Quantum State Discrimination and Quantum Cloning Schemes: Optimization and Implementation

by

Andi Shehu

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2015

This manuscript has been read and accepted for the
Graduate Faculty in Physics in satisfaction of the
dissertation requirement for the degree of Doctor of Philosophy.

Date	Prof. János A. Bergou Chair of Examining Committee
Date	Prof. Igor L. Kuskovsky Executive Officer

Supervisory Committee

Prof. Mark Hillery

Prof. Christopher C. Gerry

Prof. Ed Fieldman

Prof. Neepa T. Maitra

THE CITY UNIVERSITY OF NEW YORK

Contents

List of Tables

List of Figures

CHAPTER 1

Introduction

state discrimination. One of the most vital tools of quantum information theory is reading out the information. It is the probabilistic nature of quantum mechanics that one cannot simply obtain information encoded in states, the state is not an observable in quantum mechanics. When a quantum circuit or processor has acted on the input states to perform a task, the output needs to be read out. Thus after the processing occurs the task is to determine the state of the system. If the input states are orthogonal the process is trivial. Simply setting up detectors along the orthogonal directions and a click in those detectors will determine the state of the system. On the other hand discriminating among non orthogonal quantum states is not trivial. Since quantum mechanics does not allow for perfect discrimination of non orthogonal states the task becomes that of a measurement optimization problem. Not being able to perfectly discriminate quantum states is key to various quantum cryptographic schemes and quantum computing. The origin of the measurement optimization field dates back to the 70's by the works of Helstrom and Holevo. They minimized the average error rate of discriminating the input states. The field however gained momentum in the 90's as quantum information theory become very active primarily due to the factorization work of Peter Schor and quantum key distribution protocols such as B92.

Various optimum state discrimination measurement strategies have been developed with respect to some figure of merit. Two of those methods which we focus on are optimum Unambiguous Discrimination (UD) and Minimum Error (ME). In UD the observer is not allowed to make an error. Whenever he is handed a state $|\psi_i\rangle$ he cannot conclude that he was given $|\psi_j\rangle$. We will show that this cannot be done

with 100% success rate and that the observer must allow for inconclusive results and find an optimum measurement strategy which minimizes the average rate of inconclusive results. In the Minimum Error strategy the observer is not allowed to have inconclusive results. Thus errors are allowed and the task is to find optimum measurements that minimize the average error rate. ME and UD seem to be a special case of a more general scheme of optimum state discrimination measurement which can be approached by relaxing the conditions at either end. In the ME scheme the optimal error rate can be further reduced by allowing for some rate of inconclusive results. Thus the optimal average error rate, P_E, becomes a function of a given rate of allowed inconclusive results Q, $P_E(Q)$. On the other hand, in UD, the optimal rate of the average inconclusive outcomes, Q, may be reduced by allowing for some error rate P_E. The failure rate becomes a function of a given error rate $Q(P_E)$. The full analytical solution to this interpolation scheme with a Fixed Rate of Inconclusive Outcome, $FRIO$, was first developed by Bagan *et al* [ref here]. We provide a different approach to the $FRIO$ scheme using Neumark theorem. This approach has the advantage that it lends itself into an experimental realization which can be carried out using beam splitters and phase shifters. We have also extended this method to a different class of mixed states, states whose spectral form is such that the eigenvectors form a Jordan basis for the Hilbert space of the problem. The UD of such mixed states has been investigated and the optimal solution has been obtained in [21, 22]. Thus, we know both the UD and the ME limit for these states. In this paper we will derive the optimal strategy with a Fixed Rate of Inconclusive Outcomes ($FRIO$) that optimally interpolates between these two known limits. In particular, as the main finding of our paper, we will show that the optimal distribution of the fixed rate of inconclusive outcomes, Q, among the 2-dimensional subspaces spanned by the pair of Jordan basis vectors is highly non-trivial and an interesting threshold-like structure emerges: As we start increasing Q from $Q = 0$, first only one subspace receives the entire inconclusive rate. Then, as we increase Q further, at a certain threshold a second subspace

starts sharing the inconclusive rate. If we increase Q further, at another threshold a third subspace also starts sharing Q , and so on, until above a last threshold all subspaces share the available inconclusive rate.

Cloning. As we mentioned above, one of the reasons we need to develop optimum state discrimination measurement schemes is due to the no cloning theorem of Wootters, Zurek[**1**] and Dieks[]. If one could copy non orthogonal quantum states then by making infinitely many copies, it would be possible to distinguish the states. Cloning machines which optimize some criteria with a limited degree of success have been developed. Those cloning machines fall under two categories: universal and state dependent. Universal cloning machines, which make copies of a completely unknown quantum state, were first developed first by Bŭzek and Hillery[]. This scheme makes approximate copies of an unknown quantum state while optimizing the local fidelity, square overlap of the one of the approximate clones and the original state it is suppose to clone. The other category of cloning machine is state dependent. In this scheme the observer has full knowledge of the prepared states but does not know which is the state he is given. There are two subcategories within this scheme: approximate and exact cloning. Approximate state-dependent cloning machines deterministically generate approximate clones from a finite set of non-orthogonal quantum states while optimizing the local or global fidelity (the average square overlap between full set of approximate clones and the states to be cloned). Again Hillery and Buzek[] are the pioneers of this subcategory of cloning machines. In Exact state-dependent cloning machines, the other subcategory of quantum cloning machines, the task is to probabilistically make exact copies of the incoming non-orthogonal quantum states. This comes at the expense of allowing for failure results where the scheme fails to produce a copy altogether. Duan and Guo[] were first to develop probabilistic exact cloning machines for the two state input where the states are prepared with equal a-priori probabilities. We recently extended this method for the more general case where the a-priori probabilities of the incoming states are different. This extension not only solves the full problem

but gives new insight into the nature of quantum cloning. The symmetry of the equal priors case completely solves the problem and no further optimization can be done. This symmetry however hides the true nature of the exact cloning machines which show up in the asymmetric case. This can be shown through the two step process: exact cloning then optimal UD. First procedure makes exact clones of the incoming two states. The exact clones are then sent to an optimal UD machine where the average failure rate is minimized. To combine the two step process the inconclusive rate from the cloning process is added to the inconclusive rate, weighed with the new a-priori probabilities, from the optimal UD. When the input states are prepared with equal a-priors the total amount of the inconclusive rate reaches the IDP limit. Hence cloning then performing optimal UD is equivalent to simply performing the optimal UD first, then prepare the clones. However this is not true for when the a-priors are different. After the two step process the total inconclusive rate is higher then the IDP limit. This suggests that during the cloning process some information is being leaked due to the asymmetry of the failure rate operators. When performing exact cloning, clones are produced but no measurement has been made, hence we do not know which states are being cloned. We simply know whether the procedure was successful or it failed. When it fails, the states are discarded. This is where the information leakage comes in. The state which is prepared most often shall have a higher rate of failure. This does not happen in the equal priors case because the failure rate are symmetric.

Implementation. Theoretically, the problem of Unambiguously discriminating between nonorthogonal but linearly independent quantum states has been solved when allowing for failure outcomes. The idea is to map the incoming nonorthogonal quantum states onto orthogonal states which can then be perfectly distinguished. This mapping is a non unitary transformation as the inner product is not preserved. These non unitary transformation can be performed with some probability of success by allowing for inconclusive outcomes. The task is to find an implementation scheme which minimizes the average probability of failure. Choosing a physical

system to realize quantum information processes, which have otherwise been solved theoretically, is central challenge to building a quantum computer. Some of the systems in use today are: energy levels of ions, the orientation of nuclear spin, the presence or absence of a photon in a cavity [enter ref] and dual rail representation of a qubit[] We will realize the implementation of our works of interpolation and ME using the dual rail representation of photons combined with a six-port, which is a linear device with three input and three output ports. The six-port can be realized with beamsplitters and phase shifters. First we will demonstrate the power and simplicity of this system by working out the implementation of UD by JA. Bergou *et al* [ ]. Then from solving the ME and $FRIO$ via Neumark we show how that by choosing the dual rail representation of the photon combined with a six-port the implementation followes naturally.

CHAPTER 2

Quantum State Discrimination

2.1. Unambiguous Discrimination

In this section we give a review of the existing schemes of Unambiguous Discrimination (UD). Particularly that of two pure states as it is directly related with our work. When performing UD the detectors are not allowed to make an error. We first show that it is not possible to succeed at discriminating quantum states with 100% success rate. Another detector must be added which accounts for inconclusive results. The task is to minimize this rate of inconclusive results.

2.1.1. Two pure states via POVM. An ensemble of quantum states is prepared with two possible pure states $|\psi_1\rangle$ or $|\psi_2\rangle$. Each state is prepared with an a priori probability η_1 or η_2 such that $\eta_1 + \eta_2 = 1$. The observer has full knowledge of the states and their priors. The preparer, Alice, picks up a state and hands it over to the observer, Bob. Bob's task it to determine which state he is given by performing a single measurement or a POVM on the individual system he is given.

The observer is not allowed to make an error when making a measurement. Let us assume Bob can indeed discriminate the given states with 100% success rate. Let Π_1 and Π_2 be detectors which cover the full Hilbert space spanned by the states $|\psi_1\rangle$and $|\psi_2\rangle$.

$$\Pi_1 + \Pi_2 = 1 \tag{2.1.1}$$

In the UD scheme the detector Π_i detects only the state $|\psi_i\rangle$ and never clicks for $|\psi_j\rangle$, such that $\Pi_i|\psi_j\rangle = 0$. Multiplying (1.1.1) by $|\psi_1\rangle$ from the right and $\langle\psi_1|$ from the left we get $\langle\psi_1|\Pi_1|\psi_1\rangle = 1$ which is the probability of successfully identifying $|\psi_1\rangle$, p_1 . Similarly we get that we can also successfully identify $|\psi_2\rangle$

with a success rate of $p_2 = \langle\psi_2|\Pi_2|\psi_2\rangle = 1$. Seems like one can indeed discriminate two non-orthogonal quantum states with a 100% success rate. However multiplying (1.1.1) by $\langle\psi_1|$ from the left and $|\psi_2\rangle$ from the right one gets $\langle\psi_1|\psi_2\rangle = 0$, where we use the fact that $\Pi_i|\psi_j\rangle = 0$. This means that the input states are orthogonal, but we started with nonorthogonal quantum states. Thus one cannot discriminate non-orthogonal quantum state with 100% success rate.

One can still perform Unambiguous Discrimination but with a modified scheme. We can modify (1.1.1) by adding a third detector Π_0 which can click for both states $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$\Pi_1 + \Pi_2 + \Pi_0 = 1 \tag{2.1.2}$$

The clicks from Π_0 are all inconclusive, i.e we gain no information from Π_0. Defining $q_1 = \langle\psi_1|\Pi_0|\psi_1\rangle$and $q_2 = \langle\psi_2|\Pi_0|\psi_2\rangle$ as the failure probabilities, we want to minimize the overall failure rate $Q = \eta_1 q_1 + \eta_2 q_2$.

Let us now determine the POVM operators explicitly. First lets define the states to be in a plane, such that

$|\psi_1\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

$|\psi_2\rangle = \cos\theta|0\rangle - \sin\theta|1\rangle$

The detectors have to be orthogonal with the states for which they are not supposed to click, i.e $\Pi_i|\psi_j\rangle = 0$.

$$\Pi_1 = c_1|\psi_2^{\perp}\rangle\langle\psi_2^{\perp}|,$$

$$\Pi_2 = c_2|\psi_1^{\perp}\rangle\langle\psi_1^{\perp}|,$$

where $|\psi_2^{\perp}\rangle = \sin\theta|0\rangle + \cos\theta|1\rangle$, $|\psi_1^{\perp}\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle$

The coefficients c_i are yet to be determined based on the optimum strategies.

Using the definition of success probabilities $p_i = \langle\psi_i|\Pi_i|\psi_i\rangle$ we can replace the constants c_i

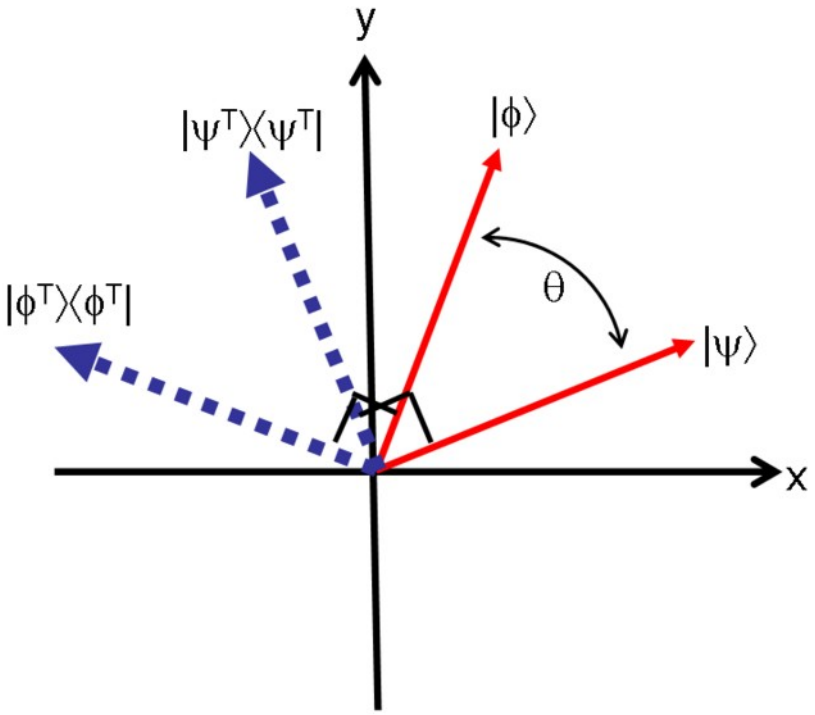


FIGURE 2.1.1. Projection

$$\Pi_1 = \frac{p_1}{|\langle\psi_1|\psi_2^\perp\rangle|^2}|\psi_2^\perp\rangle\langle\psi_2^\perp|$$

$$\Pi_2 = \frac{p_2}{|\langle\psi_2|\psi_1^\perp\rangle|^2}|\psi_1^\perp\rangle\langle\psi_1^\perp|$$

To determine what the failure operator is, plug in Π_1 and Π_2 into 2.1.2:

$$\Pi_0 = I - \Pi_1 - \Pi_2 = I - \frac{p_1}{|\langle\psi_1|\psi_2^\perp\rangle|^2}|\psi_2^\perp\rangle\langle\psi_2^\perp| - \frac{p_2}{|\langle\psi_2|\psi_1^\perp\rangle|^2}|\psi_1^\perp\rangle\langle\psi_1^\perp| \tag{2.1.3}$$

after writing everything explicitly, the positivity constraint of the eigenvalues of Π_0 gives the condition

$$q_1 q_2 \geq |\langle\psi_1|\psi_2\rangle|^2, \tag{2.1.4}$$

where we used $q_i = 1 - p_i$.

Using the above condition we can now optimize the failure rate

$$Q \quad = \quad \eta_1 q_1 + \eta_2 q_2 \tag{2.1.5}$$

from the constraint we use the equality $q_1 = s^2/q_2$, plug this into Q then optimize with respect to q_2,

$$Q = \frac{\eta_1 s^2}{q_2} + \eta_2 q_2$$

$$0 = \frac{\partial Q}{\partial q_2} = -\frac{\eta_1 s^2}{q_2^2} + \eta_2$$

This gives $q_1 = \sqrt{\frac{\eta_2}{\eta_1}} s$ and $q_2 = \sqrt{\frac{\eta_1}{\eta_2}} s$, plugging them back into Equation (2.1.4) gives the optimal Q

$$Q = 2\sqrt{\eta_1 \eta_2}\, s \tag{2.1.6}$$

Let us now check the conditions where this result holds. The individual error rates must be smaller or equal to one $q_i \leq 1$. Hence $q_1 = \sqrt{\frac{1-\eta_1}{\eta_1}} s \leq 1$ gives the lower bound on the a-prior probabilities, $\eta_1 \geq \frac{s^2}{1+s^2}$. Similarly the condition that $q_2 \leq 1$ gives the upper bound on the a-priors $\eta_1 \leq \frac{1}{1+s^2}$. Putting the two conditions together the POVM regime is valid in this range:

$$\frac{s^2}{1+s^2} \leq \eta_1 \leq \frac{1}{1+s^2} \tag{2.1.7}$$

Outside of this range it is interesting to see that the measurement strategy merges into the projective measurement.

If one one the incoming states is prepared with a much higher probability say $\eta_1 \gg \eta_2$ we design an experiment where we have only two detection operators. One of them, D_0, the failure operator, simply projects into state $|\psi_2\rangle$, the detector D_1 projects onto the orthogonality of $|\psi_2\rangle$, thus it never clicks for $|\psi_2\rangle$, so that a click

on D_1 is associated with the state $|\psi_1\rangle$. A click along D_2 is failure. The total failure rate is:

$$Q_1 = \eta_1 |\langle\psi_1|\psi_2\rangle|^2 + \eta_2 \tag{2.1.8}$$

Similarly for $\eta_2 \gg \eta_1$

$$Q_2 = \eta_1 + \eta_2 |\langle\psi_1|\psi_2\rangle|^2 \tag{2.1.9}$$

Putting the pieces together, the minimum value of Q for the three different regimes can be written as:

$$Q = \begin{cases} 2\sqrt{\eta_1\eta_2}s & \text{if } \frac{s^2}{1+s^2} \leq \eta_1 \leq \frac{1}{1+s^2} \\ \eta_1 |\langle\psi_1|\psi_2\rangle|^2 + \eta_2 \text{ if} & \eta_1 > \frac{1}{1+s^2} \\ \eta_1 + \eta_2 |\langle\psi_1|\psi_2\rangle|^2 \text{ if} & \eta_1 < \frac{s^2}{1+s^2} \end{cases} \tag{2.1.10}$$

It is very interesting that the POVM gives the minimum Q when it is valid. As soon as we step outside the boundaries it merges with the von Neumann projective measurement. Then the optimal failure is given by projecting the failure operator on the state which is least prepared. All of this arises very smoothly as can be seen by the graph.

2.1.2. Two pure states via Neumark. Theoretically the problem of minimizing the failure rate for two pure states has been solved in the above section. However to be able to implement those schemes we resort to Neumark's theorem which states that any POVM operator can be realized by generalized measurements. [fix it] The system where the incoming states live are embedded in a larger Hilbert space called Ancilla. Then a Unitary operator entangles the degrees of freedom of the system with those of the Ancilla. Then projective measurements are performed in this larger system. These measurements will also transform the system states in the original Hilbert space because of the entanglement.

To show the power of Neumark's theorem we will rederive the optimal failure rate of two nonorthogonal states. The incoming states $\{|\psi_1\rangle_s, |\psi_2\rangle_s\}$ which live in

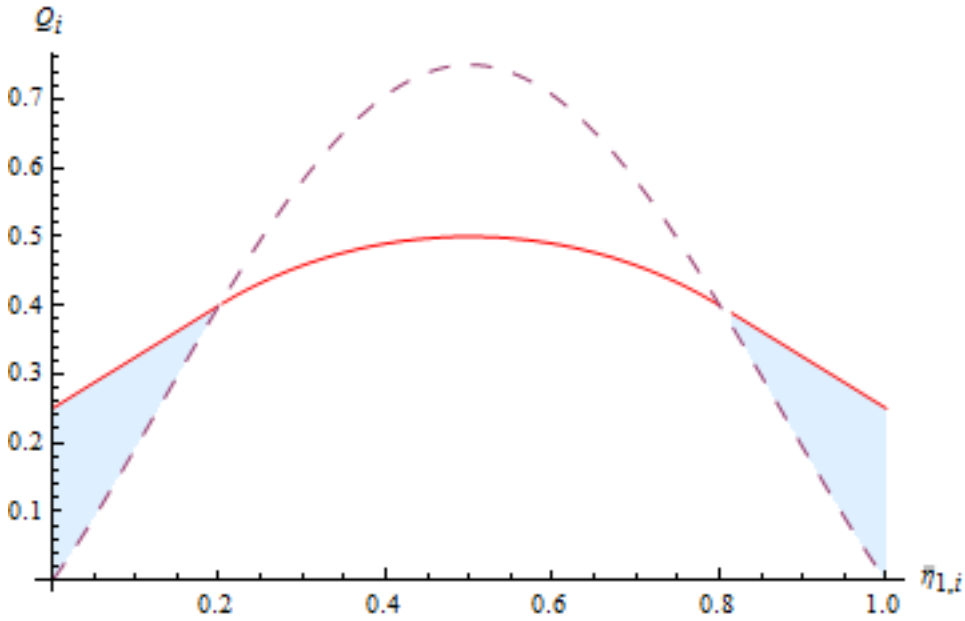


FIGURE 2.1.2. Failure probability, Q, *vs*. the prior probability η_1

the state Hilbert space S are embedded with the ancilla $|i\rangle_a$ which live in the ancilla Hilbert space A. Now the system and the ancilla live in the larger Hilbert space $H = S \otimes A$. The incoming states in this larger Hilbert space can be written in the product form $\{|\psi_1\rangle_s|i\rangle_a, |\psi_2\rangle_s|i\rangle_a\}$.

$$U|\psi_1\rangle_s|i\rangle_a = \sqrt{p_1}|\psi_1'\rangle_s|1\rangle_a + \sqrt{q_1}|\phi\rangle_s|0\rangle_a \tag{2.1.11}$$

$$U|\psi_2\rangle_s|i\rangle_a = \sqrt{p_2}|\psi_2'\rangle_s|1\rangle_a + \sqrt{q_2}|\phi\rangle_s|0\rangle_a \tag{2.1.12}$$

Where p_i is the probability of successfully discriminating the state $|\psi_i\rangle_s$, q_i is the probability of failing to discriminate $|\psi_i\rangle_s$,$p_i + q_i = 1$. The unitary operator looks to take the two incoming states and make them orthogonal. When there is a click on the ancilla $|1\rangle_a$ the input states have been separated and and output states $|\psi_i'\rangle_s$ are orthogonal and thus fully distinguishable. If there is a click along the ancilla $|0\rangle_a$ the incoming states have been collapsed into a single state which carries no information. That is why the choice on the setup of having the failed

$|\phi\rangle_s$ state be the same, there should be absolutely no information left in the failed state, otherwise it is not optimal.

