Quantum State Discrimination and Quantum Cloning: Optimization and Implementation

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Motivation Behind Quantum Info Science

- When quantum information is received, the text must be read out.
- Applications to quantum cryptography.
- Quantum Simulations.

Quantum State Discrimination: 2 pure states

Two non-orthogonal pure states can be represented in 2D

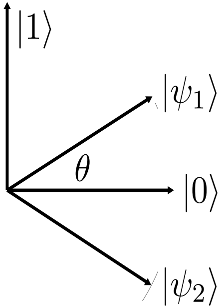

Figure: 2 pure states,

$$|\psi_1\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle \qquad |\psi_2\rangle = \cos\theta\,|0\rangle - \sin\theta\,|1\rangle$$

Quantum State Discrimination: 2 pure states

- Two detectors Π_1 and Π_2 which unambiguously detect the two pure states. The detectors span the Hilbert space:

$$\Pi_1 + \Pi_2 = I \tag{1}$$

- Π_i only clicks for $|\psi_i\rangle$ $i = 1, 2$, such that $\Pi_i |\psi_j\rangle = 0$, $i \neq j$

$$\begin{aligned} \langle\psi_1| (\Pi_1 + \Pi_2) |\psi_1\rangle &= \langle\psi_1 \mid \psi_1\rangle \\ p_1 &= 1 \end{aligned}$$

- Similarly it can be shown that $p_2 = 1$. However multiplying Eq. (1) l.h.s by $\langle\psi_1|$ and r.h.s by $|\psi_2\rangle$:

$$\begin{aligned} \langle\psi_1| (\Pi_1 + \Pi_2) |\psi_2\rangle &= \langle\psi_1 \mid \psi_2\rangle \\ 0 &= \langle\psi_1 \mid \psi_2\rangle \end{aligned}$$

Unambiguous Discrimination

- Unambiguous State Discrimination is still possible if a third detector is added:

$$\Pi_1 + \Pi_2 + \Pi_0 = I \tag{2}$$

- The condition $\Pi_i |\psi_j\rangle = 0$ still holds. Multiplying 2 by $\langle\psi_i|$ from l.h.s and $|\psi_i\rangle$:

$$\begin{aligned} \langle\psi_i| (\Pi_1 + \Pi_2 + \Pi_0) |\psi_i\rangle &= \langle\psi_i \mid \psi_i\rangle\,, \\ p_i + q_i &= 1, \end{aligned}$$

  where: p_i rate of successfully identifying the state, q_i failure rate.

- The task is to minimize the average failure rate: $Q = \eta_1 q_1 + \eta_2 q_2, \quad \eta_i$ the prior rates of input states.

Unambiguous Discrimination

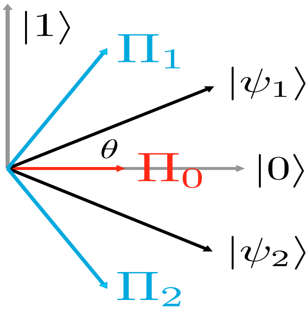

Figure: UD with three detectors

- The optimum Q is found to be: $Q_0^{IDP} = 2\sqrt{\eta_1 \eta_2} s$

Minimum Error

- The detectors can make a mistake, but are not allowed to abstain from giving an answer.

$$\Pi_1 + \Pi_2 = I \tag{3}$$

Multiplying Eq. (3) by $\langle\psi_i|$ from l.h.s and r.h.s by $|\psi_i\rangle$:
$\langle\psi_i| (\Pi_1 + \Pi_2) |\psi_i\rangle = \langle\psi_i \mid \psi_i\rangle \Rightarrow \qquad p_i + r_i = 1$

- r_i error rate, q_i failure rate.
- Minimize the average error rate[1]:
$P_E^{min} = \eta_1 r_1 + \eta_2 r_2 = \frac{1}{2}\left[1 - \sqrt{1 - 4\eta_1\eta_2 s}\right]$

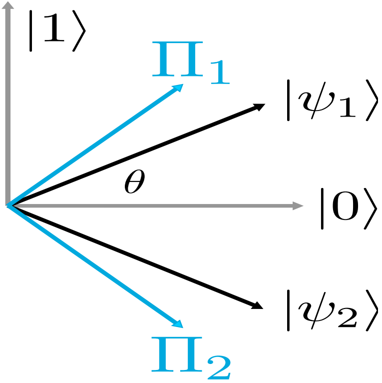

Figure: ME

Fixed Rate of Inconclusive Outcomes (FRIO)

