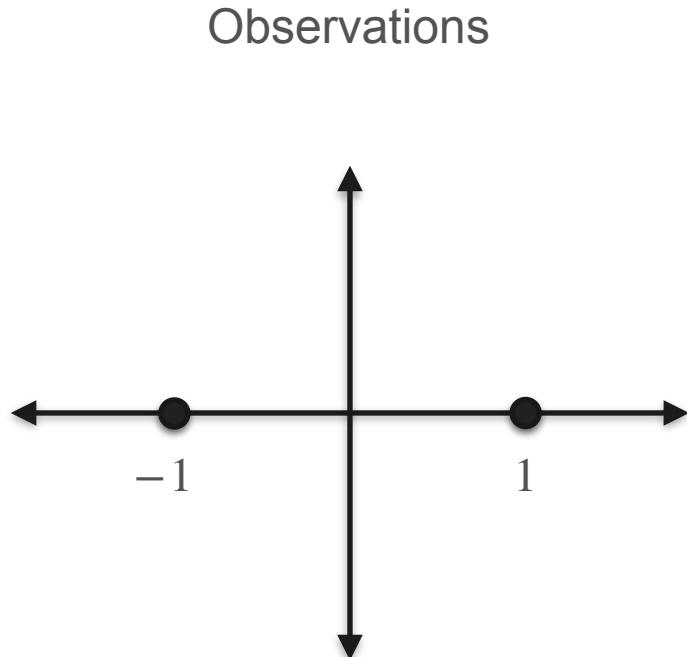
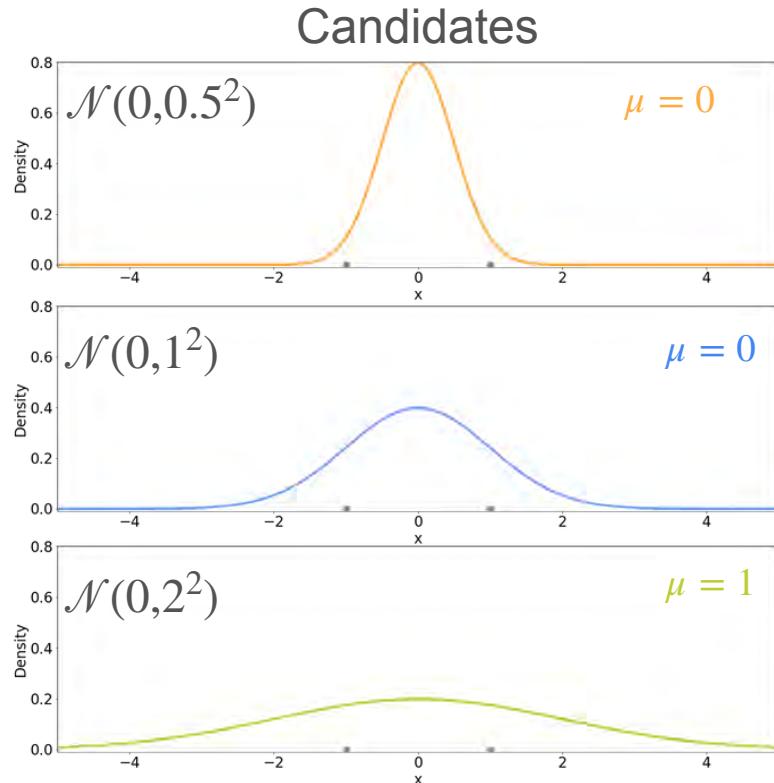
Observations



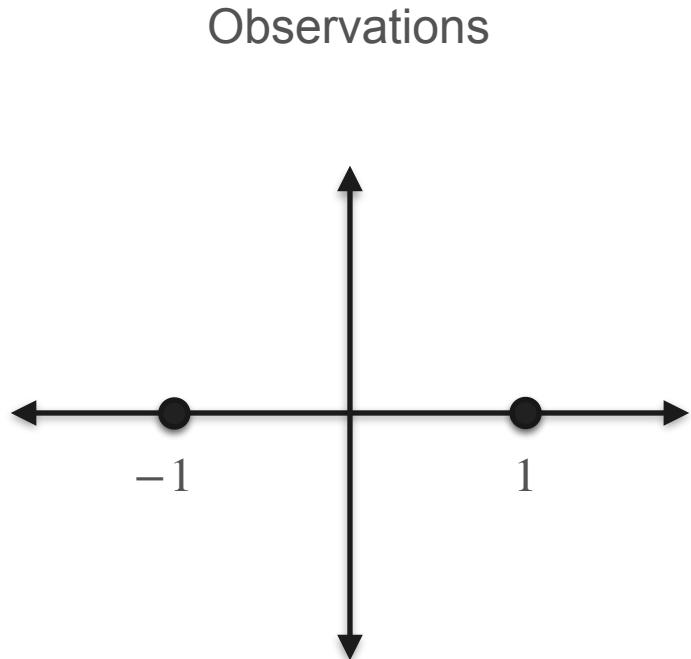
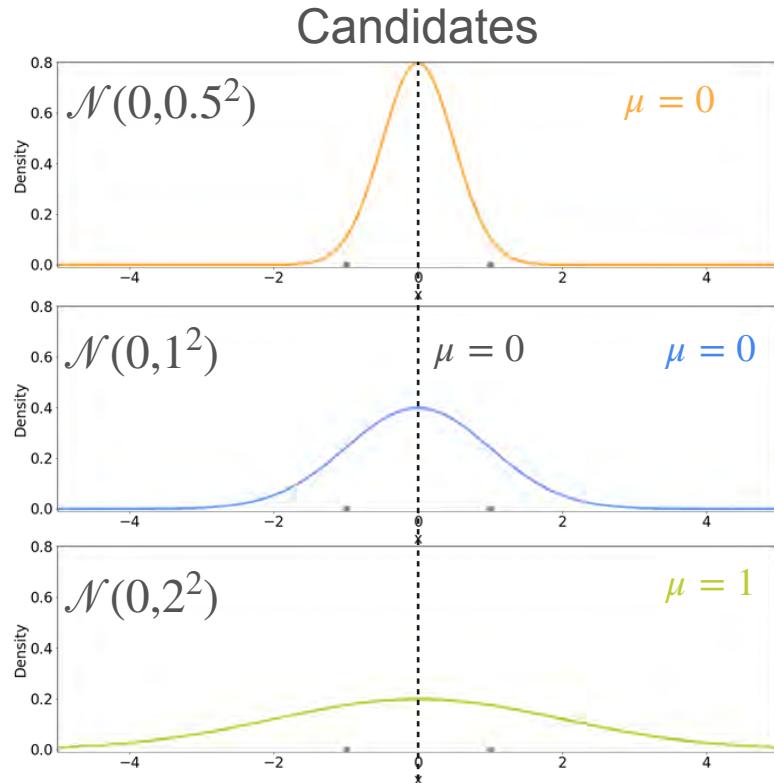
# Gaussians With Three Different Variance



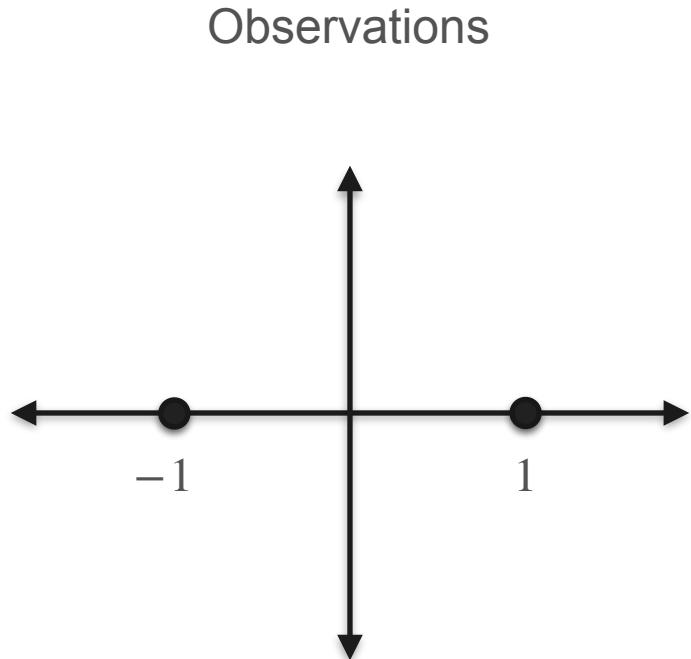
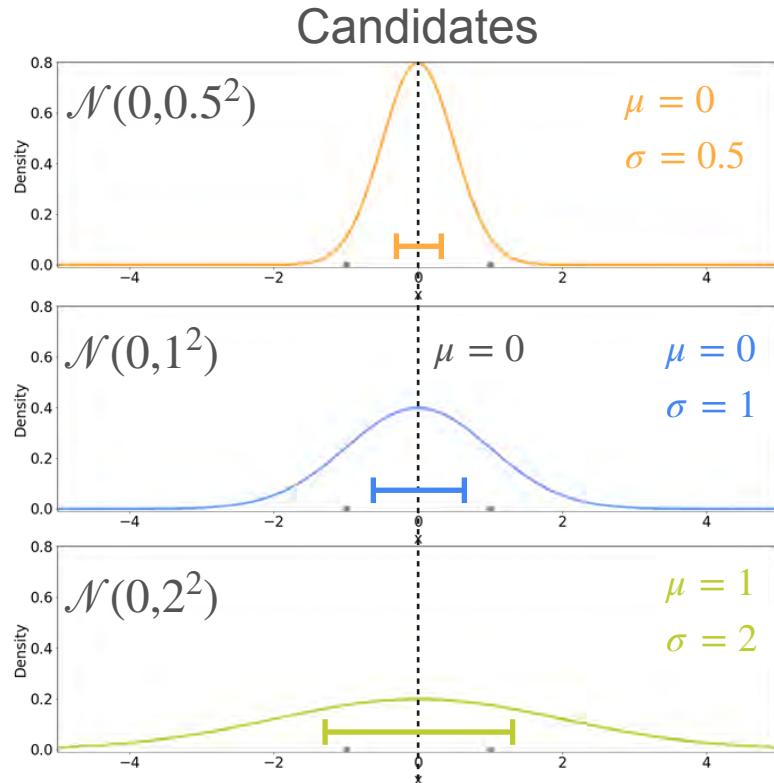
# Gaussians With Three Different Variance



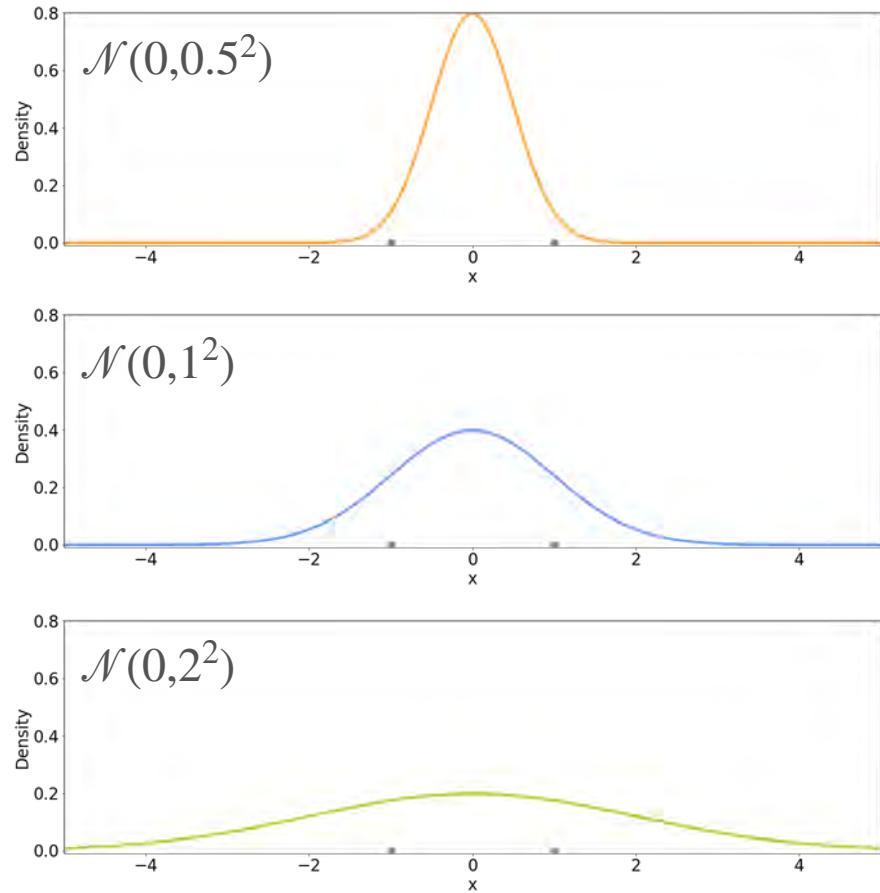
# Gaussians With Three Different Variance



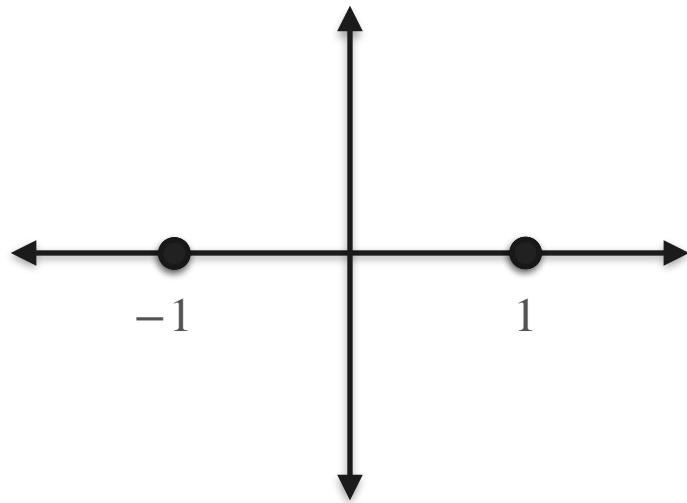
# Gaussians With Three Different Variance



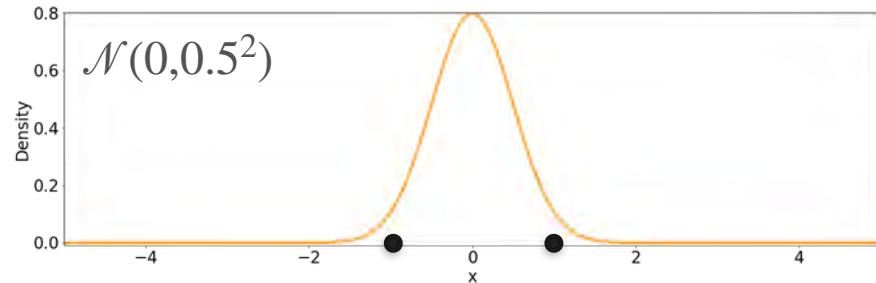
## Candidates



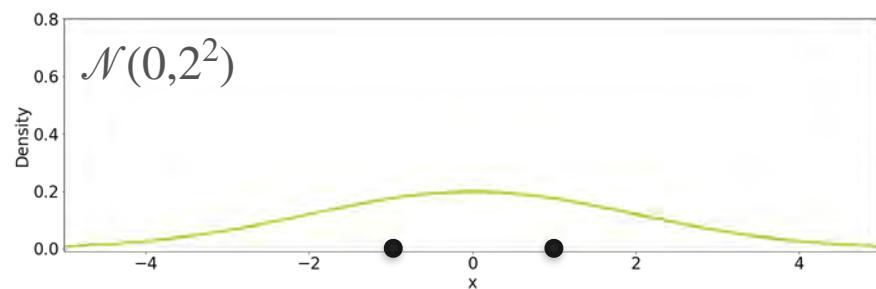
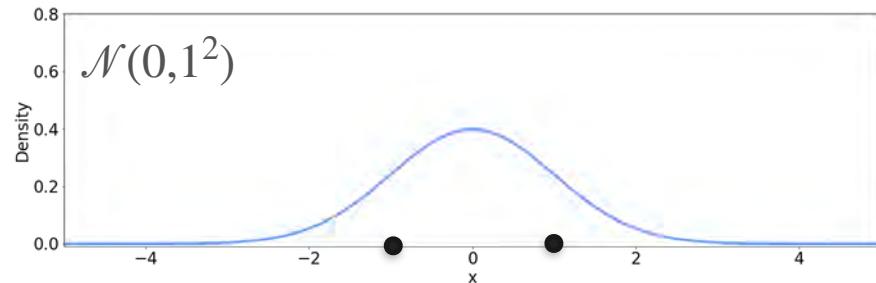
## Observations



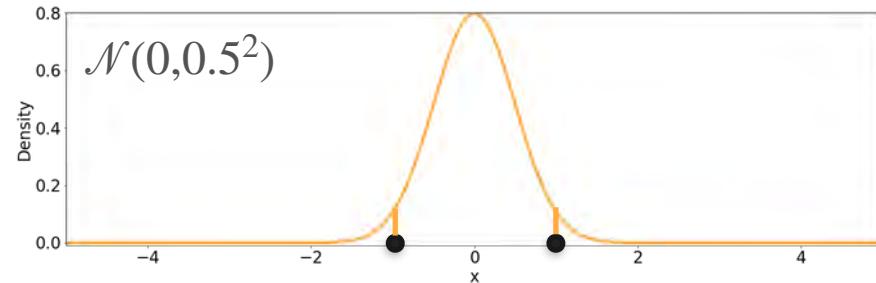
## Candidates



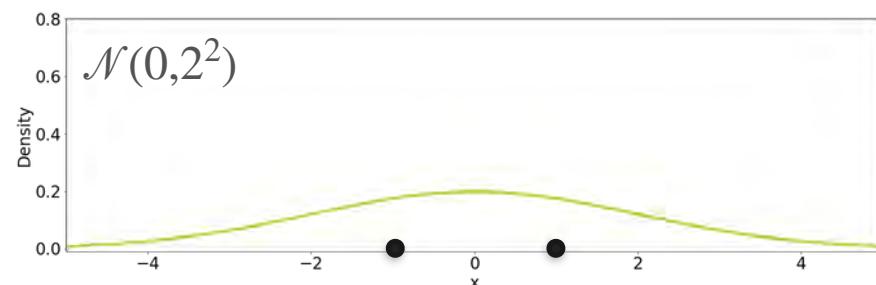
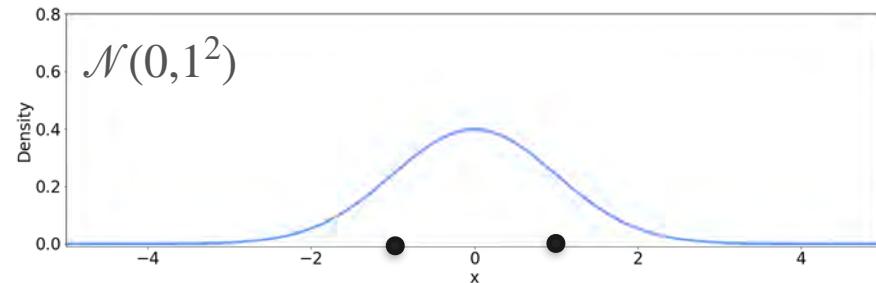
## Observations



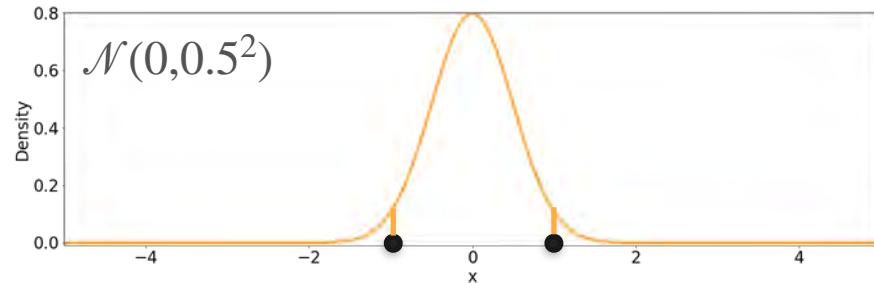
## Candidates



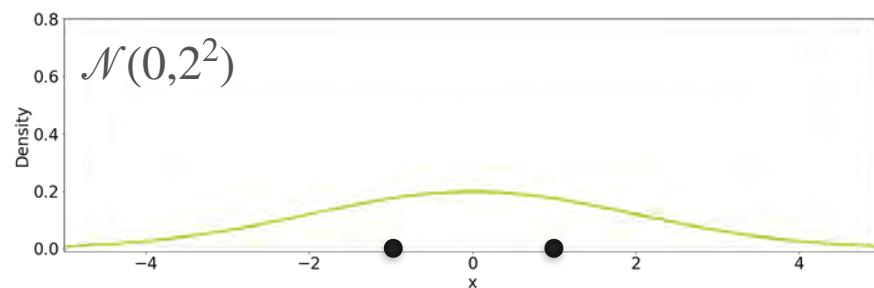
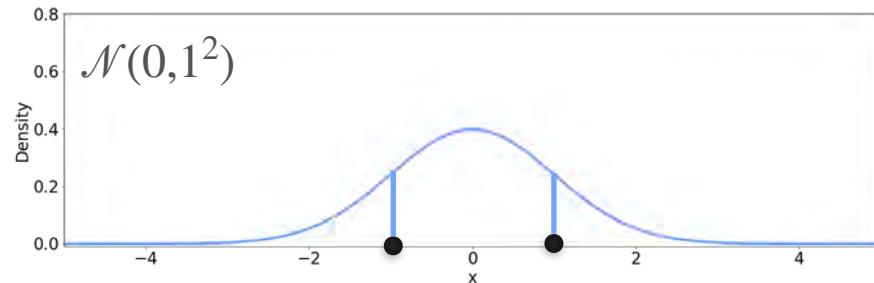
## Observations



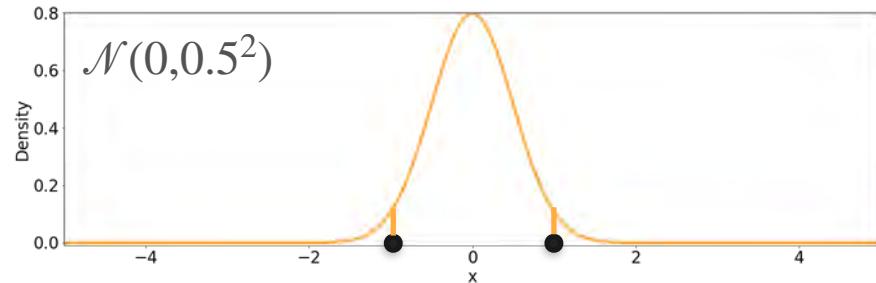
## Candidates



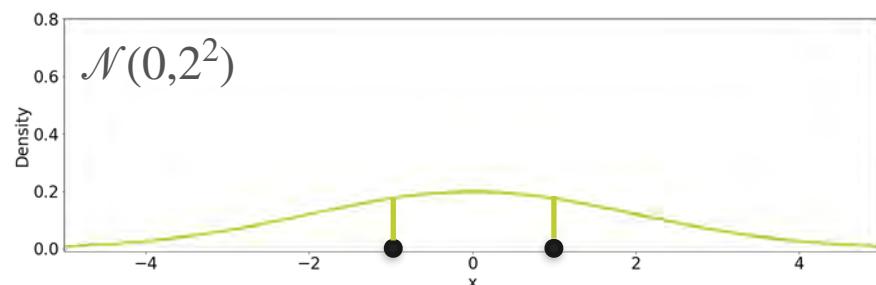
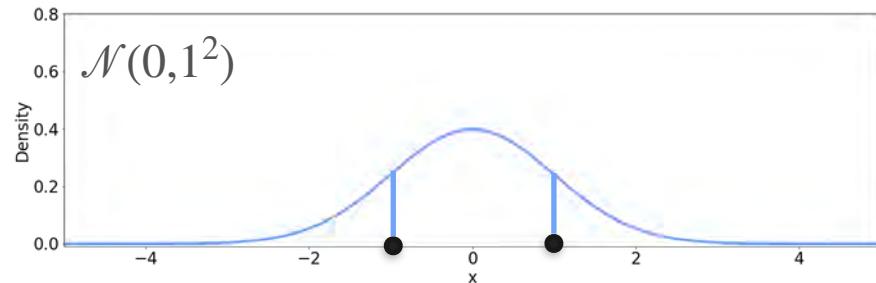
## Observations



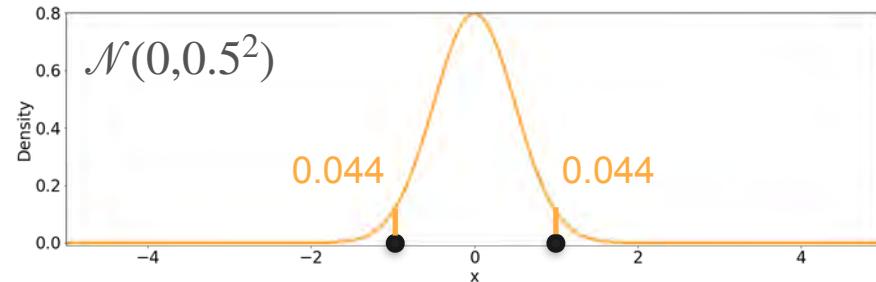
## Candidates



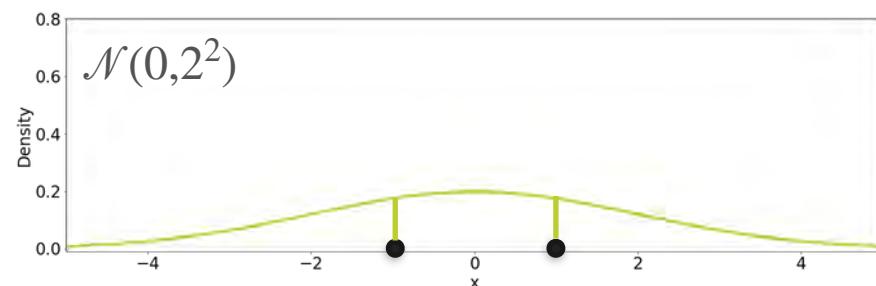
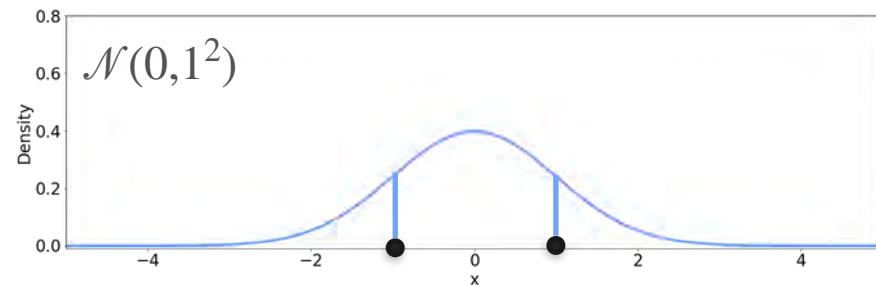
## Observations



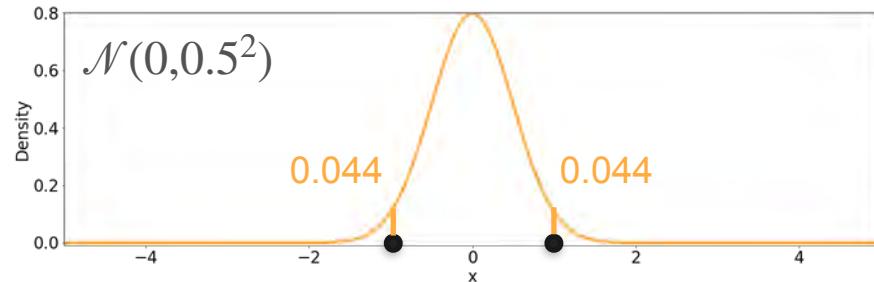
## Candidates



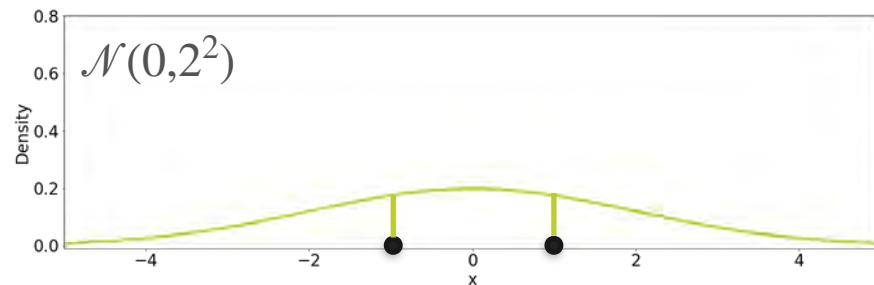
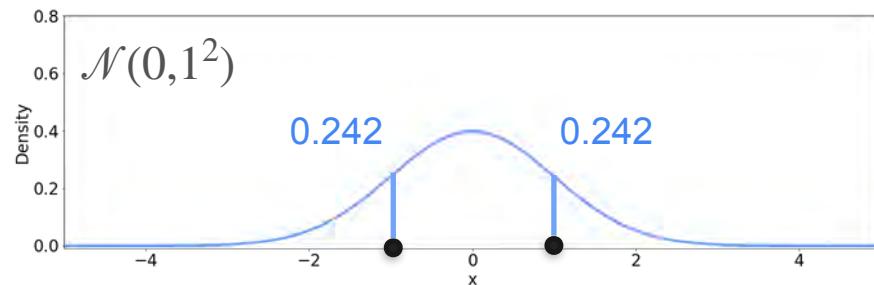
## Observations



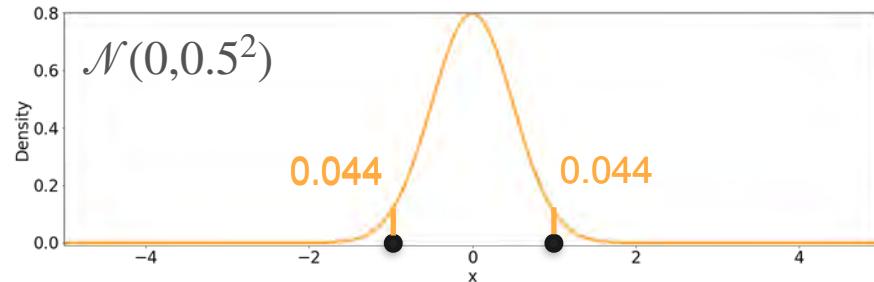
## Candidates



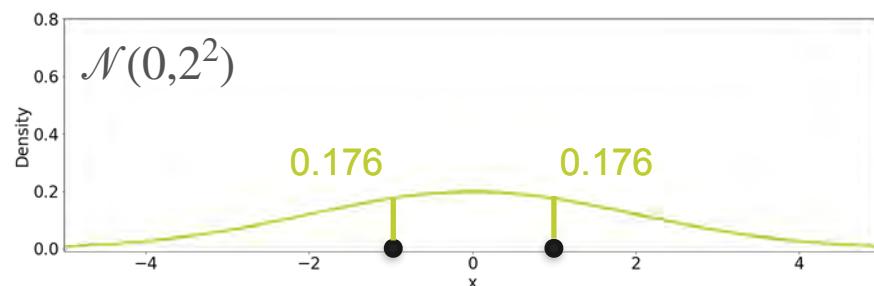
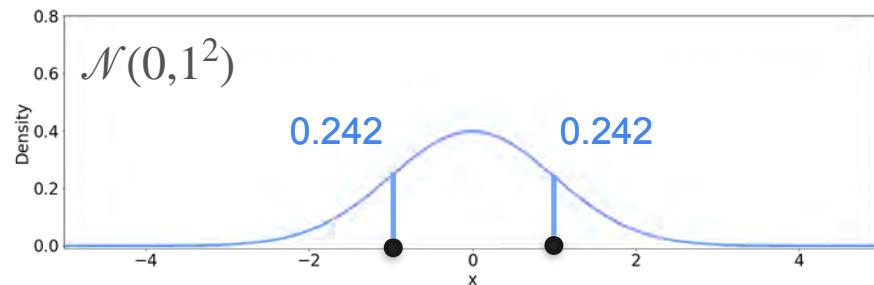
## Observations



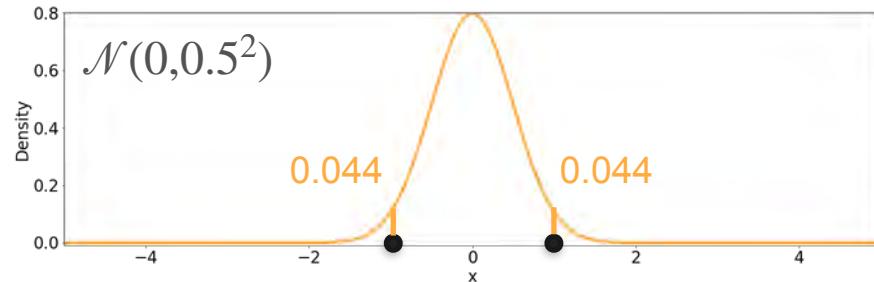
## Candidates



## Observations

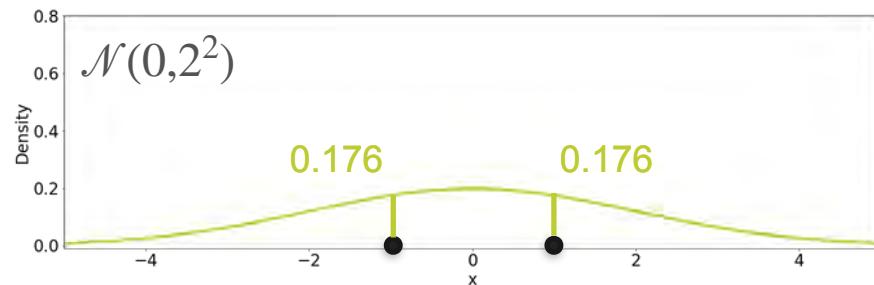
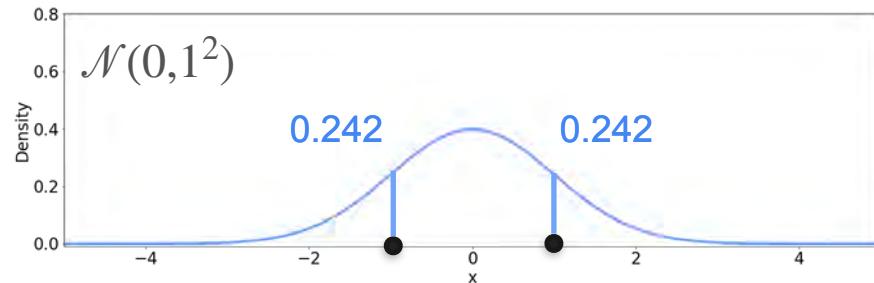


## Candidates

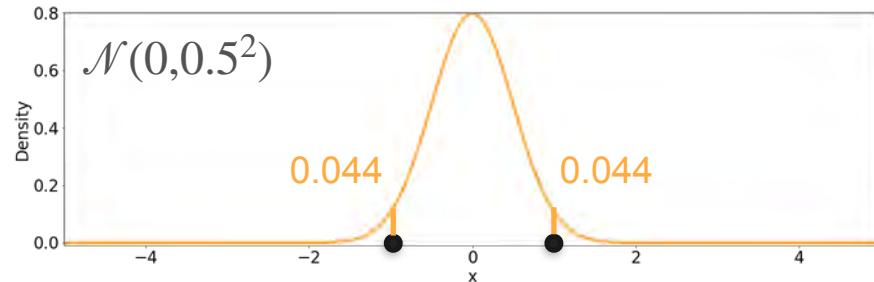


## Observations

$$0.044 \cdot 0.044 = 0.002$$

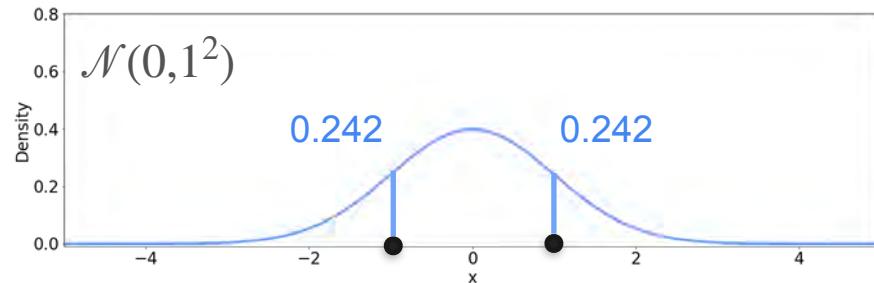


## Candidates

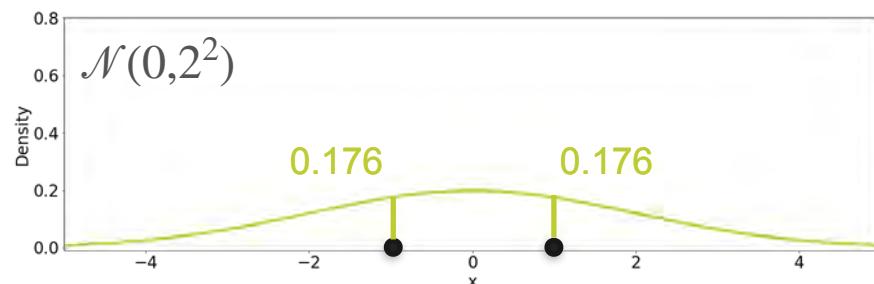


## Observations

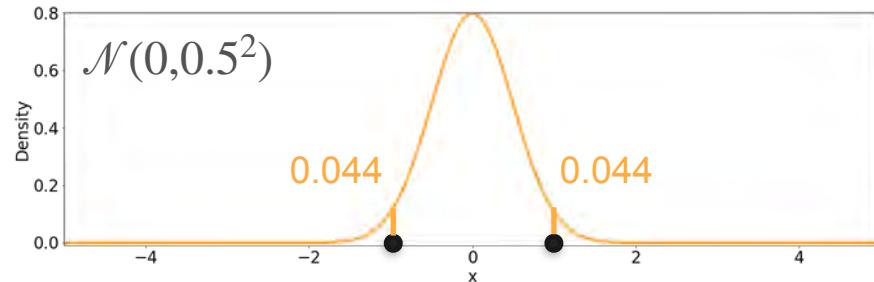
$$0.044 \cdot 0.044 = 0.002$$



$$0.242 \cdot 0.242 = 0.059$$

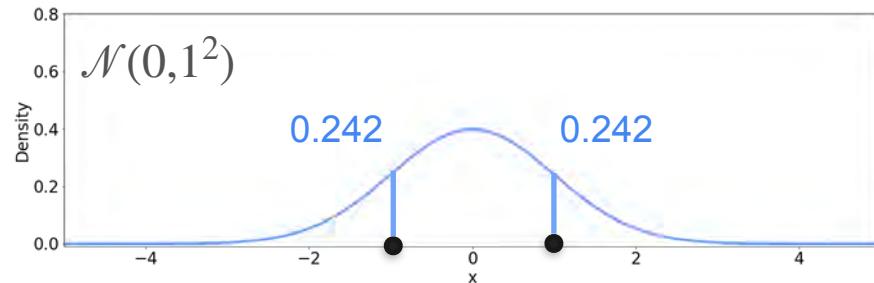


## Candidates

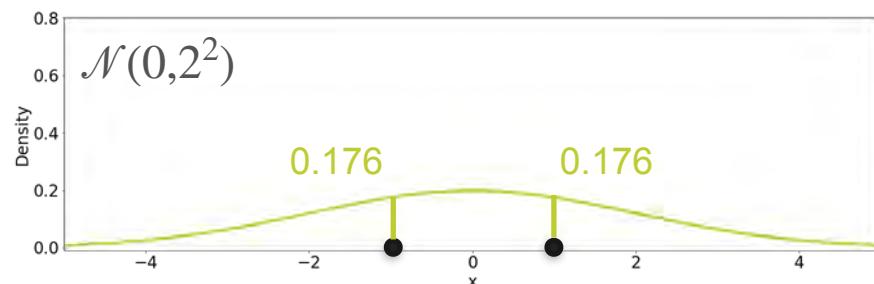


## Observations

$$0.044 \cdot 0.044 = 0.002$$

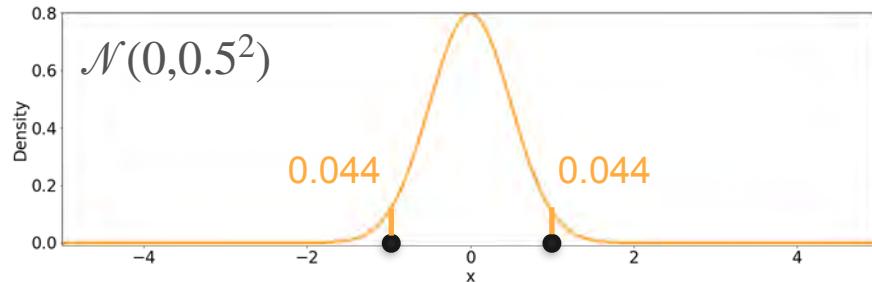


$$0.242 \cdot 0.242 = 0.059$$



$$0.176 \cdot 0.176 = 0.031$$

## Candidates



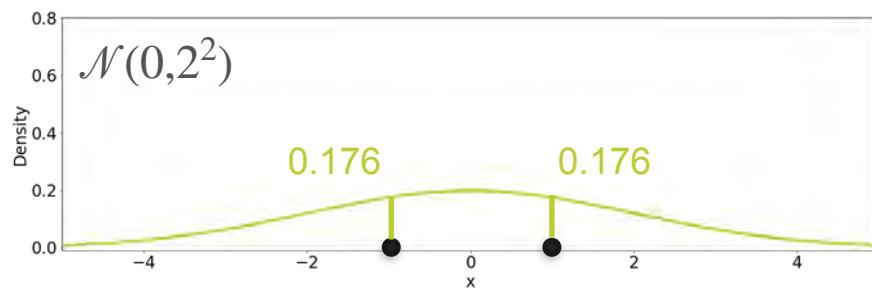
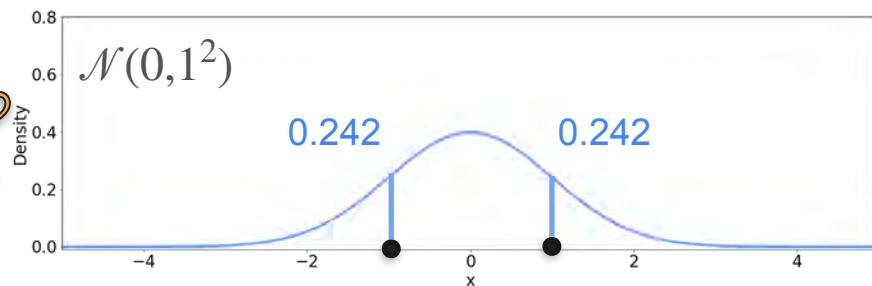
## Observations

$$0.044 \cdot 0.044 = 0.002$$



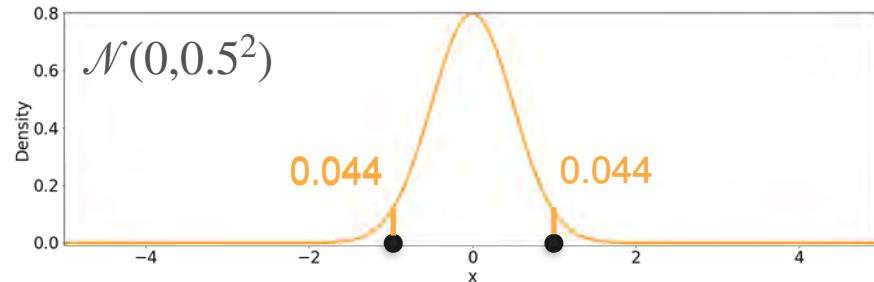
The  $\mathcal{N}(0, 1^2)$  is more likely!

$$0.242 \cdot 0.242 = 0.059$$

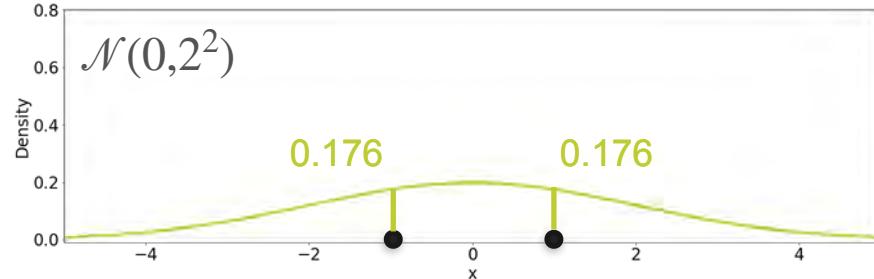
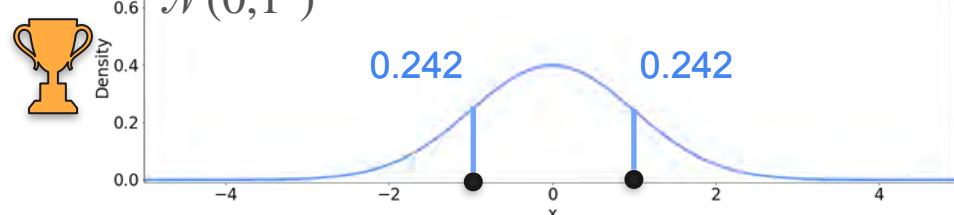


$$0.176 \cdot 0.176 = 0.031$$

## Candidates



## Observations

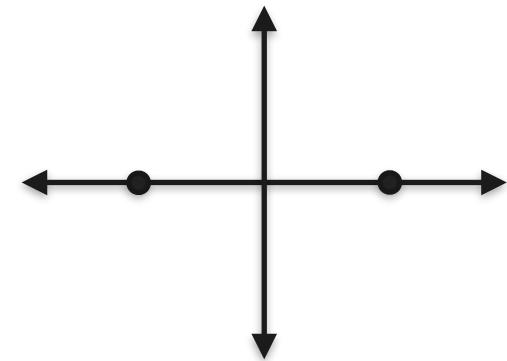


Likelihood = 0.002

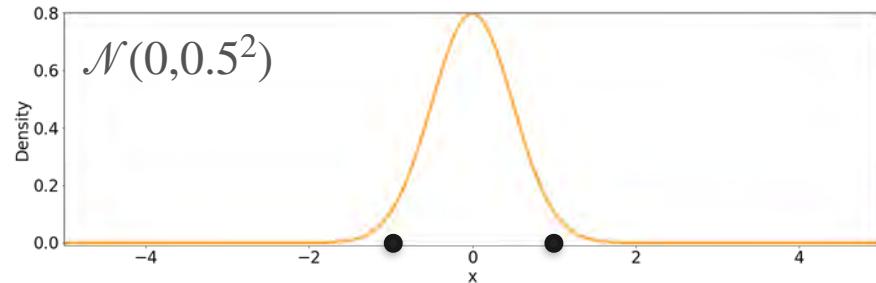
Likelihood = 0.059

Likelihood = 0.031

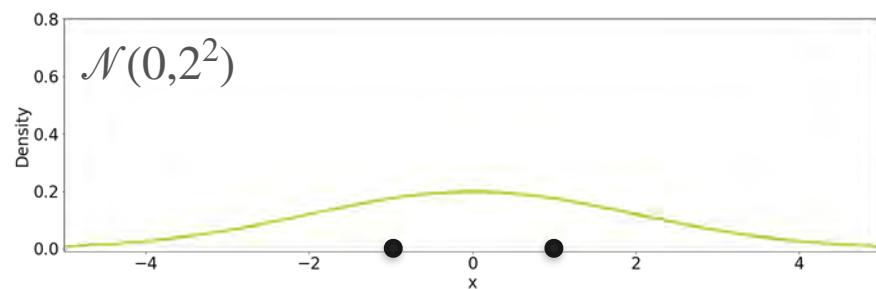
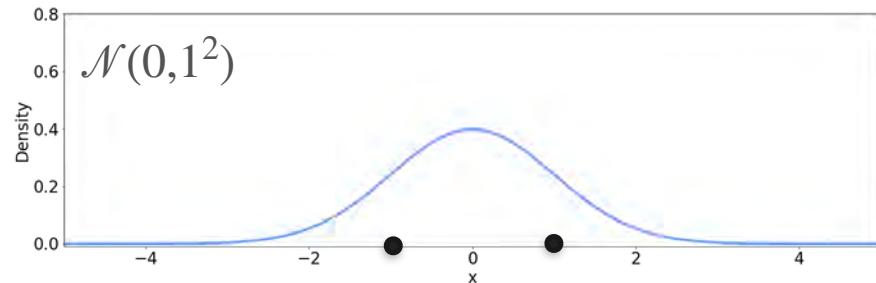
## Observations



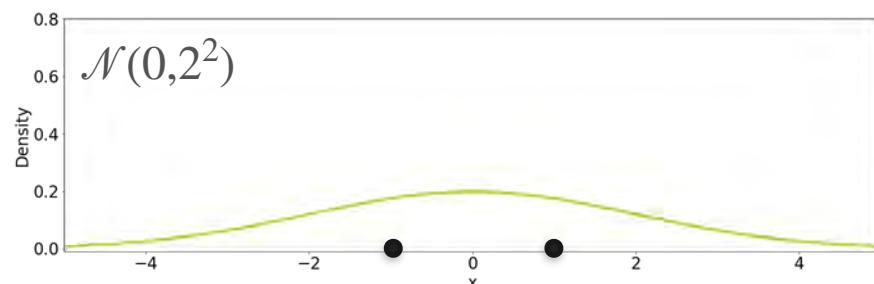
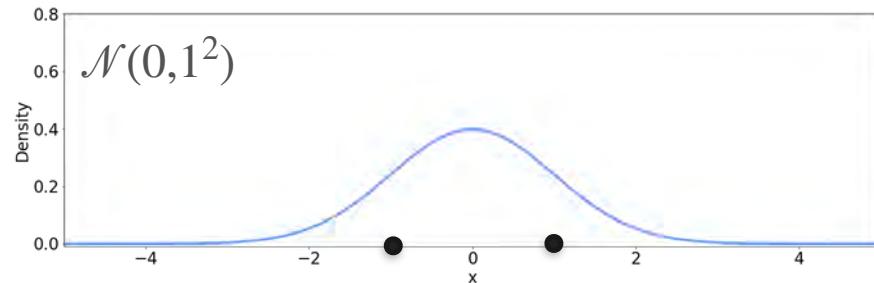
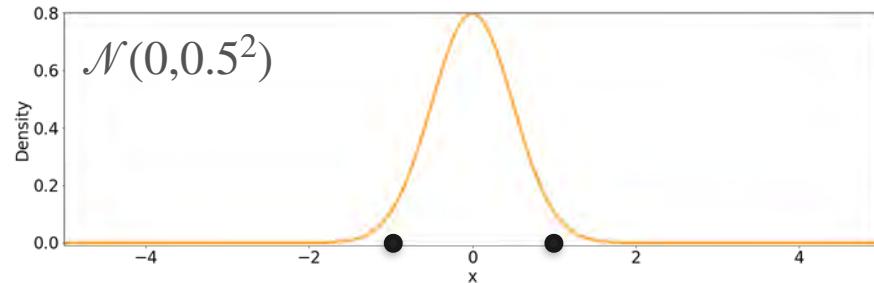
## Candidates



## Observations



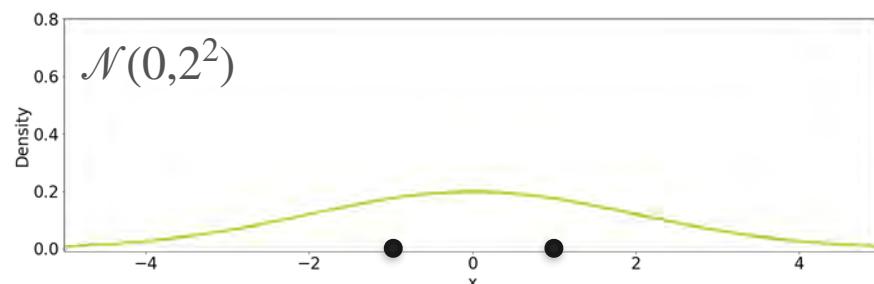
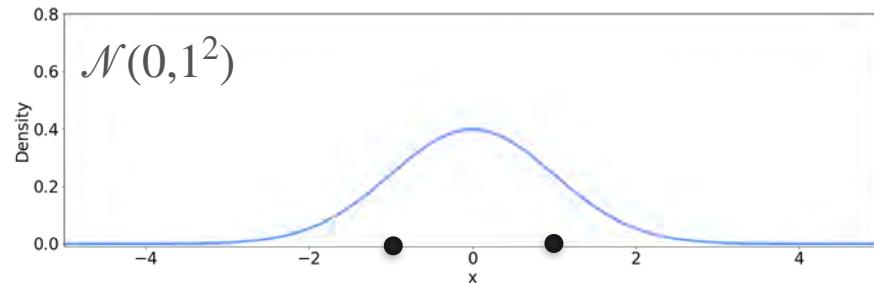
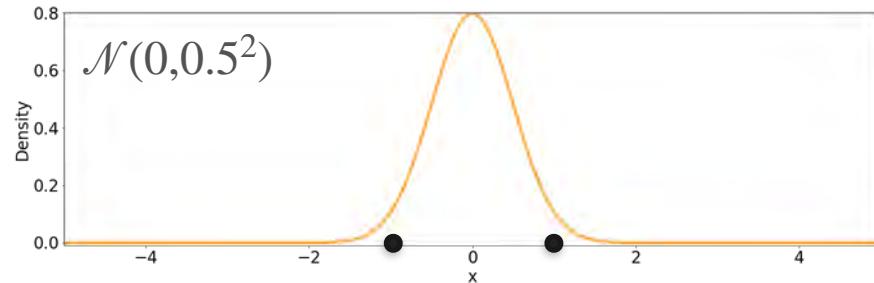
## Candidates



## Observations

Variance of the observations

## Candidates

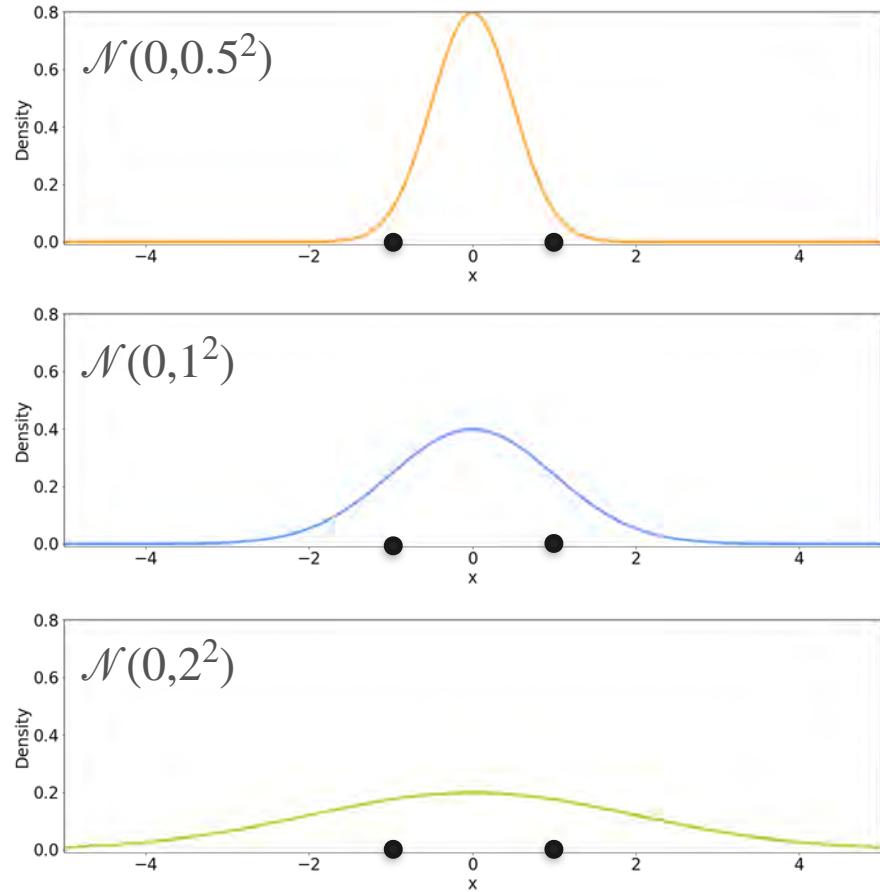


## Observations

Variance of the observations

$$\widehat{\sigma}^2 = \frac{1}{2} ((0 - 1)^2 + (0 + 1)^2)$$

## Candidates

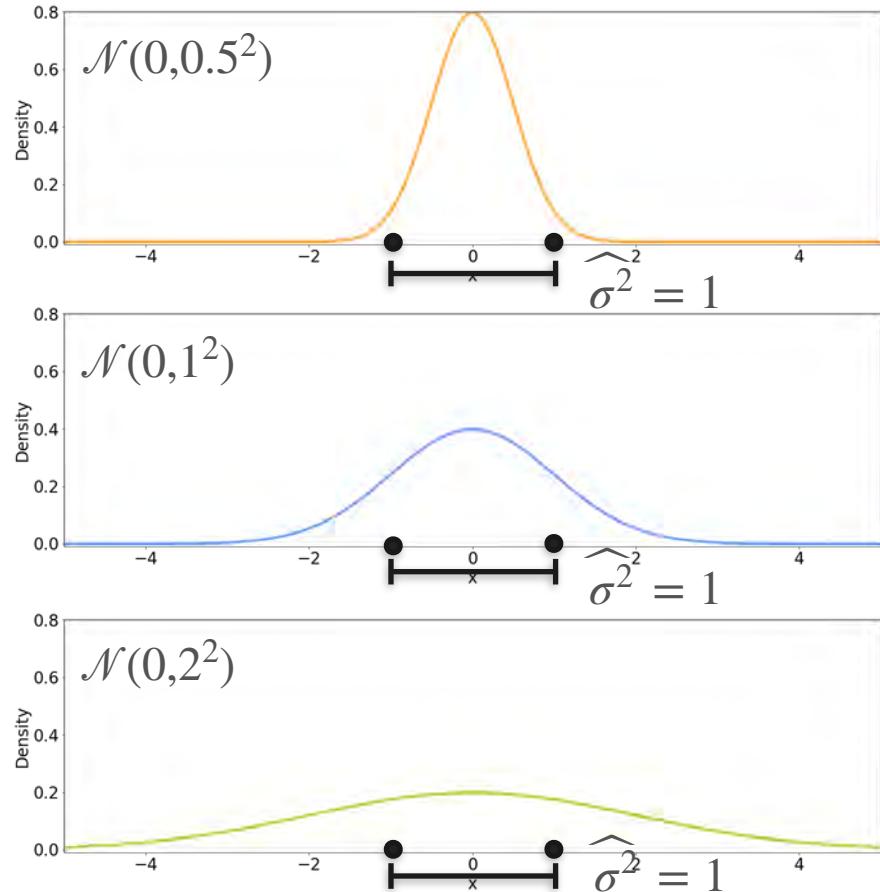


## Observations

Variance of the observations

$$\widehat{\sigma}^2 = \frac{1}{2} ((0 - 1)^2 + (0 + 1)^2) = 1$$

## Candidates

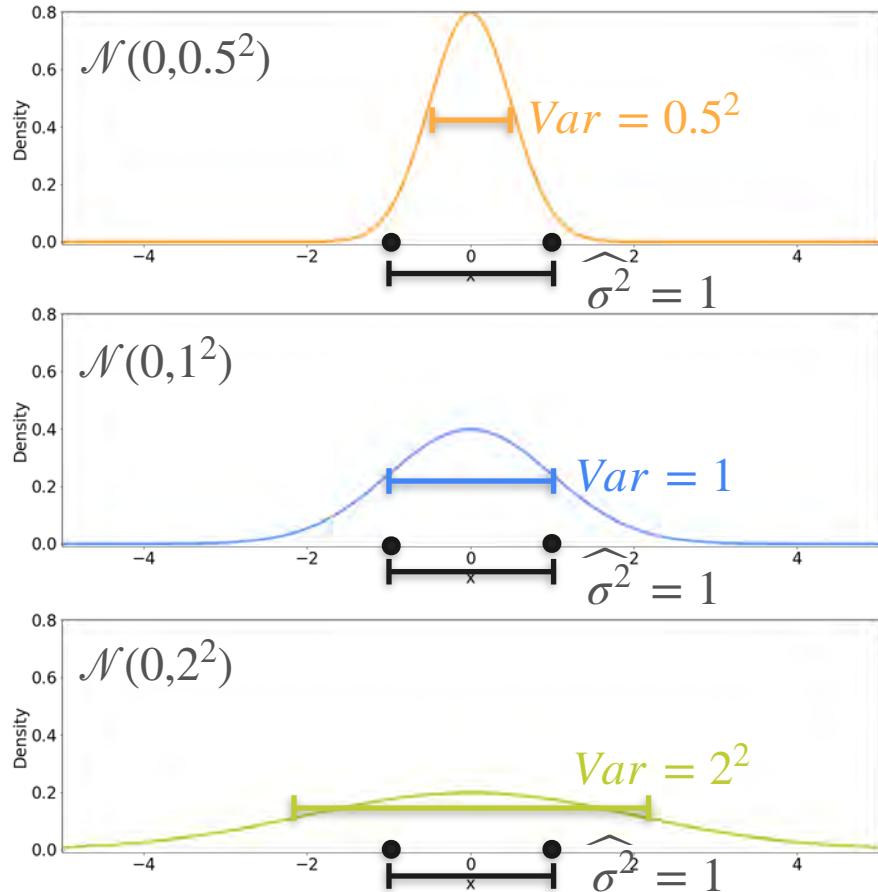


## Observations

Variance of the observations

$$\widehat{\sigma}^2 = \frac{1}{2} ((0 - 1)^2 + (0 + 1)^2) = 1$$

## Candidates

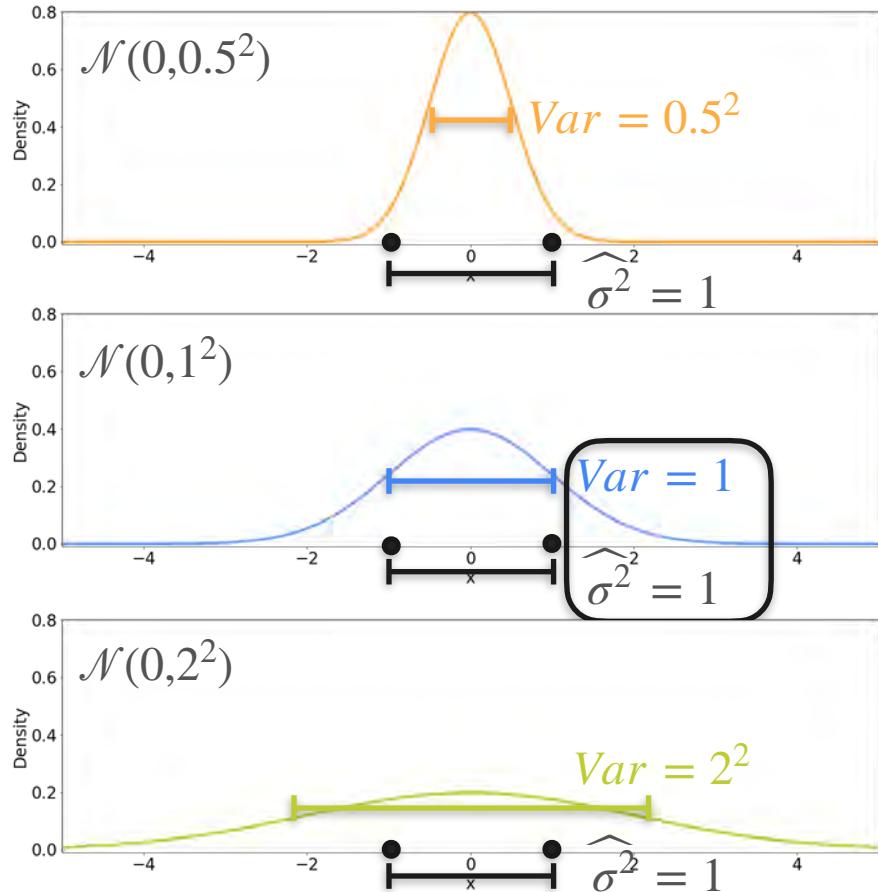


## Observations

Variance of the observations

$$\widehat{\sigma}^2 = \frac{1}{2} ((0 - 1)^2 + (0 + 1)^2) = 1$$

## Candidates

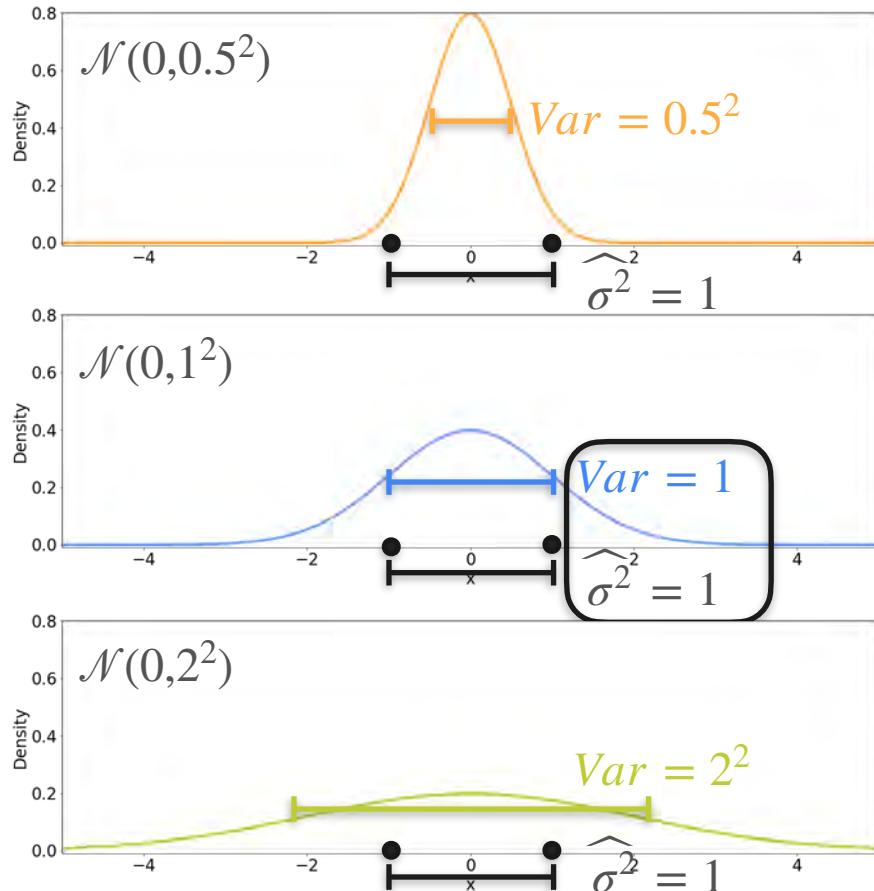


## Observations

Variance of the observations

$$\hat{\sigma}^2 = \frac{1}{2} ((0 - 1)^2 + (0 + 1)^2) = 1$$

## Candidates



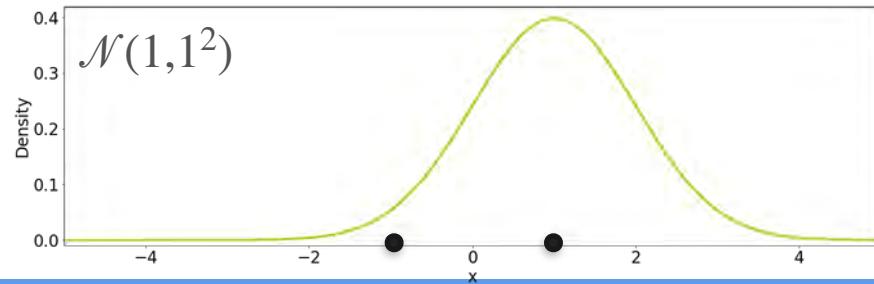
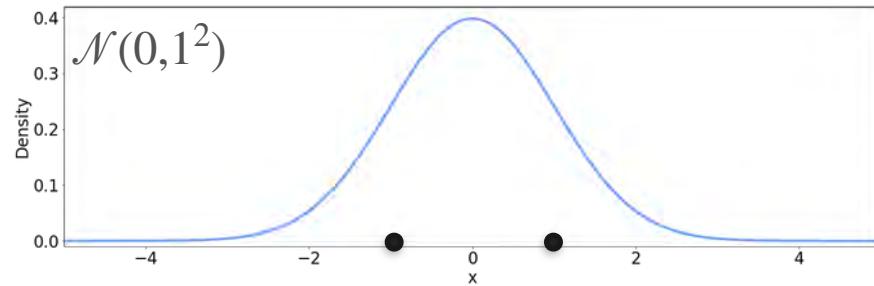
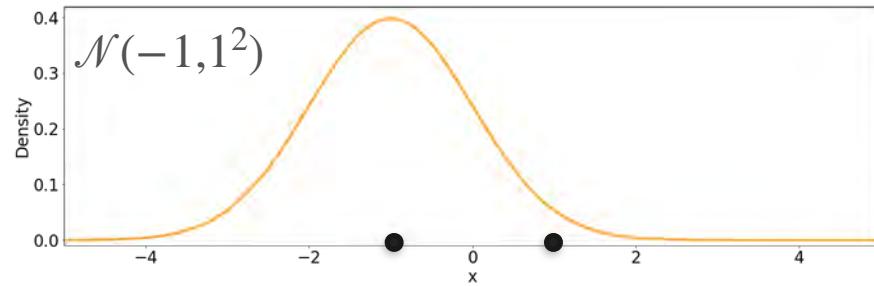
## Observations