To solve the problem we take the inner product of the two equations which gives the constraint.

$$s = \sqrt{q_1 q_2}, \tag{2.1.13}$$

where s is the overlap of the incoming states $s \equiv \langle\psi_1|\psi_2\rangle$. The quantity we want to minimize is

$$Q = \eta_1 q_1 + \eta_2 q_2 \tag{2.1.14}$$

The rest of the calculations are the same as in the POVM section and we do not need to repeat here. Implementation methods have been derived and we will show an example in the Implementation Chapter.

2.2. Minimum Error

In the Minimum Error (ME) strategy one is not allowed to abstain from identifying an incoming state. Since it was shown that perfect discrimination is not possible then the detectors are allowed to err. A click in a detector can only identify a state with some probability of success and misidentify the state with some probability of error.

2.2.1. POVM (Helstrom). Discriminating two incoming mixed states by minimizing the error rate was first developed by Helstrom.

Given an ensemble of two mixed states $\{\rho_1, \rho_2\}$ prepared with different a priori probabilities $\{\eta_1, \eta_2\}$ the task is to minimize the rate for which the detectors misidentify a state. When the detector Π_i clicks for state ρ_j we consider that an error, $r_i = Tr(\rho_i \Pi_j)$. Thus for two states we want to minimize the following expression.

$$P_E = \eta_1 Tr(\rho_1 \Pi_2) + \eta_2 Tr(\rho_2 \Pi_1) \tag{2.2.1}$$

Using the relation $\eta_1 + \eta_2 = 1$ and $\Pi_1 + \Pi_2 = I$ Equation 2.2.2 can be rewritten as:

$$P_E = \eta_1 Tr(\rho_1(I - \Pi_1)) + \eta_2 Tr(\rho_2 \Pi_1)$$

$$\begin{aligned} P_E &= \eta_1 + Tr[(\eta_2\rho_2 - \eta_1\rho_1)\Pi_1] \\ &= \eta_2 - Tr[(\eta_2\rho_2 - \eta_1\rho_1)\Pi_2] \end{aligned}$$

Let $\Lambda = \eta_2\rho_2 - \eta_1\rho_1$

$$P_E = \eta_1 + Tr(\Lambda\Pi_1) = \eta_2 - Tr(\Lambda\Pi_2) \tag{2.2.2}$$

To minimize P_E, Π_1 should project onto the eigenvectors of the negative eigenvalues of Λ, on the other hand Π_2 should project onto the positive eigenvectors. Let us write Λ into its spectral decomposition.

$$\Lambda = \eta_2\rho_2 - \eta_1\rho_1 = \sum_{i=1}^{d} \lambda_i |\lambda_i\rangle\langle\lambda_i| \tag{2.2.3}$$

To implement the projection of the POVM operators onto the positive (or negative) eigenvectors the eigenvalues λ_i can be split into three categories without any loss of generality: negative, positive and zero:

$$\begin{aligned} \lambda_i &< 0 \text{ for } 1 \leq i < i_o \\ \lambda_i &> 0 \text{ for } i_o \leq i < d \\ \lambda_i &= 0 \text{ for } d \leq i < d_s \end{aligned} \tag{2.2.4}$$

Then from the spectral decomposition we can rewrite (2.2.6) in terms of the optimal POVM.

$$P_E = \eta_1 + \sum_{i=1}^{i_o-1} \lambda_i \langle\lambda_i|\Pi_1|\lambda_i\rangle = \eta_2 - \sum_{i=i_o}^{d_s} \lambda_i \langle\lambda_i|\Pi_2|\lambda_i\rangle \tag{2.2.5}$$

Where $\Pi_1 = \sum_{i=1}^{i_o-1} \lambda_i|\lambda_i\rangle\langle\lambda_i|$ and $\Pi_2 = \sum_{i=i_o}^{d_s} \lambda_i|\lambda_i\rangle\langle\lambda_i|$.

The POVMs need to satisfy the condition $0 \leq \langle\lambda_i|\Pi_j|\lambda_i\rangle \leq 1$ which comes from the definition of the normalized probabilities $r_i = Tr(\rho_i\Pi_j)$. These POVMs are basically von Neuman projectors onto the corresponding eigenvectors. If we now replace the detection operators by the optimal detectors the minimum error can be expressed just in terms of the eigenvalues of Λ.

$$\begin{aligned} P_E &= \eta_1 - \sum_{i=1}^{i_o-1} |\lambda_i| = \eta_2 - \sum_{i=1}^{d_s} |\lambda_i| \\ P_E &= \frac{1}{2}[1 - \sum_{i=1}^{d_s} |\lambda_i|] = \frac{1}{2}[1 - Tr|\Lambda|] \\ P_E &= \frac{1}{2}[1 - Tr|\eta_2\rho_2 - \eta_1\rho_1|] \end{aligned} \tag{2.2.6}$$

When the states to be discriminated are pure, $\{|\psi_1\rangle, |\psi_2\rangle\}$, the minimum error can be reduced to.

$$P = \frac{1}{2}[1 - \sqrt{1 - 4\eta_1\eta_2|\langle\psi_1|\psi_2\rangle|^2}] \tag{2.2.7}$$

2.2.2. Two pure states via Neumark. Just as we did in the UD case, we will solve the ME problem of discriminating two pure states via the Neumark setup because of it lends itself into an optical implementation.

$$U|\psi_1\rangle|0\rangle = \sqrt{p_1}|\psi_1\rangle|1\rangle + \sqrt{r_1}|\psi_2\rangle|2\rangle, \tag{2.2.8}$$

$$U|\psi_2\rangle|0\rangle = \sqrt{r_2}|\psi_1\rangle|1\rangle + \sqrt{p_2}|\psi_2\rangle|2\rangle, \tag{2.2.9}$$

Taking the inner product gives the constraint

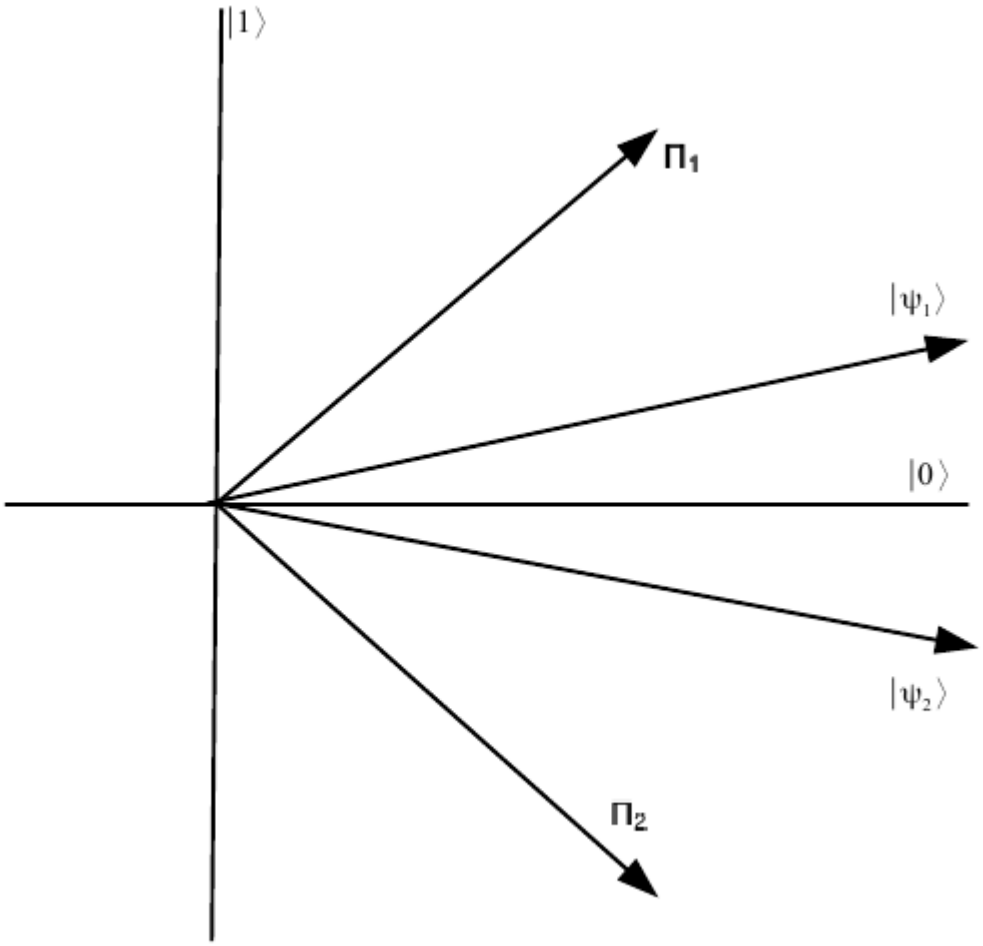


FIGURE 2.2.1. Min Error

$$s = \sqrt{p_1 r_2} + \sqrt{p_2 r_1} \tag{2.2.10}$$

The quantity we are looking to minimize in the total error rate:

$$P_E = \eta_1 r_1 + \eta_2 r_2 \tag{2.2.11}$$

subject to the above constraint. Setting up the Lagrangian and using $p_i + r_i = 1$

$$F_E = \eta_1 r_1 + \eta_2 r_2 + \lambda[s - \sqrt{(1 - r_1) r_2} - \sqrt{(1 - r_2) r_1}] \tag{2.2.12}$$

Differentiating with respect to r_i the setting equal to zero.

$$\frac{\partial F_E}{\partial r_1} = \eta_1 + \frac{1}{2}[\sqrt{\frac{r_2}{1-r_1}} - \sqrt{\frac{1-r_2}{r_1}}] = 0$$

$$\frac{\partial F_E}{\partial r_2} = \eta_2 + \frac{1}{2}[-\sqrt{\frac{r_1}{1-r_2}} + \sqrt{\frac{1-r_1}{r_2}}] = 0$$

Rearranging we get

$$\frac{2\eta_1}{\lambda}\sqrt{r_1(1-r_1)} = \sqrt{r_1 r_2} - \sqrt{(1-r_1)(1-r_2)} \tag{2.2.13}$$

$$\frac{2\eta_2}{\lambda}\sqrt{r_2(1-r_2)} = \sqrt{r_1 r_2} - \sqrt{(1-r_1)(1-r_2)} \tag{2.2.14}$$

The left hand sides can be set to a constant, then we find the constant

$$\frac{2\eta_i}{\lambda}\sqrt{r_i(1-r_i)} \equiv C \tag{2.2.15}$$

$$r_i = \frac{1}{2}(1 \pm \sqrt{1 - \frac{\lambda^2 c^2}{\eta_i^2}}) = \frac{1}{2}(1 - \sqrt{1 - \frac{\delta^2}{\eta_i^2}}) \tag{2.2.16}$$

$$r_i = \frac{1}{2}[1 - A_i] \tag{2.2.17}$$

where $A_i = \sqrt{1 - \frac{\delta^2}{\eta_i^2}}$. The smaller r_i is picked as this represents error rate, which should be minimized. Now plug r_i into the constraint and solve for δ:

$$\begin{aligned}
s &= \sqrt{(1-r_1)r_2} + \sqrt{(1-r_2)r_1} \\
2s &= \sqrt{(1+A_1)(1-A_2)} + \sqrt{(1-A_1)(1+A_2)} \\
2s^2 &= 1 - A_1 A_2 + \sqrt{(1-A_1^2)(1-A_2^2)} \\
2s^2 &= 1 - A_1 A_2 + \frac{\delta^2}{\eta_1 \eta_2} \\
(2s^2 - 1 \quad -\tfrac{\delta^2}{\eta_1\eta_2})^2 &= 1 - \frac{\delta^2}{\eta_1^2} - \frac{\delta^2}{\eta_2^2} + \frac{\delta^4}{\eta_1^2 \eta_2^2}
\end{aligned}$$

After some tedious but trivial algebra:

$$\delta^2 = \frac{4s^2(1-s^2)\eta_1^2\eta_2^2}{1-4\eta_1\eta_2 s^2} \tag{2.2.18}$$

Let's plug this back into (2.2.23) to get the explicit form of the individual error rates,

$$r_1 = \frac{1}{2}[1 - \frac{1-2\eta_2 s^2}{\sqrt{1-4\eta_1\eta_2 s^2}}] \tag{2.2.19}$$

$$r_2 = \frac{1}{2}[1 - \frac{1-2\eta_1 s^2}{\sqrt{1-4\eta_1\eta_2 s^2}}] \tag{2.2.20}$$

Replacing r_1 and r_2 into 2.2.16 we retrieve Helstrom bound

$$P_E = \frac{1}{2}[1 - \frac{\eta_1 - 2\eta_1\eta_2 s^2}{\sqrt{1-4\eta_1\eta_2 s^2}} - \frac{\eta_2 - 2\eta_1\eta_2 s^2}{\sqrt{1-4\eta_1\eta_2 s^2}}] \tag{2.2.21}$$

$$P_E = \frac{1}{2}[1 - \sqrt{1-4\eta_1\eta_2 s^2}] \tag{2.2.22}$$

As we mentioned above the advantage of solving the ME problem this way is that we now have explicit expression to individual error rates, r_1 and r_2. In the Implementation chapter it is shown that the unitary operator can be written in terms of r_i and then be decomposed into beam splitters and phase shifters.

2.3. Optimal discrimination of a certain class of mixed states with a fixed rate of inconclusive outcome(FRIO)

2.3.1. Review of the FRIO solution for two pure states.

2.4. Introduction

The two fundamental state discrimination strategies are the so-called minimum-error (ME) strategy and the unambiguous discrimination (UD) strategy. In ME, every time a system is given and a measurement is performed on it a conclusion must be drawn about its state. Errors are permitted and in the optimal strategy

the probability of making an error is minimized [**?**]. In UD, no errors are permitted at the expense of allowing an inconclusive measurement outcome to occur. When it occurs we don't learn anything about the state of the system and in the optimal strategy the probability of the inconclusive outcome is minimized [**?, ?, ?, ?**]. It has been recognized that states can be discriminated unambiguously only if they are linearly independent [**?**]. Discrimination with maximum confidence (MC) can be applied to states that are not necessarily independent and for linearly independent states it reduces to unambiguous discrimination [**?, ?**], so it can be regarded as a generalized UD strategy. In the MC strategy Bayes' rule is used to maximize the chance of the appropriate detector clicking for its desired state.

It is clear that UD (or MC for linearly dependent states) and ME are the two extremes of a more general strategy and one can approach this more general strategy from either end. Starting from UD one can allow a fixed inconclusive rate Q to occur that is smaller than the optimal (minimal) inconclusive rate Q_C for UD, $0 \leq Q \leq Q_C$. Then one also needs to allow for errors to occur and one can minimize the error rate for the fixed value of Q. This scheme was introduced by Chefles and Barnett [**?**] where they immediately provided the optimal solution for two pure states with equal prior probabilities, although at the time they did not prove full optimality of their solution. Later, Fiurášek and Ježek [**?**] generalized this result for mixed states. Further useful bounds were obtained in Refs. [**?, ?**], using the method of Semidefinite Programming (SDP). Alternatively, one can choose a certain fixed rate P_e of errors to occur that is smaller than the optimal minimum error rate P_E, $0 \leq P_e \leq P_E$. Then one needs to allow for inconclusive outcomes to occur and one can minimize the inconclusive rate for a fixed value of P_e [**?**]. The first approach yields the minimal error rate as a function of the given inconclusive rate, $P_e(Q)$, while the second yields the minimum inconclusive rate as a function of the given error rate $Q(P_e)$. Since the two expressions result from optimizations under different conditions, it is not immediately obvious that they are, in fact, the same (in the sense of being inverses of each other). It is relatively straightforward

to prove, however, that this must be true and the proof is based on showing that the opposite leads to contradiction [**?**].

In a recent burst of activity both approaches were solved analytically for two pure states and for other, highly symmetric situations. Actually, the second approach was solved earlier, first for two pure states with equal priors [**?**] and then for general occurrence probabilities [**?**]. However the connection with the first approach remained unnoticed in these two works. Quite recently, the first approach has been solved completely by two groups, using entirely different methods [**?, ?**]. Namely, Bagan *et al.* [**?**] solved the problem introducing a transformation which formally eliminates the inconclusive part of the problem and reduces it to a ME problem with an extra optimization parameter. On the other hand, Herzog employed the SDP approach to obtain the solution for two pure states with arbitrary priors and for classes of special mixed states [**?**].

The goal of this paper is to extend the method developed in Ref. [**?**] to a class of mixed states whose spectral form is such that the eigenvectors form a Jordan basis for the Hilbert space of the problem. The UD of such mixed states has been investigated and the optimal solution has been obtained in [**?, ?**]. Later this was extended to programmable quantum state discriminators, where in a different context the ME of such mixed states has been solved [**?**]. Thus, we know both the UD and the ME limit for these states. In this paper we will derive the optimal strategy with a Fixed Rate of Inconclusive Outcomes (FRIO) that optimally interpolates between these two known limits. In particular, as the main finding of our paper, we will show that the optimal distribution of the fixed rate of inconclusive outcomes, Q, among the 2-dimensional subspaces spanned by the pair of Jordan basis vectors is highly non-trivial and an interesting threshold-like structure emerges: As we start increasing Q from $Q = 0$, first only one subspace receives the entire inconclusive rate. Then, as we increase Q further, at a certain threshold a second subspace starts sharing the inconclusive rate. If we increase Q further, at another threshold

a third subspace also starts sharing Q, and so on, until above a last threshold all subspaces share the available inconclusive rate.

The paper is organized as follows. In Sec. 2.5 we briefly review the FRIO solution for two pure states, viz. the transformation solution found in [**?**], because the treatment presented in the body of the paper relies heavily on this method. In Sec. 2.6 we extend the FRIO problem to 2 subspaces, each two-dimensional. Then, in Sec. IV we consider a general Jordan Basis case, consisting of N two-dimensional subspaces, where a Lagrangian optimization to minimize the error rate for fixed Q over all subspaces leads to the threshold structure. We illustrate the results on the example where the two mixed states to be discriminated are each of rank 3 and together they span a 6-dimensional Hilbert space. Because of the Jordan structure, the Hilbert space consists of three mutually orthogonal 2-dimensional Jordan subspaces. In Sec. V we deal with the projective regimes separately. Finally, in Sec. VI we conclude with a discussion.

2.5. Review of the FRIO solution for two pure states

We first present a brief review of the method developed in [**?**] for the two pure state optimal FRIO problem since the rest of the paper relies heavily on this method. We derive the maximum probability of success or, equivalently, the minimum probability of error in identifying the states, when a certain fixed rate of inconclusive outcomes is allowed. By varying the inconclusive rate, the scheme optimally interpolates between Unambiguous and Minimum Error discrimination (UD and ME), the two standard approaches to quantum state discrimination.

In all of these scenarios (UD, ME or FRIO) one is given a system which is promised to be prepared in one of two known pure states, $\rho_1 = |\psi_1\rangle\langle\psi_1|$ or $\rho_2 = |\psi_2\rangle\langle\psi_2|$, but we don't know which. The pure states are prepared with prior probabilities η_1 and η_2, respectively, such that $\eta_1 + \eta_2 = 1$. It is well known that two pure states can be discriminated both unambiguously and with minimum error. The optimal (minimal) inconclusive rate, Q_c for UD has been derived in [**?, ?, ?, ?**],

and is given by

$$Q_{\rm c} = \begin{cases} \eta_1 + \eta_2 \cos^2\theta, \text{if } \eta_1 < \dfrac{\cos^2\theta}{1+\cos^2\theta} \equiv \eta_1^{(l)}, \\ \eta_2 + \eta_1 \cos^2\theta, \text{if } \eta_1 > \dfrac{1}{1+\cos^2\theta} \equiv \eta_1^{(r)}, \\ 2\sqrt{\eta_1\eta_2}\cos\theta \equiv Q_0, \text{if } \eta_1^{(l)} \le \eta_1 \le \eta_1^{(r)} \ , \end{cases} \tag{2.5.1}$$

where $|\langle\psi_1|\psi_2\rangle| \equiv \cos\theta$ is the overlap of the states.

The optimal (minimal) error rate for ME has been derived in [**?**] and is given by

$$P_{\rm E} = \frac{1}{2}\left(1 - \sqrt{1 - 4\eta_1\eta_2|\langle\psi_1|\psi_2\rangle|^2}\,\right). \tag{2.5.2}$$

For a compact derivation of these basic results see [**?**].

