Determinants

• **Determinant** of an $N \times N$ matrix: **Leibniz** formula

$$\det(A) = \sum_{\sigma}^{N!} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} a_{\sigma(i),i}$$

The summation runs over all $N!$ possible permutations σ of indices.

Example: a 3×3 matrix A there are 3! = 6 permutations:

$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\mathrm{sgn}(\sigma)$	$a_{\sigma(1),1}$	$a_{\sigma(2),2}$	$a_{\sigma(1),2}$
1	2	3	$+$	a_{11}	a_{22}	a_{33}
2	1	3	$-$	a_{21}	a_{12}	a_{33}
1	3	2	$-$	a_{11}	a_{32}	a_{23}
2	3	1	$+$	a_{21}	a_{32}	a_{13}
3	1	2	$+$	a_{31}	a_{12}	a_{23}
3	2	1	$-$	a_{31}	a_{22}	a_{13}

$$\det(\mathbf{A}) = a_{11}a_{22}a_{33} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

• **Determinant** of an $N \times N$ matrix: **Laplace** expansion (by j-th column or by i-th row):

$$\det(A) = \sum_{i=1}^{N} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{N} (-1)^{i+j} a_{ij} M_{ij}$$

where M_{ij} ("minor") is the determinant obtained by deleting the i-th row and the j-th column. *Example*: Expansion of a 3×3 determinant by the 1-st column:

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}; \qquad M_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Elementary transformations

- Interchanging two rows (or two columns) of the matrix

  $\Rightarrow$ The deteminant **remains changes the sign**;

- Multiplying a row (or a column) of the matrix by an arbitrary value α

  $\Rightarrow$The determinant **is multiplied by** $\boldsymbol{\alpha}$;

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{\alpha} a_{21} & \boldsymbol{\alpha} a_{22} & \boldsymbol{\alpha} a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \boldsymbol{\alpha} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Adding of j-th row multiplied by an arbitrary value α to the i-th row, putting the result to the i-th row

  $\Rightarrow$ The determinant **remains unchanged**. The same is true for columns

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{\alpha} a_{11} + a_{21} & \boldsymbol{\alpha} a_{12} + a_{22} & \boldsymbol{\alpha} a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Note. Since the **Gauss elimination** method used elementary transformations, it can be employed to calculated the determinant (Back-substitution is not needed).

Properties of determinants

- The determinant of a **triangular** matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1N} \\ 0 & a_{22} & a_{23} & \dots & a_{2N} \\ 0 & 0 & a_{33} & \dots & a_{3N} \\ 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & a_{NN} \end{vmatrix} = a_{11}a_{22}a_{33}\dots a_{NN} = \prod_{i=1}^{N} a_{ii}$$

- The determinant of the matrix **transpose**: $\det(\mathbf{A}^T) = \det(\mathbf{A})$

  $\Rightarrow$ The determinant of the **adjoint** matrix is the complex conjugate:
  $\det(\mathbf{A}^\dagger) = \det(\mathbf{A})$

- The determinant of the matrix product: $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{BA})$

  $\Rightarrow$ The determinant of the **inverse** matrix: $\det(\mathbf{A}^{-1}) = 1 / \det(\mathbf{A})$

  $\Rightarrow$ **Similar** matrices have equal determinants.

- The determinant is **zero** if and only if the matrix is **singular**: $\det(\mathbf{A}) = 0$