Optimal Measurements Tasks and Their Physical Realizations

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Outline

1. Introduction
   - Information Theory
   - Pure State Discrimination Strategies
2. State Separation
3. Cloning Pure States
   - Deterministic Approximate Cloning
   - Probabilistic Exact Cloning
   - Intermediate Cloning
4. Linear Optical Implementations

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- Quantum computing-Shor's algoritm allows cracking of modern communications security based on prime decomposition
- Quantum communication- B92 protocol allows for completely secure communication

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Our Games

- Someone prepares one of two different pure states $|\psi_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle$ or one of two ensembles $\rho_i = \sum_i c_i|\psi_i\rangle\langle\psi_i|$, for $i = 1, 2$.
- Each outcome has a different likelihood η_i that sum to 1.
- One particle (state) is sent at a time. Our job is to:
  - Guess as best we can what state was sent (discrimination)
  - Make the state more orthogonal to its complement (separation)
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Unitary Time Evolution

The Schrodinger equation for ensembles is

$$i\hbar\frac{\partial\rho}{\partial t} = [H, \rho].$$

Solving this for evolution of initial state $\rho(t = 0)$ we get

$$\rho(t) = U(t)\rho(0)U(t)^{\dagger}$$

where the unitary matrix U obeys $UU^{\dagger} = I$. There are several ways to view this formula.

Neumark's theorem

- The first is as the time evolution of a pure system evolving with pure ancilla

$$U(|\psi_A\rangle \otimes |\phi_B\rangle) = \sum_i A_i|\psi_A\rangle \otimes |i_B\rangle.$$

using the Kraus operators A_i such that $\sum A_i A_i^\dagger = \sum \Pi_i = I$, the Π_i's representing different measurement outcomes.

- The other is simply as a unitary acting on a pure state ψ to make state ϕ, as in $U|\psi\rangle = |\phi\rangle$.

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No-go

If perfect discrimination were possible we could write it as

$$\Pi_1|\psi_2\rangle = 0,$$

$$\Pi_2|\psi_1\rangle = 0,$$

then using $\Pi_1 + \Pi_2 = I$ and the inner product of these equations, we get the same result:

$$0 = \langle\psi_2|\Pi_1 + \Pi_2|\psi_1\rangle = \langle\psi_1|\psi_2\rangle$$

Since the two constraints of measurement, orthogonality of the measurement vectors and their spanning the space, proved contradictory, we must give up one of these two functions in order to perform a physical measurement.

Minimum Error Discrimination (ME)

- Two orthogonal projectors, each clicks for a state.
- There is a success rate and an error rate if the states are not orthogonal.

$$P_s = \eta_1 Tr(\rho_1 \Pi_1) + \eta_2 Tr(\rho_2 \Pi_2)$$

$$P_e = \eta_2 Tr(\rho_2 \Pi_1) + \eta_1 Tr(\rho_1 \Pi_2)$$

- The minimum error rate for pure states is achieved by the Helstrom bound.

$$P_E = \frac{1}{2}(1 - \sqrt{1 - 4\eta_1\eta_2 \left|\langle\psi_1|\psi_2\rangle\right|^2})$$

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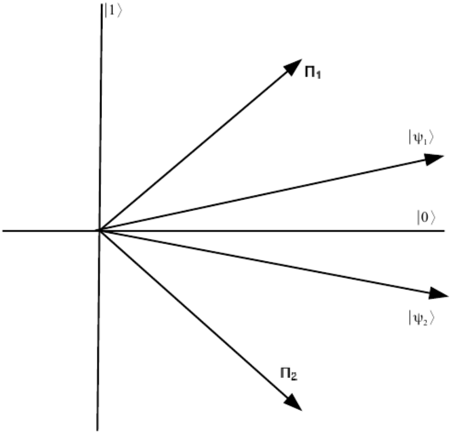
Minimum Error Graph



Unambigious State Discrimination (UD)

- Make the measurement operators orthogonal to the state that we don't want to measure.
- Since they are no longer orthogonal they don't sum to the identity. A third, inconclusive outcome is necessary.
- The detector corresponding to the inconclusive outcome we call Π_0.
- The failure probability we call Q:

$$Q = \eta_1 Tr(\rho_1 \Pi_0) + \eta_2 Tr(\rho_2 \Pi_0) = Tr(\rho \Pi_0).$$

- $Q_0 = 2\sqrt{\eta_1 \eta_2} cos\theta$ is the failure rate that corresponds to the best measurement.