Variance of the observations

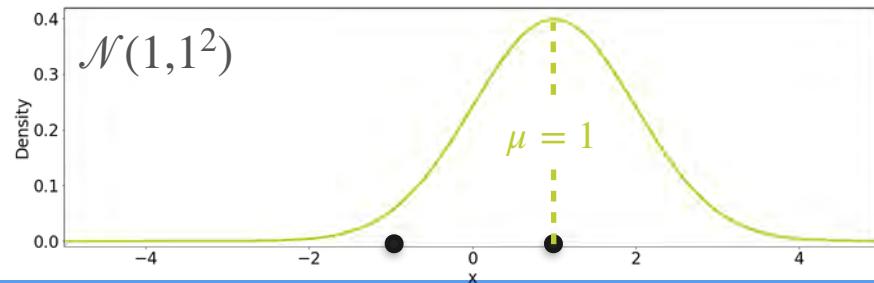
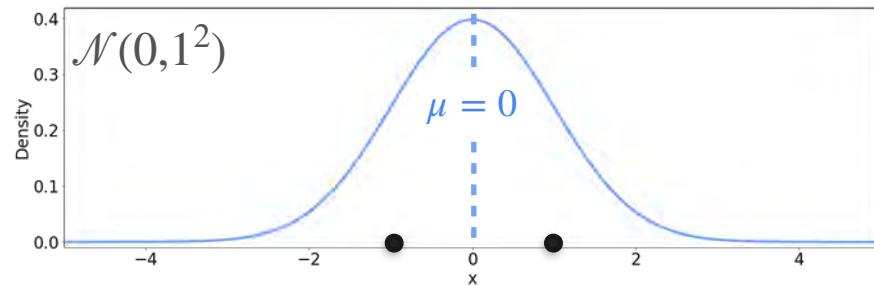
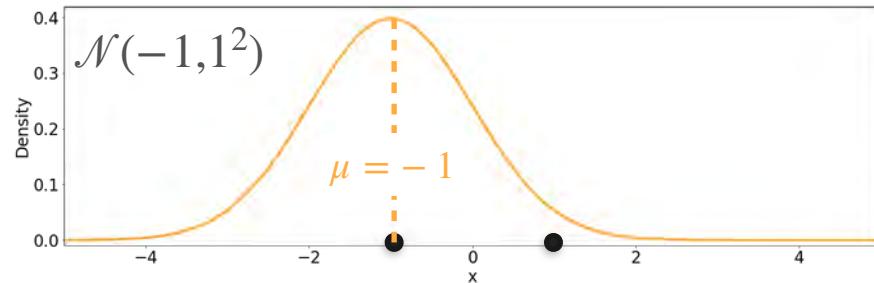
$$\widehat{\sigma}^2 = \frac{1}{2} ((0 - 1)^2 + (0 + 1)^2) = 1$$

The best distribution is the one where the **variance** of the distribution is the **variance** of the sample

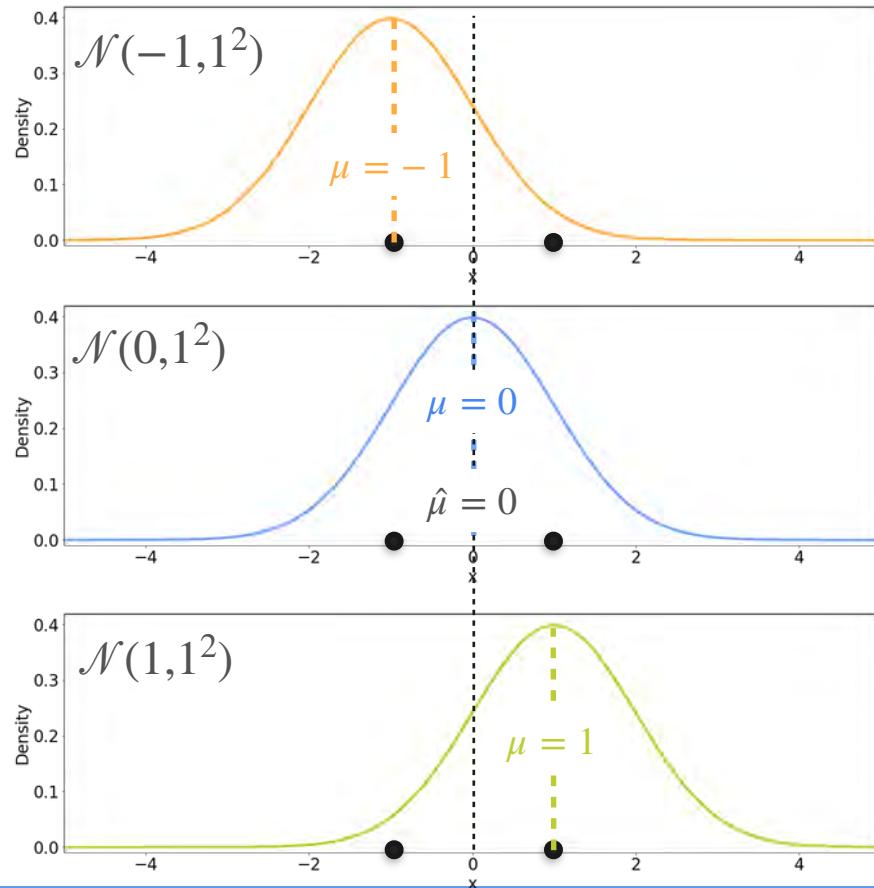
# Candidates



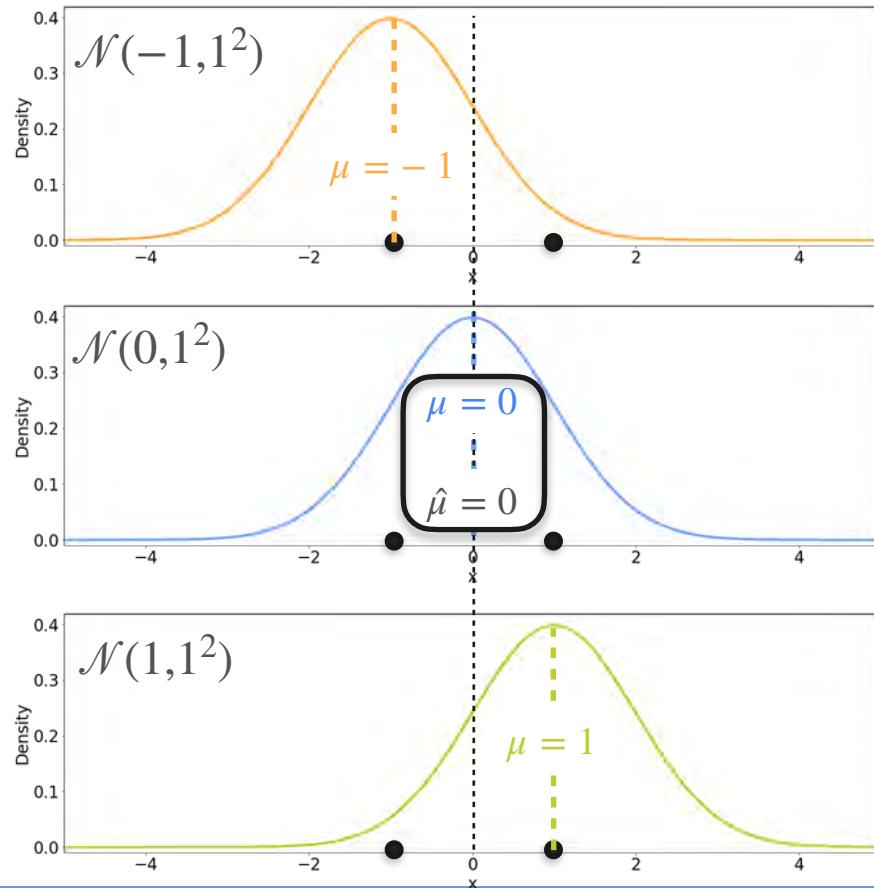
# Candidates



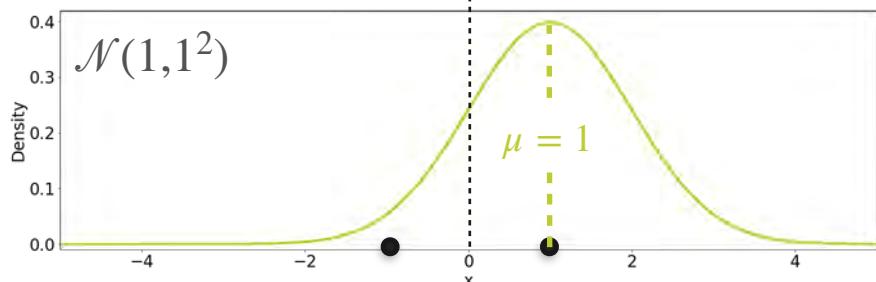
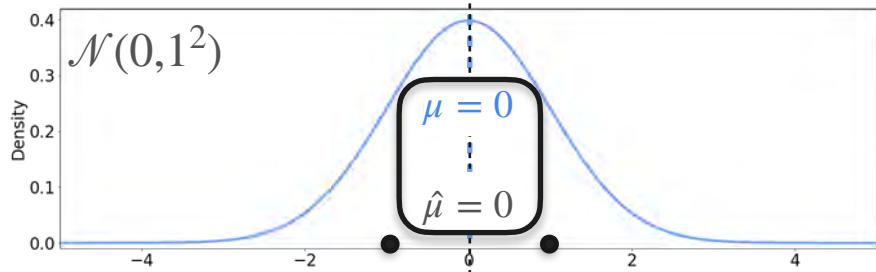
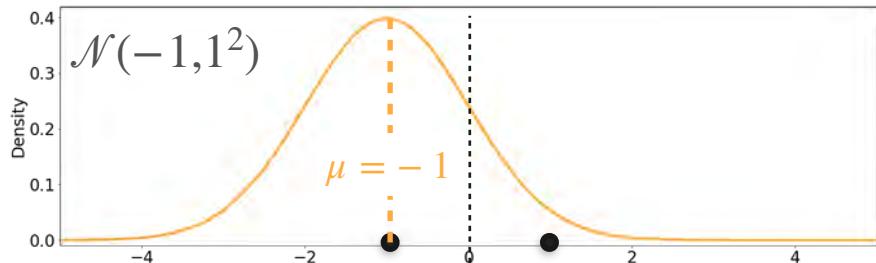
# Candidates



# Candidates

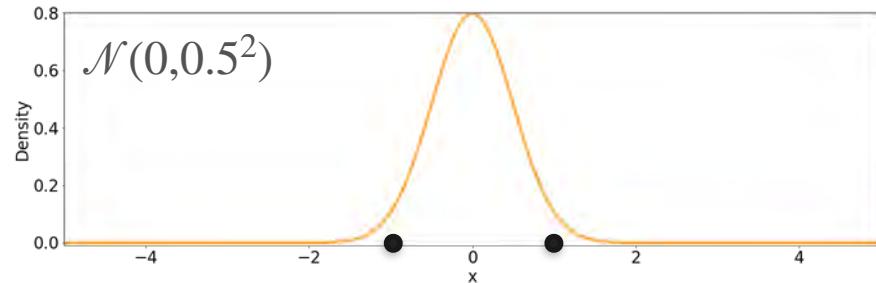


## Candidates

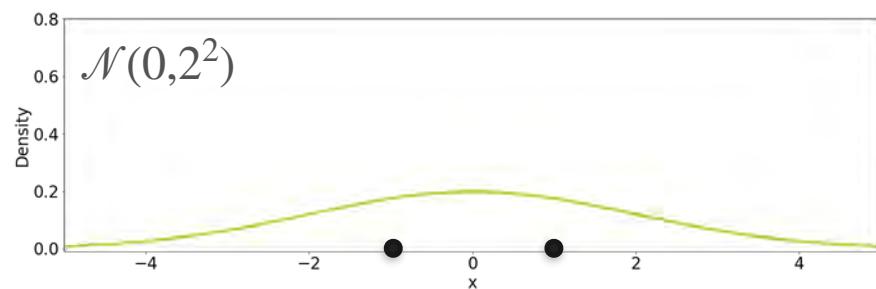
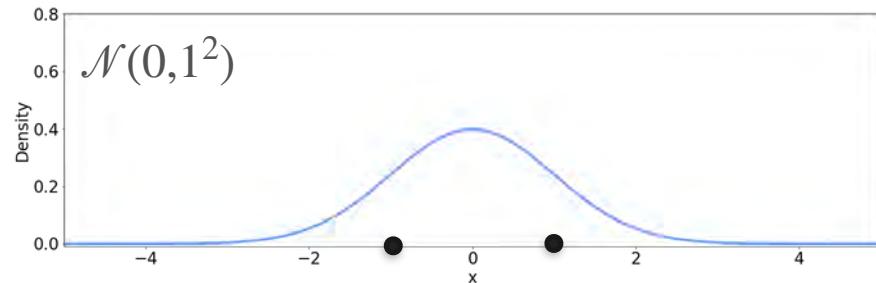


The best distribution is the one where  
the **mean** of the distribution is the  
**mean** of the sample

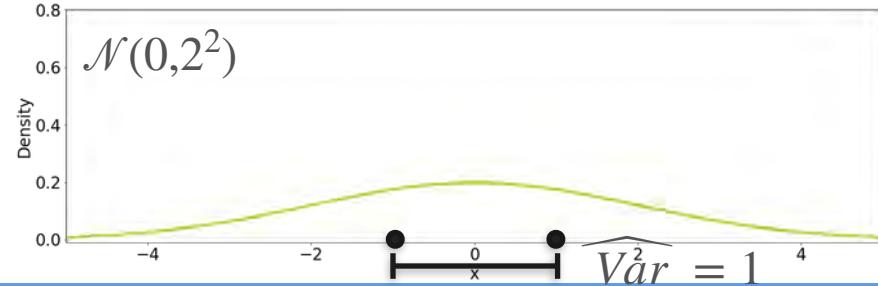
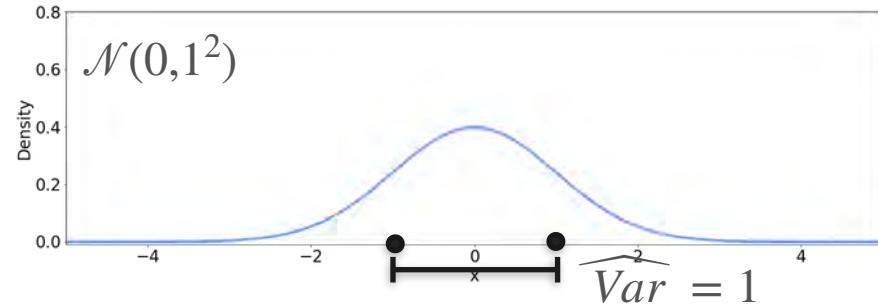
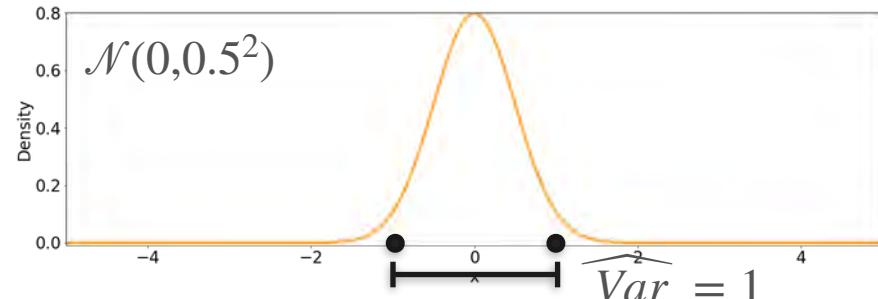
## Candidates



## Observations

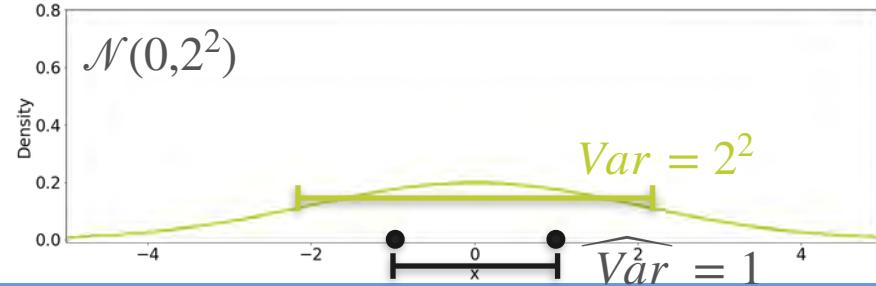
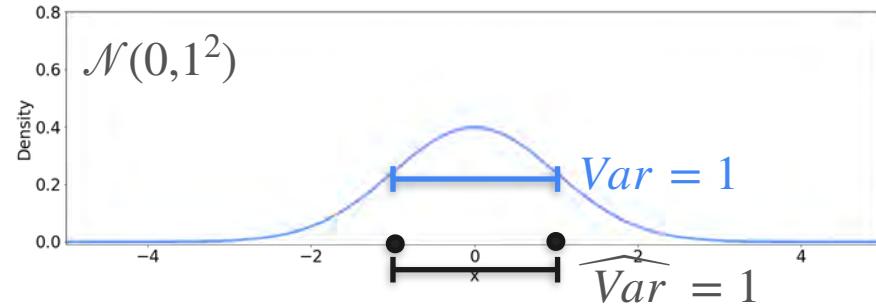
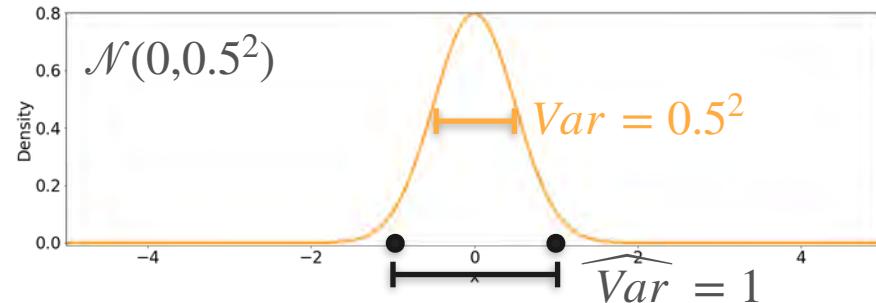


## Candidates



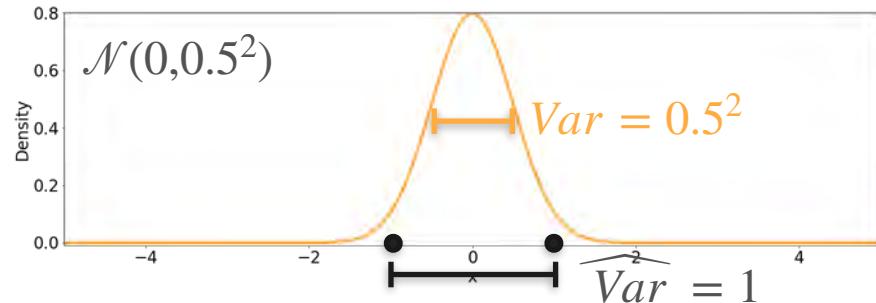
## Observations

## Candidates

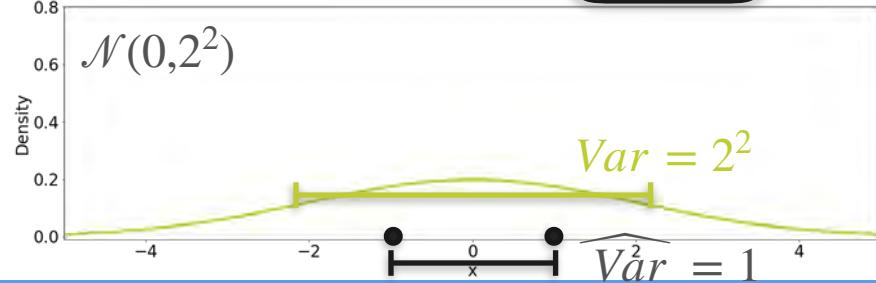
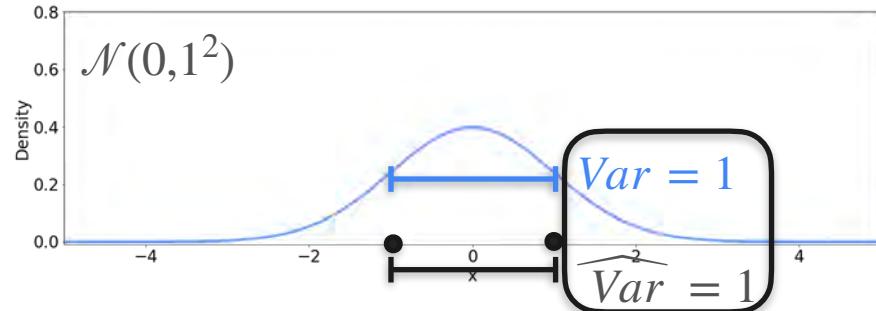


## Observations

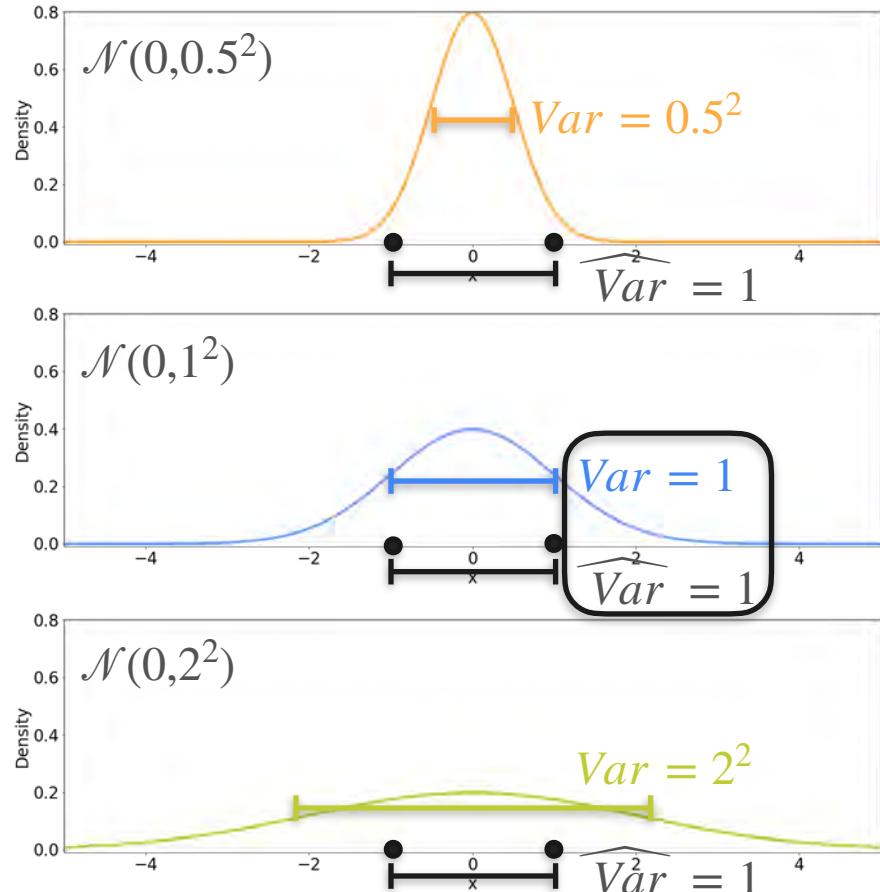
## Candidates



## Observations



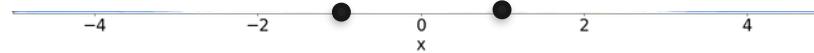
## Candidates



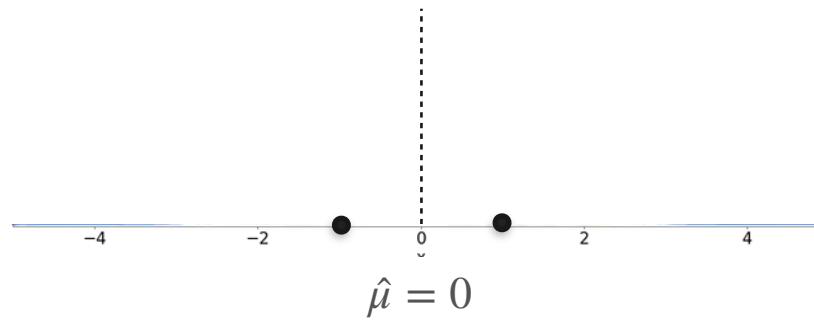
## Observations

The best distribution is the one where the **variance** of the distribution is the **variance** of the sample

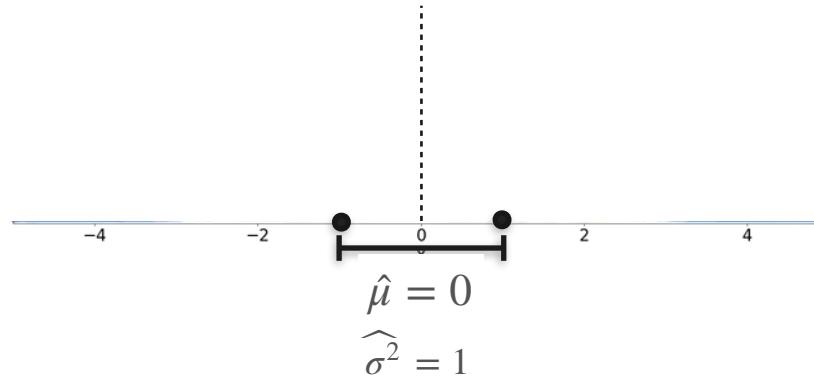
# Best Gaussian Fit



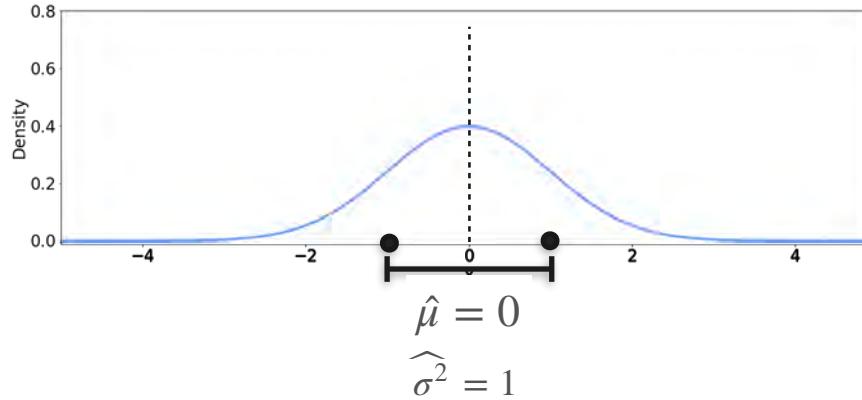
# Best Gaussian Fit



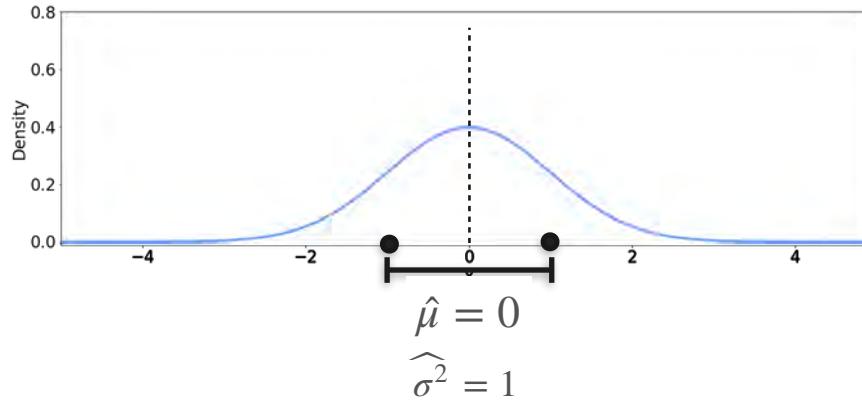
# Best Gaussian Fit



# Best Gaussian Fit

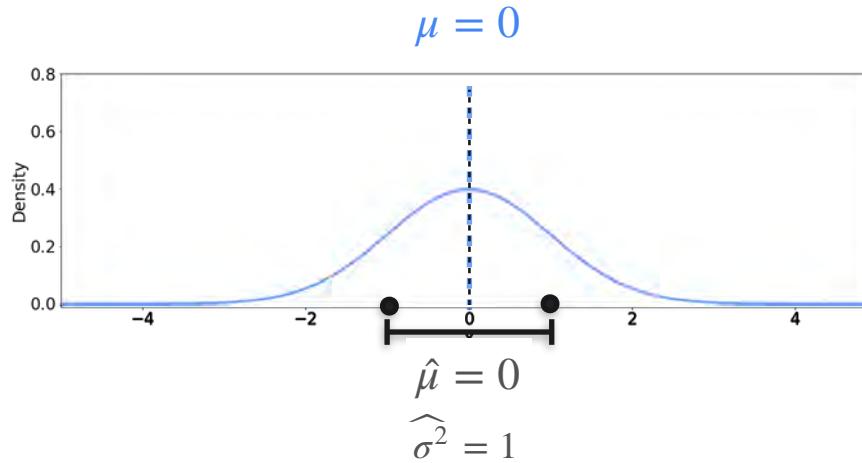


# Best Gaussian Fit



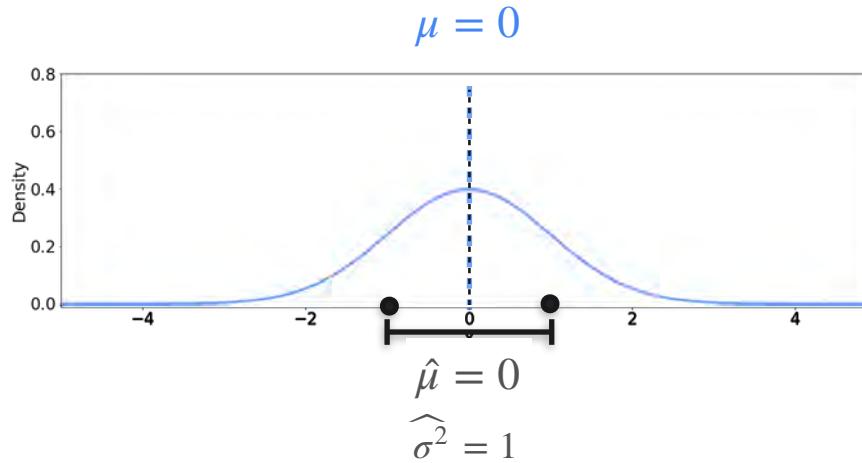
The best distribution is the one where  
the **mean** of the distribution is the  
**mean** of the sample

# Best Gaussian Fit



The best distribution is the one where  
the **mean** of the distribution is the  
**mean** of the sample

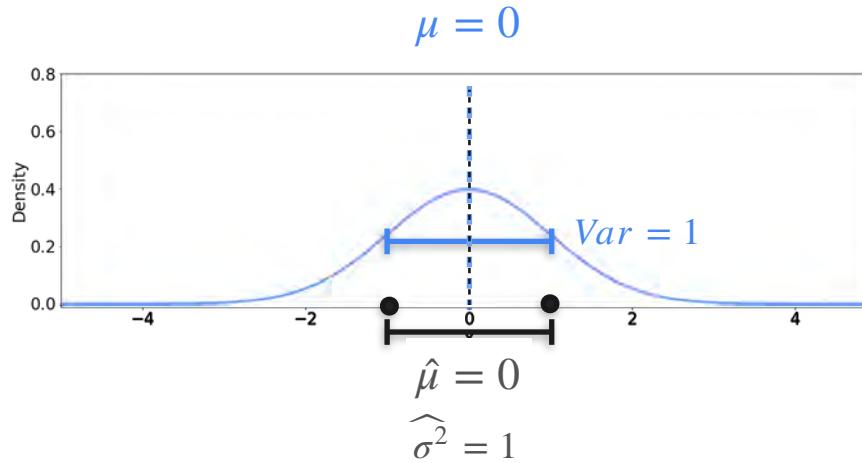
# Best Gaussian Fit



The best distribution is the one where  
the **mean** of the distribution is the  
**mean** of the sample

The best distribution is the one where  
the **variance** of the distribution is the  
**variance** of the sample

# Best Gaussian Fit



The best distribution is the one where  
the **mean** of the distribution is the  
**mean** of the sample

The best distribution is the one where  
the **variance** of the distribution is the  
**variance** of the sample

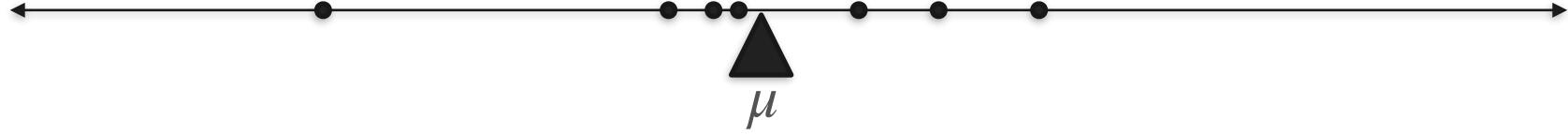
# Best Gaussian Fit



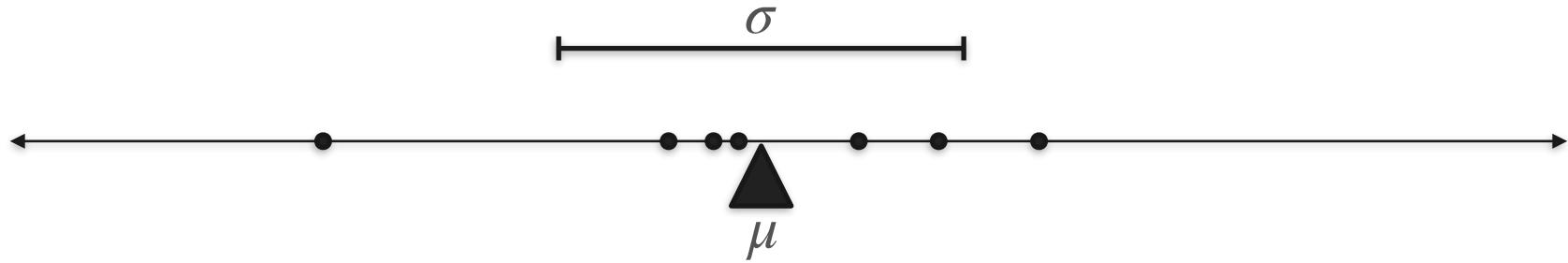
# Best Gaussian Fit



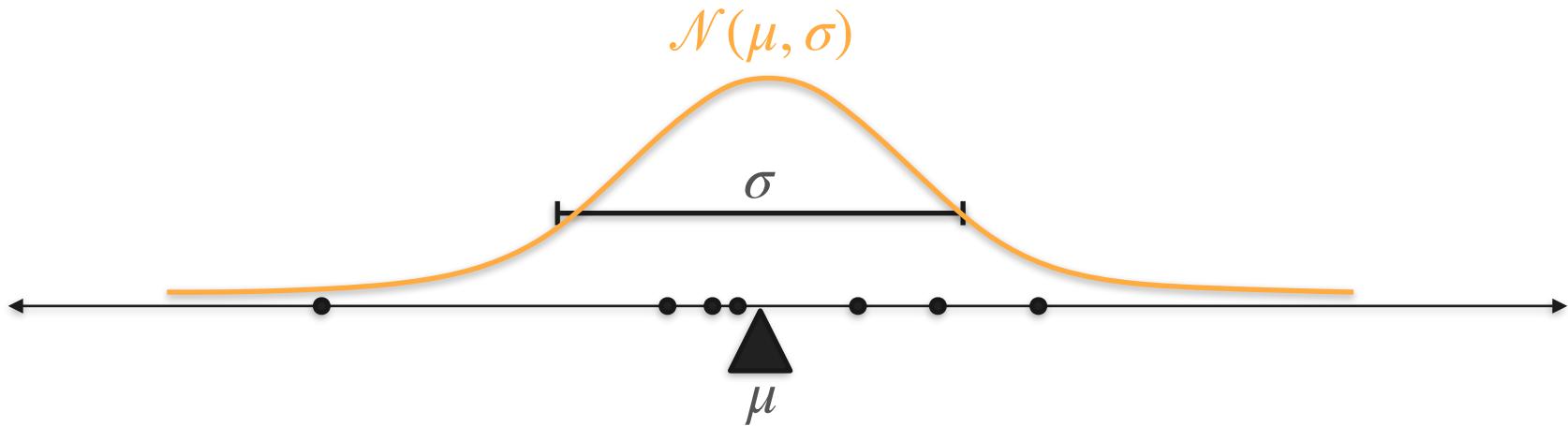
# Best Gaussian Fit



# Best Gaussian Fit



# Best Gaussian Fit



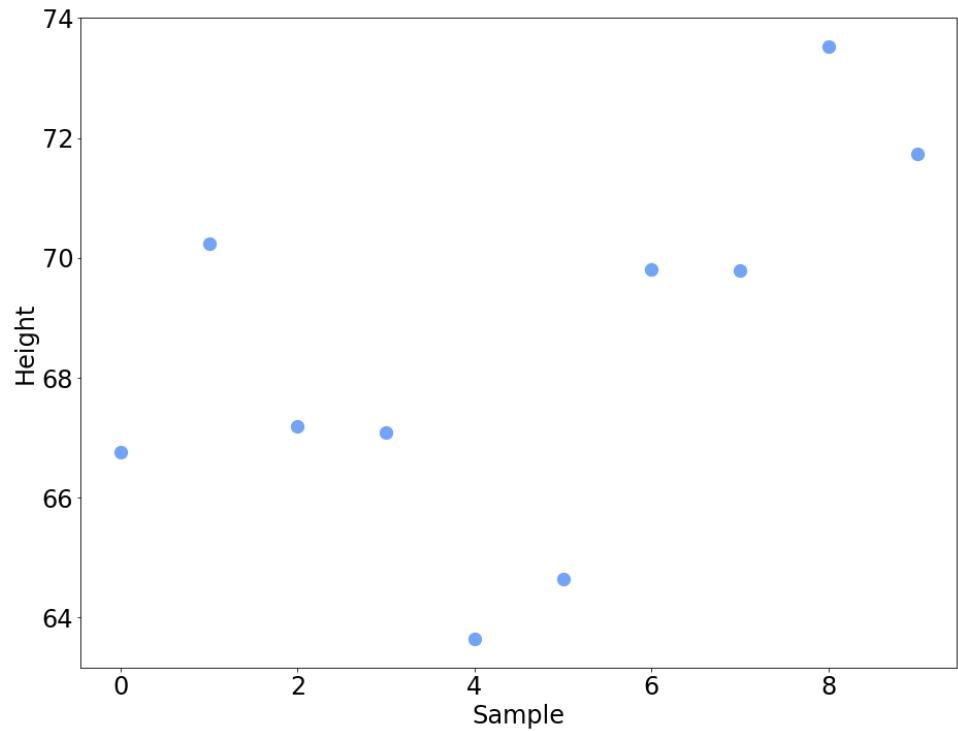
# Maximum Likelihood: Gaussian Example

# Maximum Likelihood: Gaussian Example

$X$  = "Height of an 18 year old"

You measure 10 people  
(i.e. sample size =10)

$\mathbf{X} = (X_1, X_2, \dots, X_{10})$



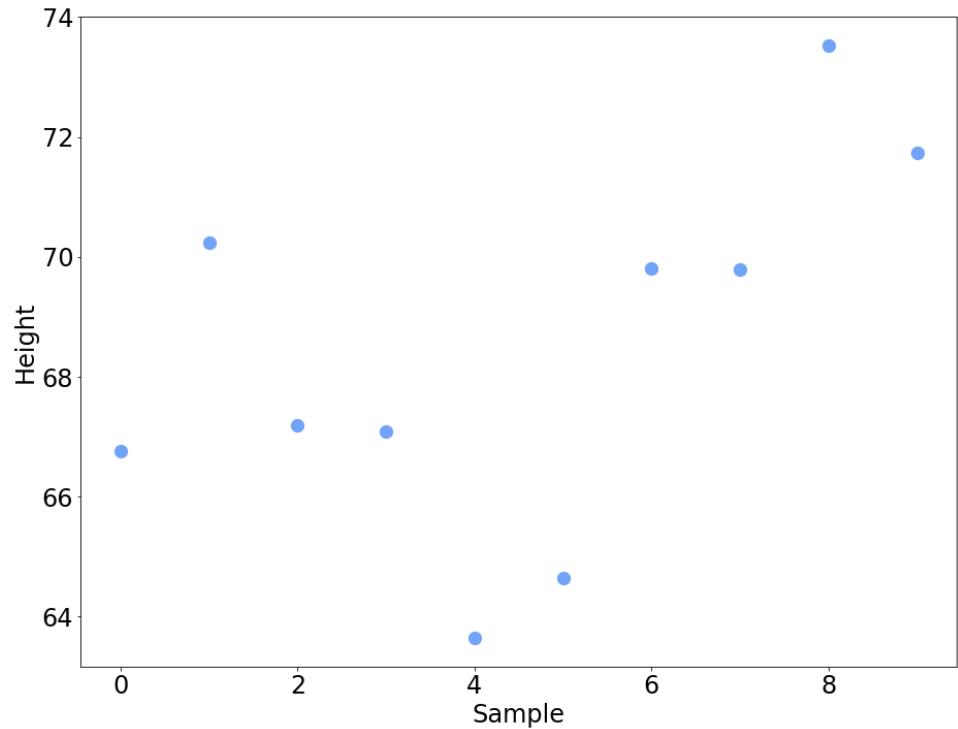
# Maximum Likelihood: Gaussian Example

$X$  = "Height of an 18 year old"

You measure 10 people  
(i.e. sample size =10)

$\mathbf{X} = (X_1, X_2, \dots, X_{10})$

$X_1, X_2, \dots, X_{10} \stackrel{i.i.d}{\sim} \mathcal{N}(\mu, \sigma)$



# Maximum Likelihood: Gaussian Example

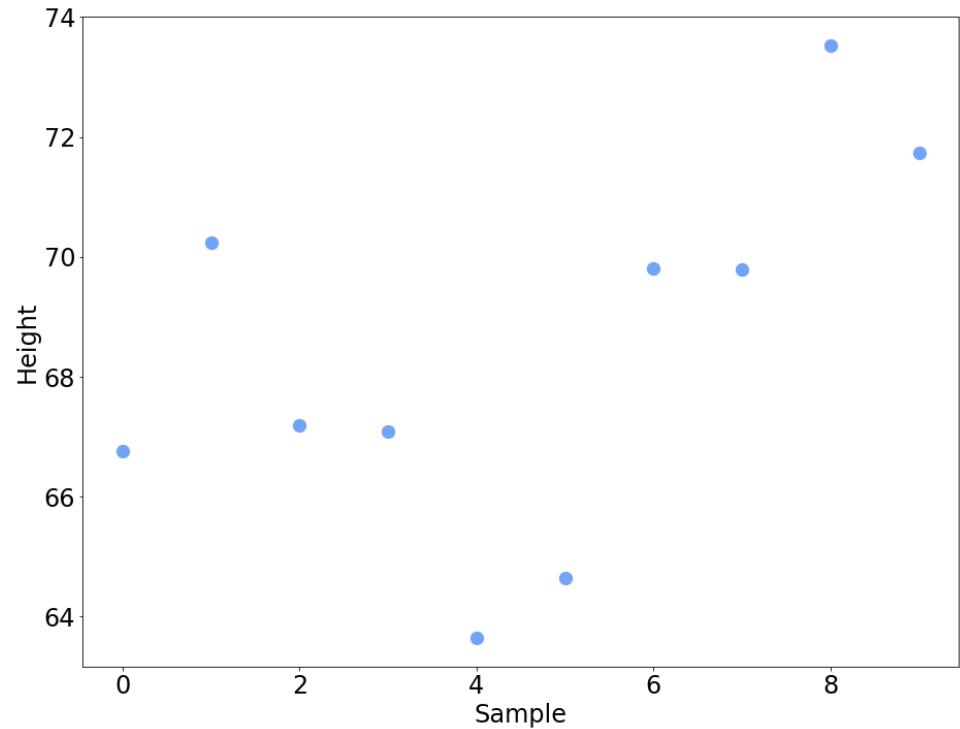
$X$  = "Height of an 18 year old"

You measure 10 people  
(i.e. sample size =10)

$\mathbf{X} = (X_1, X_2, \dots, X_{10})$

$X_1, X_2, \dots, X_{10} \stackrel{i.i.d}{\sim} \mathcal{N}(\mu, \sigma)$

$\mu?$



# Maximum Likelihood: Gaussian Example

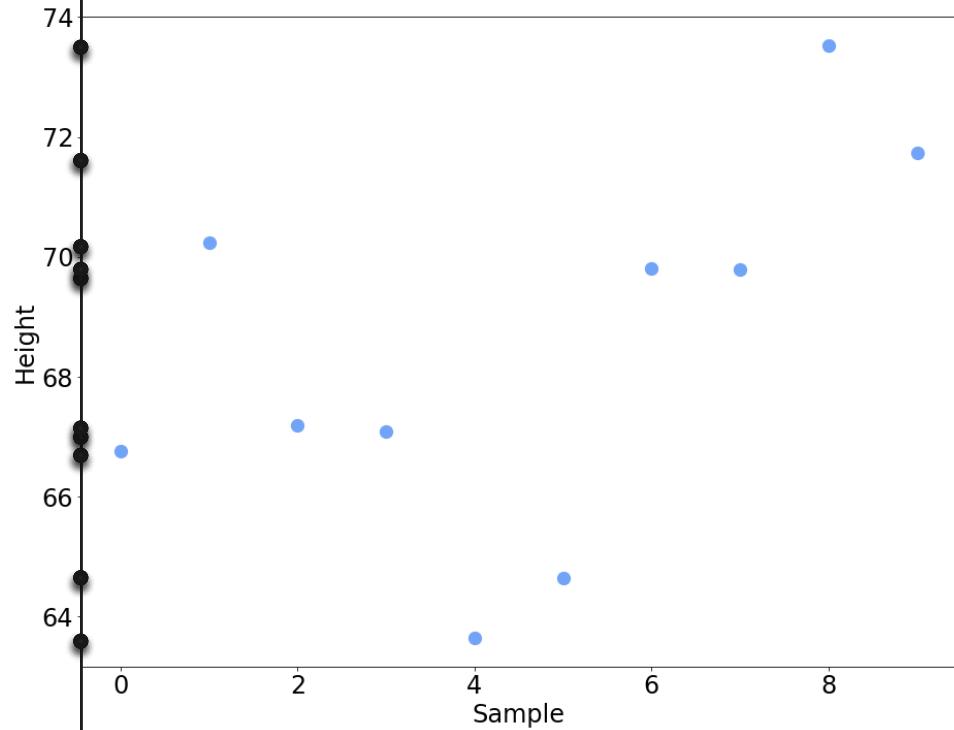
$X$  = "Height of an 18 year old"

You measure 10 people  
(i.e. sample size =10)

$\mathbf{X} = (X_1, X_2, \dots, X_{10})$

$X_1, X_2, \dots, X_{10} \stackrel{i.i.d}{\sim} \mathcal{N}(\mu, \sigma)$

$\mu?$





$\mu = 64$  $\mu = 68$  $\mu = 66$  $\mu = 70$ 

$$\begin{aligned}\mu &= 64 \\ \sigma &= 3.11\end{aligned}$$



$$\begin{aligned}\mu &= 68 \\ \sigma &= 3.11\end{aligned}$$

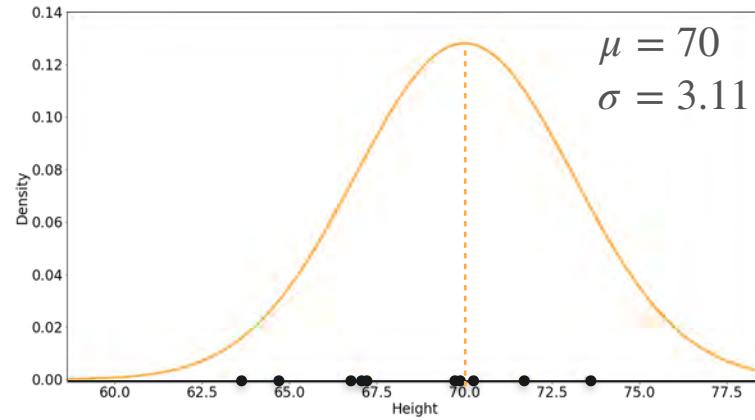
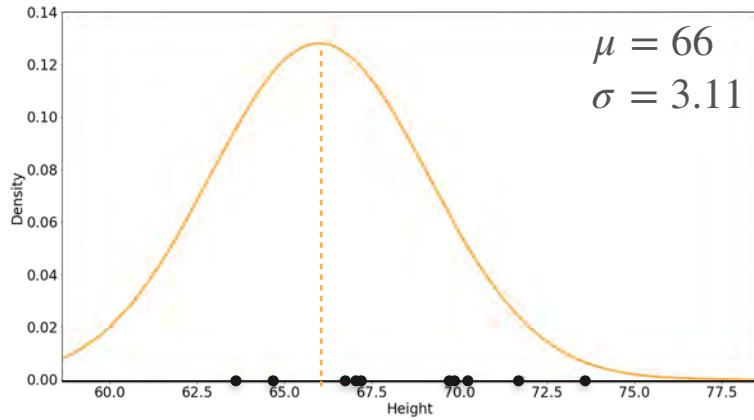
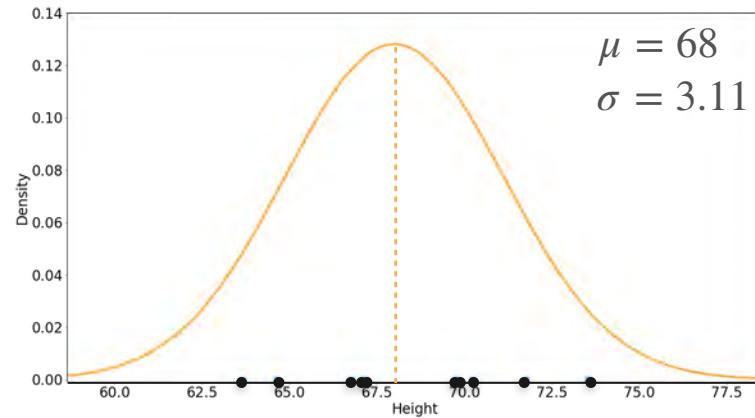
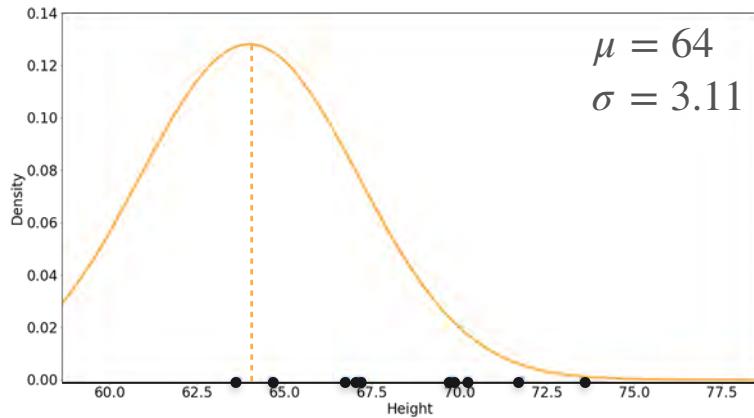


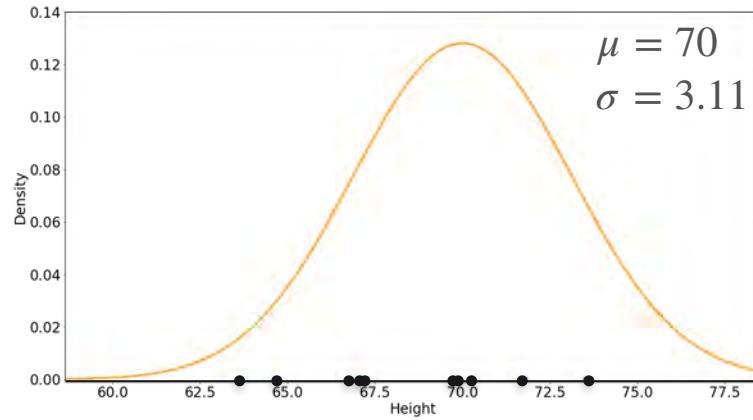
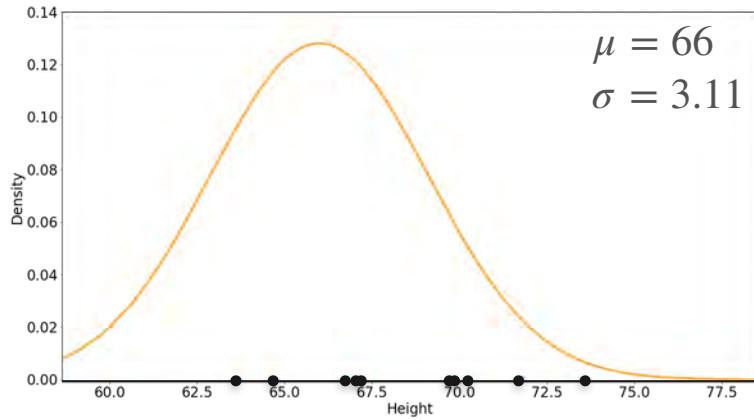
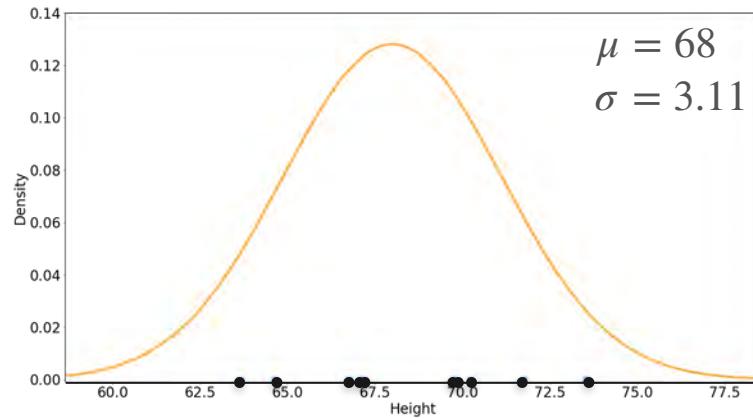
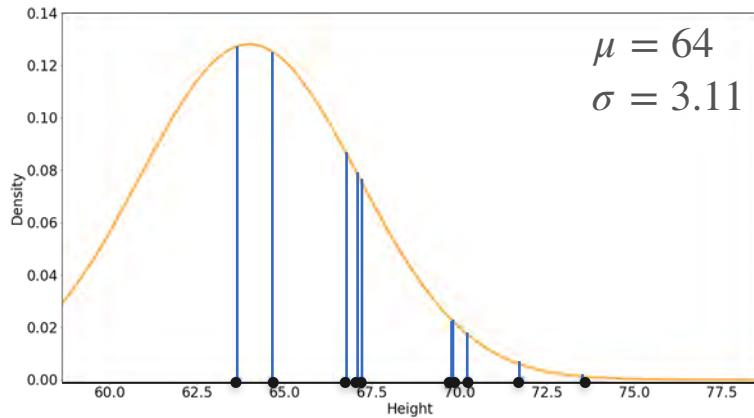
$$\begin{aligned}\mu &= 66 \\ \sigma &= 3.11\end{aligned}$$

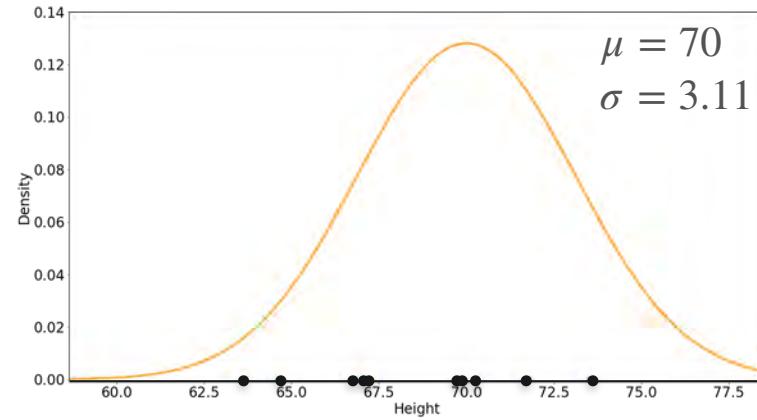
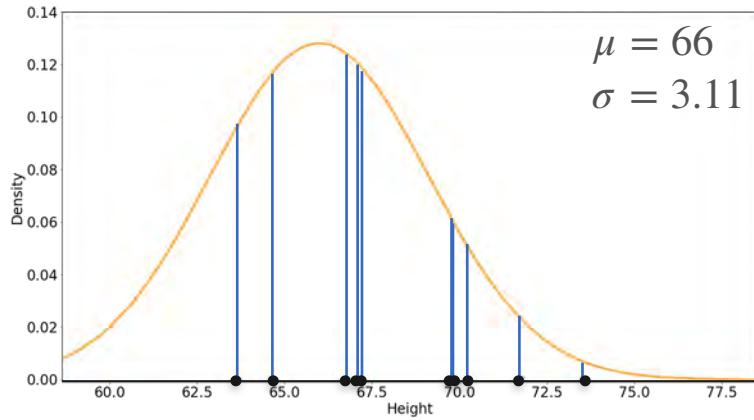
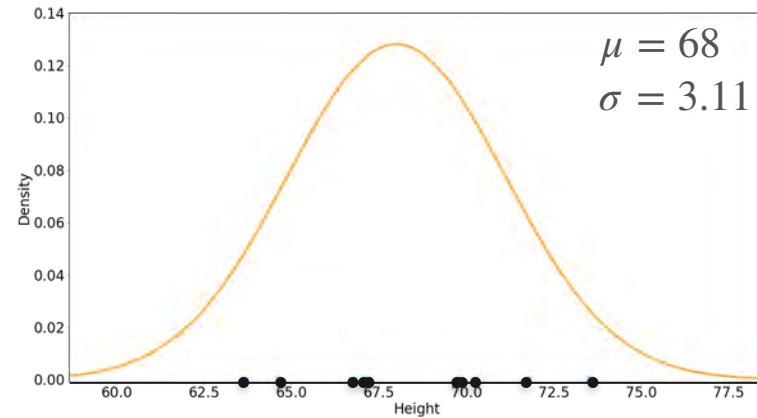
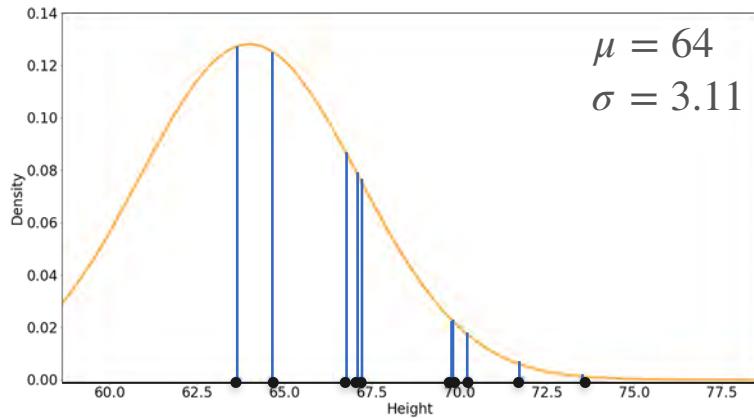


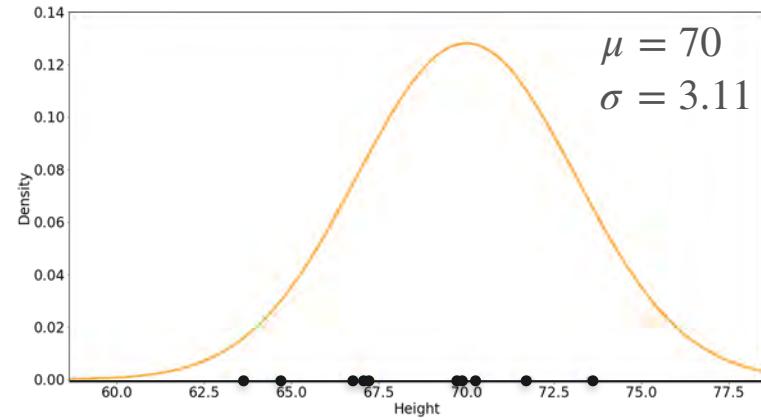
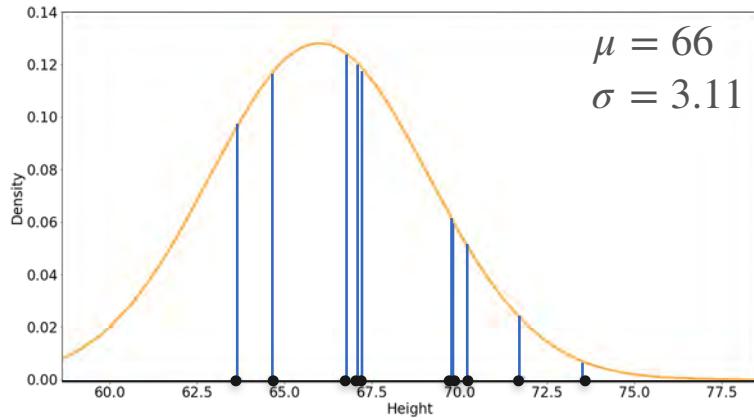
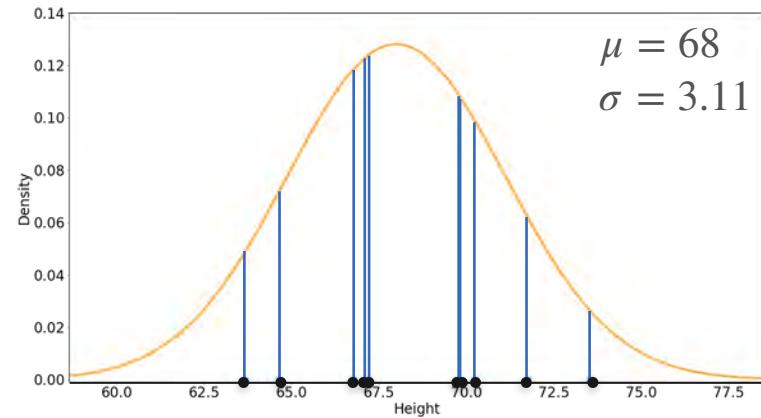
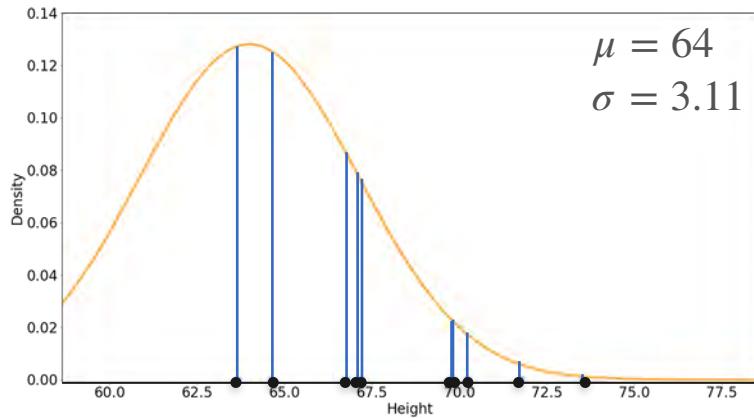
$$\begin{aligned}\mu &= 70 \\ \sigma &= 3.11\end{aligned}$$

