Quantum State Discrimination and Quantum Cloning Schemes: Optimization and Implementation

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Motivation Behind Quantum Info Science

- When you receive information, you'd like to be able to read out the text.
- Inability to fully distinguish quantum states has been exploited in quantum cryptography
- Quantum Simulations.

Quantum State Discrimination: 2 pure states

Two non-orthogonal pure states can be represented in 2D

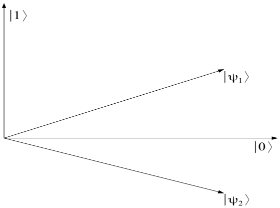


Figure: 2 pure states

$$
\begin{aligned}
|\psi_1\rangle &= \cos\theta\,|0\rangle + \sin\theta\,|1\rangle \\
|\psi_2\rangle &= \cos\theta\,|0\rangle - \sin\theta\,|1\rangle
\end{aligned}
$$

Quantum State Discrimination: 2 pure states

- Imagine two detectors Π_1 and Π_2 which unambiguously detect the two pure states. The detectors span the Hilbert space:
$$\Pi_1 + \Pi_2 = I \tag{1}$$
- The detector Π_i unambiguously detects the state $|\psi_i\rangle$ $i = 1, 2$, such that $\Pi_i |\psi_j\rangle = 0$
$$\begin{aligned} \langle\psi_1| (\Pi_1 + \Pi_2) |\psi_1\rangle &= \langle\psi_1 \mid \psi_1\rangle \\ p_1 &= 1 \end{aligned}$$
- Similarly it can be shown that $p_2 = 1$. However multiplying l.h.s by $\langle\psi_1|$ and r.h.s by $|\psi_2\rangle$ results in $\langle\psi_1 \mid \psi_2\rangle = 0$, orthogonal states.

Unambiguous Discrimination

- Unambiguous State Discrimination is still possible if a third detector is added:

$$\Pi_1 + \Pi_2 + \Pi_0 = I \tag{2}$$

- The condition $\Pi_i |\psi_j\rangle = 0$ still holds. Multiplying 2 by $\langle\psi_i|$ from l.h.s and $|\psi_i\rangle$:

$$\begin{aligned} \langle\psi_i| (\Pi_1 + \Pi_2 + \Pi_0) |\psi_i\rangle &= \langle\psi_i \mid \psi_i\rangle , \\ p_i + q_i &= 1, \end{aligned}$$

where: p_i rate of successfully identifying the state, q_i failure rate.

Unambiguous Discrimination

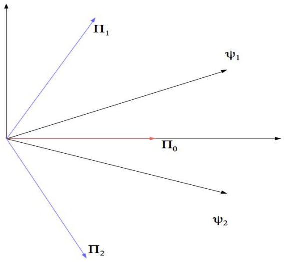


Figure: UD with three detectors

- The task is to minimize the average failure rate: $Q = \eta_1 q_1 + \eta_2 q_2$, η_i the a-priori rates of input states. The optimum Q is:

$$Q_0^{IDP} = 2\sqrt{\eta_1 \eta_2} s \tag{3}$$

Minimum Error

- The detectors can make a mistake, but are not allowed to abstain from giving an answer.

$$\Pi_1 + \Pi_2 = I \tag{4}$$

Multiplying Eq. (4) by $\langle\psi_i|$ from l.h.s and $|\psi_i\rangle$:
$\langle\psi_i| (\Pi_1 + \Pi_2) |\psi_i\rangle = \langle\psi_i \mid \psi_i\rangle \Rightarrow \qquad p_i + r_i = 1$

- Minimize the average failure rate[1]:
$P_E^{min} = \eta_1 r_1 + \eta_2 r_2 = \frac{1}{2}[1 - \sqrt{1 - 4\eta_1\eta_2|\langle\psi_1|\psi_2\rangle|^2}$

