Optimal Measurements

Vadim Yerokhin

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Outline

1. Introduction
   - Why Measurement Theory?
   - Our Game
   - The Math of Measurement Theory
   - Discrimination Strategies
2. Subspace Measurements
   - Two Subspaces
   - General Solution
3. Future Work
   - 2D Mixed States

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- Classical bits versus quantum bits: instead of just a 0 or 1, quantum bits can maintain a superposition state
- The probabalistic nature of detection: only orthogonal states can be discriminated perfectly
- Quantum Computing
- Quantum Communication

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- Someone prepares one two different pure states $|\psi_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle$
- Each state has a different likelihood η_i
- One particle (state) is sent at a time. Our job is to guess as best we can what state was sent.
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Density Matrices

A density matrix is a generalized state that is a statistical collection of pure states defined by four properties:

- 1. $\rho = \sum_i \eta_i |\psi_i\rangle\langle\psi_i|$ where $\sum \eta_i = 1$
- 2. It is Hermitian
- 3. $Tr\rho = 1$
- 4. $\langle\phi_i|\rho|\phi_i\rangle \geq 0$

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Measurement Operators

- Measurements are a decomposition of the identity in terms of positive semi-definite matrices.
- A measurement can be either a projector or a generalized measurement. In the latter case it must fulfill only 2 properties:
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  2. They are Hermitian. This corresponds to real eigenvalues (measurement outcomes).

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Expectation Values as Traces

Since non-orthogonal states cannot be discriminated perfectly, we can speak of the probability of a given outcome:

- $\langle \Pi_i \rangle = \sum_j \eta_j \langle \psi_j | \Pi_i | \psi_j \rangle = Tr(\Pi_i \rho)$

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Minimum Error Discrimination

The first measurement strategy was Minimum Error Discrimination.

- Two orthogonal projectors, each clicks for a state.
- There is a success rate and an error rate if the states are not orthogonal.

$$P_s = \eta_1 Tr(\rho_1 \Pi_1) + \eta_2 Tr(\rho_2 \Pi_2)$$
$$P_e = \eta_2 Tr(\rho_2 \Pi_1) + \eta_1 Tr(\rho_1 \Pi_2)$$

- The minimum error rate for pure states is achieved by the Helstrom bound.

$$P_E = \frac{1}{2}(1 - \sqrt{1 - 4\eta_1\eta_2 \left|\langle\psi_1|\psi_2\rangle\right|^2})$$

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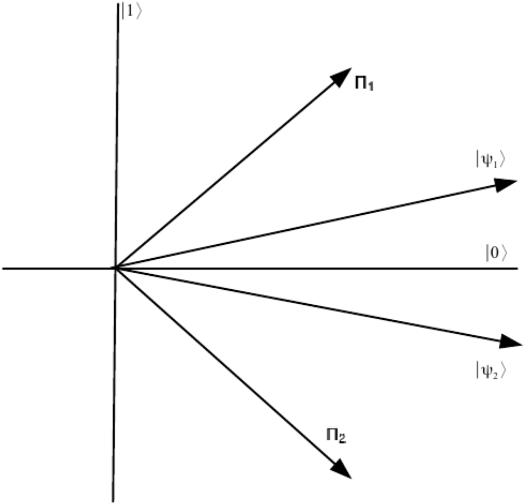
Minimum Error Graph



Unambigious State Discrimination

- Make the measurement operators orthogonal to the state that we don't want to measure.
- Since they are no longer orthogonal they don't sum to the identity. A third, inconclusive outcome is necessary.
- The detector corresponding to the inconclusive outcome we call Π_0
- The failure probability we call Q:

$$Q = \eta_1 Tr(\rho_1 \Pi_0) + \eta_2 Tr(\rho_2 \Pi_0) = Tr(\rho \Pi_0)$$

- $Q_0 = 2\sqrt{\eta_1 \eta_2} cos\theta$ is the failure rate that corresponds to the best measurement.