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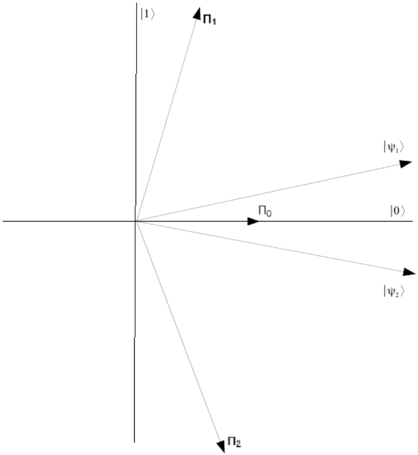
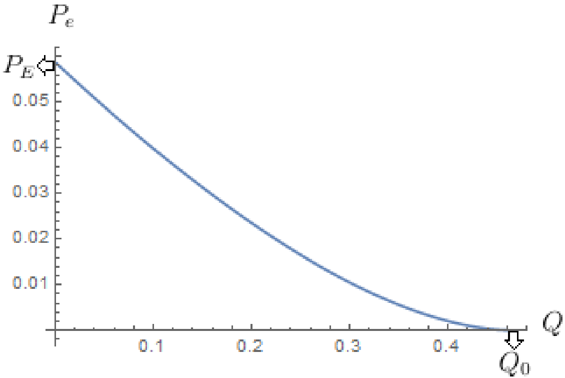
Unambiguous State Discrimination Graph



Intermediate Discrimination (IM)



Lagrange Multipliers method

We can cast the problem via Neumark's theorem as

$$U|\psi_1\rangle = \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle + \sqrt{q_1}|0\rangle,$$
$$U|\psi_2\rangle = \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle + \sqrt{q_2}|0\rangle.$$

The inner product of these two equations is useful. We call it the unitarity constraint:

$$s = \langle\psi_1|\psi_2\rangle \ = \sqrt{p_1 r_2} + \sqrt{p_2 r_1} + \sqrt{q_1 q_2}.$$

We now use the Lagrange multiplier method to minimize P_e subject to the inner-product constraint and fixed failure rate Q.

$$F_e = \eta_1 r_1 + \eta_2 r_2 + \lambda(s - \text{Unitarity Constraint}).$$

A lot of algebra gives us the optimal individual error rates as:

$$r_1 = \frac{1}{2}[\alpha_1 - \frac{[2\eta_2\omega - \alpha_1(1-Q)]}{\sqrt{(1-Q)^2 - 4\omega\eta_1\eta_2}}]$$

and

$$r_2 = \frac{1}{2}[\alpha_2 - \frac{[2\eta_1\omega - \alpha_2(1-Q)]}{\sqrt{(1-Q)^2 - 4\omega\eta_1\eta_2}}]$$

where $\alpha_i \equiv 1 - q_i$ and $\omega \equiv (s - \sqrt{q_1 q_2})^2$ and finally

$$P_e = \frac{1}{2}(1 - Q - \sqrt{(1-Q)^2 - (Q - Q_0)^2}),$$

We've solved some state discrimination problems involving mixed states but we've presented the results before so we move on to other, more interesting topics!

We can consider an operation such as UD or IM as a two-step measurement.

- In the first step we probabilistically separate the states.
- In the second step, if the first was successful, orthogonal detection operators (ME) discriminate the states.

What if we only wanted to perform the first step?

We want to make the pure states ψ_i probabilistically more distinguishable, via equations

$$\begin{aligned} U|\psi_1\rangle|0\rangle &= \sqrt{p_1}|\phi_1\rangle|1\rangle + \sqrt{q_1}|0\rangle, \\ U|\psi_2\rangle|0\rangle &= \sqrt{p_2}|\phi_2\rangle|1\rangle + \sqrt{q_2}|0\rangle. \end{aligned}$$

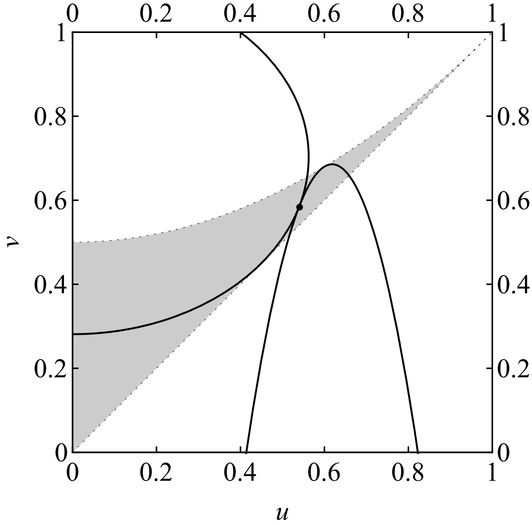
Here the inner product equation is $s = \sqrt{p_1 p_2} s' + \sqrt{q_1 q_2}$ where $s' = \langle\phi_1|\phi_2\rangle$.

