Stabilization of Double Link Inverted Pendulum Using LQR

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***Abstract*— An double inverted pendulum on cart is an object which is a nonlinear, unstable system that is used as a standard for designing the control methods and finds most versatile application in the field of control theory. In this paper the mathematical modeling and control strategy of double inverted pendulum-cart dynamic system with disturbance input using PID and LQR have been stated. The states of the system are given to LQR controller which is obtained using linear state-feedback controller. The MATLAB-SIMULINK environment have been used for simulation of the control strategies. The aim of this work is to have a comparative study of two different control strategies and the analysis of performance of two different types of controller for double inverted pendulum on cart system have been obtained. The result shows and validate the comparative lead of LQR control strategy over conventional PID control strategy.**

***Keywords*— Inverted pendulum; double inverted pendulum; LQR; PID.**

I. INTRODUCTION

Double Inverted Pendulum on Cart is a good platform for researchers for validation of different control strategies in the field of control system. Most of the modern day technologies use the concept of Inverted Pendulum, such as attitude control of satellites and rockets, aircraft landing, ship balancing against turbulent tide, Seismometer etc. An inverted pendulum has its mass above it's pivoted point is placed on a cart which may be moved horizontally. The double inverted pendulum gets stabilized when it hangs downward, but an inverted pendulum is inherently nonlinear, unstable system that needs to be stabilized Fig.2. In this case the double inverted pendulum system has one input - the force is being applied to the cart, and two output –position of the cart and angle of the pendulums, making it as a Single Input Multiple Output System Fig. 3. There are mainly three ways of balancing an inverted pendulum i.e.

i) The application of a torque at the pivot point
ii) The movement of the cart in horizontal direction.
iii) Swinging the support continuously up and down.

For achieving the three ways, proper control strategy is required. An Inverted Pendulum is a non-linear, time variant, non-holonomic, unstable open loop system. The linear mathematical methods cannot model the non-linear system which makes the system more critical for analysis [1].The non-linear system can be approached as a linear system if the area of operation is little, i.e. difference of pendulum angle from the equilibrium position is little. The Control System is a system which provides the dynamics of other plant.The performance comparison between LQR and PID controller Fig.1 for a double inverted pendulum system. Performance of both control strategies with respect to angle of the pendulums and position of the cart have been compared [2]. The mathematical model and design of the controller for non-linear double inverted pendulum-cart dynamic system in presence of disturbance input using conventional controller Fig. 4,Fig.5,Fig.6 and LQR. The nonlinear system states are given to LQR which is obtained using linear state-feedback controller. Here two control strategies have been implemented to control the position of the cart and balance the inverted pendulum and get the desired response by controlling the output. [3].A robust LQR controller is obtained to balance the pendulum in vertical balanced position and make the cart to follow the set point in purturbationFig.7 and Fig. 8 . The aim of this work is to design the LQR controller which can adapt the perturbations present in the system[4].

A. PID CONTROLLER

PID is acronym of Proportional-Integral-Derivative. This is a conventional feedback controller whose output is made of a variable which are being controlled based on the error, difference between set point and some plant variable which are being measured. Each element of the PID controller stands for a particular control action is put on error then the error is in use for adjusting some input to the process in order to its desired value. The schematic for the controller is shown below.

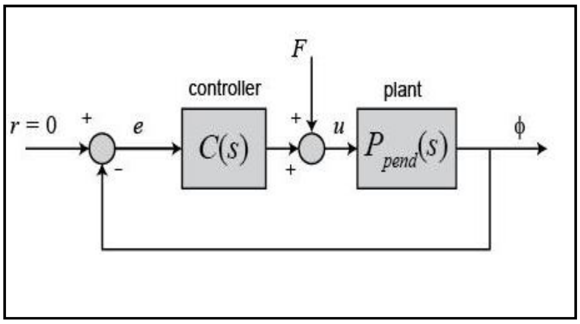


Fig. 1: Block Diagram Of PID Controller

Three parameters of PID controller and their impact on error. The transfer function of the PID controller can be written as:

$$G_{PID} = K_p + \frac{K_i}{s} + K_d.s \quad (1)$$

Where, K_p is the proportional controller gain, K_i is the integral controller gain and $K_d$is the derivative controller gain. It is necessary to choose proper parameters which guarantees the stability and the performance of the system with reduced error. There are few methods for adjusting PID controller parameters. However, in this work three parameters of PID controller values are obtained by the method of experimentation .