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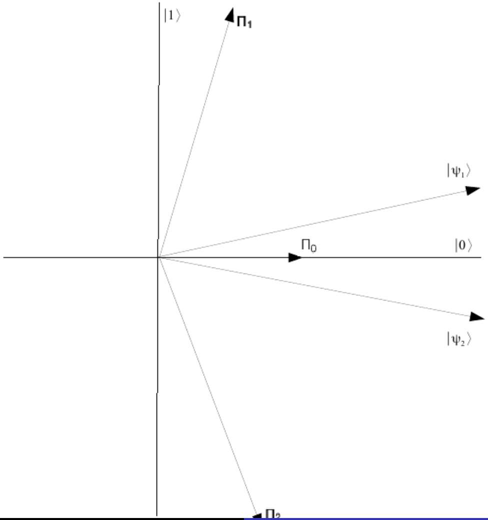

Intermediate Measurement Transformation

The Intermediate Measurement scheme interpolates optimally between ME and UD. The transformation we want takes us from 3 detection operators down to two:

$$\Pi_1 + \Pi_2 + \Pi_0 = I$$

$$\Omega^{-1/2}[\Pi_1 + \Pi_2]\Omega^{-1/2} = \widetilde{\Pi}_1 + \widetilde{\Pi}_2 = I$$

where $\Omega = I - \Pi_0$

Intermediate Measurement Optimization

The transformation changes the density matrices and a-priori probabilities, which formulate a new ME problem. The solution is a Helstrom bound in which a further optimization is required. The final result is:

$$P_e = \frac{1}{2}(1 - Q - \sqrt{(1-Q)^2 - (Q - Q_0)^2})$$

$0 \leq Q \leq Q_0$ where $Q_0 = 2\sqrt{\eta_1 \eta_2} cos\theta$

Outline

Two Subspaces Setup

We consider the case of two mixed states that form two two-dimensional Hilbert spaces. Hence our density matrices are in four dimensions consisting of two tensor product spaces:

$$\rho_1 = r_1|r_1\rangle\langle r_1| + r_2|r_2\rangle\langle r_2|$$

$$\rho_2 = s_1|s_1\rangle\langle s_1| + s_2|s_2\rangle\langle s_2|$$

With our full density matrix is $\rho = \eta_1\rho_1 + \eta_2\rho_2$
We begin by focusing on the first subspace and define our measurement operators for the first subspace as

$$\widetilde{\Pi}_{1,1} + \widetilde{\Pi}_{2,1} = I_1$$

Where I_1 is the identity matrix of the first subspace

We define $\Pi_{0,1} = \xi_1 |0\rangle_{11}\langle 0|$and we find the failure rate to be

$$Q_1 = \xi_1[\eta_{1,1} cos^2\phi_1 + \eta_{2,1} cos^2(\theta_1 - \phi_1)]$$

Where θ_1 is the overlap angle between the two states in subspace 1 , ϕ_1 is the angle $|r_1\rangle$ makes with respect to $|0\rangle_1$, and $\eta_{1,i} = \eta_1 r_i$, $\eta_{2,i} = \eta_2 s_i$

The error rate in that subspace is

$$P_{e,1} = \eta_{1,1}\langle r_1|\Pi_{2,1}|r_1\rangle + \eta_{2,1}\langle s_1|\Pi_{1,1}|s_1\rangle$$

We introduce the normalized state vector

$$|\widetilde{r_1}\rangle = \frac{\Omega^{1/2}|r_1\rangle}{\sqrt{\langle r_1|\Omega|r_1\rangle}}$$

and normalized coefficients

$$\widetilde{\eta_{1,1}} = \frac{\eta_{1,1}\langle r_1|\Omega|r_1\rangle}{\eta_{1,1}\langle r_1|\Omega|r_1\rangle + \eta_{2,1}\langle s_1|\Omega|s_1\rangle}$$

We get

$$P_{e,1} = \eta_{1,1}\langle r_1|\Omega|r_1\rangle\langle\tilde{r}_1|\widetilde{\Pi_2}|\tilde{r}_1\rangle + \eta_{2,1}\langle s_1|\Omega|s_1\rangle\langle\tilde{s}_1|\widetilde{\Pi_1}|\tilde{s}_1\rangle$$

$$=$$

$$[\eta_{1.1}\langle r_1|\Omega|r_1\rangle + \eta_{2,1}\langle s_1|\Omega|s_1\rangle](\widetilde{\eta_{1,1}}\langle\tilde{r}_1|\widetilde{\Pi_2}|\tilde{r}_1\rangle + \widetilde{\eta_{2,1}}\langle\tilde{s}_1|\widetilde{\Pi_1}|\tilde{s}_1\rangle)$$