A particularly attractive geometric formulation of this constraint comes from choosing the change of variables
$u \equiv \sqrt{q_1 q_2}, \quad v \equiv \frac{1}{2}(q_1 + q_2)$.

- The unitarity constraint becomes parabola $v = \frac{1+u^2}{2} - \frac{(u-s)^2}{2s'^2}$.
- The failure rate curve beomces parametrized ellipse

$$\begin{aligned} u &= \frac{Q}{\sqrt{1-\Delta^2}} \cos\theta, \\ v &= \frac{Q}{1-\Delta^2} + \frac{Q\Delta}{1-\Delta^2} \sin\theta, \end{aligned} \tag{1}$$

where $\Delta = \eta_2 - \eta_1$. Then both the failure rate and unitarity constraint become conic sections (an ellipse and parabola). The optimality condition is their tangency, and we are able to derive all bounds parametrically.


0
0.2
0.4
0.6
0.8
1
0.8
0.6
0.4
0.2
0
v
u

No-go

Suppose we wrote the equations describing the unitary that cloned one of two input pure states ψ_i with a-priori probabilities η_i for $i = 1, 2$. If we could perform this measurement perfectly we would write

$$U|\psi_1\rangle|0\rangle = |\psi_1\rangle|\psi_1\rangle, \tag{2}$$

$$U|\psi_2\rangle|0\rangle = |\psi_2\rangle|\psi_2\rangle. \tag{3}$$

However, taking the inner product of the two equations we find $\langle\psi_1|\psi_2\rangle = \langle\psi_1|\psi_2\rangle^2$, restricting the functionality of this unitary to the trivial case when the two states are orthogonal and $\langle\psi_1|\psi_2\rangle = 0$. Therefore there does not exist in general a unitary to perform this ideal cloning task. As with state discrimination, we must choose a figure of merit to maximize.

Outline

The 'minimum error' cloning method was devised by Chefles and Barnett to make two imperfect copies of one of two input states. The unitary representation of this transformation is

$$U|\psi_1\rangle|0\rangle = |\phi_1\rangle|\phi_1\rangle$$
$$U|\psi_2\rangle|0\rangle = |\phi_2\rangle|\phi_2\rangle$$

Where we want to maximize is the average fidelity

$$F = \eta_1 \langle\psi_1|\phi_1\rangle^2 + \eta_2\langle\psi_2|\phi_2\rangle^2$$

The optimal fidelity is

$$F = \frac{1}{2}(1 + \sqrt{1 - 4\eta_1\eta_2 \sin^2 2\alpha}).$$

where $\alpha = \theta - \frac{\phi_1 - \phi_2}{2}$.

Outline

Here we take 'unambiguous' method to solving the problem and sometimes make perfect clones. We consider m input states turned into n output states.

$$U|\psi_i^m\rangle|0\rangle = \sqrt{p_i}|\psi_i^n\rangle|1\rangle + \sqrt{q_i}|\Phi\rangle|0\rangle, \quad i = 1, 2.$$

The overlap equation now reads

$$s^m = \langle\psi_1^m|\psi_2^m\rangle = \sqrt{p_1 p_2}\langle\psi_1^n|\psi_2^n\rangle + \sqrt{q_1 q_2} = \sqrt{p_1 p_2}s^n + \sqrt{q_1 q_2}.$$

If at this point you think this equation looks remarkably similar to everything else, you'd be right. But particularly to the separation unitarity condition with $s \rightarrow s^M$,$s' \rightarrow s^N$. Load Clip

Geometric Picture

We want optimize the success rate given the unitary curve

$$s^m = \sqrt{p_1 p_2} s^n + \sqrt{q_1 q_2}$$

and subject to a fixed failure rate

$$Q = \eta_1 q_1 + \eta_2 q_2.$$

In the q_1 vs q_2 coordinate system, these two equations appear as a complex curve and a straight line. To satisfy both conditions means the curves must intersect. It can be proven that tangency is the requirement.