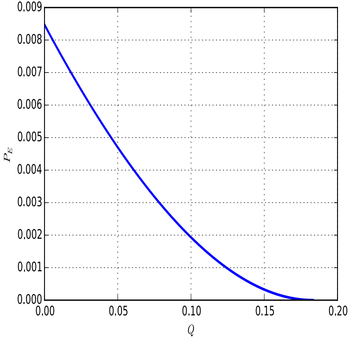

Figure: FRIO: Error rate interpolates between 0, for failure rate $Q = Q_0$, and the Helstrom P_E for zero failure rate $Q = 0$. The graph has prior probabilities $\eta_1 = 0.3$, $\eta_2 = 0.7$ and overlap $s = 0.2$.

FRIO

The error rate in the Helstrom bound can be lowered if we relax a constrained and allow for some FRIO.
Bagan et al [2] solved the problem by defining: $I - \Pi_0 \equiv \Omega$, then eliminating it by mult. l.h.s by $\Omega^{-1/2}$ and r.h.s by $\Omega^{-1/2}$

$$\begin{aligned} \Pi_1 + \Pi_2 &= \Omega \\ \tilde{\Pi}_1 + \tilde{\Pi}_2 &= I \end{aligned} \tag{4}$$

where $\tilde{\Pi}_i \equiv \Omega^{-1/2} \Pi_i \, \Omega^{-1/2}$. The optimization to Eq. (4) is that of Helstrom with new normalized prior probabilities and normalized states:

$$\begin{aligned} \tilde{P}_E &= \frac{1}{2}\left[1 - \sqrt{1 - 4\tilde{\eta}_1\tilde{\eta}_2 |\langle\tilde{\psi}_1|\tilde{\psi}_2\rangle|^2}\right] \\ P_E^{min} &= \frac{1}{2}\left\{(1-Q) - \sqrt{(1-Q)^2 - (Q_0 - Q)^2}\right\} \end{aligned}$$

Implementation of FRIO

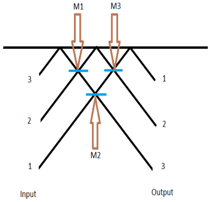

Figure: Six-port interferometer

Choosing the basis of our Hilbert space: $a_1^\dagger\,|000\rangle = |100\rangle \equiv |1\rangle$
$a_2^\dagger\,|000\rangle = |010\rangle \equiv |2\rangle$.
Two non-orthogonal input states can be expressed as:
$|\psi_1\rangle_{in} = |1\rangle, \;\; |\psi_2\rangle_{in} = \cos\theta|1\rangle + \sin\theta|2\rangle.$

Unitary

In Schrodinger picture the input and output states are related by: $U\,|in\rangle = |out\rangle$.

$$\begin{aligned} U|1\rangle &= \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle + \sqrt{q_1}|3\rangle \\ U(\cos\theta|1\rangle + \sin\theta|2\rangle) &= \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle + \sqrt{q_2}|3\rangle \end{aligned}$$

where $p_i + q_i + r_i = 1$.

First column:

$\langle 1|U|1\rangle = U_{11} = \sqrt{p_1}$

$\langle 1|U|1\rangle = U_{11} = \sqrt{p_1}$

$\langle 2|U|1\rangle = U_{21} = \sqrt{q_1}$,

Second Column

$U_{12} = \frac{\sqrt{r_2} - \sqrt{p_1}\cos\theta}{\sin\theta}$

$U_{22} = \frac{\sqrt{p_2} - \sqrt{r_1}\cos\theta}{\sin\theta}$

$U_{32} = \frac{\sqrt{q_2} - \sqrt{q_1}\cos\theta}{\sin\theta}$

Third Column calculated from unitary condition $U^\dagger U = I$.

Unitary

$$U = \begin{pmatrix} \sqrt{p_1} & \frac{\sqrt{r_2}-\sqrt{p_1}\cos\theta}{\sin\theta} & -\frac{\sqrt{\sin^2\theta-p_1-r_2+2\sqrt{p_1 r_2}\cos\theta}}{\sin\theta} \\ \sqrt{r_1} & \frac{\sqrt{p_2}-\sqrt{r_1}\cos\theta}{\sin\theta} & -\frac{\sqrt{\sin^2\theta-r_1-p_2+2\sqrt{p_2 r_1}\cos\theta}}{\sin\theta} \\ \sqrt{q_1} & \frac{\sqrt{q_2}-\sqrt{q_1}\cos\theta}{\sin\theta} & +\frac{\sqrt{sin^2\theta-q_1-q_2+2\sqrt{q_1 q_2}\cos\theta}}{\sin\theta} \end{pmatrix}. \tag{5}$$

The coefficients r_i and p_i however are not determined in the Bagan solution. We solve the FRIO problem using Neumark setup.