We notice that the second set of ( ) with all tildes contains a pure state minimum error problem, while with the notation $\eta_{1,1} + \eta_{2,1} = \omega_1$ the left hand set of [ ] can be reworked into $\omega_1 - Q_1$

$$= \frac{1}{2}[\omega_1 - Q_1](1 - \sqrt{1 - 4\widetilde{\eta_{1,1}}\widetilde{\eta_{2,1}}|\langle\tilde{r}_1|\tilde{s}_1\rangle|^2})$$

$$P_{e,1} = \frac{1}{2}(\omega_1 - Q_1 - \sqrt{(\omega_1 - Q_1)^2 - (Q_{0,1} - Q_1 sin2\phi)^2})$$

where we used the notation $Q_{0,1} = 2\sqrt{\eta_{1,1}\eta_{2,1}}cos\theta_1$ and $sin\phi = \frac{\sqrt{\eta_{2,1}}cos(\theta_1 - \phi_1)}{\sqrt{\eta_{1,1}cos^2(\phi_1) + \eta_{2,1}cos^2(\theta_1 - \phi_1)}}$
Minimization on the expression as a function of ϕ tells us to set $\phi = \frac{\pi}{4}$ so finally

$$P_{e,1} = \frac{1}{2}(\omega_1 - Q_1 - \sqrt{(\omega_1 - Q_1)^2 - (Q_{0,1} - Q_1)^2})$$

This result agrees with the single subspace limit and in fact is simply the optimized solution for that subspace alone. We can derive a similar result for the other subspace, so are now ready to consider an optimal distribution of failure among the two subspaces. We want to treat this distribution problem for N subspaces so we first generalize our preceding solution to 2n dimensions.

Outline

Normalization

We want to generalize the formalisim to 2n dimensions. Recognizing the measurement probabilities in each subspace don't add up to 1, we want to normalize the new operators so that we can solve them like the 2d case where we had

$$P_e + P_s + Q = 1$$

Instead, now we have

$$P_{e,i} + P_{s,i} + Q_i = \eta_{1,i} + \eta_{2,i} = \omega_i$$

Where we call $P_{e,i}$ the probability of error in subspace i and $P_{s,i}$, Q_i the success and the failure probabilities in that subspace too. And ω_i is the weight of that subspace, or the total probabilitiy of a partilcle being found therein.

So we redefefine

$$\bar{P_{e,i}} + \bar{P_{s,i}} + \bar{Q_i} = 1$$

with $\bar{\bullet} = \frac{\bullet}{\omega_i}$
We redefine the weights $\eta_{\bar{1},i} = \frac{\eta_1 r_i}{\omega_i}$ and $\eta_{\bar{2},i} = \frac{\eta_2 s_i}{\omega_i}$ that still sum 1, so that the states and measurements don't change.

Now it is straightforward to apply the result of the renormalized minimization to each subspace:
namely,

$$\bar{P}_{e,i} = \frac{1}{2}(1 - \bar{Q}_i - \sqrt{(1 - \bar{Q}_i)^2 - (\bar{Q}_{0,i} - \bar{Q}_i)^2})$$

where

$$\bar{Q}_{0,i} = 2\sqrt{\bar{\eta}_{1,i}\bar{\eta}_{2,i}}cos\theta_i$$

is also

$$P_{e,i} = \frac{1}{2}(\omega_i - Q_i - \sqrt{(\omega_i - Q_i)^2 - (Q_{0,i} - Q_i)^2})$$