- Proportional: Error gets multiplied by proportional gain, K_p. In other systems process stability depends on proportional gain. Very low value of proportional gain indicates that the process variable may drift away and very high value of the proportional gain indicates oscillation in process variable.
- Integral: Taking the integral of the error which is multiplied by integral gain, K_i. In other systems K_i is cause of making the error to zero, but to set K_i very high value of integral gain welcome instability.
- Derivative: Taking the derivative of the error gets multiplied by derivative gain, K_d. In many systems $K_d$is cause of faster system response. Very high value of derivative gain indicates that the process variable will oscillate and very low value of derivative gain indicates that the process variable will respond slowly.

B. LQR CONTROLLER

To get rid of some problems that are faced by conventional PID controller, the another control strategy is used, which is Linear-Quadratic Regulator (LQR) optimal control. LQR is a control strategy which operates the system with minimum cost when the system dynamics is expressed by differential equations which are linear. The performance measurement is expressed with a cost function which is quadratic in nature and made of state vector and control input. LQR control defines the pole location which are optimal in location depends on two cost function. For obtaining the gains, which are optimal. The optimal performance index must be expressed first and then solve the State Dependent Algebraic Riccati Equation(ARE). The LQR design method consists of obtaining a state feedback controller K such a way that the quadratic performance index J is minimized. In the control strategy a feedback gain matrix is obtained which aids to minimize the quadratic performance index and makes the system stable[5].

For a continuous-time linear system described by:

$$\dot{x} = Ax + Bu \quad (2)$$

Quadratic cost function is written below:

$$J = \int (x^T Q x + u^T R u) dt \quad (3)$$

Where Q and R are weight matrices, Q must be positive definite or positive semi-definite symmetric matrix. R must be positive definite symmetric matrix. Weighting matrices should be diagonal matrix. The value of the elements of the weighting matrices are linked to their impact on the performance index J. The feedback control law is:

$$u = -Kx, \quad (4)$$

Where, K is expressed as $K = R^{-1}B^T P$ (5)

And P can be obtained by the solution of State Dependent AlgebricRiccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (6)$$

LQR is just an automatic method to find out an appropriate state. This is common to find that control engineers prefer different methods as full state feedback controller to find a controller over the LQR controller. With these the researcher has a much clearer link between adjusted parameters and the resulting changes in the behaviour of the controller. Difficulties are faced in finding out the right weighting matrices, which limits the application of the LQR Controller design [5].

C. SYSTEM MODELLING (DOUBLE INVERTED PENDULUM)

Double inverted pendulum system is made of cart, two pendulums, guide etc as shown in fig 3.

Before the derivation few assumptions has been made:

1. The whole system is a rigid body.
2. There is no relative sliding presents in the whole system.
3. Friction which exits between cart and rail is directly proportional to the speed of the cart.
4. Resistance torque that exists between cart and pendulum 1 and pendulum 2 are directly proportional to angular velocity of pendulum 1 and pendulum 2.