In the FRIO strategy we want three measurement outcomes, one that we identify with the first state, one that we identify with the second state and one that we do not identify with a state, corresponding to the inconclusive outcome. Therefore, we look for the solution as a three-element POVM (Positive Operator Valued Measure),

$$\Pi_1 + \Pi_2 + \Pi_0 = I \ , \tag{2.5.3}$$

where I is the identity operator in the Hilbert space of the states. We identify a click in Π_1 (Π_2) with the first (second) state, while a click in Π_0 corresponds to the inconclusive outcome. We wish to optimize (maximize) the success rate, $P_s = \eta_1 tr[\Pi_1\rho_1] + \eta_2 tr[\Pi_2\rho_2]$ or, equivalently, minimize the error rate, $P_e = \eta_1 tr[\Pi_2\rho_1] + \eta_2 tr[\Pi_1\rho_2]$, for a fixed failure rate,

$$Q = tr[\Pi_0(\eta_1\rho_1 + \eta_2\rho_2)] \ . \tag{2.5.4}$$

Obviously, $0 \le Q \le Q_{\rm c}$ and for $Q = Q_{\rm c}$ the FRIO strategy reduces to UD with $P_e = 0$ while for $Q = 0$ the FRIO strategy reduces to ME with the error rate given by (2.5.2). For a given intermediate value of Q the strategy minimizes P_e.

The method of solution is based on the observation that, for optimal FRIO, Π_0 must be a rank one operator. Namely, if we assume the opposite, i.e., Π_0 is a

rank two operator then it maps the two input states into a pair of other two states that can be further discriminated, contradicting that the strategy is optimal. The starting point is to write $\Pi_1 + \Pi_2 = I - \Pi_0 \equiv \Omega$. Multiplying both sides of this equation by $\Omega^{-1/2}$ we have $\tilde{\Pi}_1 + \tilde{\Pi}_2 = I$, where

$$\tilde{\Pi}_i = \Omega^{-1/2} \Pi_i \, \Omega^{-1/2} \geq 0. \tag{2.5.5}$$

So $\{\tilde{\Pi}_1, \tilde{\Pi}_2\}$ is a POVM. Note that $\Omega^{-1/2}$ exists unless Π_0 has a unit eigenvalue, in which case we address the problem differently (see later). Defining the normalized transformed states $\tilde{\rho}_i$ and *a priori* probabilities $\tilde{\eta}_i$ as

$$\tilde{\rho}_i = \frac{\Omega^{1/2} \rho_i \Omega^{1/2}}{Tr(\Omega \rho_i))}, \qquad \tilde{\eta}_i = \frac{\eta_i Tr(\Omega \rho_i)}{1 - Q}, \tag{2.5.6}$$

the error probability, $P_e = (1 - Q)\tilde{P}_e$, becomes

$$\tilde{P}_e \;\; = \;\; Tr(\tilde{\eta}_1 \tilde{\rho}_1 \tilde{\Pi}_2) + Tr(\tilde{\eta}_2 \tilde{\rho}_2 \tilde{\Pi}_1), \tag{2.5.7}$$

and $\tilde{P}_s = 1 - \tilde{P}_e$. Equation (2.5.7) and $\tilde{\Pi}_1 + \tilde{\Pi}_2 = I$ define a ME discrimination problem for the transformed states and *priors* given in Eq. (2.5.6). The optimal solution to this ME discrimination problem immediately follows by using the tilde quantities in (2.5.2), with being the properly normalized transformed states.

Since the optimal Π_0 is a positive rank one operator it can be written as $\Pi_0 = \xi |0\rangle\langle 0|$, where ξ is its eigenvalue, $0 \leq \xi \leq 1$, and the eigenstate belonging to ξ is $|0\rangle$ and the orthogonal state is $|1\rangle$. In this basis $\Omega = (1 - \xi)|0\rangle\langle 0| + |1\rangle\langle 1|$. Writing the input states also in this basis, $|\psi_i\rangle = c_i|0\rangle + s_i|1\rangle$, where $c_i \equiv \cos\theta_i$, $s_i \equiv \sin\theta_i$, and obtaining the transformed states and priors from Eq. (2.5.6), after some simplifications we get

$$\tilde{P}_e^{\rm ME} = \frac{1}{2}\left\{1 - \sqrt{1 - 4\eta_1\eta_2(\cos\theta - \xi c_1 c_2)^2/(1 - Q)^2}\right\}, \tag{2.5.8}$$

where we used that $\theta_1 - \theta_2 \equiv \theta$. It follows from (2.5.4) that

$$\xi = \frac{Q}{\eta_1 c_1^2 + \eta_2 c_2^2}. \tag{2.5.9}$$

Hence Eq. (2.5.8) depends only on one parameter, say θ_1, which determines the orientation of Π_0 relative to that of the two pure states. The minimization over θ_1 simplifies considerably, using Eq. (2.5.9) and defining $c_1\,\eta_1^{1/2}(\eta_1 c_1^2+\eta_2 c_2^2)^{-1/2} \equiv \cos\varphi$, and $c_2\,\eta_2^{1/2}(\eta_1 c_1^2+\eta_2 c_2^2)^{-1/2} \equiv \sin\varphi$. The resulting expression is minimum for $\varphi=\pi/4$, yielding

$$P_e^{\min}=\frac{1}{2}\left\{1-Q-\sqrt{(1-Q)^2-(Q_0-Q)^2}\right\}\ , \tag{2.5.10}$$

for all $Q \le Q_0$, where Q_0 was introduced in the third line of (2.5.1). This is the optimal error rate for an intermediate range of the prior probabilities.

For the validity of (2.5.10), $\xi \le 1$ must hold. definitions of $\cos\varphi$ and $\sin\varphi$ after Eq. (2.5.9) give $\sqrt{\eta}_2 c_2 = \sqrt{\eta}_1 c_1$ for $\varphi=\pi/4$ which, in turn, leads to $\eta_1 c_1^2 = \eta_2 c_2^2 = \eta_1\eta_2\sin^2\theta/(1-Q_0)$, and Eq. (2.5.9) yields $\xi = (1-Q_0)Q/(2\eta_1\eta_2\sin^2\theta)$. Setting $\xi=1$ defines the boundary Q_b, between the projective and POVM regimes,

$$Q_{\rm b} \equiv 2\eta_1\eta_2\sin^2\theta/(1-Q_0). \tag{2.5.11}$$

Hence $\xi \le 1$ if $Q \le Q_{\rm b}$ and $\xi = 1$ if $Q > Q_{\rm b}$.

In Fig. **??** we plot $Q_{\rm c}$ and $Q_{\rm b}$ vs. η_1 together for a fixed overlap, $\cos\theta = 0.5$ ($\theta=\pi/3$). The two curves intersect at $\eta_1=\eta_1^{(l)}$ and $\eta_1=\eta_1^{(r)}$, the same points as in Eq. (2.5.1). The interval $0 \le \eta_1 \le 1$ is thus divided into three regions. In regions I and III, we have $Q_{\rm b} < Q_{\rm c}$ and the solution (2.5.10) is valid for $0 \le Q < Q_{\rm b}$ only. In Region II, $\eta_1^{(l)} \le \eta_1 \le \eta_1^{(r)}$, we have $Q_{\rm c} = Q_0 < Q_{\rm b}$ and the solution (2.5.10) is valid for the entire $0 \le Q \le Q_{\rm c}$ range.

"$Q_{\rm c}$ (dashed line, Eq. (2.5.1)) and $Q_{\rm b}$ (solid line, Eq. (2.5.11)) vs. η_1 for $\theta=\pi/3$. Measurements can be optimized in the area under the dashed line, Q_c. Measurements in the area above Q_c are suboptimal. In the shaded areas between Q_c and Q_b (regions I, left, and III, right) the optimal FRIO measurement is a projective measurement, in the unshaded area below Q_c (region II) the optimal measurement is a POVM."

In the shaded parts of regions I and III one has $Q_{\rm b} \leq Q \leq Q_{\rm c}$ and, necessarily, $\xi = 1$. Hence, $\Pi_0 = |0\rangle\langle 0|$ and $\Omega = |1\rangle\langle 1|$ are projectors. Therefore, $\Omega^{-1/2}$ does not exist in these areas and the case needs special consideration.

The calculation of the error probability is most easily performed by realizing that Π_1 and Π_2 become degenerate, both must be proportional to $\Pi_d = |1\rangle\langle 1|$. The three-element POVM becomes a standard two element projective measurement, $\{\Pi_d = |1\rangle\langle 1|,\ \Pi_0 = |0\rangle\langle 0|\}$. We identify a click in Π_d with ρ_1 (ρ_2) if $\eta_1 \geq \eta_2$ ($\eta_2 \geq \eta_1$), so $P_{e(s)} = \eta_2 s_2^2, \quad P_{s(e)} = \eta_1 s_1^2$, with $Q = 1 - P_e - P_s$. These equations completely determine the solution. There is nothing to optimize here, so we drop the superscript min in what follows. $\theta_1 - \theta_2 = \theta$ immediately gives $Q(P_e)$ as

$$Q = 1 - P_e - \eta_{1(2)} \left(\sqrt{\frac{P_e}{\eta_{2(1)}}} \cos\theta \pm \sqrt{1 - \frac{P_e}{\eta_{2(1)}}} \sin\theta \right)^2 . \tag{2.5.12}$$

Inverting this equation gives $P_e(Q)$ yields

$$P_e = \eta_2 \frac{2\eta_1 \cos^2\theta (1 - Q - Q_2) - (\eta_1 - \eta_2)(Q - Q_2) - 2\eta_1\eta_2 \sin\theta\cos\theta \sqrt{Q(1-Q) - \eta_1\eta_2 \sin^2\theta}}{1 - 4\eta_1\eta_2 \sin^2\theta} , \tag{2.5.13}$$

but the resulting expression is not particularly insightful. However, we note that for $P_e = 0$ (UD limit) one has $Q = Q_2$, given by the second line in Eq. (2.5.1), and for $Q = Q_{\rm b}$, P_e reduces to (2.5.10), as it should.

2.6. Subspaces formalism: FRIO discrimination of two Rank 2 mixed states

In this and the next section we consider the FRIO discrimination of two mixed states that exhibit a Jordan structure. The two states, with their respective *prior* probabilities η_1 and η_2, are given in spectral (diagonal) form as,

$$\begin{aligned} \rho_1 &= \sum_{i=1}^{N} r_i |r_i\rangle\langle r_i| \\ \rho_2 &= \sum_{i=1}^{N} s_i |s_i\rangle\langle s_i| \ . \end{aligned} \tag{2.6.1}$$

where $\langle r_i|r_j\rangle = \delta_{ij}$ and $\langle s_i|s_j\rangle = \delta_{ij}$. Each density matrix has rank N and together they span a $2N$ dimensional Hilbert space. We assume that the Jordan condition,

$$\langle r_i|s_j\rangle = \delta_{ij}\cos\theta_i \ , \tag{2.6.2}$$

holds so the set $\{|r_i\rangle, |s_i\rangle | i = 1, \ldots, N\rangle\}$ forms a Jordan basis for the $2N$ dimensional Hilbert space.

Due to the Jordan structure, instead of one $2N$ dimensional problem we have N mutually orthogonal 2-dimensional subspaces. The i^{th} subspace is spanned by $|r_i\rangle$ and $|s_i\rangle$ with *prior* probabilities $\eta_1 r_i$ and $\eta_2 s_i$, respectively. The full discrimination of ρ_1 and ρ_2 can be accomplished by performing the discrimination in each subspace independently of the others, i.e., by discriminating $|r_i\rangle$ and $|s_i\rangle$ in each subspace which is much simpler than the original problem. A physical implementation of this scheme is provided, e.g., by the transmission of two input states over several optical fibers. Each fiber contains two degrees of freedom, that could be two orthogonal polarizations.

In the present section we will focus on the $N = 2$ case when we have to deal with two 2-dimensional subspaces only. We leave the general case of arbitrary N to the next section. discussed above, in this case we are dealing with two orthogonal 2-dimensional subspaces. The i^{th} subspace is spanned by $\{|r_i\rangle, |s_i\rangle\}$ for $i = 1, 2$ and our task is to perform the optimal FRIO discrimination of $\{|r_i\rangle$ and $|s_i\rangle\}$ within subspace i. The prior probability of $|r_i\rangle$ is $\eta_1 r_i$ while the prior probability of $|s_i\rangle$ is $\eta_2 s_i$. We define their normalized probabilities as

$$\begin{aligned} \eta_{1,i} &\equiv \frac{\eta_1 r_i}{\eta_1 r_i + \eta_2 s_i}, \\ \eta_{2,i} &\equiv \frac{\eta_2 s_i}{\eta_1 r_i + \eta_2 s_i} \ , \end{aligned} \tag{2.6.3}$$

such that $\eta_{1,i} + \eta_{2,i} = 1$.

We now introduce a fixed rate of inconclusive outcomes for subspace i, Q_i, such that $0 \leq Q_i \leq Q_{c,i}$ where $Q_{c,i}$ is given by Eq. (2.5.1) with the obvious substitutions $\eta_1 \to \eta_{1,i}$, $\eta_2 \to \eta_{2,i}$ and $\cos\theta \to \cos\theta_i = \langle r_i|s_i\rangle$. We note that this

reduces the problem to the FRIO discrimination of two pure states in subspace i, with prior probabilities given above. It follows immediately that the solution is given by (2.5.10) in the POVM regime of Q_i, $Q_i \leq Q_{c,i}, Q_{th,i}$, and (2.5.12) in the projective regime of Q_i, $Q_{th,i} < Q_i \leq Q_{c,i}$, again with the above substitutions.

In the following we will focus mainly on the POVM regime where the solution in each subspace is given explicitly by Eq. (2.5.10),

$$P_{e,i} = \frac{1}{2}(1 - Q_i - \sqrt{(1-Q_i)^2 - (Q_{0,i} - Q_i)^2}) \ , \tag{2.6.4}$$

where $Q_{0,i} = 2\sqrt{\eta_{1,i}\eta_{2,i}}\langle r_i|s_i\rangle$ $(i = 1, 2)$. Q_0 was introduced in Eq. (2.5.1) for the single subspace (two pure states) case, $Q_{0,i}$ is its generalization for the case of two (or many) subspaces. We introduce the weight ω_i of subspace i as

$$\eta_1 r_i + \eta_2 s_i = \omega_i \ , \tag{2.6.5}$$

where, obviously,

$$\omega_1 + \omega_2 = 1 \ . \tag{2.6.6}$$

Then the total error rate can be written as a weighted sum of the error rates of the individual subspaces

$$P_e = \omega_1 P_{e,1} + \omega_2 P_{e,2} \ . \tag{2.6.7}$$

Similarly, the total inconclusive rate is a weighted sum of the inconclusive rates of the individual subspaces,

$$Q = \omega_1 Q_1 + \omega_2 Q_2 \ . \tag{2.6.8}$$

We notice now that the total inconclusive rate Q is fixed, with the fixed value satisfying $0 \leq Q \leq Q_0$, where

$$Q_0 \equiv \omega_1 Q_{0,1} + \omega_2 Q_{0,2} \ , \tag{2.6.9}$$

is the maximal inconclusive rate that the two subspaces can accommodate.

The task is to determine the optimal distribution of Q between the two subspaces, under the constraint that Q in Eq. (2.6.8) is fixed. (2.6.8) tells us that only one of the Q_is, say Q_1, is an independent variable, since $Q_2 = (Q - \omega_1 Q_1)/\omega_2$. Inserting this into Eq. (2.6.7) it becomes a function of the independent variable Q_1. It can then be optimized with respect to this variable, yielding an optimal distribution of the fixed inconclusive outcomes between the two subspaces. Without loss of generality in what follows we assume the hierarchy

$$Q_{0,1} \geq Q_{0,2} \ . \tag{2.6.10}$$

The optimization corresponds to finding the solution to $\partial P_e / \partial Q_1 = 0$. If we introduce the notation

$$Q_{th}^{(1)} = 0 \quad Q_{th}^{(2)} \equiv \frac{\omega_1 (Q_{0,1} - Q_{0,2})}{1 - Q_{0,2}} \ , \tag{2.6.11}$$

the result of this straightforward optimization can be written as

$$Q_1^{opt} = \begin{cases} \frac{Q}{\omega_1} & \text{if } Q_{th}^{(1)} \leq Q \leq Q_{th}^{(2)} \ , \\ \frac{1-Q_{0,1}}{1-Q_0}(Q - Q_{th}^{(2)}) + \frac{Q_{th}^{(2)}}{\omega_1} & \text{if } Q_{th}^{(2)} < Q \leq Q_0 \ , \end{cases} \tag{2.6.12}$$

$$Q_2^{opt} = \begin{cases} 0 & \text{if } 0 \leq Q \leq Q_{th}^{(2)} \ , \\ \frac{1-Q_{0,2}}{1-Q_0}(Q - Q_{th}^{(2)}) & \text{if } Q_{th}^{(2)} < Q \leq Q_0 \ . \end{cases} \tag{2.6.13}$$

solutions but there is no range where the naive expectation, $Q_1 = Q_2 = Q$, would hold. The subspace with the larger $Q_{0,i}$ which is the first subspace in our specific case, always accommodates a higher rate of inconclusive outcomes than the subspace with the lower $Q_{0,i}$.

The most interesting aspect of the optimal solution is that a threshold structure emerges. For $Q < Q_{th}^{(2)}$ the second subspace does not accommodate any inconclusive outcome at all and it operates at the Minimum Error level. In this region of Q all the inconclusive outcome is accommodated by the first subspace only. Above this threshold the two subspaces share the available inconclusive rate. If

$\omega_1 Q_{0,1} > \omega_2 Q_{0,2}$, the first subspace always accommodates more inconclusive rate than the second, otherwise above some value of the total Q the second subspace will accommodate more inconclusive rate than the first. The situation is depicted in Fig. **??**, for some specific values of the parameters.

Finally, inserting (**??**) and (2.6.13) into Eq. (2.6.4) and using the resulting expressions in (2.6.7) yields the optimal error probability, P_E, with a fixed rate of inconclusive outcomes, $begin\{widetext\}$

$$P_E = \begin{cases} \frac{1}{2}\left\{1 - Q - \sqrt{(\omega_1 - Q)^2 - (\omega_1 Q_{0,1} - Q)^2} - \sqrt{(\omega_2)^2 - (\omega_2 Q_{0,2})^2}\right\} & \text{if } Q_{th}^{(1)} \leq Q \leq Q_{th}^{(2)} , \\ \frac{1}{2}\left\{1 - Q - \sqrt{(1-Q)^2 - (Q_0 - Q)^2}\right\} & \text{if } Q_{th}^{(2)} < Q \leq Q_0 , \end{cases} \tag{2.6.14}$$

which is the main result of this section. It is valid in all regions of the parameter Q. In particular, for the maximum allowable inconclusive rate, $Q = \omega_1 Q_{0,1} + \omega_2 Q_{2,0}$, Eq. (2.6.14) reduces to $P_E = 0$, corresponding to optimal Unambiguous Discrimination of two subspaces [**?, ?**], while for $Q = 0$ it reduces to the Minimum Error expression for two subspaces [**?**], as it should. We close this section by displaying $P_{e,i}$ vs. Q_i in Fig. 2.6.1, for some specific values of the parameters.

FIGURE 2.6.1. $Q_{\rm c}$ (solid line) and $Q_{\rm b}$ (dashed line) vs. η_1 for $\theta = \pi/3$. The area of interest lies under the solid line. The regions I, II and III are defined in Eq. (2.5.1).

2.7. The general case: FRIO discrimination of two Rank N mixed states, POVM regime

In this section we study the general case of two Rank N density matrices, with N being an arbitrary but fixed integer. The two density matrices together span a $2N$ dimensional Hilbert space and their eigenvectors form a Jordan basis for the space. They are given in Eq. (2.6.1). definitions and properties presented in Eqs. (2.6.2)–(2.6.5) remain in effect but the part starting with Eq. (2.6.6) has to be

modified accordingly. weights of the subspaces, introduced in (2.6.5) now satisfy

$$\sum_{i=1}^{N} \omega_i = 1 \ . \tag{2.7.1}$$

Then the total error rate can be written as a weighted sum of the error rates of the individual subspaces

$$P_e = \sum_{i=1}^{N} \omega_i P_{e,i} \ . \tag{2.7.2}$$

The total inconclusive rate also remains a weighted sum of the inconclusive rates of the individual subspaces,

$$Q = \sum_{i=1}^{N} \omega_i Q_i \ . \tag{2.7.3}$$

We notice now that the total inconclusive rate Q is again fixed, with the fixed value satisfying

$$0 \leq Q \leq Q_0 \ , \tag{2.7.4}$$

where now

$$Q_0 \equiv \sum_{i=1}^{N} \omega_i Q_{0,i} \ , \tag{2.7.5}$$

is the maximal inconclusive rate that the N subspaces can accommodate.

remaining task is to determine the optimal distribution of Q between the N subspaces, under the constraint of Eq. (2.7.4), i.e., to determine the optimal values of Q_i as a function of the fixed Q.