Lagrangian Optimization

Since each subspace can vary independently we are interested in the optimal values for Q_i as a function of fixed Q
If we consider this a Lagrange Multiplier problem with $P_{e,i}$ with constraint $\sum Q_i = Q$ then we get the constrained function

$$F = P_{e,i} - \lambda(\sum Q_i - Q)$$

$$= \frac{1}{2}(\omega_i - Q_i - \sqrt{(\omega_i - Q_{0,i})(\omega_i + Q_{0,i} - 2Q_i)} - \lambda(\sum Q_i - Q)$$

$$\frac{dF}{dQ_i} = 0$$

when

$$Q_i = \frac{1}{2}(\omega_i + Q_{0,i} - \frac{\omega_i - Q_{0,i}}{(2\lambda + 1)^2})$$

Since $\sum Q_i = Q$ we have

$$Q = \sum Q_i = \frac{1}{2}(1 + Q_0 - \frac{1 - Q_0}{(2\lambda + 1)^2})$$

where $Q_0 = \sum Q_{0,i} = 2\sqrt{\eta_1\eta_2}\sum\sqrt{r_i s_i} cos\theta_i$

Solving for λ we find

$$\frac{2Q-1-Q_0}{1-Q_0} = \frac{-1}{(2\lambda+1)^2}$$

We insert this expression for λ into the expression for Q_i to eliminate the Lagrange Multiplier, and find its optimal value as

$$Q_i = \frac{1}{2}(\omega_i + Q_{0,i} + (\omega_i - Q_{0,i})(\frac{2Q-1-Q_0}{1-Q_0}))$$

Now the optimized subspace error rate is

$$P_{e,i} = \frac{1}{2}(\omega_i - Q_i - (\omega_i - Q_{0,i})\sqrt{\frac{1+Q_0-2Q}{1-Q_0}})$$

with the total optimal error rate $P_e = \sum P_{e,i}$

$$P_e = \frac{1}{2}(1 - Q - \sqrt{(1-Q)^2 - (Q-Q_0)^2})$$

Threshold Structure

The range of the subspace solution is valid strictly for more than one subspace and is limited by the positivity of Q_i. If we re-write it as

$$Q_i = \frac{Q_{0,i} - \omega_i Q_0 + Q(\omega_i - Q_{0,i})}{1 - Q_0}$$

and set it to zero we find the critical value of Q for that subspace to be

$$Q = Q_c^i = \frac{\omega_i Q_0 - Q_{0,i}}{\omega_i - Q_{0,i}}$$

So when Q falls below Q_c^j we fix $Q_j = 0$ and discard that subspace from our optimization.

We discover the threshold behavior of this solution and realize that after the first threshold we must re-do the optimization with the remaining subspaces.

It is worthwile to consider also the positivity of the Q_c^i . Since $\omega_i - Q_{0,i} \geq 0$ we analyze the positivity of $\omega_i Q_0 - Q_{0,i}$. For this to be positive we need $\omega_i Q_0 \geq Q_{0,i}$ or $Q_0 \geq \bar{Q}_{0,i}$ which means that the UD failure rate of that normalized subspace should be smaller than the total UD failure rate of the system of subspaces.

First Iteration

After the first state is discarded from the failure optimization the set of subspaces decreases causing changes in the formula. To elucidate suppose we order the subspaces such that the highest has the largest Q_c^i, and have discarded the *Nth* subspace associated with the Q_N and ω_N . Now

$$Q_i^{(1)} = \frac{Q_{0,i}\Lambda_{N-1} - \omega_i F_{N-1} + Q(\omega_i - Q_{0,i})}{\Lambda_{N-1} - F_{N-1}}$$

between $Q_c^N \geq Q \geq Q_c^{(1)N-1}$ where we've introduced the notation $\Lambda_k = \sum_1^k \omega_i$ and $F_k = \sum_1^k Q_{0,i}$, and the number in parenthesis in $Q_i^{(1)}$ indicates the number of subspaces removed from the Lagrangian optimization.

Now setting the new $Q_i = 0$ we find the Q_c^i have changed as well since we have redone the optimization. We can call the new (second order) set of critical values $Q_c^{(1)i}$ as

$$Q_c^{(1)i} = \frac{\omega_i F_{N-1} - Q_{0,i}\Lambda_{N-1}}{\omega_i - Q_{0,i}}$$

For this to be positive we need $\omega_i F_{N-1} \geq \Lambda_{N-1} Q_{0,i}$ or we can consider this inequality as

$$\frac{\sum_1^{N-1} Q_{0,i}}{\sum_1^{N-1} \omega_i} \geq \frac{Q_{0,i}}{\omega_i}$$

which states that the weighted maximum failure rate for the single subspace must be less than the average over all remaining subspaces.