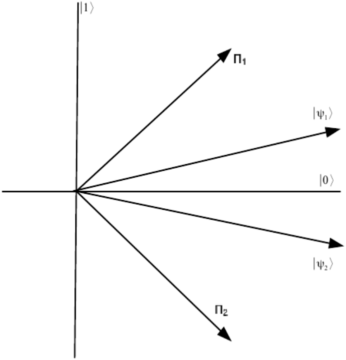


Figure: ME

FRIO

- The error rate in the Helstrom bound can be lowered if we relax a constrained and allow for some FRIO.
- Bagan et al [2] solved the problem first by transforming the three out
-

$$\begin{aligned} \Pi_1 + \Pi_2 &= I - \Pi_0 \equiv \Omega \\ \tilde{\Pi}_1 + \tilde{\Pi}_2 &= I \end{aligned} \tag{5}$$

where $\tilde{\Pi}_i \equiv \Omega^{-1/2} \Pi_i \, \Omega^{-1/2}$. The optimization to Eq. (5) is that of Helstrom with new normalized probabilities

$$\begin{aligned} \tilde{P}_E &= \frac{1}{2}\left[1 - \sqrt{1 - 4\tilde{\eta}_1\tilde{\eta}_2 |\langle\tilde{\psi}_1|\tilde{\psi}_2\rangle|^2}\right] \\ P_E^{min} &= \frac{1}{2}\left\{(1-Q) - \sqrt{(1-Q)^2 - (Q_0 - Q)^2}\right\} \end{aligned}$$

FRIO

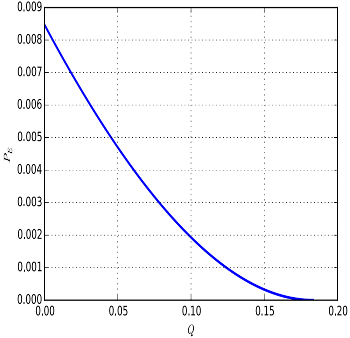


Figure: FRIO: Error rate interpolates between 0, for failure rate $Q = Q_0$, and the Helstrom $P_E^{helstrom}$ for zero failure rate $Q = 0$. The graph has prior probabilities $\eta_1 = 0.3$, $\eta_2 = 0.7$ and overlap $s = 0.2$.

Implementation of FRIO

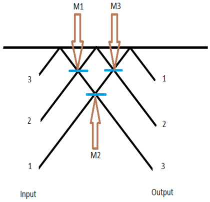


Figure: Dual rail representation of a photon and six port

Choosing the basis of our Hilbert space: $a_1^\dagger\,|000\rangle = |100\rangle \equiv |1\rangle$
$a_2^\dagger\,|000\rangle = |010\rangle \equiv |2\rangle$.
Two non-orthogonal input states can be expressed as:
$|\psi_1\rangle_{in} = |1\rangle,\ \ |\psi_2\rangle_{in} = \cos\theta|1\rangle + \sin\theta|2\rangle.$

Unitary

In Schrodinger picture the in and out states are related by:
$U\,|in\rangle = |out\rangle$.

$$\begin{aligned} U|1\rangle &= \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle + \sqrt{q_1}|3\rangle, \\ U(\cos\theta|1\rangle + \sin\theta|2\rangle) &= \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle + \sqrt{q_2}|3\rangle \end{aligned}$$

The first column is:
$\langle 1|U|1\rangle = U_{11} = \sqrt{p_1}$,
$\langle 2|U|1\rangle = U_{21} = \sqrt{q_1}$,
$\langle 3|U|1\rangle = U_{31} = \sqrt{r_1}$.
Second Column:
$U_{12} = \frac{\sqrt{r_2} - \sqrt{p_1}\cos\theta}{\sin\theta}$,
$U_{22} = \frac{\sqrt{p_2} - \sqrt{r_1}\cos\theta}{\sin\theta}$,
$U_{32} = \frac{\sqrt{q_2} - \sqrt{q_1}\cos\theta}{\sin\theta}$,

Unitary

$$U = \begin{pmatrix} \sqrt{p_1} & \frac{\sqrt{r_2}-\sqrt{p_1}\cos\theta}{\sin\theta} & -\frac{\sqrt{\sin^2\theta - p_1 - r_2 + 2\sqrt{p_1 r_2}\cos\theta}}{\sin\theta} \\ \sqrt{r_1} & \frac{\sqrt{p_2}-\sqrt{r_1}\cos\theta}{\sin\theta} & -\frac{\sqrt{\sin^2\theta - r_1 - p_2 + 2\sqrt{p_2 r_1}\cos\theta}}{\sin\theta} \\ \sqrt{q_1} & \frac{\sqrt{q_2}-\sqrt{q_1}\cos\theta}{\sin\theta} & +\frac{\sqrt{sin^2\theta - q_1 - q_2 + 2\sqrt{q_1 q_2}\cos\theta}}{\sin\theta} \end{pmatrix}. \tag{6}$$

The coefficients r_i and p_i however are not determined in the Bagan solution. We solve the FRIO problem using Neumark setup.