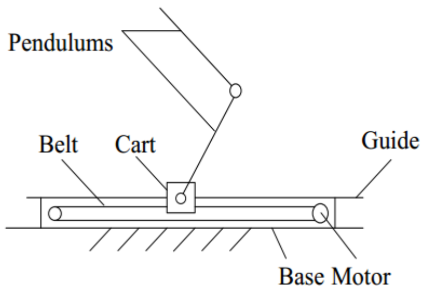


Fig. 2: Configuration of double inverted pendulum

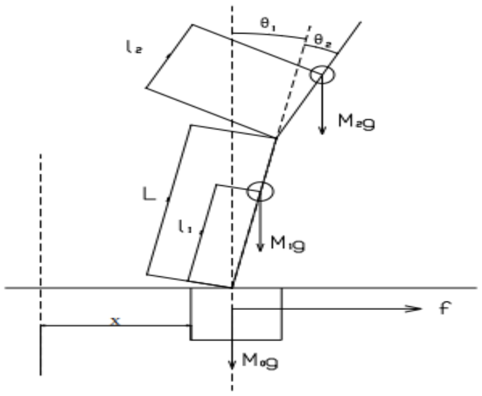


Fig. 3 : Force analysis for double inverted pendulum system

For Pendulum 1

$$x_1 = x + l_1 \sin\theta_1 \qquad (7)$$
$$y_1 = l_1 \cos\theta_1$$

For Pendulum 2

$$x_2 = x + L_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \qquad (8)$$
$$y_2 = L_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$

Lagrange Equation (9)

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i$$

$$\ddot{x} = \frac{1}{M}\big[f - (M_1 l_1 \cos\theta_1 + M_2 L \cos\theta_1 + M_2 l_2 \cos(\theta_1 + \theta_2))\ddot{\theta}_1 + \{M_1 l_1 \sin\theta_1 + M_2 L \sin\theta_1 + M_2 l_2 \sin(\theta_1 + \theta_2)\}\dot{\theta}_1^2 - \{M_2 l_2 \cos(\theta_1 + \theta_2)\}\ddot{\theta}_2 + 2\{M_2 l_2 \sin(\theta_1 + \theta_2)\dot{\theta}_1\dot{\theta}_2\} + \{M_2 l_2 \sin(\theta_1 + \theta_2)\}\dot{\theta}_2^2 - R_0\dot{x}\big] \qquad (10)$$

$$L\left(q, \dot{q}\right) = T\left(q, \dot{q}\right) - V\left(q, \dot{q}\right) \qquad (11)$$

T is the Total Kinetic Energy; V is the Total Potential Energyand D is the Total Dissipation Energy, from (9) we get,

When $q_i = x$ *and* $Q_i = f$

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} + \frac{\partial D}{\partial \dot{x}} = f \qquad (12)$$

When $q_i = \theta_1, Q_i = 0$ from (Eq. 9) we get,

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \dot{\theta}_1} - \frac{\partial D}{\partial \dot{\theta}_1} = 0 \qquad (13)$$

When $q_i = \theta_2, Q_i = 0$ from (Eq. 9) we get,

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \dot{\theta}_2} + \frac{\partial D}{\partial \dot{\theta}_2} = 0 \qquad (14)$$

By calculating we get,

$$(M_0 + M_1 + M_2)\ddot{x} + (M_1 l_1 C_1 + M_2 L C_1 + M_2 l_2 C_{12})\ddot{\theta}_1 + (-M_1 l_1 S_1 - M_2 l S_1 - M_2 l_2 S_{12})\dot{\theta}_1^2 + M_2 l_2 C_{12}\ddot{\theta}_2 - 2M_2 l_2 S_{12}\dot{\theta}_1\dot{\theta}_2 - M_2 l_2 S_{12}\dot{\theta}_1^2 + R_0\dot{x} = f \qquad (15)$$

$$(J_1 + M_1 l_1^2 + J_2 + M_2 L^2 + M_2 l_2^2 + 2M_2 L l_2 C_2)\ddot{\theta}_1 + (-2M_2 L l_2 S_2)\dot{\theta}_1\dot{\theta}_2 + (M_1 l_1 C_1 + M_2 L l_2 C_2)\ddot{\theta}_2 + (-M_2 L l_2 S_2)\dot{\theta}_2^2 - M_1 g l_1 S_1 - M_2 g(L S_1 + l_2 S_{12}) + R_1\dot{\theta}_1 = 0 \qquad (16)$$

$$M_2 l_2 C_{12}\ddot{x} + (J_2 + M_2 l_2^2 + M_2 L l_2 C_2)\ddot{\theta}_1 + (J_2 + M_2 l_2^2)\ddot{\theta}_2 + (M_2 L l_2 S_2)\dot{\theta}_1^2 - M_2 g l_2 S_{12} + R_2\dot{\theta}_2 = 0 \qquad (17)$$