In order to perform the optimization we employ the Lagrange multiplier method because it leads to more symmetric and easily tractable equations. To minimize the overall error rate $P_E = \sum_{Pi=1}^{N} \frac{\omega_i}{2}(1 - Q_i - \sqrt{(1-Q_i)^2 - (Q_{0,i} - Q_i)^2})$ subject to the constraint in 2.6.8 we form the function with one constraint λ.

$$F = \sum_{i=1}^{N} \frac{\omega_i}{2}(1 - Q_i - \sqrt{(1-Q_i)^2 - (Q_{0,i} - Q_i)^2}) + \lambda(Q - \sum_{i=1}^{N} \omega_i Q_i) \ , \tag{2.7.6}$$

We vary the function F treating Q_i as the independent variables. Solving the resulting equations together with the constraint $Q = \sum_{i=1}^{N} \omega_i Q_i$ determines the value of the Lagrange multiplier which minimizes the error rate $P_e = \sum_{i=1}^{N} \omega_i P_{e,i}$.

Before we present the results it will prove useful to introduce a hierarchy of the subspaces. So, in what follows we will assume

$$Q_{0,1} \geq Q_{0,2} \geq \ldots \geq Q_{0,N} \ , \tag{2.7.7}$$

which generalizes the ordering used in the case of two subspaces in the previous section.

Then the optimal Q_i can be written as

$$Q_i^{opt} = \begin{cases} \frac{1-Q_{0,i}}{\sum_{i=1}^{k}\omega_i - \sum_{i=1}^{k}\omega_i Q_{0,i}} Q + \frac{Q_{0,i}\sum_{i=1}^{k}\omega_i - \sum_{i=1}^{k}\omega_i Q_{0,i}}{\sum_{i=1}^{k}\omega_i - \sum_{i=1}^{k}\omega_i Q_{0,i}} & \text{if } Q_{th}^{(k)} \leq Q \leq Q_{th}^{(k+1)} \text{ and } i \leq k \\ 0 & \text{if } i > k \ . \end{cases} \tag{2.7.8}$$

Here $k = 1, 2, \ldots, N$ and we introduced the notation

$$Q_{th}^{(k)} = \frac{\sum_{i=1}^{k}\omega_i Q_{0,i} - Q_{0,k}\sum_{i=1}^{k}\omega_i}{1 - Q_{0,k}} \ . \tag{2.7.9}$$

and also $Q_{th}^{(N+1)} = Q_0 = Q_{max}$, cf. (2.7.5). Obviously, for $k = 1, 2$ these results reproduce the results for two subspaces, Eqs. (2.6.11)–(2.6.13).

Inserting the optimal failure rates into the subspace-error rates $P_{e,i}$, (2.6.4), gives the optimal error rates for the subspaces, $P_{e,i}^{opt}$. Then using these optimal subspace-error rates in (2.7.2) the total optimal error rate $P_E = \sum_{i=1}^{N} P_{e,i}^{opt}$,

$$P_E = \begin{cases} \frac{1}{2}\left[1 - Q - \sqrt{\left(\sum_{i=1}^{k}\omega_i - Q\right)^2 - \left(\sum_{i=1}^{k}\omega_i Q_{0,i} - Q\right)^2} - \sum_{i=k+1}^{N}\sqrt{(\omega_i)^2 - (\omega_i Q_{0,i})^2}\right] & \text{if } Q_{th}^{(k)} \leq Q \leq \\ & \text{and } k < N \ , \\ \frac{1}{2}\left[1 - Q - \sqrt{(1-Q)^2 - (Q_0 - Q)^2}\right] & \text{if } Q_{th}^{(N)} < Q \leq \end{cases}$$

which is valid in all regions of the parameter Q. For $N = 2$, (**??**) reduces to the two-subspaces solution, (2.6.14). For the maximum allowable inconclusive rate,

$Q = Q_0$, the second line in Eq. (**??**) holds and it reduces to $P_E = 0$, corresponding to optimal Unambiguous Discrimination of the subspaces [**?, ?**], while for $Q = 0$ the first line holds and it reduces to the Minimum Error expression for N subspaces [**?**], as expected.

Again, the most interesting aspect of the optimal solution is that a structure with multiple thresholds emerges. For $Q_{th}^{(1)} = 0 \leq Q < Q_{th}^{(2)}$ the total available inconclusive rate is accommodated by the first subspace only and all others operate at the Minimum Error level. According to the hierarchy introduced in Eq. (2.7.7), the first subspace is the one with the largest $Q_{0,i}$. Then between $Q_{th}^{(2)} \leq Q < Q_{th}^{(3)}$ the second subspace, the one with the second largest $Q_{0,,i}$ will also participate in sharing the available inconclusive rate, while the remaining $N-2$ subspaces continue to operate at the minimum error level. In general, in the interval $Q_{th}^{(k)} \leq Q < Q_{th}^{(k+1)}$ the first k subspaces share the available inconclusive rate and the remaining $N-k$ subspaces remain at the minimum error level. Finally, in the range $Q_{th}^{(N)} \leq Q \leq Q_0 = Q_{max}$ all N subspaces participate in sharing the available inconclusive rate.

It is easy to show that the expressions are continuous at the threshold, i.e. the expressions valid below the threshold and the ones valid above the threshold tend to the same values at the threshold, although their slopes are, in general, different below and above the threshold. Furthermore, if $\omega_i Q_{0,i} > \omega_{i+1} Q_{0,i+1}$, the i^{th} subspace always accommodates more inconclusive rate than the $i+1^{st}$ for all $i = 1, 2, \ldots, N$, otherwise above some value of the total Q the $i+1^{st}$ subspace will accommodate more inconclusive rate than the i^{th} subspace, the $Q_i^{opt}(Q)$ curves will intersect.

We illustrate these results on the example of $N = 3$. In Fig. **??**, we plot the optimal failure rates Q_i^{opt} and in Fig. **??**, we plot the optimal error rates $P_{e,i}^{\text{opt}}$ for the three subspaces as a function of the total failure rate Q for some specific values of the parameters.

The results presented so far, in Sections III and IV, are valid if the parameters are such that in all subspaces we are in the POVM regime, i.e., Q_i is in the unshaded region (region II) of Fig. 1 for all i. When the parameters are such that in some subspaces we are in the projective regime, i.e., Q_i falls in the shaded regions of Fig. 1 (regions I and III) for some i we have to modify the treatment to account for the fact that the error expression for the corresponding subspace, $P_{e,i}$, is no longer given by Eq. (2.6.4) but by (2.5.12). We will study this case in the next section.

2.8. Projective regime

We have seen in Section II for the single subspace case that the POVM solution is valid if the inconclusive rate is smaller than a boundary value, $Q \leq Q_b$, where Q_b is given by Eq. (2.5.11). With an obvious generalization, the POVM solution holds in subspace i if in that subspace $Q_i \leq Q_{b,i}$ holds where the subspace boundary value is given by

$$Q_{\rm b,i} \equiv 2\eta_{1,i}\eta_{2,i}\sin^2\theta_i/(1-Q_{0,i}). \tag{2.8.1}$$

In Fig. **??** we plot $Q_{\rm c,i}$ and $Q_{\rm b,i}$ vs. η_1 together for a fixed overlap, $\cos\theta = 0.5$ ($\theta = \pi/3$).

$$Q_i \textbf{ vs. } \eta_{1,i}$$

The two curves intersect at $\eta_{1,i} = \eta_{1,i}^{(l)}$ and $\eta_{1,i} = \eta_{1,i}^{(r)}$, the same points as in Eq. (2.5.1). The interval $0 \leq \eta_{1,i} \leq 1$ is thus divided into three regions. In regions I and III, we have $Q_{\rm b,i} < Q_{\rm c,i}$ and the solution (2.5.10) is valid for $0 \leq Q_i < Q_{\rm b}$ only. In Region II, $\eta_{1,i}^{(l)} \leq \eta_{1,i} \leq \eta_{1,i}^{(r)}$, we have $Q_{\rm c,i} = Q_{0,i} < Q_{\rm b,i}$ and the solution (2.5.10) is valid for the entire $0 \leq Q_i \leq Q_{\rm c,i}$ range.

Thus, $Q_i > Q_{b,i}$ occurs in regions I and III and in the shaded areas the optimal FRIO measurement is a standard projective measurement while in the unshaded area it is a POVM. We now illustrate the case when the FRIO measurement is a POVM in some and a Projection in other subspaces on a numerical example. We

consider three subspaces with $\eta_1 = \eta_2$ and the other parameters are listed in Table 1.

TABLE 1. Alice's measurement outcomes and Bob's subsequent operations.

Subspace i	r_i	s_i	θ_i	ω_i	$Q_{0,i}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{\pi}{4}$	$\frac{1}{4}$	$Q_{0,1} = \sqrt{2}/8 \approx .17$
2	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{\pi}{4}$	$\frac{1}{4}$	$Q_2^< = 7/32 \approx .21$
3	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$Q_3^> = 29/64 \approx .45$

Subspace 1 has its maximum failure rate as $Q_{0,1} = \sqrt{2}/8 \approx .17$. For subspace 2, the maximum failure rate isn't $Q_{0,2} = \sqrt{6}/16 \approx .15$ because it fails one of the SSD conditions and instead $Q_2^< = 7/32 \approx .21$, and for subspace 3 the maximum failure rate isn't $Q_{0,3} = 3\sqrt{5}/16 \approx .42$, because it fails the other SSD condition and instead $Q_3^> = 29/64 \approx .45$. The failure rate maximum $Q^{MAX} = Q_{0,1} + Q_2^< + Q_3^> \approx .87$ while $Q_0 \approx .75$ and $\sum \omega_i = 1$ as it should.

2.8.1. First elimination. To find which subspace to discard first we find the critical Q's: $Q_c^1 \approx .14$, $Q_c^2 \approx .35$, and $Q_c^3 < 0$, so subspace 2 is discarded first when $Q \approx .35$. This means that $Q_2 = 0$ when $Q = Q_c^2$ and we do not allow the value of Q_2 to vary afterward. At Q_c^2 we find the values of the other failure rates to be $Q_1 \approx .06$ and $Q_3 \approx .29$

2.8.2. Second elimination. It may be clear that Q_1 will reach 0 first and indeed this is so. Before we find the second set of critical values we find our new constants as: $\Lambda_2 = \sum^{1,3} \omega_i = 3/4$; $F_2 = \sum^{1,3} Q_{0,i} \approx .6$. Now the critical values read $Q_c^{(1)1} \approx .22$ and $Q_c^{(1)3} < 0$ so when $Q = Q_c^{(1)1}$ we discard subspace 1 and reduce the optimization problem to the single subspace case, where $Q^{(2)3} = Q$. This process is depicted in Fig. **??**.

DO WE NEED THESE GRAPHS? THEY DON'T COME OUT ON THE RIGHT PAGE!!

We include a comparison of the error and failure rates for a different set of initial conditions for two subspaces in Fig. **??**.

2.9. summary and conclusion

We have found analytic solutions for the optimal measurements for a fixed failure rate for a class of mixed states that can be expressed using a Jordan Basis structure. This solution interpolates between the minimum error and unambiguous discrimination bounds. We found several surprising and unexpected conclusions. The first is that the form of the error rate remains the same over all subspaces. This implies a deep relationship between the measurement strategies of pure and mixed states. The second is the threshold behavior in the optimization that shuts off successive subspaces as the total failure rate decreases. This is a new behavior and allows for interesting experimental opportunities. The most straightforward is an optical experiment where a single photon is split between different fiber optic cables and a separate set of measurements is performed on each cable. Applications could be considered in cryptography where a key is shared over different lines to enhance security without sacrificing overall error rate.

FIGURE 2.8.1. Parametrized curve of (a)The error rates for subspace 1 vs subspace 2 as a function of Q (b) Q_1 vs Q_2 as a function of Q, truncated at $Q_2 = 0$, both for $\eta_1 = 3/4$, $r_i = s_i = 1/2$, $\cos\theta_1 = 1/2$, $\cos\theta_2 = \sqrt{3}/2$ and $0 \leq Q \leq 1$

CHAPTER 3

Quantum Cloning

3.1. No-Cloning Theorem

Some of the schemes described above to discriminate quantum states would not be necessary if one could make copies of the non-orthogonal quantum states as can be done with classical states. If this were possible then the receiver, Bob, after receiving the state $|\psi_1\rangle$ or $|\psi_2\rangle$ from the preparer, Alice, makes n number of copies. After a large set of copies the states become nearly orthogonal and almost fully distinguishable. The average inconclusive rate of failing to distinguish the n copies of $|\psi_1\rangle$ or $|\psi_2\rangle$ is $Q_o = 2\sqrt{\eta_1\eta_2}s^n$. For a large n the inconclusive rate is very small and Bob can discriminate nearly all incoming states.

Thus while in classical information it is possible to make exact copies of information, as this dissertation is printed on this paper, multiple times by a printer. The no-cloning theorem restricts the receiver for doing the same with quantum states. More specifically it is non-orthogonal quantum states which cannot be copied, as classical states are a special case of quantum states, that of orthogonal states.

Let us now show a proof by contradiction of why such a quantum cloning machine cannot exist. Suppose there is such a cloning machines with an input and an output port. Inside the machine there are two slots: S slot for the system state,$|\psi_i\rangle$, to be copied, and an A slot for the ancilla space where the input state is to be copied. Let the ancilla space be in some blank space $|0\rangle$, then the initial state of the copying machine would be: $|\psi_i\rangle|0\rangle$. A unitary operator would copy the state $|\psi_i\rangle$ into $|0\rangle$:

$$U|\psi_i\rangle|0\rangle = |\psi_i\rangle|\psi_i\rangle \tag{3.1.1}$$

Let there be two possible input states to be copied, $\{|\psi_1\rangle, |\psi_2\rangle\}$ and we are interested in a quantum cloning machine which produces $|\psi_1\rangle|\psi_1\rangle$ when $|\psi_1\rangle$ is sent and $|\psi_2\rangle|\psi_2\rangle$ when $|\psi_2\rangle$ is sent. The unitary operator would do the following.

$$U|\psi_1\rangle|0\rangle = |\psi_1\rangle|\psi_1\rangle \tag{3.1.2}$$

$$U|\psi_2\rangle|0\rangle = |\psi_2\rangle|\psi_2\rangle \tag{3.1.3}$$

The inner product of these equations gives $\langle\psi_2|\psi_1\rangle\langle 0|0\rangle = |\langle\psi_2|\psi_1\rangle|^2 \Rightarrow s = s^2$. This condition can be satisfied only if $s = 0$, states are orthogonal, or $s = 1$, the two states are the same. But we said that the two states are distinct and non-orthogonal. Thus one cannot design a unitary device which makes perfect clones of an unknown quantum system deterministically. Here we only proved that non-orthogonal pure states cannot be copied through a unitary process. Other proofs exist which show that this holds for mixed states and also for other non-unitary processes.

Since such a machine cannot be designed the next logical step is how to build a quantum machine which produces clones similar to the input states while allowing for some fidelity or inconclusive results. We derive some previous results considering such quantum cloning machines and also provide new results for optimal exact cloning with some inconclusive rate allowed. In a follow up paper we interpolate between the deterministic cloning scheme and exact cloning by relaxing some of the conditions.

3.2. Exact cloning with failure rate

Here we are concerned with making exact copies of the given quantum state. It is indeed possible to make exact copies but only probabilistically. This is very similar to the Unambiguous Discrimination where one discriminates between a given set of non orthogonal quantum states unambiguously.

We begin by setting up Neumark scheme where the Unitary operator acts on the input pure states $\{|\psi_1\rangle, |\psi_2\rangle\}$ to produce perfect clones or failure states, which are then discarded.

$$U|\psi_1\rangle|i\rangle = \sqrt{p_1}|\Psi_1\rangle|\alpha_1\rangle + \sqrt{q_1}|\Phi_1\rangle, \tag{3.2.1}$$

$$U|\psi_2\rangle|i\rangle = \sqrt{p_2}|\Psi_2\rangle|\alpha_2\rangle + \sqrt{q_2}|\Phi_2\rangle, \tag{3.2.2}$$

Where p_i is the rate of having successfully produced a perfect clone, q_i is the rate of having failed to do so. The inner product of each of the above equation with its own transpose gives the normalized probabilities $p_i + q_i = 1$. $|\psi_i\rangle$ are the input states we wish to clone and their corresponding ancillas are $|\alpha_i\rangle$. Getting a click in $|\Phi_i\rangle$ means we have failed to clone and discard the state.

Taking the inner product of the transpose of equation (1) with (2) we get the constraint of success and failure rate in terms of the overlap of the input states.

$$s^M = \sqrt{p_1 p_2} s^N \alpha + \sqrt{q_1 q_2}\phi \tag{3.2.3}$$

where:

$$s = \langle\psi_1|\psi_2\rangle, \quad s' = \langle\Psi_1|\Psi_2\rangle, \quad \alpha = \langle\alpha_1|\alpha_2\rangle, \quad \phi = \langle\Phi_1|\Phi_2\rangle$$

In the above setup the ancilla corresponding to successfully making a clone is distinct. The failed states are also distinct. That is the most general case. In order for the success rates to be optimal we take the ancilla to be the same. Looking at the constraint we absorb α into s, meaning there is more copies which would come at the expense of a lower p_i . More explicitly this can be shown for the equal priors case shown below. We will now show the the failed states should also be the same. If $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are different that means that there is still information left in the failed states and we can still perform an Unambiguous State discrimination and

probabilistically determine whether we received $|\psi_1\rangle$ or $|\psi_2\rangle$. The optimal strategy is one which leaves no information at all in the failed states and the overlap $\langle\Phi_2|\Phi_1\rangle = 1$. Any click from the failed state is simply discarded as failure. The revised Neumark setup reduces to:

$$U|\phi_1\rangle|i\rangle = \sqrt{p_1}|\Psi_1\rangle|\alpha\rangle + \sqrt{q_1}|\Phi\rangle, \tag{3.2.4}$$

$$U|\phi_2\rangle|i\rangle = \sqrt{p_2}|\Psi_2\rangle|\alpha\rangle + \sqrt{q_2}|\Phi\rangle, \tag{3.2.5}$$

The constraint simplifies into:

$$s = \sqrt{p_1 p_2}s' + \sqrt{q_1 q_2} \tag{3.2.6}$$

3.2.1. Equal a-priori. The problem is trivial for equal a-priori probabilities $\eta_1 = \eta_2 = \frac{1}{2}$, which leads to $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Equation (1) reduces into: $s^M = ps^N + q = ps^N + (1-p) \Rightarrow p = \frac{1-s^M}{1-s^N}$,where $s \equiv \langle\Psi_1|\Psi_2\rangle$. In the limit $N \to inf$, $p = 1 - s^M$ which is the IDP limit in unambiguous discrimination.

3.2.2. General Case I. We now seek to solve the general problem for when the incoming states are prepared with different a priori probabilities. In this section we show a geometric/numeric solution using the Lagrange multipliers method with the constraint given in (3). We want to optimize the average rate of failing to produce a copy , $Q = \eta_1 q_1 + \eta_2 q_2$. Adding the constraint for the one to two cloning

$$s = \sqrt{(1-q_1)(1-q_2)}s^2 + \sqrt{q_1 q_2}] \tag{3.2.7}$$

the function to be maximized is:

$$F_S = \eta_1 q_1 + \eta_2 q_2 + \lambda[s - \sqrt{(1-q_1)(1-q_2)}s^2 - \sqrt{q_1 q_2}] \tag{3.2.8}$$

where λ is the Lagrange multiplier to be determined which optimizes Q.

Step 1: Set $\partial F/\partial q_i = 0$, solve for

$$\frac{2\eta_1}{\lambda} = \sqrt{\frac{q_2}{q_1}} - \sqrt{\frac{1-q_2}{1-q_1}}s^2$$

$$\frac{2\eta_2}{\lambda} = \sqrt{\frac{q_1}{q_2}} - \sqrt{\frac{1-q_1}{1-q_2}}s^2$$

$$\sqrt{\frac{q_1}{q_2}} = \frac{\lambda}{2\eta_1}[1 - \frac{s^2}{A}]$$

$$\sqrt{\frac{q_2}{q_1}} = \frac{\lambda}{2\eta_2}[1 - As^2]$$

where $A = \sqrt{\frac{q_2(1-q_1)}{q_1(1-q_2)}}$.

Multiplying the above two equations and setting $\delta = \frac{4\eta\eta_2}{\lambda^2}$ and $C = \frac{s^4-\delta+1}{s^2}$.

$$\delta - 1 = s^2[s^2 - A - \frac{1}{A}] \tag{3.2.9}$$

$$0 = A^2 - CA + 1 \tag{3.2.10}$$

$$\tag{3.2.11}$$

This quadratic equations is one of the equations we will use to obtain the value of λ.