Unitary

$$
\begin{aligned}
U|1\rangle &= \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle + \sqrt{q_1}|3\rangle, \\
U(\cos\theta|1\rangle + \sin\theta|2\rangle) &= \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle + \sqrt{q_2}|3\rangle
\end{aligned}
$$

The inner product :

$$
s = \sqrt{p_1 r_2} + \sqrt{p_2 r_1} + \sqrt{q_1 q_2}, \tag{7}
$$

Minimize $P_E = \eta_1 r_1 + \eta_2 r_2$ subject to the constraint in (7) can be solved with the use of Lagrange multipliers

$$
\begin{aligned}
F &= \eta_1 r_1 + \eta_2 r_2 + \\
&\quad \lambda(s - \sqrt{(1 - r_1 - q_1) r_2} - \sqrt{(1 - r_2 - q_2) r_1} - \sqrt{q_1 q_2})
\end{aligned}
$$

Solution to Lagrange Multipliers

$$r_i = \frac{1}{2}\left[\left(1 - \frac{Q}{2\eta_i}\right) - \frac{\left(1 - \frac{Q}{2\eta_i}\right)(1-Q) - \frac{1}{2\eta_i}(Q_o - Q)^2}{\sqrt{(1-Q)^2 - (Q-Q_o)^2}}\right], \tag{8}$$

$$p_i = \frac{1}{2}\left[\left(1 - \frac{Q}{2\eta_i}\right) + \frac{\left(1 - \frac{Q}{2\eta_i}\right)(1-Q) - \frac{1}{2\eta_i}(Q_o - Q)^2}{\sqrt{(1-Q)^2 - (Q-Q_o)^2}}\right]. \tag{9}$$

Reck-Zeilinger Algorithm

Any discrete finite-dimensional unitary operator can be can be constructed in the lab using optical devices [3]. Following the Reck-Zeilinger algorithm the unitary calculated in (6) can be decomposed in terms of three beam splitters $U = M_1 M_2 M_3$ where:

$$M_1 = \begin{pmatrix} \sin\omega_1 & \cos\omega_1 & 0 \\ \cos\omega_1 & -\sin\omega_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \sin\omega_2 & 0 & \cos\omega_2 \\ 0 & 1 & 0 \\ \cos\omega_2 & 0 & -\sin\omega_2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\omega_3 & \cos\omega_3 \\ 0 & \cos\omega_3 & -\sin\omega_3 \end{pmatrix}.$$

Reck-Zeilinger Algorithm

The reflective and transmittance coefficients are calculated and expressed in terms of η_i, s, and FRIO Q.

$$\cos\omega_1 = \sqrt{\frac{r_1}{1-Q/2\eta_1}}\ , \ \ \sin\omega_1 = \sqrt{\frac{p_1}{1-Q/2\eta_1}},$$

$$\cos\omega_2 = \sqrt{Q/2\eta_1}, \sin\omega_2 = \sqrt{1-Q/2\eta_1},$$

$$\cos\omega_3 = -\frac{\sqrt{Q/2\eta_2} - \frac{Q_o}{2\eta_1}\sqrt{Q/2\eta_2}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}},$$

$$\sin\omega_3 = \frac{\sqrt{1-Q_o^2/4\eta_1\eta_2 - Q/(2\eta_1\eta_2) + QQ_o/(2\eta_1\eta_2)}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}}.$$

where r_i and p_i are given in (8) and (9).