Unitary

$$
\begin{aligned}
U|1\rangle &= \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle + \sqrt{q_1}|3\rangle \\
U(\cos\theta|1\rangle + \sin\theta|2\rangle) &= \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle + \sqrt{q_2}|3\rangle
\end{aligned}
$$

Optimization starts with six parameters $p_1, r_1, q_1, r_2, p_2, q_2$.
The unitary condition $p_i + r_i + q_i = 1 \Rightarrow p_i = 1 - r_i - q_i$, reduces the problem into 4 independent variables
The constraint from the inner product:

$$
\begin{aligned}
s &= \sqrt{p_1 r_2} + \sqrt{p_2 r_1} + \sqrt{q_1 q_2} \\
&= \sqrt{r_2\,(1 - r_1 - q_1)} + \sqrt{r_1\,(1 - r_2 - q_2)} + \sqrt{q_1 q_2} \quad (6)
\end{aligned}
$$

eliminates one more variable, three independent variables left.
The fixed average failure $Q = \eta_1 q_1 + \eta_2 q_2$ further reduces the problem into two independent variables.
Minimizing the average error rate $P_E = \eta_1 r_1 + \eta_2 r_2$ subject to

Solution to Lagrange Multipliers

To minimize $P_E(r_1, r_2)$:

- Set $dF_{(r_1,r_2,\lambda)}/dr_i$ to zero and solve for $r_i(\lambda)$
- Substitute $r_i(\lambda)$ into the constraint ??
- Solve for λ
- Substitute λ into $r_i(\lambda)$
- Substitute $r_i(\lambda)$ into $P_E(r_1, r_2)$
- Optimize $P_E(r_1, r_2)$

$$r_i = \frac{1}{2}\left[\left(1 - \frac{Q}{2\eta_i}\right) - \frac{\left(1 - \frac{Q}{2\eta_i}\right)(1-Q) - \frac{1}{2\eta_i}(Q_o - Q)^2}{\sqrt{(1-Q)^2 - (Q - Q_o)^2}}\right], \tag{7}$$

$$p_i = \frac{1}{2}\left[\left(1 - \frac{Q}{2\eta_i}\right) + \frac{\left(1 - \frac{Q}{2\eta_i}\right)(1-Q) - \frac{1}{2\eta_i}(Q_o - Q)^2}{\sqrt{(1-Q)^2 - (Q - Q_o)^2}}\right]. \tag{8}$$

Reck-Zeilinger Algorithm

- Any discrete finite-dimensional unitary operator can be can be constructed in the lab using optical devices.
- Following the Reck-Zeilinger algorithm the unitary calculated in (5) can be decomposed in terms of three beam splitters $U = M_1 M_2 M_3$ where: [3].

$$M_1 = \begin{pmatrix} \sin\omega_1 & \cos\omega_1 & 0 \\ \cos\omega_1 & -\sin\omega_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \sin\omega_2 & 0 & \cos\omega_2 \\ 0 & 1 & 0 \\ \cos\omega_2 & 0 & -\sin\omega_2 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\omega_3 & \cos\omega_3 \\ 0 & \cos\omega_3 & -\sin\omega_3 \end{pmatrix}.$$

Reck-Zeilinger Algorithm

The reflective and transmittance coefficients are calculated and expressed in terms of η_i, s, and FRIO Q.

$$\cos\omega_1 = \sqrt{\frac{r_1}{1-Q/2\eta_1}}, \qquad \sin\omega_1 = \sqrt{\frac{p_1}{1-Q/2\eta_1}}$$

$$\cos\omega_2 = \sqrt{Q/2\eta_1}, \qquad \sin\omega_2 = \sqrt{1-Q/2\eta_1}$$

$$\cos\omega_3 = -\frac{\sqrt{Q/2\eta_2} - \frac{Q_o}{2\eta_1}\sqrt{Q/2\eta_2}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}}$$

$$\sin\omega_3 = \frac{\sqrt{1-Q_o^2/4\eta_1\eta_2 - Q/(2\eta_1\eta_2) + QQ_o/(2\eta_1\eta_2)}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}}$$

r_i and p_i are explicitly given in (7) and (8).