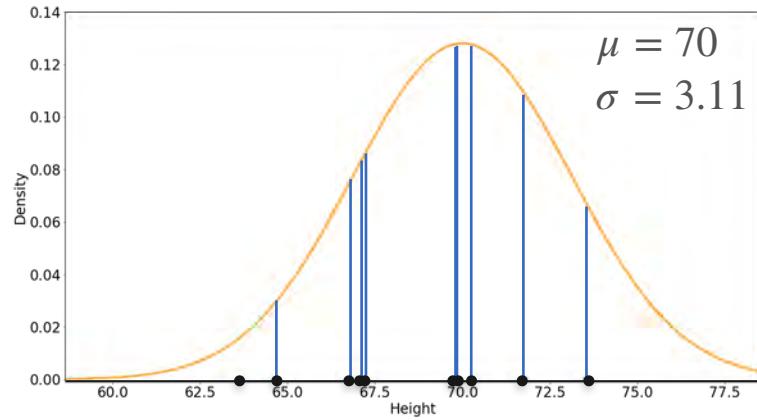
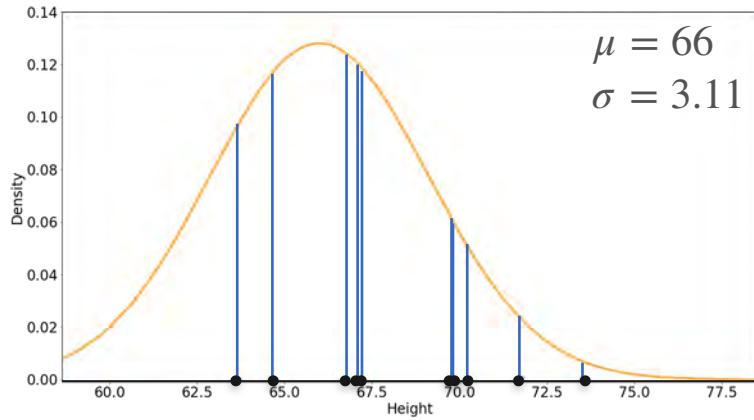
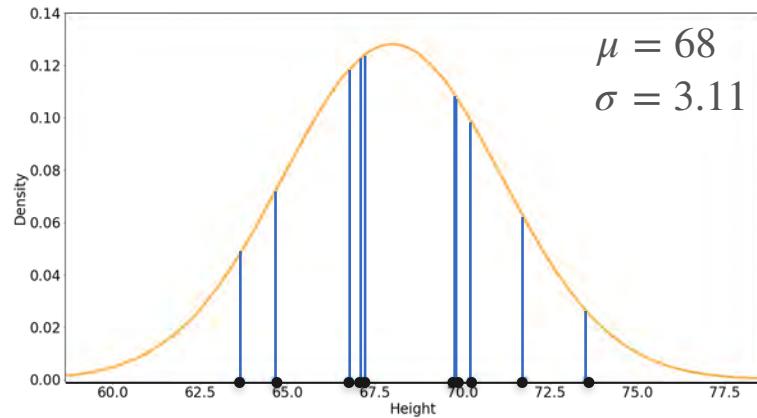
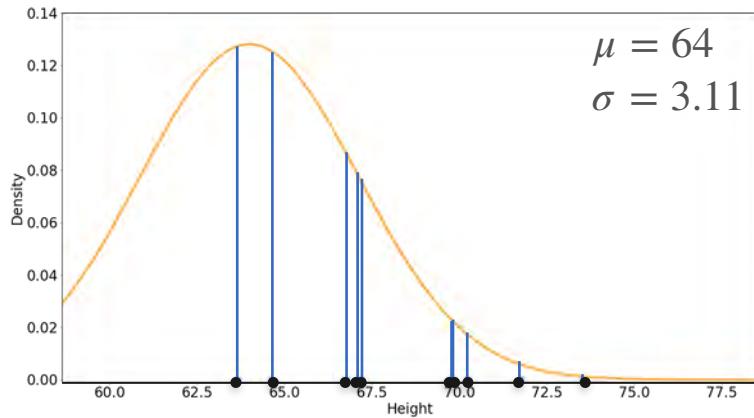


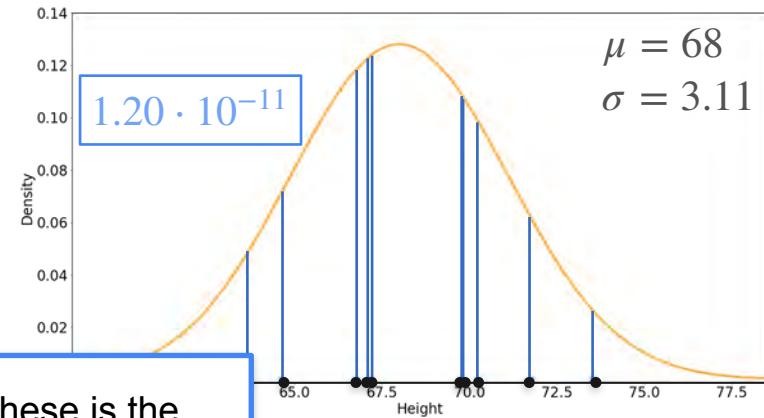
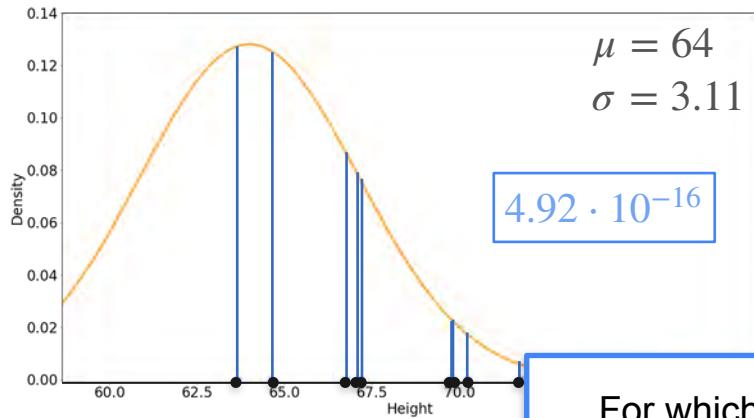




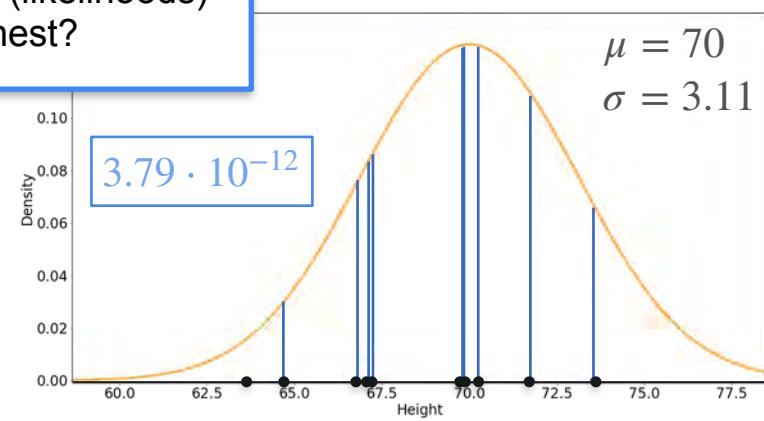
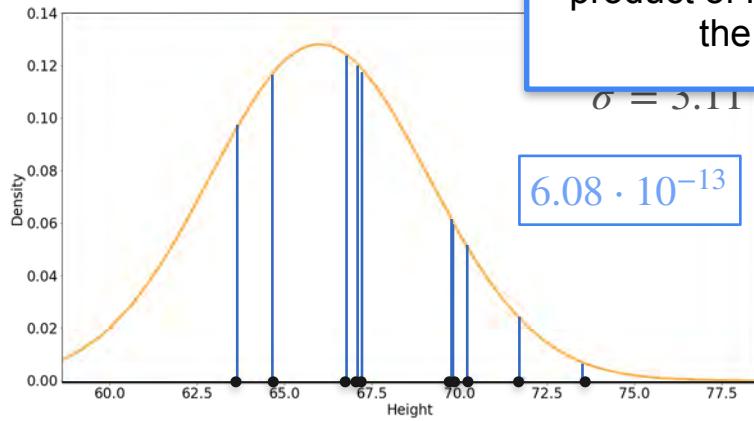


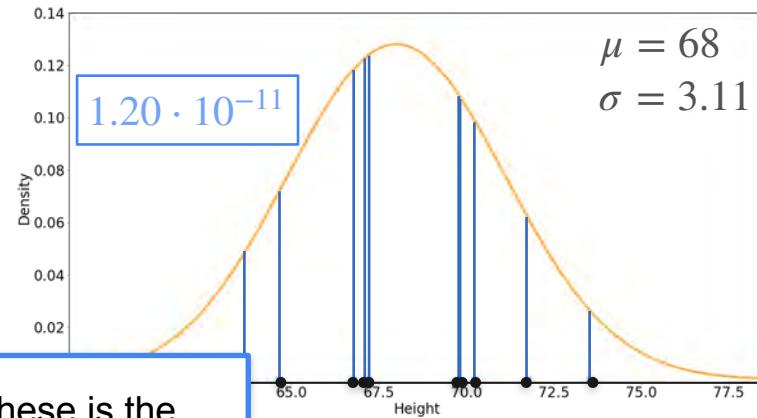
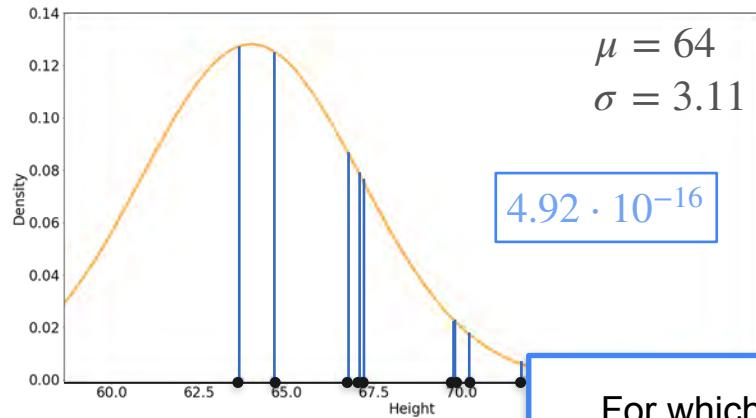




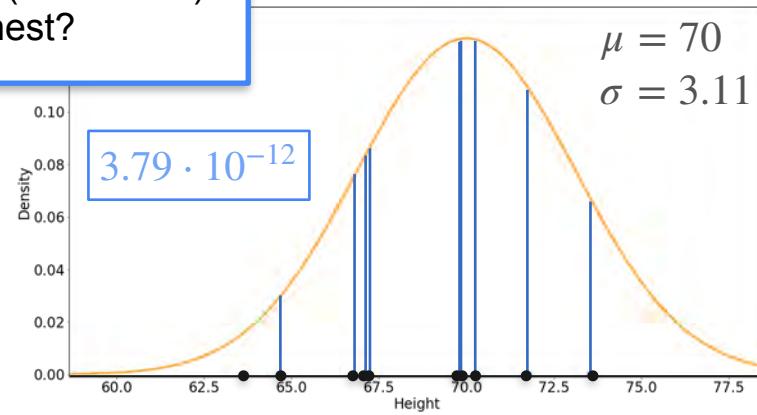
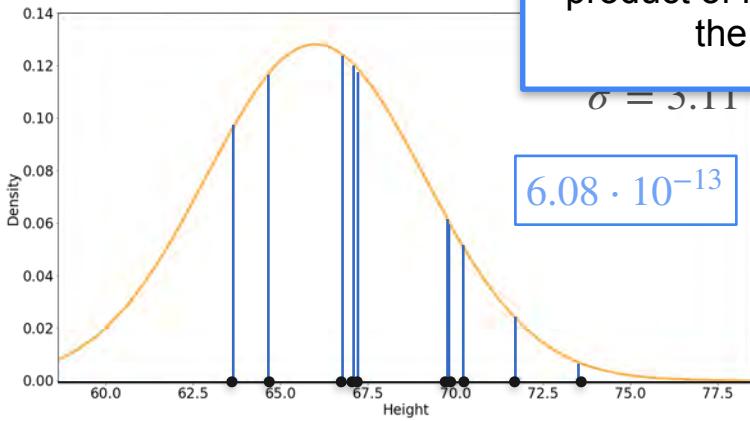


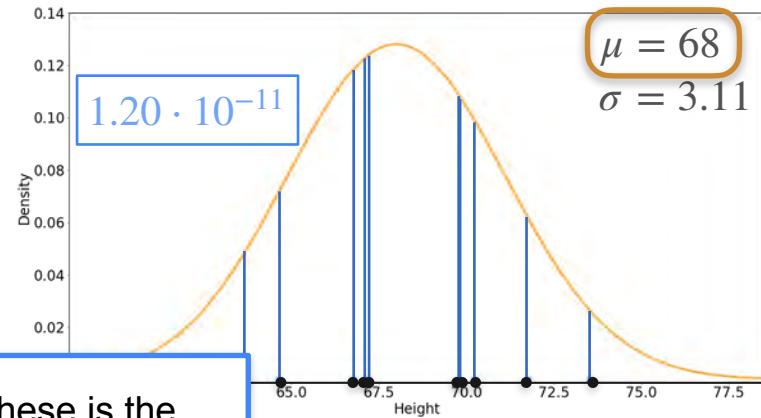
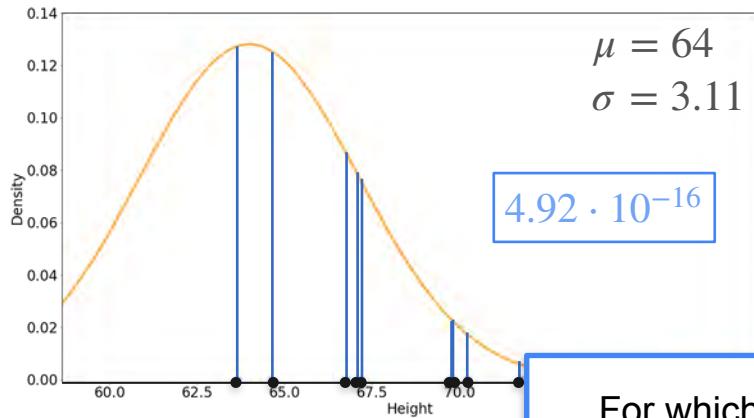
For which of these is the product of lines (likelihoods) the highest?



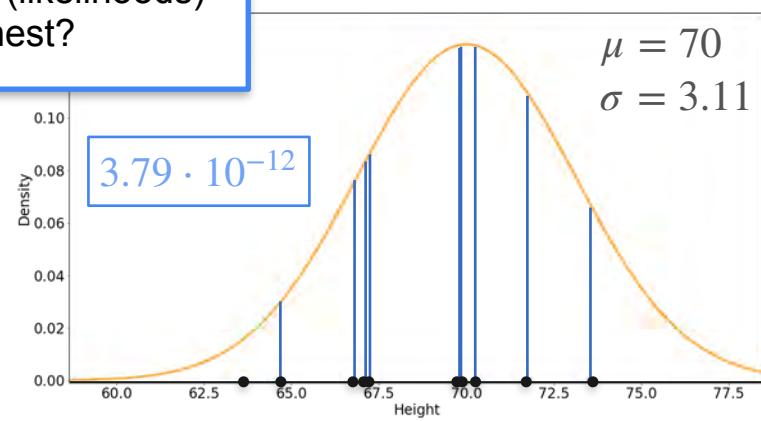
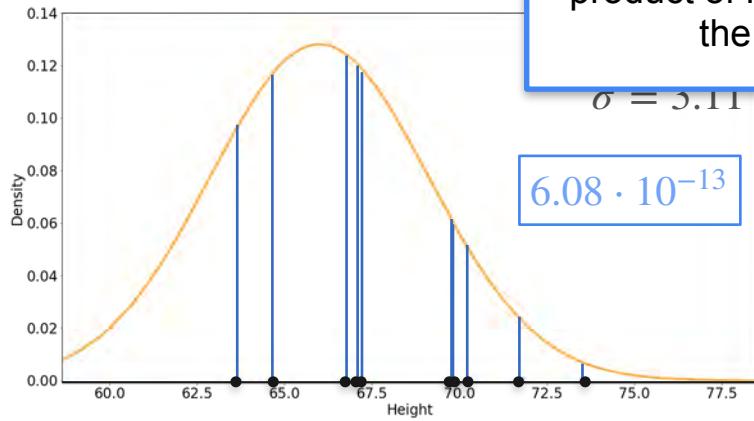


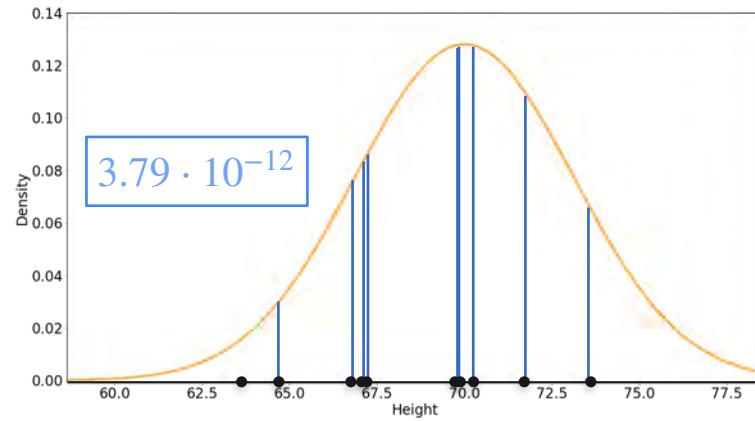
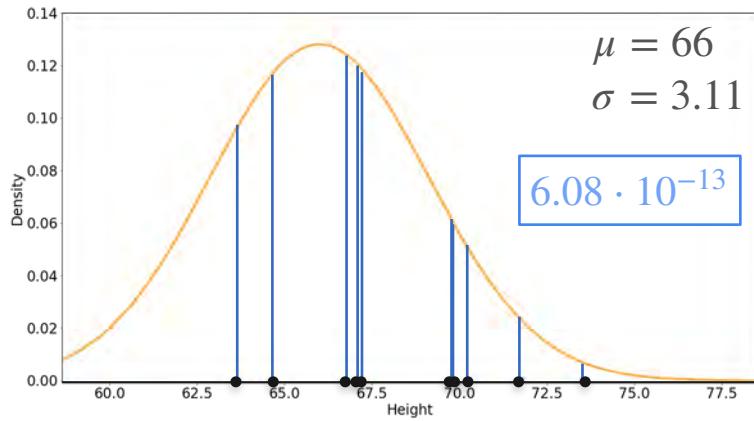
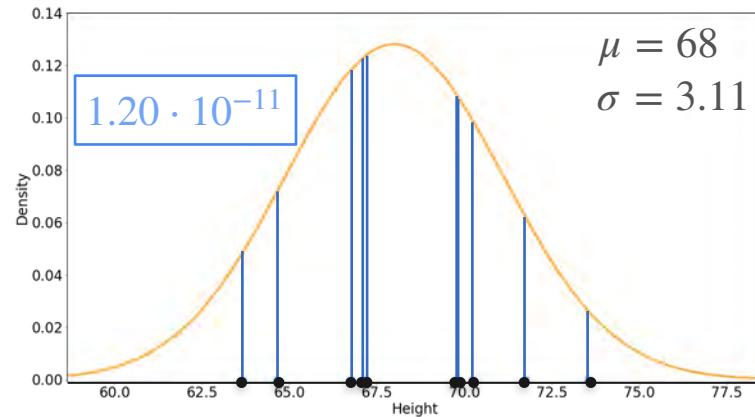
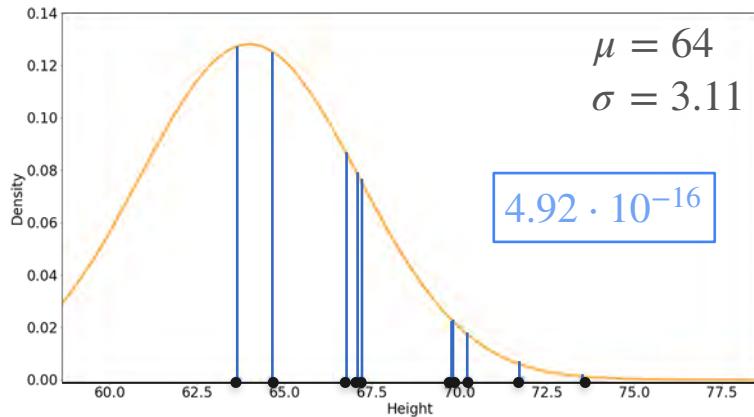
For which of these is the product of lines (likelihoods) the highest?

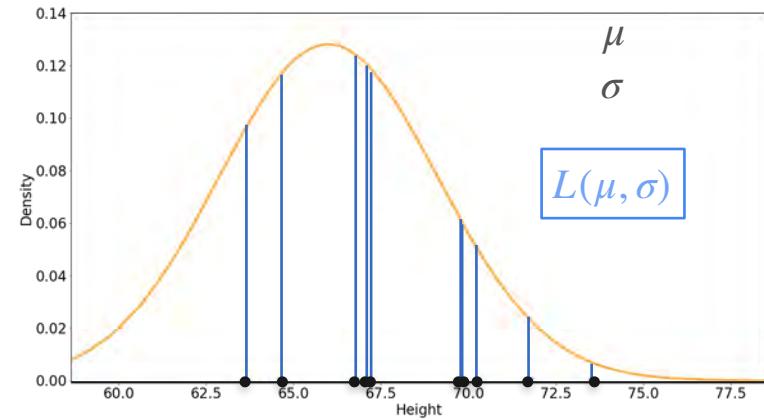




For which of these is the product of lines (likelihoods) the highest?

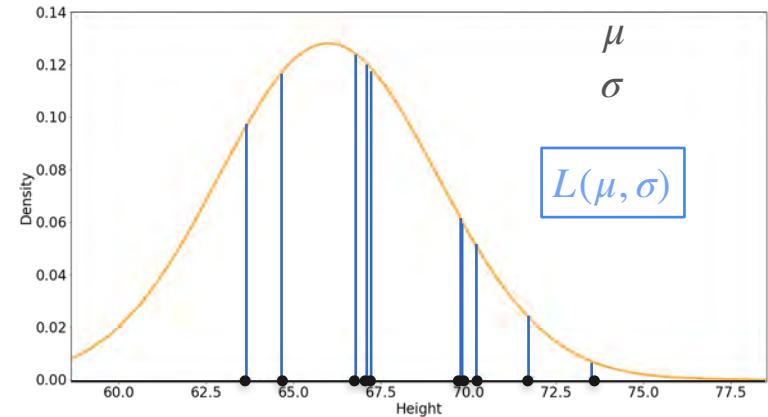






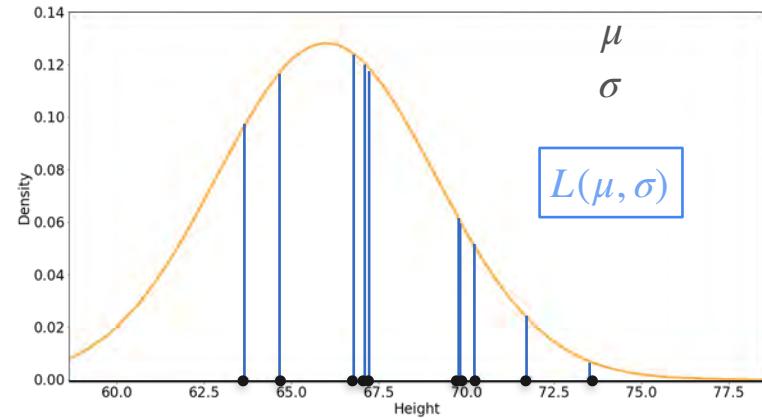
$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log \left( \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^{10} \right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

$$\text{Likelihood } L(\mu; \underline{x}) = \prod_{i=1}^{10} f_\mu(x_i)$$



$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log \left( \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^{10} \right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

$$\begin{aligned}
 \text{Likelihood } L(\mu; \underline{x}) &= \prod_{i=1}^{10} f_\mu(x_i) \\
 &= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right)
 \end{aligned}$$

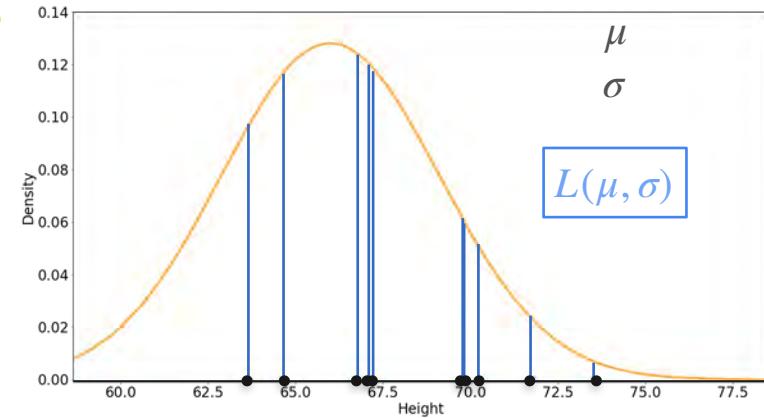


$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log \left( \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

$$\text{Likelihood } L(\mu; \underline{x}) = \prod_{i=1}^{10} f_\mu(x_i)$$

$$= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp \left( -\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2} \right)$$

Observations  
 $x_i$  are fixed!



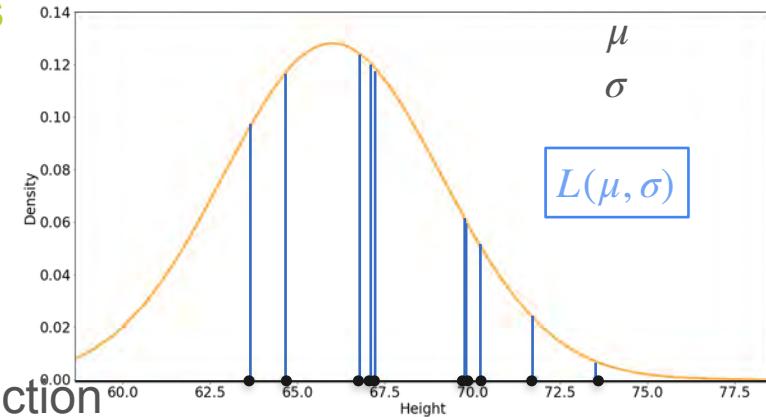
$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log \left( \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

$$\text{Likelihood } L(\mu; \underline{x}) = \prod_{i=1}^{10} f_\mu(x_i)$$

$$= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp \left( -\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2} \right)$$

Observations  
 $x_i$  are fixed!

It's a function  
of  $\mu$

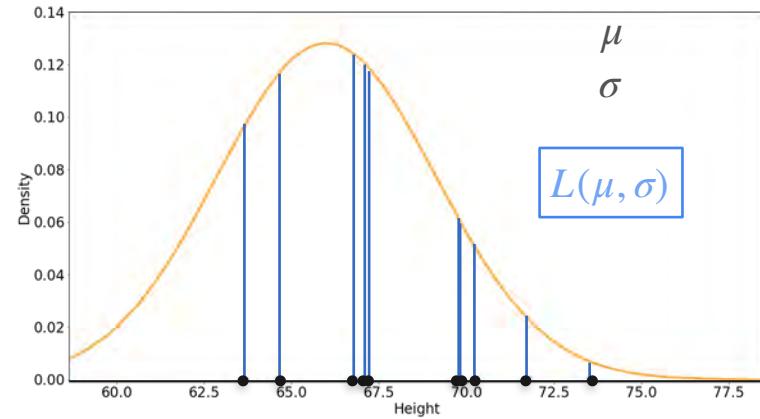


$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log \left( \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

$$\text{Likelihood } L(\mu; \underline{x}) = \prod_{i=1}^{10} f_\mu(x_i)$$

$$= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right)$$

$$\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}$$

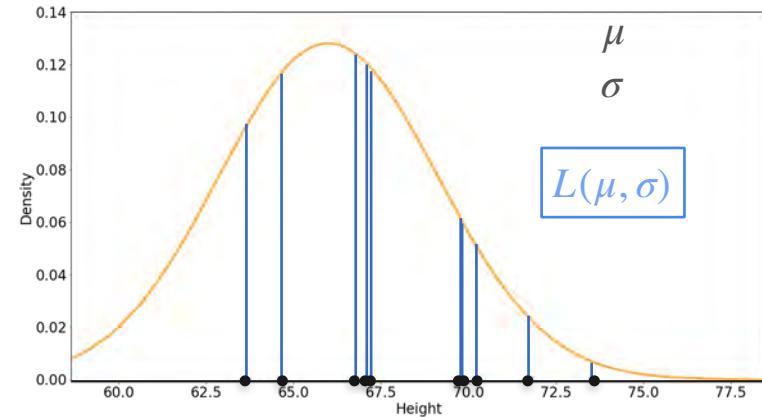


$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log \left( \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

$$\text{Likelihood } L(\mu; \underline{x}) = \prod_{i=1}^{10} f_\mu(x_i)$$

$$= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp \left( -\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2} \right)$$

$$\left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \exp \left( -\frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right)$$

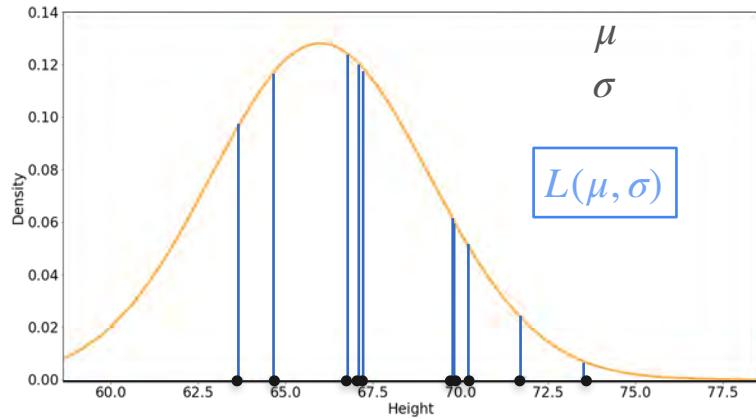


$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log \left( \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

Likelihood  $L(\mu; \underline{x})$  =  $\prod_{i=1}^{10} f_\mu(x_i)$

$$= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right)$$

$$\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10} \exp\left(-\frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}\right)$$

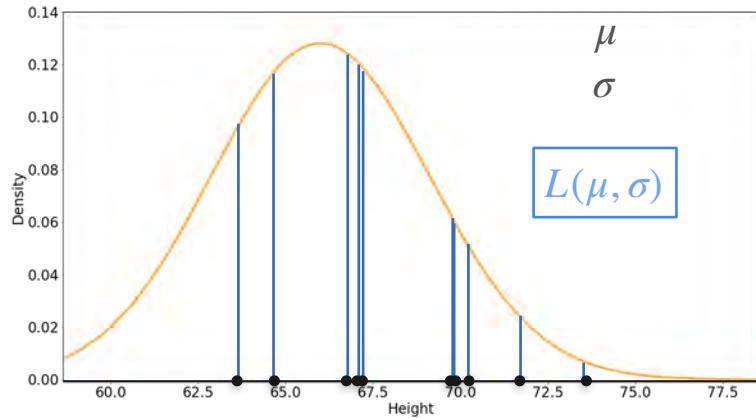


$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

Likelihood  $L(\mu; \underline{x})$  =  $\prod_{i=1}^{10} f_\mu(x_i)$

$$= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right)$$

$$\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10} \exp\left(-\frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}\right)$$



$$\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x}))$$

$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$

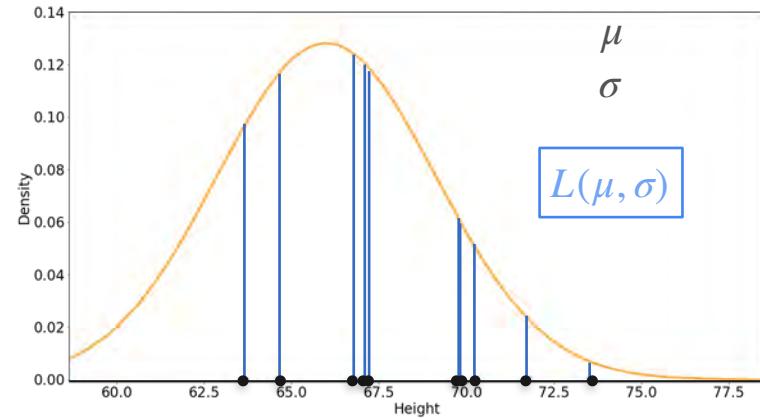
Likelihood  $L(\mu; \underline{x})$  =  $\prod_{i=1}^{10} f_\mu(x_i)$

$$= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right)$$

$$\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10} \exp\left(-\frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}\right)$$

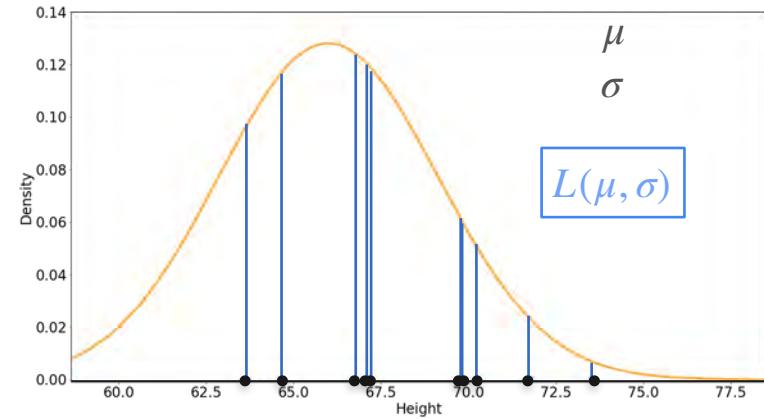
$$\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$$

$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$



$$\begin{aligned}
 \text{Likelihood } L(\mu; \underline{x}) &= \prod_{i=1}^{10} f_\mu(x_i) \\
 &= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right) \\
 &\quad \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \exp\left(-\frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Log-Likelihood } \ell(\mu; \mathbf{x}) &= \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \\
 \frac{d}{d\mu} \ell(\mu; \mathbf{x}) &= \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]
 \end{aligned}$$

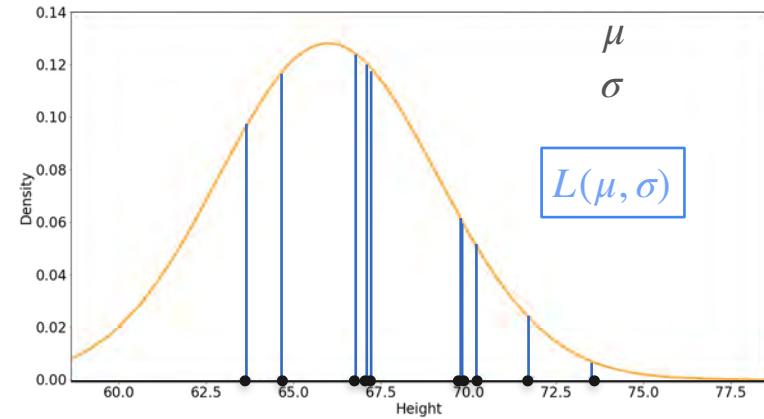


$$\begin{aligned}
 \text{Likelihood } L(\mu; \underline{x}) &= \prod_{i=1}^{10} f_\mu(x_i) \\
 &= \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right) \\
 &\quad \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{10} \exp\left(-\frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}\right)
 \end{aligned}$$

$$\text{Log-Likelihood } \ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$$

Derivative

$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi} \sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$$



Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right]$

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

$=$  0

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

$$= 0 - \frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^{10} 2(x_i - \mu)(-1)$$

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

$$= 0$$

$$-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^{10} 2(x_i - \mu)(-1)$$

This needs to be zero

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

$$= 0$$

$$\boxed{-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^{10} 2(x_i - \mu)(-1)}$$

This needs to be zero

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

$$= 0$$

$$-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^{10} 2(x_i - \mu)(-1)$$

This needs to be zero

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

$$= 0$$

$$-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^{10} 2(x_i - \mu)(-1)$$

This needs to be zero

$$\sum_{i=1}^{10} (x_i - \mu) = 0$$

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

$= \quad \quad \quad 0$

$$-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^{10} 2(x_i - \mu)(-1)$$

$$\sum_{i=1}^{10} (x_i - \mu) = 0$$

$$\left( \sum_{i=1}^{10} x_i \right) - 10\mu = 0$$

This needs to be zero

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{10}\right) - \frac{1}{2} \frac{\sum_{i=1}^{10} (x_i - \mu)^2}{\sigma^2} \right] = 0$

= 0

$$-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^{10} 2(x_i - \mu)(-1)$$

$$\sum_{i=1}^{10} (x_i - \mu) = 0$$

$$\left( \sum_{i=1}^{10} x_i \right) - 10\mu = 0 \quad \rightarrow \quad \mu = \frac{1}{10} \sum_{i=1}^{10} x_i$$

This needs to be zero

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n\right) - \frac{1}{2} \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^2}$

Derivative  $\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n\right) - \frac{1}{2} \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^2} \right] = 0$

$= 0$

$$\boxed{-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^n 2(x_i - \mu)(-1)}$$

$$\sum_{i=1}^n (x_i - \mu) = 0$$

$$\left( \sum_{i=1}^n x_i \right) - n \mu = 0 \quad \rightarrow \quad \mu = \frac{1}{n} \sum_{i=1}^n x_i$$

This needs to be zero

Log-Likelihood  $\ell(\mu; \mathbf{x}) = \log(L(\mu; \mathbf{x})) = \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n\right) - \frac{1}{2} \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^2}$

Derivative

$$\frac{d}{d\mu} \ell(\mu; \mathbf{x}) = \frac{d}{d\mu} \left[ \log\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n\right) - \frac{1}{2} \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^2} \right] = 0$$

$$= 0$$

$$\boxed{-\frac{1}{2} \frac{1}{\sigma^2} \sum_{i=1}^n 2(x_i - \mu)(-\cancel{1})}$$

$$\sum_{i=1}^n (x_i - \mu) = 0$$

$$\left( \sum_{i=1}^n x_i \right) - n \mu = 0 \quad \rightarrow \quad \mu = \frac{1}{n} \sum_{i=1}^n x_i$$

This needs to be zero

The best distribution is the one where the **mean** of the distribution is the **mean** of the sample

# Maximum Likelihood: Gaussian Example

# Maximum Likelihood: Gaussian Example

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

# Maximum Likelihood: Gaussian Example

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

$$\hat{\mu} = \frac{66.75 + 70.24 + 67.19 + 67.09 + 63.65 + 64.64 + 69.81 + 69.79 + 73.52 + 71.74}{10}$$

# Maximum Likelihood: Gaussian Example

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

$$\hat{\mu} = \frac{66.75 + 70.24 + 67.19 + 67.09 + 63.65 + 64.64 + 69.81 + 69.79 + 73.52 + 71.74}{10}$$
$$= 68.442$$

# Maximum Likelihood: Gaussian Example

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

# Maximum Likelihood: Gaussian Example

What about the variance?

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

# Maximum Likelihood: Gaussian Example

What about the variance?

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

If you repeat the process, assuming unknown variance, you'll get that

# Maximum Likelihood: Gaussian Example

What about the variance?

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

If you repeat the process, assuming unknown variance, you'll get that

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{10} (x_i - \bar{x})^2}{10}$$

# Maximum Likelihood: Gaussian Example

What about the variance?

Heights	66.75	70.24	67.19	67.09	63.65	64.64	69.81	69.79	73.52	71.74
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------

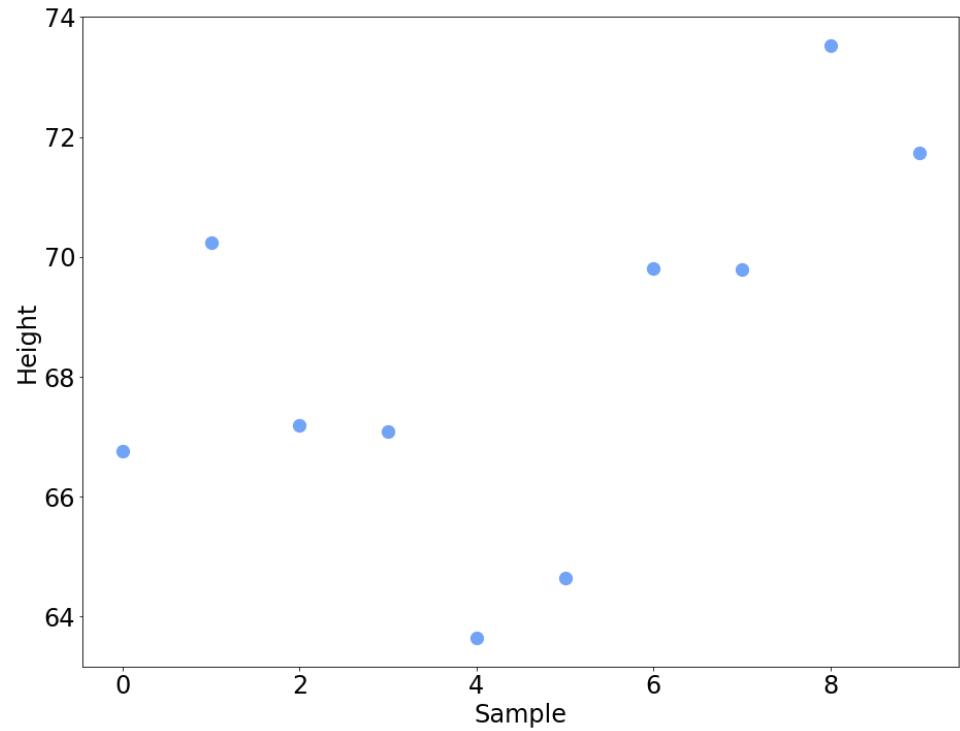
If you repeat the process, assuming unknown variance, you'll get that

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{10} (x_i - \bar{x})^2}{10} = \frac{1}{10} ((66.75 - 68.442)^2 + (70.24 - 68.442)^2 + (67.19 - 68.442)^2 + (67.09 - 68.442)^2 + (63.65 - 68.442)^2 + (64.64 - 68.442)^2 + (69.81 - 68.442)^2 + (69.79 - 68.442)^2 + (73.52 - 68.442)^2 + (71.74 - 68.442)^2) = 8.72$$

# Maximum Likelihood: Gaussian Example

$X$  = "Height of an 18 year old"

$\mathcal{N}(\mu, \sigma)$

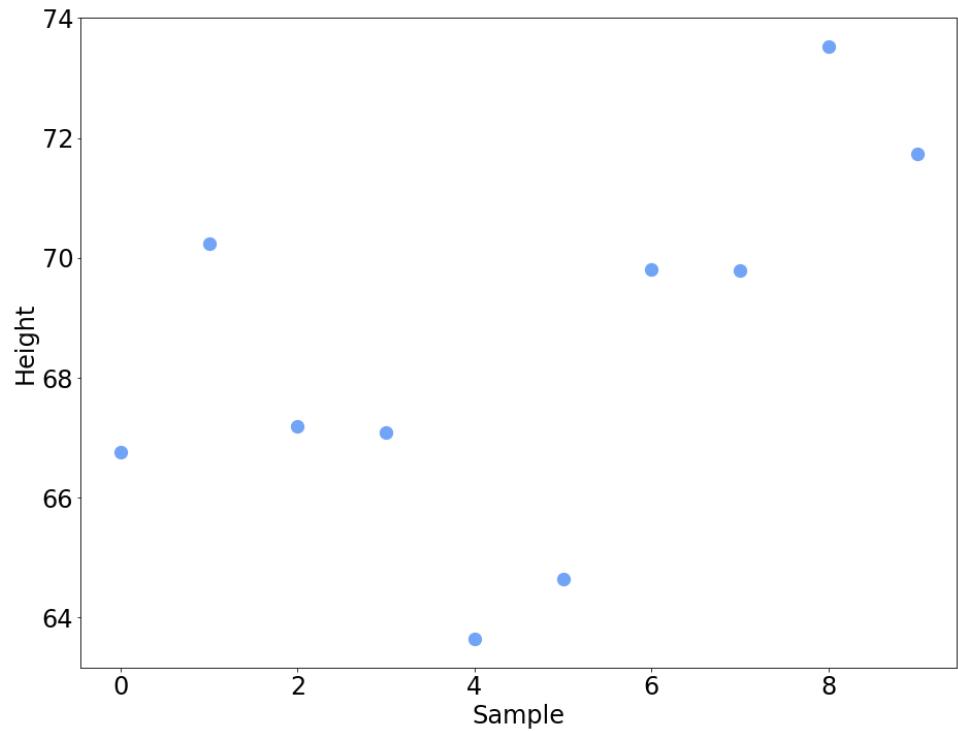


# Maximum Likelihood: Gaussian Example

$X$  = "Height of an 18 year old"

$\mathcal{N}(\mu, \sigma)$

$\hat{\mu} = 68.442$



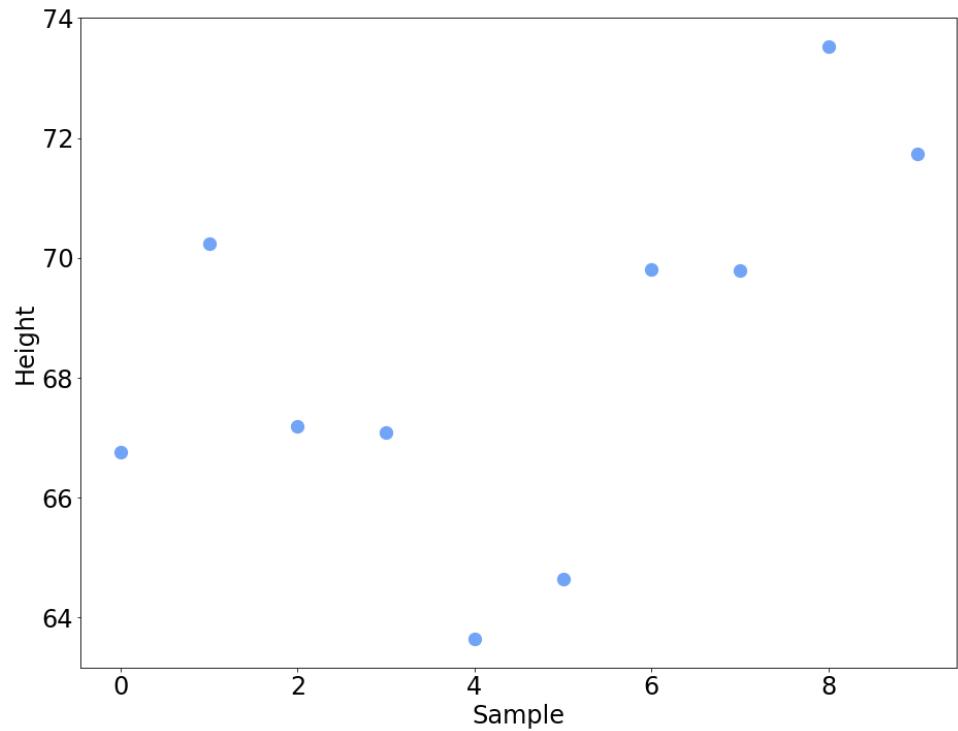
# Maximum Likelihood: Gaussian Example

$X$  = "Height of an 18 year old"

$$\mathcal{N}(\mu, \sigma)$$

$$\hat{\mu} = 68.442$$

$$\hat{\sigma} = 8.72$$





DeepLearning.AI

## Point Estimation

---

## MLE: Linear Regression

# Maximum Likelihood

# Maximum Likelihood



Data

# Maximum Likelihood



Model 1



Data

# Maximum Likelihood



Model 1



Model 2



Data

# Maximum Likelihood



Model 1



Model 2

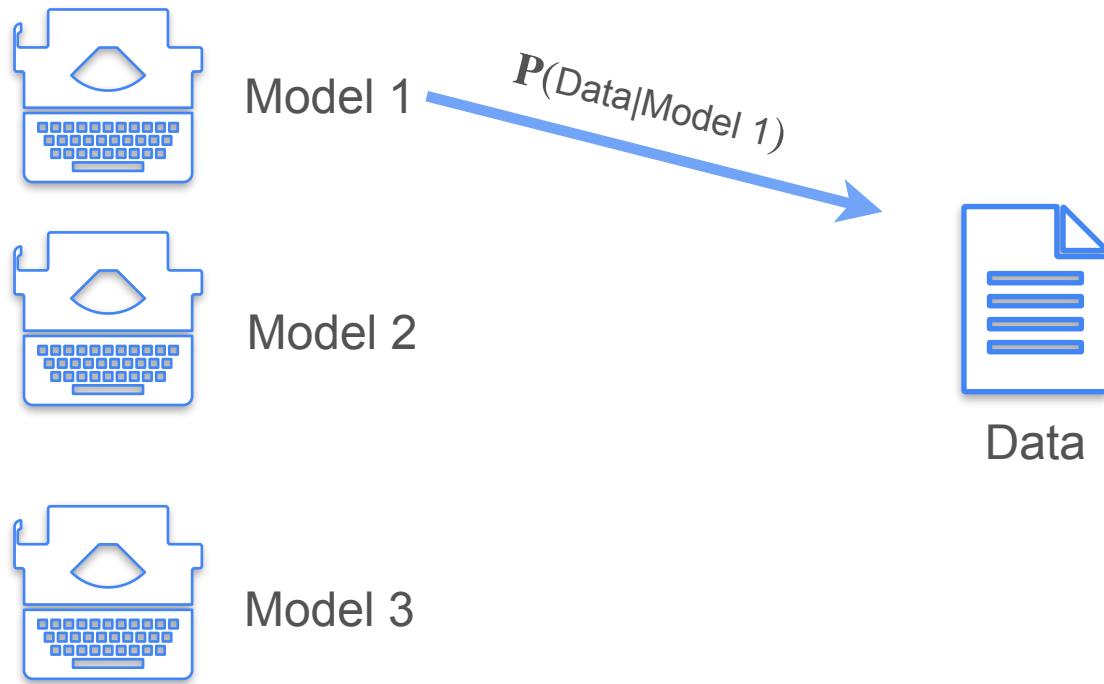


Model 3

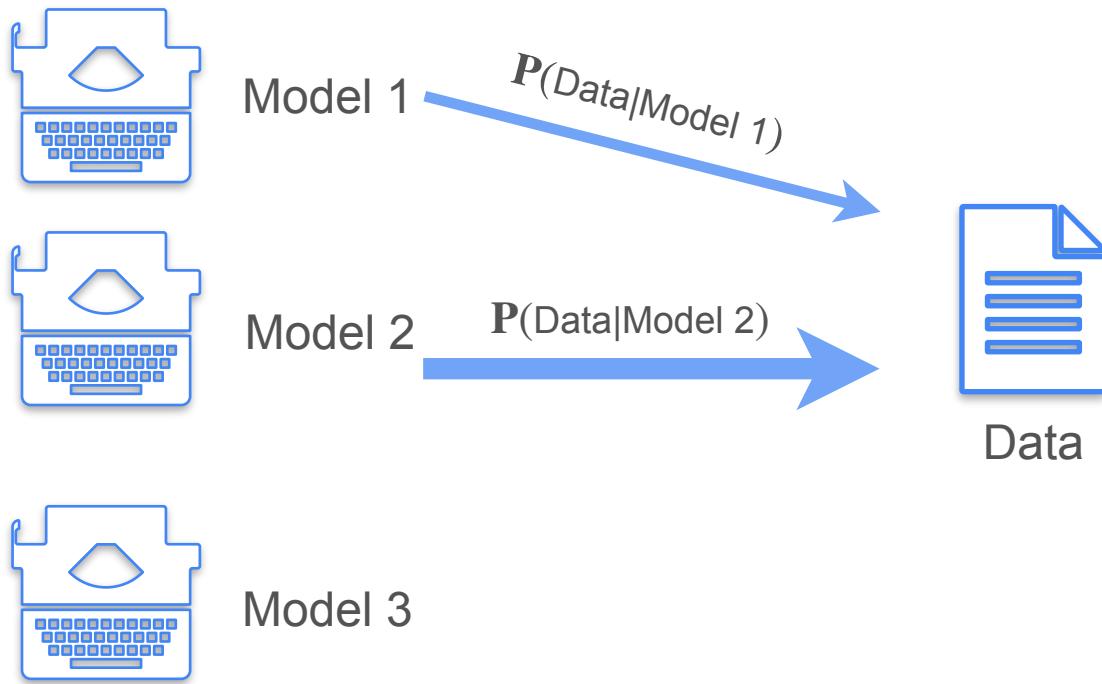


Data

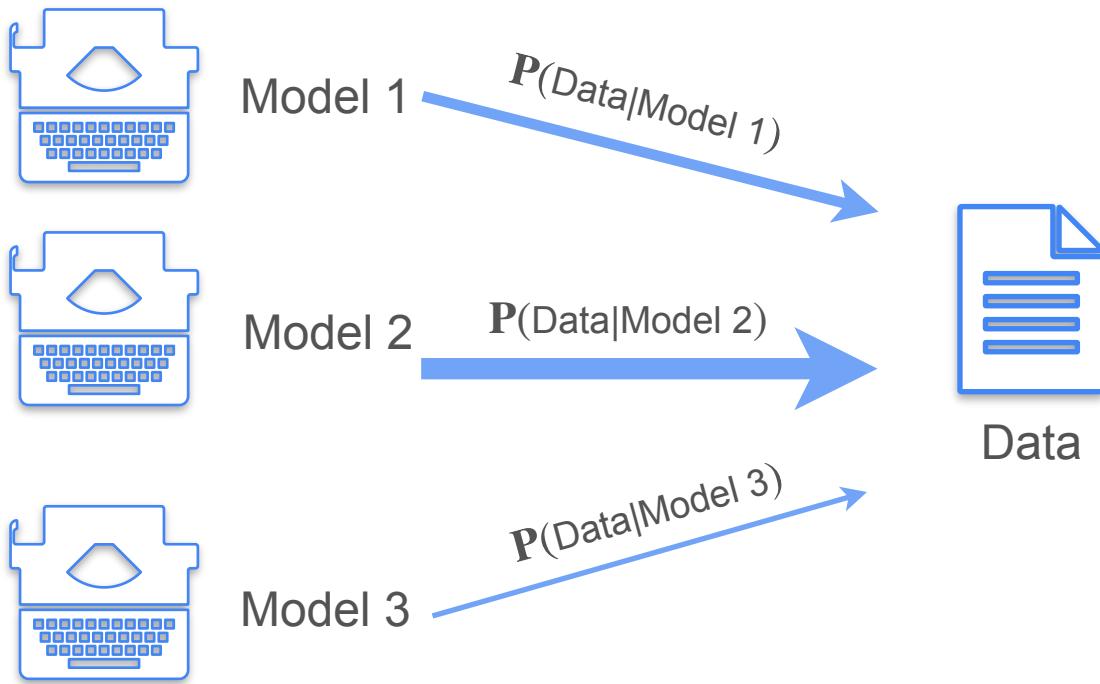
# Maximum Likelihood



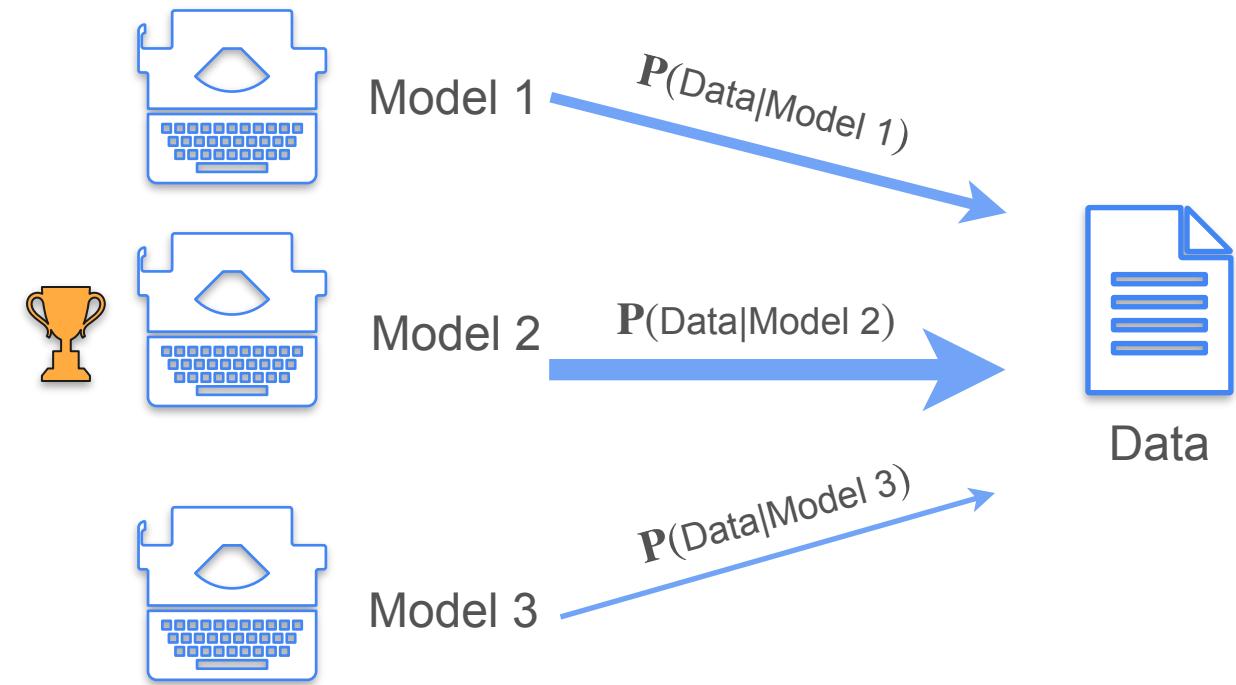
# Maximum Likelihood



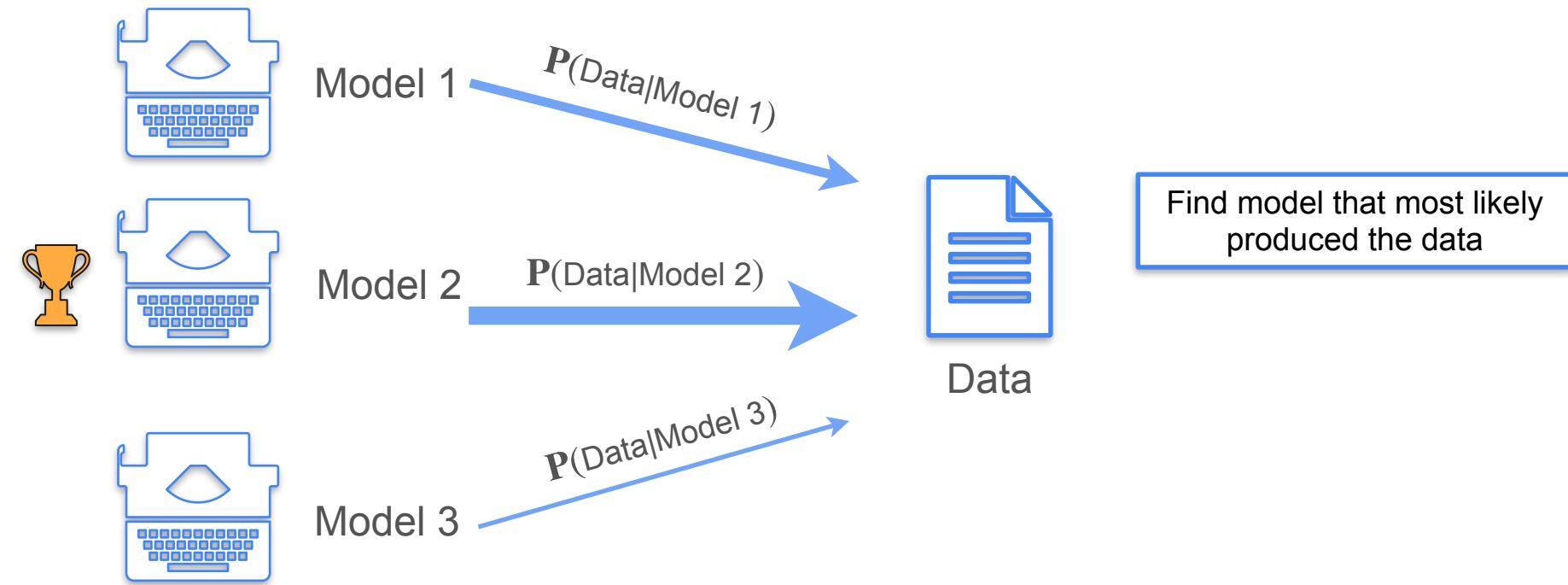
# Maximum Likelihood



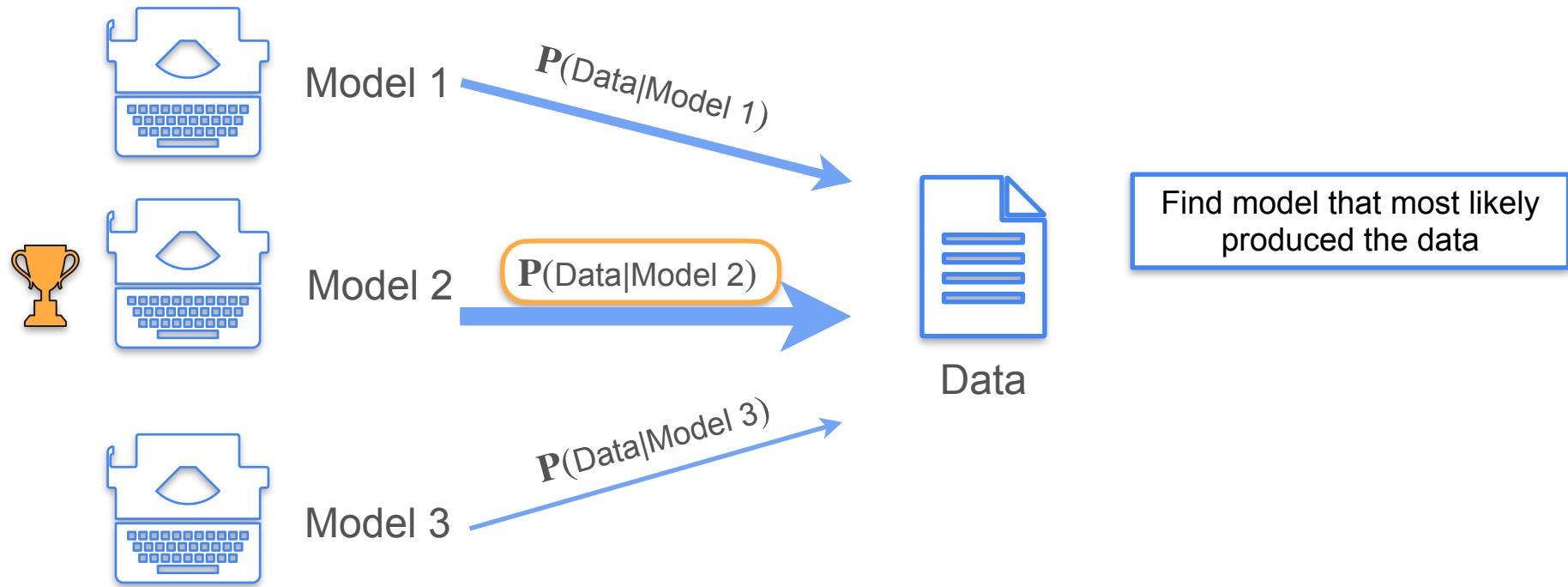
# Maximum Likelihood



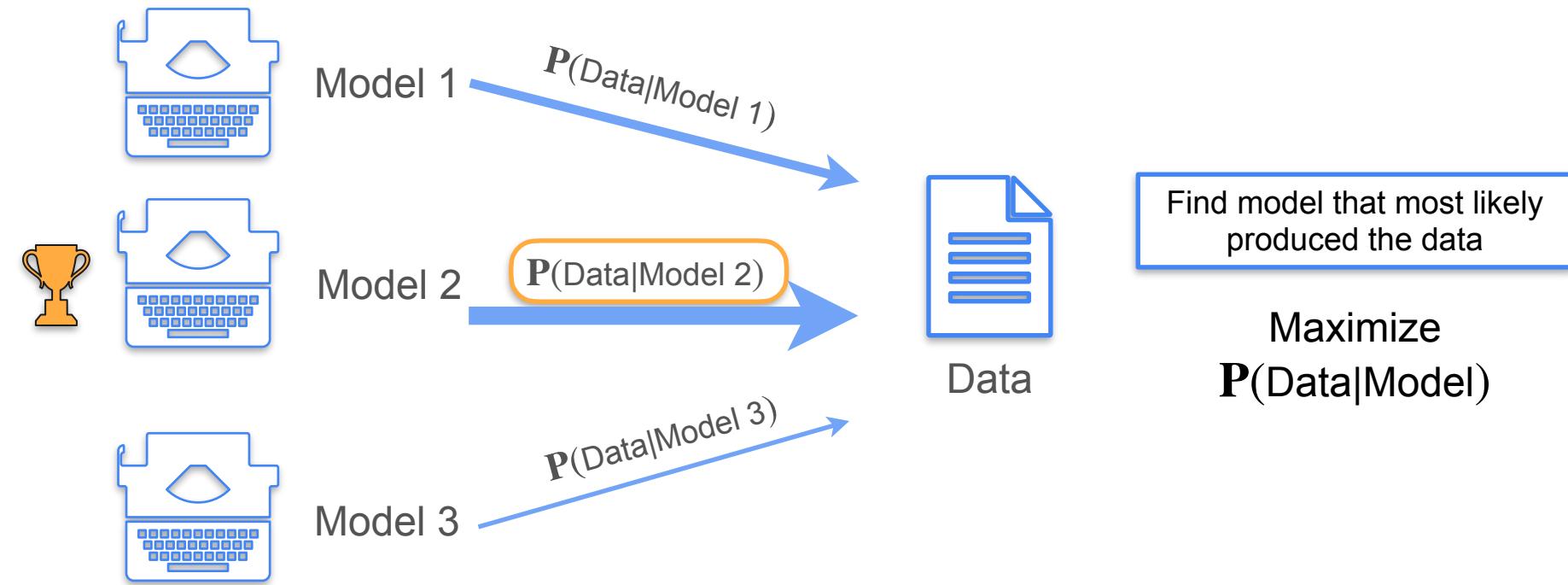
# Maximum Likelihood



# Maximum Likelihood



# Maximum Likelihood

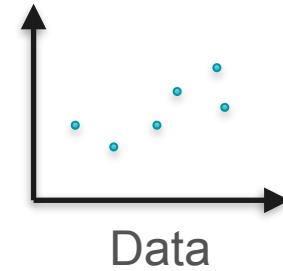
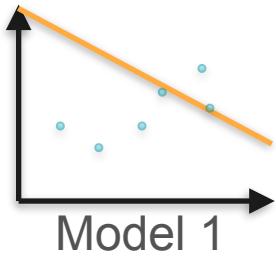