Another quadratic equation emerges using the two equations in ( ). First let $\alpha \equiv \frac{\lambda}{2\eta_1}[1 - \frac{s^2}{A}]$ and $\frac{1}{\alpha} = \frac{\lambda}{2\eta_2}[1 - As^2]$ the relationship between the two failure rates becomes:

$$\sqrt{q_1} = \alpha\sqrt{q_2} \tag{3.2.12}$$

using this and the definition of A q_i can be expressed explicilety in terms of the fixed constants: $\{\eta_1, \eta_2, s\}$ and the parameter λ which is yet to be determined. The derivation starts from the definition at A, $A^2 = \frac{q_2(1-q_1)}{q_1(1-q_2)} \Rightarrow \frac{q_1}{1-q_1}A^2 = \frac{q_2}{1-q_2}$ replace $q_1 = \alpha^2 q_2$

$\frac{\alpha^2 q_2}{1-\alpha^2 q_2}A^2 = \frac{q_2}{1-q_2}$ solving for q_2, then using the relationship $q_1 = \alpha^2 q_2$

$$q_2 = \frac{1-\alpha^2 A^2}{\alpha^2(1-A^2)} \tag{3.2.13}$$

$$q_1 = \frac{1-\alpha^2 A^2}{(1-A^2)} \tag{3.2.14}$$

This is the expression of the individual failure rates which are yet to be optimized subject to the constraint. Now q_1 and q_2 is replaced in the constrained given in 3.2.7 and from there the optimal value of λ can be obtained.

Some prior calculations before replacing q_i into the constraint will simplify the overall algebra:

$1 - q_1 = \frac{A^2(\alpha^2-1)}{(1-A^2)}$

$1 - q_2 = \frac{\alpha^2-1}{\alpha^2(1-A^2)}$

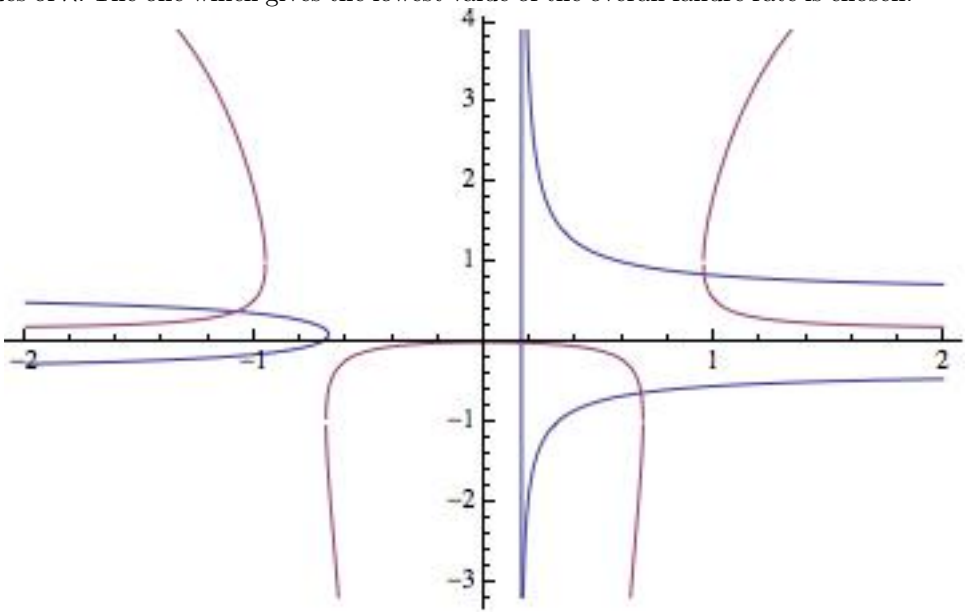
Now we are ready to replace q_i and $1 - q_i$ into the constraint:

$$\begin{aligned} s &= \frac{A(\alpha^2-1)}{\alpha(1-A^2)}s^2 + \frac{1-\alpha^2A^2}{\alpha(1-A^2)} \\ (1-A^2)s &= \frac{1}{\alpha}(1-s^2A^2) - \alpha A^2(1-\frac{s^2}{A}) \\ (1-A^2)s &= \frac{\lambda}{2\eta_2}(1-s^2A^2)^2 - \frac{\lambda A^2}{2\eta_1}(1-\frac{s^2}{A})^2 \end{aligned}$$

Where we replaced $\alpha = \frac{\lambda}{2\eta_1}[1-\frac{s^2}{A}]$ and $\frac{1}{\alpha} = \frac{\lambda}{2\eta_2}[1-As^2]$. After some trivial algebra another quadratic equation and A emerges:

$$A^2 - \frac{2s^2(\eta_1-\eta_2)}{\eta_1 s^4 - \eta_2 + 2\eta_1\eta_2 s/\lambda}A + \frac{\eta_1 - \eta_2 s^4 + 2\eta_1\eta_2 s/\lambda}{\eta_1 s^4 - \eta_2 + 2\eta_1\eta_2 s/\lambda} = 0 \tag{3.2.15}$$

The combination of 3.2.10 and 3.2.15 should give the value of λ which in turn gives explicit solution to the minimum individual failure rates, q_i, and overall optimal failure rate $Q = \eta_1 q_1 + \eta_2 q_2$. Analytically such a solution is hard to achieve as one would have to solve a sixth order quadratic equation. Simply by plotting the two quadratic equations as a function of λ the solution can be obtained at the intersection of the two graphs. There will be multiple intersections, hence multiple values of λ. The one which gives the lowest value of the overall failure rate is chosen.



.

In our future work we would like to obtain a closed form solution of q_i. One approach would be by making some educated guesses for λ.

(My guess is that the general UD should be similar to the equal priors and should look like $P = \frac{1-2\sqrt{\eta_1\eta_2}s}{1-4\eta_1\eta_2 s^2} = \frac{1-Q_o}{1-Q_o^2}$)

Bergou's guesses:

$q_1 = s(\sqrt{\eta_2/\eta_1} - s)/(1 - s^2)$,

$q_2 = s(\sqrt{\eta_2/\eta_1} - s)/(1 - s^2)$

$\lambda = \sqrt{\eta_1\eta_2}/(1 - s^2)$

$Q = \frac{s}{1+s}\frac{2\sqrt{\eta_1\eta_2}-s}{1-s}$

$Q = \frac{s}{1+s}\frac{2\sqrt{eta_1\eta_2}-s}{1-s}$

$Q = \eta_1 q_1 + \eta_2 q_2 = \frac{s(2\sqrt{\eta_1\eta_2}-s)}{1-s^2}$

3.2.3. General case II. (will probably not use this version)

We try a different approach here derived by Emili Bagan.

Let us start by parametrizing the constraint: $s^M = \sqrt{p_1 p_2}s^N + \sqrt{q_1 q_2}$

Let $\sqrt{p_i} = \cos\theta_i$, $\sqrt{q_i} = \sin\theta_i$ for $0 \leq \theta_i \leq \pi/2$ the constraint becomes:

$$s^M = \cos\theta_1\cos\theta_2 s^N + \sin\theta_1\sin\theta_2 \tag{3.2.16}$$

$$s^M = \frac{1}{2}[\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)]s^N - \frac{1}{2}[\cos(\theta_1+\theta_2)-\cos(\theta_1-\theta_2)]$$

Let $x \equiv \cos(\theta_1+\theta_2), y \equiv \cos(\theta_1-\theta_2)]$

$$s^M = \frac{1}{2}[(x+y)s^N - (x-y)] \tag{3.2.17}$$

$$s^M = \frac{1}{2}[-(1-s^N)x + (1+s^N)y] \tag{3.2.18}$$

$-1 \leq x \leq 1, -1 \leq y \leq 1$

As this is the equation of a straight line we can write this in parametric form.

$x = \frac{1\text{-}(1+s^N)t}{s^{N-M}}, y = \frac{1\text{-}(1-s^N)t}{s^{N-M}}$ where $\frac{1-s^{N-M}}{1-s^N} \leq t \leq \frac{1}{1-s^N}$ and $\frac{1-s^{N-M}}{1-s^N} \leq t \leq \frac{1+s^{N+M}}{1-s^N}$

thus $\frac{1-s^{N-M}}{1-s^N} \leq t \leq min\{\frac{1}{1-s^N}, \frac{1+s^{N+M}}{1-s^N}\}$

We can now express the success and failure probabilities in terms of the new parametrization

$\sqrt{p_1} = \cos\frac{1}{2}\arccos\frac{1\text{-}(1+s^N)t}{s^{N-M}} + \frac{1}{2}\arccos\frac{1\text{-}(1-s^N)t}{s^{N-M}}$

$\sqrt{p_2} = \cos\frac{1}{2}\arccos\frac{1\text{-}(1+s^N)t}{s^{N-M}} - \frac{1}{2}\arccos\frac{1\text{-}(1-s^N)t}{s^{N-M}}$

$\sqrt{q_1} = \cos\frac{1}{2}\arcsin\frac{1\text{-}(1+s^N)t}{s^{N-M}} + \frac{1}{2}\arcsin\frac{1\text{-}(1-s^N)t}{s^{N-M}}$

$\sqrt{q_2} = \cos\frac{1}{2}\arcsin\frac{1\text{-}(1+s^N)t}{s^{N-M}} - \frac{1}{2}\arcsin\frac{1\text{-}(1-s^N)t}{s^{N-M}}$

The range of t changes into $\frac{1-s^{N-M}}{1-s^N} \leq t \leq 1$ as the values of p_i and q_i need to be positive. For example $t = 1$

$\sqrt{p_1} = \cos\frac{1}{2}\arccos(-s^M) + \frac{1}{2}\arccos(s^M) = 1$ for any $s^M \geq 0$ and

$\sqrt{q_1} = \cos\frac{1}{2}\arcsin(-s^M) + \frac{1}{2}\arcsin(s^M)$

As we are looking to minimize the average failure rate

$$Q = \eta_1 q_1 + \eta_2 q_2 \tag{3.2.19}$$

the minimum of Q is equivalent to $\eta_1 q_1' + \eta_2 q_2' = 0$ and we can express the priors in the new parametrization form $\eta_1 = \frac{q_2'}{q_2' - q_1'}$ and $\eta_2 = \frac{q_1'}{q_1' - q_2'}$. Since the problem is symmetric in $\eta_1 \Leftrightarrow \eta_2$ we can analyze the problem for $\eta_1 \leq 1/2$. For the extreme point of $\eta_1 = 0$ we have $q_2' = 0 \Rightarrow \frac{d}{dt}\sqrt{q_2} = 0$ and we can obtain the maximum value of

$\frac{d}{dt}\sqrt{q_2} = \frac{\sqrt{p_2}}{2s^{N-M}}(\frac{1+s^N}{\sqrt{1-x^2}} - \frac{1-s^N}{\sqrt{1-y^2}}) = 0 \Rightarrow t_{max} = \frac{1-s^{2(N-M)}}{1-s^{2N}}$ thus $Q_{min} = \frac{s^{2M}-s^{2N}}{1-s^{2N}}$ and $Q_{max} = \frac{s^M - s^N}{1-s^N}$ which is true for $\eta_1 = \eta_2$

3.3. Exact Cloning then Unambiguous Discrimination.

It is interesting to see the connection between quantum cloning and UD. We consider the two step process where we first clone then make a measurement. The other way, UD first then cloning is less interesting because once we have succeed in discriminating a state we can make as many copies as we wish by simply preparing them as we have full knowledge of the states. On the other hand making a clone then performing UD is quite interesting. For the case when the incoming states are prepared with equal a priors we show that cloning first then performing UD on the cloned states which come with some new a priori probability p' reaches the IDP limit.

Given an ensemble of M quantum states $\{|\Psi_1\rangle, |\Psi_2\rangle\}$ we first make N exact clones and then perform an unambiguous discrimination on the successfully cloned states.

Equal priors.

Step one: Exact Cloning. We showed that for equal priors and a a set of M copies of two non-orthogonal states we could successfully make exact copies with a probability $P_{clone} = \frac{1-s^M}{1-s^N}$. The average failure rate, failing to clone a state is $Q_{clone} = 1 - P_{clone} = \frac{s^M - s^N}{1-s^N}$.

Step two: Unambiguous Discrimination. After the input states are sent through the exact cloning machines N states of $\{|\Psi_1\rangle, |\Psi_2\rangle\}$ come out through the output port. The cloning machine only makes copies and does not say which state it has made a copy of. The clones are now sent through an unambiguous discriminating machine to distinguish the incoming states, each comes with a probability $P_{clone} = \frac{1-s^M}{1-s^N}$. The standard Neumark theorem is used to discriminate the incoming clones:

$$\begin{aligned} U|\Psi_1\rangle^{\oplus N} &= \sqrt{p}|\Psi_1\rangle|1\rangle + \sqrt{q}|\Phi\rangle|0\rangle \\ U|\Psi_2\rangle^{\oplus N} &= \sqrt{p}|\Psi_2\rangle|2\rangle + \sqrt{q}|\Phi\rangle|0\rangle \end{aligned}$$

The inner product gives $s^N = q$. We have to take into consideration the fact that the states come in with a priori probability of $\frac{1}{2}p = \frac{1}{2}\frac{1-s^M}{1-s^N}$, the average inconclusive rate for discriminating the incoming pair of states is

$Q = \tilde{\eta}_1 q_1 + \tilde{\eta}_2 q_2 = \eta_1 p_1 q_1 + \eta_2 p_2 q_2 = \frac{1-s^M}{1-s^N} s^N$,where $\eta_1 = \eta_2 = \frac{1}{2}, q_1 = q_2 = s^N$ and success rate is $p_1 = p_2 = \frac{s^M - s^N}{1-s^N}$.

The total inconclusive rate for the two step process: cloning failure rate plus unambiguous discrimination failure rate:

$$\begin{aligned} Q_{total} &= Q_{clone} + Q_{UD} \\ &= \frac{s^M - s^N}{1-s^N} + \frac{1-s^M}{1-s^N} s^N = s^M \end{aligned} \tag{3.3.1}$$

After this two step process the IDP limit is reached for UD of M non-orthogonal states . In equation 3.3.1 .

3.4. Deterministic State Dependent Quantum Cloning

In this section we re-derive the works of Chefles and Barnett in designing an approximate quantum cloning machine for two possible input states while maximizing the global fidelity.

Consider a set of K non orthogonal quantum states with M copies each $|\psi_i\rangle^{\otimes M} = |\psi_1\rangle|\psi_2\rangle...|\psi_M\rangle$. The states are unknown and our task is to produce $N > M$ copies, as best as we can. Introducing an ancilla state $|\chi\rangle$ which is $N - M$ dimensional the goal is to transform the state $|\psi_j\rangle|\chi\rangle$ into the state which approximates the N exact copies of the input state $|\psi_j^N\rangle$.

This is deterministic cloning, although imperfect, clones are generated on demand. The authors choose the global fidelity rate to improve the quality of the clones so they resemble the given copies as best as possible. This measure was introduced by Brub et al [enter citation]. Thus given a set K of non orthogonal states $|\psi_i\rangle^{\otimes M}$, we wish to produce a set K of N clones $|\phi_j\rangle^{\otimes N}$ while optimizing the global fidelity:

$$F_{MN} = \sum_{j=1}^{K} \eta_j |\langle \psi_j^N | \phi_j \rangle|^2 \tag{3.4.1}$$

Other figures of merit can also be used to improve the quality of the clones, such as local fidelity. The local fidelity is the average fidelity of each of the individual clones of each of the N subsystems measured against the input states $|\psi_j\rangle$. The authors choose the global fidelity due to its close connection to state discrimination. We are also very interested in the connection between cloning and state discrimination, particularly in the two step process where clones are first produced then they are unambiguously discriminated. We discover some very interesting features which are described in another section.

The global fidelity can be expressed differently if unitary operator acts on the input states $U|\psi_j^M\rangle|\chi\rangle = |\phi_j^N\rangle$.

$$F_{MN} = \sum_{j=1}^{K} \eta_j |\langle \psi_j^N | U | \psi_j^M \rangle | \chi \rangle|^2 \tag{3.4.2}$$

Maximizing the fidelity can be explicitly solved for an M set of two possible input states, $\{|\psi_1\rangle, |\psi_2\rangle\}$. It was originally solved by Brub *et al* for the case when the the incoming states are prepared with equal a priori probabilities.They noticed that the optimum clones $\{|\Phi_1\rangle, |\Phi_2\rangle\}$ lie in the subspace spanned by the input states to be cloned $\{|\psi_1\rangle, |\psi_2\rangle\}$.

A unitary produces N copies $|\Phi_1\rangle$ or $|\Phi_2\rangle$, to resemble the original states as best as possible.

$$U|\psi_1^M\rangle|i\rangle = |\Phi_1^N\rangle \tag{3.4.3}$$

$$U|\psi_2^M\rangle|i\rangle = |\Phi_2^N\rangle \tag{3.4.4}$$

The inner product of the above two equations gives a relationship between the input and the output states.

$$|\langle \psi_1 | \psi_2 \rangle|^M = |\langle \Phi_1 | \Phi_2 \rangle|^N \tag{3.4.5}$$

$$s^M = t^N \tag{3.4.6}$$

Where $s^{M=}|\langle \psi_1 | \psi_2 \rangle|^M$, and $t^N = |\langle \Phi_1 | \Phi_2 \rangle|$

The input states can be expressed as:

$$|\psi_1^N\rangle = \cos\theta|1\rangle + \sin\theta|0\rangle \tag{3.4.7}$$

$$|\psi_2^N\rangle = \cos\theta|1\rangle - \sin\theta|0\rangle \tag{3.4.8}$$

similarly the clones yet to be optimized can be expressed as:

$$|\phi_1\rangle = \cos\phi_1|1\rangle + \sin\phi_1|0\rangle \tag{3.4.9}$$

$$|\phi_2\rangle = \cos\phi_2|1\rangle - \sin\phi_2|0\rangle \tag{3.4.10}$$

Using this general representation of input and output states and using the overlap relation in 3.4.6 we see that the sum of the output angles is fixed as $|\langle\psi_1|\psi_2\rangle|^M = \cos^M 2\theta$ and $|\langle\phi_1|\phi_2\rangle|^N = \cos^N(\phi_1+\phi_2) \Rightarrow \cos^M 2\theta = \cos^N(\phi_1+\phi_2)$.

The global fidelity in terms of the angles becomes:

$$F_{MN} = \eta_1|\langle\psi_1|\phi_1\rangle|^2 + \eta_2|\langle\psi_2|\phi_2\rangle|^2 \tag{3.4.11}$$

$$= \eta_1(\cos\theta\cos\phi_1 + \sin\theta\sin\phi_1) + \eta_2(\cos\theta\cos\phi_2 + \sin\theta\sin\phi_2) \tag{3.4.12}$$

$$= \eta_1\cos^2(\theta-\phi_1) + \eta_2\cos^2(\theta-\phi_2) \tag{3.4.13}$$

Rewriting the fidelity in terms of the sum and difference of the output angles,

$$\begin{aligned} F_{MN} &= \eta_1\cos^2(\theta - \frac{\phi_1+\phi_2}{2} - \frac{\phi_1-\phi_2}{2}) + \eta_2\cos^2(\theta - \frac{\phi_1+\phi_2}{2} + \frac{\phi_1-\phi_2}{2}) \\ &= \eta_1\cos^2(\alpha - x) + \eta_2\cos^2(\alpha + x) \\ &= \frac{1}{2}[\eta_1(\cos 2(\alpha-x)+1] + \frac{1}{2}[\eta_2(\cos 2(\alpha+x)+1] \\ &= \frac{1}{2}[1 + \eta_1(\cos 2(\alpha-x) + \eta_2(\cos 2(\alpha+x)] \end{aligned}$$

where $\alpha = \theta - \frac{\phi_1+\phi_2}{2}$ is fixed and $x = \frac{\phi_1-\phi_2}{2}$ is the only variable to optimize with. Differentiating with respect to x we get

$$\eta_1 \sin(\alpha - x) = \eta_2 \sin(\alpha - x) \quad (3.4.14)$$

$$\eta_1[\sin 2\alpha \cos 2x - \cos 2\alpha \cos 2x] = \eta_2[\sin 2\alpha \cos 2x + \cos 2\alpha \cos 2x] \quad (3.4.15)$$

$$(\eta_1 - \eta_2) \sin 2\alpha \cos 2x = \cos 2\alpha \cos 2x \quad (3.4.16)$$

$$(\eta_1 - \eta_2) \tan 2\alpha = \tan 2x \quad (3.4.17)$$

This is the relationship that gives the optimal clones and x should be replaced in F_{MN}. To be able to use this relationship we re-express F_{MN} in a different way:

$$\begin{aligned} F_{MN} &= \frac{1}{2}[1 + \eta_1(\cos 2(\alpha - x) + \eta_2(\cos 2(\alpha + x)] \\ &= \frac{1}{2}[\cos 2\alpha \cos 2x\{1 + (\eta_1 - \eta_2) \tan 2\alpha \tan 2x\}] + \frac{1}{2} \end{aligned}$$

we can rewrite 3.4.17 $(\eta_1 - \eta_2) \tan 2\alpha = \frac{\sin 2x}{\cos 2x} .... \cos 2x = \frac{\sqrt{1-\cos^2 2x}}{(\eta_1 - \eta_2) \tan 2\alpha} = \frac{1}{\sqrt{1+(\eta_1 - \eta_2)^2 \tan^2(2\alpha)}}$, finally the optimal fidelity can be expressed in terms of the a-priori probabilities and the overlap of the input states only .

$$\begin{aligned} F_{MN} &= \frac{1}{2}[1 + \cos 2\alpha \cos 2x\{1 + (\eta_1 - \eta_2) \tan 2\alpha \tan 2x\}] \\ &= \frac{1}{2}[1 + \frac{\cos 2\alpha}{\sqrt{1 + (\eta_1 - \eta_2)^2 \tan^2(2\alpha)}}\{1 + (\eta_1 - \eta_2)^2 \tan^2 2\alpha] \\ &= \frac{1}{2}[1 + \cos 2\alpha \sqrt{1 + (\eta_1 - \eta_2)^2 \tan^2 2\alpha)}] \\ &= \frac{1}{2}[1 + \sqrt{\cos^2 2\alpha + (1 - 4\eta_1\eta_2) \sin^2 2\alpha)}] \\ &= \frac{1}{2}[1 + \sqrt{1 - 4\eta_1\eta_2 \sin^2 2\alpha}] \\ &= \frac{1}{2}[1 + \sqrt{1 - 4\eta_1\eta_2 \sin^2(2\theta - (\phi_1 + \phi_2))}] \end{aligned} \quad (3.4.18)$$

Where $\cos^M 2\theta = \cos^N(\phi_1 + \phi_2)$ in the limit of producing infinite copies $N \to infinity$ fidelity merges into the helstrom bound in discriminating of two pure states.

$$F_{M\infty} = \frac{1}{2}[1 + \sqrt{1 - 4\eta_1\eta_2\cos^2(2\theta)}] \tag{3.4.19}$$

This is one of the connections between cloning and state discrimination, it emerges in the infinite number of clones being produced.