Quantum Cloning

A different approach to state discrimination: Clone the states first then perform a measurement:
Given a state from a set of non-orthogonal quantum states $\{|\psi_1\rangle, |\psi_2\rangle\}$ make a large number of copies:

$$U|\psi_1\rangle|i\rangle^{N-1} = |\psi_1\rangle|\psi_1\rangle....|\psi_1\rangle = |\psi_1\rangle^N$$
$$U|\psi_2\rangle|i\rangle^{N-1} = |\psi_2\rangle|\psi_2\rangle....|\psi_2\rangle = |\psi_2\rangle^N$$

Now perform a measurement scheme: ME or UD
$P_E = \frac{1}{2}\left[1 - \sqrt{1 - 4\eta_1\eta_2 s^{2N}}\right]$, $Q = 2\sqrt{\eta_1\eta_2}s^N$
In the asymptotic limit the error rate and the failure rate reduce to zero

No-Cloning Theorem

It was shown by Wootters, Zurek [4] and Dieks [5] that deterministic quantum cloning is not possible. Imagine a unitary operator which would copy the state $|\psi_i\rangle$ into $|0\rangle$:

$$\begin{aligned} U|\psi_1\rangle|0\rangle &= |\psi_1\rangle|\psi_1\rangle \\ U|\psi_2\rangle|0\rangle &= |\psi_2\rangle|\psi_2\rangle \end{aligned} \tag{9}$$

Inner product: $\langle\psi_2|\psi_1\rangle\langle 0|0\rangle = |\langle\psi_2|\psi_1\rangle|^2 \Rightarrow s = s^2$. The condition can be satisfied only if $s = 0$, states are orthogonal, or $s = 1$, the two states are the same.

Beyond the no-cloning theorem

- M.Hillery and V. Bužek make approximate clones [6, 7, 8]!
- Two main quantum cloning machines (QCM): Universal and State Dependent
- Universal QCM: independent of the input
- State Dependent QCM: Probabilistic or Deterministic.
- Deterministic SD-QCM: produce approximate clones on demand while optimizing the fidelity between clones and input states.
- Probabilistic SD-QCM: produce exact clones with some rate of abstention.

Probabilistic SD-QCM

Given a pair non-orthogonal quantum states $\{|\psi_1\rangle, |\psi_2\rangle\}$ a probabilistic QCM produces $\{|\psi_1\rangle|\psi_1\rangle, |\psi_2\rangle|\psi_2\rangle\}$

$$\begin{aligned} U|\psi_1\rangle|i\rangle &= \sqrt{p_1}|\psi_1\rangle|\psi_1\rangle\,|1\rangle + \sqrt{q_1}|\Phi\rangle\,|0\rangle\,, \\ U|\psi_2\rangle|i\rangle &= \sqrt{p_2}|\psi_2\rangle|\psi_2\rangle|1\rangle + \sqrt{q_2}|\Phi\rangle\,|0\rangle\,, \end{aligned}$$

$p_i + q_i = 1$.
Constraint from scalar product: $s = \sqrt{p_1 p_2} s^2 + \sqrt{q_1 q_2}$.
Solution to equal priors [9]: $s = ps^2 + q \Rightarrow q = \frac{s}{1+s}$

State Separation

Probabilistic cloning as a special case of state separation

$$\begin{aligned} U|\psi_1\rangle|i\rangle &= \sqrt{p_1}|\phi_1\rangle|\alpha\rangle + \sqrt{q_1}|\Phi\rangle|0\rangle, \\ U|\psi_2\rangle|i\rangle &= \sqrt{p_2}|\phi_2\rangle|\alpha\rangle + \sqrt{q_2}|\Phi\rangle|0\rangle, \end{aligned} \quad (10)$$

The constraint: $s = \sqrt{p_1 p_2} s' + \sqrt{q_1 q_2}$. (1 to 2 exact cloning is equivalent to setting $s' = s^2$)
Optimize the average failure rate $Q = \eta_1 q_1 + \eta_2 q_2$, failure rate to separate the input states.

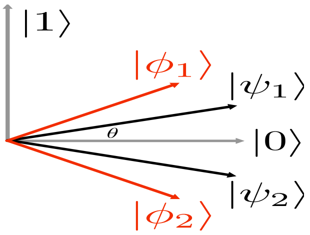

Figure: The input states $\{|\psi_1\rangle, |\psi_2\rangle\}$ are separated into a pair of

Geometric solution to state separation

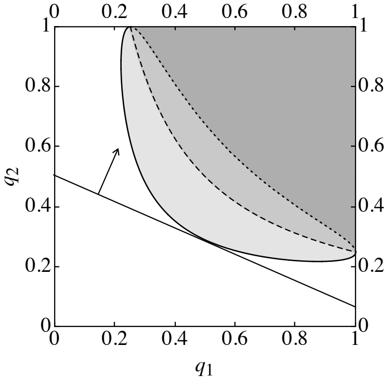

Figure: Unitarity curves $s = \sqrt{p_1 p_2}\, s' + \sqrt{q_1 q_2}$ the optimal straight segment $Q = \eta_1 q_1 + \eta_2 q_2$ and its normal vector (η_1, η_2).