Quantum Cloning

Let's try a different approach to state discrimination: Clone the states first then perform a measurement:
Given a state from a set of non-orthogonal quantum states $\{|\psi_1\rangle, |\psi_2\rangle\}$ make a large number of copies:

$$\begin{aligned} U|\psi_1\rangle &= |\psi_1\rangle|\psi_1\rangle \ldots |\psi_1\rangle = |\psi_1\rangle^N \\ U|\psi_2\rangle &= |\psi_2\rangle|\psi_2\rangle \ldots |\psi_2\rangle = |\psi_2\rangle^N \end{aligned}$$

Now perform a measurement scheme: ME or UD
$P_E = \frac{1}{2}\left[1 - \sqrt{1 - 4\eta_1\eta_2|\langle\psi_1|\psi_2\rangle|^{2N}}\right]$,
$Q = 2\sqrt{\eta_1\eta_2}|\langle\psi_1|\psi_2\rangle|^N$
In the asymptotic limit the error rate and the failure rate reduce to zero

No-Cloning Theorem

It was shown by Wootters, Zurek [4] and Dieks [5] that deterministic quantum cloning is not possible. Imagine a unitary operator which would copy the state $|\psi_i\rangle$ into $|0\rangle$:

$$\begin{aligned} U|\psi_1\rangle|0\rangle &= |\psi_1\rangle|\psi_1\rangle \\ U|\psi_2\rangle|0\rangle &= |\psi_2\rangle|\psi_2\rangle \end{aligned} \tag{10}$$

Inner product: $\langle\psi_2|\psi_1\rangle\langle 0|0\rangle = |\langle\psi_2|\psi_1\rangle|^2 \Rightarrow s = s^2$. The condition can be satisfied only if $s = 0$, states are orthogonal, or $s = 1$, the two states are the same.

Beyond the no-cloning theorem

- M.Hillery and V. Bužek make clones [6, 7, 8]!
- Two main quantum cloning machines (QCM): Universal and State Dependent
- Universal QCM: independent of the input
- State Dependent QCM: Probabilistic or Deterministic.
- Deterministic SD-QCM: produce approximate clones on demand while optimizing the fidelity between clones and input states.
- Probabilistic SD-QCM: produce exact clones with some rate of abstention.

Probabilistic SD-QCM

Given a pair non-orthogonal quantum states $\{|\psi_1\rangle, |\psi_2\rangle\}$ a probabilistic QCM produces $\{|\psi_1\rangle|\psi_1\rangle, |\psi_2\rangle|\psi_2\rangle\}$

$$\begin{aligned} U|\psi_1\rangle|i\rangle &= \sqrt{p_1}|\psi_1\rangle^N |\alpha\rangle + \sqrt{q_1}|\Phi\rangle |0\rangle , \\ U|\psi_2\rangle|i\rangle &= \sqrt{p_2}|\psi_2^N\rangle|\alpha\rangle + \sqrt{q_2}|\Phi\rangle |0\rangle , \end{aligned}$$

where $p_i + q_i = 1$.
Unitarity constrained: $s = \sqrt{p_1 p_2} s^2 + \sqrt{q_1 q_2}$.
Equal priors [9]: $s = ps^2 + q \Rightarrow q = \frac{s}{1+s}$

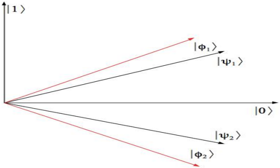
State Separation

Probabilistic cloning turns out to be a special case of state separation:

$$
\begin{aligned}
U|\psi_1\rangle|i\rangle &= \sqrt{p_1}|\phi_1\rangle|\alpha\rangle + \sqrt{q_1}|\Phi\rangle|0\rangle, \\
U|\psi_2\rangle|i\rangle &= \sqrt{p_2}|\phi_2\rangle|\alpha\rangle + \sqrt{q_2}|\Phi\rangle|0\rangle,
\end{aligned} \tag{11}
$$

The unitarity constraint: $s = \sqrt{p_1 p_2} s' + \sqrt{q_1 q_2}$. (one to two exact cloning is equivalent to setting $s' = s^2$)

Optimize the average rate of failing to separate the input states: $Q = \eta_1 q_1 + \eta_2 q_2$



Geometric solution to state separation

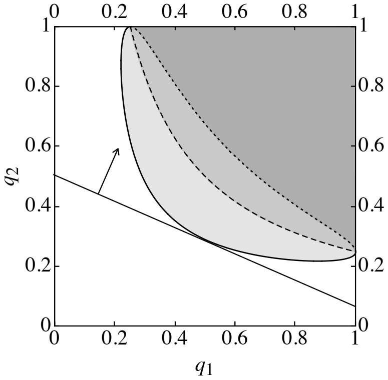


Figure: Unitarity curves $s^m = \sqrt{p_1 p_2}\, s^n \alpha + \sqrt{q_1 q_2}$ and the associated sets $S_\alpha = \{(q_1, q_2) : \sqrt{p_1 p_2}\, s^n \alpha + \sqrt{q_1 q_2} - s^m \geq 0\}$ for values of α positive (solid/light gray), zero (dashed/medium gray), and negative (dotted/dark gray). The figure also shows the optimal straight segment $Q = \eta_1 q_1 + \eta_2 q_2$ and its normal vector (η_1, η_2).