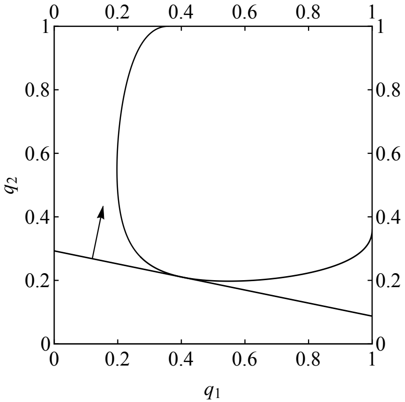


Figure: The figure shows the overlap constraint and the optimal straight segment $Q = \eta_1 q_1 + \eta_2 q_2$ and its normal vector (η_1, η_2). Plotted for $s = 0.6$, $\eta_1 = 0.17$, $\eta_2 = 0.83$ and $Q = 0.24$.

Parametrization

We use the change of variables $\sqrt{q_i} = \sin\theta_i$ then $x = cos(\theta_1 + \theta_2)$, $y = (cos\theta_1 - \theta_2)$ to write the unitary constraint as

$$x = \frac{1-(1+s^n)t}{s^{n-m}}, \qquad y = \frac{1-(1-s^n)t}{s^{n-m}}.$$

In this form we can write the failure rates as

$$q_i = \frac{1 - xy - (-1)^i\sqrt{1-x^2}\sqrt{1-y^2}}{2}. \tag{4}$$

Optimization

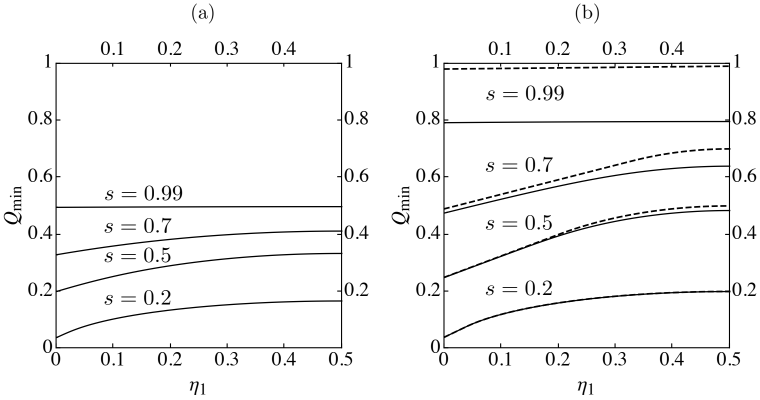
We can characterize our result by expressing the necessery a-priori η_i for any point on the unitary curve. These take the form

$$\eta_1 = \frac{q_2'}{q_2' - q_1'}, \quad Q_{\min} = \frac{q_2' q_1 - q_1' q_2}{q_2' - q_1'}, \quad t_{-1} \leq t \leq t_0,$$

where the derivative of q is

$$q_i' = \frac{\sqrt{q_i(1-q_i)}}{s^{n-m}} \left\{ \frac{1+s^n}{\sqrt{1-x^2}} - (-1)^i \frac{1-s^n}{\sqrt{1-y^2}} \right\}. \qquad (5)$$

Relation to State Discrimination



Optimal cloning is always better than discrimination, though this difference decreases with more clones. In the limit $n \to \infty$ there is perfect agreement. In geometric terms, the unitary curve $s^m = \sqrt{p_1 p_2} s^n + \sqrt{q_1 q_2}$ collapses onto the hyperbola $s^m = \sqrt{q_1 q_2}$, which is the constraint for UD state discrimination. Therefore the limiting solution to our cloning procedure is unambiguous discrimination.

Outline

Chefles and Barnett solved the equal priors case in 1998, giving us the optimal global fidelity as

$$F_{MN} = \frac{1}{2}\left[1 + |\langle\psi_1^N|\psi_2^N\rangle|\left(1 - \frac{P_{IDP}}{P_S}\right)\right] +$$

$$\frac{1}{2P_S}\left(\left(1 - |\langle\psi_1^N|\psi_2^N\rangle|^2\right)\left(P_S^2 - (P_S - P_{IDP})^2\right)\right)^{1/2}$$

As you may imagine the unequal priors case cannot be solved in the same manner. This formula reduces to intermediate discrimination in the $n \to \infty$ limit.