# Example: Linear Regression

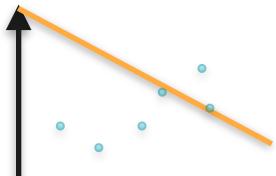
# Example: Linear Regression



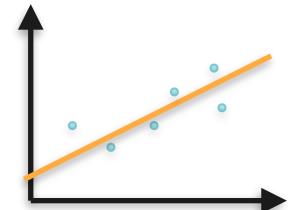
# Example: Linear Regression



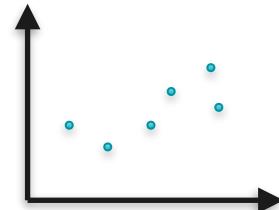
# Example: Linear Regression



Model 1

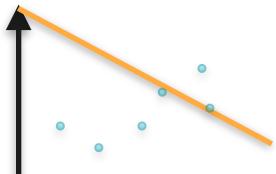


Model 2

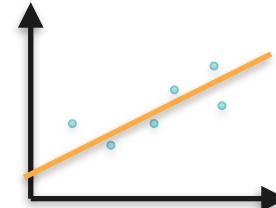


Data

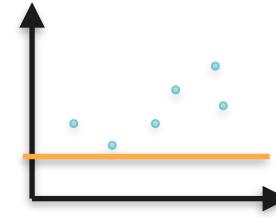
# Example: Linear Regression



Model 1



Model 2

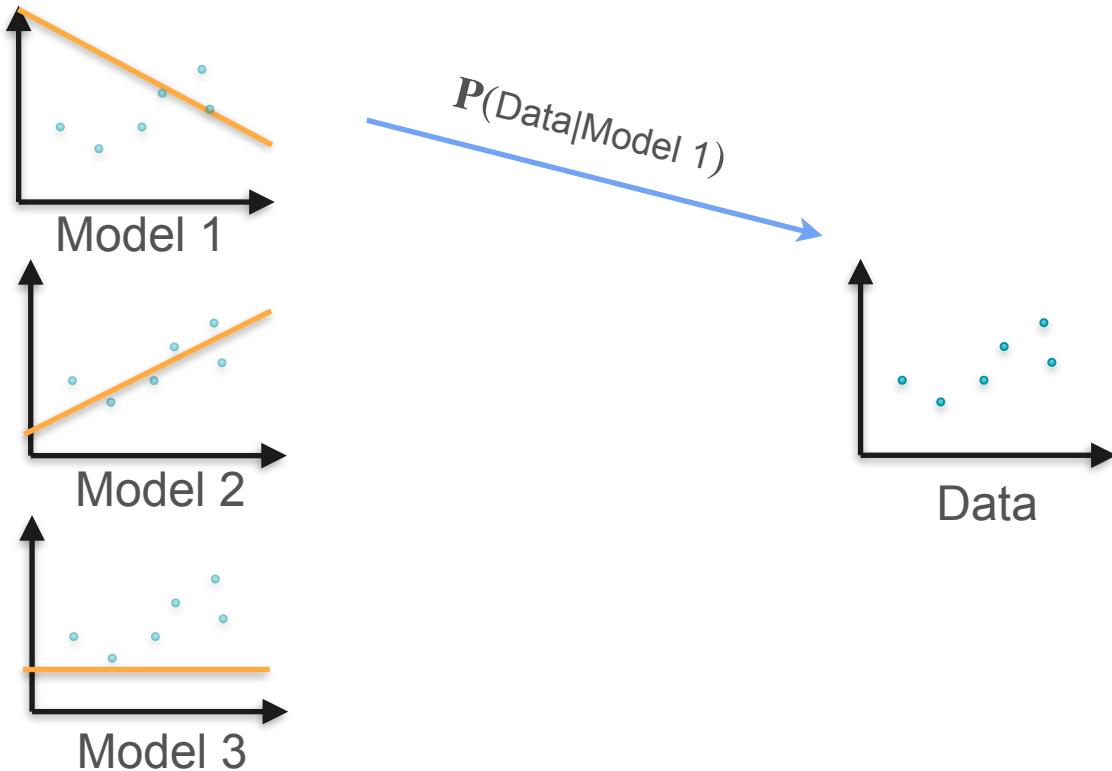


Model 3

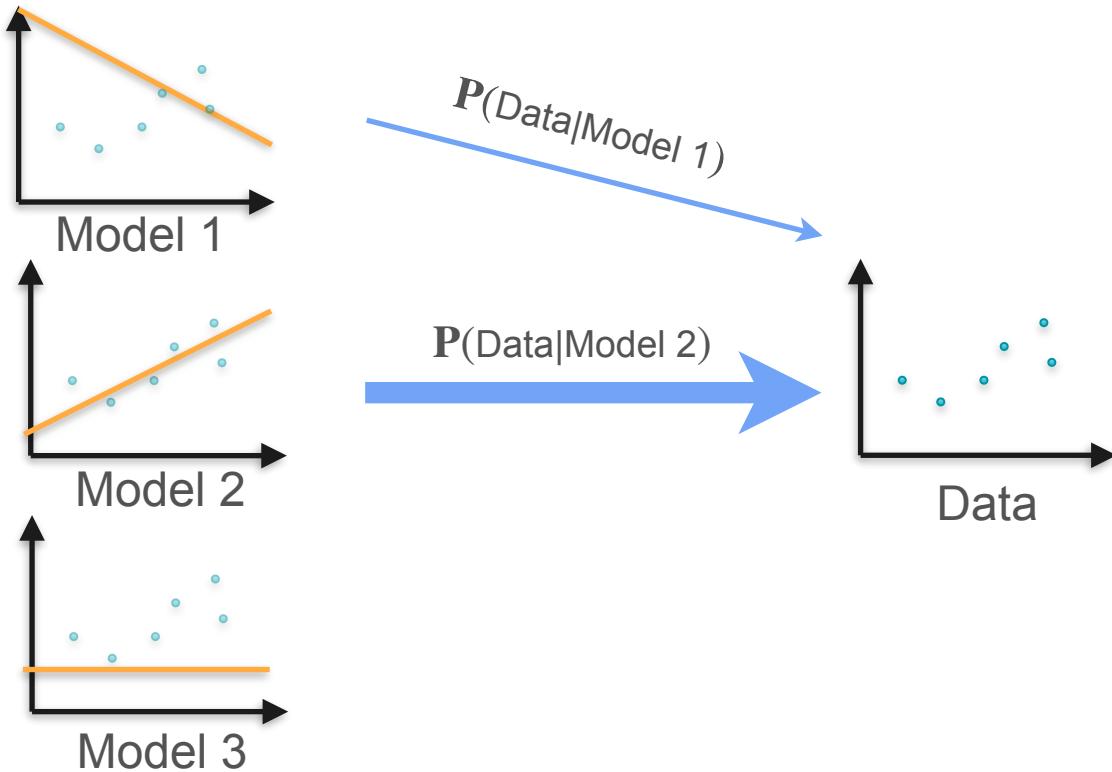


Data

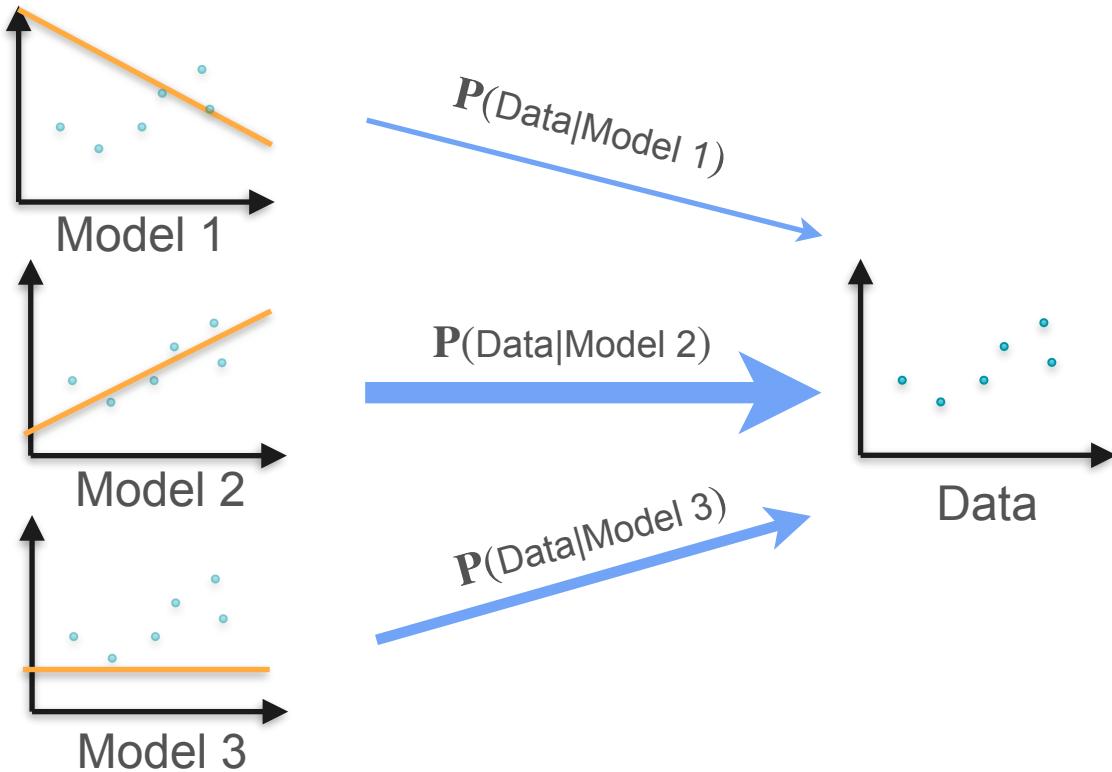
# Example: Linear Regression



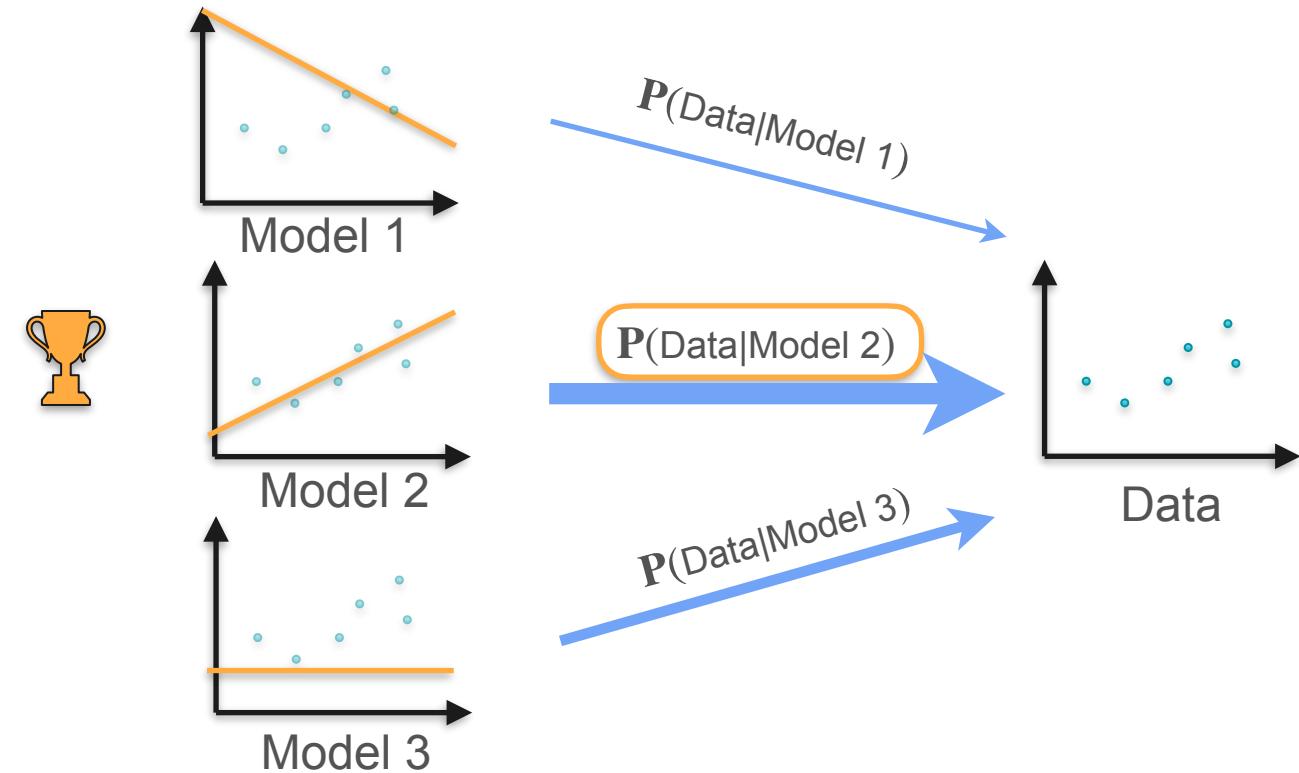
# Example: Linear Regression



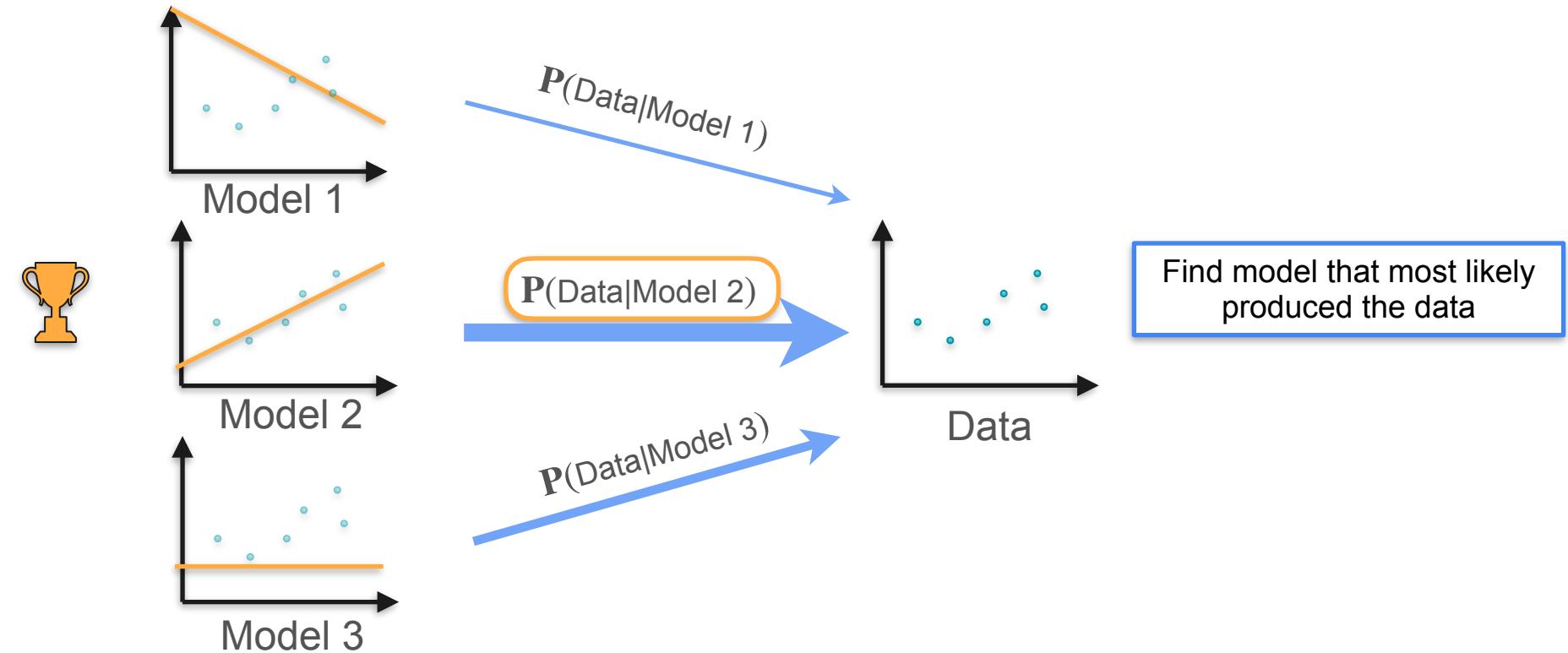
# Example: Linear Regression



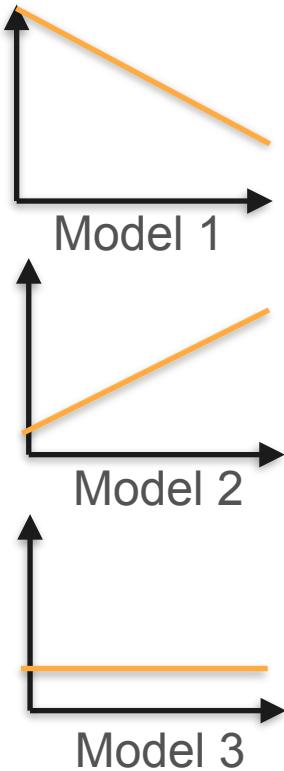
# Example: Linear Regression



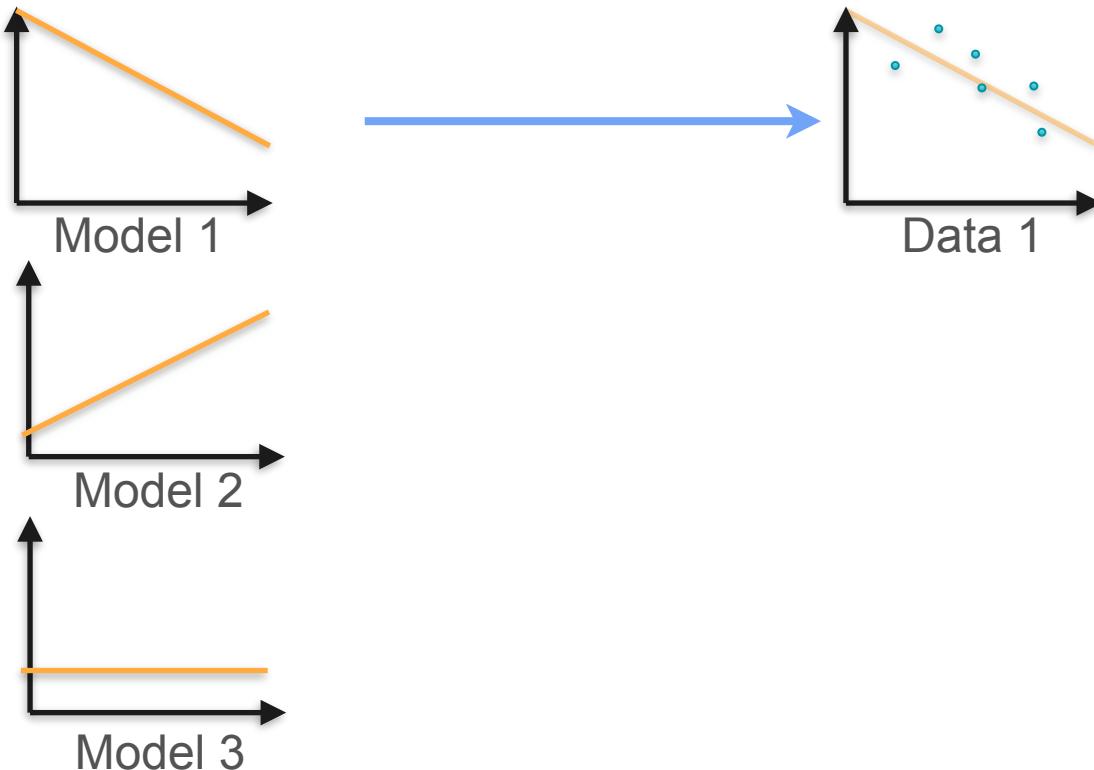
# Example: Linear Regression



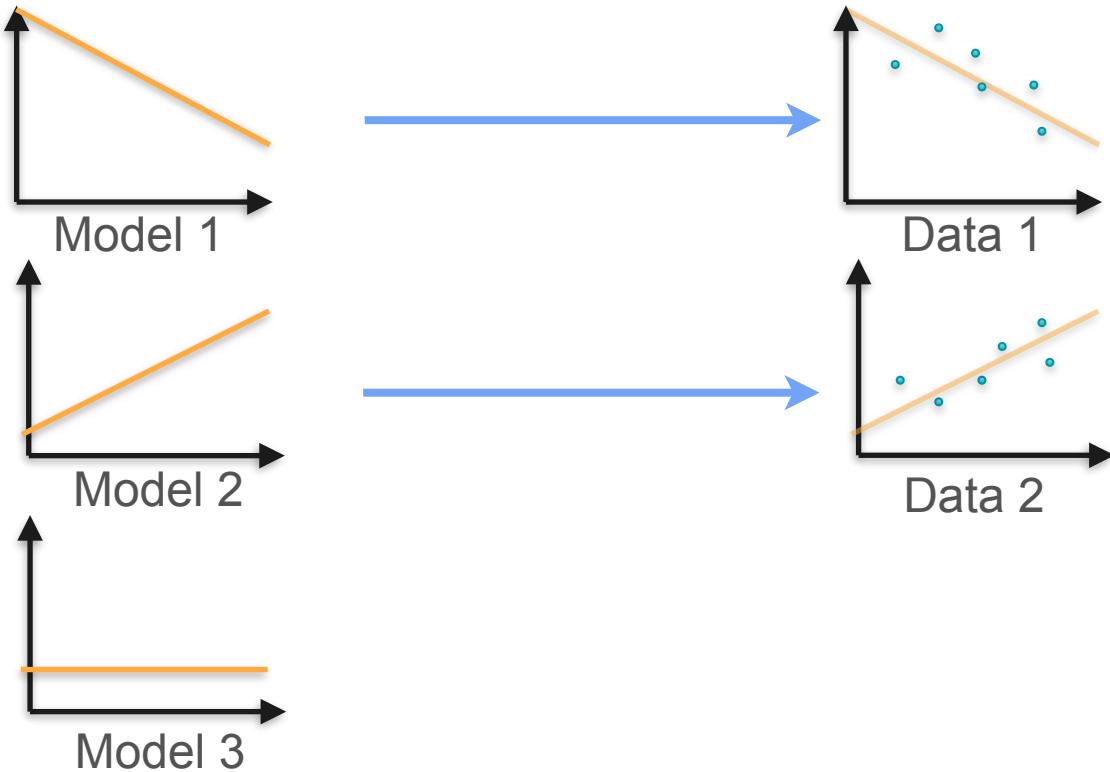
# How Exactly Does a Line Produce Points?



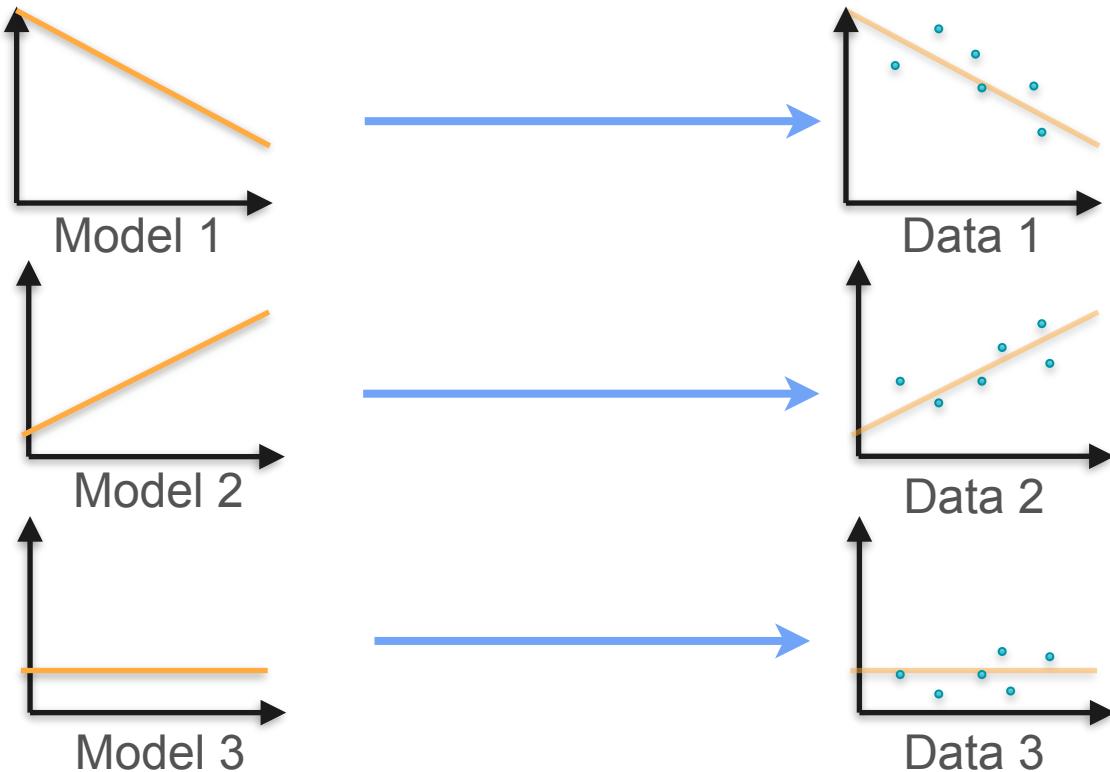
# How Exactly Does a Line Produce Points?



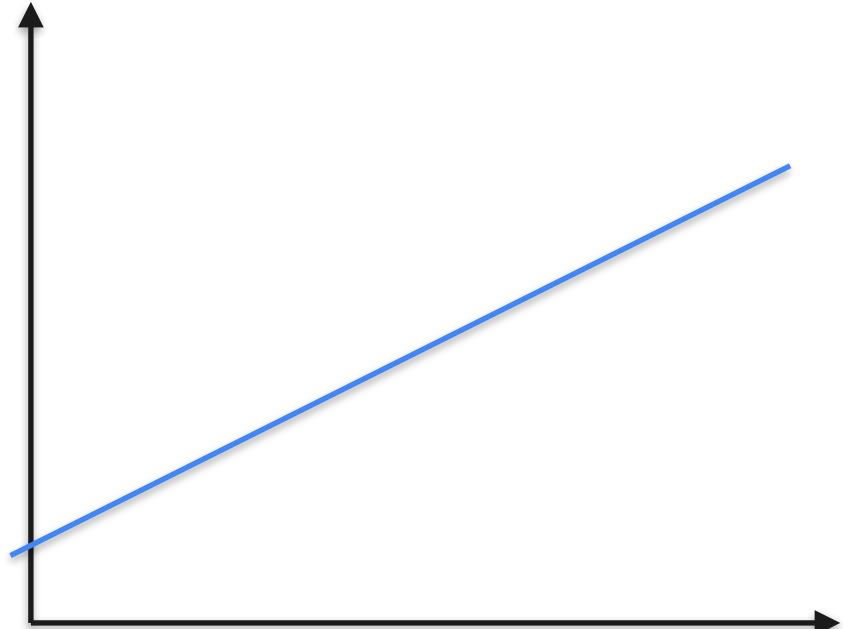
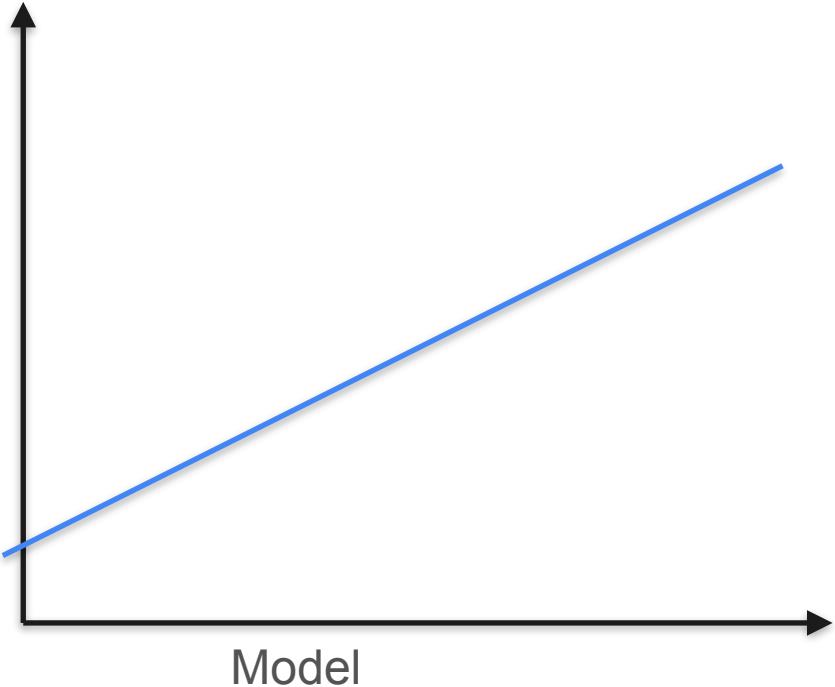
# How Exactly Does a Line Produce Points?



# How Exactly Does a Line Produce Points?

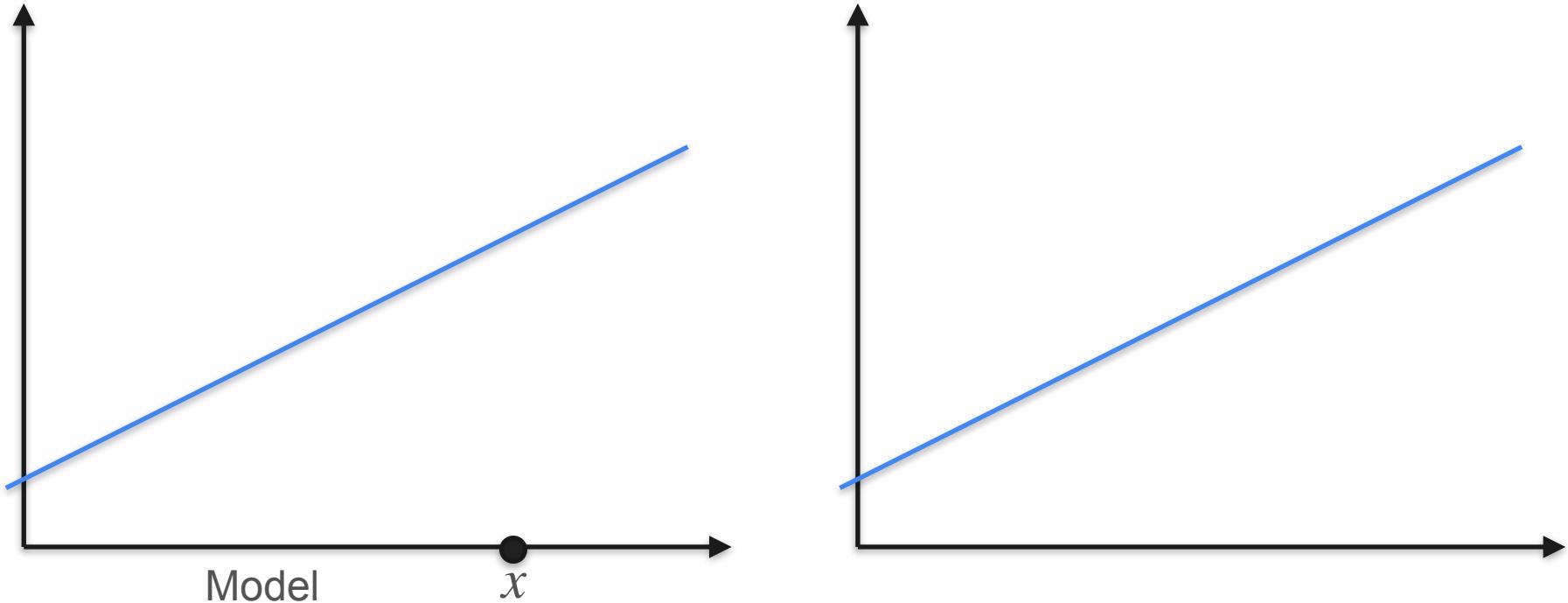


# How Exactly Does a Line Produce Points?

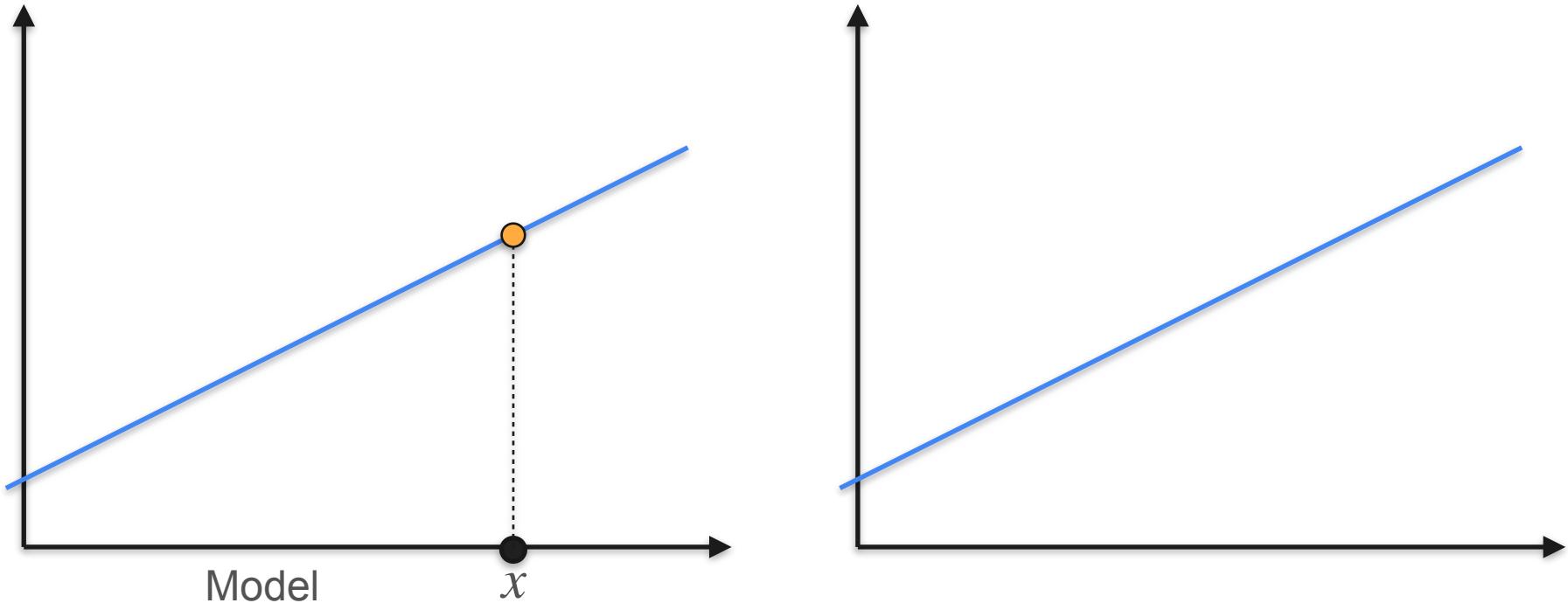


Model

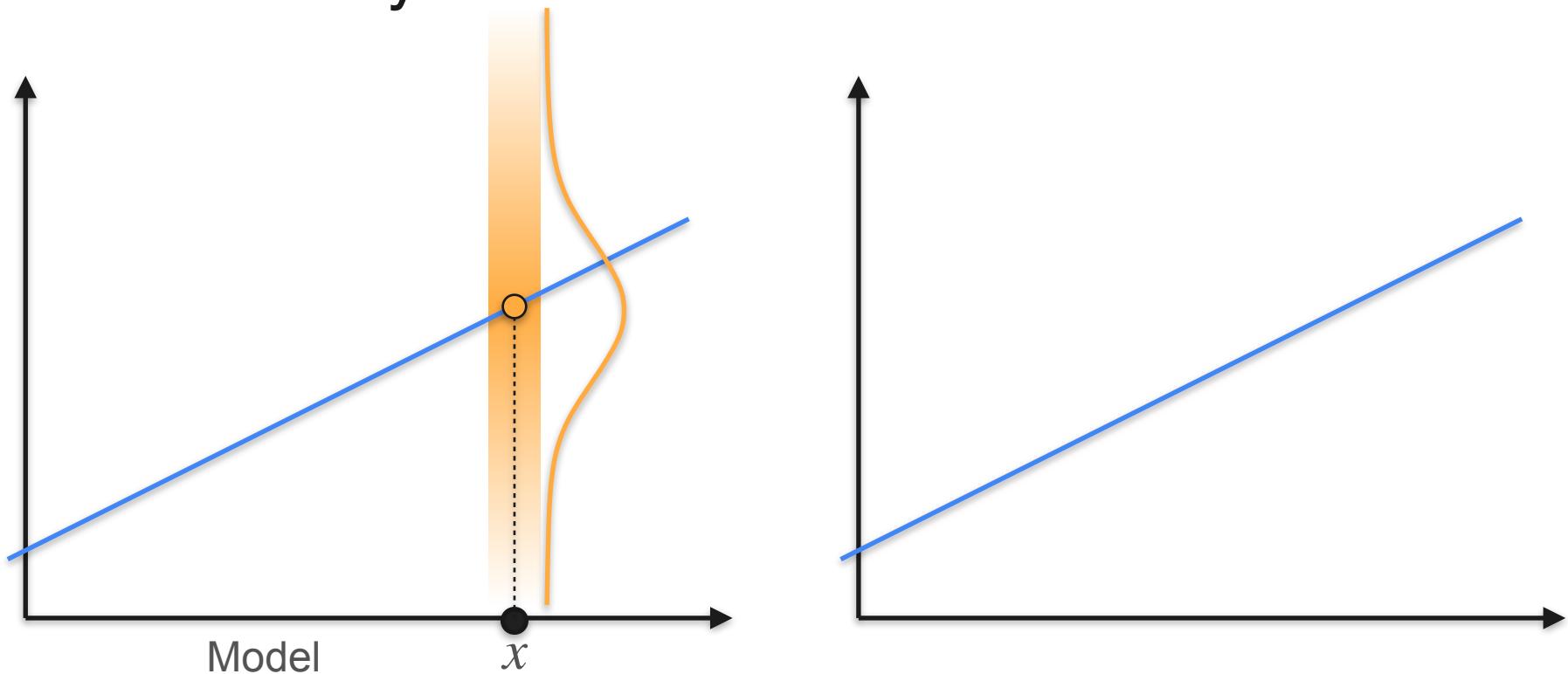
# How Exactly Does a Line Produce Points?



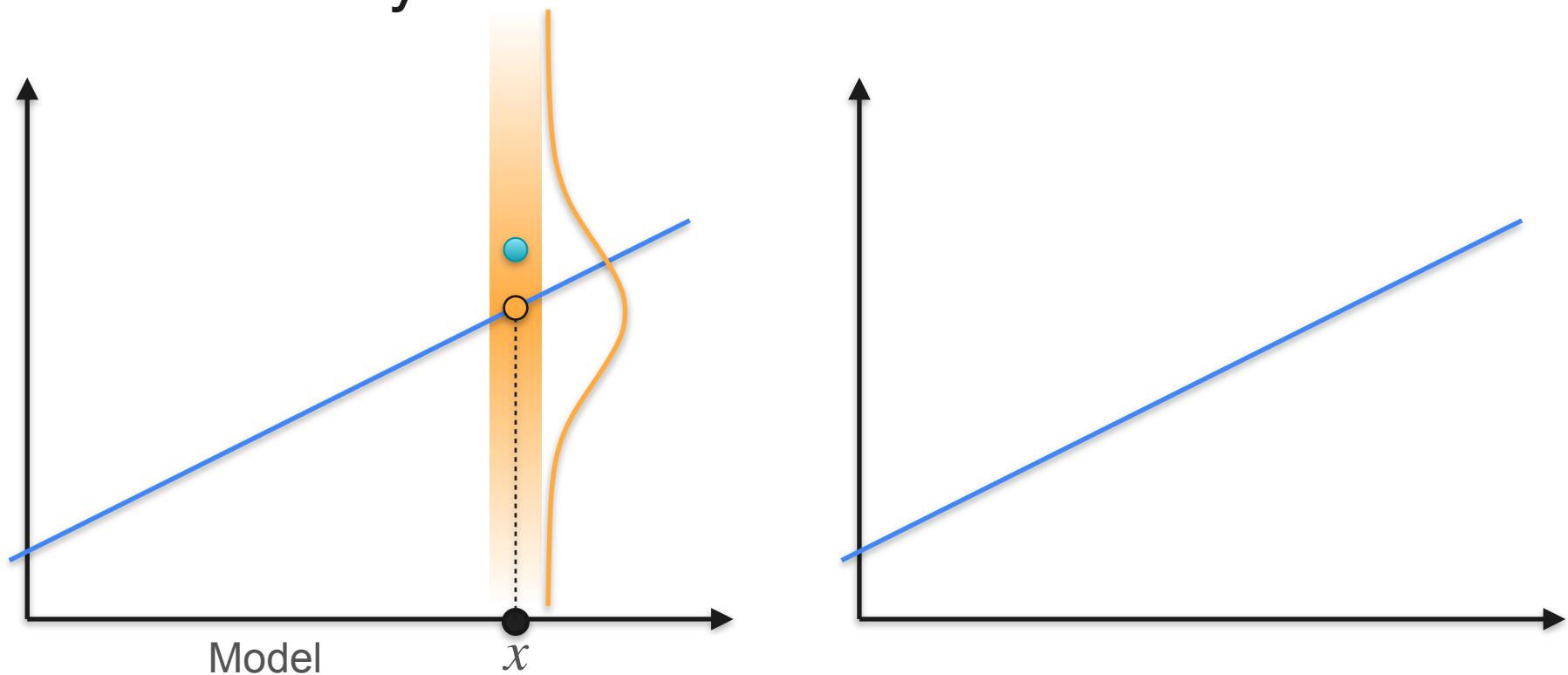
# How Exactly Does a Line Produce Points?



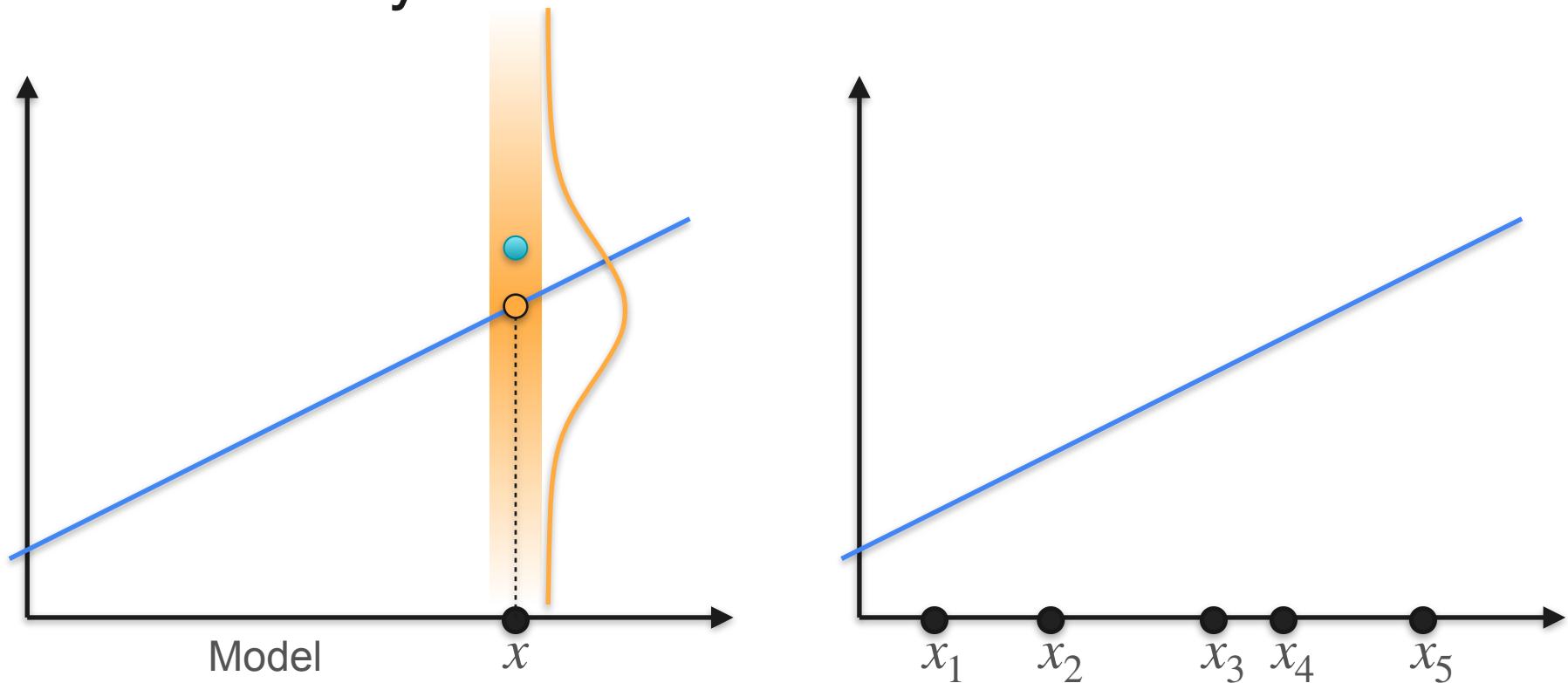
# How Exactly Does a Line Produce Points?



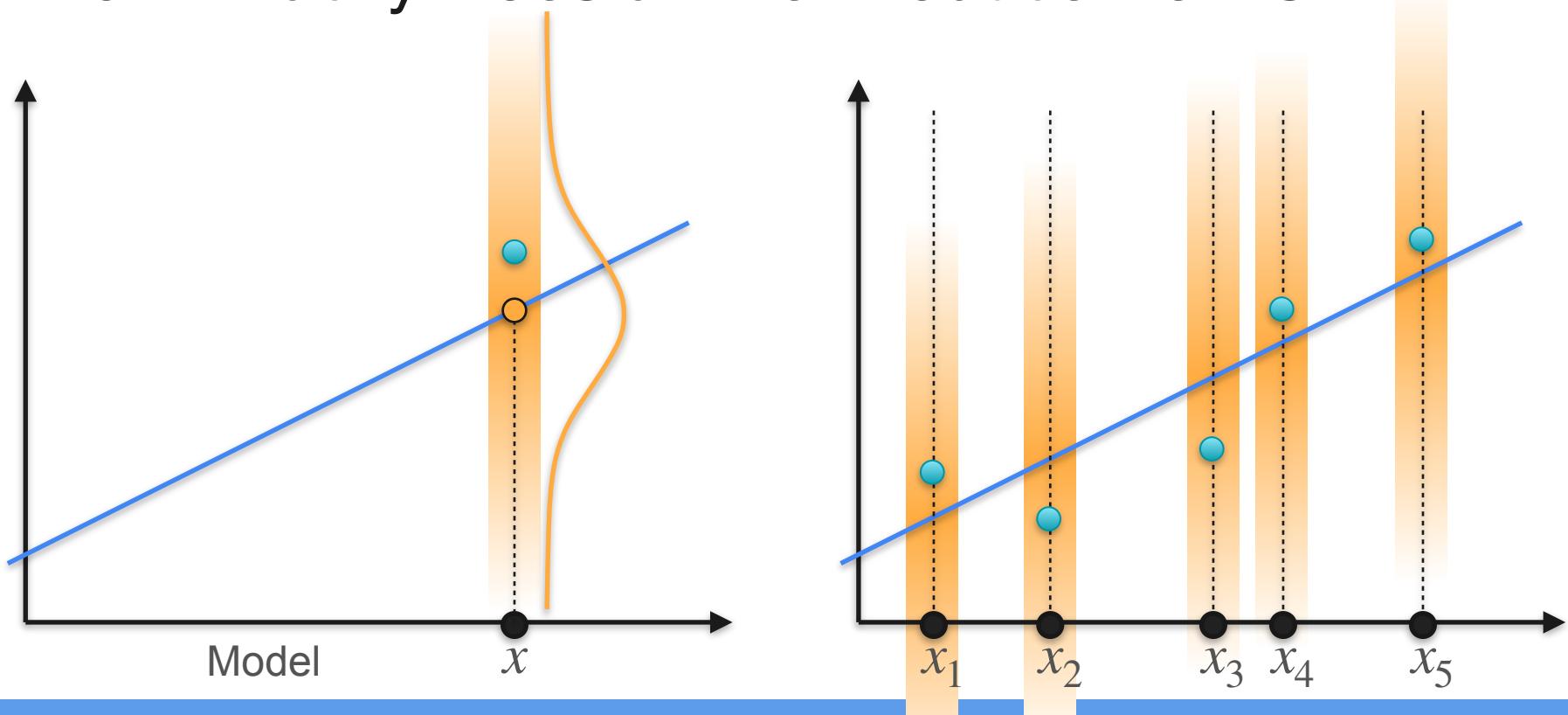
# How Exactly Does a Line Produce Points?



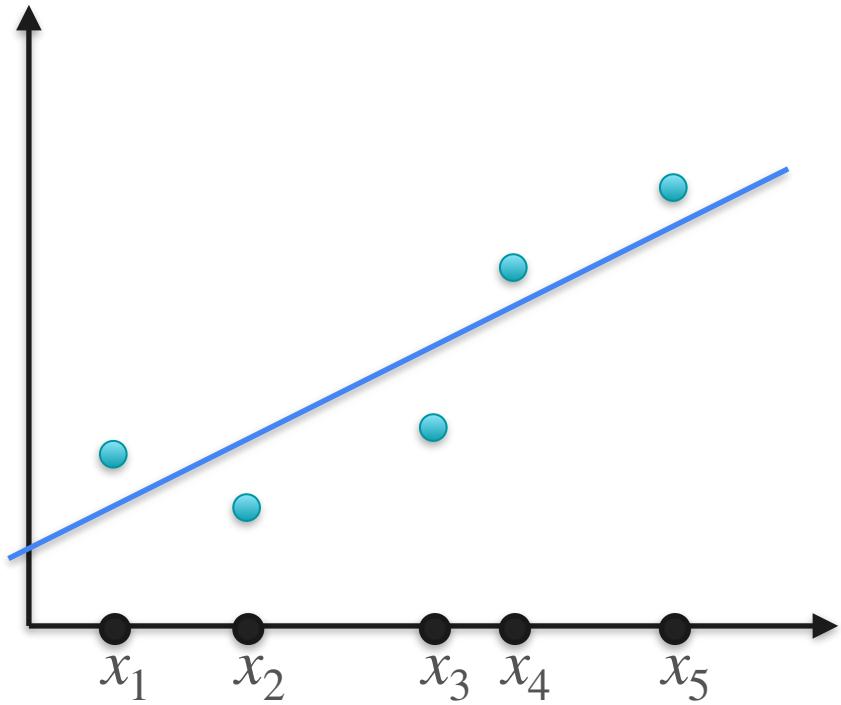
# How Exactly Does a Line Produce Points?



# How Exactly Does a Line Produce Points?

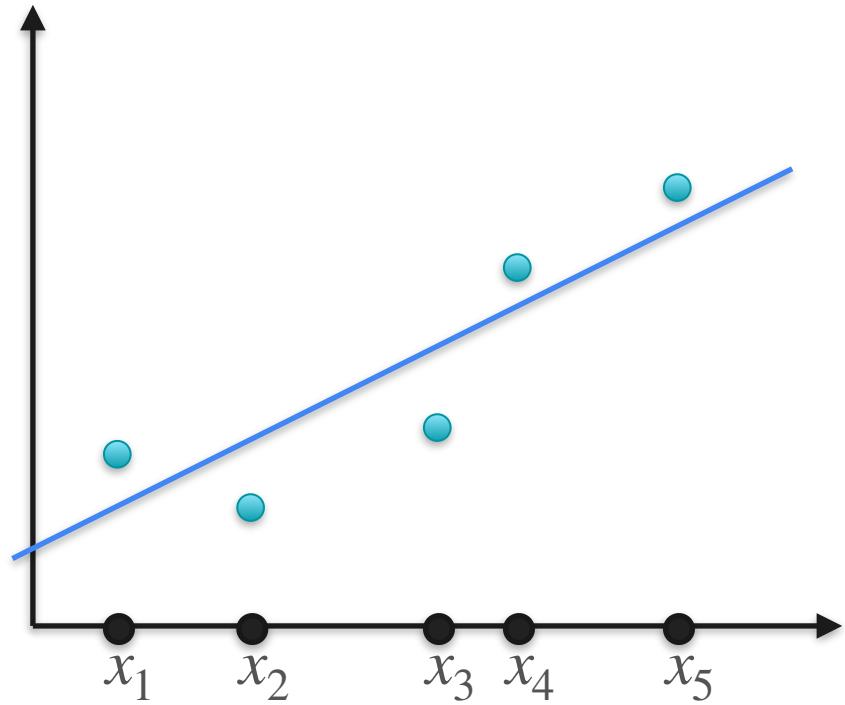


# Linear Regression



# Linear Regression

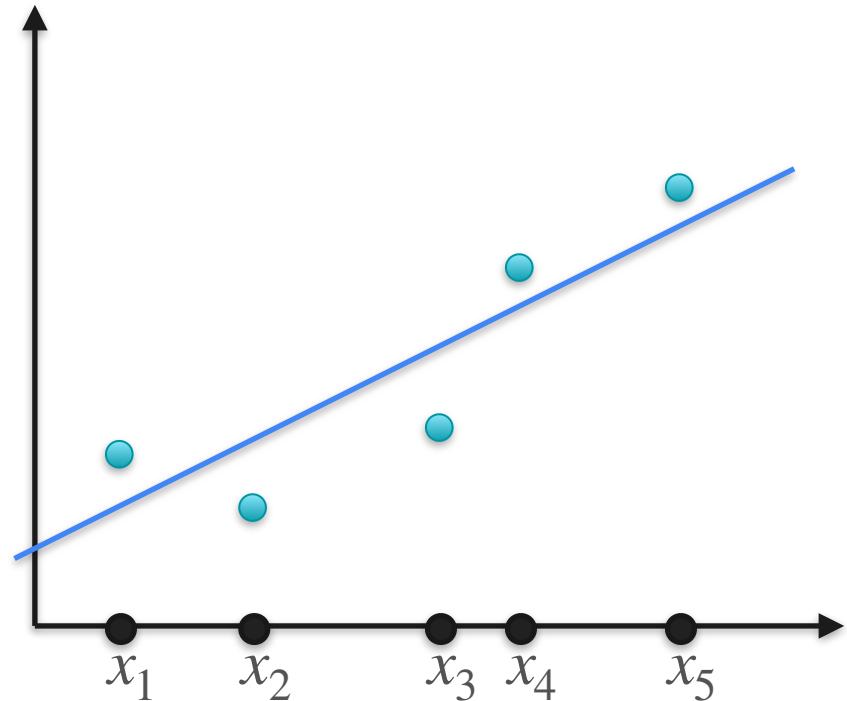
Line that best produced the points



# Linear Regression

Line that best produced the points

Line that best fits the data  
(linear regression)

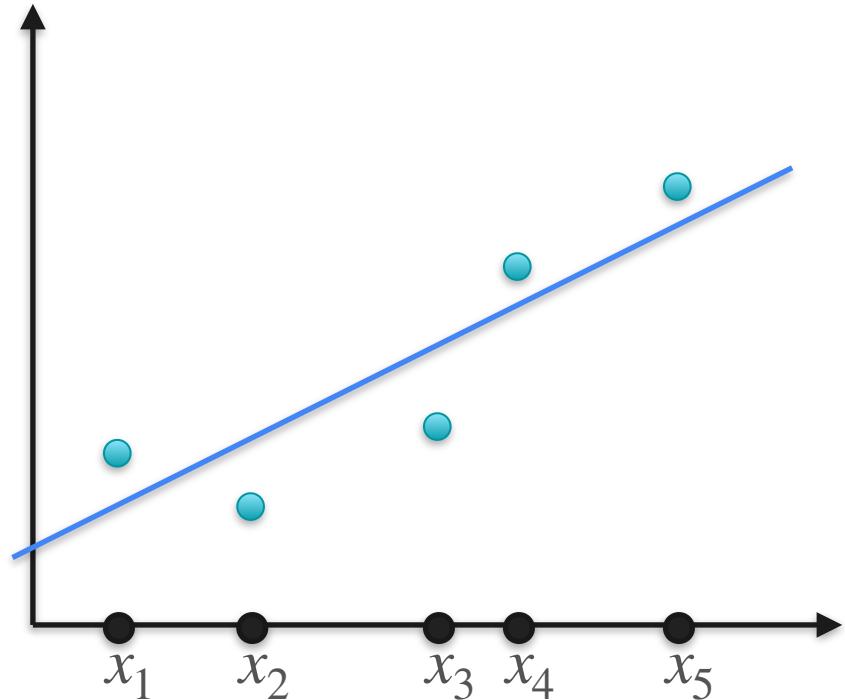


# Linear Regression

Line that best produced the points



Line that best fits the data  
(linear regression)



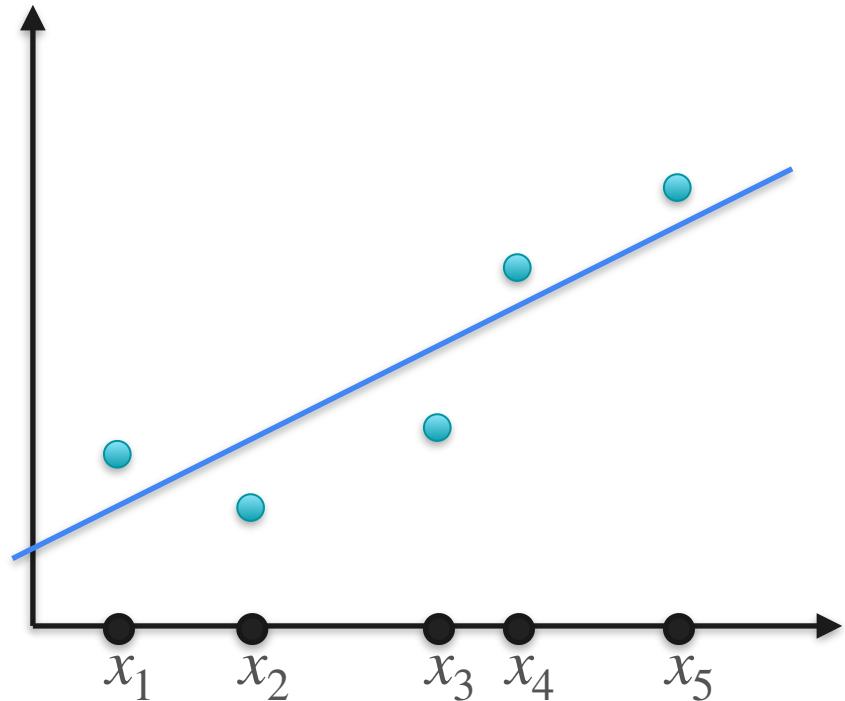
# Linear Regression

Line that best produced the points

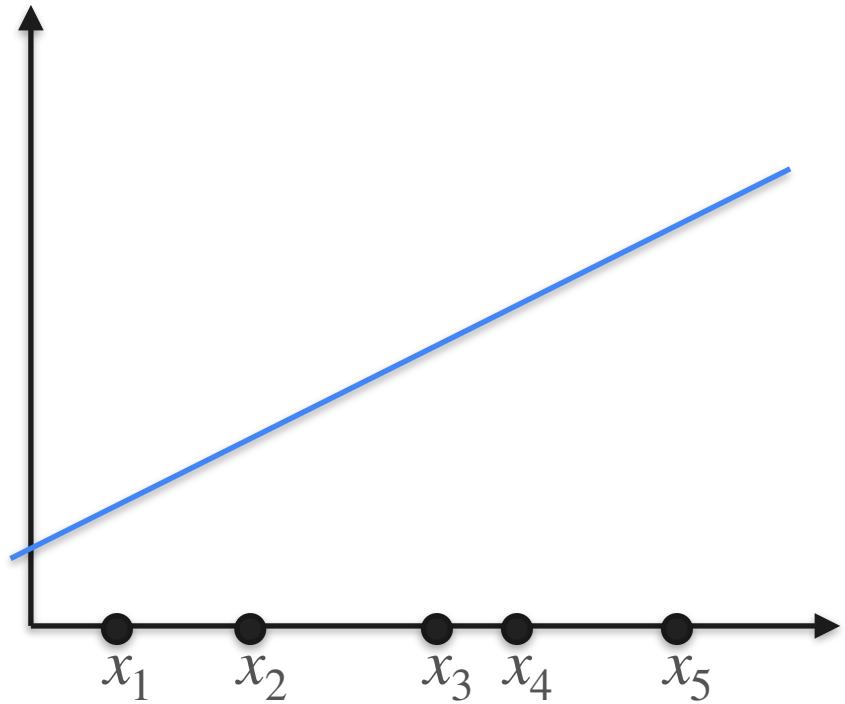


How?