General Iteration

We can iterate this process to find the *n*th order set of equations as

$$Q_i^{(n)} = \frac{Q_{0,i}\Lambda_{N-n} - \omega_i F_{N-n} + Q(\omega_i - Q_{0,i})}{\Lambda_{N-n} - F_{N-n}}$$

$$Q_c^{(n)i} = \frac{\omega_i F_{N-n} - Q_{0,i}\Lambda_{N-n}}{\omega_i - Q_{0,i}}$$

Final Iteration

For only two remaining subspaces we can conclude the equation for the critical points will be

$$Q_c^{(N-2)i} = \frac{\omega_i(Q_{0,1} + Q_{0,2}) - (\omega_1 + \omega_2)Q_{0,i}}{\omega_i - Q_{0,i}}$$

and the inequality for consideration for the final elimination is

$$\frac{Q_{0,1} + Q_{0,2}}{\omega_1 + \omega_2} \geq \frac{Q_{0,i}}{\omega_i}$$

Graph of 3 Subspace Example

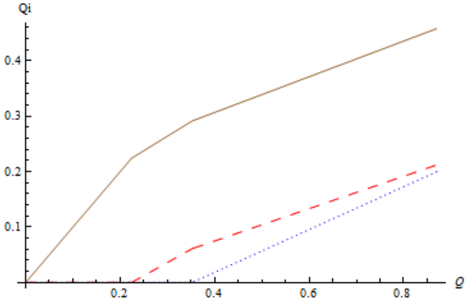

Figure: Q vs Qi for all iterations with Q_3, Q_1, Q_2 from top to bottom

Continuity

It is worthwile to demonstrate the continuity of our solutions for the Q_i's. To do this we need to show that the optimal solutions match at the boundaries where a subspace is discarded. Hence we want to show

$$Q_i^{(n)}(Q = Q_c^{(n)N-n}) = Q_i^{(n+1)}(Q = Q_c^{(n)N-n})$$

Where we have chosen to consider the nth iteration of the solution and now have decided to discard the $N - n$th subspace.

Now the equality we want to demonstrate is

$$\frac{Q_{0,i}\Lambda_{N-n} - \omega_i F_{N-n} + Q^{(n)N-n}(\omega_i - Q_{0,i})}{\Lambda_{N-n} - F_{N_n}} =$$

$$= \frac{Q_{0,i}\Lambda_{N-n-1} - \omega_i F_{N-n-1} + Q^{(n)N-n}(\omega_i - Q_{0,i})}{\Lambda_{N-n-1} - F_{N_n-1}}$$

If we multiply through by the denominators, group and eliminate like terms we get

$$F_{N-n}\Lambda_{N-n-1} - \Lambda_{N-n}F_{N-n-1} + Q_c^{(n)N-n}[\omega_{N-n} - Q_{0,N-n}] = 0$$

Failure Rate Intersection

The question of whether the failure rates can intersect is interesting and deserves some attention. Let us start discussing it in the context of two subspaces. We realize that if $Q_{0,1} > Q_{0,2}$ and $Q_c^1 < Q_c^2$ then the values of the failure rates never coincide. The second inequality can be restated thusly:

$$\frac{Q_0 - \bar{Q}_{0,1}}{1 - \bar{Q}_{0,1}} < \frac{Q_0 - \bar{Q}_{0,2}}{1 - \bar{Q}_{0,2}}$$

