Quantum State Separation and Interpolation Between Approximate and Probabilistic Exact Cloning

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We solve the problem of state separation of two known pure states in the general case where they have arbitrary a-priori probabilities. The solution emerges from a geometric formulation of the problem. This formulation also reveals a deeper connection between cloning and state discrimination. The results are then applied in designing a scheme for hybrid cloning which interpolates between approximate and probabilistic exact cloning. State separation and hybrid cloning are generalized schemes to well established state discrimination and cloning strategies.

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I. INTRODUCTION

The strange nature of the microscopic world allows for communications (cite security) and computational(cite shor) tasks impossible without quantum mechanics. We aim to investigate the ultimate limits imposed by quantum mechanics on communication and information processing in the more general context of probabilistic protocols. Examples are quantum cloning and quantum discrimination, which we describe below. The important common feature of these examples is that the output states must have a smaller overlap, thereby more distinguishable, than the input states. A process by which the overlap of the output states is smaller than that of the input states is known as state separation. State separation cannot be done deterministically, since it would imply that quantum states become more distinguishable. No result concerning general prior probabilities are known so far. For reasons analogous to those discussed in our PRL (symmetry,...), the equal prior solution may not be representative of general case. We provide the general solution here and use it to obtain the optimal interpolating scheme for cloning (perfect-max fidelity).

While it is impossible to always clone quantum states perfectly [? ?], this leads to some unexpected advantages for quantum systems over classical ones in some communications protocols of practical relevance. A common example is stronger security in cryptographic key distribution, with recent works showing more applications [? ?]. Developments in cloning, including probabilistically perfect [?] and approximate cloning [?] [] provide anchors for better understanding quantum theory as a whole, such as the relationship between the no-cloning and no-signaling theorems [?], and fundamental limits on quantum measurements . In particular, cloning's relationship with the fundamental limits of state discrimination will be central to this paper.

In Exact Cloning one prepares perfect clones of the input states while allowing for some failure rate in which no clones have been produced and the states are discarded. In optimal unambigious discrimination (UD) the input states are made orthogonal and hence fully distinguishable. It was shown by Chefles and Barnett [?] that these strategies are a special case of a more general scheme. Both have two outcomes: failure and success. In each strategy the overlap of the input states is decreased, in UD the overlap becomes zer, in Exact Cloning it is the overlap of the input states raised to the power of the desired number of the clones to be made, s^N. State separation unifies the two schemes as it produces states with an overlap in the range zero to s^N. The authors showed the results for the case when the states are preapared with equal apriori probabilities. In liu of our recent work on probabilistic Exact Cloning [ ] where a geometric picture emerges we use similar tools to solve the more general state separation when the input states are prepared with different a priori probabilities.

The results are then applied to hybrid cloning, interpolation between probabilistic exact cloning and approximate cloning non-orthogonal quantum states which are also prepared with different a priori probabilities. Strategies for hybrid cloning when the incoming states are prepared with equal apriori probabilities are derived in [?]. In the limit of infinite number of copies the scheme reproduces that of minimum error state discrimination with a fixed rate of inconclusive results in [?].

Both state separation and hybrid cloning can be seen as generalization schemes of well established state discrimination and cloning strategies. In essence state separation unified two well established strategies: The optimal UD and probabilistic Exact Cloning. Hybrid cloning unifies approximate cloning with probabilistic exact cloning. In the limit of producing infinite copies in unifies ME with UD.

II. STATE SEPARATION

We approach state separation via the Neumark formulation. The system is embedded in a larger Hilbert space where the extra degrees of freedom are customarily called the ancilla. Then a unitary transformation entangles the system degrees of freedom with those of the ancilla. The input states $\{|\psi_1\rangle_s, |\psi_2\rangle_s\}$ which live in the state Hilbert space S are embedded with the ancilla $|i\rangle_a$ which live in the Hilbert space A. Now the system and the ancilla live in the larger Hilbert space $H = S \otimes A$. The incoming states in this larger Hilbert space can be written in the product form $\{|\psi_1\rangle_s|i\rangle_a, |\psi_2\rangle_s|i\rangle_a\}$. The unitary should do the following:

$$U|\psi_1\rangle|i\rangle = \sqrt{p_1}|\phi_1\rangle|\alpha\rangle + \sqrt{q_1}|\Phi_o\rangle|f\rangle, \tag{2.1}$$

$$U|\psi_2\rangle|i\rangle = \sqrt{p_2}|\phi_2\rangle|\alpha\rangle + \sqrt{q_2}|\Phi_o\rangle|f\rangle, \tag{2.2}$$

were $|\alpha\rangle$ and $|f\rangle$ are orthogonal. A projective measurement along the ancilla $|\alpha\rangle$ means that the states have succesfully become more distinguishable with a success rate of p_i, otherwise a measurement in the $|f\rangle$ space means that the process has failed to produce more distinguishable states and the states are discarded with a probability of q_i.