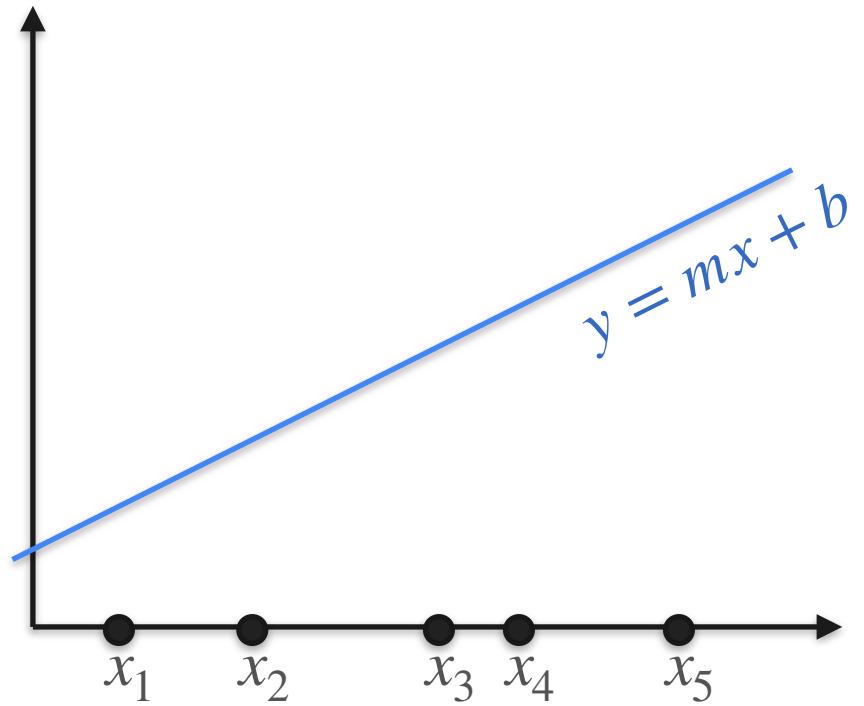
Line that best fits the data  
(linear regression)



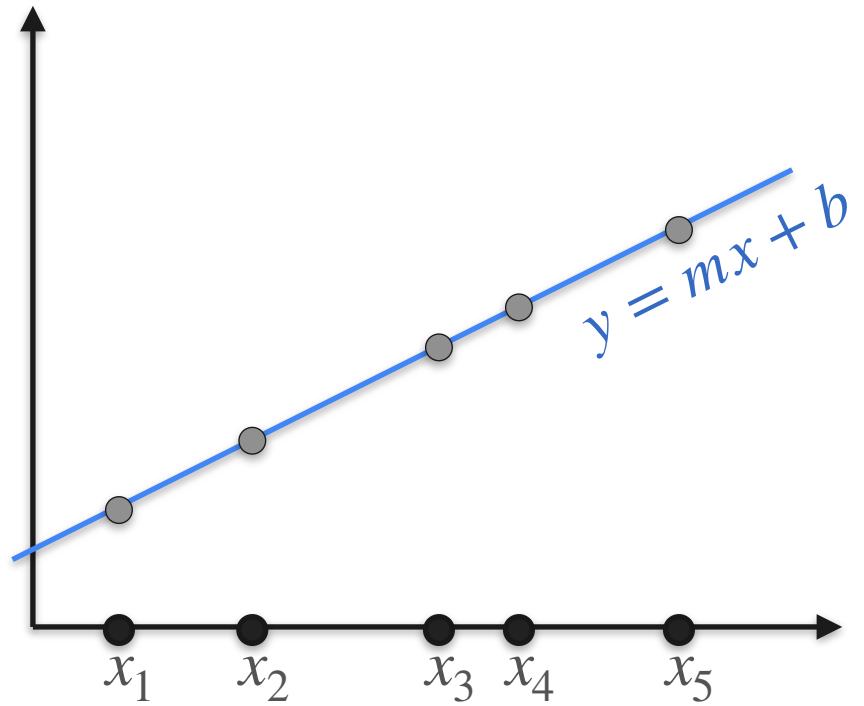
# Linear Regression and Likelihood



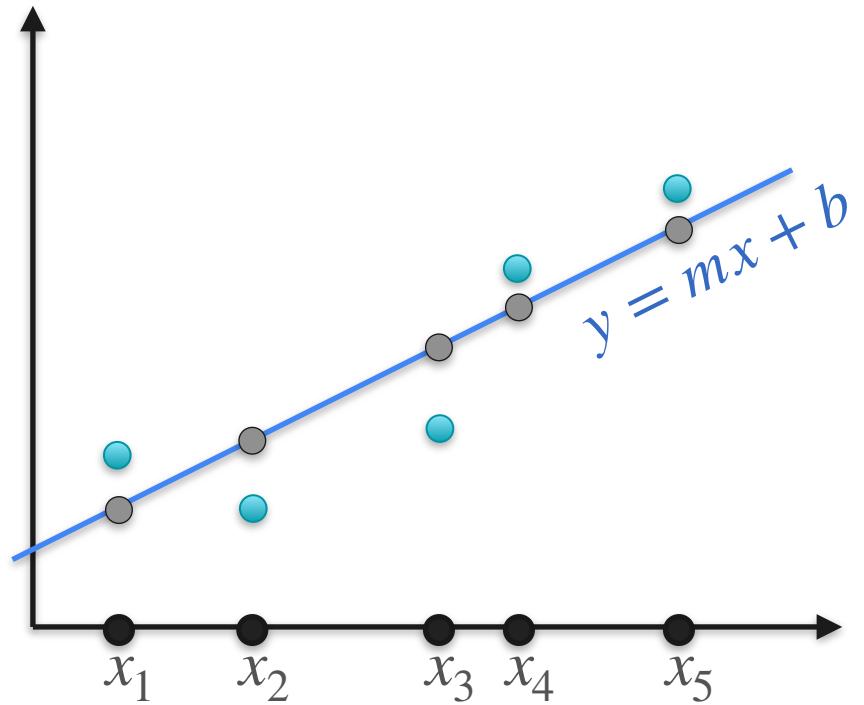
# Linear Regression and Likelihood



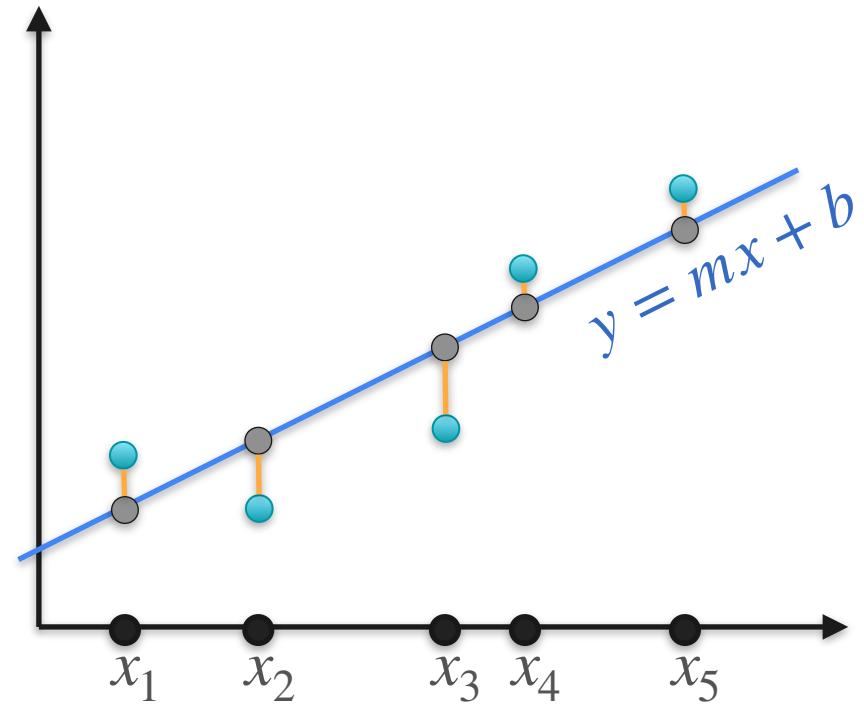
# Linear Regression and Likelihood



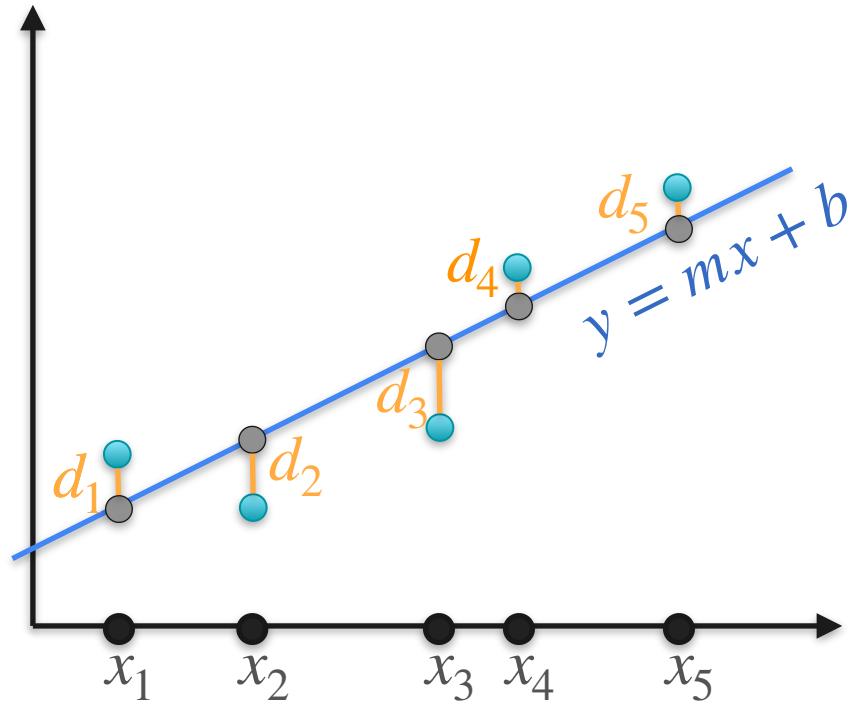
# Linear Regression and Likelihood



# Linear Regression and Likelihood

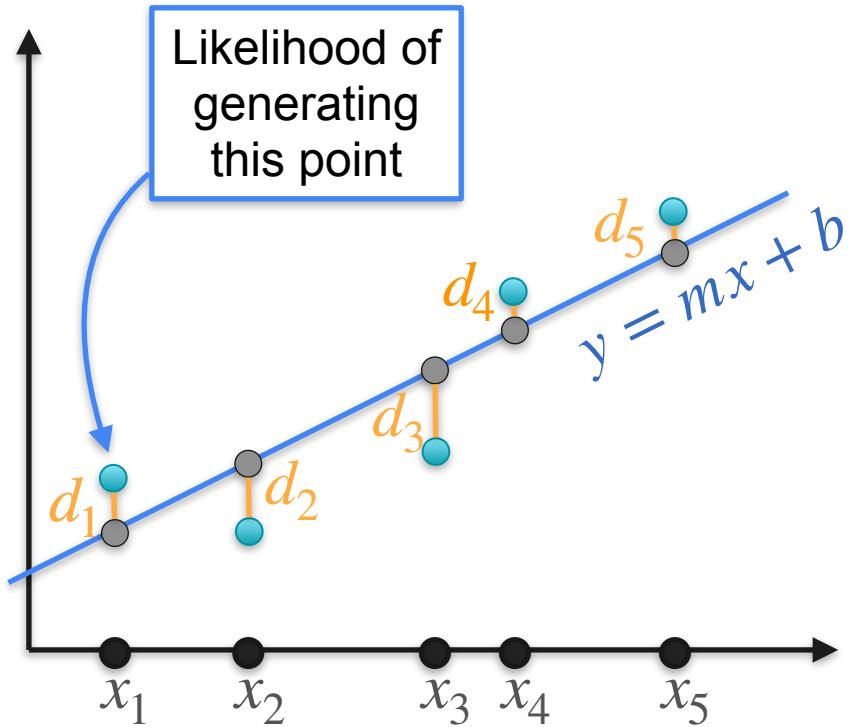


# Linear Regression and Likelihood



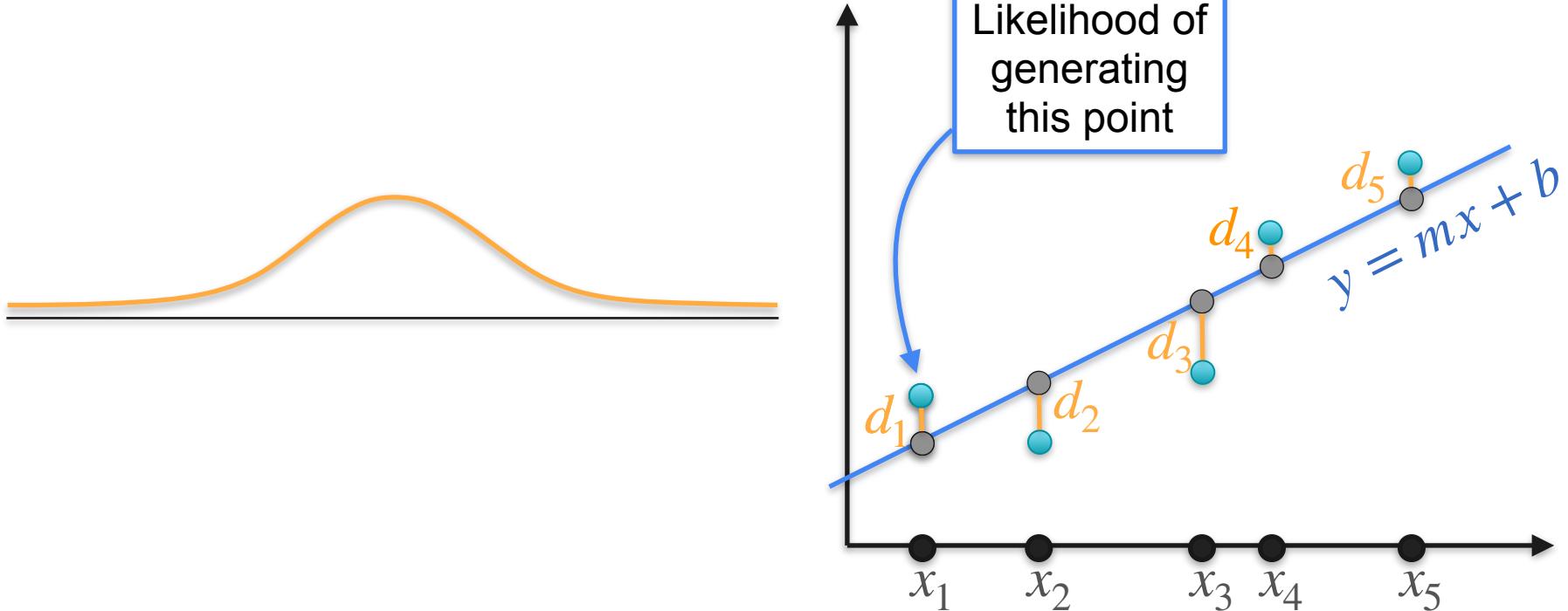
# Linear Regression and Likelihood

Likelihood:



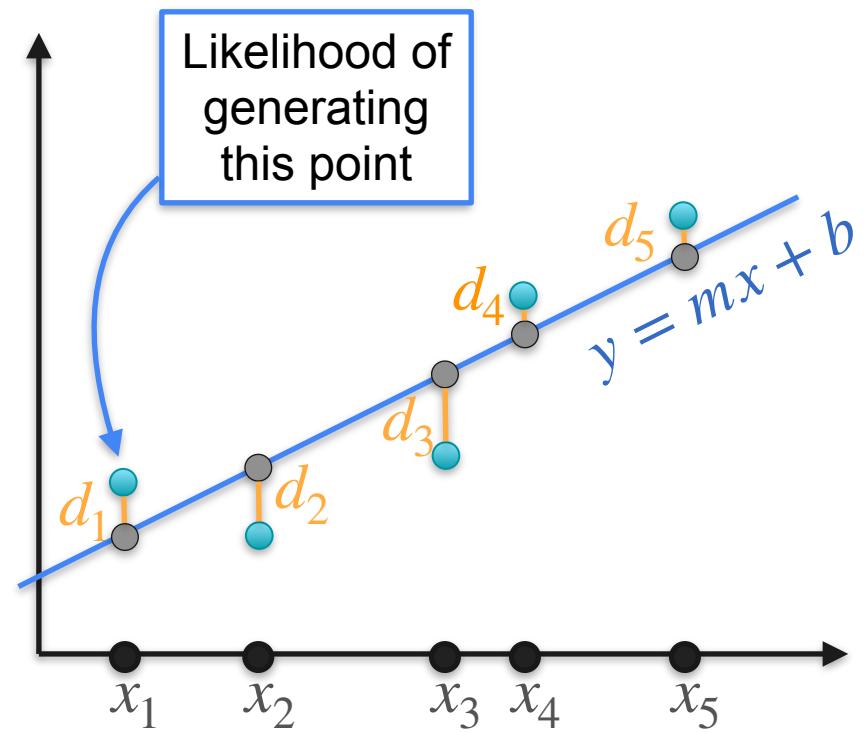
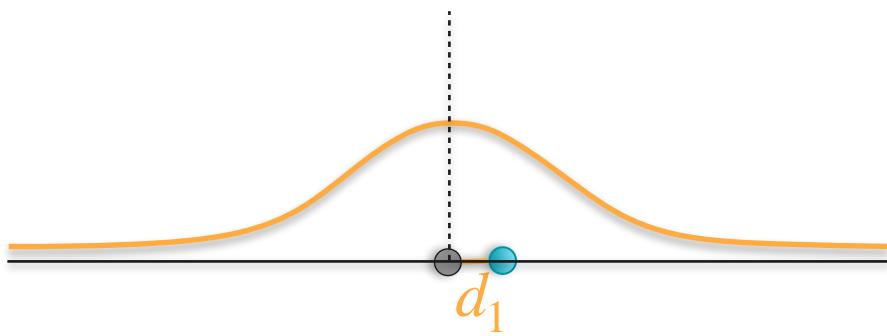
# Linear Regression and Likelihood

Likelihood:



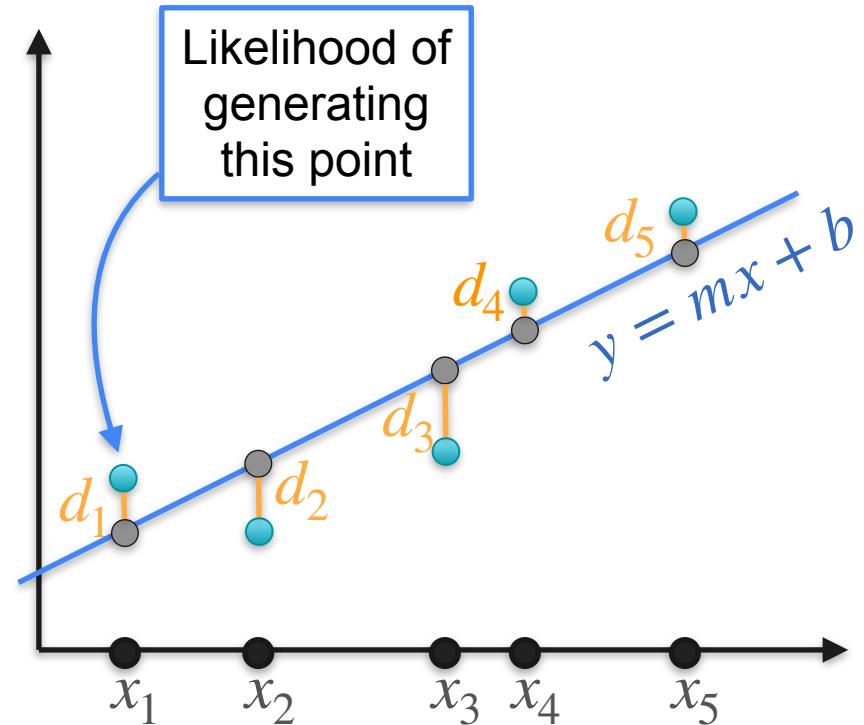
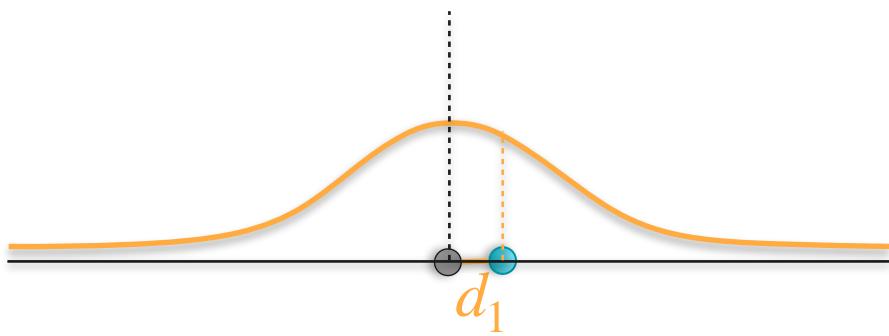
# Linear Regression and Likelihood

Likelihood:



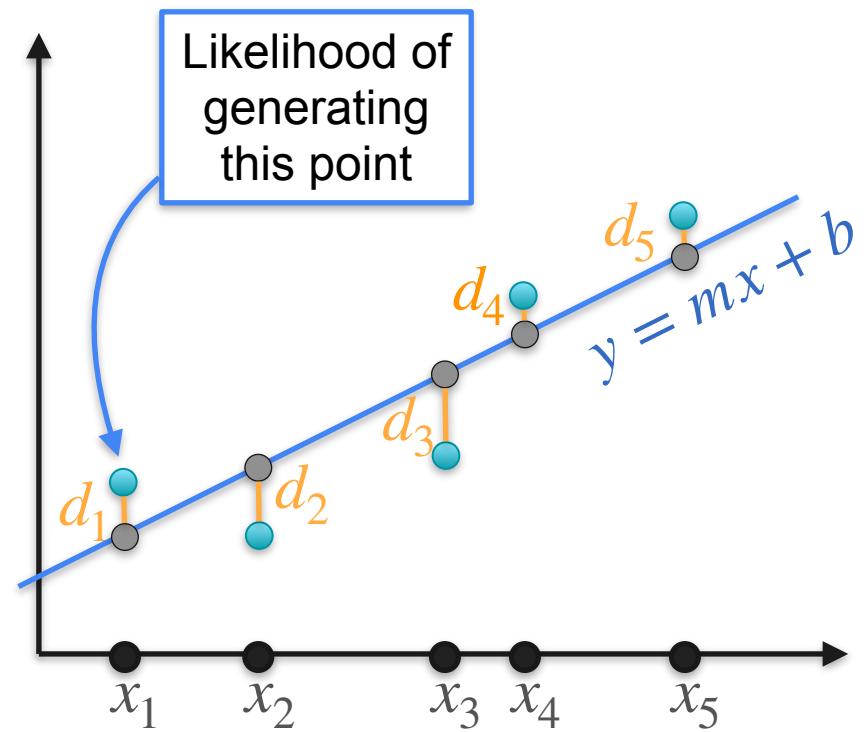
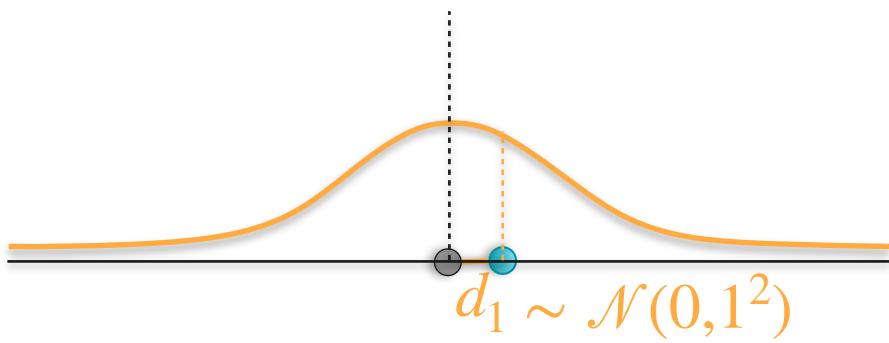
# Linear Regression and Likelihood

Likelihood:



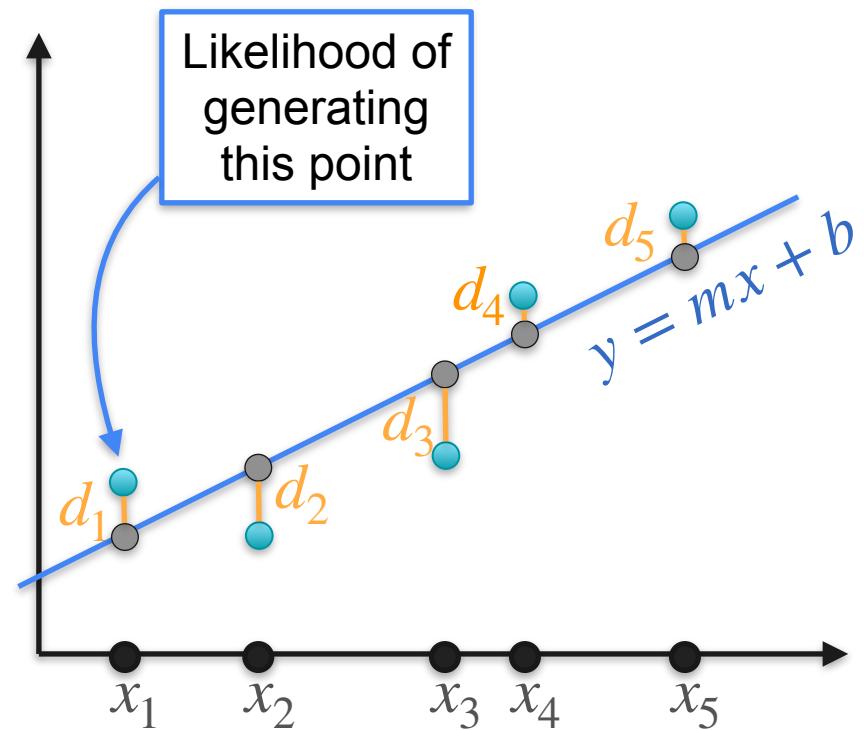
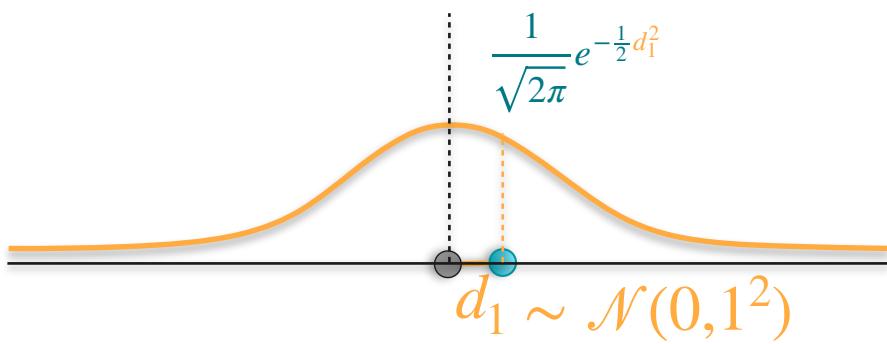
# Linear Regression and Likelihood

Likelihood:



# Linear Regression and Likelihood

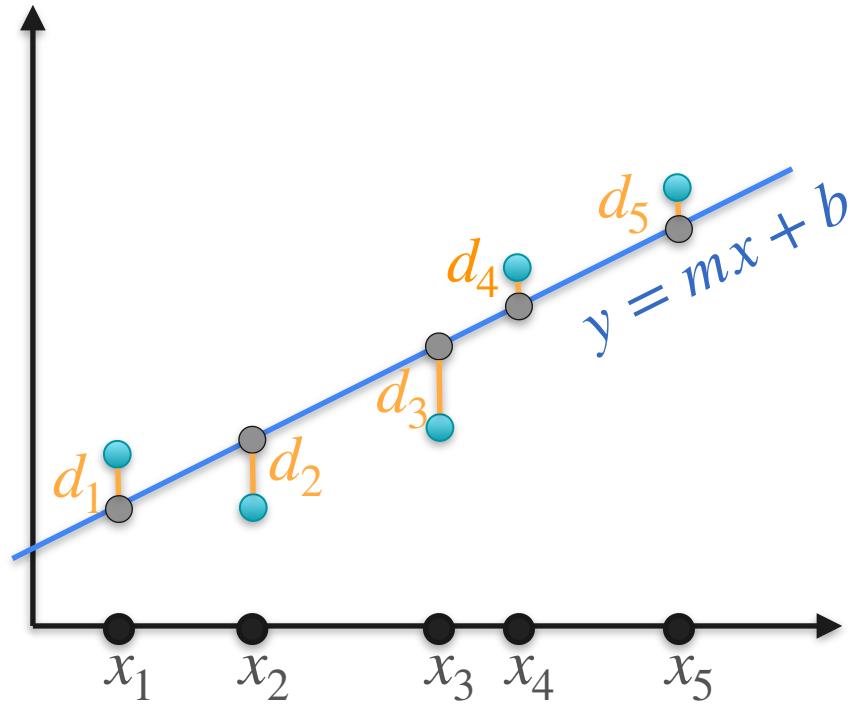
Likelihood:



# Linear Regression and Likelihood

Likelihood:

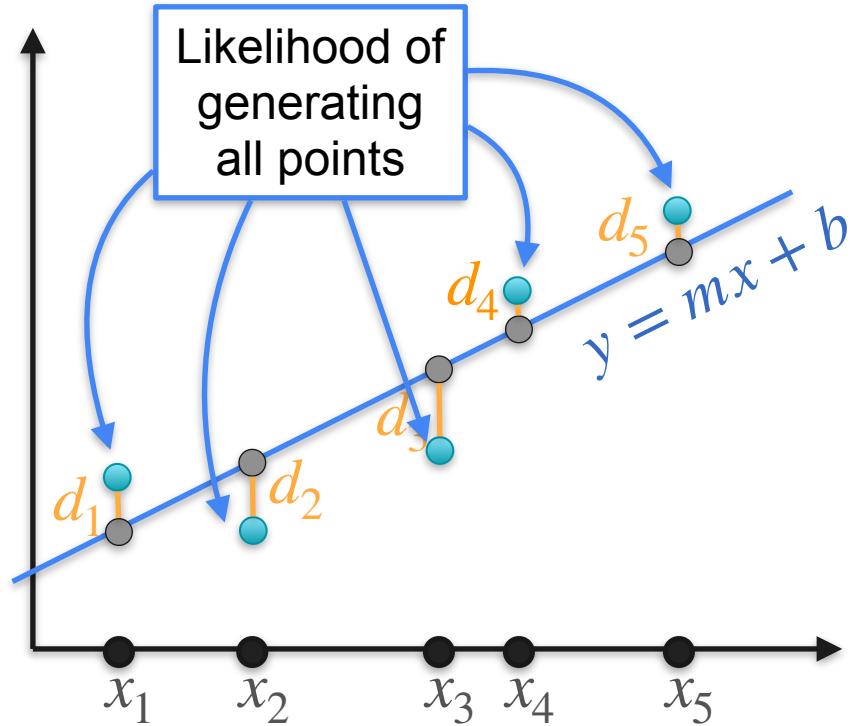
$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_i^2}$$



# Linear Regression and Likelihood

Likelihood:

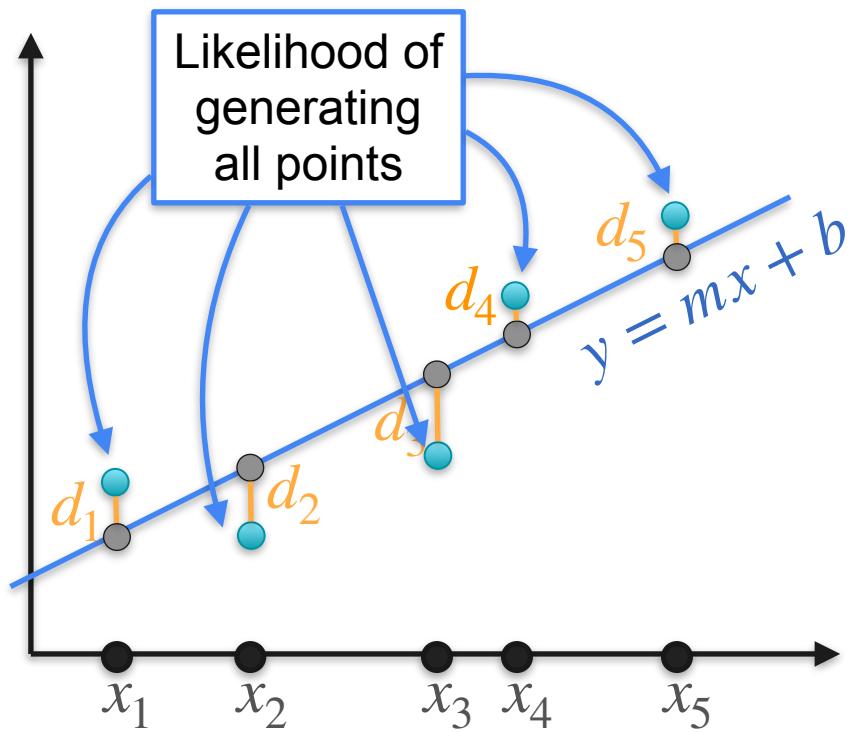
$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_i^2}$$



# Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

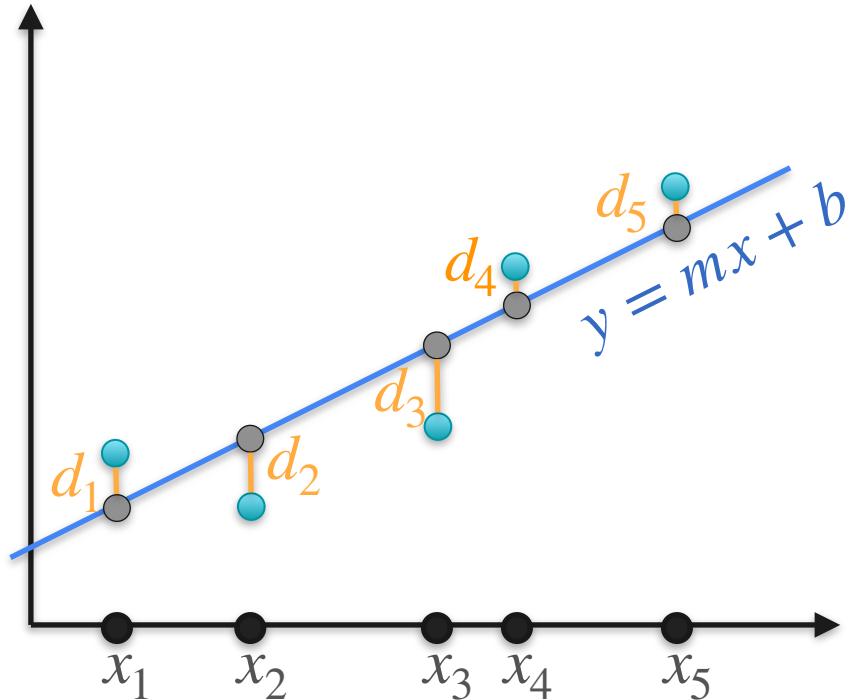


# Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

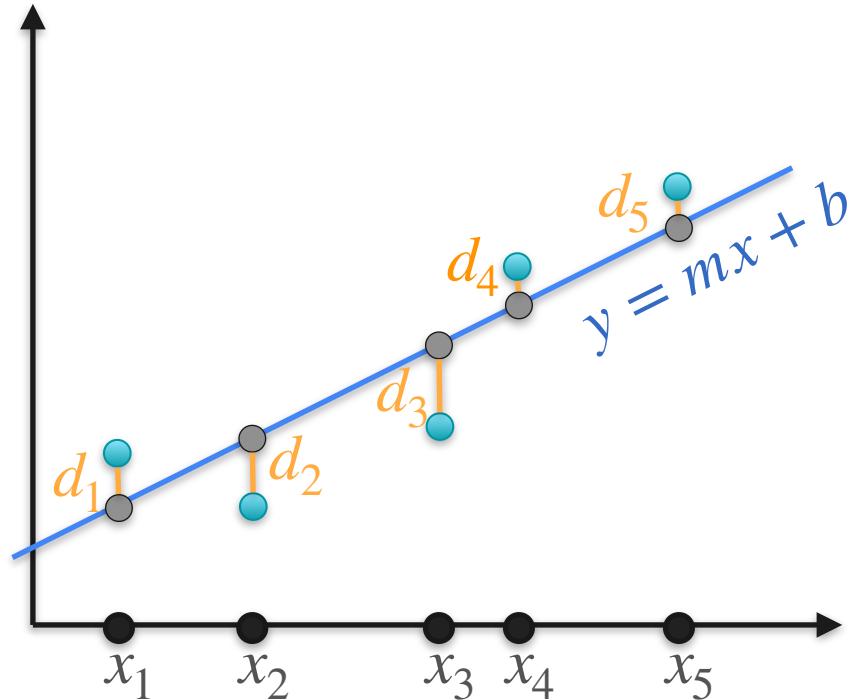


# Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize



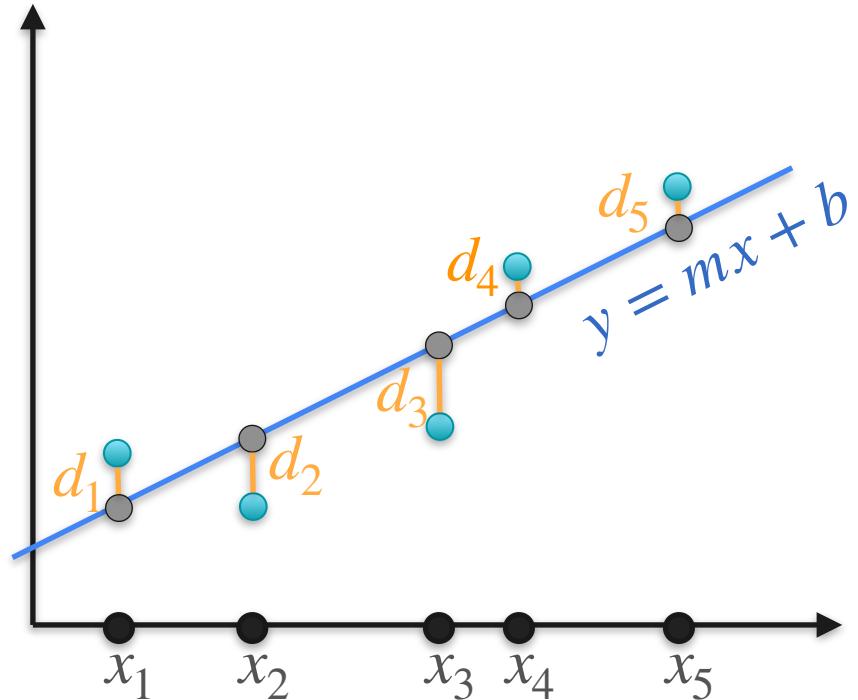
# Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$



# Linear Regression and Likelihood

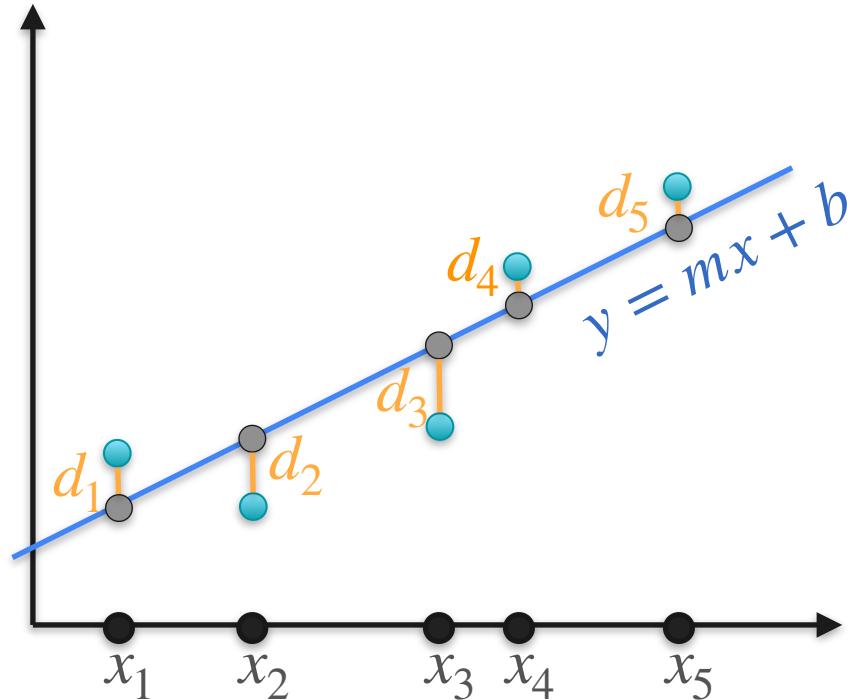
Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$

$$e^{-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)}$$



# Linear Regression and Likelihood

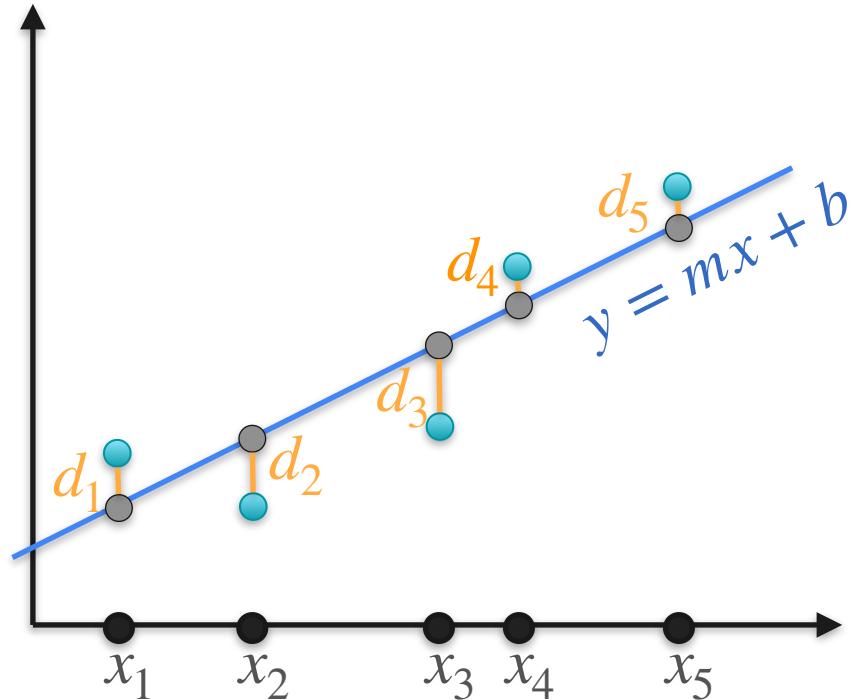
Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$

$$e^{-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)}$$



# Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

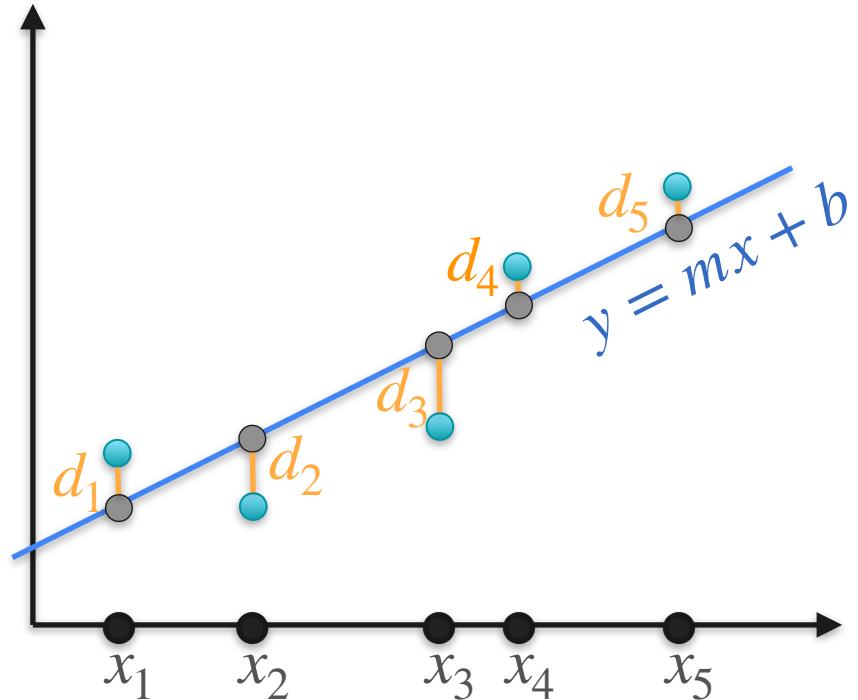
Maximize

$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$

$$\cancel{e^{-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)}}$$

Minimize

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$



# Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

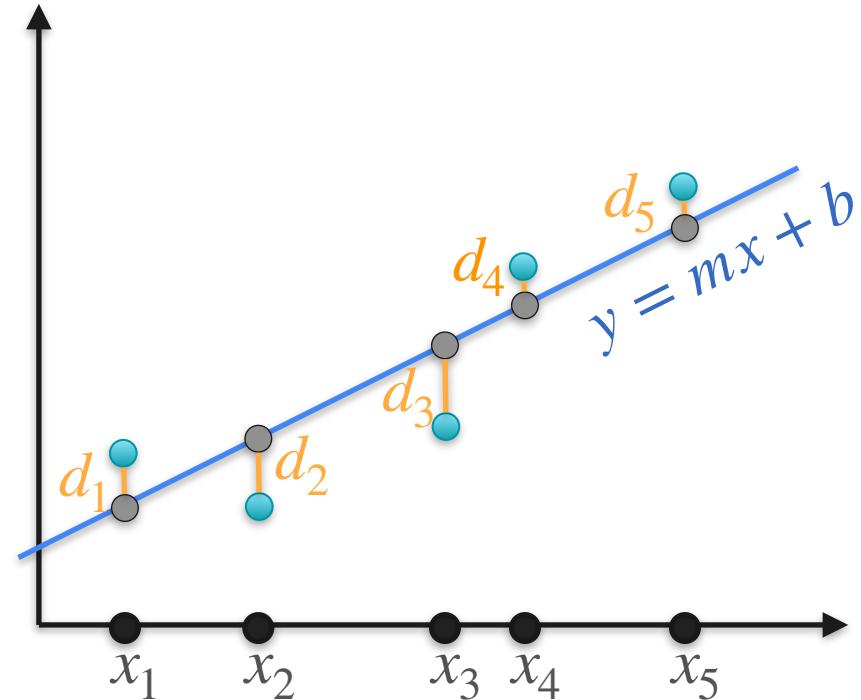
$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$

$$\cancel{e^{-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)}}$$

Minimize

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

Least squares error!



# Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$

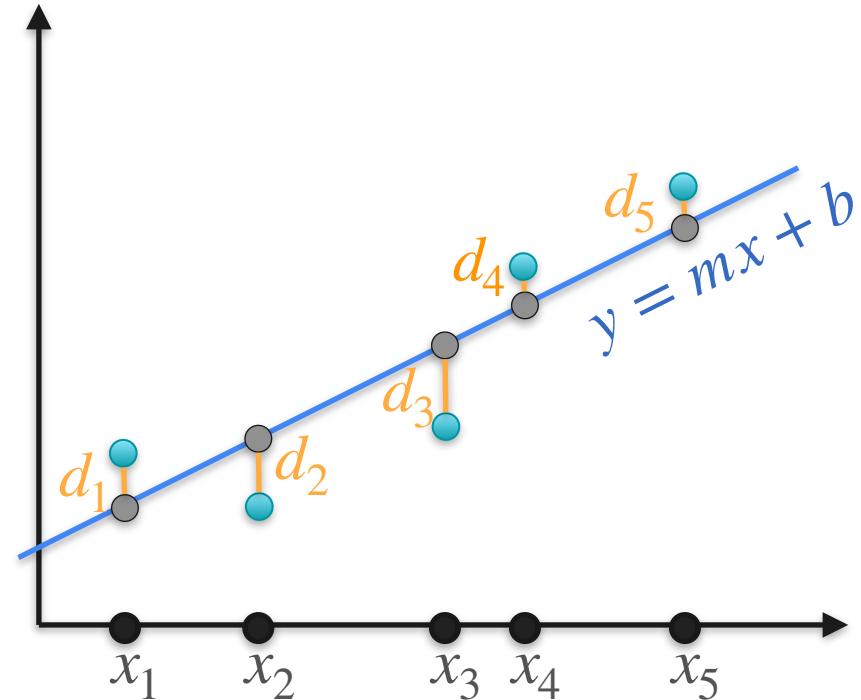
$$\cancel{e^{-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)}}$$

Minimize

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

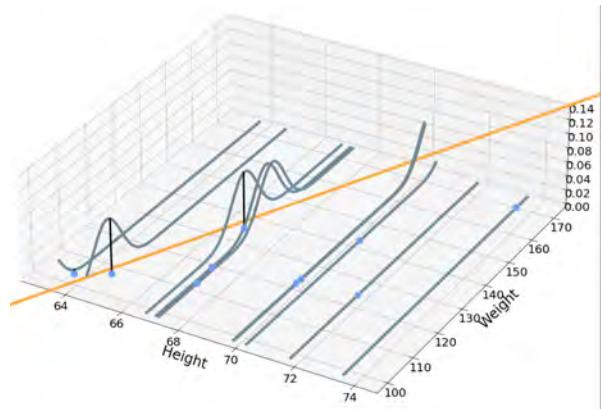
Linear regression!

Least squares error!

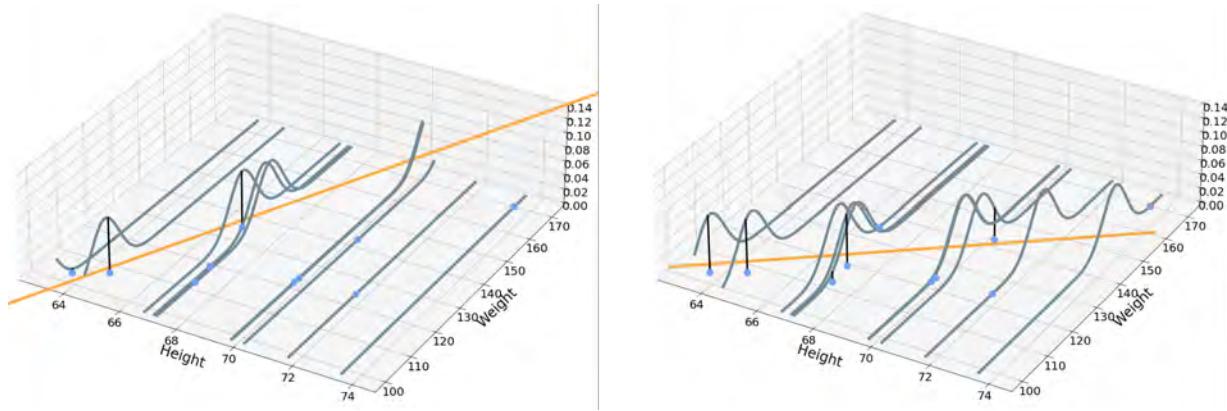


# Picking the Right Model

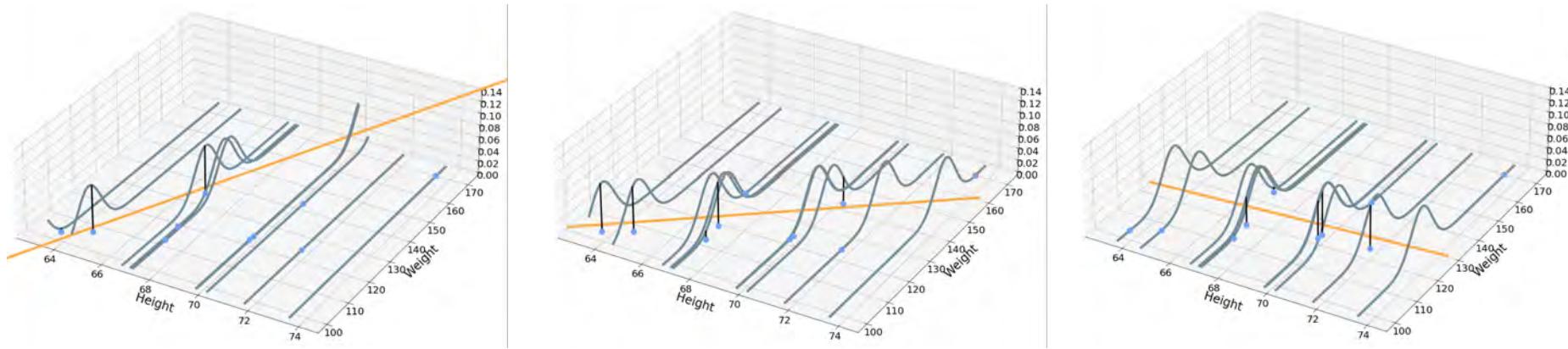
# Picking the Right Model



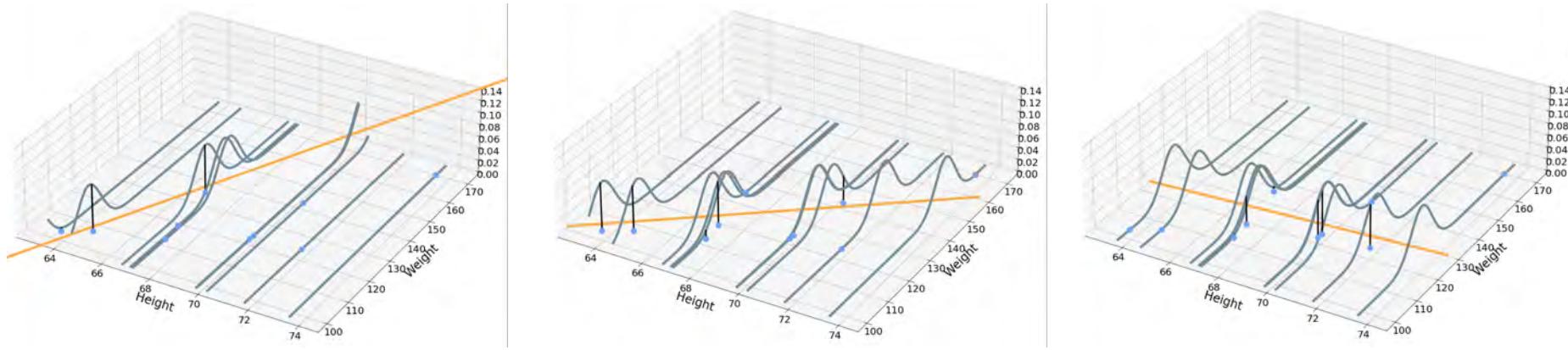
# Picking the Right Model



# Picking the Right Model



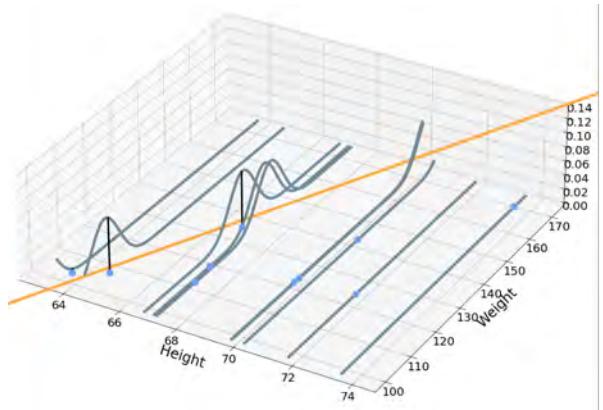
# Picking the Right Model



**Model 1:**

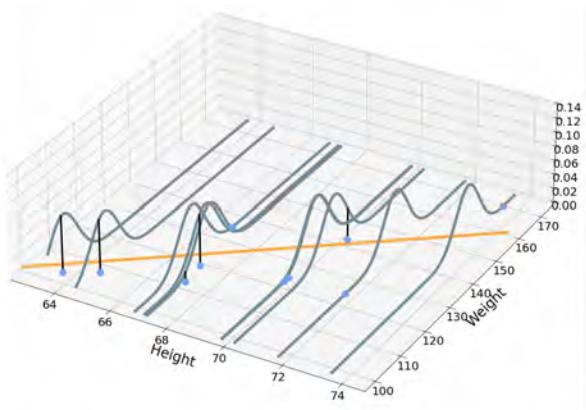
$$\text{Likelihood} = 4.91 \cdot 10^{-260}$$

# Picking the Right Model



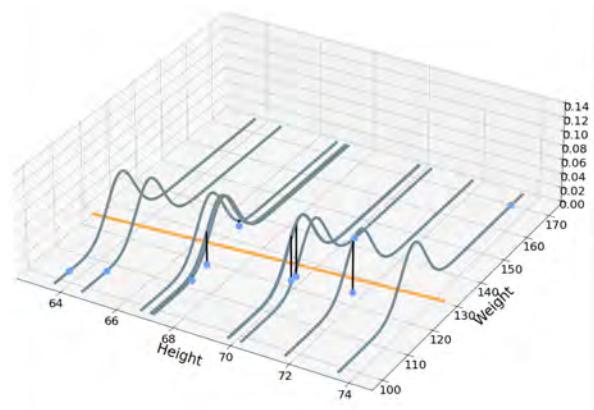
**Model 1:**

$$\text{Likelihood} = 4.91 \cdot 10^{-260}$$

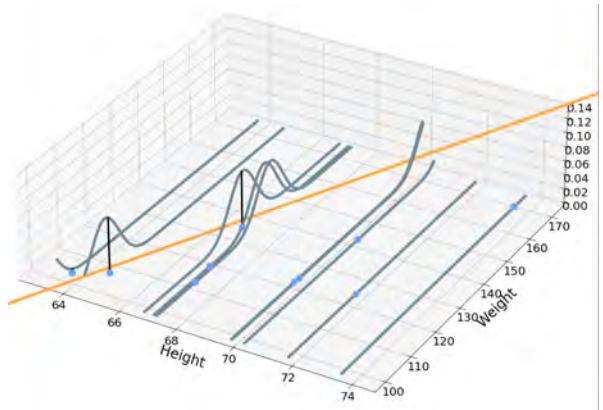


**Model 2:**

$$\text{Likelihood} = 8.16 \cdot 10^{-28}$$

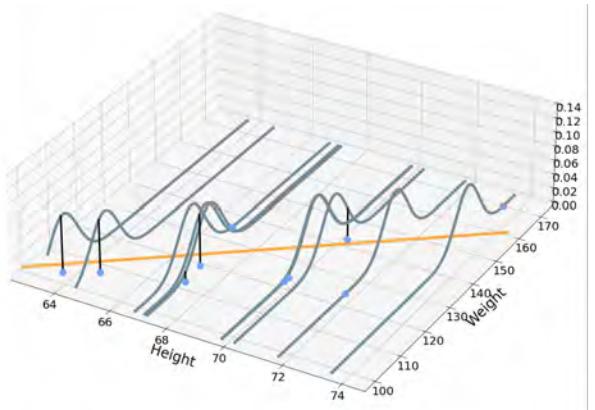


# Picking the Right Model



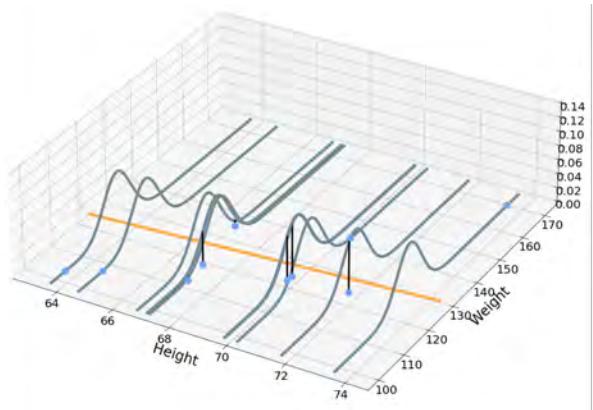
**Model 1:**

$$\text{Likelihood} = 4.91 \cdot 10^{-260}$$



**Model 2:**

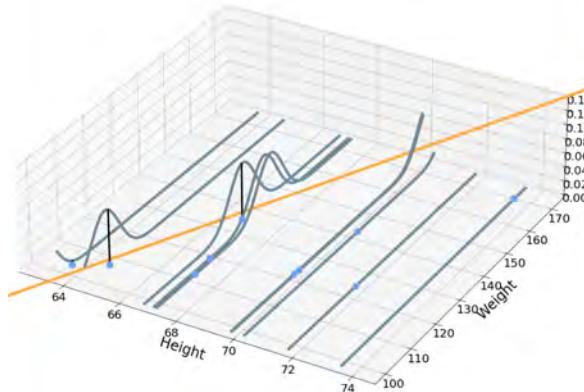
$$\text{Likelihood} = 8.16 \cdot 10^{-28}$$



**Model 3:**

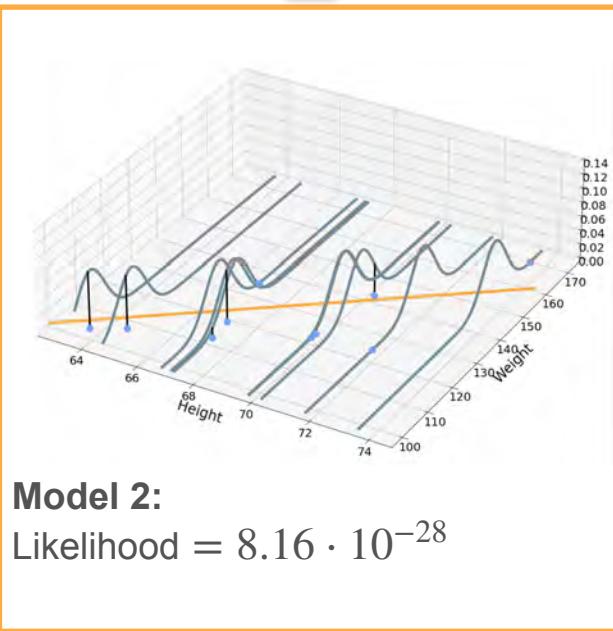
$$\text{Likelihood} = 3.48 \cdot 10^{-49}$$

# Picking the Right Model



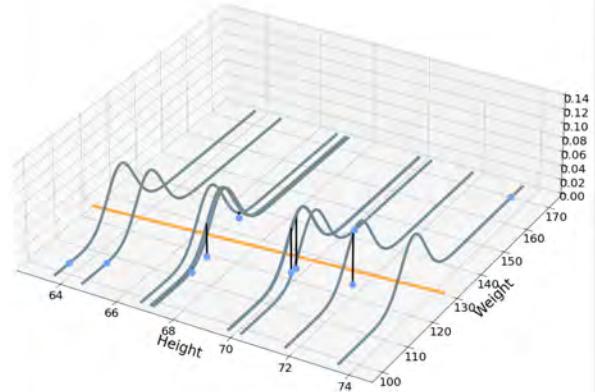
**Model 1:**

$$\text{Likelihood} = 4.91 \cdot 10^{-260}$$



**Model 2:**

$$\text{Likelihood} = 8.16 \cdot 10^{-28}$$



**Model 3:**

$$\text{Likelihood} = 3.48 \cdot 10^{-49}$$



DeepLearning.AI

## Point Estimation

---

**Frequentist vs Bayesian  
Statistics**

# Frequentist Vs. Bayesian Statistics

# Frequentist Vs. Bayesian Statistics

Frequentists

Bayesians

# Frequentist Vs. Bayesian Statistics

## Frequentists

- Probabilities represent long term frequency of events

## Bayesians

# Frequentist Vs. Bayesian Statistics

## Frequentists

- Probabilities represent long term frequency of events

## Bayesians

- Probabilities represent the degree of belief (or certainty)

# Frequentist Vs. Bayesian Statistics

## Frequentists

- Probabilities represent long term frequency of events
- Concept of Likelihood

## Bayesians

- Probabilities represent the degree of belief (or certainty)

# Frequentist Vs. Bayesian Statistics

## Frequentists

- Probabilities represent long term frequency of events
- Concept of Likelihood

## Bayesians

- Probabilities represent the degree of belief (or certainty)
- Concept of Prior

# Frequentist Vs. Bayesian Statistics

## Frequentists

- Probabilities represent long term frequency of events
- Concept of Likelihood

## Bayesians

- Probabilities represent the degree of belief (or certainty)
- Concept of Prior
- Goal: update prior belief based on observations

# Frequentist Vs. Bayesian Statistics

## Frequentists

- Probabilities represent long term frequency of events
- Concept of Likelihood
- Goal: Find the model that most likely generated the observed data

## Bayesians

- Probabilities represent the degree of belief (or certainty)
- Concept of Prior
- Goal: update prior belief based on observations

# Frequentist Approach

# Frequentist Approach



# Frequentist Approach



$p$  = probability of heads

# Frequentist Approach



$p$  = probability of heads

$$p = 0.8$$

# Frequentist Approach



$p$  = probability of heads

$$p = 0.8$$

Maximum likelihood

# Bayesian Approach

$p$  = probability of heads



# Bayesian Approach

$p$  = probability of heads



$p$  could be anything between 0 and 1

# Bayesian Approach

$p$  = probability of heads



$p$  could be anything between 0 and 1

$p$  could still be anything between 0 and 1, but it may be closer to 1

# Bayesian Approach

$p$  = probability of heads



$p$  could be anything between 0 and 1

$p$  could still be anything between 0 and 1, but it may be closer to 1

$p$  could still be anything between 0 and 1, but it's probably around 0.5

# Bayesian Approach

$p$  = probability of heads



$p$  could be anything between 0 and 1

$p$  could still be anything between 0 and 1, but it may be closer to 1

$p$  could still be anything between 0 and 1, but it's probably around 0.5

$p$  could still be anything between 0 and 1, but it's more likely closer to 0.66

# Bayesian Approach

$p$  = probability of heads



$p$  could be anything between 0 and 1

$p$  could still be anything between 0 and 1, but it may be closer to 1

$p$  could still be anything between 0 and 1, but it's probably around 0.5

$p$  could still be anything between 0 and 1, but it's more likely closer to 0.66

...

# Bayesian Approach

$p$  = probability of heads



$p$  could be anything between 0 and 1

$p$  could still be anything between 0 and 1, but it may be closer to 1

$p$  could still be anything between 0 and 1, but it's probably around 0.5

$p$  could still be anything between 0 and 1, but it's more likely closer to 0.66

...

$p$  could still be anything between 0 and 1, but it's very probably close to 0.8

# Bayesian Approach

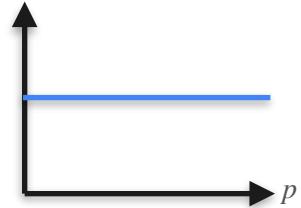


$p$  = probability of heads

# Bayesian Approach



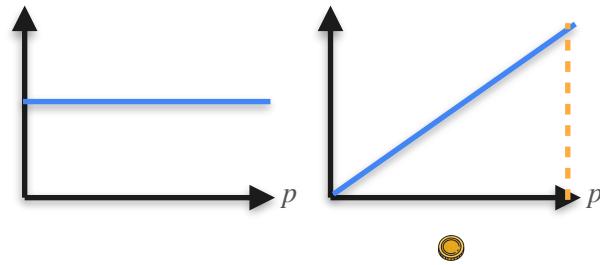
$p$  = probability of heads



# Bayesian Approach



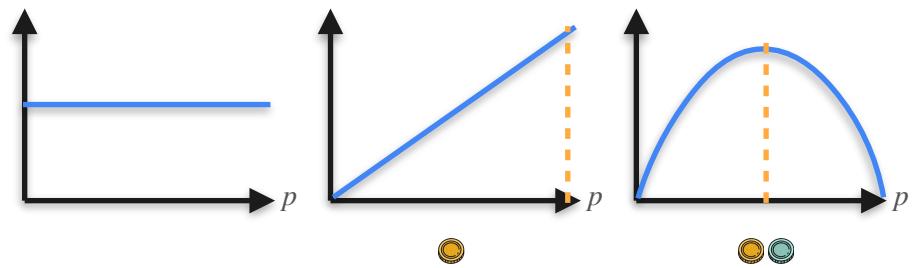
$p$  = probability of heads



# Bayesian Approach



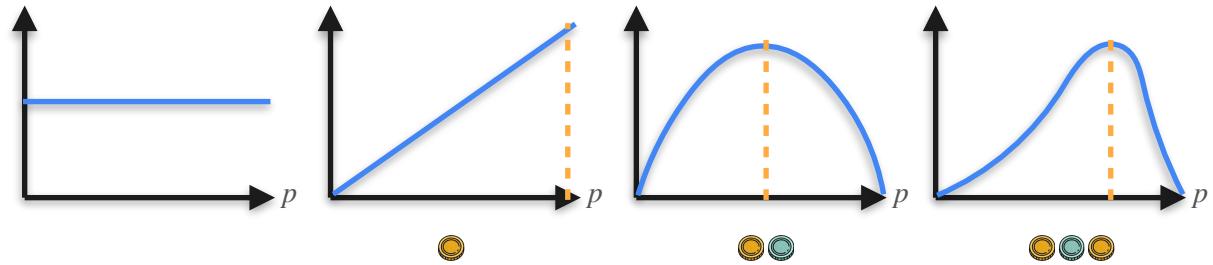
$p$  = probability of heads



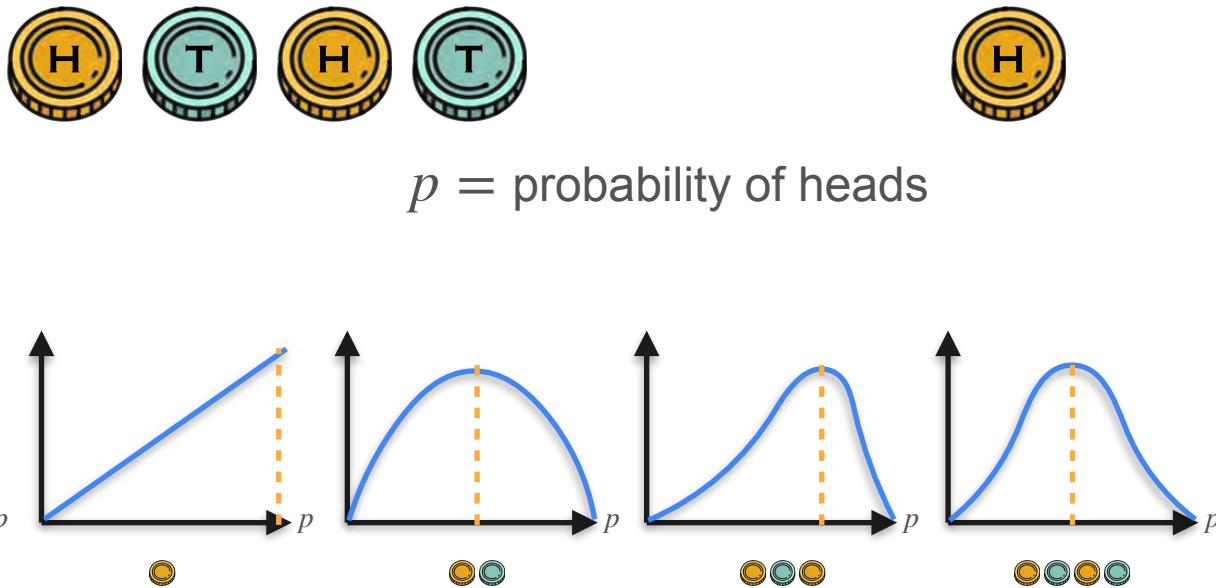
# Bayesian Approach



$p$  = probability of heads



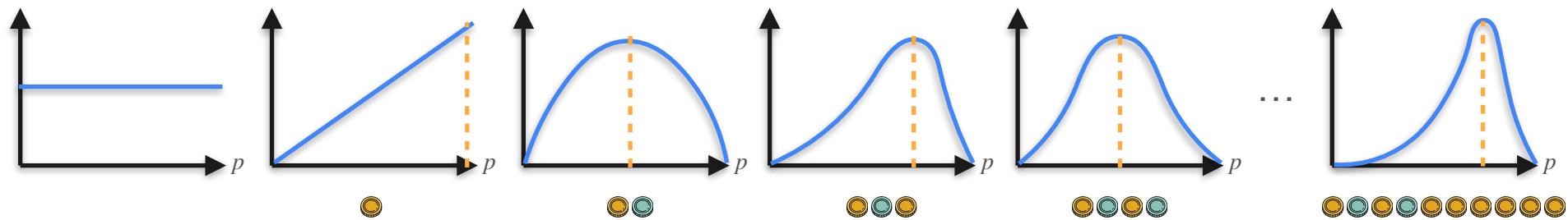
# Bayesian Approach



# Bayesian Approach



$p$  = probability of heads



# Bayesian Statistics

# Bayesian Statistics

Remember Bayes' theorem?  $\mathbf{P}(B|A) = \frac{\mathbf{P}(A|B)\mathbf{P}(B)}{\mathbf{P}(A)}$

# Bayesian Statistics

Remember Bayes' theorem?

$$\boxed{P(B|A)} = \frac{P(A|B)P(B)}{P(A)}$$

↑  
Posterior

# Bayesian Statistics

Remember Bayes' theorem?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

↑  
Posterior

← Prior

# Bayesian Statistics

Remember Bayes' theorem?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

The diagram illustrates the components of Bayes' theorem:

- Posterior:**  $P(B|A)$  is highlighted with an orange box and has an orange arrow pointing up to it from the text "Posterior".
- Prior:**  $P(B)$  is highlighted with a blue box and has a blue arrow pointing left to it from the text "Prior".
- Normalizing constant:**  $P(A)$  is highlighted with a green box and has a green arrow pointing up to it from the text "Normalizing constant".