Multiplying thorugh and simplifying we find this equals

$$Q_0(\bar{Q}_{0,1} - \bar{Q}_{0,2}) < (\bar{Q}_{0,1} - \bar{Q}_{0,2})$$

This equality holds if

$$\bar{Q}_{0,1} > \bar{Q}_{0,2}$$

Unfortunately this does not prove the desired first inequality. In fact, it is possible to construct a counterexample by these two conditions. Specifically, if we choose $cos\theta_1 = \frac{1}{4}$ and $cos\theta_2 = \frac{1}{2}$, $\eta_1 = \eta_2$, $r_1 = s_1\frac{3}{4}$, $r_2 = s_2 = \frac{1}{4}$, we find that $Q_{0,1} = \frac{\sqrt{3}}{16}$ and $Q_{0,2} = \frac{1}{8}$. With these values we find $Q_c^1 \approx .1$ and $Q_c^2 < 0$:

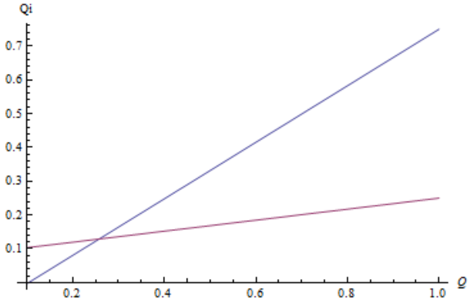

Figure: Q vs Qi with Q1 starting higher and going to the origin

If there are more than two subspaces, we can generalize this idea: assume $Q_{0,i} > Q_{0,j}$ and find that for $Q_c^i < Q_c^j$ we need $\bar{Q}_{0,i} > \bar{Q}_{0,j}$. Furthermore, we can consider what happens at later critical points: Since

$$Q_c^{(n)i} = \frac{\omega_i F_{N-n} - Q_{0,i}\Lambda_{N-n}}{\omega_i - Q_{0,i}}$$

then the condition for $Q_c^{(n)i} < Q_c^{(n)j}$ we find the requisite inequality to be

$$\Lambda_{N-n}(\bar{Q}_{0,i} - \bar{Q}_{0,j}) > F_{N-n}(\bar{Q}_{0,i} - \bar{Q}_{0,j})$$

which is clearly satisfied by the same requirement as the first critical point. This creates a structure on the intersection of failure rates: if $Q_{0,i} > Q_{0,j}$ and $\bar{Q}_{0,i} \leq \bar{Q}_{0,j}$, the two failure rates Q_i and Q_j will intersect at some iteration.

Single-State Domain

Each subspace failure rate also has a ceiling. For the majority of initial conditions the UD failure rate $Q_{0,i}$ sets this upper bound. For the other cases, we find it from the constraint that $\Pi_{0,i} \leq |0\rangle_{ii}\langle 0|$. The equality limit is a full projector which eliminates another measurement and moves us from the POVM to the single-state domain (SSD).

For the single subspace case the equation for the critical ceiling was

$$Q = Q_c = \frac{2\eta_1\eta_2 sin^2\theta}{1 - Q_0}$$

This result was derived from the constraint that $\xi \leq 1$ where $\Pi_0 = \xi|0\rangle\langle 0|$. Evaluating ξ for the optimal solution gave us

$$\xi \leq \frac{1 - Q_0}{sin^2\theta}\frac{Q_0}{2\eta_1\eta_2}$$

where we take the equality limit and set $\xi = 1$ to find the region in which the POVM strategy outperforms the projector measurement.

There are two regions that this occurs.

- For $\eta_1 \geq \eta_2$, the SSD overlaps with the interpolation measurement in the region $\frac{1}{1+cos^2\theta} \leq \eta_1$ and when $Q \geq Q_c$.
- For $\eta_2 \geq \eta_1$ this happens when $\frac{cos^2\theta}{1+cos^2\theta} \geq \eta_1$ and $Q \geq Q_c$.

Because the failure operator points directly onto the less likely state in either of these cases, we find the failure rates to be simply $Q^< = \eta_2 + \eta_1 cos^2\theta$ and $Q^> = \eta_1 + \eta_2 cos^2\theta$ respectively.