Parametric solution to State Separation

Parametrize the unitary constraint: $s = \sqrt{p_1 p_2} s' + \sqrt{q_1 q_2}$ using $p_1 p_1 = t^2, \quad q_1 q_2 = z^2.$
The condition becomes:
$z = s - s't, \qquad 0 \leq t, z \leq 1, \qquad 0 \leq s' \leq s.$

- From the first equation $t^2 = (1 - q_1)(1 - q_2) = 1 + z^2 - q_1 - q_2$. We now solve for q_1 (similarly for q_2) and obtain
$$q_{1,2} = \frac{1 + z^2 - t^2 \pm \sqrt{(1 + z^2 - t^2)^2 - 4z^2}}{2}.$$
- The condition $Q = \eta_1 q_1 + \eta_2 q_2$ becomes $2Q = 1 + z^2 - t^2 \pm (\eta_1 - \eta_2)\sqrt{(1 + z^2 - t^2)^2 - 4z^2}$. Solve for z^2
$$\begin{aligned} z^2 &= \frac{2\eta_1\eta_2(1+\tau) - 1 + Q + \sqrt{(1 - 4\eta_1\eta_2)\left[(1-Q)^2 - 4\right.}}{2\eta_1\eta_2} \\ &\equiv \zeta(\tau) \end{aligned}$$

Parametric solution to State Separation

The derivative $d\zeta/d\tau$ is immediate. We find that the extrema is located at

$$t_{\min} = \begin{cases} \sqrt{\left(1-\frac{Q}{2\eta_1}\right)\left(1-\frac{Q}{2\eta_2}\right)}, & \text{if} \quad 0 \leq Q \leq 2\eta_1 \\ 0, & \text{if} \quad 2\eta_1 < Q \leq 1. \end{cases}$$

The corresponding values of z are

$$z_{\min} = \begin{cases} \frac{Q}{2\sqrt{\eta_1\eta_2}}, & \text{if} \quad 0 \leq Q \leq 2\eta_1 \\ \sqrt{\frac{Q-\eta_1}{\eta_2}}, & \text{if} \quad 2\eta_1 < Q \leq 1. \end{cases}$$

Parametric solution to State Separation

$$s'(t) = -\frac{dz}{dt} = -\frac{t\,\zeta'(t^2)}{\sqrt{\zeta(t^2)}}, \qquad t_{\min} \leq t \leq 1 - Q,$$

and next *ii.* define

$$s(t) = z + ts'(t) = \sqrt{\zeta(t^2)} + ts'(t), \qquad t_{\min} \leq t \leq 1 - Q.$$

where

$$\zeta'(t^2) = 1 - \frac{\sqrt{1 - 4\eta_1\eta_2}}{\sqrt{(1-Q)^2 - 4\eta_1\eta_2 t^2}}$$

For $s < z_{\min}$ it is always possible to separate the initial states, i.e., $|\psi_1\rangle$ and $|\psi_2\rangle$ can be made orthogonal. We note that the condition $s = z_{\min}$ is equivalent to the unambiguous discrimination result

$$Q = 2\sqrt{\eta_1\eta_2}s, \quad Q = \eta_1 + \eta_2 s^2.$$

Parametric solution to State Separation

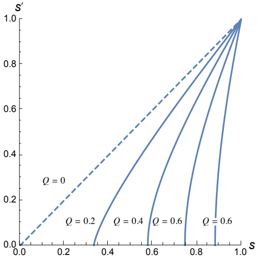

Figure: s vs. s' for variant fixed Q. The plot is for $\eta_1 = 0.1$. As η_1 approaches $1/2$ the curves approach a straight line. The difference is more noticeable for very small values of η_1.

Conclusion

- Optical implementation to FRIO using a optical fibers, beam splitters and mirrors.
- The implementation scheme is readily realizable in a lab, all coefficients of beam splitters are explicitly calculated.
- Full geometric and parametric solution to state separation.
- Probabilistic exact cloning as a special case to state separation.

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