# Bayesian Statistics

The parameters  
(or model) you  
want to estimate

Remember Bayes' theorem?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Prior

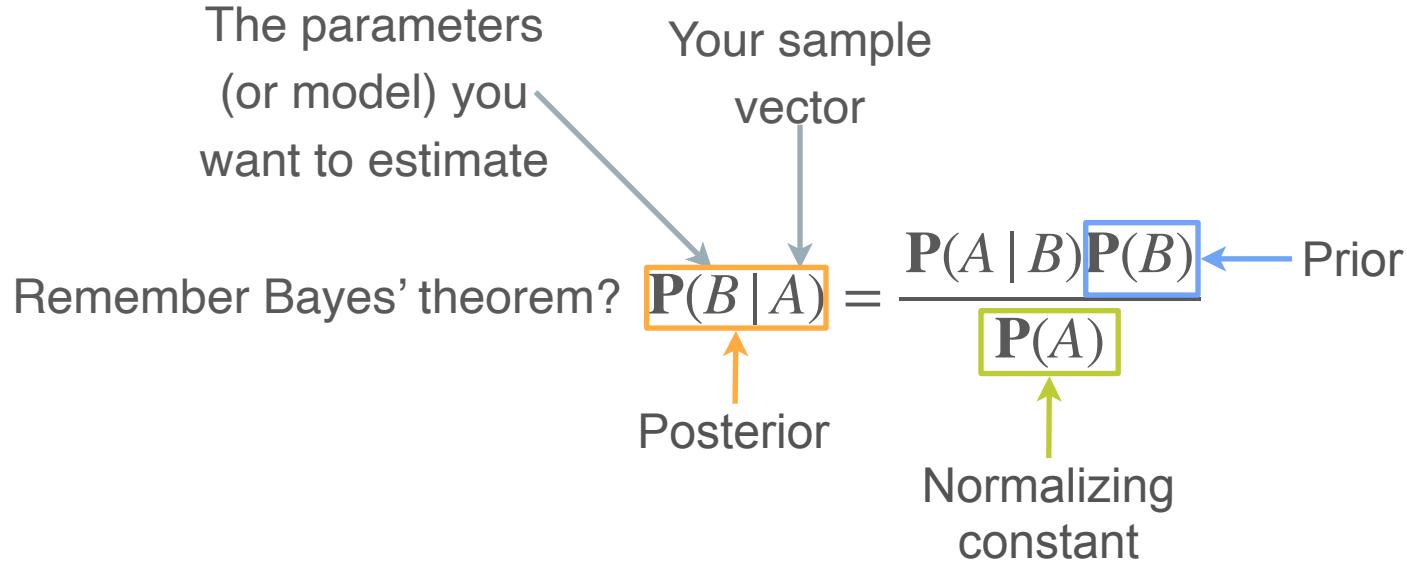
Posterior

Normalizing constant

The diagram illustrates Bayes' theorem with the following components and arrows:

- Posterior:**  $P(B|A)$  is highlighted with an orange box and has an orange arrow pointing up to it from the text "The parameters (or model) you want to estimate".
- Prior:**  $P(B)$  is highlighted with a blue box and has a blue arrow pointing left to it from the text "Remember Bayes' theorem?".
- Normalizing constant:**  $P(A)$  is highlighted with a green box and has a green arrow pointing up to it from the text "Normalizing constant".

# Bayesian Statistics



# Bayesian Statistics

The parameters  
(or model) you  
want to estimate

Your sample  
vector

Remember Bayes' theorem?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Prior

Posterior

Normalizing  
constant

The diagram illustrates the components of Bayes' theorem. At the top left, text asks 'Remember Bayes' theorem?'. To its right, an orange box contains the term  $P(B|A)$ . Above this box, a grey arrow points from the text 'The parameters (or model) you want to estimate'. Another grey arrow points from the text 'Your sample vector'. Below the orange box, an orange arrow points upwards to the text 'Posterior'. To the right of the equation, a blue box contains the terms  $P(A|B)P(B)$ , with a blue arrow pointing from the text 'Probability of samples given model  $B$ '. Below this blue box, a green box contains the term  $P(A)$ , with a green arrow pointing from the text 'Normalizing constant'. A blue arrow also points from the text 'Prior' to the blue box.

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$$\Theta = P(H)$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$        $X_i = 1$  if  $H, 0$  if  $T$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$   $X_i = 1$  if  $H, 0$  if  $T$

Observations:  $\sum_{i=1}^{10} X_i = 8$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$   $X_i = 1$  if  $H, 0$  if  $T$

Observations:  $\sum_{i=1}^{10} X_i = 8$

Suppose that  $\Theta$  takes on the particular value  $\theta$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$   $X_i = 1$  if  $H, 0$  if  $T$

Observations:  $\sum_{i=1}^{10} X_i = 8$

Suppose that  $\Theta$  takes on the particular value  $\theta$

$$P(A | B)$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$        $X_i = 1$  if  $H, 0$  if  $T$

Observations:  $\sum_{i=1}^{10} X_i = 8 \leftarrow A$

Suppose that  $\Theta$  takes on the particular value  $\theta$

$$P(A|B)$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$        $X_i = 1$  if  $H, 0$  if  $T$

Observations:  $\sum_{i=1}^{10} X_i = 8 \leftarrow A$

Suppose that  $\Theta$  takes on the particular value  $\theta \leftarrow B$

$$P(A|B)$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$        $X_i = 1$  if  $H, 0$  if  $T$

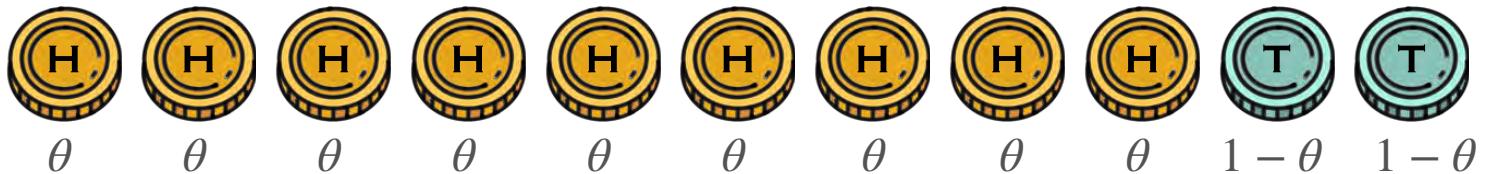
Observations:  $\sum_{i=1}^{10} X_i = 8 \leftarrow A$

Suppose that  $\Theta$  takes on the particular value  $\theta \leftarrow B$

$P(A|B) \rightarrow P\left(\sum_{i=1}^{10} X_i = 8 | \Theta = \theta\right) =$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$        $X_i = 1$  if  $H, 0$  if  $T$

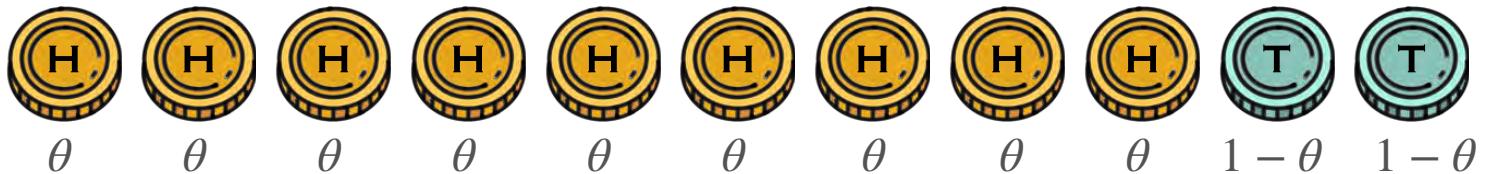
Observations:  $\sum_{i=1}^{10} X_i = 8 \leftarrow A$

Suppose that  $\Theta$  takes on the particular value  $\theta \leftarrow B$

$$P(A|B) \rightarrow P\left(\sum_{i=1}^{10} X_i = 8 | \Theta = \theta\right) = \theta^8(1 - \theta)^2$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$\Theta$  is a random variable  $\rightarrow \Theta = P(H)$        $X_i = 1$  if  $H, 0$  if  $T$

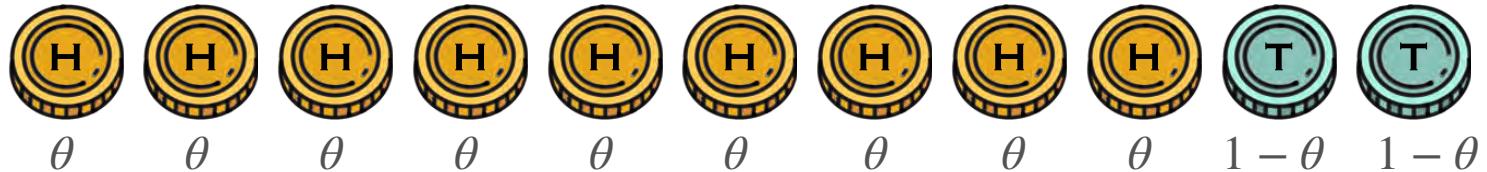
Observations:  $\sum_{i=1}^{10} X_i = 8 \leftarrow A$

Suppose that  $\Theta$  takes on the particular value  $\theta \leftarrow B$

$$P(A|B) \rightarrow P\left(\sum_{i=1}^{10} X_i = 8 | \Theta = \theta\right) = \theta^8(1 - \theta)^2 \quad X_i | \Theta = \theta \sim Bernoulli(\theta)$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

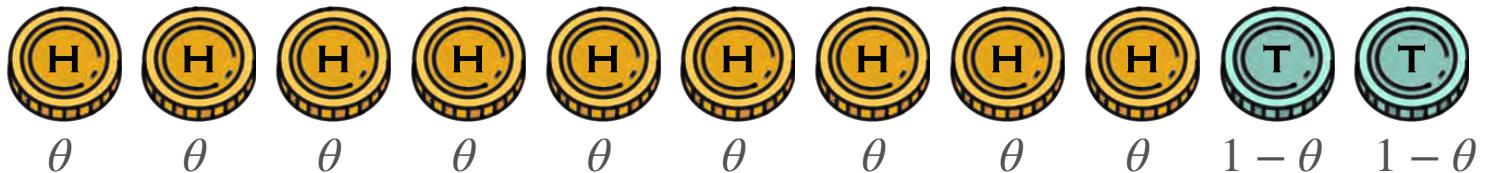


$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta)$$

How do you choose the prior?

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta)$$

How do you choose the prior?

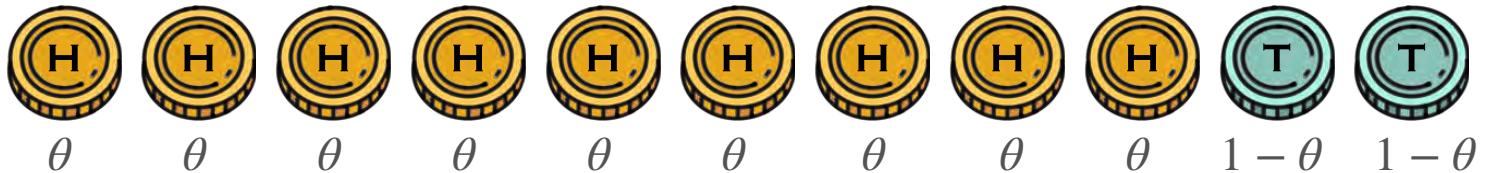
If before tossing the coins you have no idea whether the coin is fair or not, then your prior belief about  $\Theta$  should not favor any value.

You can model this with a uniform distribution

$$\Theta \sim \text{Uniform}(0,1) \quad f_{\Theta}(\theta) = 1, \quad 0 \leq \theta \leq 1$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta)$$

How do you choose the prior?

If before tossing the coins you have no idea whether the coin is fair or not, then your prior belief about  $\Theta$  should not favor any value.

You can model this with a uniform distribution

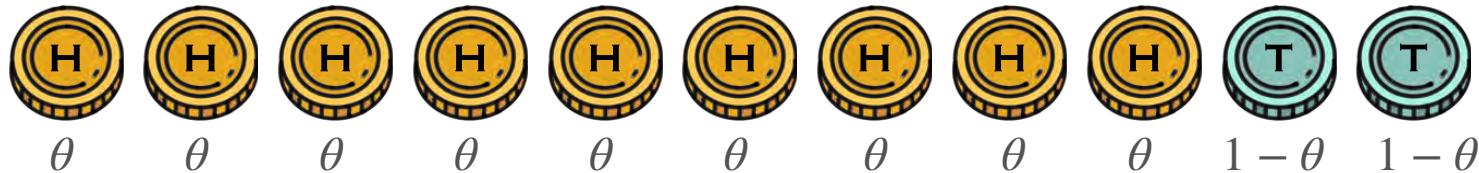
$$\Theta \sim \text{Uniform}(0,1)$$

$$f_{\Theta}(\theta) = 1, \quad 0 \leq \theta \leq 1$$

← Prior ( $P(B)$ )

# Bayesian Statistics: Bernoulli Example

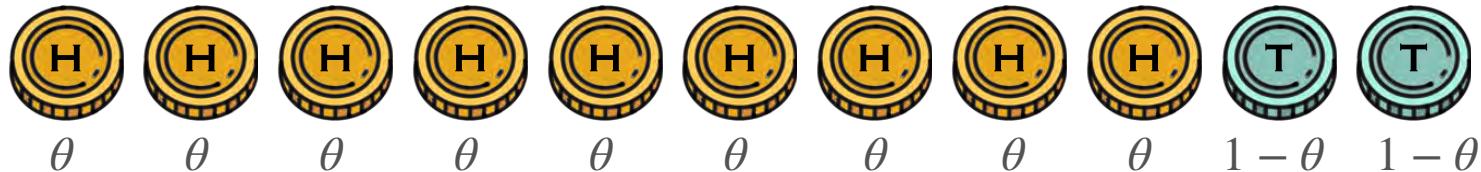
Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta) \xleftarrow{\text{(P(A | B))}} \quad \Theta \sim \text{Uniform}(0,1) \xleftarrow{\text{Prior (P(B))}}$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

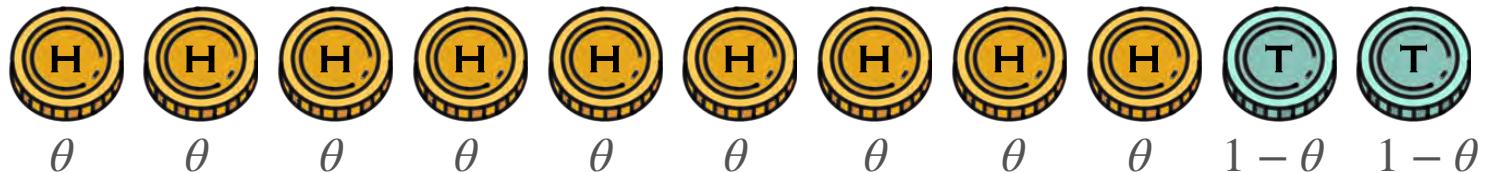


$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta) \xleftarrow{\text{(P(A | B))}} \quad \Theta \sim \text{Uniform}(0,1) \xleftarrow{\text{Prior (P(B))}}$$

How do you get the posterior? Bayes!

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



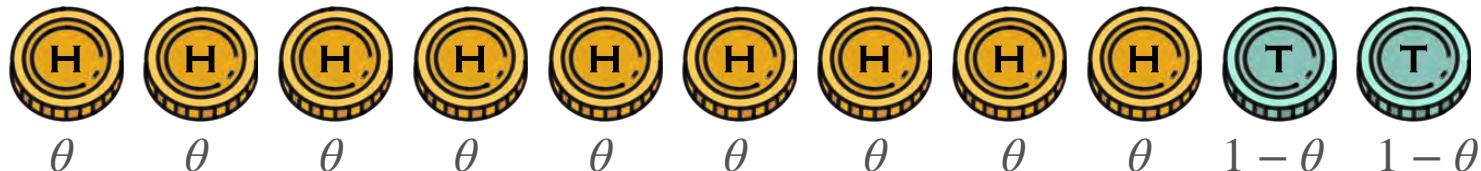
$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta) \xleftarrow{\text{(P(A | B))}} \quad \Theta \sim \text{Uniform}(0,1) \xleftarrow{\text{Prior (P(B))}}$$

How do you get the posterior? Bayes!

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{p_{\mathbf{X}|\Theta=\theta}(\mathbf{x})f_{\Theta}(\theta)}{p_{\mathbf{X}}(\mathbf{x})}$$

# Bayesian Statistics: Bernoulli Example

Remember Bayes' theorem?  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$



$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta) \xleftarrow{\text{(P(A | B))}} \quad \Theta \sim \text{Uniform}(0,1) \xleftarrow{\text{Prior (P(B))}}$$

How do you get the posterior? Bayes!

$$f_{\Theta|X=x}(\theta) = \frac{p_{X|\Theta=\theta}(x)f_{\Theta}(\theta)}{p_X(x)}$$

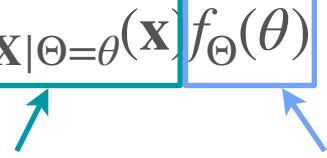
↑  
Posterior ( $P(B|A)$ )

Normalizing constant

# Bayesian Statistics: MAP Estimator

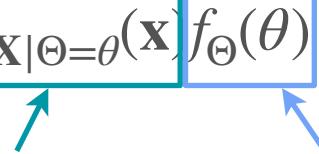
$$f_{\Theta|X=x}(\theta) \propto p_{X|\Theta=\theta}(x)f_{\Theta}(\theta)$$

# Bayesian Statistics: MAP Estimator

$$f_{\Theta|X=x}(\theta) \propto p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$


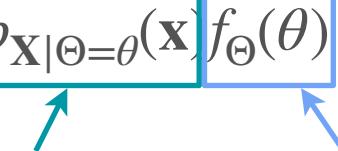
P(data | model)      P(model)

# Bayesian Statistics: MAP Estimator

$$f_{\Theta|X=x}(\theta) \propto p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$


Does this look familiar?     $\mathbf{P}(\text{data} \mid \text{model})$      $\mathbf{P}(\text{model})$

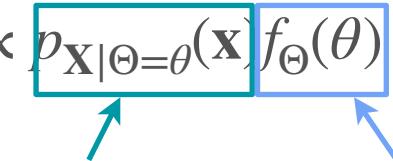
# Bayesian Statistics: MAP Estimator

$$f_{\Theta|X=x}(\theta) \propto p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$


Does this look familiar?     $\mathbf{P}(\text{data} \mid \text{model})$      $\mathbf{P}(\text{model})$

Point estimators?

# Bayesian Statistics: MAP Estimator

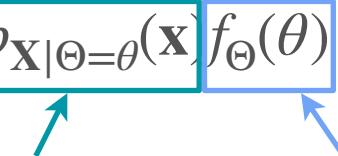
$$f_{\Theta|X=x}(\theta) \propto p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$


Does this look familiar?     $\text{P}(\text{data} | \text{model})$      $\text{P}(\text{model})$

Point estimators?

$$\hat{\theta} = \arg \max_{\theta} f_{\Theta|X=x}(\theta) = \arg \max_{\theta} p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$

# Bayesian Statistics: MAP Estimator

$$f_{\Theta|X=x}(\theta) \propto p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$


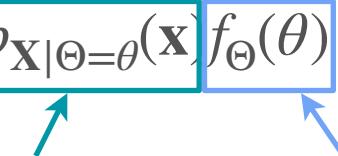
Does this look familiar?     $\text{P}(\text{data} | \text{model})$      $\text{P}(\text{model})$

Point estimators?

$$\hat{\theta} = \arg \max_{\theta} f_{\Theta|X=x}(\theta) = \arg \max_{\theta} p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$

This is called the Maximum a Posteriori (MAP) estimator

# Bayesian Statistics: MAP Estimator

$$f_{\Theta|X=x}(\theta) \propto p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$


Does this look familiar?     $\text{P}(\text{data} | \text{model})$      $\text{P}(\text{model})$

Point estimators?

$$\hat{\theta} = \arg \max_{\theta} f_{\Theta|X=x}(\theta) = \arg \max_{\theta} p_{X|\Theta=\theta}(x) f_{\Theta}(\theta)$$

This is called the Maximum a Posteriori (MAP) estimator

Notice that if the prior is non informative, then MAP = MLE



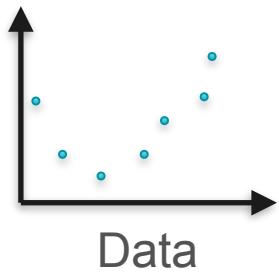
DeepLearning.AI

## Point Estimation

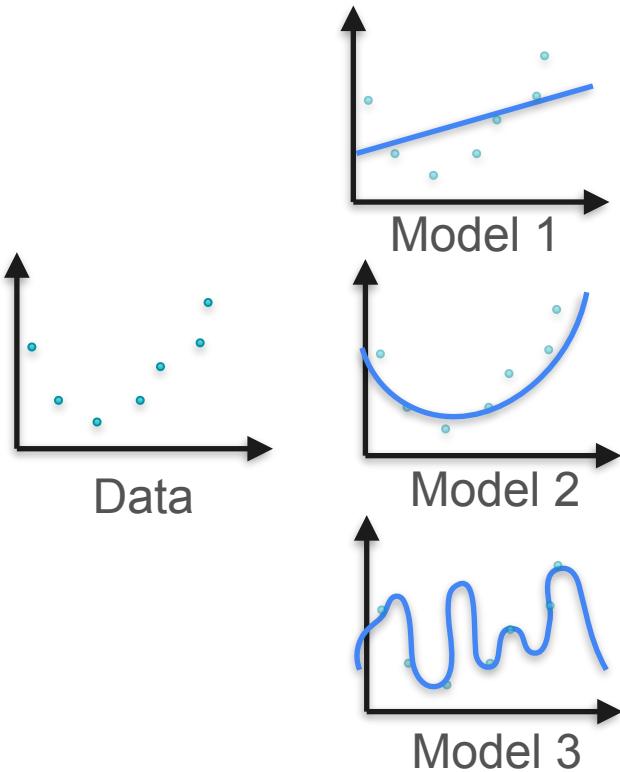
---

## Regularization

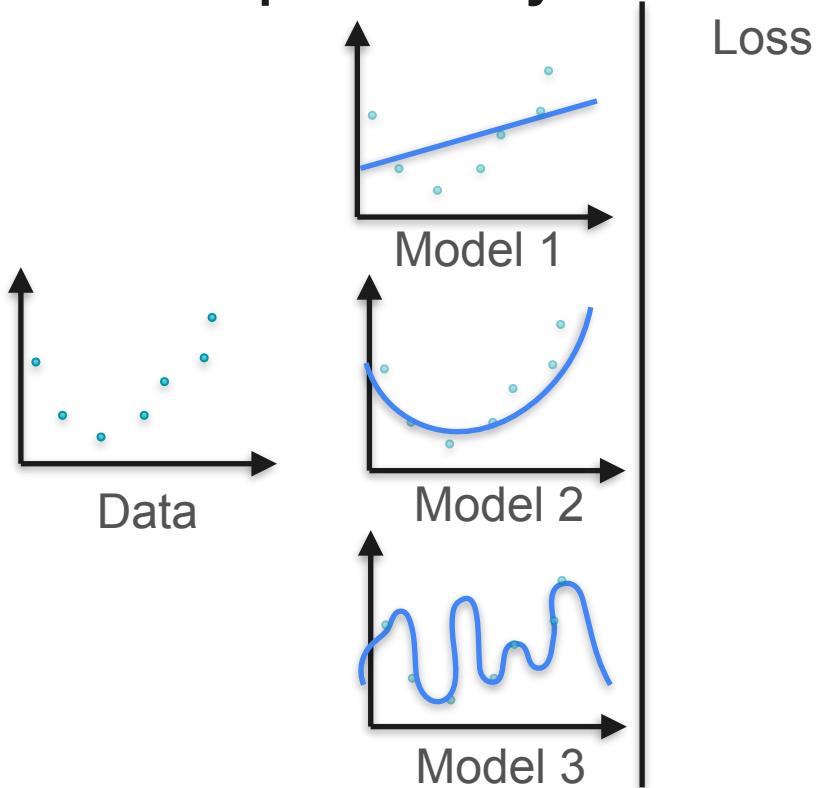
# Example: Polynomial Regression



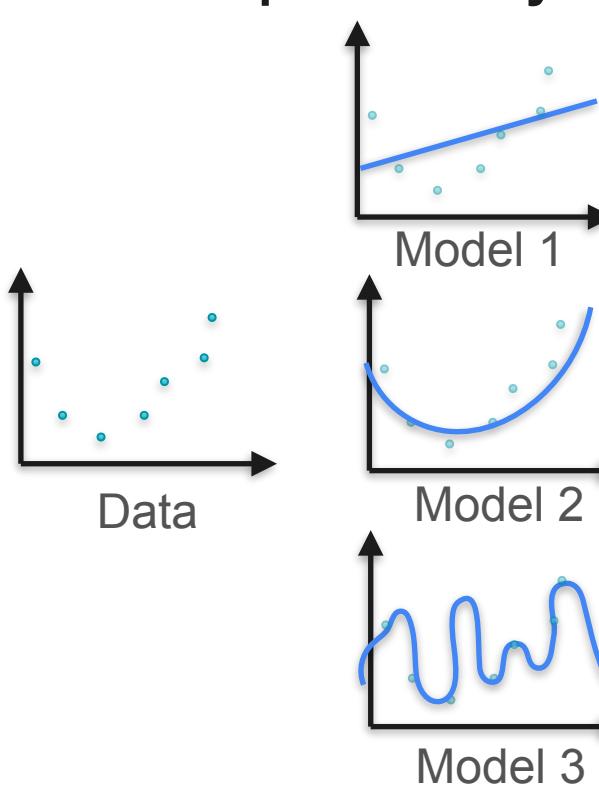
# Example: Polynomial Regression



# Example: Polynomial Regression



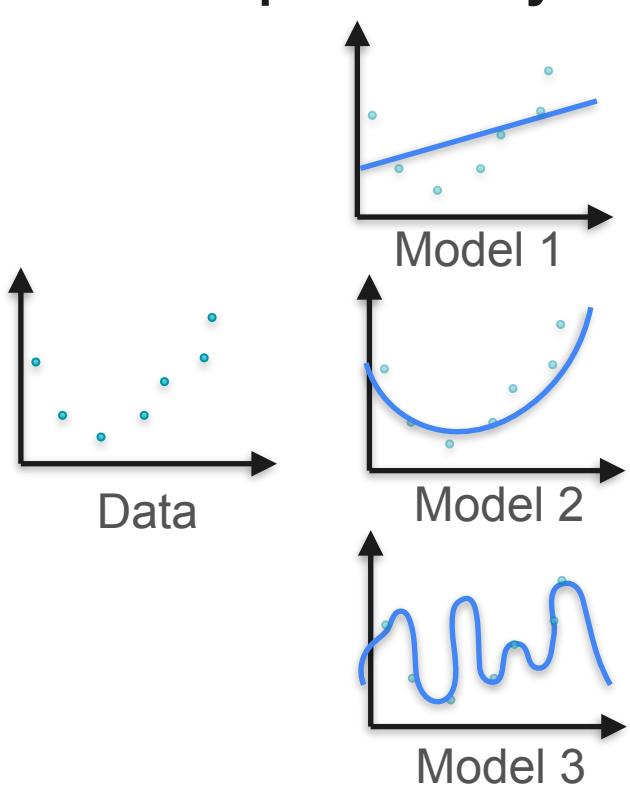
# Example: Polynomial Regression



Loss

10

# Example: Polynomial Regression

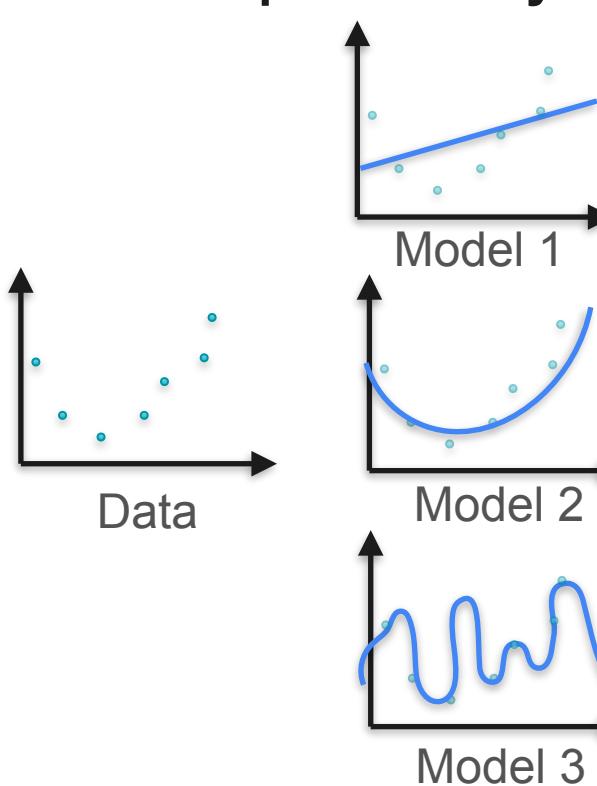


Loss

10

2

# Example: Polynomial Regression



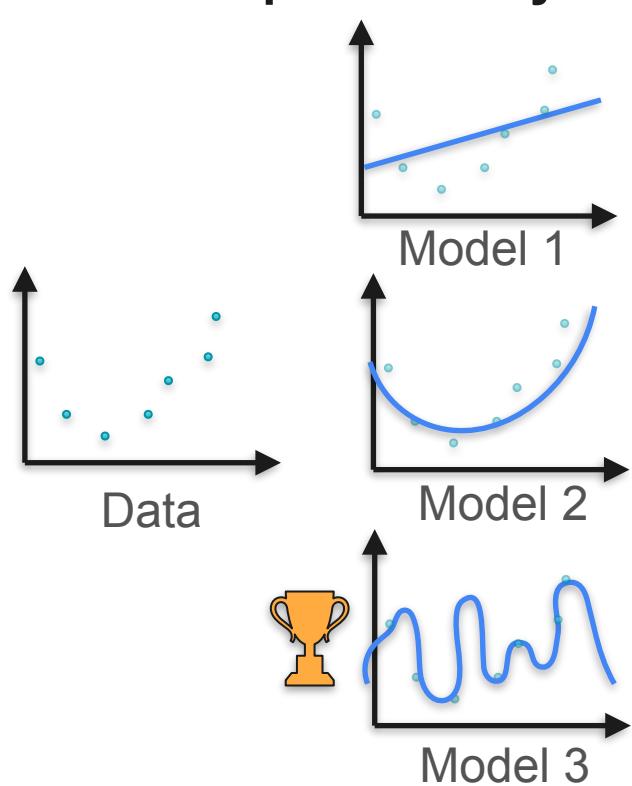
Loss

10

2

0.1

# Example: Polynomial Regression



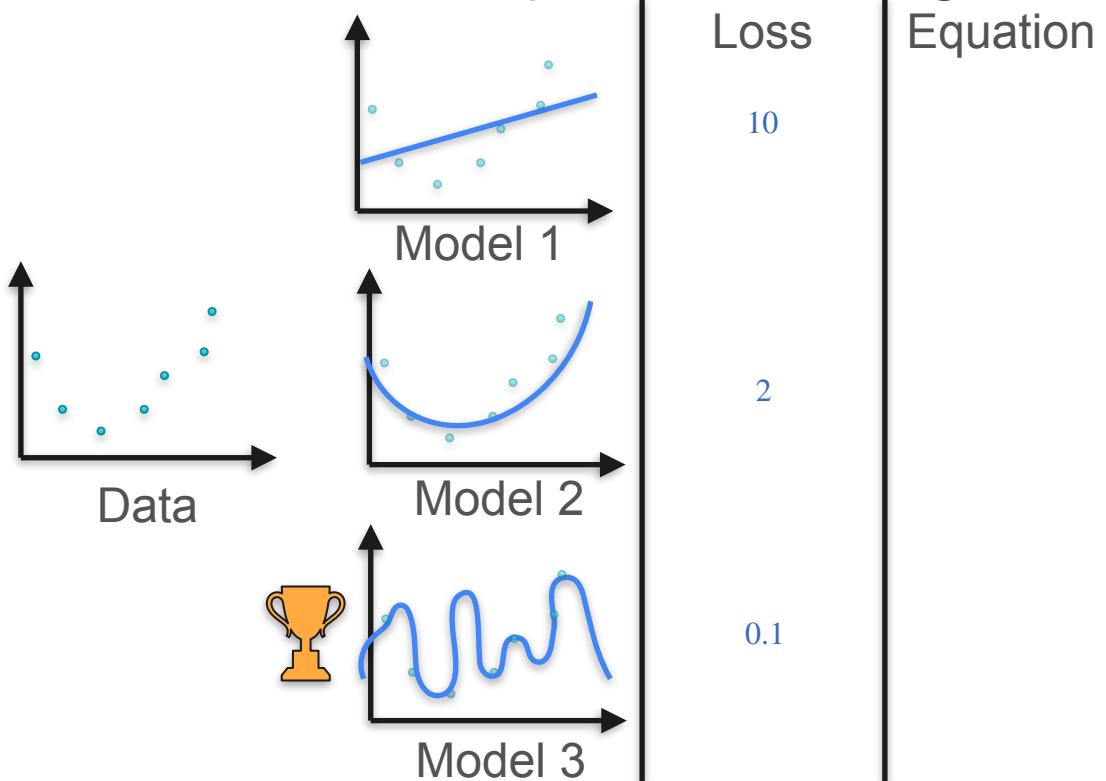
Loss

10

2

0.1

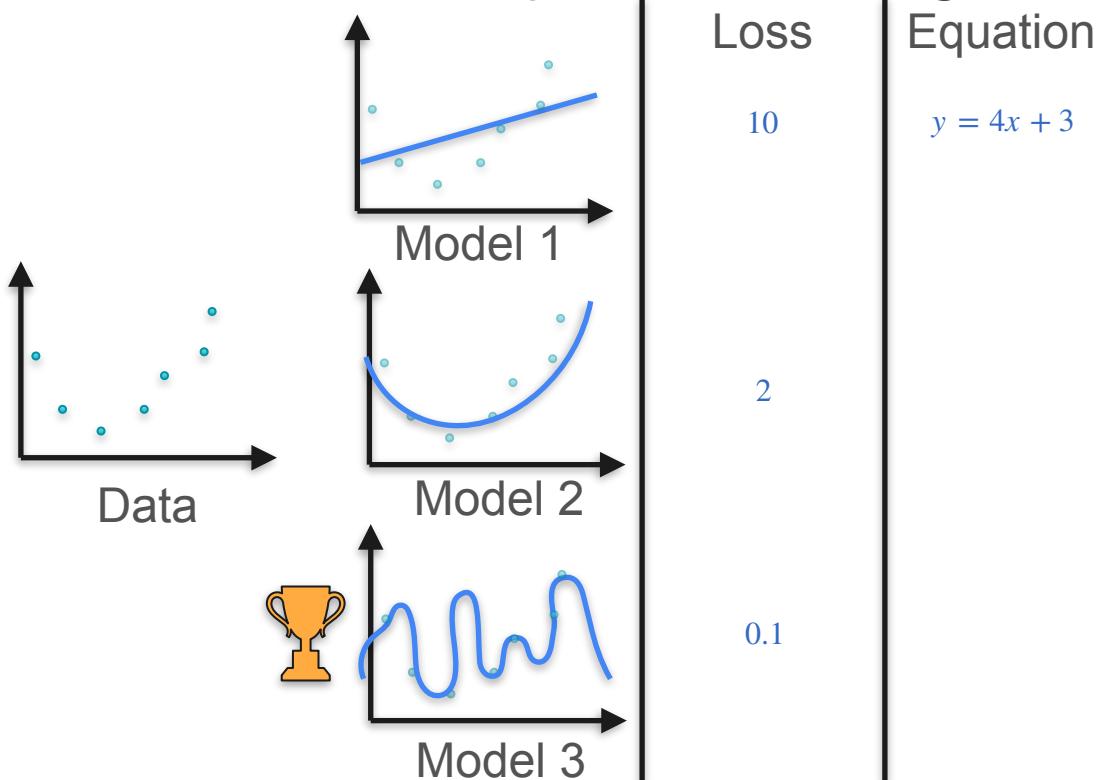
# Example: Polynomial Regression



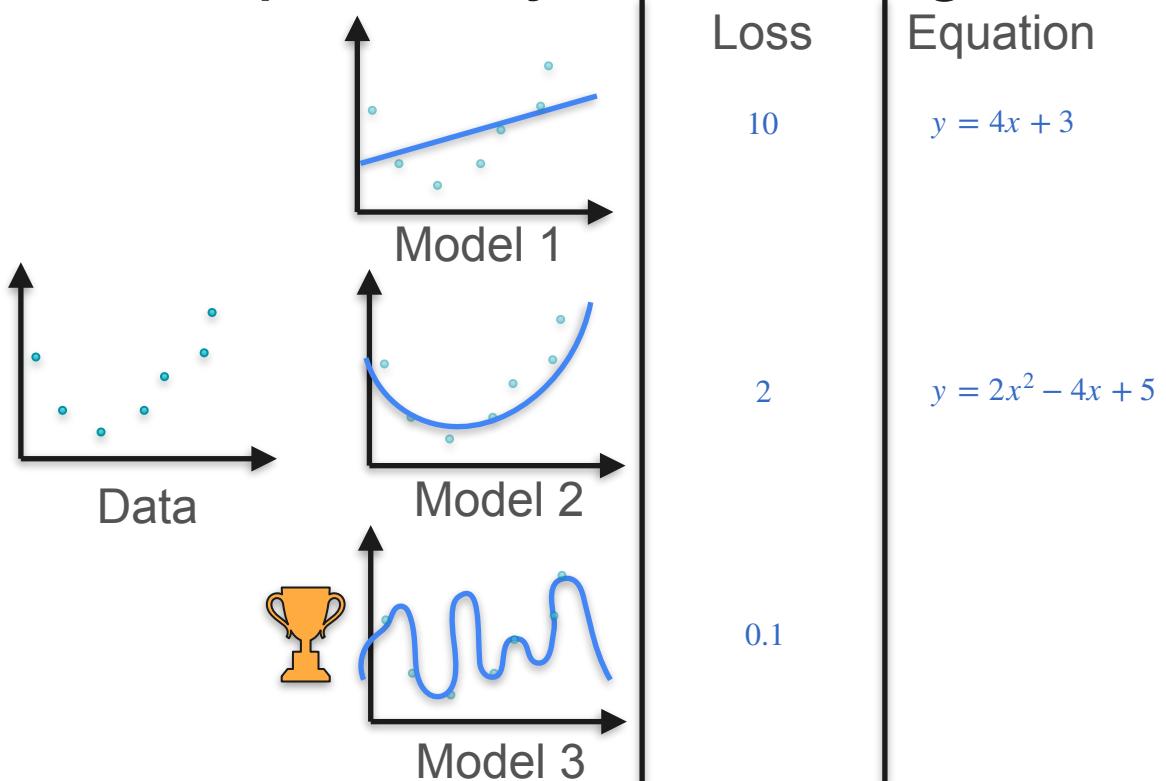
Loss

Equation

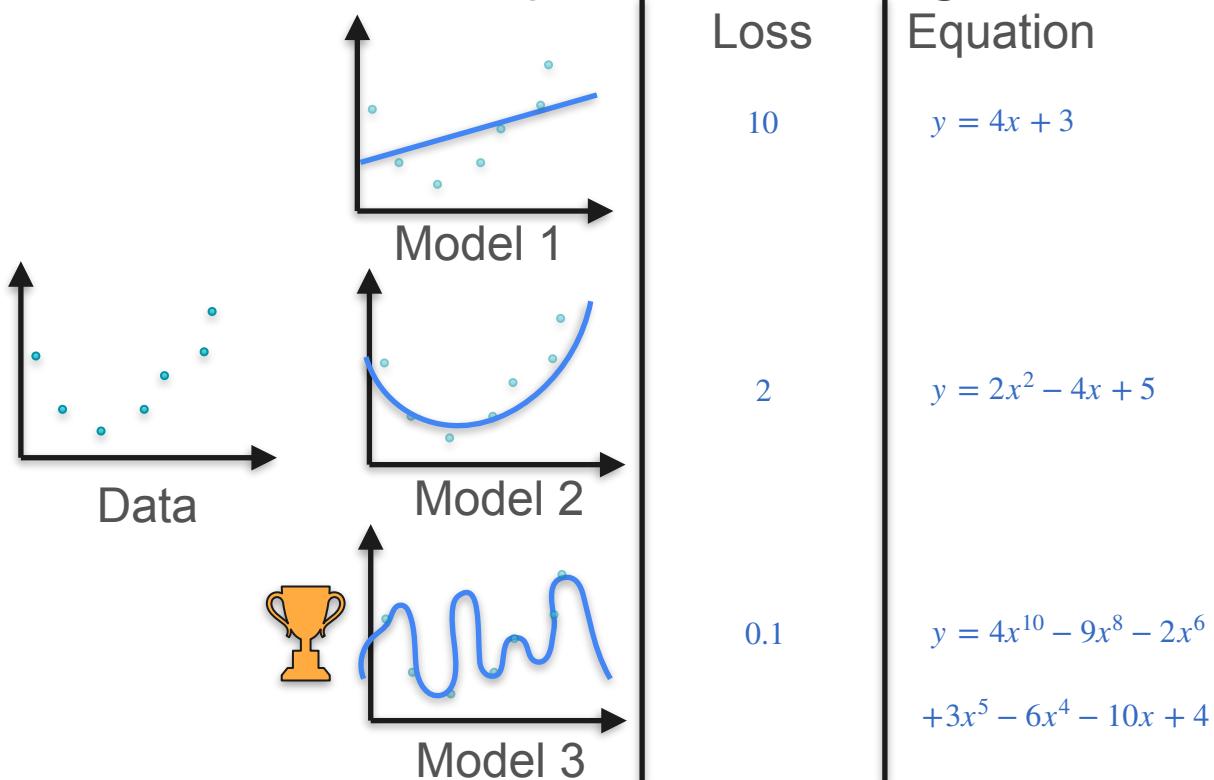
# Example: Polynomial Regression



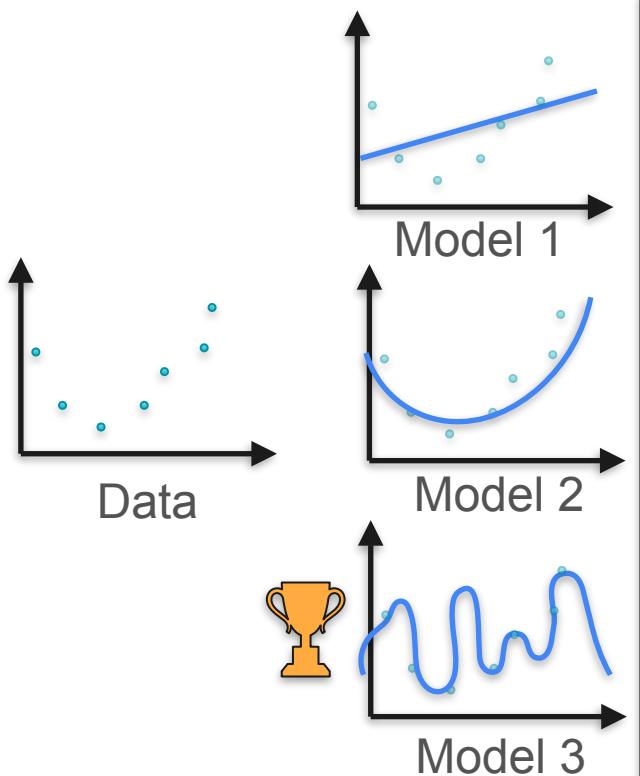
# Example: Polynomial Regression



# Example: Polynomial Regression

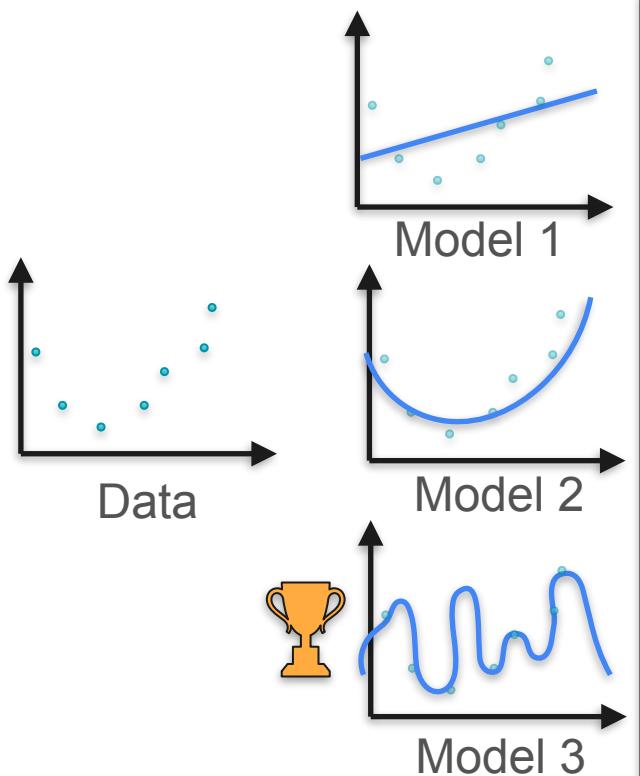


# Example: Polynomial Regression



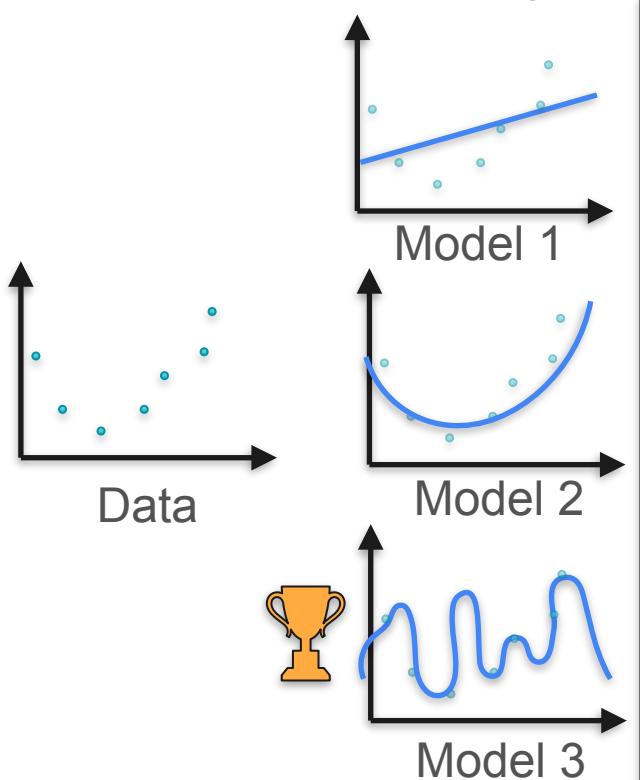
Loss	Equation	Penalty
10	$y = 4x + 3$	
2	$y = 2x^2 - 4x + 5$	
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	

# Example: Polynomial Regression



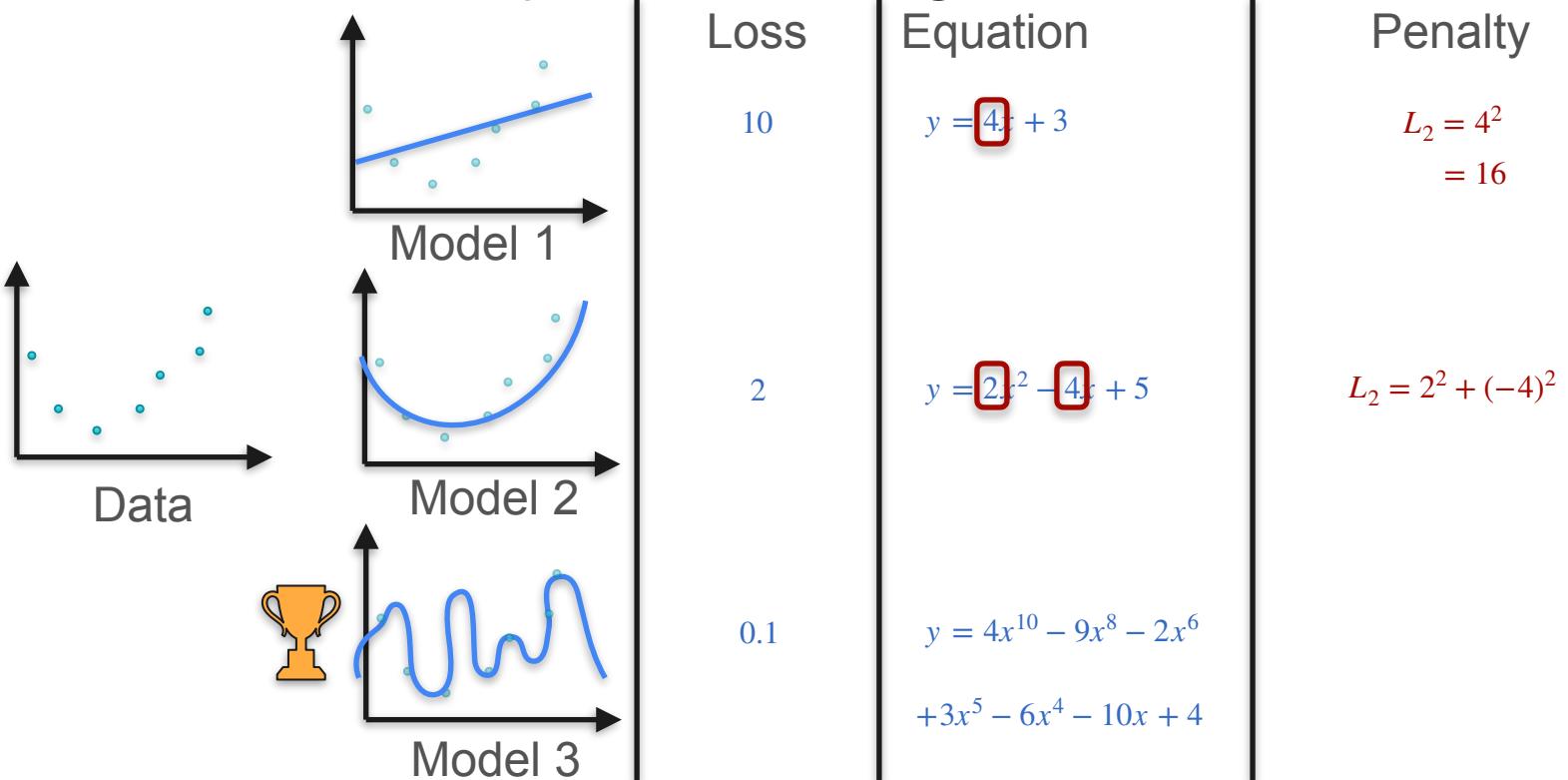
Loss	Equation	Penalty
10	$y = 4x + 3$	$L_2 = 4^2$
2	$y = 2x^2 - 4x + 5$	
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	

# Example: Polynomial Regression

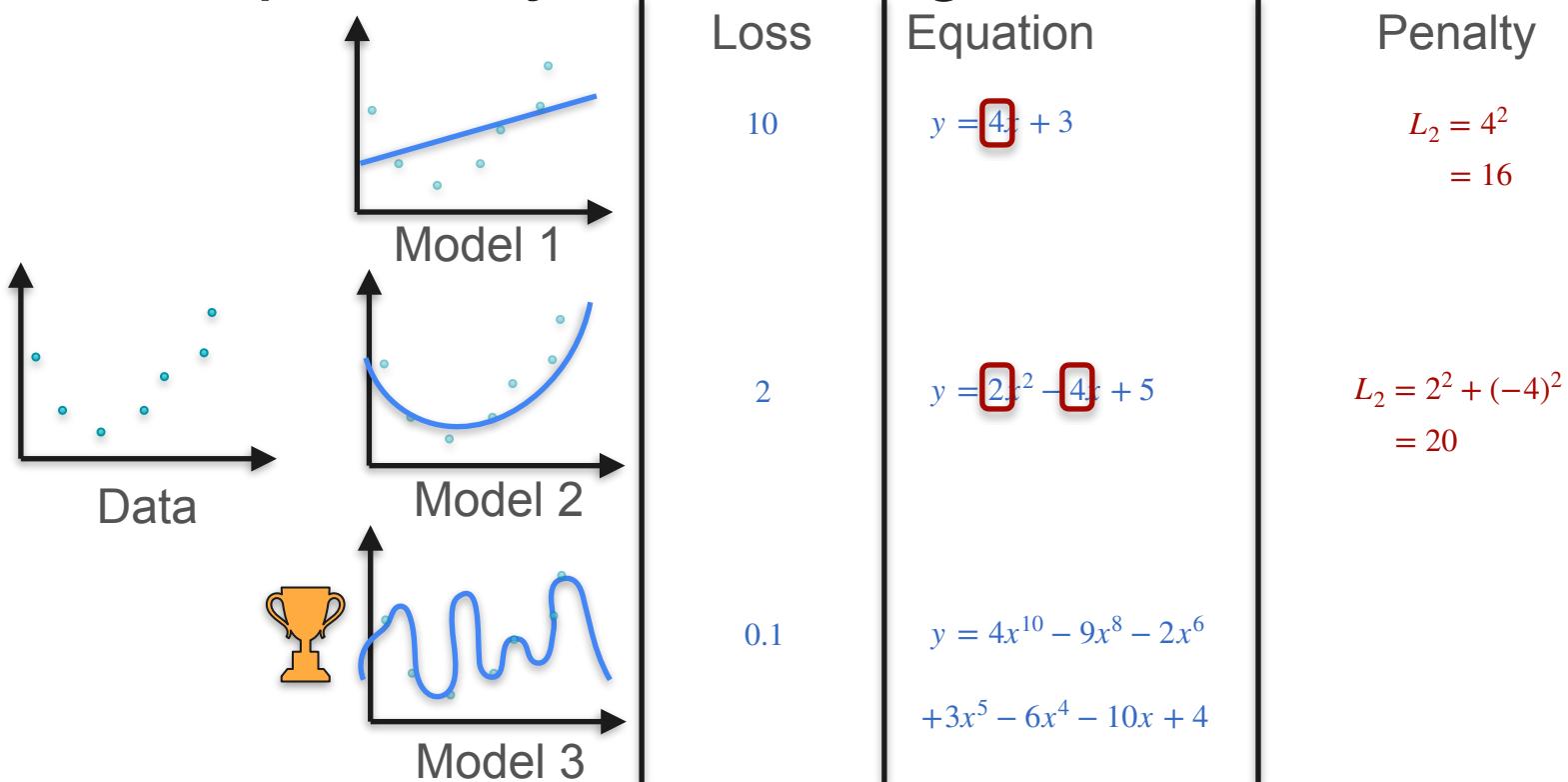


Loss	Equation	Penalty
10	$y = 4x + 3$	$L_2 = 4^2 = 16$
2	$y = 2x^2 - 4x + 5$	
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	

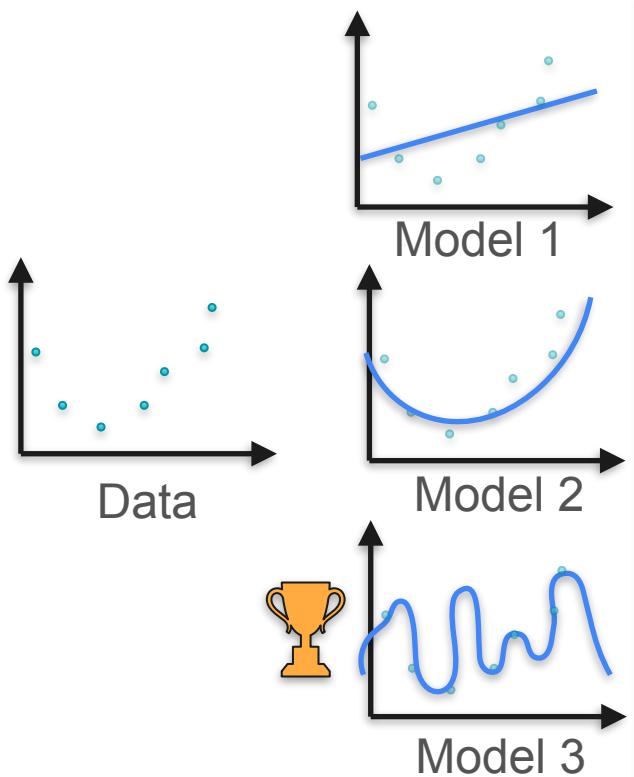
# Example: Polynomial Regression



# Example: Polynomial Regression

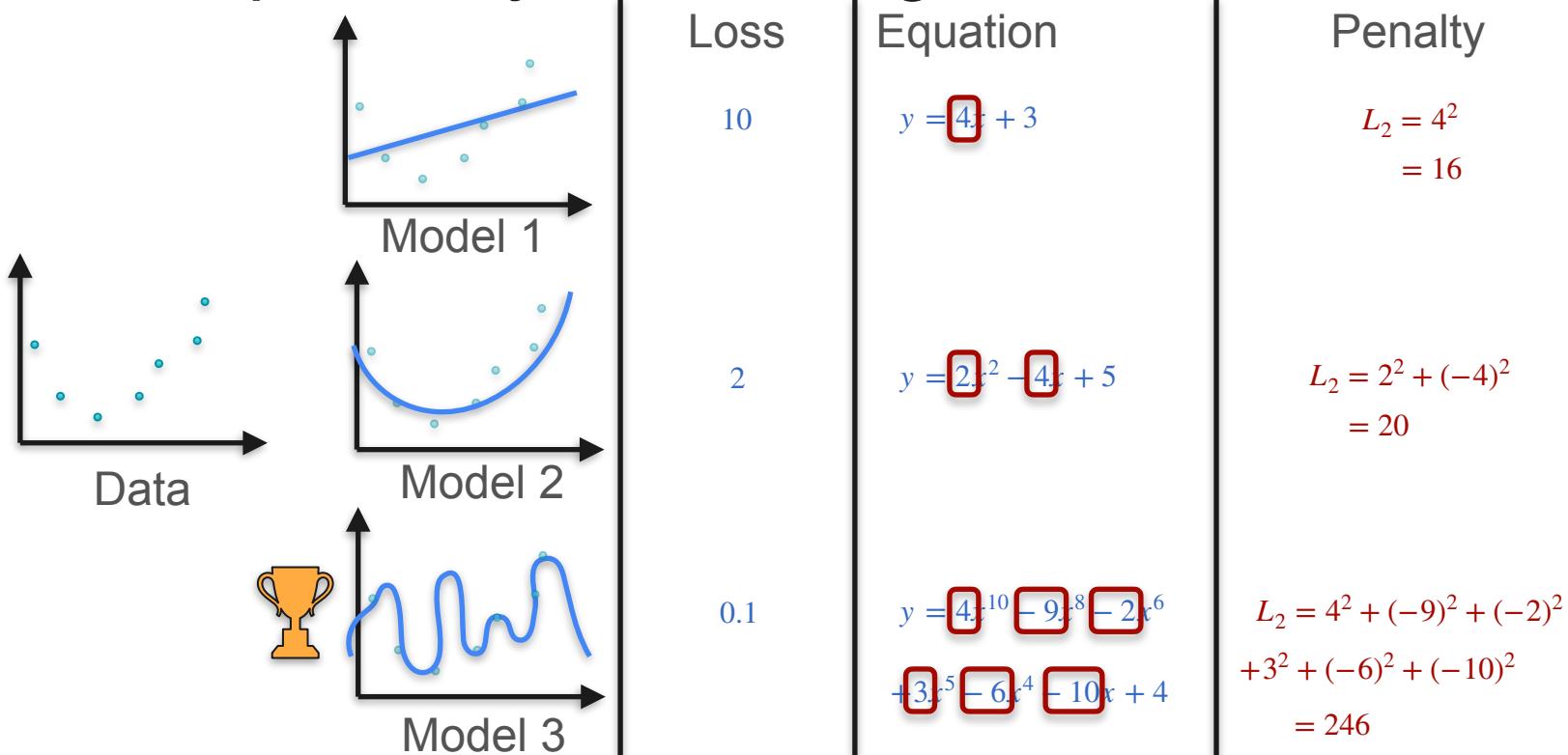


# Example: Polynomial Regression

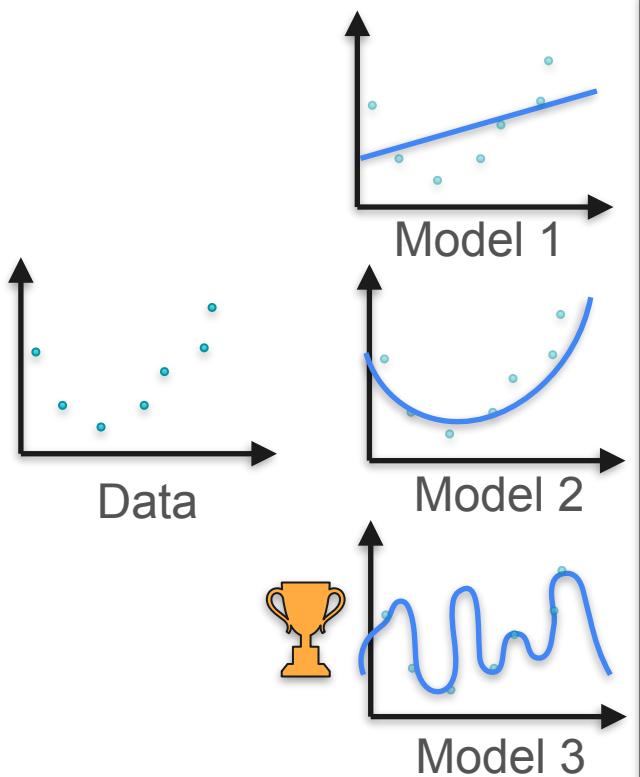


Loss	Equation	Penalty
10	$y = 4x + 3$	$L_2 = 4^2 = 16$
2	$y = 2x^2 - 4x + 5$	$L_2 = 2^2 + (-4)^2 = 20$
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	$L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2$

# Example: Polynomial Regression

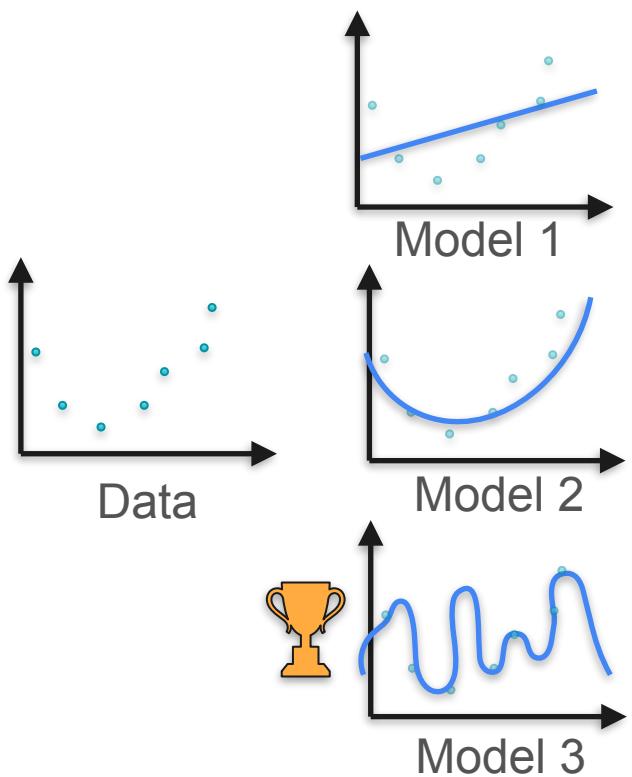


# Example: Polynomial Regression



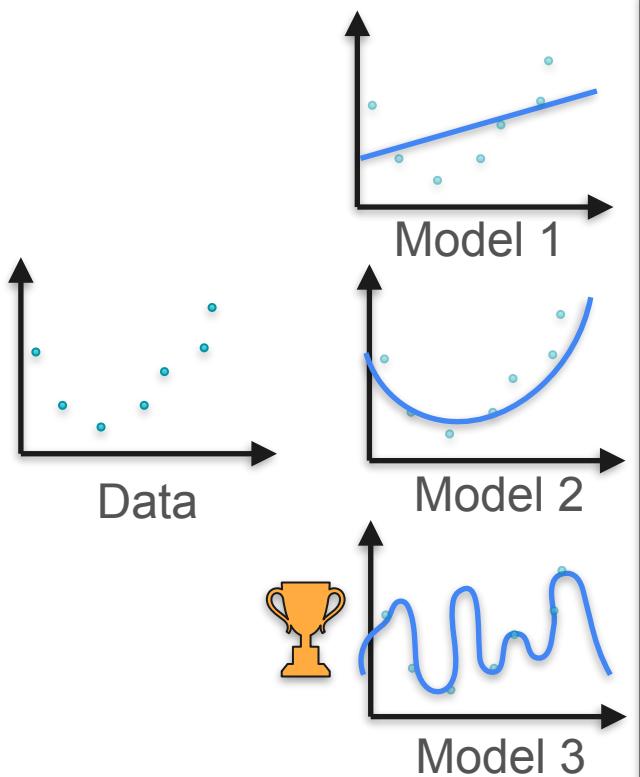
Loss	Equation	Penalty	New loss
10	$y = 4x + 3$	$L_2 = 4^2 = 16$	
2	$y = 2x^2 - 4x + 5$	$L_2 = 2^2 + (-4)^2 = 20$	
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	$L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2 = 246$	

# Example: Polynomial Regression



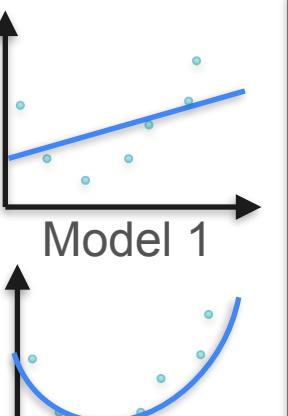
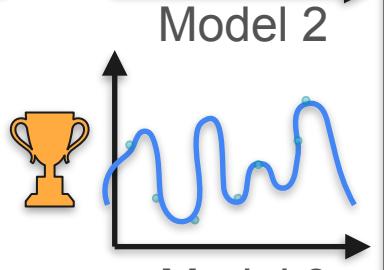
Loss	Equation	Penalty	New loss
10	$y = 4x + 3$	$L_2 = 4^2 = 16$	26
2	$y = 2x^2 - 4x + 5$	$L_2 = 2^2 + (-4)^2 = 20$	
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	$L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2 = 246$	

# Example: Polynomial Regression



Loss	Equation	Penalty	New loss
10	$y = 4x + 3$	$L_2 = 4^2 = 16$	26
2	$y = 2x^2 - 4x + 5$	$L_2 = 2^2 + (-4)^2 = 20$	22
0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	$L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2 = 246$	

# Example: Polynomial Regression

	Loss	Equation	Penalty	New loss
 Data	10	$y = 4x + 3$	$L_2 = 4^2 = 16$	26
 Model 2	2	$y = 2x^2 - 4x + 5$	$L_2 = 2^2 + (-4)^2 = 20$	22
 Model 3	0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	$L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2 = 246$	246.1

# Example: Polynomial Regression

Model	Loss	Equation	Penalty	New loss
Model 1	10	$y = 4x + 3$	$L_2 = 4^2 = 16$	26
Model 2	2	$y = 2x^2 - 4x + 5$	$L_2 = 2^2 + (-4)^2 = 20$	22
Model 3	0.1	$y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$	$L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2 = 246$	246.1

# Regularization Term

# Regularization Term

Model:  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

# Regularization Term

Model:  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Log-loss:  $\ell\ell$

# Regularization Term

Model:  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Log-loss:  $\ell\ell$

L2 Regularization Error:  $a_n^2 + a_{n-1}^2 + \dots + a_1^2$

# Regularization Term

Model:  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Log-loss:  $\ell\ell$

L2 Regularization Error:  $a_n^2 + a_{n-1}^2 + \dots + a_1^2$

Regularization parameter:  $\lambda$

# Regularization Term

Model:  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Log-loss:  $\ell\ell$

L2 Regularization Error:  $a_n^2 + a_{n-1}^2 + \dots + a_1^2$

Regularization parameter:  $\lambda$

# Regularization Term

Model:  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Log-loss:  $\ell\ell$

L2 Regularization Error:  $a_n^2 + a_{n-1}^2 + \dots + a_1^2$

Regularization parameter:  $\lambda$

Regularized error:

# Regularization Term

Model:  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Log-loss:  $\ell \ell$

L2 Regularization Error:  $a_n^2 + a_{n-1}^2 + \dots + a_1^2$

Regularization parameter:  $\lambda$

Regularized error:  $\ell \ell + \lambda (a_n^2 + a_{n-1}^2 + \dots + a_1^2)$



DeepLearning.AI

## Point Estimation

---

## Back to Bayesics

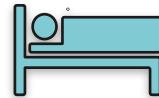
# There's Popcorn on the Floor. What Happened?