3.5. Hybrid Cloning:Interpolation between exact and approximate cloning

3.5.1. Equal priors. The solution to the interpolation of cloning for equal a priori probabilities has been derived by Chefles et al [**2**]. The authors develop a scheme which depending on how exact requiring clones to be can interpolate between exact cloning with inconclusive results in one extreme and optimal approximate cloning on the other extreme. In our work this scheme has been generalized for the case when the input states are prepared with different a priori probabilities. First we show the derivation of the equal priors as it will help to better understand the general case.

For $\eta_1 = \eta_2 = 1/2$, the output states are symmetric, $\phi_1 = \phi_2 = \phi$, and the optimal global fidelity, F_{MN}, derived in section() reduces to

$$\begin{aligned} F_{MN} &= \frac{1}{2}[1 + \sqrt{1 - \sin^2(2\theta - 2\phi)}] \\ &= \frac{1}{2}[1 + \cos^2(2\theta - 2\phi)] \end{aligned} \tag{3.5.1}$$

Duan and Guo [**3**] showed that the maximum success probability of obtaining N exact clones from M given copies of non orthogonal quantum states $\{|\psi_1\rangle, |\psi_2\rangle\}$ which are prepared with equal a priori probabilities is:

$$P_{MN} = \frac{1 - s^M}{1 - s^N}, \tag{3.5.2}$$

where s is the overlap of the input states $s = \langle\psi_1|\psi_2\rangle$. The success rate for one to two cloning, $M = 1, N = 2$, reduces into:

$$P_{12} = \frac{1}{1+s}. \tag{3.5.3}$$

The interpolation takes us from optimal exact cloning to maximum fidelity. Given a set M of two non-orthogonal quantum states, $\{|\psi_1\rangle, |\psi_2\rangle\}$ the goal is to make N clones $\{|\phi_1\rangle, |\phi_2\rangle\}$, which are similar to the input states but not perfect. The Neumark setup is:

$$U|\psi_1\rangle^{\otimes M}|i\rangle = \sqrt{p}|\phi_1\rangle^{\oplus N}|1\rangle + \sqrt{q}|f\rangle|0\rangle \tag{3.5.4}$$

$$U|\psi_2\rangle^{\oplus M}|i\rangle = \sqrt{p}|\phi_2\rangle^{\oplus N}|1\rangle + \sqrt{q}|f\rangle|0\rangle \tag{3.5.5}$$

The input states are prepared with equal a priori probabilities. A click in the $|1\rangle$ direction means that we succeed in making the clones and the probability of success is p. A click in the $|0\rangle$ direction means that we failed to create a clone with a probability q. The inner product or 3.5.4 and 3.5.5 gives the constraint:

$$s^M = pt^N + q \tag{3.5.6}$$

Using the unitarity condition $p+q = 1$, the average rate of successfully making a clone is:

$$p = \frac{1-s^M}{1-t^N} \tag{3.5.7}$$

t is the overlap of the clones $t = \langle\phi_1|\phi_2\rangle$. If the final states are orthogonal, $t = 0$ then the state separation reaches the IDP limit and $P_S = P_{IDP} = 1 - |\langle\psi_1|\psi_2\rangle|^M$.

First we express the overlap of the output states in terms on the success rates and the overlap of input states, $\cos 2\theta = |\langle\psi_1|\psi_2\rangle|^N$

$$|\langle\phi_1|\phi_2\rangle|^N = 1 - \frac{1 - |\langle\psi_1|\psi_2\rangle|^M}{P_S} \tag{3.5.8}$$

$$\cos^N(\phi_1 + \phi_2) = 1 - \frac{P_{IDP}}{P_S} \tag{3.5.9}$$

Where the exact clones live in an N dimensional space

$|\psi_{1,2}^N\rangle = \cos\theta|1\rangle \pm \sin\theta|0\rangle$

The approximate clones can be expressed as

$|\phi_{1,2}\rangle = \cos\phi_1|1\rangle \pm \sin\phi_1|0\rangle$

The fidelity rate for equal priors is:

$$F_{MN} = \frac{1}{2}[1 + \cos(2\theta - (\phi_1 + \phi_2)] \tag{3.5.10}$$

and we want to use the relationship in 3.5.9. Let us expand the cosine term

$\cos(2\theta - (\phi_1 + \phi_2) = \cos 2\theta\cos(\phi_1 + \phi_2) + \sin 2\theta\sin(\phi_1 + \phi_2)$.

The fidelity becomes:

$$F_{MN} = \frac{1}{2}[1 + |\langle\psi_1^N|\psi_2^N\rangle|(1 - \frac{P_{IDP}}{P_S}) + \frac{1}{P_S}((1 - |\langle\psi_1^N|\psi_2^N\rangle|^2)(P_S^2 - (P_S - P_{IDP})^2)^{1/2})]$$

As $N \to \inf, |\langle\psi_1|\psi_2\rangle|^N \to 0$ and F_{MN} reduces to

$$F_{MN} = \frac{1}{2}[1 + \frac{1}{P_S}\sqrt{P_S^2 - (P_S - P_{IDP})^2}].$$

We can also express the fidelity in terms of fixed failure rate $Q = 1 - P_S$ which serves as the parameter by which we are interpolating and the optimal failure rate $Q_o = |\langle\psi_1|\psi_2\rangle$

$$\begin{aligned} F_{MN} &= \frac{1}{2}[1+|\langle\psi_1^N|\psi_2^N\rangle|(1-\frac{1-Q_o}{1-Q})+\frac{1}{1-Q}((1-|\langle\psi_1^N|\psi_2^N\rangle|^2)((1-Q)^2-(Q-Q_o)^2)^{1/2})] \\ &= \frac{1}{2(1-Q)}[(1-Q)+Q_o^N(Q_o-Q)+\sqrt{(1-Q_o^{2N})[(1-Q)^2-(Q-Q_o)^2]}] \end{aligned}$$

in the limit $N \to \inf, |\langle\psi_1|\psi_2\rangle|^N \to 0$

$$\begin{aligned} F_{MN} &= \frac{1}{2}[1+\frac{1}{1-Q}\sqrt{(1-Q)^2-(Q-Q_o)^2}]. \\ (1-Q)F_{MN} &= \frac{1}{2}[(1-Q)+\sqrt{(1-Q)^2-(Q-Q_o)^2}] \end{aligned}$$

$(1-Q)F_{MN} = P_{succes}$, the probability of successfully identifying a state. (This is a different success rate then the P_S defined above, the P_S was defined as the rate of successfully carrying out a state separation.)

$$P_{success} = \frac{1}{2}[(1-Q)+\sqrt{(1-Q)^2-(Q-Q_o)^2}]$$

This formula describes the relationship between the discrimination of states with a fixed rate of inconclusive outcome. When $Q = 0$ it reaches the Helstrom bound of minimum error, $Q = Q_o$ reaches the IDP limit in UD.

3.6. State Separation

3.6.1. Equal Priors.

3.6.2. General State separation. The unitary entangling the input states with the ancilla should do the following:

$$U|\psi_1\rangle^{\otimes M}|i\rangle = \sqrt{p_1}|\phi_1\rangle^{\oplus N}|\alpha\rangle + \sqrt{q_1}|\Phi\rangle|f\rangle \quad (3.6.1)$$

$$U|\psi_2\rangle^{\oplus M}|i\rangle = \sqrt{p_2}|\phi_2\rangle^{\oplus N}|\alpha\rangle + \sqrt{q_2}|\Phi\rangle|f\rangle \quad (3.6.2)$$

Where a projective measurement along $|\alpha\rangle$ means that the states have been successfully become more distinguishable with a success probability of p_i, otherwise a measurement in the $|f\rangle$ space means that the process has failed to produce more distinguishable states and the states are discarded with a probability of q_i.

The inner product of 3.6.1 with 3.6.2 gives the unitarity constraint:

$$s = \sqrt{p_1 p_2}\, s' + \sqrt{q_1 q_2} \tag{3.6.3}$$

where $s = |\langle \psi_1 \mid \psi_2 \rangle|^M$ and $s' = |\langle \phi_1 \mid \phi_2 \rangle|^N$.

For given η_1, η_2 and average failure probability Q, we wish to find out the minimum value of the final overlap s' as a function of the initial overlap s. The same difficulties encountered in perfect probabilistic cloning show up here and we will proceed along the same lines, giving the curve $(s, \min s')$ in parametric form, rather than attempting to give $\min s'$ as a explicit function of s, which would require solving a high degree polynomial equation. To this end, we linearize the unitarity condition by means of the same change of variables used in exact probabilistic cloning, but in addition we write

$$x = t - z, \qquad y = t + z,$$

or equivalently

$$p_1 p_1 = t^2, \quad q_1 q_2 = z^2. \tag{3.6.4}$$

The condition becomes

$$z = s - s' t, \qquad 0 \le t, z \le 1, \qquad 0 \le s' \le s.$$

It is a straight line in the (first quadrant of the) plane t-z with (negative) slope $-s'$ and crossing the z-axis at $z = s$. From the first Eq. (3.6.4) we have

$$t^2 = (1 - q_1)(1 - q_2) = 1 + z^2 - q_1 - q_2.$$

Solving for q_2 and substituting back in the second Eq. (3.6.4) we have

$$q_1(q_1 - 1 + t^2 - z^2) + z^2 = 0.$$

We now solve for q_1 and obtain

$$q_1 = \frac{1 + z^2 - t^2 \pm \sqrt{(1 + z^2 - t^2)^2 - 4z^2}}{2}.$$

Similarly, for q_2 we obtain

$$q_2 = \frac{1 + z^2 - t^2 \mp \sqrt{(1 + z^2 - t^2)^2 - 4z^2}}{2}.$$

Therefore, the condition $Q = \eta_1 q_1 + \eta_2 q_2$ becomes

$$2Q = 1 + z^2 - t^2 \pm (\eta_1 - \eta_2)\sqrt{(1 + z^2 - t^2)^2 - 4z^2}.$$

We now solve for z^2. After a bit of algebra we obtain

$$z^2 = \frac{2\eta_1\eta_2(1 + \tau) - 1 + Q + \sqrt{(1 - 4\eta_1\eta_2)\left[(1 - Q)^2 - 4\eta_1\eta_2\tau\right]}}{2\eta_1\eta_2} \equiv \zeta(\tau),$$

where $\tau \equiv t^2$. z^2 cannot be less than zero, we picked up the plus sign for the root. us assume that $0 \leq \eta_1 \leq 1/2$ to simplify the analysis. need to locate the maximum of z. For that,

$$\frac{dz}{dt} = \frac{d\sqrt{\zeta}}{d\tau} 2t = \frac{d\zeta}{d\tau} \frac{t}{z}.$$

The derivative $d\zeta/d\tau$ is immediate. We find that the maximum is located at

$$t_{\min} = \begin{cases} \sqrt{\left(1 - \dfrac{Q}{2\eta_1}\right)\left(1 - \dfrac{Q}{2\eta_2}\right)}, & \text{if} \quad 0 \leq Q \leq 2\eta_1 \\ \\ 0, & \text{if} \quad 2\eta_1 < Q \leq 1. \end{cases}$$

The corresponding values of z are

$$z_{\min} = \begin{cases} \dfrac{Q}{2\sqrt{\eta_1\eta_2}}, & \text{if} \quad 0 \le Q \le 2\eta_1 \\ \\ \sqrt{\dfrac{Q-\eta_1}{\eta_2}}, & \text{if} \quad 2\eta_1 < Q \le 1. \end{cases}$$

There is a second point that we need to define. We first note that for equal priors the curve is simply the hyperbola

$$z^2 = t^2 + 2Q - 1,$$

which intersects the straight line

$$z = 1 - t$$

at the point

$$(z, t) = (Q, 1 - Q).$$

By trying this solution in $z^2 = \zeta(t^2)$ we note that it is actually a general solution for any η_1, η_2. Moreover, the straight line $z = 1 - t$ is tangent to $z^2 = \zeta(t^2)$ at $(Q, 1-Q)$ for any values of η_1, η_2, as can be checked by substituting in the formula $dz/dt = (t/z)(d\zeta/d\tau)$. Note also that $z = 1 - t$ is the limiting line for the family $z = s - s't$. Hence, an obvious parametrization for the curve (s, s') is obtained as follows: *i.* define

$$s'(t) = -\frac{dz}{dt} = -\frac{t\,\zeta'(t^2)}{\sqrt{\zeta(t^2)}}, \qquad t_{\min} \le t \le 1 - Q,$$

and next *ii.* define

$$s(t) = z + ts'(t) = \sqrt{\zeta(t^2)} + ts'(t), \qquad t_{\min} \le t \le 1 - Q.$$

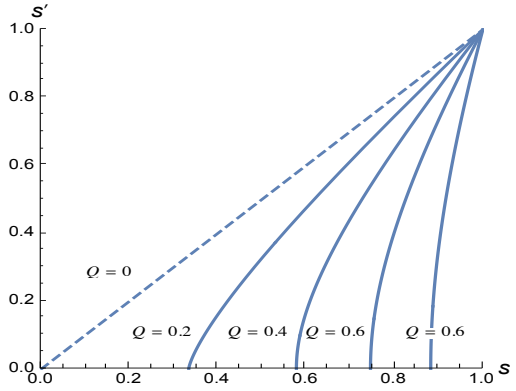
where

$$\zeta'(\tau) = 1 - \frac{\sqrt{1 - 4\eta_1\eta_2}}{\sqrt{(1-Q)^2 - 4\eta_1\eta_2\tau}}$$

For $s < z_{\min}$ is is always possible to separate the initial states, i.e., $|\Psi_1\rangle$ and $|\Psi_2\rangle$ can be made orthogonal. We note that the condition $s = z_{\min}$ is equivalent to the unambiguous discrimination result

$$Q = 2\sqrt{\eta_1\eta_2}s, \quad Q = \eta_1 + \eta_2 s^2.$$

The next plot is for $\eta_1 = 0.1$. As η_1 approaches $1/2$ the curves approach a straight line. The difference is more noticeable for very small values of η_1.



CHAPTER 4

Experimental realization using optical devices

4.0.3. Introduction. Since the interpolation scheme of a general measurement with a Fixed Rate of Inconclusive Outcome (FRIO) has been solved by *Bagan et al* [**4**] there has been a quest for a physical realization. The authors solved the problem using an operator transformation technique that reformulated the intermediate problem into a ME problem with an extra optimization parameter. Essentially they reduced the problem from a three element POVM to two element POVM similar to ME. As is the case with POVM solutions, they are not naturally realizable in an experimental setting. Inspired by the work of *Reck et al* [**5**] in which they prove that any finite dimensional unitary operator can be constructed in the lab using optical devices. We set out to solve the intermediate discrimination problem using the Neumark setup. This gives us a closed form solution as in *Bagan et al*. In addition it gives a three dimensional unitary operator. Using the *Reck at al* algorithm we express the unitary in terms of beam splitters with corresponding coefficients of transmittance and reflectivity. Similarly we give a solution to the Minimum error problem and provide the necessary beamsplitters with the corresponding coefficients.

4.1. Analytical Solution of Interpolation

In this section we solve the intermediate scheme of ME with a Fixed Rate of Inconclusive Outcome (FRIO) in two different ways. In the first method, through a parametrization, the problem is converted into a minimum error whose solution is well known. Then there is one last optimization with respect to a $FRIO$. This method generates the same solution as in *Bagan et al* in a few lines. However

to obtain an experimental realization of the interpolation we solve it using Neumark's theorem. It is a generalized measurement procedure in which the system is embedded in a larger Hilbert space with extra degrees of freedom. A unitary transformation entangles the system with the extra degree of freedom known as ancilla. After this interaction has taken place, projective von Neuman measurements are carried out on the ancilla.

Our input states are qubits, which can be expressed in general as unit length vectors in the two dimensional basis spanned by $|1\rangle$ and $|2\rangle$. In the output of the transformation, we associate the basis state $|1\rangle$ with $|\psi_1\rangle$, $|2\rangle$ with $|\psi_2\rangle$, and $|0\rangle$ gives no information about the system.

The unitary transformation should do the following:

$$\begin{aligned} U|\psi_1\rangle_s|0\rangle_a &= \sqrt{p_1}|1\rangle\,|1\rangle + \sqrt{r_1}|2\rangle\,|2\rangle + \sqrt{q_1}|0\rangle\,|0\rangle\,, \\ U|\psi_2\rangle_s|0\rangle_a &= \sqrt{r_2}|1\rangle\,|1\rangle + \sqrt{p_2}|2\rangle\,|2\rangle + \sqrt{q_2}|0\rangle\,|0\rangle\,, \end{aligned} \tag{4.1.1}$$

where: p_i is the probability that state $i(i = 1, 2)$ is correctly identified, r_i is the probability that the detector mistakenly identifies state i for j, and q_i is the failure probability, the detector fails to identify the state at all. Whenever we get a click in the $|0\rangle_a$ detector means the results are inconclusive. From the unitarity conditions we obtain the normalized probabilities $p_i + r_i + q_i = 1$.

We wish to maximize the probability of success, $P_s = \eta_1 p_1 + \eta_2 p_2$, and minimize the error rate, $P_e = \eta_1 r_1 + \eta_2 r_2$, for a fixed failure rate $Q = \eta_1 q_1 + \eta_2 q_2$. Clearly $P_s + P_e + Q = 1$.

The inner product of the two equations in 4.1.1 gives the overlap of the input states in terms of r_i, p_i and q_i,

$$s \equiv \langle\psi_1|\psi_2\rangle = \sqrt{p_1 r_2} + \sqrt{p_2 r_1} + \sqrt{q_1 q_2}. \tag{4.1.2}$$

This is a constraint on the optimization.

4.1.1. Optimal solution with equal prior probabilities. Let us first present the solution where the incoming states are given with equal a-priori probabilities, $\eta_1 = \eta_2 = \frac{1}{2}$. This implies equal error, success and failure rates: $r_1 = r_2$, $p_1 = p_2$ and $q_1 = q_2$. Thus the total error and failure rates reduce to: $P_E = \eta_1 r_1 + \eta_2 r_2 = r$ and $Q = \eta_1 q_1 + \eta_2 q_2 = Q$. The Neumark set up reduces into

$$U|\psi_1\rangle_s|0\rangle = \sqrt{p}|1\rangle\,|1\rangle + \sqrt{r}|2\rangle\,|2\rangle + \sqrt{q}|0\rangle\,|0\rangle\,, \tag{4.1.3}$$

$$U|\psi_2\rangle_s|0\rangle = \sqrt{r}|1\rangle\,|1\rangle + \sqrt{p}|2\rangle\,|2\rangle + \sqrt{q}|0\rangle\,|0\rangle\,. \tag{4.1.4}$$

We can immediately solve the constraint equation by replacing $p = 1 - r - Q$, solving the quadratic equation for the error rate in terms of the failure rate and overlap s, which is also the overall failure rate in the IDP limit for the equal priors: $Q_o \equiv 2\sqrt{\eta_1\eta_2}s = s$.

$$\begin{aligned} s &= \sqrt{pr} + \sqrt{pr} + Q \\ Q_o &= 2\sqrt{r(1-r-Q)} + Q \\ (Q_o - Q)^2 &= 4r(1-r-Q) \\ \frac{1}{4}(Q_o - Q)^2 &= r(1-Q) - r^2 \\ 0 &= r^2 - (1-Q)r + \frac{1}{4}(Q_o - Q)^2 \end{aligned} \tag{4.1.5}$$

Solving the quadratic equation and taking the smaller root, *i.e* the smaller error rate:

$$r = P_E = \frac{1}{2}[(1-Q) - \sqrt{(1-Q)^2 - (Q_o - Q)^2}], \tag{4.1.6}$$

$$p = P_S = \frac{1}{2}[(1-Q) + \sqrt{(1-Q)^2 - (Q_o - Q)^2}]. \tag{4.1.7}$$

The equal priors case requires no further optimization. Simply solving the quadratic equation in 4.1.5 derived from the constraint the optimal solution is carried out. By varying Q from zero to Q_o we recover the Helstrom and IDP bounds. In the Helstrom bound one is not allowed to have inconclusive results, hence setting failure rate to zero, $Q = 0$, we get

$$\begin{aligned} P_E &= \frac{1}{2}[1 - \sqrt{1 - Q_o^2}] \\ P_E &= \frac{1}{2}[1 - \sqrt{1 - s^2}] \end{aligned} \tag{4.1.8}$$

In the IDP limit one is not allowed to make an error, $r = 0$, while allowing for inconclusive outcomes:

$$\begin{aligned} 0 &= \frac{1}{2}[(1 - Q) - \sqrt{(1 - Q)^2 - (Q_o - Q)^2}] \\ Q &= Q_o \\ Q &= s^2 \end{aligned} \tag{4.1.9}$$

4.1.2. Full Solution. Because of the recent interest in this problem, we feel it is beneficial to show two new and different approaches to its solution. The first is more conceptual: a renormalization inspired by E. Bagan et al. [**4**] allows us to rewrite the problem as a ME problem with an implicit dependence on the last free parameter, the failure rate of one state with relation to the other. This greatly simplifies the problem as the solution to the first part is well known and the second a straight-forward derivative. The second solution is a Lagrange multiplier method that is algebraically difficult but useful in its explicit results of individual error rate. This in turn is useful in designing an implementation scheme.