The inner product of (**??**) with (**??**) gives the unitarity constraint:

$$s = \sqrt{p_1 p_2} s' + \sqrt{q_1 q_2}, \tag{2.3}$$

where $s = |\langle\psi_1 \mid \psi_2\rangle|$ and $s' = |\langle\phi_1 \mid \phi_2\rangle|$.

When the input states are prepared with equal priors $\eta_1 = \eta_2$, the solution is directly derived from Eq. (**??**) and no further optimization is neccessary: $s = ps' + q \Rightarrow p = (1-s)/(1-s')$. For $s' = 0$ the IDP limit is reached, while for $s' = s^N$ probabilistic exact cloning limit is reached.

For given a priori probabilities η_1, η_2 of our states $|\psi_1\rangle$ and $|\psi_2\rangle$, and average failure probability

$$Q = \eta_1 q_1 + \eta_2 q_2, \tag{2.4}$$

we wish to find the minimum value of the final overlap s' as a function of the initial overlap s. For general a-priori probabilities, rather than attempting to write the optimal output overlap s' as an explicit function of s, which would require solving a high degree polynomial equation, we will give the curve $(s, \min s')$ in parametric form.

We solve the problem by linearizing the constraint, (**??**), and finding the point at which it is tangent to the failure rate curve (**??**). We choose a change of variables

$$p_1 p_1 = t^2, \quad q_1 q_2 = z^2 \tag{2.5}$$

which linearizes the unitarity constraint (**??**) as $z = s - s't$ where $0 \le t, z \le 1$ and $0 \le s' \le s$. From the first of these substitutions we have $t^2 = (1-q_1)(1-q_2) = 1 + z^2 - q_1 - q_2$. Solving for q_2 and substituting back in the second equation results in a quadratic equation which can be solved for q_1, giving the individual failure rates as

$$q_i = \frac{1 + z^2 - t^2 \mp (-1)^i \sqrt{(1+z^2-t^2)^2 - 4z^2}}{2}.$$

Therefore, the failure rate (**??**) becomes

$$Q = \frac{1}{2}\left(1 + z^2 - t^2 \pm (\eta_1 - \eta_2)\sqrt{(1+z^2-t^2)^2 - 4z^2}\right).$$

We now solve for z^2. After a bit of algebra we obtain

$$z^2 = \frac{2\eta_1\eta_2(1+\tau) - (1-Q) + \sqrt{(1-4\eta_1\eta_2)\left[(1-Q)^2 - 4\eta_1\eta_2\tau\right]}}{2\eta_1\eta_2} \equiv \zeta(\tau), \tag{2.6}$$

where $\tau \equiv t^2$. Since z^2 cannot be less than zero, we picked up the plus sign for the root.

We assume that $0 \le \eta_1 \le 1/2$ to simplify the analysis. To locate the extrema of z, we find $dz/dt = (d\zeta/d\tau)(t/z)$. The derivative $d\zeta/d\tau$ is immediate, given by

$$\zeta'(\tau) = 1 - \frac{\sqrt{1-4\eta_1\eta_2}}{\sqrt{(1-Q)^2 - 4\eta_1\eta_2\tau}}.$$

We find that the minimum is located at

$$t_{\min} = \begin{cases} \sqrt{\left(1 - \dfrac{Q}{2\eta_1}\right)\left(1 - \dfrac{Q}{2\eta_2}\right)}, & \text{if} \quad 0 \le Q \le 2\eta_1 \\ 0, & \text{if} \quad 2\eta_1 < Q \le 1. \end{cases}$$

The corresponding values of z are

$$z_{\min} = \begin{cases} \dfrac{Q}{2\sqrt{\eta_1\eta_2}}, & \text{if} \quad 0 \le Q \le 2\eta_1 \\ \sqrt{\dfrac{Q-\eta_1}{\eta_2}}, & \text{if} \quad 2\eta_1 < Q \le 1. \end{cases}$$

Now that we have mapped out the shape of the failure rate (??), we want to know where it is tangent to the constraint (??). We first note that for equal priors the curve (??) is simply the hyperbola $z^2 = t^2 + 2Q - 1$, which intersects the straight line $z = 1 - t$ at the point $(z,t) = (Q, 1-Q)$.