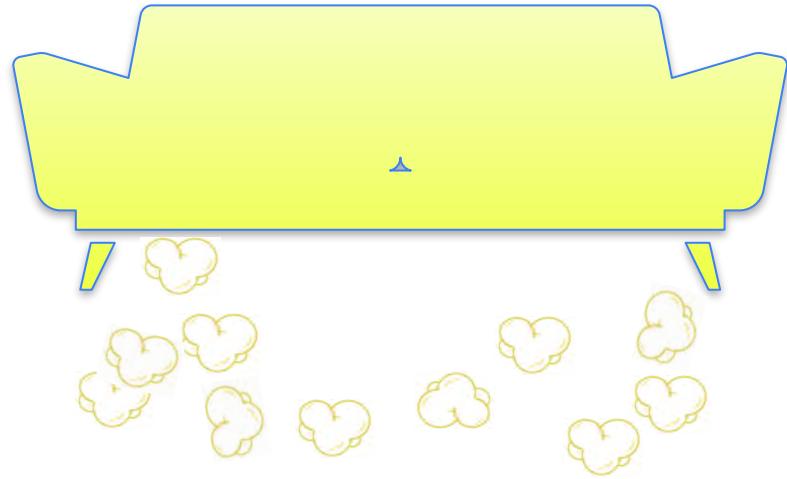
Movies



Board  
Games



Nap



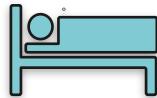
# There's Popcorn on the Floor. What Happened?



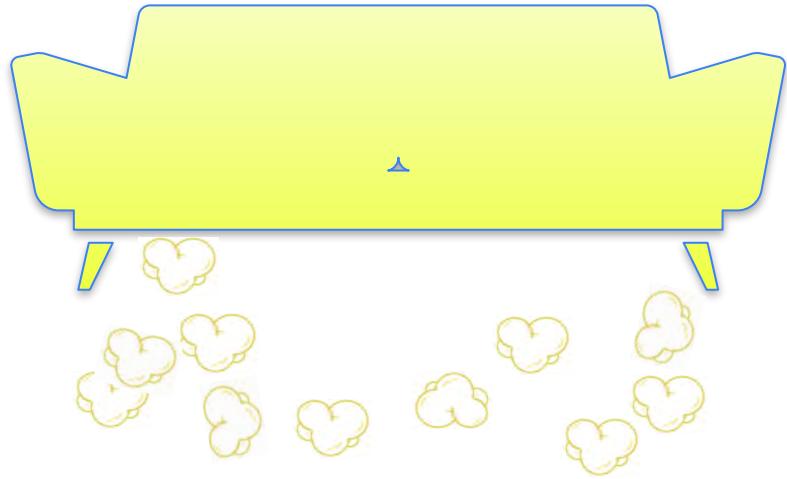
Movies *High*



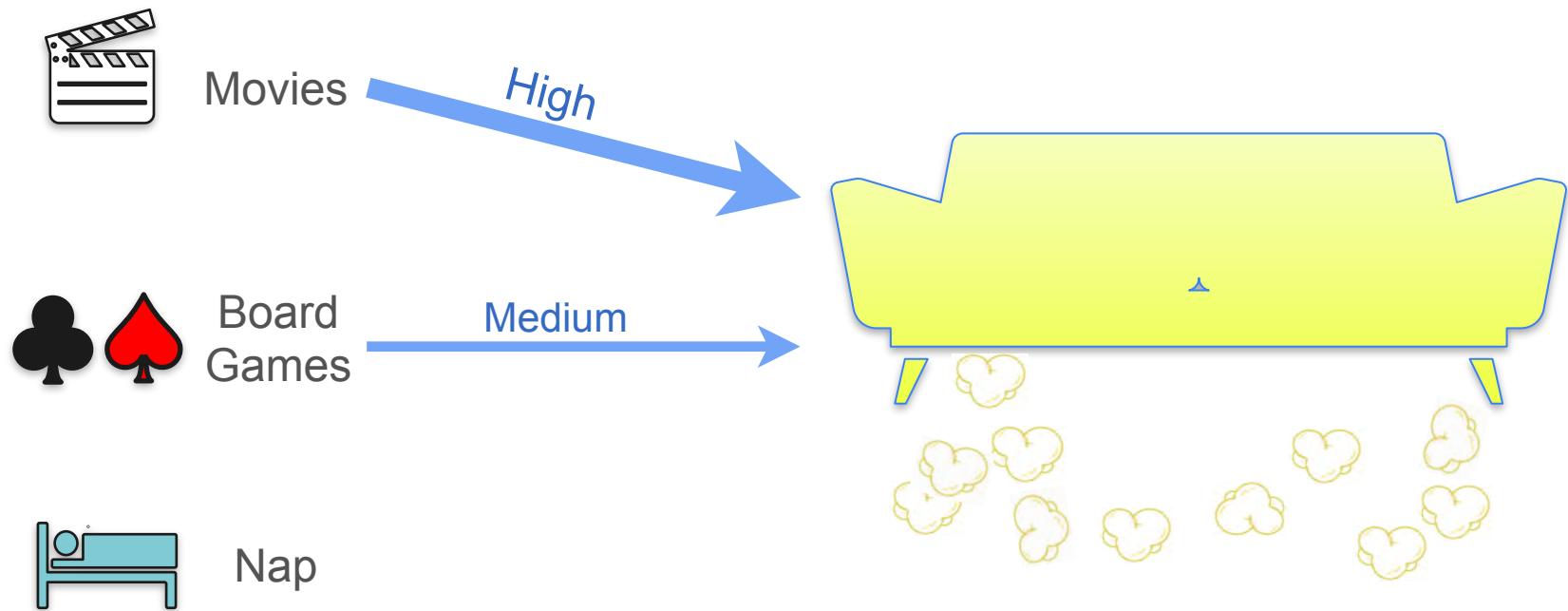
Board Games



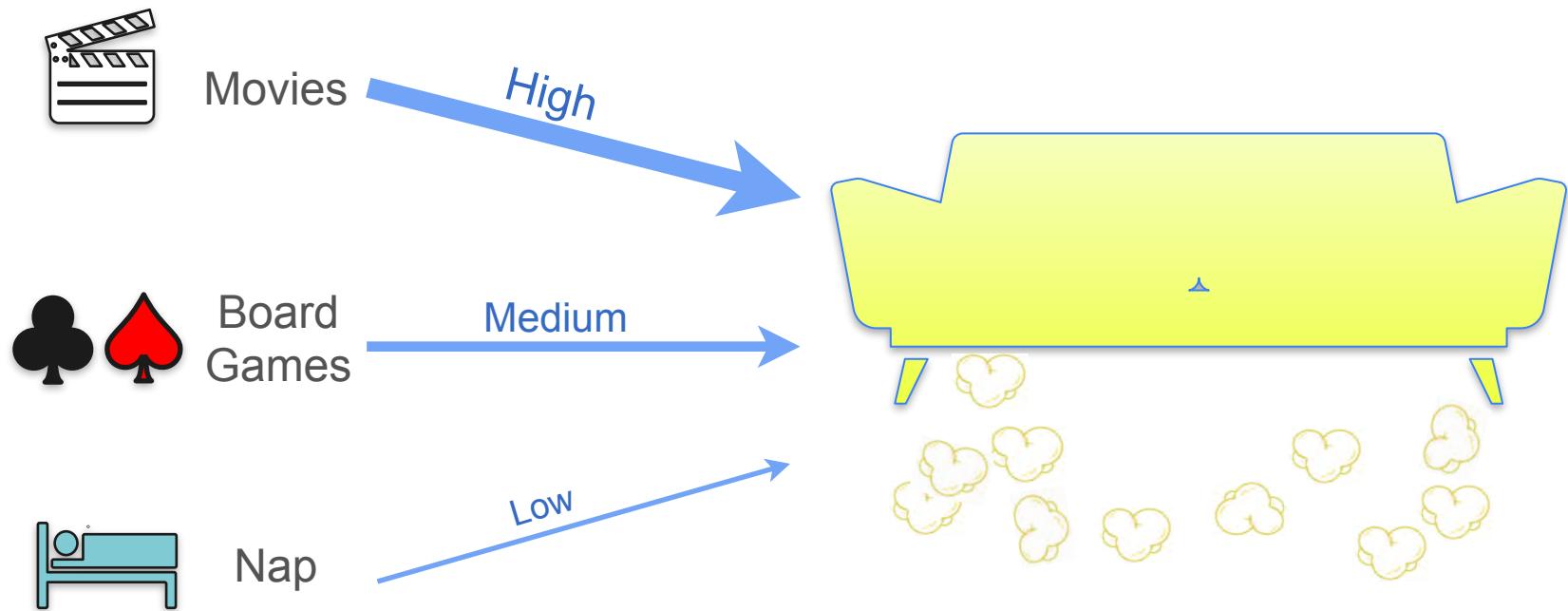
Nap



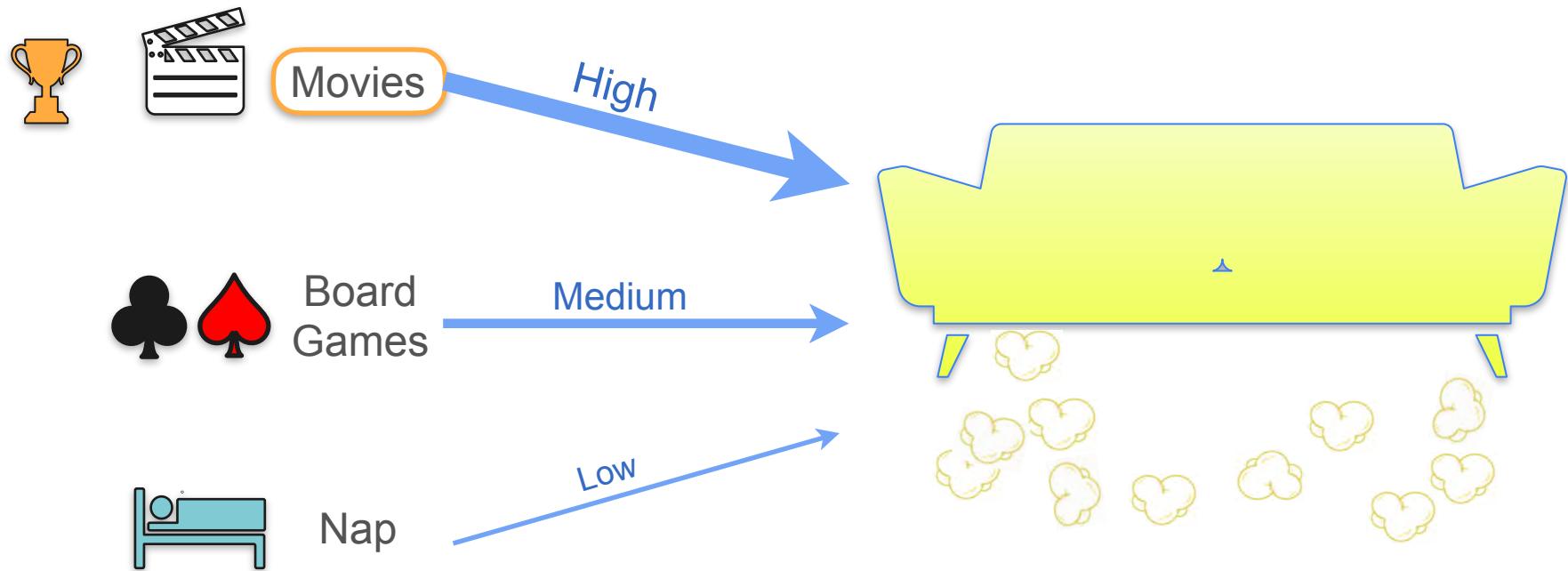
# There's Popcorn on the Floor. What Happened?



# There's Popcorn on the Floor. What Happened?



# There's Popcorn on the Floor. What Happened?



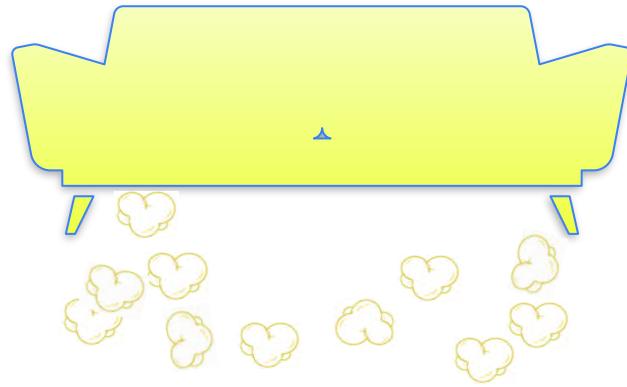
# There's Popcorn on the Floor. What Happened?



Movies



Popcorn  
throwing  
contest



# There's Popcorn on the Floor. What Happened?



Movies

*High*



Popcorn  
throwing  
contest



# There's Popcorn on the Floor. What Happened?



Movies

*High*

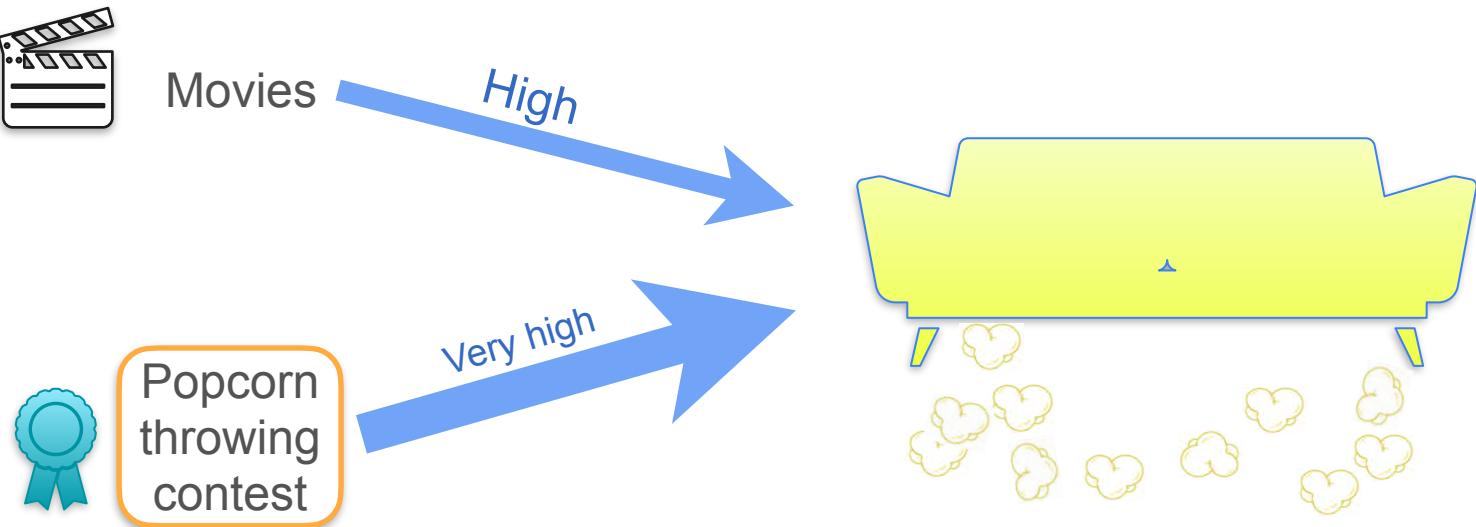


Popcorn  
throwing  
contest

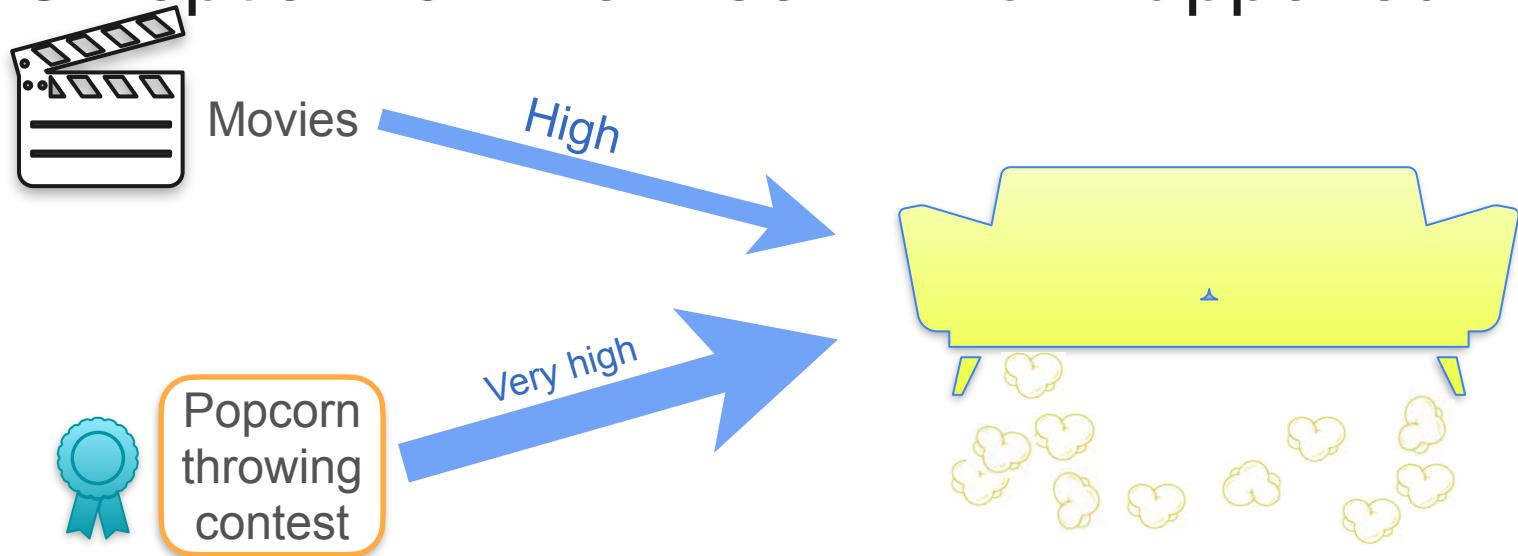
*Very high*



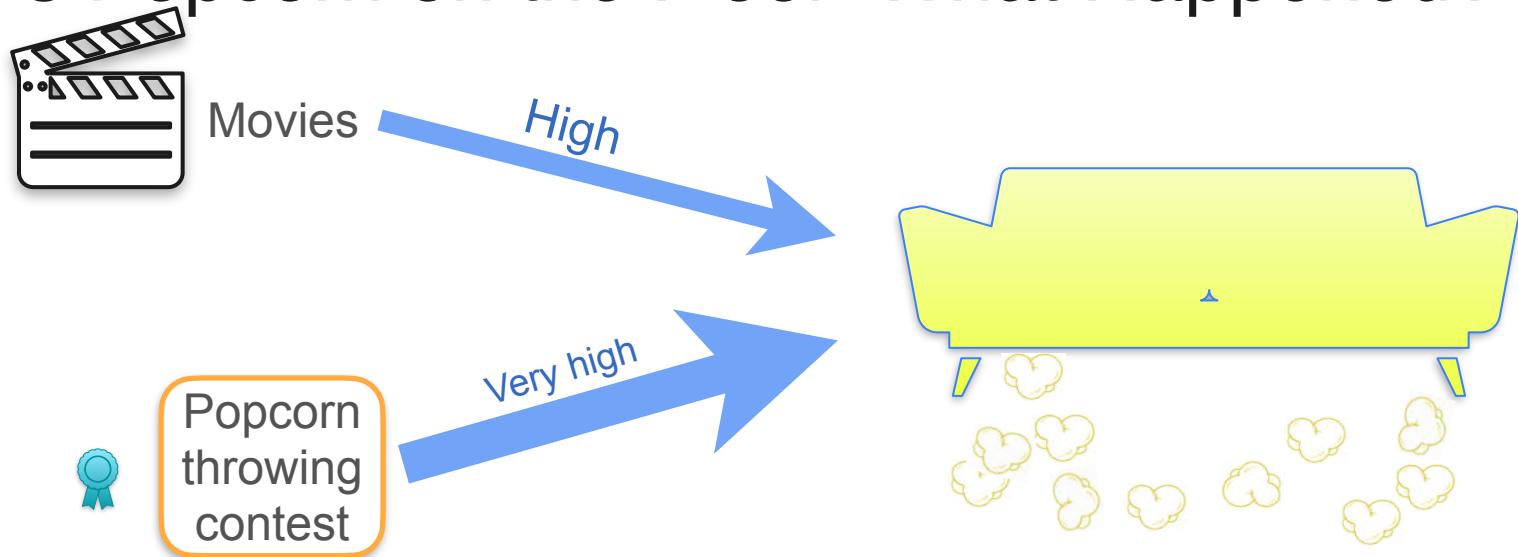
# There's Popcorn on the Floor. What Happened?



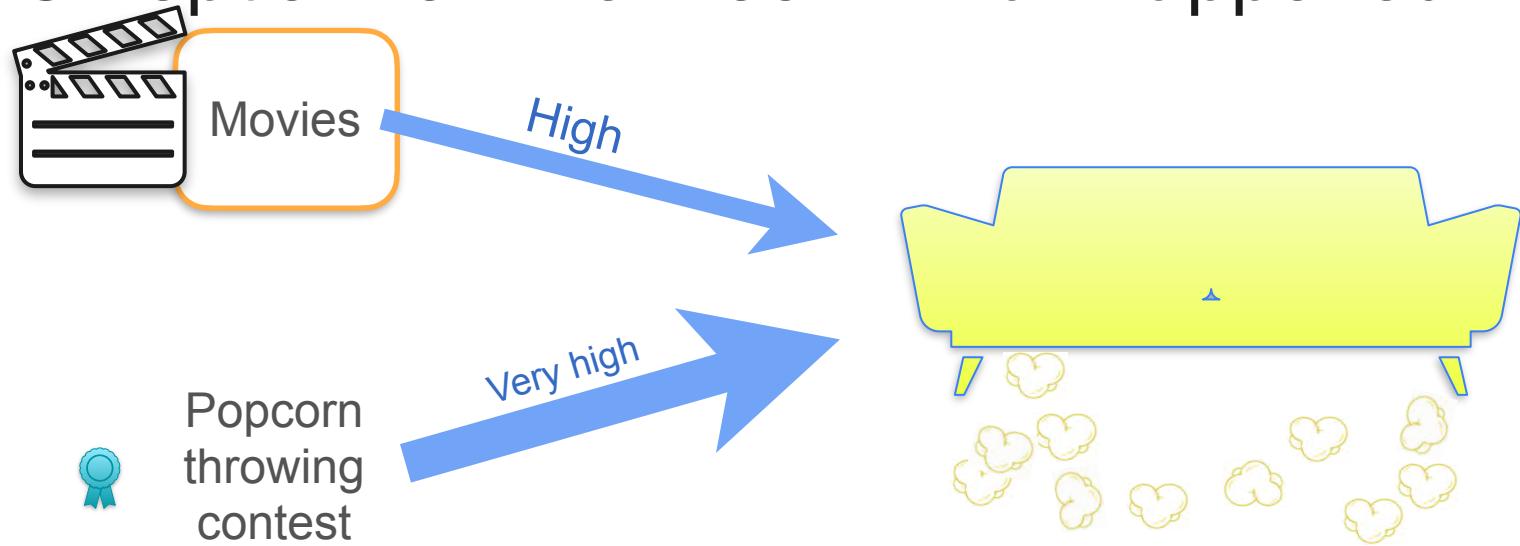
# There's Popcorn on the Floor. What Happened?



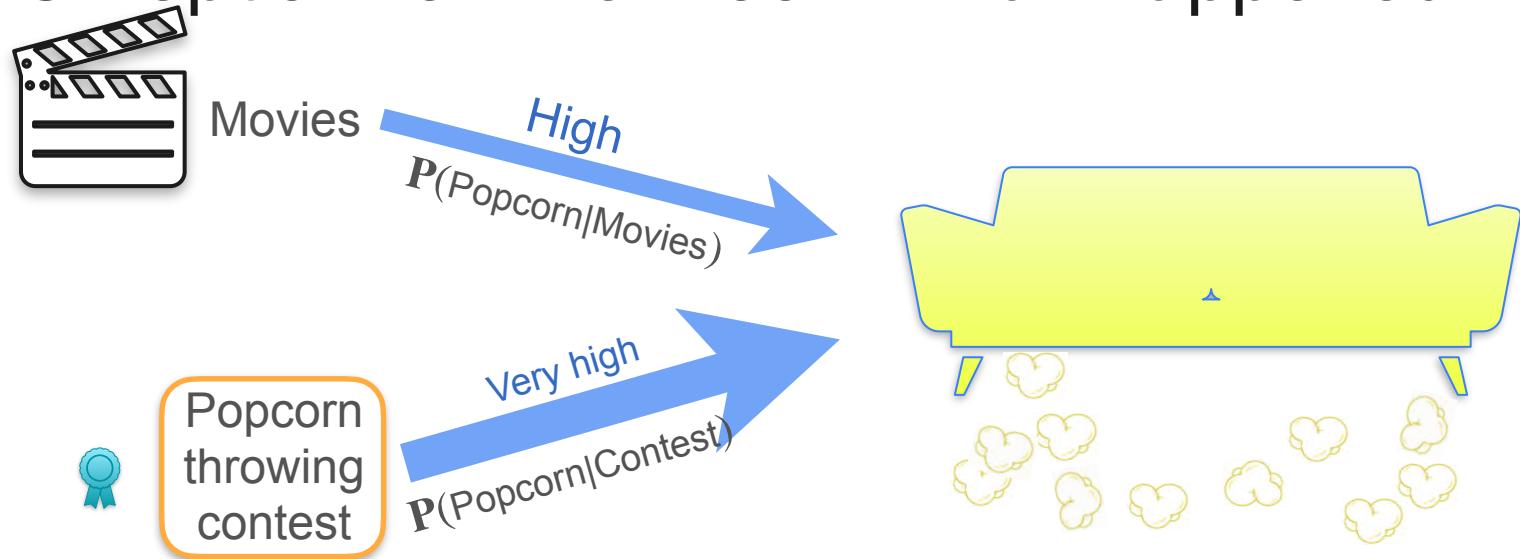
# There's Popcorn on the Floor. What Happened?



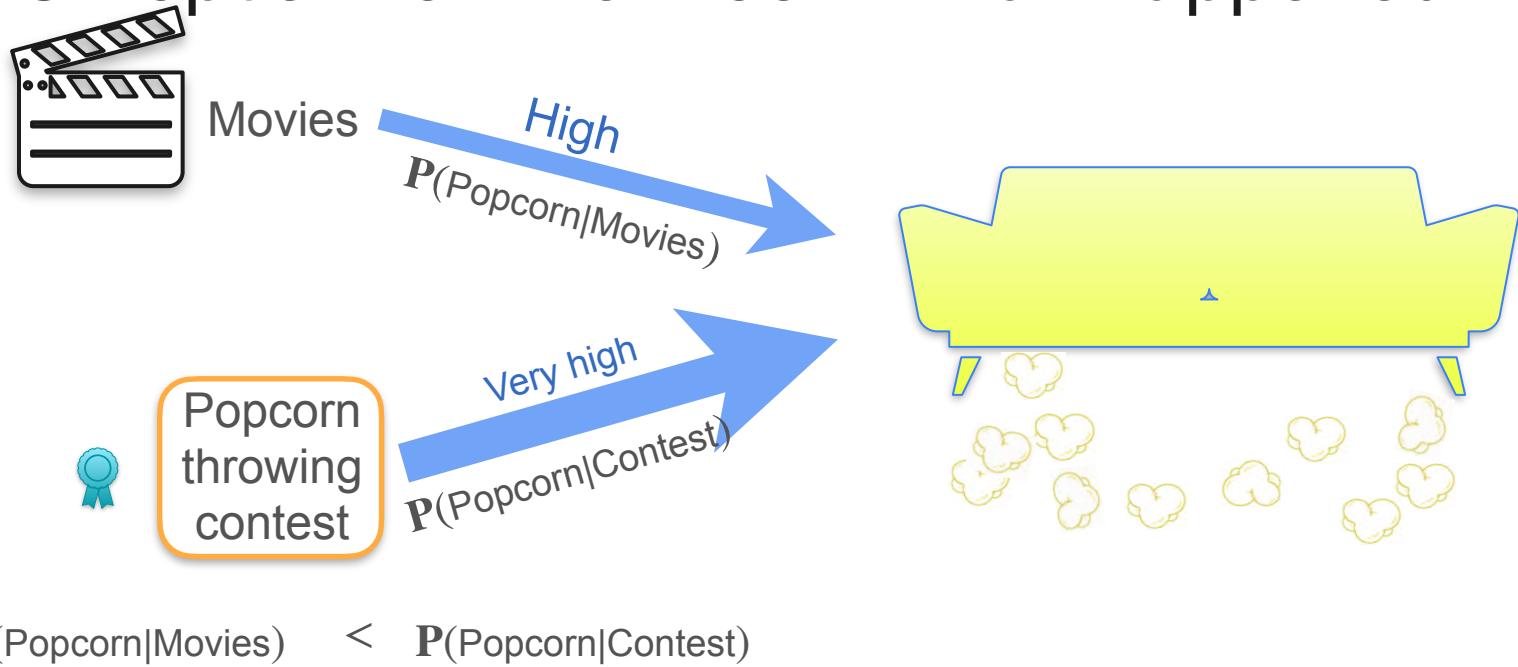
# There's Popcorn on the Floor. What Happened?



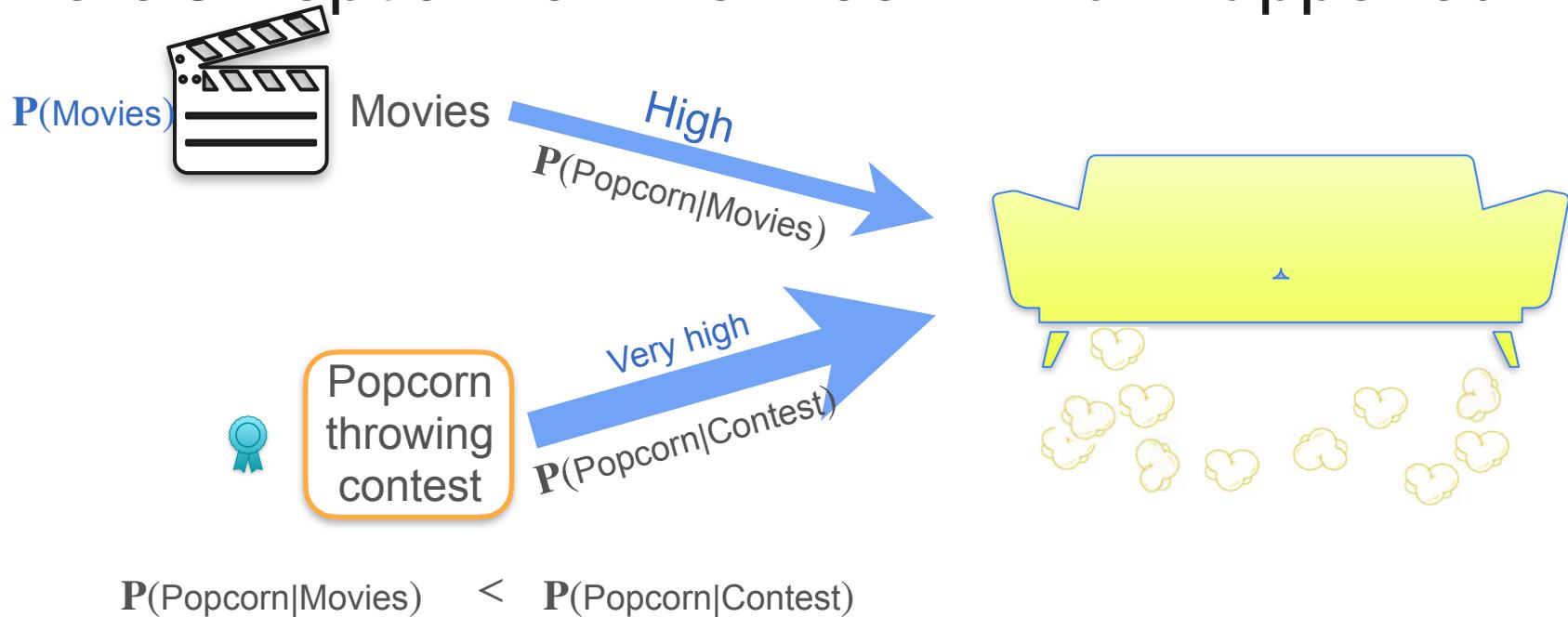
# There's Popcorn on the Floor. What Happened?



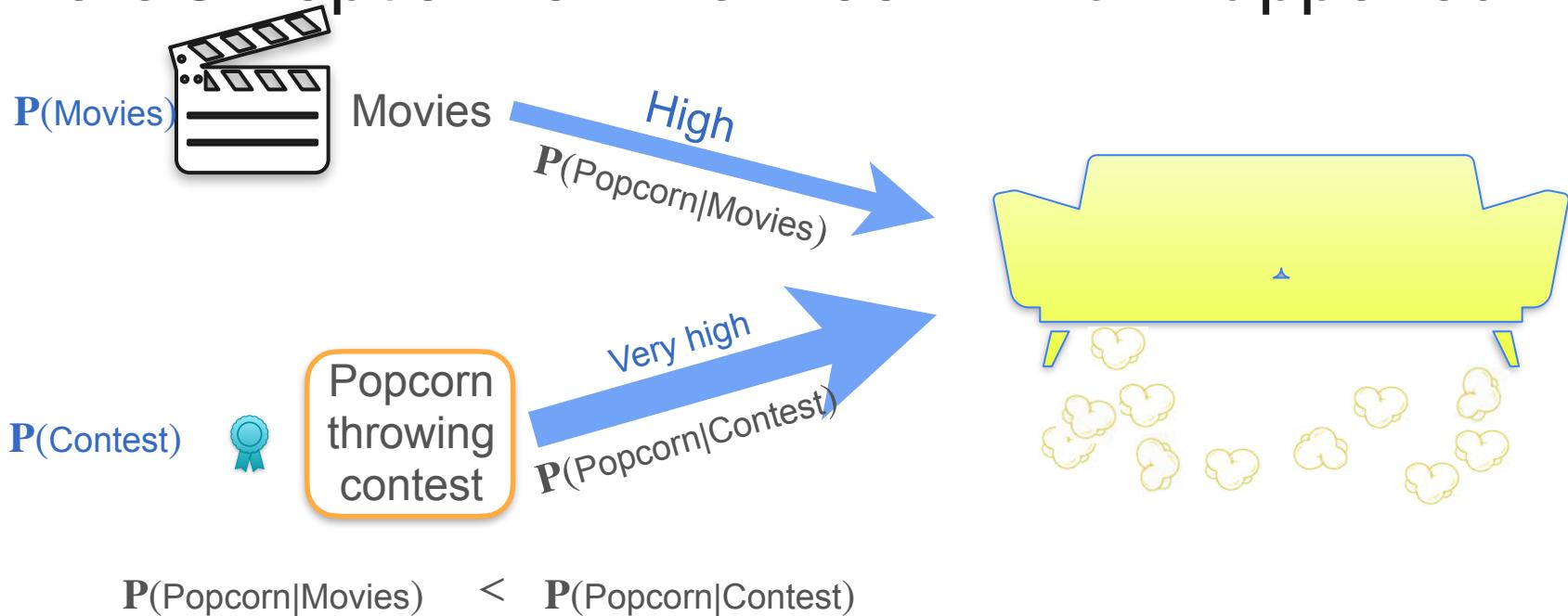
# There's Popcorn on the Floor. What Happened?



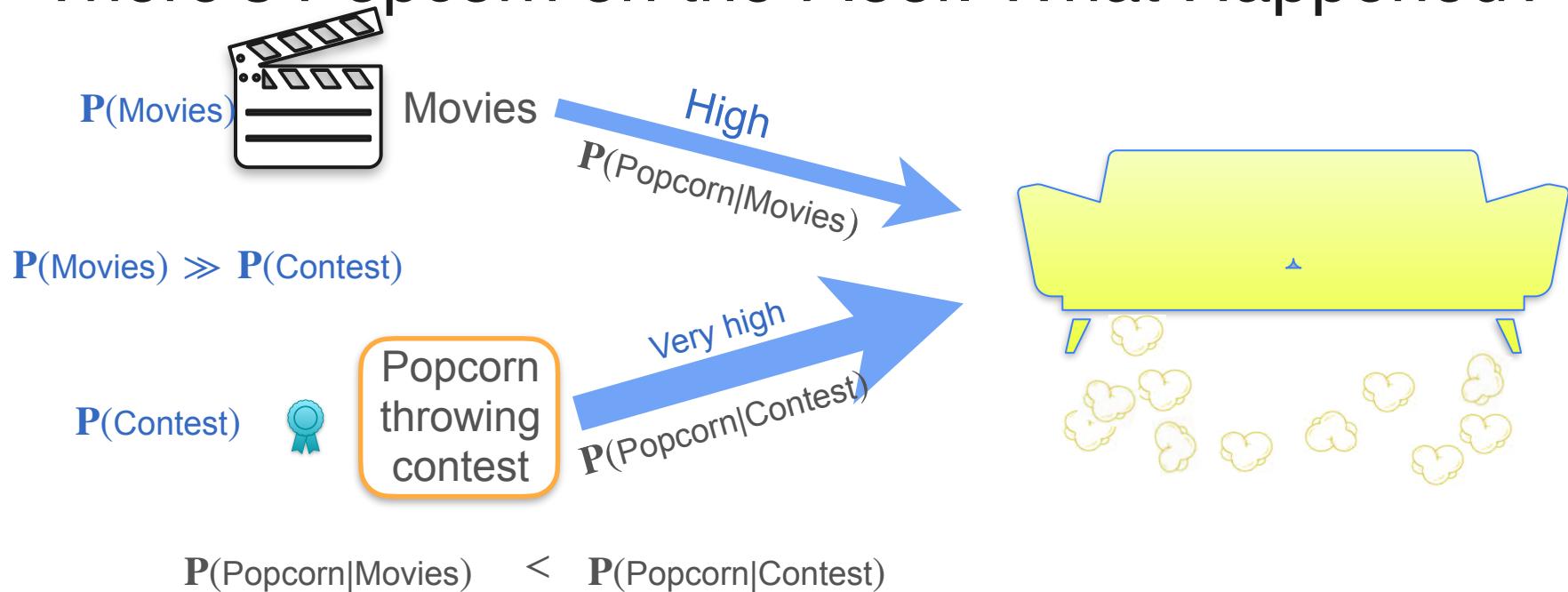
# There's Popcorn on the Floor. What Happened?



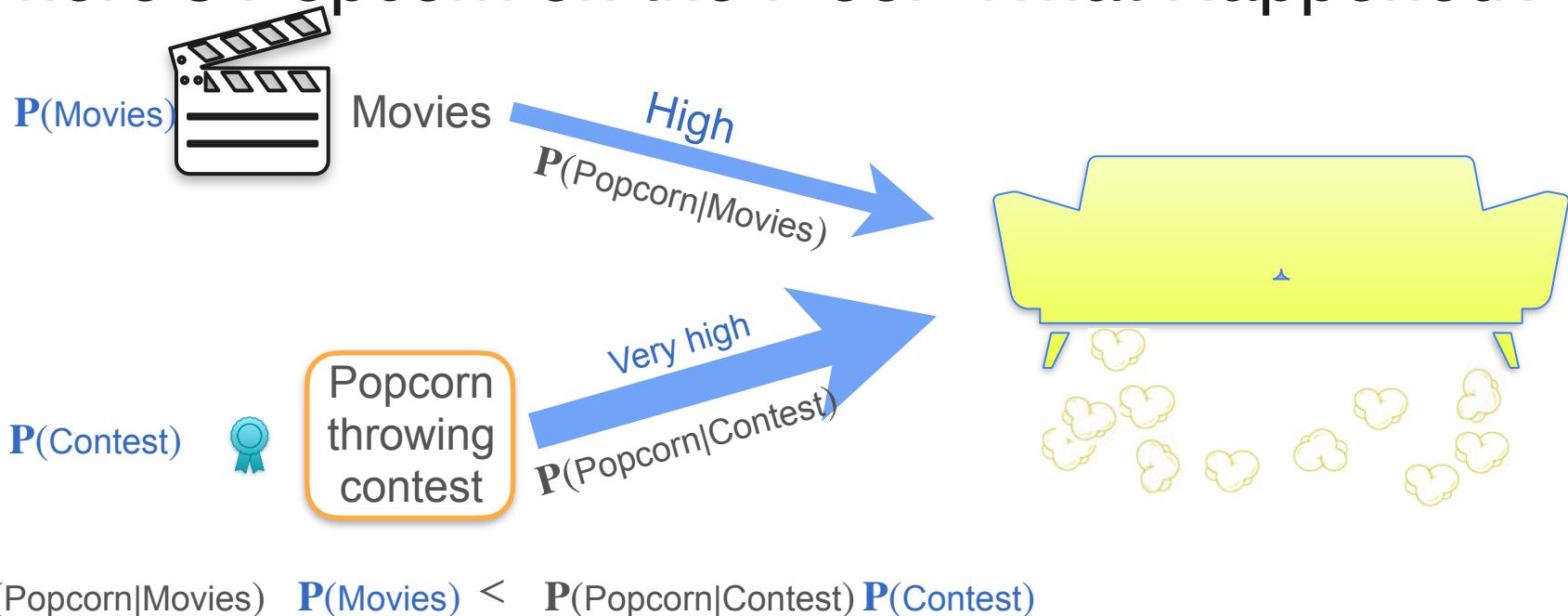
# There's Popcorn on the Floor. What Happened?



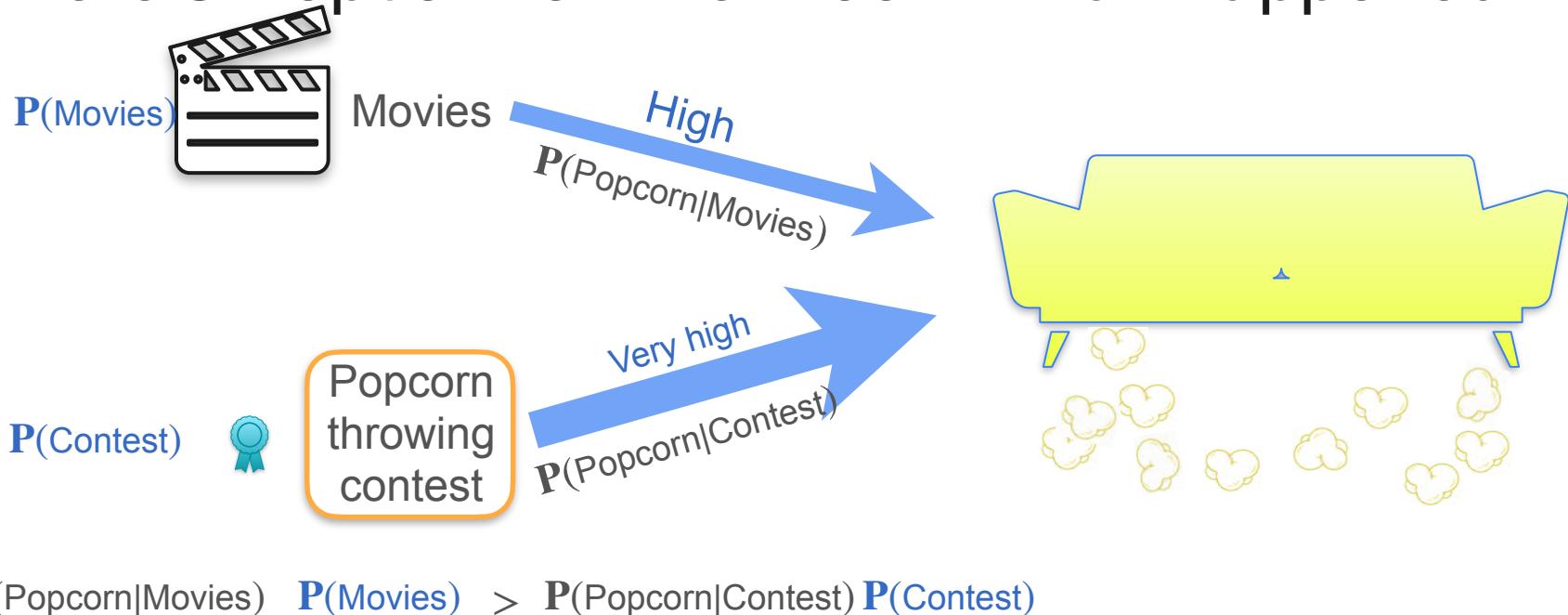
# There's Popcorn on the Floor. What Happened?



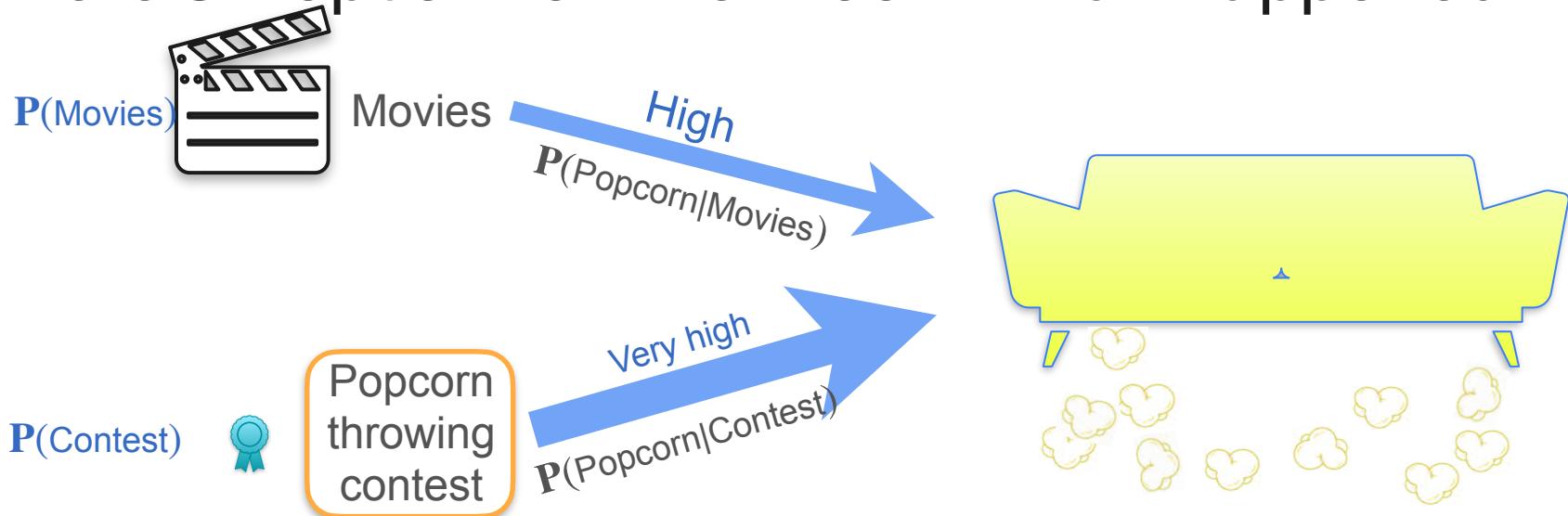
# There's Popcorn on the Floor. What Happened?



# There's Popcorn on the Floor. What Happened?



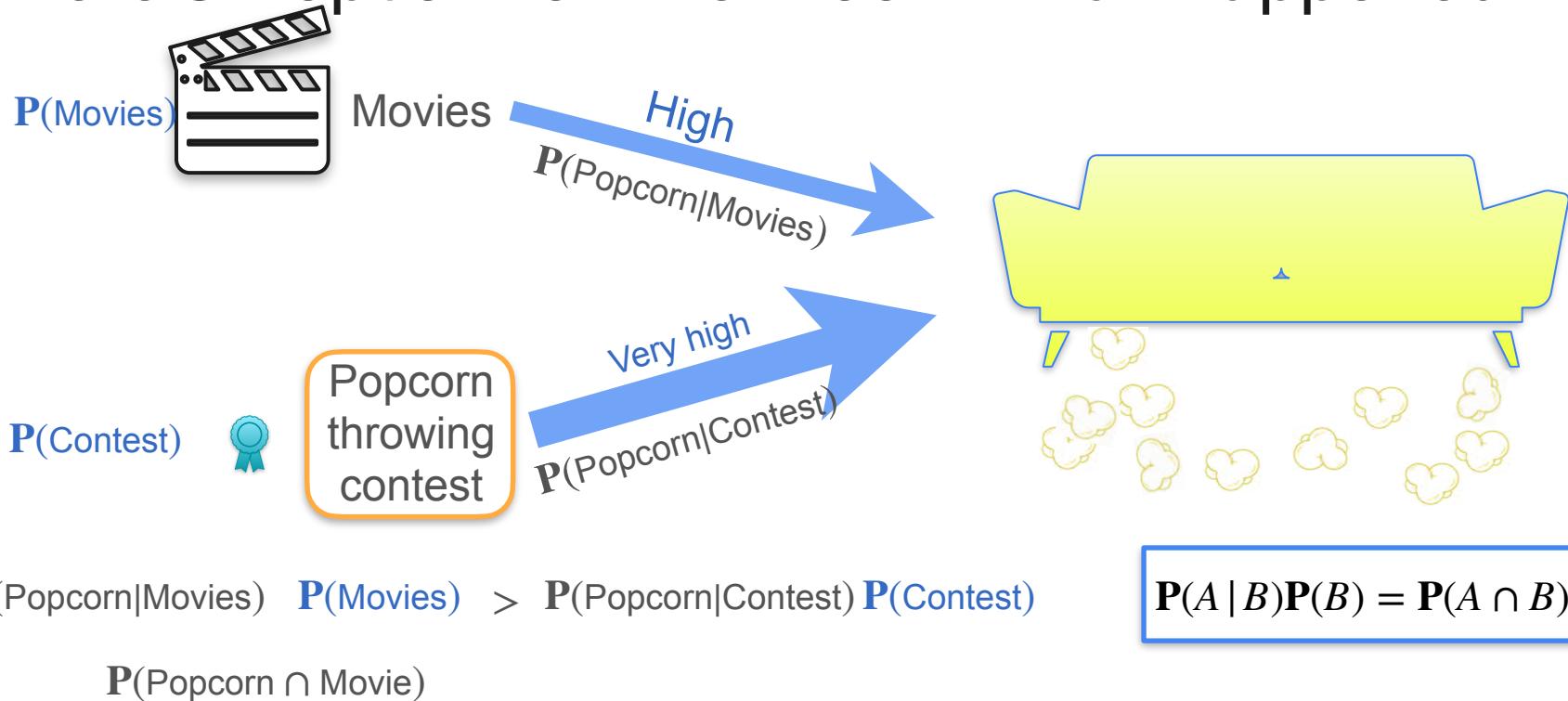
# There's Popcorn on the Floor. What Happened?



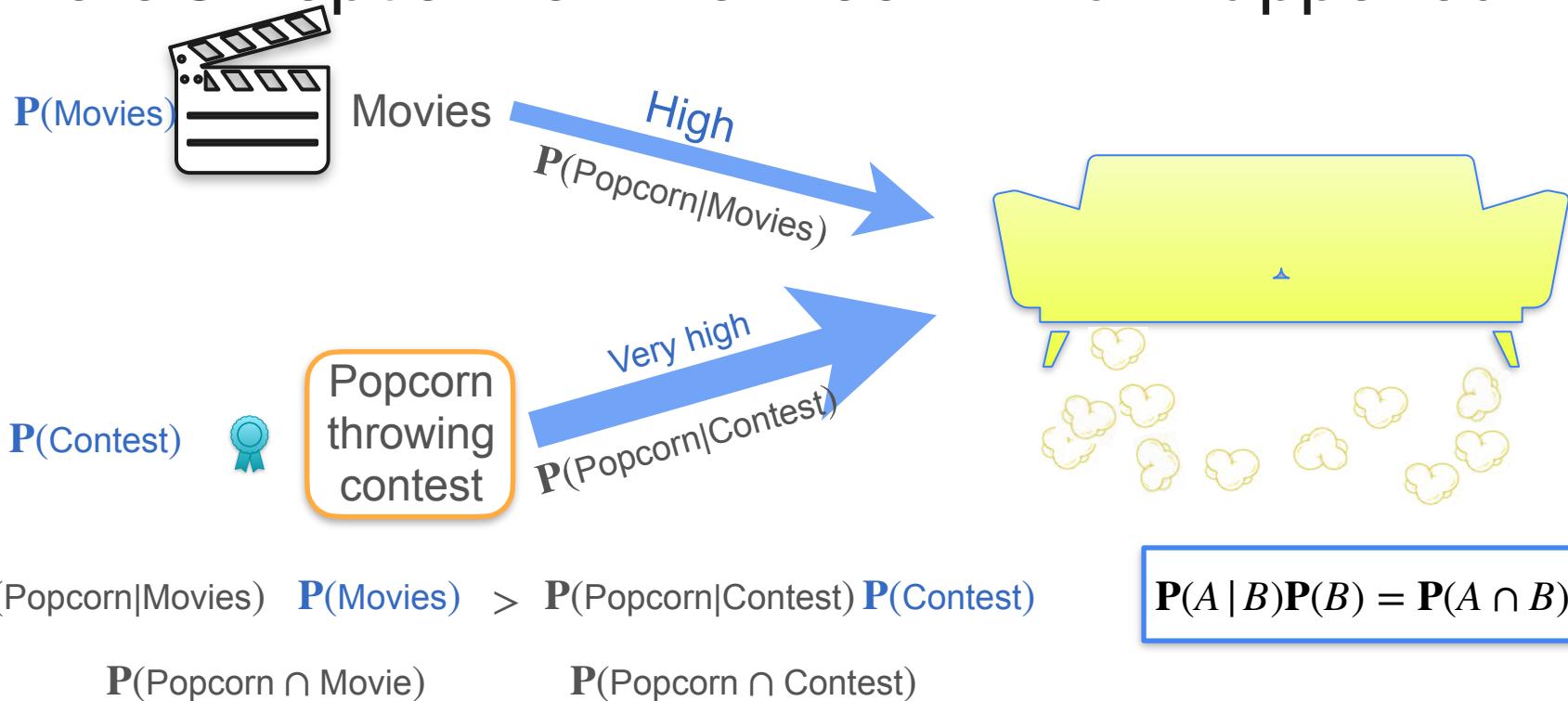
$$P(\text{Popcorn}|\text{Movies}) P(\text{Movies}) > P(\text{Popcorn}|\text{Contest}) P(\text{Contest})$$

$$P(A | B)P(B) = P(A \cap B)$$

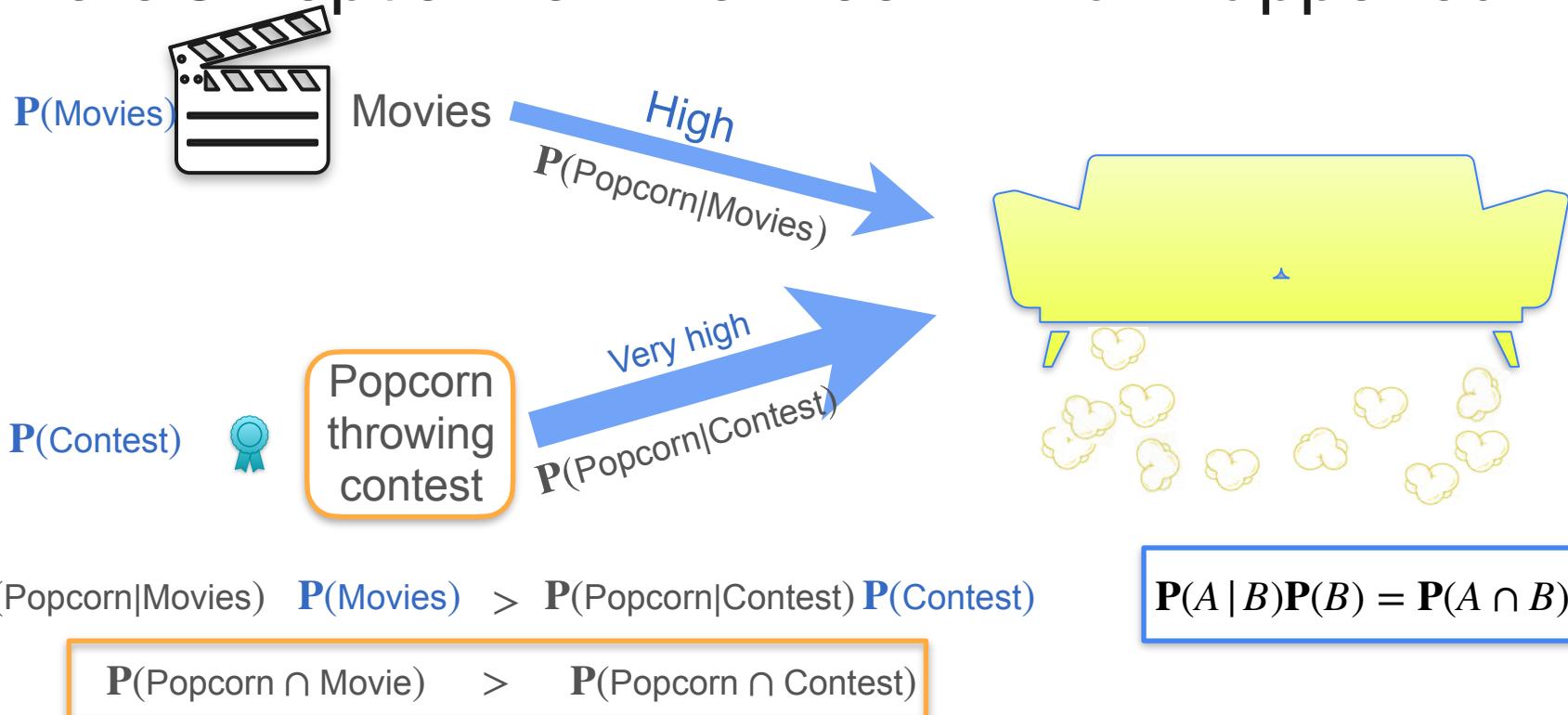
# There's Popcorn on the Floor. What Happened?