4.1.3. Transformation of the problem into the Helstrom form. We choose the following transformation to convert our problem into the well known Helstrom form.

First we define the useful quantity $\omega \equiv s - \sqrt{q_1 q_2}$ which will serve as normalized overlap:

$$\omega = \sqrt{p_1 r_2} + \sqrt{p_2 r_1}$$

Next we normalize all probabilities in the problem:

$$\begin{aligned} \tilde{p_i} &= \frac{p_i}{\alpha_i}, \\ \tilde{r_i} &= \frac{r_i}{\alpha_i}, \\ \tilde{\omega} &= \frac{\omega}{\alpha_1 \alpha_2}, \end{aligned} \tag{4.1.10}$$

where $\alpha_i = 1 - q_i$. Now $\widetilde{r_i} + \tilde{p_i} = 1$, and we have the overlap in terms of $\tilde{r_i}$ and $\tilde{p_i} : \tilde{\omega} = \sqrt{\tilde{p_1}\tilde{r_2}} + \sqrt{\tilde{p_2}\tilde{r_1}}$.

Using the above transformation of r_i the error rate can be expressed as:

$$\tilde{P_E} = \tilde{\eta_1}\tilde{r_1} + \tilde{\eta_2}\tilde{r_2}, \tag{4.1.11}$$

where $\tilde{P_E} = \frac{P_E}{\eta_1\alpha_1 + \eta_2\alpha_2} = \frac{P_E}{1-Q}, \tilde{\eta_i} = \frac{\eta_i \alpha_i}{\eta_1\alpha_1 + \eta_2\alpha_2} = \frac{\eta_i \alpha_i}{1-Q}$ and $\tilde{\eta_1} + \tilde{\eta_2} = 1$.

We have transformed the problem into a discrimination between two pure states with overlap $\tilde{\omega}$ and no explicit failure rate. Hence we can simply write down the expression to solution of minimum error of two pure states (the Helstrom bound), and then replace the normalized quantities with the original expressions:

$$\begin{aligned} \tilde{P_E} &= \frac{1}{2}[1 - \sqrt{1 - 4\tilde{\eta_1}\tilde{\eta_2}\tilde{\omega}^2}], \\ P_E &= \frac{1}{2}[(1-Q) - \sqrt{(1-Q)^2 - 4\eta_1\eta_2(s - \sqrt{q_1 q_2})^2}]. \end{aligned} \tag{4.1.12}$$

We now have one last optimization. Given a fixed rate of the average of inconclusive outcomes, Q, what are the individual failure rates q_i. To minimize 4.1.12 we maximize the square root term $\sqrt{(1-Q)^2 - 4\eta_1\eta_2(s - \sqrt{q_1 q_2})^2}$, which in term means to minimize $(s - \sqrt{q_1 q_2})^2 = (s^2 + q_1 q_2 - 2s\sqrt{q_1 q_2})$. The overlap term, s, is

fixed and we are left with $q_1q_2 - 2s\sqrt{q_1q_2}$. Only one of the $q_i's$ is an independent variable as we fix the overall failure rate $Q = \eta_1 q_1 + \eta_2 q_2$.

$$\begin{aligned}
\Theta &= q_1q_2 - 2s\sqrt{q_1q_2}. \\
\Theta &= \frac{q_1(Q_0 - \eta_1 q_1^2)}{\eta_2} - 2s\sqrt{\frac{q_1(Q_0 - \eta_1 q_1^2)}{\eta_2}} \\
\Theta &= \frac{\eta_1 q_1 Q_0 - \eta_1^2 q_1^2}{\eta_1\eta_2} - 2s\sqrt{\frac{\eta_1 q_1 Q_0 - \eta_1^2 q_1^2}{\eta_1\eta_2}} \\
\Theta &= \frac{\tilde{q}_1 Q_0 - \tilde{q}_1^2}{\eta_1\eta_2} - 2s\sqrt{\frac{\tilde{q}_1 Q_0 - \tilde{q}_1^2)}{\eta_1\eta_2}}
\end{aligned}$$

where $\eta_1 q_1 = \tilde{q}$. Let's optimize with respect to $\tilde{q}_1$.

$$0 = \frac{\partial\Theta}{\partial\tilde{q}_1} = \frac{Q_0 - 2\tilde{q}_1}{\eta_1\eta_2} - 2s\sqrt{\frac{Q_0 - 2\tilde{q}_1}{\eta_1\eta_2}} \tag{4.1.13}$$

this leads to the optimality condition $Q_0 = 2\tilde{q}_1 = 2\eta_1 q_1 = 2\eta_2 q_2$, giving us the minimal error rate in discriminating two pure states with a fixed rate of failure as

$$P_E = \frac{1}{2}[(1-Q) - \sqrt{(1-Q)^2 - (Q_o - Q)^2}]. \tag{4.1.14}$$

Where $Q_o = 2\sqrt{\eta_1\eta_2}s$ is the failure rate in the optimal unambiguous state discrimination, which our expression reaches when we set $P_E = 0$. On the other hand when the failure rate is zero we can recover the Helstrom bound for two pure states $P_E = \frac{1}{2}[1 - \sqrt{1 - 4\eta_1\eta_2 s^2}]$.

4.1.4. Lagrange Multipliers Method. While the above method gives a closed formed solution of the average error rate in terms of a FRIO it does not produce individual error or success rates, i.e the error rate of mistaking state $|\psi_i\rangle$ for state $|\psi_j\rangle$, which are needed for the implementation in calculating the transmittance and reflection coefficients of the beam splitters. To obtain these expressions we show another solution to the interpolation using the Lagrange multipliers method with constraint in 4.1.2.

We want to minimize the average error rate $P_E = \eta_1 r_1 + \eta_2 r_2$ subject to the constraint $s = \sqrt{(1-r_1-q_1)r_2} + \sqrt{(1-r_2-q_2)r_1} + \sqrt{q_1 q_2}$, setting up the function with one Lagrange multiplier λ :

$$F = \eta_1 r_1 + \eta_2 r_2 + \lambda(s - \sqrt{(1-r_1-q_1)r_2} - \sqrt{(1-r_2-q_2)r_1} - \sqrt{q_1 q_2}) \quad (4.1.15)$$

Setting the derivative $dF_{(r_1,r_2,\lambda)}/dr_i$ to zero then solving for $r_i(\lambda)$, we exploit the symmetry in the resulting equations to solve for the individual error rates r_i as a function of the failure rates q_i. Subsequent substitution into the constraint gives us the optimal value of λ. Then we can obtain the total minimum error by replacing the expressions of r_i into P_e and minimizing P_e under the additional constraint that $\eta_1 q_1 + \eta_2 q_2 = Q$. This gives us the optimal relationship between failure rates as $\eta_1 q_1 = \eta_2 q_2$ and the total optimal error rate as

Set $dF/dr_1 = 0$:

$$\begin{aligned}
0 &= \eta_1 + \lambda[\frac{(-1/2)(-r_2)}{\sqrt{(1-q_1-r_1)r_2}} - \frac{(1/2)(1-q_2-r_2)}{\sqrt{(1-q_2-r_2)r_1}}] \\
0 &= \eta_1 + (\lambda/2)[\frac{(r_2)}{\sqrt{(1-q_1-r_1)r_2}} - \frac{(1-q_2-r_2)}{\sqrt{(1-q_2-r_2)r_1}} \\
-(2\eta_1/\lambda) &= \frac{\sqrt{r_2}}{\sqrt{(1-q_1-r_1)}} - \frac{\sqrt{(1-q_2-r_2)}}{\sqrt{r_1}} \\
(2\eta_1/\lambda)\sqrt{(1-q_1-r_1)r_1} &= \sqrt{(1-q_2-r_2)(1-q_1-r_1)} - \sqrt{r_1 r_2} \qquad (4.1.16)
\end{aligned}$$

Similarly $dF/dr_2 = 0$ gives

$$(2\eta_2/\lambda)\sqrt{(1-q_2-r_2)r_2} = \sqrt{(1-q_2-r_2)(1-q_1-r_1)} - \sqrt{r_1 r_2} \qquad (4.1.17)$$

This step is algebraically challenging and requires the insight that the resulting equations can each be separated into two expressions, left hand side depending on only r_1 or r_2 and the right hand sides are equivalent. Because both equations have the same multivariable expression, we can set the left hand sides equal to a

constant, C, which is yet to be determined.

$$(2\eta_1/\lambda)\sqrt{(1-q_1-r_1)r_1} = (2\eta_2/\lambda)\sqrt{(1-q_2-r_2)r_2} \equiv C \tag{4.1.18}$$

This greatly simplifies the problem, turning it into a quadratic equation.

$$(2\eta_i/\lambda)\sqrt{(1-q_i-r_i)r_i} = C, \tag{4.1.19}$$

let $\alpha_i \equiv 1 - q_i$

$$\sqrt{(\alpha_1 - r_i)r_i} = (\lambda C)/(2\eta_i),$$

$$(\alpha_i - r_i)r_i = (\lambda^2 C^2)/(4\eta_i^2),$$

$$\alpha_i r_i - r_i^2 - \lambda^2 C^2/4\eta_i^2 = 0,$$

$$r_i^2 - \alpha_i r_i + \lambda^2 C^2/4\eta_i^2 = 0,$$

$$r_i(\pm) = 1/2\left(\alpha_i \pm \sqrt{\alpha_i^2 - \frac{\lambda^2 C^2}{\eta_i^2}}\right),$$

r_i is the error rate which we want to be minimized, thus we take the smaller root $r_i(-)$.

$$r_1 = 1/2\left(\alpha_1 - \sqrt{\alpha_1^2 - \frac{\delta}{\eta_1^2}}\right) = 1/2\,(\alpha_1 - A_1) \tag{4.1.20}$$

$$r_2 = 1/2\left(\alpha_2 - \sqrt{\alpha_2^2 - \frac{\delta}{\eta_2^2}}\right) = 1/2\,(\alpha_2 - A_2) \tag{4.1.21}$$

Where $\delta \equiv \lambda^2 C^2$ and $A_i = \sqrt{\alpha_i^2 - (\frac{\lambda^2 C^2}{\eta_i^2})}$.

We replace the expression of r_i from 4.1.20 and 4.1.21 into the constraint from 4.1.2. First let us rewrite the constraint so it simplifies the algebra later:

$$(s - \sqrt{q_1 q_2})^2 =$$

$$(1 - r_1 - q_1)r_2 + (1 - r_2 - q_2)r_1 + \sqrt{(1 - r_1 - q_1)(1 - r_2 - q_2)r_1 r_2}$$

we use the definition in 4.1.19 to reduce the square root term $\sqrt{(1 - r_1 - q_1)(1 - r_2 - q_2)r_1 r_2} = \frac{\lambda^2 C^2}{4\eta_1 \eta_2}$.

$$(s - \sqrt{q_1 q_2})^2 = (1 - r_1 - q_1)r_2 + (1 - r_2 - q_2)r_1 + \frac{\lambda^2 C^2}{4\eta_1 \eta_2}$$

Setting $\omega \equiv (s - \sqrt{q_1 q_2})^2$ and using $\alpha_i = 1 - q_i$, the above expression becomes:

$$\omega = (\alpha_1 - q_1)r_2 + (\alpha_2 - q_2)r_1 + \frac{\delta}{4\eta_1 \eta_2}$$

$$\omega = \alpha_1 r_2 - \alpha_2 r_1 - 2r_1 r_2 + \delta/4\eta_1 \eta_2$$

$$\omega = \alpha_1/2(\alpha_2 - A_2) + \alpha_2/2(\alpha_1 - A_1) - 1/2(\alpha_1 - A_1)(\alpha_2 - A_2) + \delta/4\eta_1 \eta_2$$

$$\omega = \alpha_1 \alpha_2 - A_1 A_2 + \delta/\eta_1 \eta_2$$

Replacing A_1, A_2 by their respective value

$$\omega = 1/2[\alpha_1 \alpha_2 - \sqrt{\alpha_1^2 - \frac{\delta}{\eta_1^2}}\sqrt{\alpha_2^2 - \frac{\delta}{\eta_2^2}} + \delta/\eta_1 \eta_2]$$

$$\sqrt{\alpha_1^2 - \frac{\delta}{\eta_1^2}}\sqrt{\alpha_2^2 - \frac{\delta}{\eta_2^2}} = \alpha_1 \alpha_2 + \delta/\eta_1 \eta_2 - 2\omega$$

Squaring both sides and after some trivial algebra we get

$$\frac{-\alpha_1^2 \delta}{\eta_2^2} - \frac{-\alpha_2^2 \delta}{\eta_1^2} - \frac{2\alpha_1 \alpha_2 \delta}{\eta_1 \eta_2} + \frac{4\omega\delta}{\eta_1 \eta_2} = 4\omega^2 - 4\omega\alpha_1 \alpha_2$$

$$\delta[\frac{4\omega}{\eta_1 \eta_2} - (\frac{\alpha_1}{\eta_2} + \frac{\alpha_2}{\eta_1})^2] = 4\omega^2 - 4\omega\alpha_1 \alpha_2$$

$$\frac{\alpha_1}{\eta_2} + \frac{\alpha_2}{\eta_1} = \frac{\alpha_1 \eta_1 + \alpha_2 \eta_2}{\eta_1 \eta_2} = \frac{\eta_1(1 - q_1) + \eta_2(1 - q_2)}{\eta_1 \eta_2} = \frac{1 - Q}{\eta_1 \eta_2}$$

$$\delta[\frac{4\omega}{\eta_1\eta_2} - \frac{(1-Q)^2}{\eta_1^2\eta_2^2}] = 4\omega^2 - 4\omega\alpha_1\alpha_2$$

$$\delta[\frac{\omega}{\eta_1\eta_2} - \frac{(1-Q)^2}{4\eta_1^2\eta_2^2}] = \omega^2 - \omega\alpha_1\alpha_2$$

$$\delta = \frac{(\omega^2 - \omega\alpha_1\alpha_2)4\eta_1^2\eta_2^2}{4\omega\eta_1\eta_2 - (1-Q)^2} \tag{4.1.22}$$

We now plug δ into r_1 in 4.1.20

$$\begin{aligned} r_1 &= 1/2(\alpha_1 - \sqrt{\alpha_1^2 - (\frac{\lambda^2 C^2}{\eta_1^2})}) \\ r_1 &= 1/2(\alpha_1 - \sqrt{\alpha_1^2 - \frac{1}{\eta_1^2}\frac{(\omega^2 - \omega\alpha_1\alpha_2)4\eta_1^2\eta_2^2}{4\omega\eta_1\eta_2 - (1-Q)^2}} \\ r_1 &= 1/2(\alpha_1 - \sqrt{\frac{\alpha_1^2[4\omega\eta_1\eta_2 - (1-Q)^2] - 4\eta[\omega^2 - \omega\alpha_1\alpha_2]}{4\omega\eta_1\eta_2 - (1-Q)^2}} \end{aligned} \tag{4.1.23}$$

The numerator can be greatly simplified

$$\begin{aligned} &= \alpha_1^2[4\omega\eta_1\eta_2 - (1-Q)^2] - 4\eta_2^2[\omega^2 - \omega\alpha_1\alpha_2] \\ &= -\alpha_1^2(1-Q)^2 - 4\eta_2^2\omega^2 + 4\omega\eta_2\alpha_1[\eta_1\alpha_1 + \eta_2\alpha_2] \\ &= -\alpha_1^2(1-Q)^2 - 4\eta_2^2\omega^2 + 4\omega\eta_2\alpha_1(1-Q) \\ &= -[\alpha_1(1-Q) - 2\eta_2\omega]^2 \\ &= -[(1-q_1)(1-Q) - 2\eta_2(s - \sqrt{q_1 q_2})^2]^2 \end{aligned} \tag{4.1.24}$$

The calculation for r_2 goes along the same line. Expression for r_1 and r_2 become:

$$r_1 = \frac{1}{2}[(1-q_1) - \frac{(1-q_1)(1-Q) - 2\eta_2(s - \sqrt{q_1 q_2})^2}{\sqrt{(1-Q)^2 - 4\eta_1\eta_2(s - \sqrt{q_1 q_2})^2}}] \tag{4.1.25}$$

$$r_2 = \frac{1}{2}[(1-q_2) - \frac{(1-q_2)(1-Q) - 2\eta_1(s - \sqrt{q_1 q_2})^2}{\sqrt{(1-Q)^2 - 4\eta_1\eta_2(s - \sqrt{q_1 q_2})^2}} \tag{4.1.26}$$

Finally we can substitute $r_1$and r_2 into the overall average error rate $P_E = \eta_1 r_1 + \eta_2 r_2$:

$$P_E = \frac{1}{2}[1-(\eta_1 q_1+\eta_2 q_2)-\frac{(1-Q)(\eta_1+\eta_2-(\eta_1 q_1+\eta_2 q_2)-4\eta_1\eta_2(s-\sqrt{q_1 q_2})^2}{\sqrt{(1-Q)^2-4\eta_1\eta_2(s-\sqrt{q_1 q_2})^2}}]$$

$$= \frac{1}{2}[(1-Q)-\frac{(1-Q)^2-4\eta_1\eta_2(s-\sqrt{q_1 q_2})^2}{\sqrt{(1-Q)^2-4\eta_1\eta_2(s-\sqrt{q_1 q_2})^2}}] \quad (4.1.27)$$

$$\frac{1}{2}[(1-Q)-\sqrt{(1-Q)^2-4\eta_1\eta_2(s-\sqrt{q_1 q_2})^2}] \quad (4.1.28)$$

We already show that 4.1.27 is optimized for a fixed value of failure rate when $\eta_1 q_1 = \eta_2 q_2$. It then reduces into the expression derived above:

$$P_E = \frac{1}{2}[(1-Q)-\sqrt{(1-Q)^2-(Q-Q_0)^2}]. \quad (4.1.29)$$

The individual error and success rates can now be expressed explicitly in terms of η_i, Q_o and most importantly the fixed failure rate Q as:

$$r_i = \frac{1}{2}[(1-\frac{Q}{2\eta_i})-\frac{(1-\frac{Q}{2\eta_i})(1-Q)-\frac{1}{2\eta_i}(Q_o-Q)^2}{\sqrt{(1-Q)^2-(Q-Q_o)^2}}], \quad (4.1.30)$$

$$p_i = \frac{1}{2}[(1-\frac{Q}{2\eta_i})+\frac{(1-\frac{Q}{2\eta_i})(1-Q)-\frac{1}{2\eta_i}(Q_o-Q)^2}{\sqrt{(1-Q)^2-(Q-Q_o)^2}}]. \quad (4.1.31)$$

This is the first time that the individual error and failure rates have been calculated. It has the advantage that we can now use them in our calculations of the beam splitters and phase shifters.

4.1.5. Choosing the physical implementation. The main reason to seek a solution using the Neumark setup is because it lends itself into an optical implementation. This implementation, as we will see, can be carried out using only linear optical elements (beamsplitters and a mirror). The possible states are represented by single photons and a photodetector will carry out the measurement process at the output.

To choose the basis we will start with the two mode vacuum state $|00\rangle$. The total number of photons in both modes is one. This two dimensional Hilbert space where our photons live can be spanned by the states $\{|10\rangle, |01\rangle\}$, where $|10\rangle = a_1^\dagger|00\rangle$ and

$|01\rangle = a_2^\dagger|00\rangle$ and $a_i^\dagger$ are different modes of the electromagnetic fields. The basis $\{|10\rangle, |01\rangle\}$ corresponds to the basis of qubit in basis $\{|0\rangle, |1\rangle\}$. The most general input state can be written as $|\psi_i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle = \alpha_i|10\rangle + \beta_i|01\rangle$ which can be produced by sending a photon into a beam splitter with some transmission and reflection coefficient.

To represent a qubit in the six-port we need to add another mode, the vacuum mode. The six-port corresponds to three input and three output ports. Our single photon input state can now be represented as $|\psi_i\rangle_{in} = \sum_{j=1}^3 c_j a_j^\dagger|000\rangle$ and the output states

$|\psi_i^\dagger\rangle = U|\psi_i\rangle_{in} = U\sum_{j=1}^3 c_j a_j^\dagger U^\dagger|000\rangle = \sum_{j=1}^3\sum_{k=1}^3 c_j M_{jk}^\dagger a_k^\dagger|000\rangle$. This is the most general form of the output states.