Arbitrary Priors

This takes two steps: first maximally separate the incoming states for a fixed rate of inconclusive result,

- Step 1: State Separation

Optimally separate the incoming states $\{|\psi_1^M\rangle, |\psi_2^M\rangle\}$ with a fixed rate of inconclusive results q_i, then prepare states $\{|\Psi_1\rangle, |\Psi_2\rangle\}$ with the corresponding success probabilities. The incoming states are separated with a success probability p_i and failed to separate the states with a failure probability q_i.

- Step 2: Optimize Deterministic Fidelity

The resulting states are rotated in such a way that their average overlap with n copies of the original state is mazimized. More precisely, for individual fidelities $F_i = |\langle\psi_i^N|\phi_i\rangle|^2$ we want to maximize the global conditional fidelilty

$$F = \frac{\eta_1 p_1 F_1 + \eta_2 p_2 F_2}{1 - Q} = \tilde{\eta_1} F_1 + \tilde{\eta_2} F_2$$

Where we've also shown the resulting normalization. This normalization means we can apply the deterministic fidelity result to this problem.

The resulting fidelity expression,

$$F_{MN} = \frac{1}{2}\left[1 + \sqrt{1 - 4\tilde{\eta}_1\tilde{\eta}_2 \sin^2\left(2\theta - (\phi_1 + \phi_2)\right)}\right],$$

has a remaining optimization under the square root, namely the term

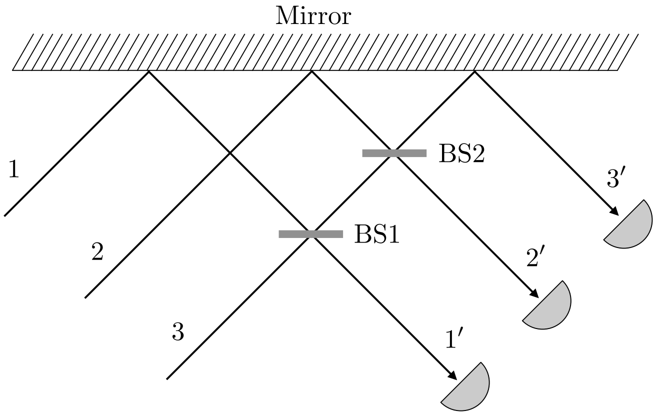
$$\Lambda = \sqrt{p_1 p_2}\sin\left(2\theta - (\phi_1 + \phi_2)\right).$$

can be written with optimal separation as

$$\Lambda = \sqrt{1 - s^{2n}}\,(s - u) - s^n\sqrt{1 - s^2 - 2v + 2sv}$$

where $u \equiv \sqrt{q_1 q_2}, \quad v \equiv \frac{1}{2}(q_1 + q_2)$, are variables we will introduce in the next section on state separation.

Every unitary can be reduced into beam splitters and phase-shifters via the Reck-Zeilinger algorithm. This has been experimentally implemented for various protocols.



Approximate Probabilistic Cloning

We can decompose the action of a probabilistic cloner into two steps, first separate the states and then clone them to the desired copies. We will show this setup for equal prior probabilities.

Separation

$$U|\psi_1\rangle = \sqrt{p_1}|\phi_1\rangle + \sqrt{q_1}|0\rangle \tag{6}$$
$$U|\psi_2\rangle = \sqrt{p_2}|\phi_2\rangle + \sqrt{q_2}|0\rangle \tag{7}$$

where $|\psi_1\rangle = c_1|1\rangle + s_1|2\rangle$, $|\psi_2\rangle = c_1|1\rangle - s_1|2\rangle$, $|\phi_1\rangle = c_2|1\rangle + s_2|2\rangle$, $|\phi_2\rangle = c_2|1\rangle - s_2|2\rangle$. then we get the equations

$$\langle 0|U|\psi_1\rangle = c_1 U_{01} + s_1 U_{02} = \sqrt{q_1} \tag{8}$$
$$\langle 1|U|\psi_1\rangle = c_1 U_{11} + s_1 U_{12} = c_2\sqrt{p_1} \tag{9}$$
$$\langle 2|U|\psi_1\rangle = c_1 U_{21} + s_1 U_{22} = s_2\sqrt{p_1} \tag{10}$$

Using the optimal sucess rate

$$p = \frac{\langle\psi_1|\psi_2\rangle - 1}{\langle\phi_1|\phi_2\rangle - 1}, \tag{11}$$

we can choose the remaining elements to be

$$[U] = \begin{pmatrix} \frac{c_2 s_1}{c_1 s_2} & \frac{\sqrt{s_2^2 - s_1^2}}{c_1 s_2} & 0 \\ -\frac{\sqrt{s_2^2 - s_1^2}}{c_1 s_2} & \frac{c_2 s_1}{c_1 s_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

and only one beam splitter is needed to perform this operation.