We note that this is actually a general solution of (??) for any η_1, η_2. Moreover, the straight line $z = 1 - t$ is tangent to (??) at $(Q, 1-Q)$ for any values of η_1, η_2, as can be checked by substituting in the formula $dz/dt = (t/z)(d\zeta/d\tau)$. Since $z = 1 - t$ is the limiting line for the family $z = s - s't$, an obvious parametrization for the curve (s, s') is obtained as follows: *i.* define

$$s'(t) = -\frac{dz}{dt} = -\frac{t\,\zeta'(t^2)}{\sqrt{\zeta(t^2)}}, \qquad t_{\min} \leq t \leq 1 - Q,$$

and next *ii.* define

$$s(t) = z + ts'(t) = \sqrt{\zeta(t^2)} + ts'(t), \qquad t_{\min} \leq t \leq 1 - Q.$$

where

For $s < z_{\min}$ is is always possible to separate the initial states, i.e., $|\Psi_1\rangle$ and $|\Psi_2\rangle$ can be made orthogonal. We note that the condition $s = z_{\min}$ is equivalent to the unambiguous discrimination result

$$Q = 2\sqrt{\eta_1\eta_2}s, \quad Q = \eta_1 + \eta_2 s^2.$$

The next plot is for $\eta_1 = 0.1$. As η_1 approaches $1/2$ the curves approach a straight line. The difference is more noticeable for very small values of η_1.

III. HYBRID CLONING

In this section we seek to interpolate between probabilistic exact cloning and approximate cloning machines using our results from state separation. Exact cloning machines produce perfect clones while allowing for some inconclusive outcomes. Approximate cloning machines produce copies on demand which resemble the input states while maximizing the fidelity. One can imagine a scheme where fidelity can be higher then maximum fidelity in the approximate cloning machine while it allowes for a fixed rate of inconlusive outcomes, $FRIO$. This scheme should reproduce exact cloning and approximate cloning machines by setting $FRIO$ to Q_o and zero respectively. Chefles and Barnett [?] solve the problem for when the input states are prepared with equal a priori probabilities. We extend the solution to the more general case when the states are prepared with different aprioris. Such a solution is possible due to our recent work in making n perfect clones from m copies of one of two known pure states with minimum failure probability in the general case where the known states have arbitrary a-priori probabilities.

We can imagine a state dependent approximate cloner as a machine with an input and an output port. The input states $|\psi_i^M\rangle = |\psi_i\rangle^{\otimes M}$, $i = 1, 2$ (m identical copies of either $|\psi_1\rangle$ or $|\psi_2\rangle$) are fed through the input port for processing. The output states are n approximate clones $|\phi_i^M\rangle = |\phi_i\rangle^{\otimes M}$. This is deterministic cloning, although imperfect, clones are generated on demand while optimizing the global fidelity. Thus given a set K of non orthogonal states $|\psi_i\rangle^{\otimes M}$, we wish to produce a set K of N clones $|\phi_j\rangle^{\otimes N}$ while optimizing the global fidelity:

$$F_{MN} = \sum_{j=1}^{K} \eta_j |\langle\psi_j^N|\phi_j\rangle|^2. \tag{3.1}$$

A unitary produces N copies $|\phi_1\rangle$ or $|\phi_2\rangle$, to resemble the original states as closely as possible:

$$U|\psi_1^M\rangle|i\rangle = |\phi_1^n\rangle, \tag{3.2}$$
$$U|\psi_2^M\rangle|i\rangle = |\phi_2^n\rangle. \tag{3.3}$$

The inner product of the above two equations gives a relationship between the input and the output states.

$$|\langle\psi_1|\psi_2\rangle|^M = |\langle\Phi_1|\Phi_2\rangle|^N \tag{3.4}$$
$$\cos^M 2\theta = \cos^N(\phi_1 + \phi_2) \tag{3.5}$$

Where the input states are expressed as $|\psi_{1,2}^N\rangle = \cos\theta|1\rangle \pm \sin\theta|0\rangle$ and the clones as $|\phi_{1,2}\rangle = \cos\phi_1|1\rangle \pm \sin\phi_1|0\rangle$.

It was shown in [?] that the maximum fidelity is:

$$F_{MN} = \frac{1}{2}[1 + \sqrt{1 - 4\eta_1\eta_2\sin^2(2\theta - (\phi_1 + \phi_2))}] \tag{3.6}$$

To see the connection with state separation expand the sin term:

$\sin(2\theta - (\phi_1 + \phi_2)) = \sin(2\theta)\cos(\phi_1 + \phi_2) + \cos(2\theta)[1 - \cos^2(\phi_1 + \phi_2)]$

The overlap of the output states $\cos(\phi_1 + \phi_2)$ can be viewd as state separation. In this case we are in the limit $s' = s$, approximate cloning. To interpolate between approximate cloning and deterministic cloning s' takes on the optimal values derived in Section II.