We can get the matrix form as:

$$\mathrm{M}[\theta_1, \theta_2]\begin{bmatrix}\ddot{x}\\ \ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + N[\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]\begin{bmatrix}\dot{x}\\ \dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = G[u, \theta_1, \theta_2] \qquad (18)$$

$$\ddot{x} = \frac{1}{M}\Big[f - (M_1 l_1 \cos\theta_1 + M_2 L\cos\theta_1 + M_2 l_2 \cos(\theta_1+\theta_2))\ddot{\theta}_1 + \{M_1 l_1 \sin\theta_1 + M_2 L \sin\theta_1 + M_2 l_2 \sin(\theta_1+\theta_2)\}\dot{\theta}_1^2 - \{M_2 l_2 \cos(\theta_1+\theta_2)\}\ddot{\theta}_2 + 2\{M_2 l_2 \sin(\theta_1+\theta_2)\dot{\theta}_1\dot{\theta}_2\} + \{M_2 l_2 \sin(\theta_1+\theta_2)\}\dot{\theta}_2^2 - R_0\dot{x}\Big] \quad (19)$$

$$\ddot{\theta}_1 = \frac{1}{Z}\Big[+\{2M_2 L l_2 \sin\theta_2 \dot{\theta}_1\dot{\theta}_2\} - \{M_1 l_1 \cos\theta_1 + M_2 L\cos\theta_1 + M_2 l_2 \cos(\theta_1+\theta_2)\}\ddot{x} - \{J_2 + M_2 l_2^2 + M_2 L l_2 \cos\theta_2\}\ddot{\theta}_2 + \{M_2 L l_2 \sin\theta_2\}\dot{\theta}_2^2 + \{M_1 g l_1 \sin\theta_1\} + M_2 g\{L\sin\theta_1 + l_2\sin(\theta_1+\theta_2)\} - R_1\dot{\theta}_1\Big] \quad (20)$$

$$\ddot{\theta}_2 = \frac{1}{Y}\Big[-\{M_2 l_2 \cos(\theta_1+\theta_2)\}\ddot{x} - \{J_2 + M_2 l_2^2 + M_2 L l_2 \cos\theta_2\}\ddot{\theta}_1 - \{M_2 L l_2 \sin\theta_2\}\dot{\theta}_1^2 + \{M_2 g l_2 \sin(\theta_1+\theta_2)\} - R_2\dot{\theta}_2\Big] \quad (21)$$

D. CONTROLLER DESIGN

Theequations of motion of Lagrange, may be formulated to a sixth-order system consists of ordinary differential equations:

From above generalized version it is easy to deduce the system equation as follow.

$$\dot{x} = \begin{pmatrix} 0 & I \\ 0 & -D^{-1}C \end{pmatrix} x + \begin{pmatrix} 0 \\ -D^{-1}G \end{pmatrix} + \begin{pmatrix} 0 \\ D^{-1}H \end{pmatrix} u \quad (22)$$

Non-Linear system is approximated as linear system. Linearization has been done using a matrix named Jacobian. A Jacobian means differentiation. Approximation has been done about an equilibrium point which is unstable. So, the linear model needed can be obtained by the linearization of the above equation around x=0, so system with non-linear dynamics has been changed to a system with linear dynamics as stated below

$$\dot{x} = Ax + Bu \quad (23)$$

Where A and B are obtained by Jacobian matrix

$$A = \frac{\partial f(x)}{\partial x} = \begin{pmatrix} 0 & I \\ -D(0)^{-1}\frac{\partial G(0)}{\partial q} & 0 \end{pmatrix}$$

$$B = \frac{\partial g(x)}{\partial x} = \begin{pmatrix} 0 \\ D(0)^{-1}H \end{pmatrix}$$

If we compute above matrices for the following data:

$M_0 = 0.42kg,\quad M_1 = 0.4kg,\quad M_2 = 0.2kg,\quad L_1 = 0.6m,\quad L_2 = 0.4m$We end up with:

$System\ Matrix\ A\ is\ given\ by$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -4.527 & 0.0200 & -2.876 & 0.01252 & -0.0070 \\ 0 & 20.050 & -6.9600 & 4.3300 & -0.6200 & 0.0700 \\ 0 & -15.140 & 34.910 & -3.7430 & 0.1140 & -0.360 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1.5380 \\ -2.3180 \\ -2.0020 \end{pmatrix}$$

$C = I_6$ where, I is an identity matrix of sixth order

$\mathrm{D} = 0_{6X1}$ i. e. D is a zero matrix of sixth order

(ii) Cost Function:

Continuous time quadratic cost function is selected for the system and will be minimized it.

$$\mathrm{J} = \int_0^\infty (\mathrm{x^T Q x + u^T R u})\mathrm{dt} \quad (24)$$

In case of infinite-horizon,,the weighting matrices are positive-semi definite and positive-definite. These additional limitations on Q and R in case of the infinite-horizon guarantees the presence of positive cost function.

(iii) Linear Quadratic Regulator (LQR)

Optimal control law which is mentioned here to meet unstable equilibrium position is the solution of the dynamic system expressed by linear differential equations and performance index is expressed by a quadratic function of state vector and control input,a state feedback controller whose equations are given as-

$$U = -R^{-1}B^T P x = -Kx \quad (25)$$

Where P is solution of State Dependent AlgebricRiccatiEquation(ARE) is written as,

$$PA + A^T P - PBR^{-1}P + Q = 0 \quad (26)$$

The value of K i.e. linear feedback gain matrix is obtained for the simulation.

II.SIMULATION RESULT OF DOUBLE LINK INVERTED PENDULUM SYSTEM USING PID CONTROLLER IN MATLAB

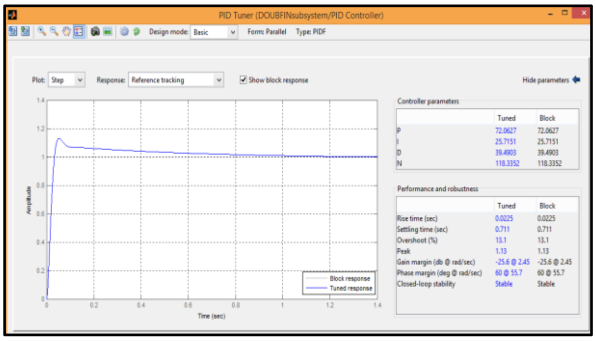


Fig.4: PID Tuning Parameter for Double Link Inverted Pendulum

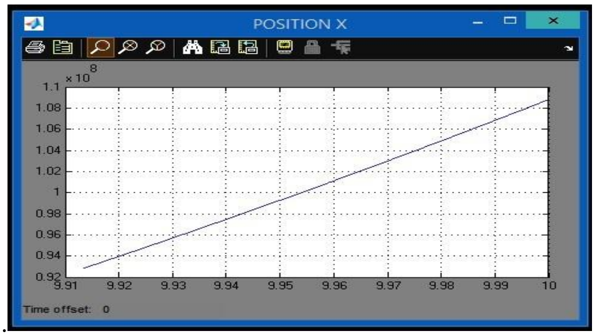
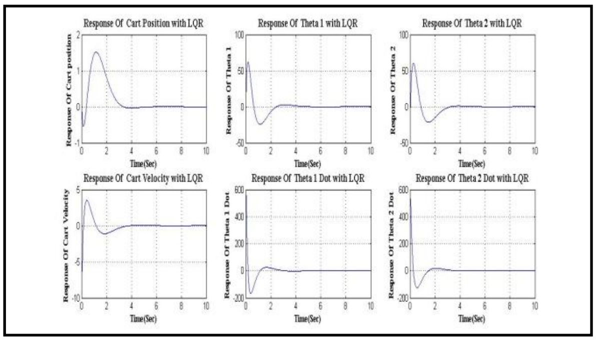