Parametric solution to State Separation

Parametrize the unitary constraint: $s = \sqrt{p_1 p_2} s' + \sqrt{q_1 q_2}$ using $p_1 p_1 = t^2, \quad q_1 q_2 = z^2$.
The condition becomes:
$z = s - s't, \qquad 0 \leq t, z \leq 1, \qquad 0 \leq s' \leq s.$

- From the first equation $t^2 = (1 - q_1)(1 - q_2) = 1 + z^2 - q_1 - q_2$. We now solve for q_1 (similarly for q_2) and obtain

$$q_{1,2} = \frac{1 + z^2 - t^2 \pm \sqrt{(1 + z^2 - t^2)^2 - 4z^2}}{2}.$$

- The condition $Q = \eta_1 q_1 + \eta_2 q_2$ becomes $2Q = 1 + z^2 - t^2 \pm (\eta_1 - \eta_2)\sqrt{(1 + z^2 - t^2)^2 - 4z^2}$. Solve for z^2

$$\begin{aligned} z^2 &= \frac{2\eta_1\eta_2(1 + \tau) - 1 + Q + \sqrt{(1 - 4\eta_1\eta_2)\left[(1 - Q)^2 - 4\right.}}{2\eta_1\eta_2} \\ &\equiv \zeta(\tau) \end{aligned}$$

Parametric solution to State Separation

The derivative $d\zeta/d\tau$ is immediate. We find that the maximum is located at

$$t_{\min} = \begin{cases} \sqrt{\left(1 - \dfrac{Q}{2\eta_1}\right)\left(1 - \dfrac{Q}{2\eta_2}\right)}, & \text{if} \quad 0 \le Q \le 2\eta_1 \\ 0, & \text{if} \quad 2\eta_1 < Q \le 1. \end{cases}$$

The corresponding values of z are

$$z_{\min} = \begin{cases} \dfrac{Q}{2\sqrt{\eta_1\eta_2}}, & \text{if} \quad 0 \le Q \le 2\eta_1 \\ \sqrt{\dfrac{Q - \eta_1}{\eta_2}}, & \text{if} \quad 2\eta_1 < Q \le 1. \end{cases}$$

Parametric solution to State Separation

$$s'(t) = -\frac{dz}{dt} = -\frac{t\,\zeta'(t^2)}{\sqrt{\zeta(t^2)}}, \qquad t_{\min} \leq t \leq 1 - Q,$$

and next *ii.* define

$$s(t) = z + ts'(t) = \sqrt{\zeta(t^2)} + ts'(t), \qquad t_{\min} \leq t \leq 1 - Q.$$

where

$$\zeta'(\tau) = 1 - \frac{\sqrt{1 - 4\eta_1\eta_2}}{\sqrt{(1-Q)^2 - 4\eta_1\eta_2\tau}}$$

For $s < z_{\min}$ is is always possible to separate the initial states, i.e., $|\psi_1\rangle$ and $|\psi_2\rangle$ can be made orthogonal. We note that the condition $s = z_{\min}$ is equivalent to the unambiguous discrimination result

$$Q = 2\sqrt{\eta_1\eta_2}s, \quad Q = \eta_1 + \eta_2 s^2.$$

Parametric solution to State Separation

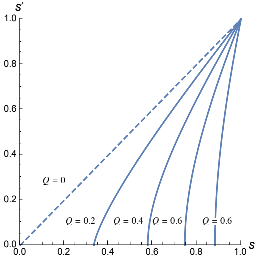


Figure: The plot is for $\eta_1 = 0.1$. As η_1 approaches $1/2$ the curves approach a straight line. The difference is more noticeable for very small values of η_1.

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