# There's Popcorn on the Floor. What Happened?



# There's Popcorn on the Floor. What Happened?



# What Does This Have To Do With Regularization?



DeepLearning.AI

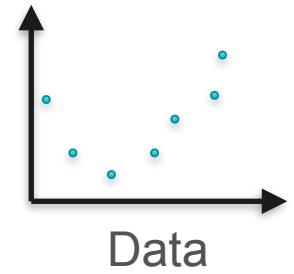
## Point Estimation

---

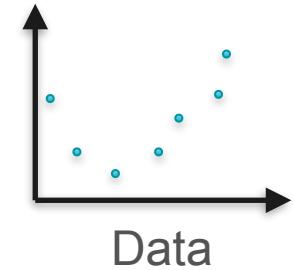
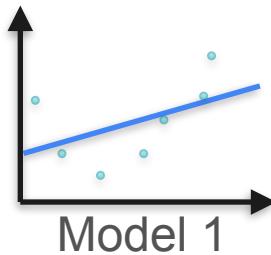
**Bayes Theorem and  
Regularization**

# Example: Polynomial Regression

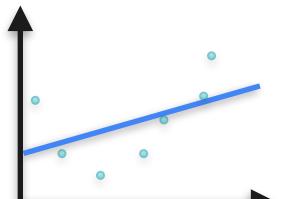
# Example: Polynomial Regression



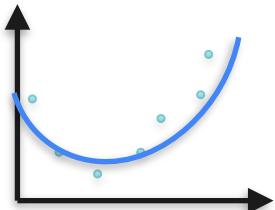
# Example: Polynomial Regression



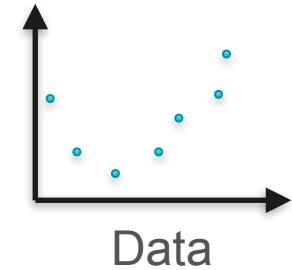
# Example: Polynomial Regression



Model 1

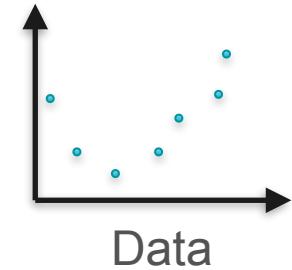
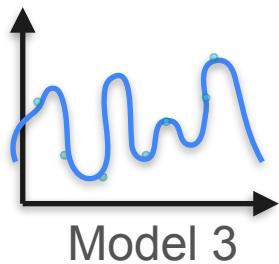
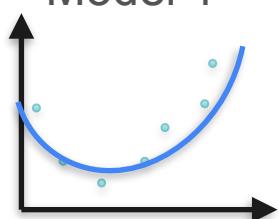
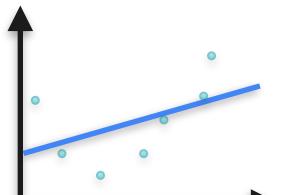


Model 2

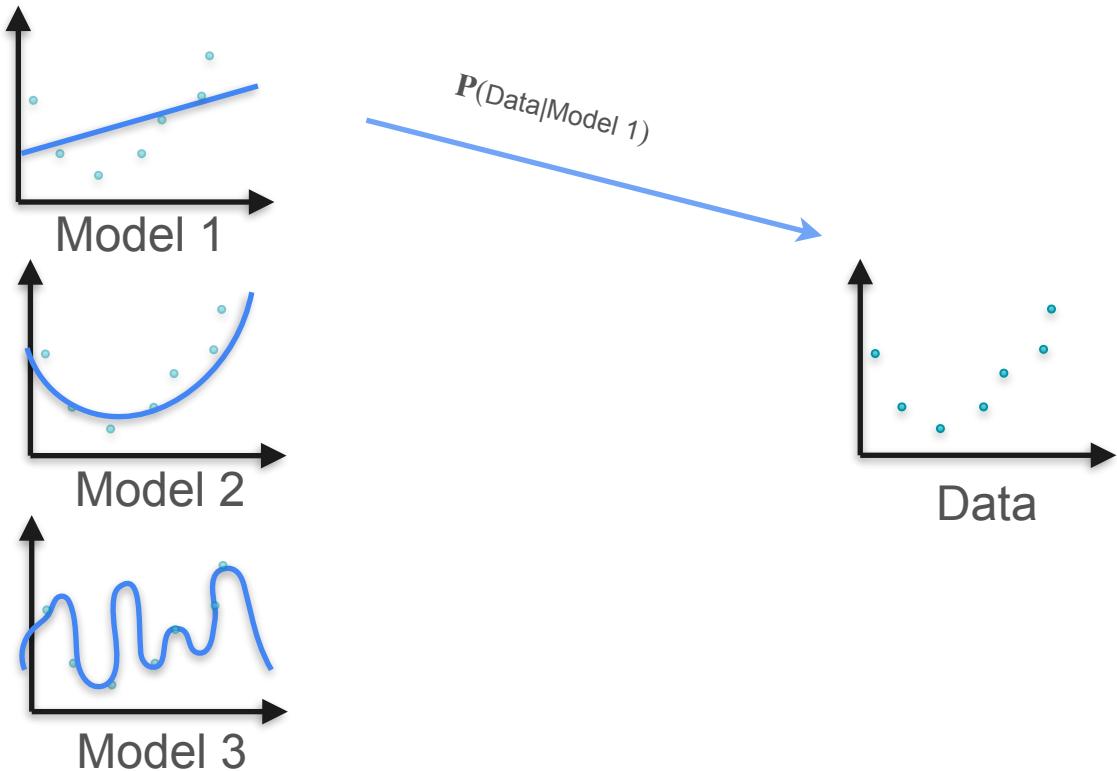


Data

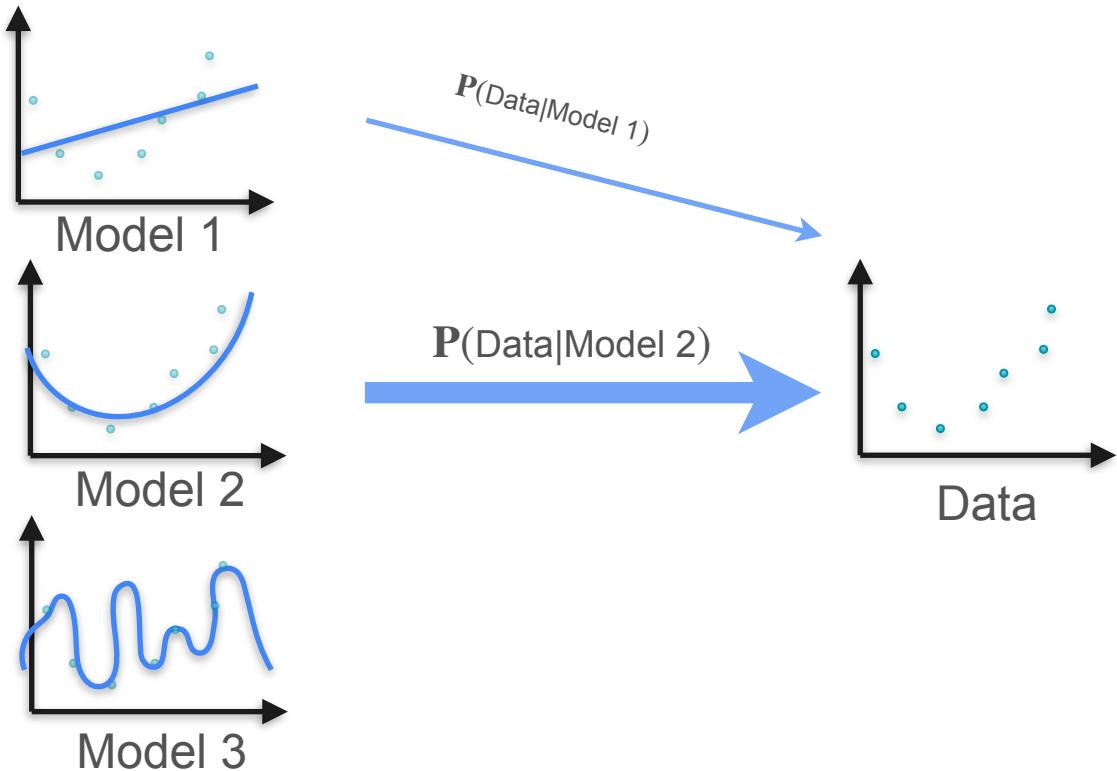
# Example: Polynomial Regression



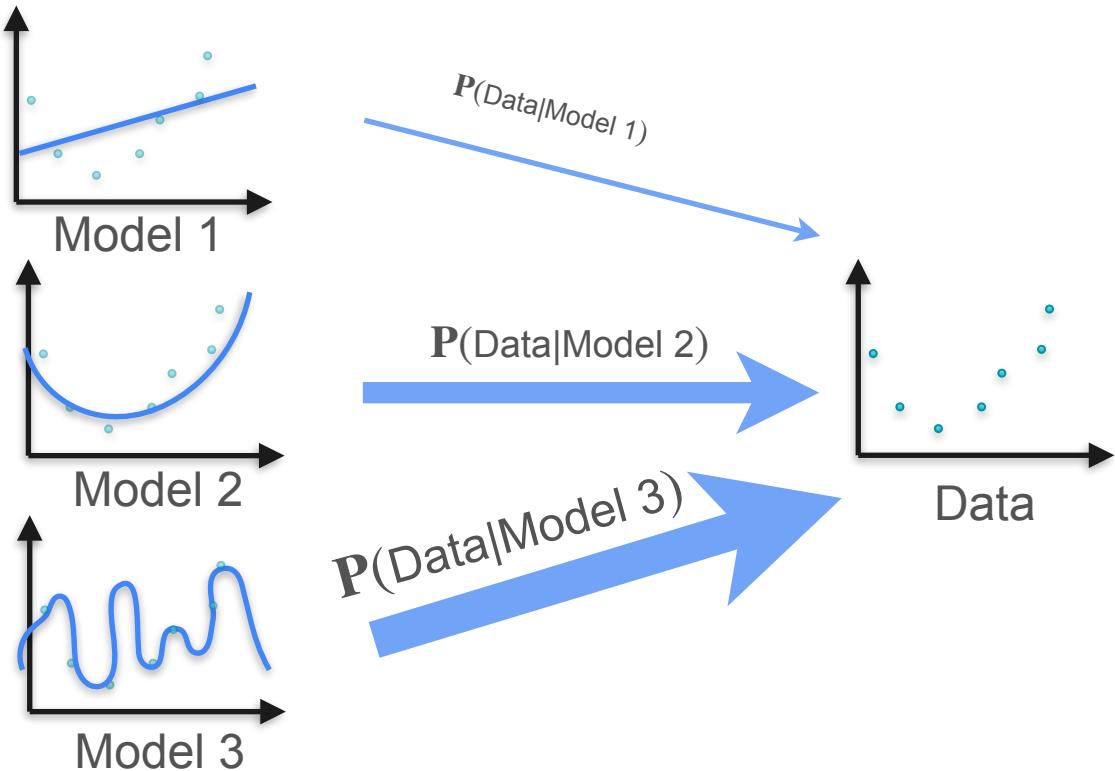
# Example: Polynomial Regression



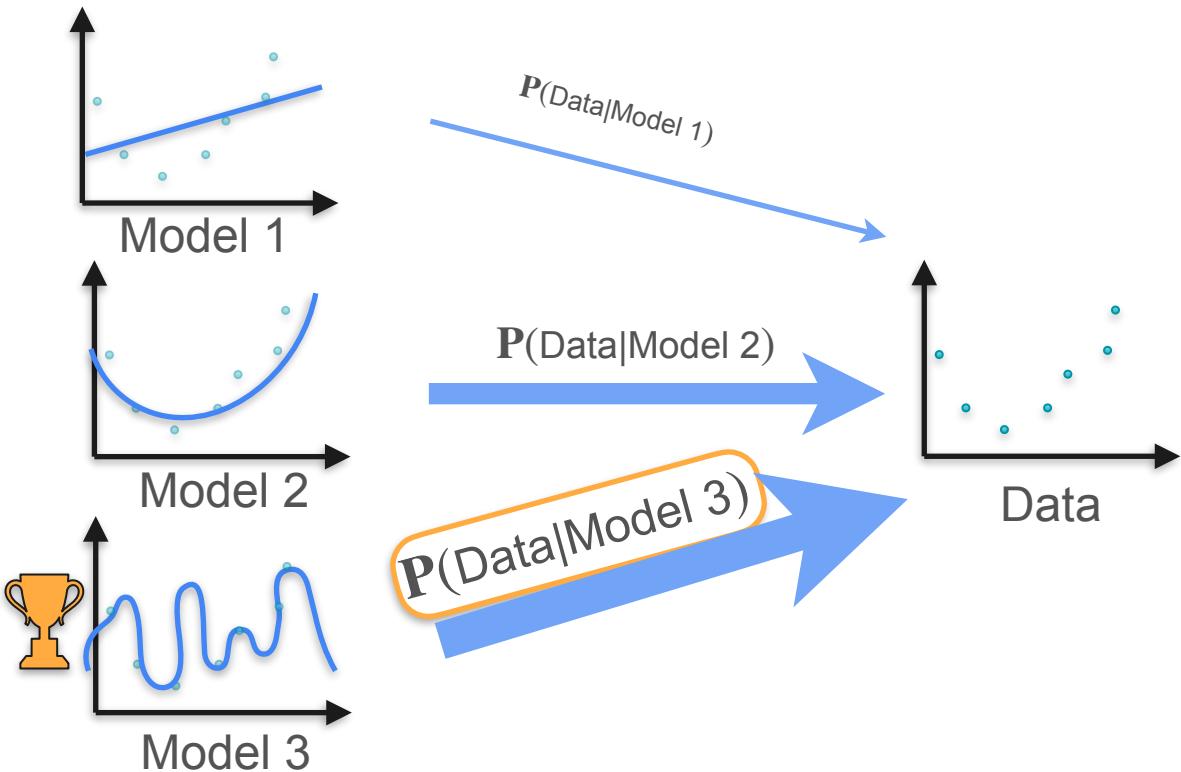
# Example: Polynomial Regression



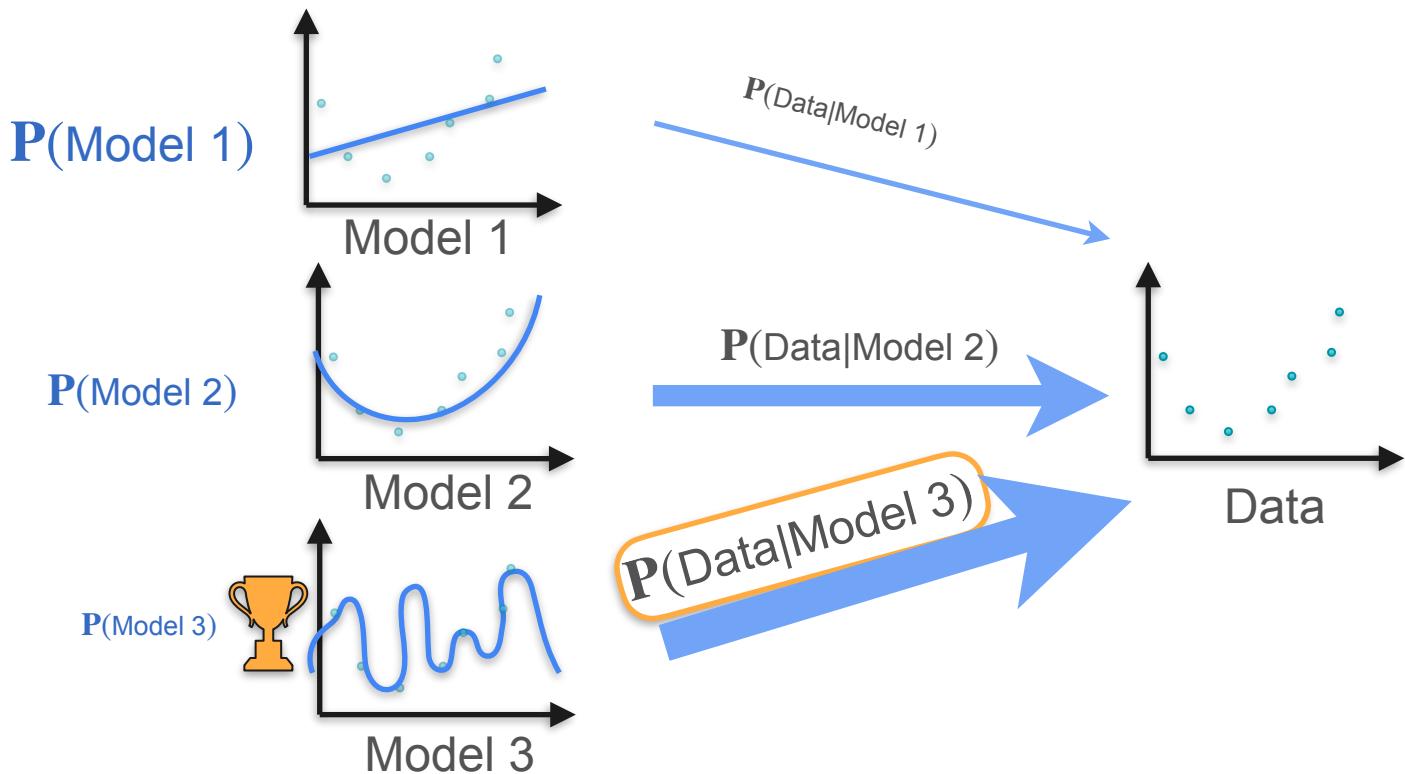
# Example: Polynomial Regression



# Example: Polynomial Regression

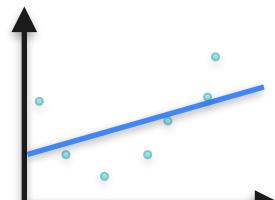


# Example: Polynomial Regression

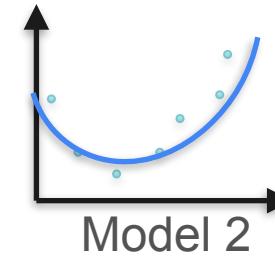


# Example: Polynomial Regression

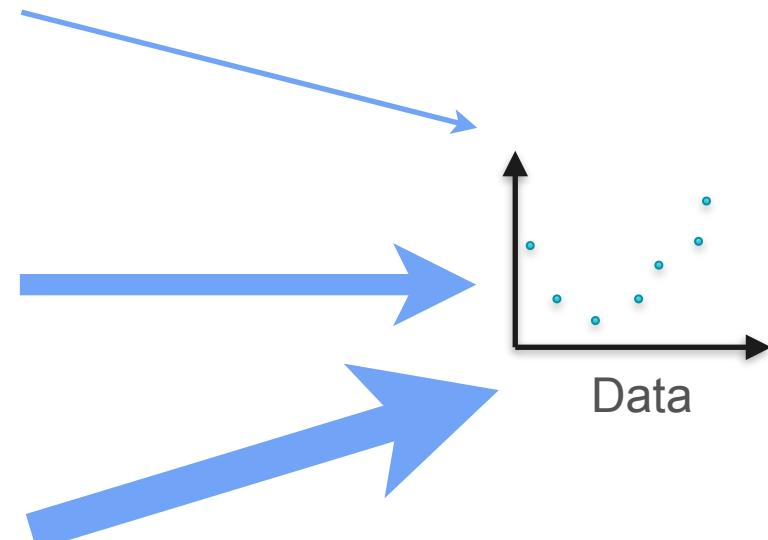
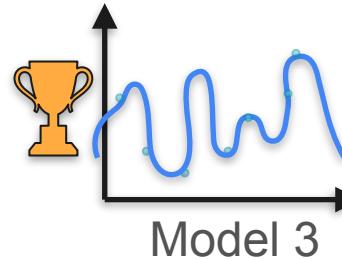
$P(\text{Model 1}) P(\text{Data}|\text{Model 1})$



$P(\text{Model 2}) P(\text{Data}|\text{Model 2})$

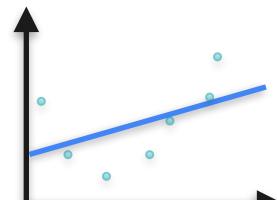


$P(\text{Model 3}) P(\text{Data}|\text{Model 3})$

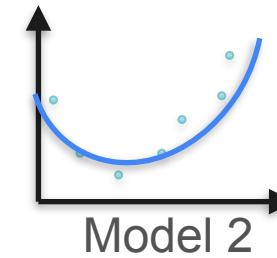


# Example: Polynomial Regression

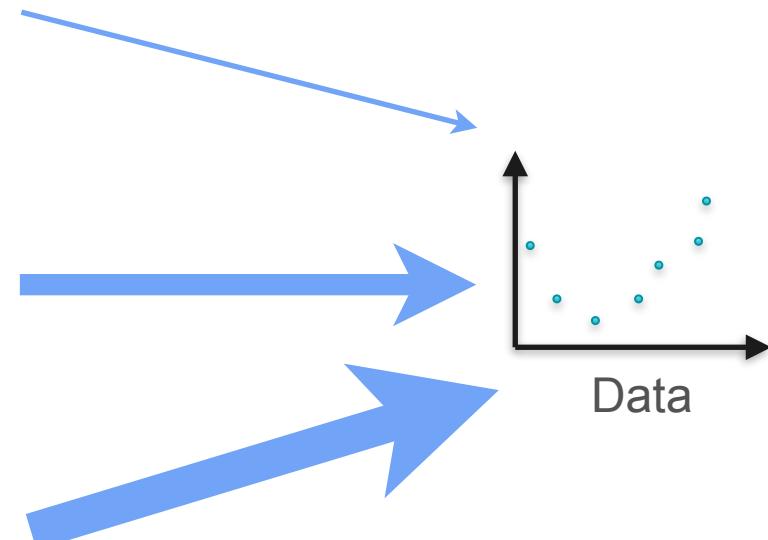
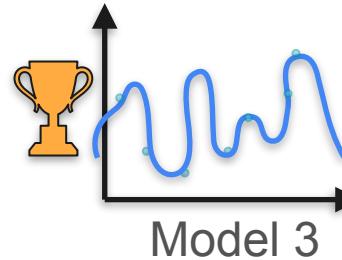
$P(\text{Model 1}) P(\text{Data}|\text{Model 1})$



$P(\text{Model 2}) P(\text{Data}|\text{Model 2})$

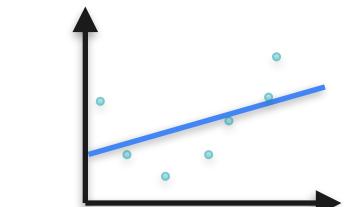


$P(\text{Model 3}) P(\text{Data}|\text{Model 3})$

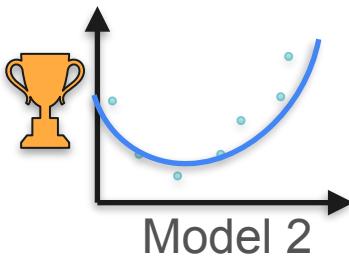


# Example: Polynomial Regression

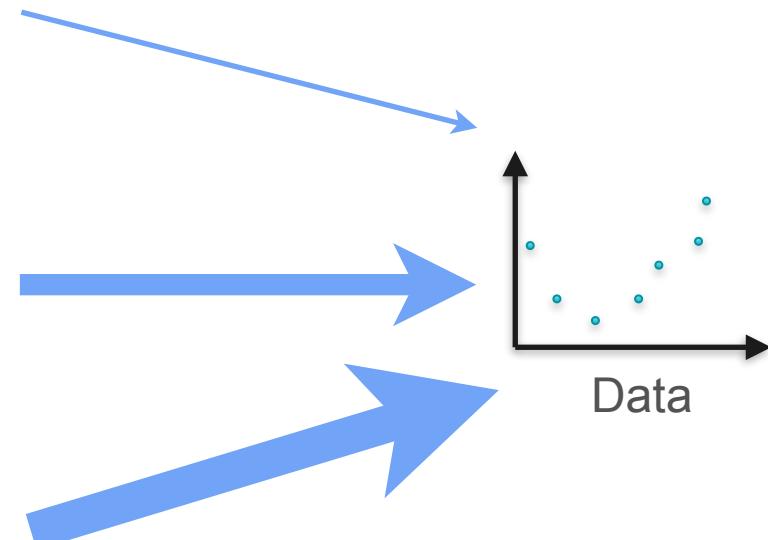
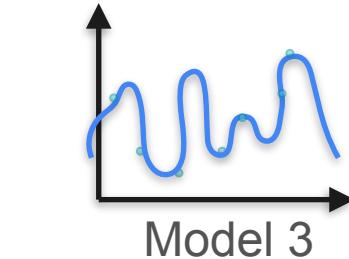
$P(\text{Model 1}) P(\text{Data}|\text{Model 1})$



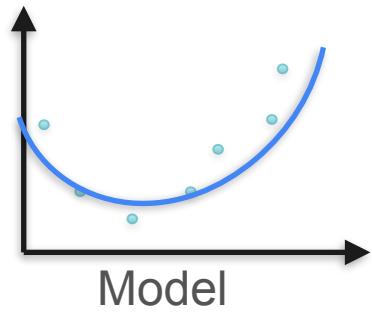
$P(\text{Model 2}) P(\text{Data}|\text{Model 2})$



$P(\text{Model 3}) P(\text{Data}|\text{Model 3})$

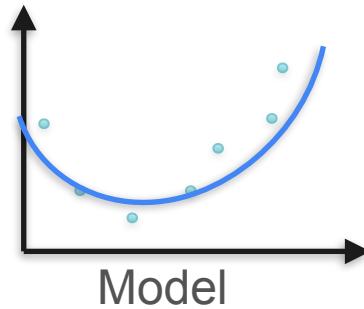


**Maximum likelihood**



**Polynomial regression**

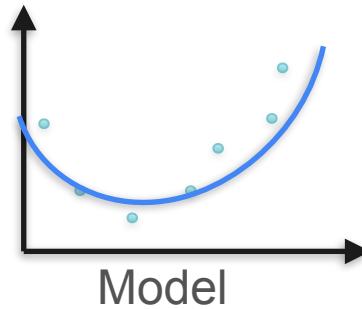
**Maximum likelihood**



**Polynomial regression**

**Log-loss**

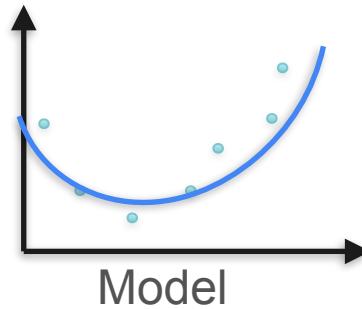
**Maximum likelihood**



**Polynomial regression**

Log-loss

**Maximum likelihood  
with Bayes**

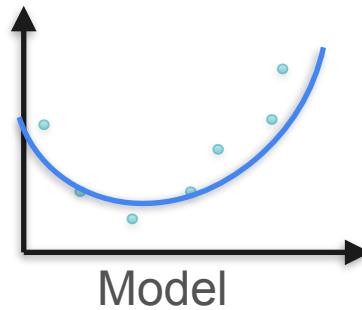


**Polynomial regression**

$P(\text{Data}|\text{Model})$

Log-loss

**Maximum likelihood  
with Bayes**



**Polynomial regression**

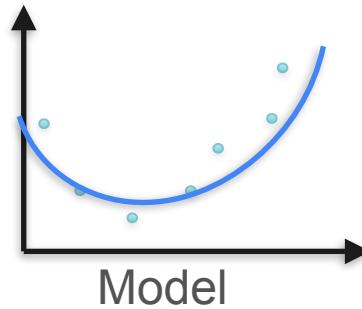
$P(\text{Data}|\text{Model})$

Log-loss

•

**$P(\text{Model})$**

**Maximum likelihood  
with Bayes**



**Polynomial regression  
with regularization**

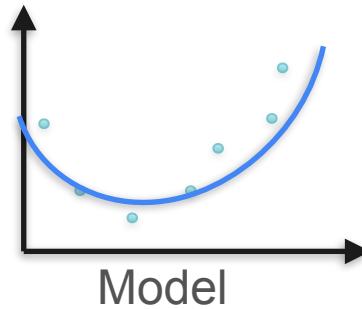
$P(\text{Data}|\text{Model})$

Log-loss

•

**$P(\text{Model})$**

**Maximum likelihood  
with Bayes**



**Polynomial regression  
with regularization**

$P(\text{Data}|\text{Model})$

•

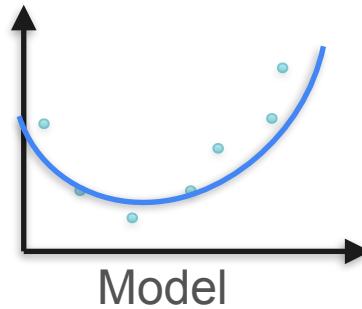
**$P(\text{Model})$**

Log-loss



Regularization term

Maximum likelihood  
with Bayes



Polynomial regression  
with regularization

$P(\text{Data}|\text{Model})$

.

$P(\text{Model})$

Take logarithms!

Log-loss

+

Regularization term

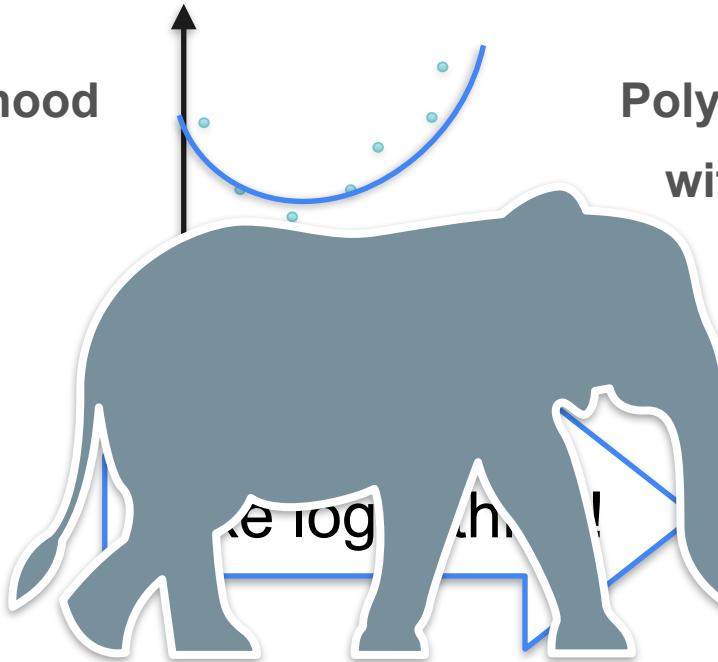
Maximum likelihood  
with Bayes

Polynomial regression  
with regularization

$P(\text{Data}|\text{Model})$

•

$P(\text{Model})$



Log-loss

+

Regularization term

Maximum likelihood  
with Bayes

Polynomial regression  
with regularization

$P(\text{Data}|\text{Model})$

$P(\text{Model})$

?

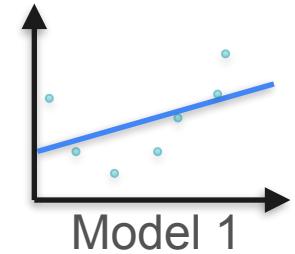
Log-loss

+

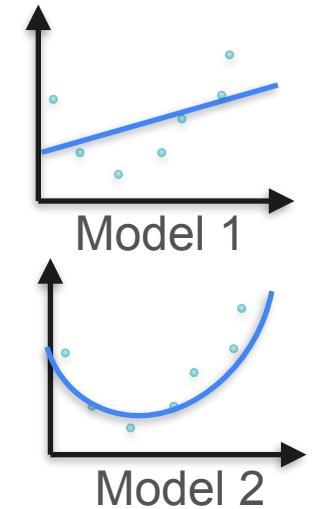
Regularization term

# What Is the Probability of a Model?

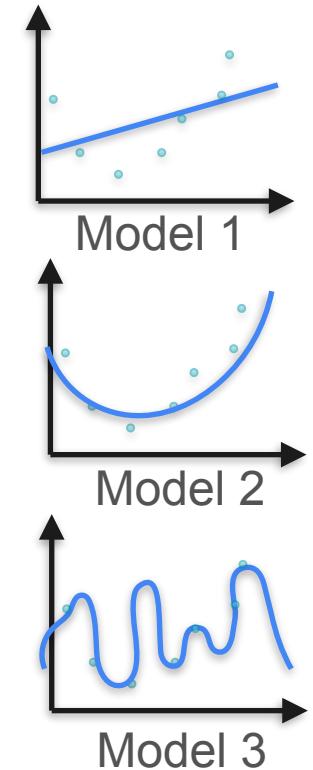
# What Is the Probability of a Model?



# What Is the Probability of a Model?



# What Is the Probability of a Model?

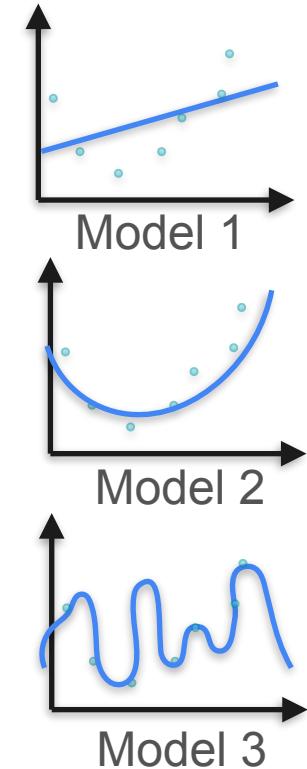


# What Is the Probability of a Model?

$P(\text{Model 1})$

$P(\text{Model 2})$

$P(\text{Model 3})$



# What Is the Probability of a Model?

$P(\text{Model 1})$

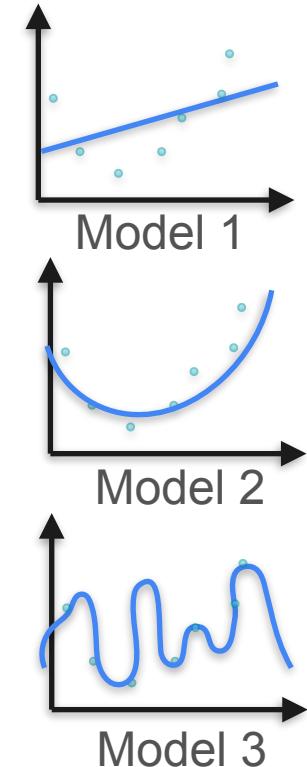
$$a_1x + b$$

$P(\text{Model 2})$

$$a_1x^2 + a_2x + b$$

$P(\text{Model 3})$

$$a_1x^{10} + a_2x^9 + \cdots + a_{10}x + b$$



# What Is the Probability of a Model?

$P(\text{Model 1})$

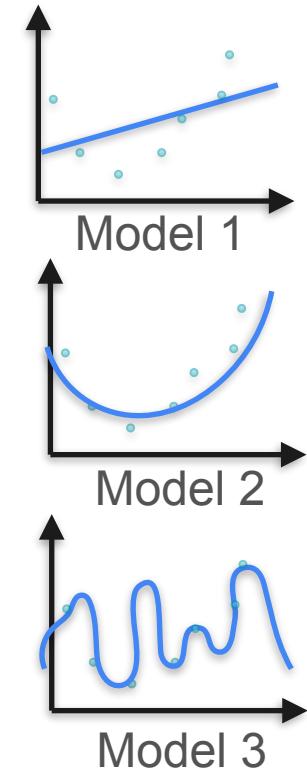
$$a_1x + b$$

$P(\text{Model 2})$

$$a_1x^2 + a_2x + b$$

$P(\text{Model 3})$

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

P(Model 1)

$$a_1x + b$$

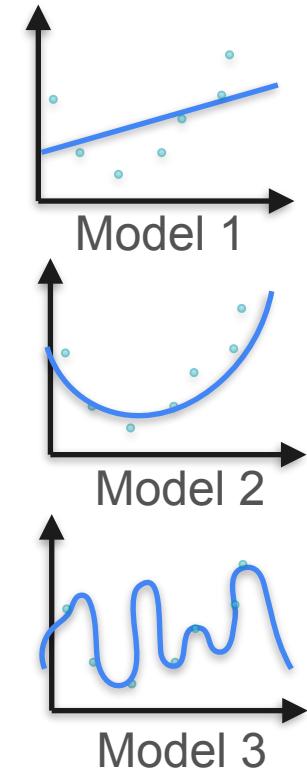
P(Model 2)

$$a_1x^2 + a_2x + b$$

P(Model 3)

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$

$$a_i \sim \mathcal{N}(0,1)$$



# What Is the Probability of a Model?

$P(\text{Model 1})$

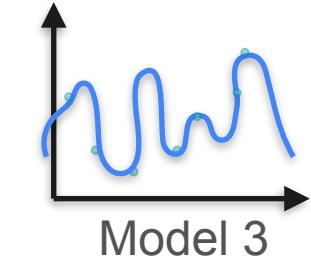
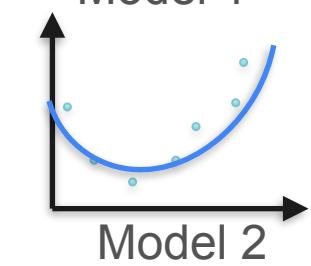
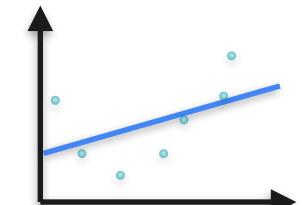
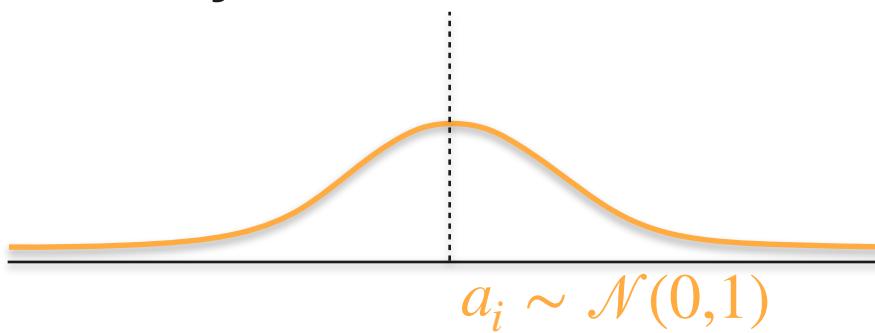
$$a_1x + b$$

$P(\text{Model 2})$

$$a_1x^2 + a_2x + b$$

$P(\text{Model 3})$

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

P(Model 1)

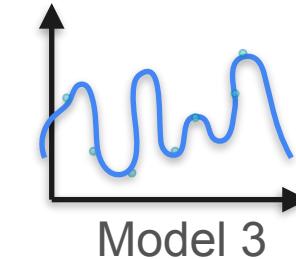
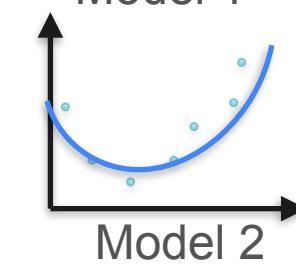
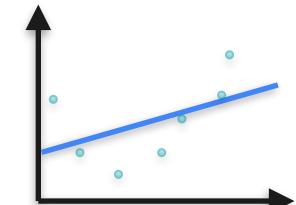
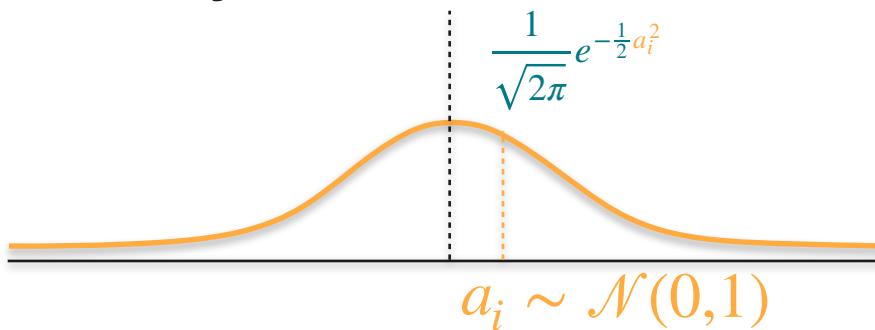
$$a_1x + b$$

P(Model 2)

$$a_1x^2 + a_2x + b$$

P(Model 3)

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

$$P(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

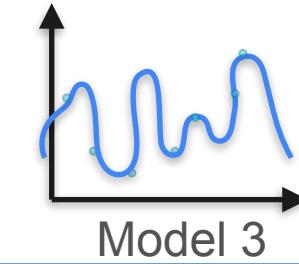
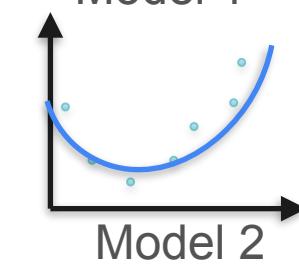
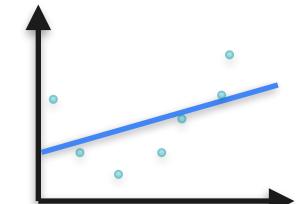
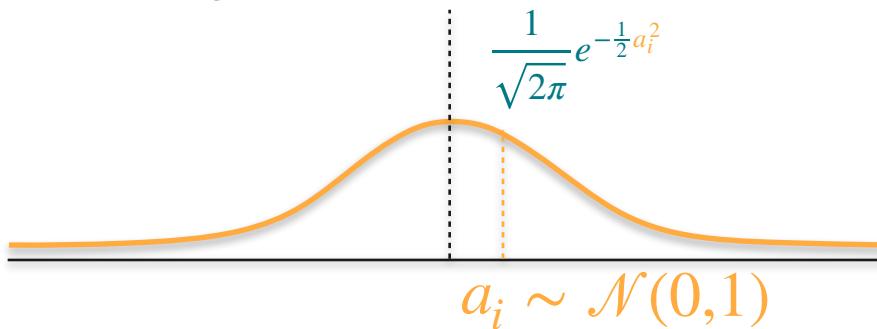
$a_1x + b$

P(Model 2)

$$a_1x^2 + a_2x + b$$

P(Model 3)

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

$$P(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

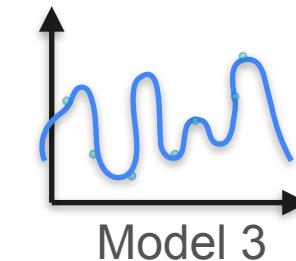
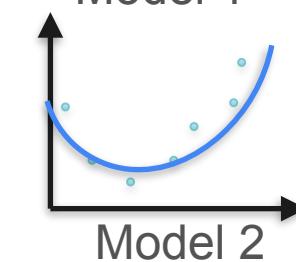
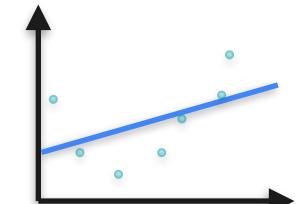
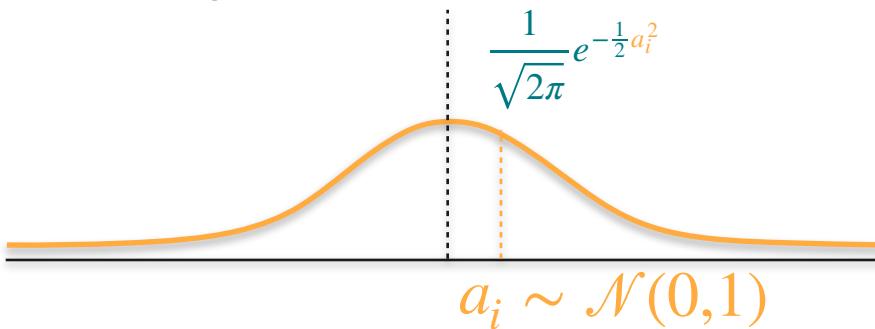
$a_1x + b$

$P(\text{Model 2})$

$$a_1x^2 + a_2x + b$$

$P(\text{Model 3})$

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

$$P(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

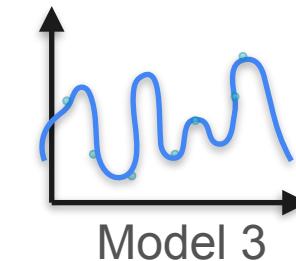
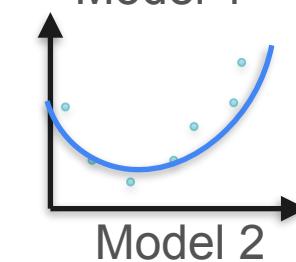
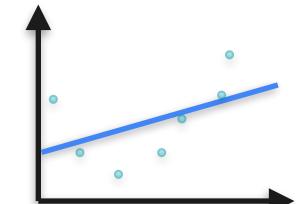
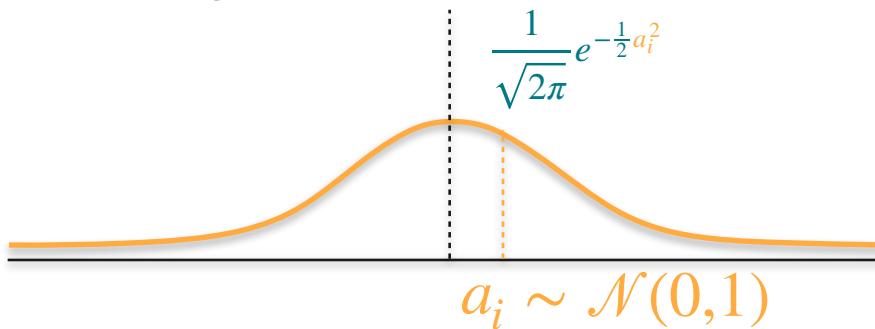
$a_1x + b$

P(Model 2)

$$a_1x^2 + a_2x + b$$

P(Model 3)

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

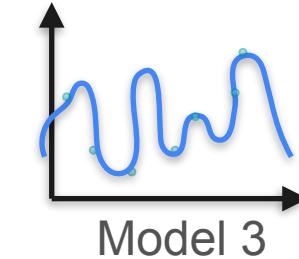
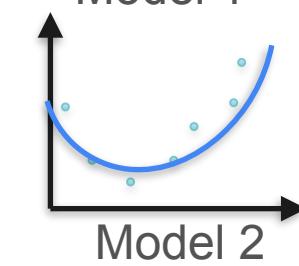
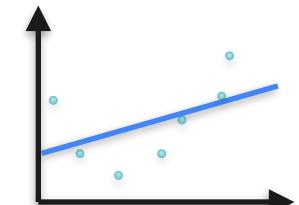
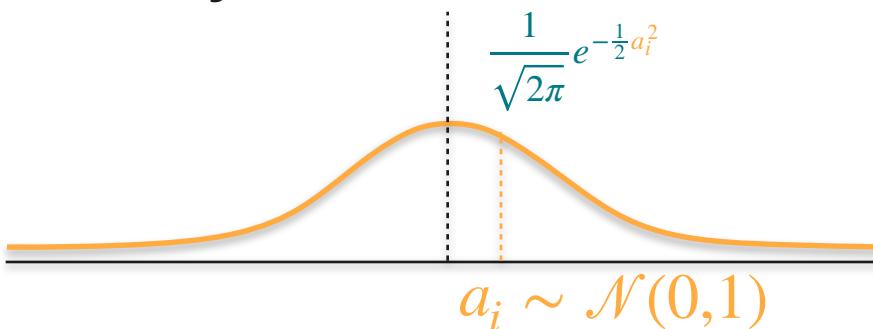
$$P(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

$a_1x + b$

$$P(\text{Model 2}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$

$a_1x^2 + a_2x + b$

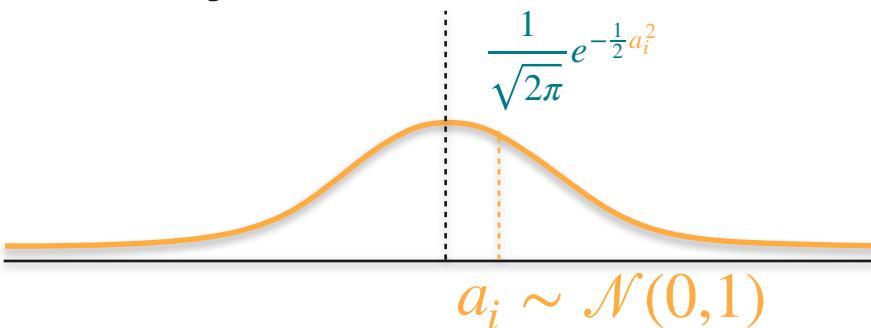
$$P(\text{Model 3}) = a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

$$P(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

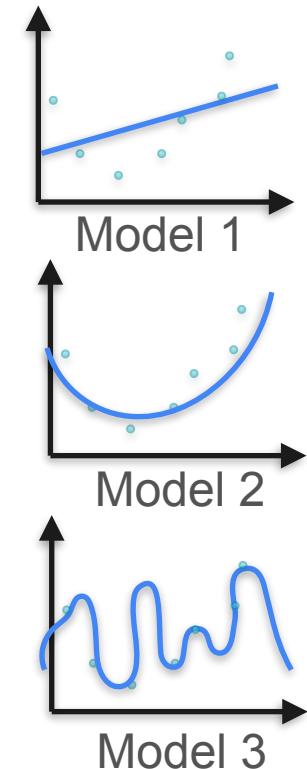
$a_1x + b$



$$P(\text{Model 2}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$

$a_1x^2 + a_2x + b$

$$P(\text{Model 3}) = a_1x^{10} + a_2x^9 + \dots + a_{10}x + b$$



# What Is the Probability of a Model?

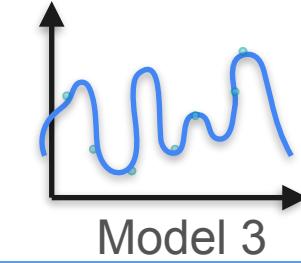
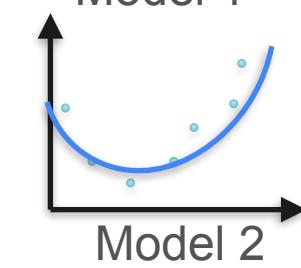
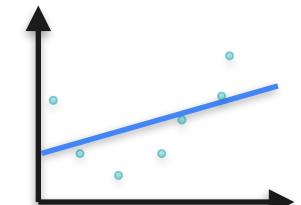
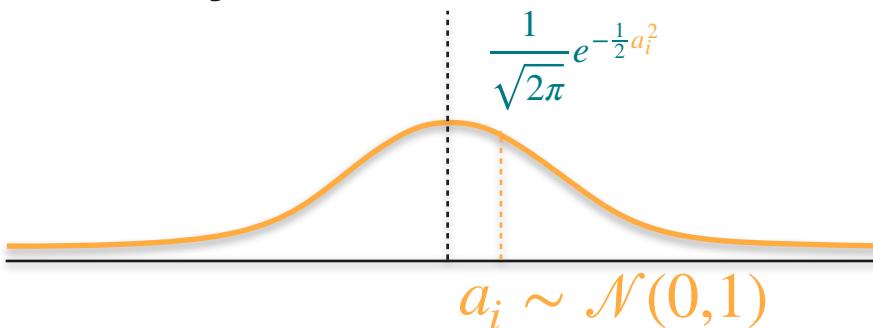
$$P(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

P(Model 2)

$$a_1x^2 + a_2x + b = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$

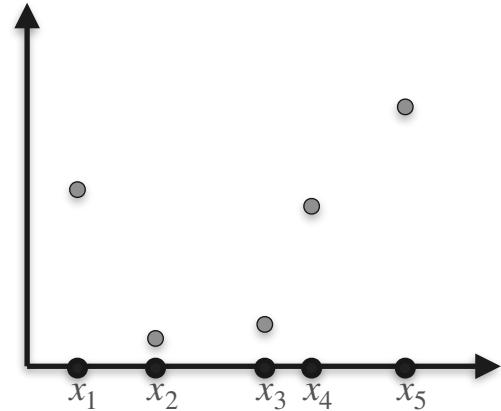
P(Model 3)

$$a_1x^{10} + a_2x^9 + \dots + a_{10}x + b = \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_i^2}$$

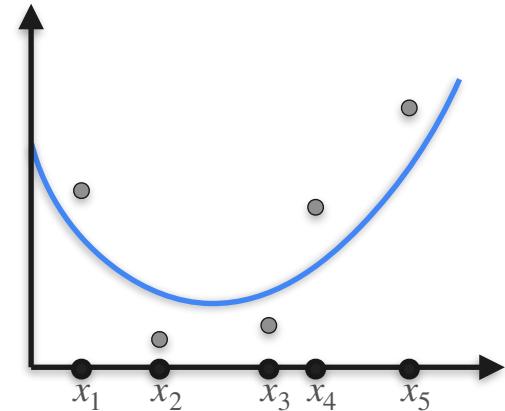


# Bayes and Regularization

# Bayes and Regularization

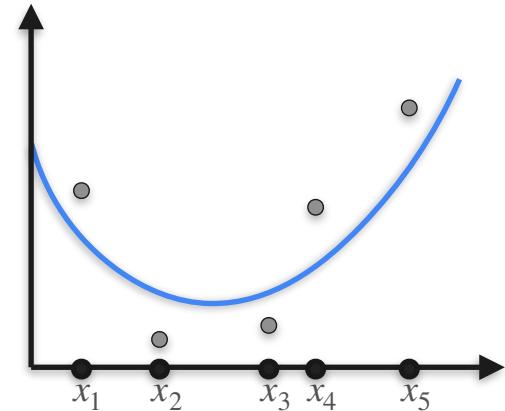


# Bayes and Regularization



# Bayes and Regularization

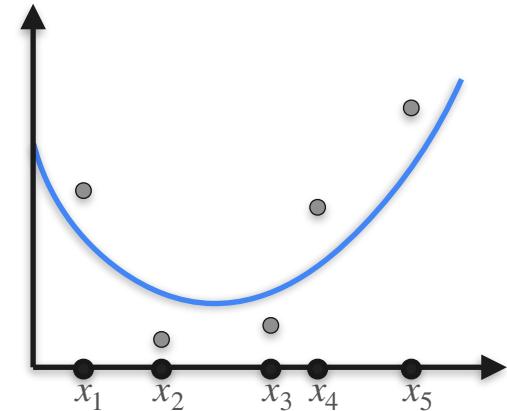
$P(\text{Data}|\text{Model})$



# Bayes and Regularization

$P(\text{Data}|\text{Model})$

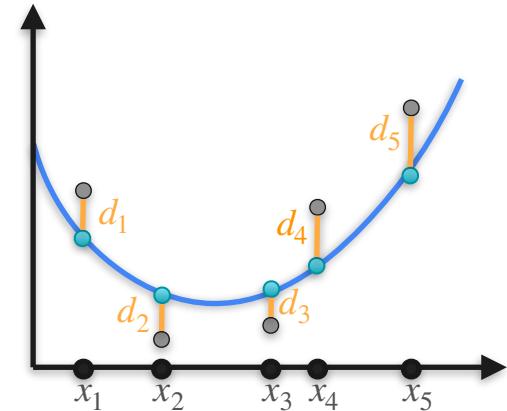
$P(\text{Model})$



# Bayes and Regularization

$P(\text{Data}|\text{Model})$

$P(\text{Model})$

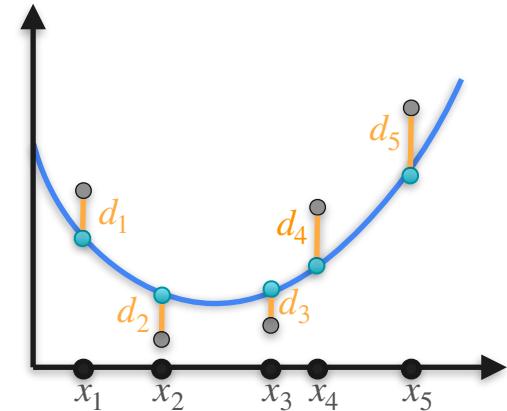


# Bayes and Regularization

$\mathbf{P}(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$\mathbf{P}(\text{Model})$

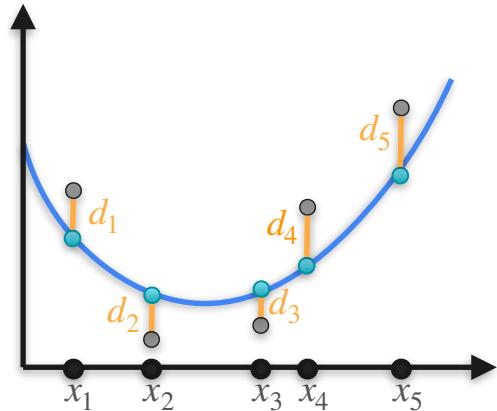


# Bayes and Regularization

$\mathbf{P}(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$\mathbf{P}(\text{Model})$



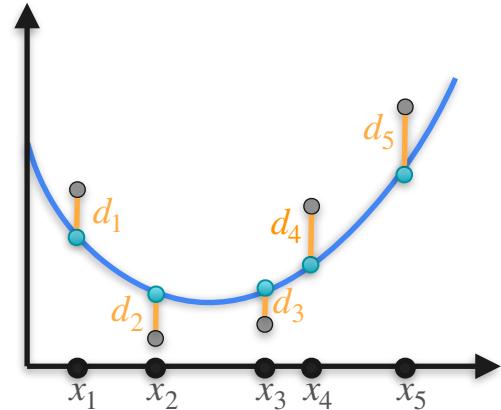
$$a_1x^2 + a_2x + b$$

# Bayes and Regularization

$\mathbf{P}(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$\mathbf{P}(\text{Model})$



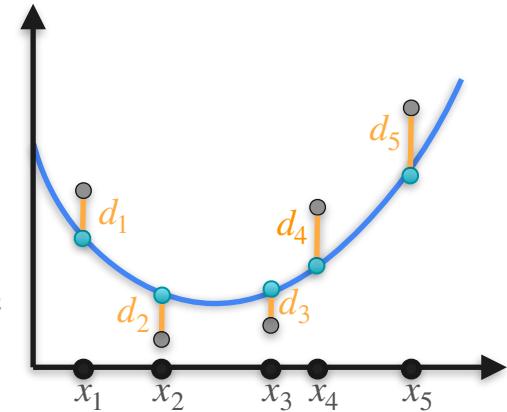
$$[a_1]x^2 + [a_2]x + b$$

# Bayes and Regularization

$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$

$P(\text{Model})$



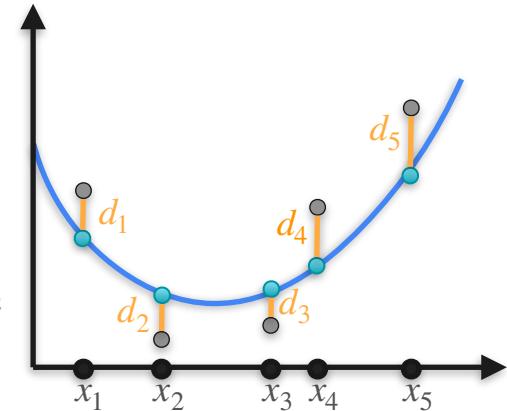
$$[a_1]x^2 + [a_2]x + b$$

# Bayes and Regularization

$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$

$P(\text{Model})$



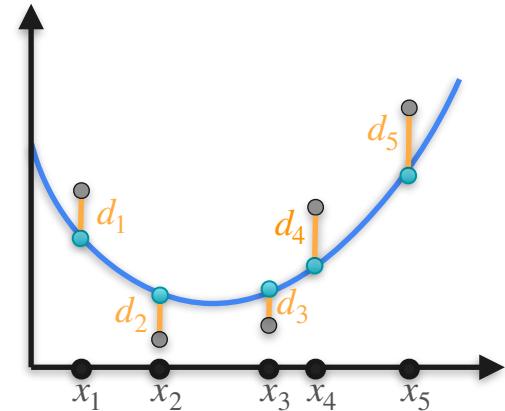
$$[a_1]x^2 + [a_2]x + b$$

# Bayes and Regularization

$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$P(\text{Model})$



$$[a_1]x^2 + [a_2]x + b$$

# Bayes and Regularization

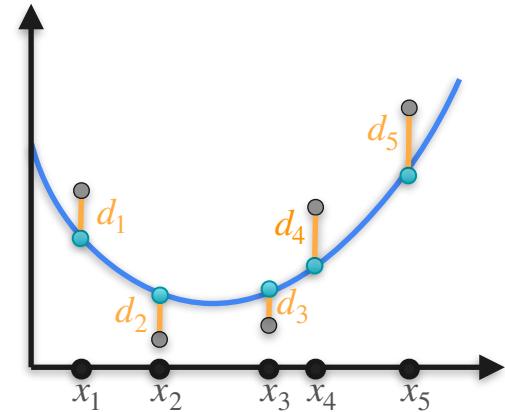
$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$

log

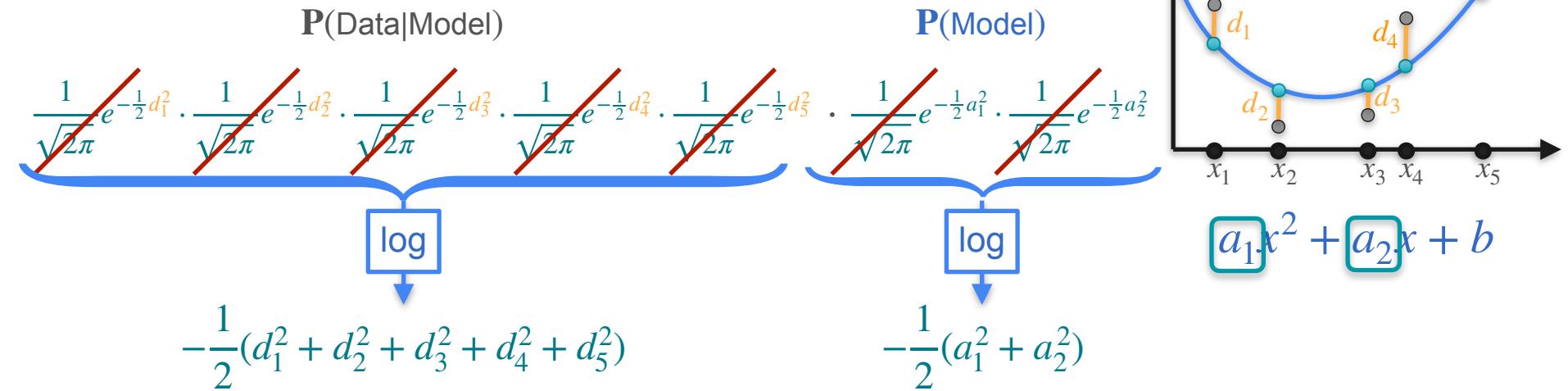
$$-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)$$

$P(\text{Model})$

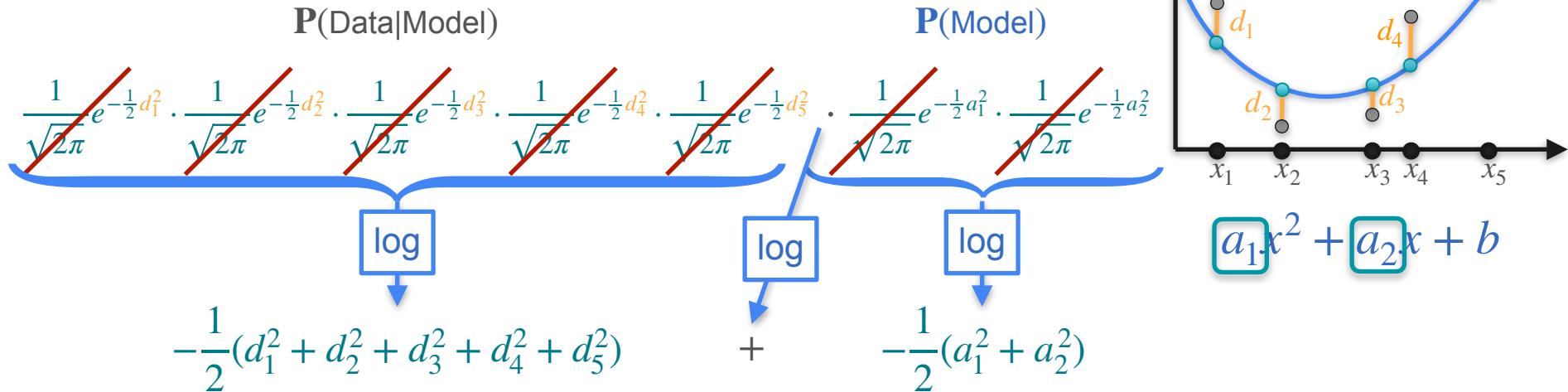


$$[a_1]x^2 + [a_2]x + b$$

# Bayes and Regularization



# Bayes and Regularization

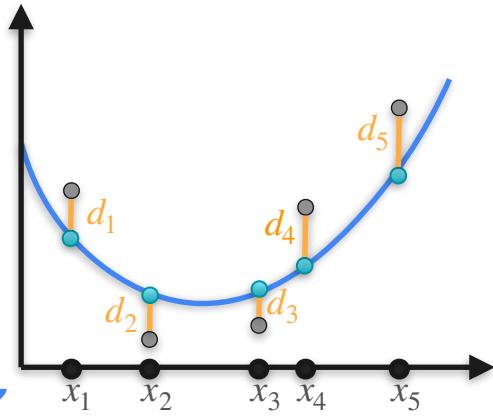


# Bayes and Regularization

$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$P(\text{Model})$



Maximize

$$-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)$$

log

log

log

+

$$-\frac{1}{2}(a_1^2 + a_2^2)$$

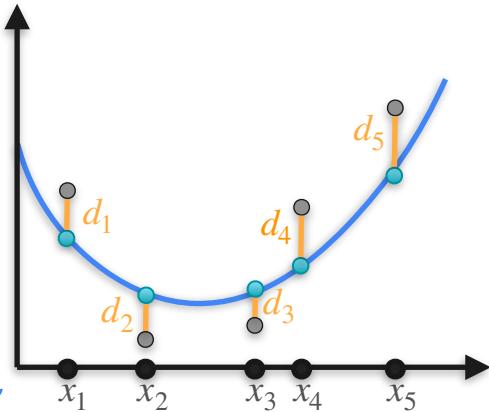
$$a_1x^2 + a_2x + b$$

# Bayes and Regularization

$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$P(\text{Model})$



Maximize

$$-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)$$

log

log

log

$$a_1x^2 + a_2x + b$$

$$+ -\frac{1}{2}(a_1^2 + a_2^2)$$

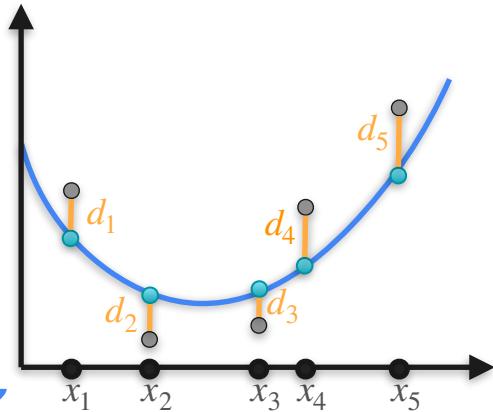
Minimize

# Bayes and Regularization

$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$P(\text{Model})$



Maximize

$$-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)$$

$\log$

$\log$

$\log$

+

$$-\frac{1}{2}(a_1^2 + a_2^2)$$

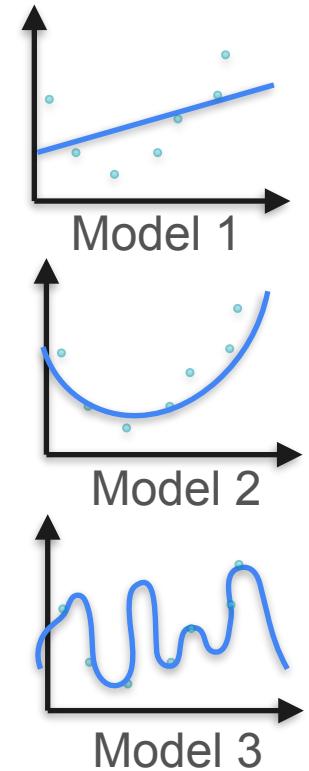
$$a_1x^2 + a_2x + b$$

Minimize

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + a_1^2 + a_2^2$$

# Regularization

# Regularization

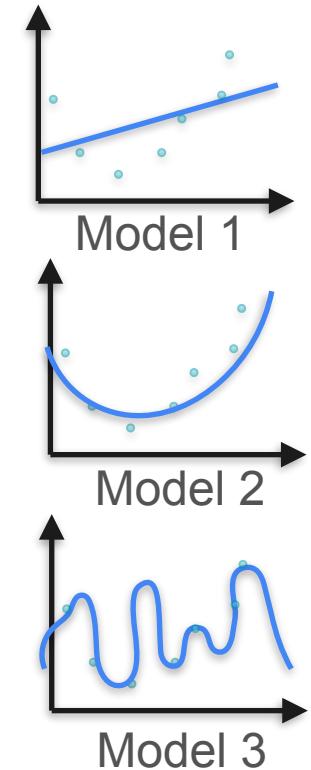


# Regularization

$P(\text{Model 1})$

$P(\text{Model 2})$

$P(\text{Model 3})$



# Regularization

**P(Model 1)**

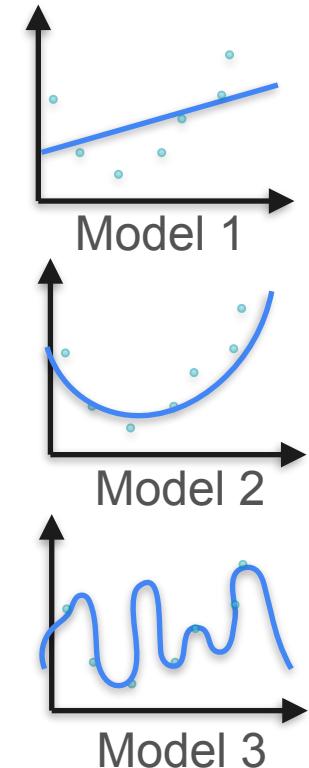
Minimize  $x_1^2$

**P(Model 2)**

Minimize  $x_1^2 + x_2^2$

**P(Model 3)**

Minimize  $x_1^2 + \dots + x_{10}^2$



# Regularization

$P(\text{Model 1})$

Minimize  $x_1^2$

$P(\text{Model 2})$

Minimize  $x_1^2 + x_2^2$

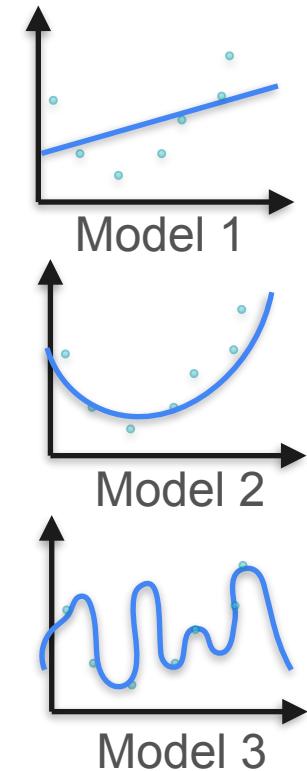
$P(\text{Model 3})$

Minimize  $x_1^2 + \dots + x_{10}^2$

$P(\text{Data}|\text{Model 1})$

$P(\text{Data}|\text{Model 2})$

$P(\text{Data}|\text{Model 3})$



# Regularization

P(Model 1)

Minimize  $x_1^2$

P(Model 2)

Minimize  $x_1^2 + x_2^2$

P(Model 3)

Minimize  $x_1^2 + \dots + x_{10}^2$

P(Data|Model 1)

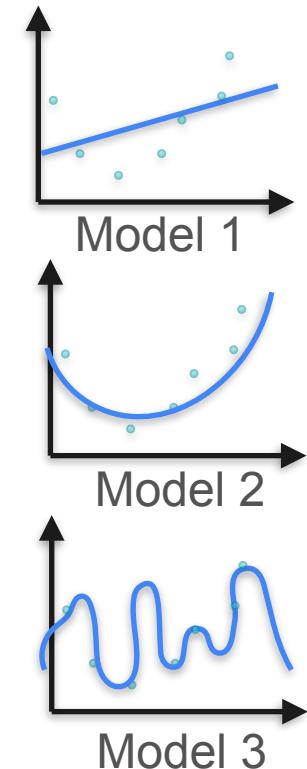
$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Data|Model 2)

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Data|Model 3)

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$



# Regularization

P(Model 1)

Minimize  $x_1^2$

P(Model 2)

Minimize  $x_1^2 + x_2^2$

P(Model 3)

Minimize  $x_1^2 + \dots + x_{10}^2$

New Loss

P(Data|Model 1)

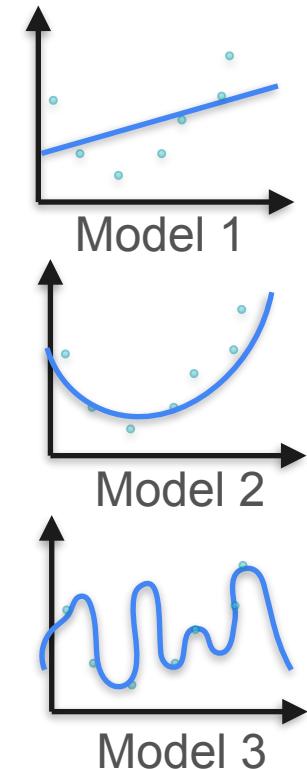
$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Data|Model 2)

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Data|Model 3)

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$



# Regularization

P(Model 1)

$$\text{Minimize } x_1^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Model 2)

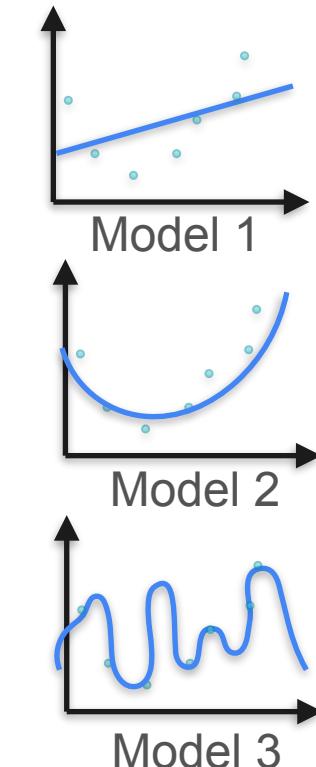
$$\text{Minimize } x_1^2 + x_2^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Model 3)

$$\text{Minimize } x_1^2 + \dots + x_{10}^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

New Loss

P(Data|Model 1)





DeepLearning.AI

## Point Estimation

---

## Conclusion