Fig.5: Response for the position x with respect to time after using PID control



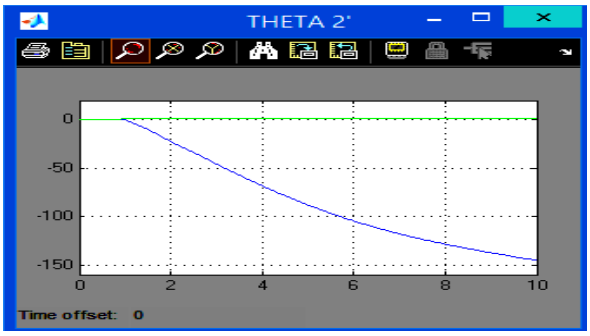


Fig.6: Response of Theta 1 With Respect To Input

Double Link Inverted Pendulum on cart is modeled in Simulink using PID control. The tuning of PID controller is done and fixed at the following parameters-

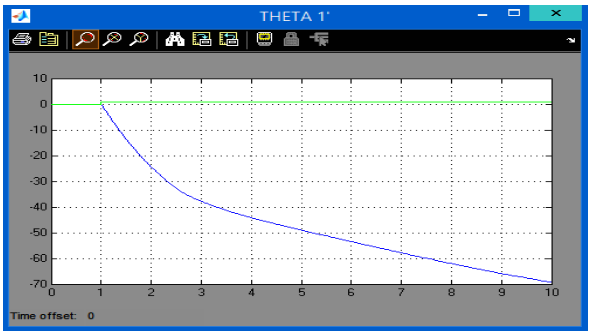


Fig. 7: Response of Theta 2 With Respect To Input

III.RESULTS USING LQR WITH AND WITHOUT DISTURBANCE

For better stability and control LQR controller is used in double pendulum system. The coding is implemented in MATLAB. The figure below shows the response of double inverted pendulum system (which is taken as plant here) without any disturbance; that means in nominal condition using LQR..Consider the response for initial conditions of states specified as zero for cart position and -10 degree and +10 degree for angle of deflections of two arms of the pendulum respectively that is, for a relatively small deflections from the equilibrium. Although we refer to the deflection angles in degrees, in MATLAB, they should be inputted in radians, as can be seen from the MATLAB code at the end. The six states can be seen in Fig. 8.

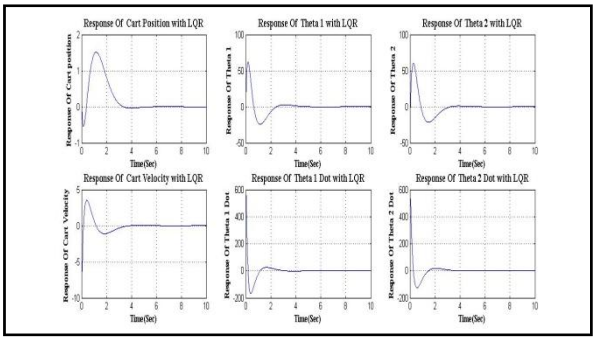


Fig. 8: Response of plant using LQR in nominal condition

But the actual situations are far from nominal. In any actual plant there will be some added perturbations. So after adding disturbances we have checked the responses using LQR as shown below.

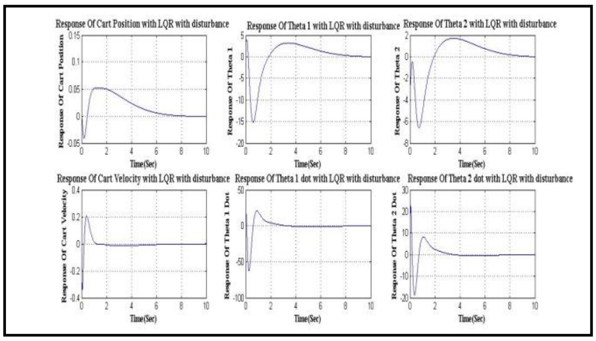


Fig. 9: Response of Plant using LQR in nominal condition after Adding Disturbances

IV. CONCLUSIONS

Thesimulation of double link inverted pendulum system using PID controller in MATLAB is done, it has been observed from the response curve that system is stable but gain margin is negative.

Then LQR controller is used on DIPC system and we get a stable response. But practical system will have disturbances. So perturbations are added to the system and responses are checked using LQR.

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