To generalize to subspaces we return to the bar normalization that renormalized the subspace probabilities to 1. Remembering that

$$Q_i = \xi_i \langle 0_i | D_i | 0_i \rangle$$

where D_i is the full density matrix of the states in the ith subspace, $D_i = \eta_{1,i}\rho_{1,i} + \eta_{2,i}\rho_{2,i}$ we can conclude that

$$\bar{Q}_i = \xi_i \langle 0_i | \bar{D}_i | 0_i \rangle$$

where $\bar{D}_i = \eta_{\bar{1},i}\rho_{1,i} + \eta_{\bar{2},i}\rho_{2,i}$

Now we have restored the summation of the a-priori probabilities for each subspace to 1 while leaving ξ_i unchanged, so the preceding arguments for the single subspace can be implemented to rewrite the inequality for ξ_i as

$$\xi_i \leq \frac{1 - 2\sqrt{\eta_{\bar{1},i}\eta_{\bar{2},i}}cos\theta_i}{1 - cos^2\theta_i} \frac{cos\theta_i}{\sqrt{\eta_{\bar{1},i}\eta_{\bar{2},i}}}$$

or

$$\xi_i \leq \frac{\omega_i - 2\sqrt{\eta_{1,i}\eta_{2,i}}cos\theta_i}{1 - cos^2\theta_i} \frac{cos\theta_i}{\sqrt{\eta_{1,i}\eta_{2,i}}}$$

we get the natural generalization of the critical ceiling to subspaces to be

$$Q_i = Q_i^{cc} = \frac{2\eta_{1,i}\eta_{2,i} sin^2\theta_i}{\omega_i - Q_{0,i}}$$

As Q_i is increased past this point we have $\Pi_{1,i} = |1\rangle_{ii}\langle 1|$ and $\Pi_{0,i} = |0\rangle_{ii}\langle 0|$

Now the condition for the overlap of the SSD onto the POVM region, assuming $\eta_{1,i} \geq \eta_{2,i}$ is

$$\frac{\omega_i}{1 + cos^2\theta_i} \leq \eta_{1,i}$$

With the maximum failure rate that can be generalized as:
$Q_i^< = \eta_{2,i} + \eta_{1,i} cos^2\theta_i$
Similarly for $\eta_{2,i} \geq \eta_{1,i}$ we get the condition

$$\frac{\omega_i cos^2\theta_i}{1 + cos^2\theta_i} \geq \eta_{1,i}$$

and the maximum failure rate as $Q_i^> = \eta_{2,i} + \eta_{1,i} cos^2\theta_i$

We notice that with more subspaces, the condition for the overlap region of SSD over the POVM does not change for individual subspaces as the bar transformation would show us. We show these elements in the following figure, where the shaded regions represents the SSD domains.

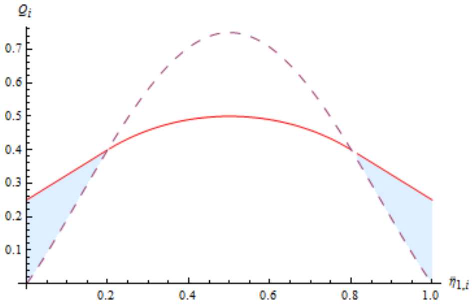

Figure: $\bar{\eta}_{1,i}$ vs Q_i The dashed line represents Q_i^{cc} and the sold line the absolute maximum Q_i, the intersection point of the two is determined by the inequalities above. Values given for $\theta_i = \pi/3$

Subspaces Summary

We found several surprising and extraordinary conclusions. The first is that the form of the error rate remains the same over all subspaces. The second is the threshold behavior in the optimization that shuts off successive subspaces as the total failure rate decreases.

Outline

2D Mixed States

A problem we are currently working on is to distinguish between two mixed states in two dimensions. We can represent them as:

$$\rho_1 = p|\psi_1\rangle\langle\psi_1| + \frac{(1-p)}{2}I$$

and

$$\rho_1 = d|\psi_2\rangle\langle\psi_2| + \frac{(1-d)}{2}I$$

Where the pure states $|\psi_1\rangle = c_1|0\rangle + s_1|1\rangle$ and $|\psi_2\rangle = c_2|0\rangle + s_2|1\rangle$ form a 2 dimensional space and p,d are the purities of the two mixed states.
The a-priori probabilities are as usual η_1 and η_2

We cannot in general distinguish perfectly between these two because are are not in general orthogonal. We also cannot distinguish them unabmiguously in general because of the identity terms. Hence it is less clear which strategy is better. An interpolative approach will allow us to see the transition between ME and MC and understand the trade-off in increasing the failure rate for different starting conditions. We use the transformation discussed earlier to tackle this problem.