Deterministic Cloning

The unitary transformation for the second step is

$$U|\phi_1\rangle|0\rangle = |\xi_1\rangle|\xi_1\rangle \tag{13}$$
$$U|\phi_2\rangle|0\rangle = |\xi_2\rangle|\xi_2\rangle \tag{14}$$

where $|\xi_1\rangle = c_3|1\rangle + s_3|2\rangle$ and $|\xi_1\rangle = c_3|1\rangle - s_3|2\rangle$ Following a similar procedure we choose the basis states $|00\rangle$,$|10\rangle$,$|01\rangle$, and$|11\rangle$, giving us the final unitary as

$$[U] = \begin{pmatrix} \frac{c_3^2}{c_2} & 0 & 0 & \frac{s_3^2}{c_2} \\ 0 & \frac{c_3 s_3}{s_2} & -\frac{c_3 s_3}{s_2} & 0 \\ 0 & \frac{c_3 s_3}{s_2} & \frac{c_3 s_3}{s_2} & 0 \\ \frac{s_3^2}{c_2} & 0 & 0 & \frac{c_3^2}{c_2} \end{pmatrix} \tag{15}$$

This is clearly the action of two separate beam splitters M_{14} and M_{23} such that

$$[M_{14}] = \frac{1}{c_2}\begin{pmatrix} c_3^2 & s_3^2 \\ s_3^2 & c_3^2 \end{pmatrix} [M_{23}] = \frac{c_3 s_3}{s_2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{16}$$

Pure State Discrimination

$$U|1\rangle = \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle + \sqrt{q_1}|3\rangle \tag{17}$$

$$U(\cos\theta|1\rangle + \sin\theta|2\rangle) = \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle + \sqrt{q_2}|3\rangle \tag{18}$$

Gives the unitary as

$$U = \begin{pmatrix} \sqrt{p_1} & \frac{\sqrt{r_2}-\sqrt{p_1}\cos\theta}{\sin\theta} & \pm\frac{\sqrt{\sin^2\theta-p_1-r_2+2\sqrt{p_1 r_2}\cos\theta}}{\sin\theta} \\ \sqrt{r_1} & \frac{\sqrt{p_2}-\sqrt{r_1}\cos\theta}{\sin\theta} & \pm\frac{\sqrt{\sin^2\theta-r_1-p_2+2\sqrt{p_2 r_1}\cos\theta}}{\sin\theta} \\ \sqrt{q_1} & \frac{\sqrt{q_2}-\sqrt{q_1}\cos\theta}{\sin\theta} & \pm\frac{\sqrt{sin^2\theta-q_1-q_2+2\sqrt{q_1 q_2}\cos\theta}}{\sin\theta} \end{pmatrix}. \tag{19}$$

We can decompose this into three beam splitters for all ranges of parameters analytically.

The UD and ME strategies require fewer beam splitters; since UD is essentially two-step it requires 2 beam splitters. ME is a projective measurement and requires only one.

$$U_{UD} = \begin{pmatrix} \sqrt{p} & -\frac{\sqrt{p}Q_o}{\sqrt{1-Q_o^2}} & \sqrt{\frac{Q_0}{1+Q_o}} \\ 0 & \frac{\sqrt{p}}{\sqrt{1-Q_o^2}} & \sqrt{\frac{Q_o}{1+Q_o}} \\ \sqrt{Q_0} & \sqrt{\frac{Q_o(1-Q_o)}{1+Q_o}} & -\sqrt{\frac{1-Q_o}{1+Q_o}} \end{pmatrix}. \tag{20}$$

$$U_{ME} = \begin{pmatrix} \sqrt{p} & \sqrt{r} & 0 \\ \sqrt{r} & -\sqrt{p} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

Thank you

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For Further Reading I

A. Author.
Handbook of Everything.
Some Press, 1990.

S. Someone.
On this and that.
Journal of This and That, 2(1):50–100, 2000.