We want to implement the transformation that gives us

$$\widetilde{\rho_1} = \frac{\Omega^{1/2}\rho_1\Omega^{1/2}}{Tr(\Omega\rho_1)}$$

$$\widetilde{\eta_1} = \frac{\eta_1 Tr(\Omega\rho_1)}{1-Q}$$

So that we still have $Tr\widetilde{\rho_1} = 1$ and $\widetilde{\eta_1} + \widetilde{\eta_2} = 1$ and the error rate is

$$\widetilde{P_e} = \frac{1}{2}(1 - Tr\left|\widetilde{\Lambda}\right|)$$

where $\widetilde{\Lambda} = \widetilde{\eta_1}\widetilde{\rho_1} - \widetilde{\eta_2}\widetilde{\rho_2}$

We work on the lambda to find:

$$Tr\left|\widetilde{\Lambda}\right| = \frac{1}{1-Q} Tr\left|\Omega^{1/2}\Lambda\Omega^{1/2}\right|$$

where $\Omega^{1/2} = \begin{pmatrix} \sqrt{1-\xi} & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\Lambda = \begin{pmatrix} p\eta_1 c_1^2 - d\eta_2 c_2^2 + \frac{\eta_1(1-p)-\eta_2(1-d)}{2} & p\eta_1 c_1 s_1 - d\eta_2 c_2 s_2 \\ p\eta_1 c_1 s_1 - d\eta_2 c_2 s_2 & p\eta_1 s_1^2 - d\eta_2 s_2^2 + \frac{\eta_1(1-p)-\eta_2(1-d)}{2} \end{pmatrix}$$

$\Omega^{1/2}\Lambda\Omega^{1/2} = \widetilde{\Lambda}$ is easy to find
To find the sum of the absolute values of its eigenvalues we first write the characteristic equation as

$$\lambda^2 - b\lambda + c = 0$$

where

$$b = \eta_1 - \eta_2 - \xi[p\eta_1 c_1^2 - d\eta_2 c_2^2 + \frac{\eta_1(1-p) - \eta_2(1-d)}{2}]$$

and

$$c = (1-\xi)[\frac{1 - 4\eta_1\eta_2 - (p\eta_1 - d\eta_2)^2}{4} - pd\eta_1\eta_2(c_1 s_2 - c_2 s_1)^2]$$

Where we can rewrite the term $(c_1 s_2 - c_2 s_1)^2 = sin^2\theta$

We can find ξ by solving

$$1 - Q = Tr\Omega\rho = 1 - \xi[p\eta_1 c_1^2 + d\eta_2 c_2^2 + \frac{1 - p\eta_1 - d\eta_2}{2}]$$

We can solve the quadratic equation $\lambda^2 - b\lambda + c$ for the two eigenvalues, but we notice that for $p = d = 1$ the value of c is $c = (1 - \xi)[-\eta_1\eta_2 sin^2\theta]$ is negative and for $p = d = 0$ the value of c is $c = (1 - \xi)[\frac{(\eta_1 - \eta_2)^2}{4}]$ is positive.
Hence the sum of the eigenvalues for $c < 0$ is

$$\sum |\lambda_i| = \sqrt{b^2 - 4c}$$

And the sum of the eigenvalues for $c \geq 0$ is

$$\sum |\lambda_i| = b$$

As we noted before, this is not the final solution. A further optimization is required to determine c_1 and c_2. Considering the complexity of the b and c terms, this is not trivial. We hope to be able to solve at least some of the specific cases if not for all of them. Simpler cases include $p = d$, when the purity of both states are equal. Another is $d = 1$, where we discriminate between one pure and one mixed state.

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - Something you haven't solved.
  - Something else you haven't solved.

For Further Reading I

A. Author.
Handbook of Everything.
Some Press, 1990.

S. Someone.
On this and that.
Journal of This and That, 2(1):50–100, 2000.