4.1.6. Equal a-priors Interpolation. We will first show full solution of the equal a-priori case where we use the direct sum to set up the Neumark.

For equal priors $\eta_1 = \eta_2$ we have $p_1 = p_2 = p, r_1 = r_2 = r, q_1 = q_2 = Q$, where $Q = q$ will serve as the fixed parameter. The two input states to be discriminated can be written as $|\psi_1\rangle_{in} = |1\rangle$ and $|\psi_2\rangle_{in} = \cos\theta|1\rangle + \sin\theta|2\rangle$

$$U|1\rangle = \sqrt{p}|1\rangle + \sqrt{r}|2\rangle + \sqrt{q}|3\rangle \tag{4.1.32}$$

$$U(\cos\theta|1\rangle + \sin\theta|2\rangle) = \sqrt{r}|1\rangle + \sqrt{p}|2\rangle + \sqrt{q}|3\rangle \tag{4.1.33}$$

Where the error and success rate was calculated in section 4.1.1:

$$r = \frac{1}{2}[(1-Q) - \sqrt{(1-Q)^2 - (Q_o - Q)^2}] \tag{4.1.34}$$

$$p = \frac{1}{2}[(1-Q) + \sqrt{(1-Q)^2 - (Q_o - Q)^2}] \tag{4.1.35}$$

From the Neumark setup we can read out six out of nine elements of the unitary matrix. 4.1.32 gives the first column, 4.1.33 gives the second column and the last column can be constructed from the conditions of unitarity.

To get first column, multiply 4.1.32 from the left with $\langle 1|$, $\langle 2|$ and $\langle 3|$:

$\langle 1|U|1\rangle = U_{11} = \sqrt{p}$

$\langle 2|U|1\rangle = U_{21} = \sqrt{q}$

$\langle 3|U|1\rangle = U_{31} = \sqrt{r}$

To get the second column, multiply 4.1.33 from the left with $\langle 1|$, $\langle 2|$ and $\langle 3|$:

$\cos\theta\langle 1|U|1\rangle + \sin\theta\langle 1|U|2\rangle = \cos\theta U_{11} + \sin\theta U_{12} = \sqrt{r}$

$\cos\theta\langle 2|U|1\rangle + \sin\theta\langle 2|U|2\rangle = \cos\theta U_{21} + \sin\theta U_{22} = \sqrt{p}$

$\cos\theta\langle 3|U|1\rangle + \sin\theta\langle 3|U|2\rangle = \cos\theta U_{31} + \sin\theta U_{32} = \sqrt{q}$

Using the solution of the first column U_{i1} ,gives the entries to second column U_{i2} where $i = 1, 2, 3$

$\cos\theta\sqrt{p}_1 + \sin\theta U_{12} = \sqrt{r_2} \Rightarrow U_{12} = \frac{\sqrt{r}-\sqrt{p}Q_o}{\sqrt{1-Q_o^2}}$

$\cos\theta\sqrt{q_1} + \sin\theta U_{22} = \sqrt{p_2} \Rightarrow U_{22} = \frac{[\sqrt{p}-\sqrt{r}Q_o]}{\sqrt{1-Q_o^2}}$

$\cos\theta\sqrt{r_1} + \sin\theta U_{32} = \sqrt{q_2} \Rightarrow U_{32} = \sqrt{\frac{Q(1-Q_o)}{1+Q_o}}$

Where $\cos\theta = Q_o$ and $\sin\theta = \sqrt{1-Q_o^2}$

The remaining elements be calculated from the conditions of the unitarity, $U^TU = I$, $U_{i1}^2 + U_{i2}^2 + U_{i3}^2 = 1$ where $i = 1, 2, 3$

$U_{13} = \pm\sqrt{1-U_{11}^2-U_{12}^2} = \pm\sqrt{1-p-\frac{r+pQ_o^2-2\sqrt{pr}Q_0}{1-Q_0^2}} = \pm\sqrt{\frac{Q-Q_o^2+2\sqrt{pr}Q_0}{1-Q_0^2}} = \pm\sqrt{\frac{Q}{1-Q_0}}$

where we used:$\sqrt{pr} = \frac{1}{2}(Q_o - Q)$, which is derived from the multiplication of 4.1.34 and 4.1.35.

$U_{23} = \pm\sqrt{1-U_{21}^2-U_{22}^2} = \pm\sqrt{1-r-\frac{p+rQ_o^2-2\sqrt{pr}Q_0}{1-Q_0^2}} = \pm\sqrt{\frac{Q}{1-Q_0}}$

$U_{33} = \pm\sqrt{1-U_{31}^2-U_{32}^2} = \pm\sqrt{1-Q-\frac{Q(1-Q_o)}{1+Q_o}} = \pm\sqrt{\frac{1+Q_o-2Q}{1+Q_o}} = \pm\frac{\sqrt{p}+\sqrt{r}}{\sqrt{1+Q_o}}$

where we used : $\sqrt{(1+Q_o-2Q)} = \sqrt{p}+\sqrt{r}$.

The full unitary with the signs of the last column chosen so that $U_{13}^2 + U_{23}^2 + U_{33}^2 = 1$.

$$U = \begin{pmatrix} \sqrt{p} & \frac{\sqrt{r}-\sqrt{p}Q_o}{\sqrt{1-Q_o^2}} & -\sqrt{\frac{Q}{1+Q_o}} \\ \sqrt{r} & \frac{\sqrt{p}-\sqrt{r}Q_o}{\sqrt{1-Q_o^2}} & -\sqrt{\frac{Q}{1+Q_o}} \\ \sqrt{Q} & \sqrt{\frac{Q(1-Q_o)}{1+Q_o}} & \frac{\sqrt{p}+\sqrt{r}}{\sqrt{1+Q_o}} \end{pmatrix} \tag{4.1.36}$$

In this presentation the above matrix can be shown that it satisfies all the unitary conditions.

The unitary in 4.1.36 interpolates between the optimal ME scheme and optimal UD with a fixed rate of inconclusive results Q. Setting the failure rate to zero, $Q = 0$, this collapses it into the unitary of optimal ME:

$$U_{ME} = \begin{pmatrix} \sqrt{p} & \frac{\sqrt{r}-\sqrt{p}Q_o}{\sqrt{1-Q_o^2}} & 0 \\ \sqrt{r} & \frac{\sqrt{p}-\sqrt{r}Q_o}{\sqrt{1-Q_o^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which can be simplified further by noticing $(U_{12})^2 = (\frac{\sqrt{r}-\sqrt{p}Q_o}{\sqrt{1-Q_o^2}})^2 = r$, similarly $(U_{22})^2 = (\frac{\sqrt{p}-\sqrt{r}Q_o}{\sqrt{1-Q_o^2}})^2 = p$ simplifying the unitary into:

$$U_{ME} = \begin{pmatrix} \sqrt{p} & \sqrt{r} & 0 \\ \sqrt{r} & -\sqrt{p} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Only one beam splitter in needed for optimal ME measurements.

On the other hand setting the error rate of the unitary 4.1.36 to zero gives the optimal UD unitary.

$$U_{UD} = \begin{pmatrix} \sqrt{p} & -\frac{\sqrt{p}Q_o}{\sqrt{1-Q_o^2}} & -\sqrt{\frac{Q_0}{1+Q_o}} \\ 0 & \frac{\sqrt{p}}{\sqrt{1-Q_o^2}} & -\sqrt{\frac{Q_o}{1+Q_o}} \\ \sqrt{Q_0} & \sqrt{\frac{Q_o(1-Q_o)}{1+Q_o}} & \sqrt{\frac{1-Q_o}{1+Q_o}} \end{pmatrix}$$

All three beamsplitters are still necessary for a general UD measurement. This is because the measurement is essentially two-step: in the first step we attempt to orthogonalize the states, and upon succeeding we perform a projective measurement.

Let us now express interpolative unitary in terms of three beamsplitters such as $U = M_1M_2M_3$. This ordering was derived using the Reck-Zeilinger algorithm

which says that any Unitary matrix can be expressed in terms of beamsplitters and phase shifters.

The beamsplitters can be written in terms of $\sin\omega_i$ and $\cos\omega_i$ as it is easy to check the unitarity condition, $\sin^2\omega_i + \cos^2\omega_i = 1$. Then the task is that of calculating ω_i.

$$M_1 = \begin{pmatrix} sin\omega_1 & cos\omega_1 & 0 \\ cos\omega_1 & -sin\omega_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} sin\omega_2 & 0 & cos\omega_2 \\ 0 & 1 & 0 \\ cos\omega_2 & 0 & -sin\omega_2 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & sin\omega_3 & cos\omega_3 \\ 0 & cos\omega_3 & -sin\omega_3 \end{pmatrix}$$

$$U = M_1M_2M_3 = \begin{pmatrix} \sqrt{p} & \frac{\sqrt{r}-\sqrt{p}Q_o}{\sqrt{1-Q_o^2}} & \sqrt{\frac{Q}{1+Q_o}} \\ \sqrt{r} & \frac{[\sqrt{p}-\sqrt{r}Q_o]}{\sqrt{1-Q_o^2}} & \sqrt{\frac{Q}{1+Q_o}} \\ \sqrt{Q} & \sqrt{\frac{Q(1-Q_o)}{1+Q_o}} & \sqrt{\frac{1+Q_o-2Q}{1+Q_o}} \end{pmatrix}$$

$$= \begin{pmatrix} sin\omega_1 sin\omega_2 & cos\omega_1 sin\omega_3 + sin\omega_1 cos\omega_2 cos\omega_3 & cos\omega_1 cos\omega_3 - sin\omega_1 cos\omega_2 sin\omega_3 \\ cos\omega_1 sin\omega_2 & -sin\omega_1 sin\omega_3 + cos\omega_1 cos\omega_2 cos\omega_3 & -sin\omega_1 cos\omega_3 - cos\omega_1 cos\omega_2 sin\omega_3 \\ cos\omega_2 & -sin\omega_2 cos\omega_3 & sin\omega_2 sin\omega_3 \end{pmatrix}$$

This gives nine equations and only three independent variables to be calculated. We can get all the elements by using just U_{31}, U_{32}, U_{21}

$\cos\omega_2 = U_{31} = \sqrt{Q}, sin\omega_2 = \sqrt{1-r^2} = \sqrt{1-Q}$

$cos\omega_3 = -\sqrt{\frac{Q(1-Q_o)}{(1+Q_o)(1-Q)}}, sin\omega_3 = \sqrt{\frac{1+Q_o-2Q}{(1-Q)(1+Q_o)}} = \frac{\sqrt{p}+\sqrt{r}}{\sqrt{(1-Q)(1+Q_o)}}$

$cos\omega_1 = \sqrt{\frac{r}{1-Q}}, sin\omega_1 = \sqrt{\frac{p}{1-Q}}$

The three beamsplitters with the proper coefficients of reflectivity and transmittance are:

$$M_1 = \begin{pmatrix} \sqrt{\frac{p}{1-Q}} & \sqrt{\frac{r}{1-Q}} & 0 \\ \sqrt{\frac{r}{1-Q}} & -\sqrt{\frac{p}{1-Q}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \sqrt{1-Q} & 0 & \sqrt{Q} \\ 0 & 1 & 0 \\ \sqrt{Q} & 0 & -\sqrt{1-Q} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1+Q_o-2Q}{(1-Q)(1+Q_o)}} & -\sqrt{\frac{Q(1-Q_o)}{(1+Q_o)(1-Q)}} \\ 0 & -\sqrt{\frac{Q(1-Q_o)}{(1+Q_o)(1-Q)}} & -\sqrt{\frac{1+Q_o-2Q}{(1-Q)(1+Q_o)}} \end{pmatrix}$$

By choosing the FRIO this matrix minimizes the error rate and maximizes the rate of success. Hence, by setting the FRIO to zero we obtain the setup to the minimum error problem on the other hand setting the error rate to zero gives the setup of the optimal unambiguous discrimination where the optimal inconclusive rate is the $Q_o = s$. This simplifies the works of the experimentalists because now they only need one setup and are not restrained to the extreme points.

4.1.7. General case: Interpolation with general priors. In this section we derive the beamsplitter coefficients necessary to interpolate minimum error measurements with a $FRIO$ when the input states are prepared with different a priori probabilities. The two input states to be discriminated are $|\psi_1\rangle_{in} = |1\rangle$ and $|\psi_2\rangle_{in} = \cos\theta|1\rangle + \sin\theta|2\rangle$

$$U|1\rangle = \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle + \sqrt{q_1}|3\rangle \tag{4.1.37}$$

$$U(\cos\theta|1\rangle + \sin\theta|2\rangle) = \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle + \sqrt{q_2}|3\rangle \tag{4.1.38}$$

The error and success rates where calculated in Section 4.1.4

$$r_i = \frac{1}{2}[(1 - \frac{Q}{2\eta_i}) - \frac{(1 - \frac{Q}{2\eta_i})(1 - Q) - \frac{1}{2\eta_i}(Q_o - Q)^2}{\sqrt{(1 - Q)^2 - (Q - Q_o)^2}}], \tag{4.1.39}$$

$$p_i = \frac{1}{2}[(1 - \frac{Q}{2\eta_i}) + \frac{(1 - \frac{Q}{2\eta_i})(1 - Q) - \frac{1}{2\eta_i}(Q_o - Q)^2}{\sqrt{(1 - Q)^2 - (Q - Q_o)^2}}]. \tag{4.1.40}$$

From these two equations we can read out six of nine elements of the three by three Unitary matrix. Multiplying 4.1.37 on the left hand side by $\{\langle 1|, \langle 2|, \langle 3|\}$ will give the elements U_{i1}, $i = 1, 2, 3$. Similarly we can obtain three more elements.

The first column is:

$\langle 1|U|1\rangle = U_{11} = \sqrt{p_1}$

$\langle 2|U|1\rangle = U_{21} = \sqrt{q_1}$

$\langle 3|U|1\rangle = U_{31} = \sqrt{r_1}$

Second Column:

$\cos\theta\langle 1|U|1\rangle + \sin\theta\langle 1|U|2\rangle = \cos\theta U_{11} + \sin\theta U_{12} = \sqrt{r_2}$

$\cos\theta\langle 2|U|1\rangle + \sin\theta\langle 2|U|2\rangle = \cos\theta U_{21} + \sin\theta U_{22} = \sqrt{p_2}$

$\cos\theta\langle 3|U|1\rangle + \sin\theta\langle 3|U|2\rangle = \cos\theta U_{31} + \sin\theta U_{32} = \sqrt{q_2}$

Using the solution of the first column U_{i1} ,gives the entries to second column U_{i2} :

$\cos\theta\sqrt{p}_1 + \sin\theta U_{12} = \sqrt{r_2} \Rightarrow U_{12} = \frac{\sqrt{r_2} - \sqrt{p_1}\cos\theta}{\sin\theta}$

$\cos\theta\sqrt{q_1} + \sin\theta U_{22} = \sqrt{p_2} \Rightarrow U_{22} = \frac{\sqrt{p_2} - \sqrt{r_1}\cos\theta}{\sin\theta}$

$\cos\theta\sqrt{r_1} + \sin\theta U_{32} = \sqrt{q_2} \Rightarrow U_{32} = \frac{\sqrt{q_2} - \sqrt{q_1}\cos\theta}{\sin\theta}$

The last column be calculated from the conditions of the unitarity, $U^T U = I$, $U_{i1}^2 + U_{i2}^2 + U_{i3}^2 = 1$ where $i = 1, 2, 3$

$U_{13} = \pm\sqrt{1 - U_{11}^2 - U_{12}^2} = \pm\sqrt{1 - p_1 - \frac{r_2 + p_1\cos\theta - 2\sqrt{p_1 r_2}\cos\theta}{\sin^2\theta}} = \pm\frac{\sqrt{\sin^2\theta - p_1 - r_2 + 2\sqrt{p_1 r_2}\cos\theta}}{\sin\theta}$

Similarly:

$U_{23} = \pm\frac{\sqrt{\sin^2\theta - r_1 - p_2 + 2\sqrt{p_2 r_1}\cos\theta}}{\sin\theta}$

$U_{33} = \pm\frac{\sqrt{sin^2\theta - q_1 - q_2 + 2\sqrt{q_1 q_2}\cos\theta}}{\sin\theta}$

Now that all the elements have been calculated the Unitary is:

$$U = \begin{pmatrix} \sqrt{p_1} & \frac{\sqrt{r_2}-\sqrt{p_1}\cos\theta}{\sin\theta} & -\frac{\sqrt{\sin^2\theta - p_1 - r_2 + 2\sqrt{p_1 r_2}\cos\theta}}{\sin\theta} \\ \sqrt{r_1} & \frac{\sqrt{p_2}-\sqrt{r_1}\cos\theta}{\sin\theta} & -\frac{\sqrt{\sin^2\theta - r_1 - p_2 + 2\sqrt{p_2 r_1}\cos\theta}}{\sin\theta} \\ \sqrt{q_1} & \frac{\sqrt{q_2}-\sqrt{q_1}\cos\theta}{\sin\theta} & +\frac{\sqrt{sin^2\theta - q_1 - q_2 + 2\sqrt{q_1 q_2}\cos\theta}}{\sin\theta} \end{pmatrix}. \tag{4.1.41}$$

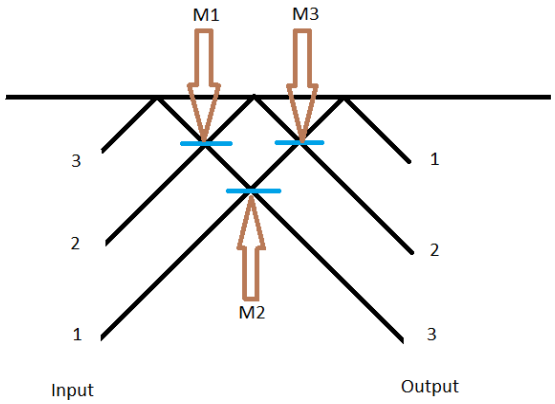
It is worth mentioning that all equations in this section referencing r_i and p_i are using the optimal values 4.1.30 and 4.1.31 derived in the previous section.

Now that we have a full unitary matrix we want to express it in terms of linear optical devices. Again we use the Reck-Zeilinger algorithm to decompose the unitary in terms of beamsplitters. The operator U is decomposed into beamsplitters in the order of $U = M_1 \cdot M_2 \cdot M_3$, and no phase shifters are needed:

$$M_1 = \begin{pmatrix} \sin\omega_1 & \cos\omega_1 & 0 \\ \cos\omega_1 & -\sin\omega_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \sin\omega_2 & 0 & \cos\omega_2 \\ 0 & 1 & 0 \\ \cos\omega_2 & 0 & -\sin\omega_2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\omega_3 & \cos\omega_3 \\ 0 & \cos\omega_3 & -\sin\omega_3 \end{pmatrix},$$



where the coefficients of reflectivity and transmittance are given by $\sqrt{R_i} = \sin\omega_i$ and $\sqrt{T_i} = \cos\omega_i$.

$$U = M_1M_2M_3 = \begin{pmatrix} \sqrt{p_1} & \frac{\sqrt{r_2}-\sqrt{p_1}cos\theta}{sin\theta} & \pm\frac{\sqrt{sin^2\theta-p_1-r_2+2\sqrt{p_1r_2}cos\theta}}{sin^2\theta} \\ \sqrt{r_1} & \frac{\sqrt{p_2}-\sqrt{r_1}cos\theta}{sin\theta} & \pm\frac{\sqrt{sin^2\theta-r_1-p_2+2\sqrt{p_2r_1}cos\theta}}{sin^2\theta} \\ \sqrt{q_1} & \frac{\sqrt{q_2}-\sqrt{q_1}cos\theta}{sin\theta} & \pm\frac{\sqrt{sin^2\theta-q_1-q_2+2\sqrt{q_1q_2}cos\theta}}{sin^2\theta} \end{pmatrix} =$$

$$\begin{pmatrix} sin\omega_1 sin\omega_2 & cos\omega_1 sin\omega_3 + sin\omega_1 cos\omega_2 cos\omega_3 & cos\omega_1 cos\omega_3 - sin\omega_1 cos\omega_2 sin\omega_3 \\ cos\omega_1 sin\omega_2 & -sin\omega_1 sin\omega_3 + cos\omega_1 cos\omega_2 cos\omega_3 & -sin\omega_1 cos\omega_3 - cos\omega_1 cos\omega_2 sin\omega_3 \\ cos\omega_2 & -sin\omega_2 cos\omega_3 & sin\omega_2 sin\omega_3 \end{pmatrix}$$

The coefficients of reflectivity and transmittance can be calculated by matching the corresponding elements of the unitary and it's decomposition. We can get all the elements by using just U_{31}, U_{32}, U_{21}

$U_{31} = \sqrt{q_1} = \cos\omega_2, \sin\omega_2 = \sqrt{1-\cos\omega_2^2} = \sqrt{1-q_1}$

$U_{32} = \frac{\sqrt{q_2}-\sqrt{q_1}cos\theta}{sin\theta} = -\sin\omega_2\cos\omega_3 = -\sqrt{1-q_1}\cos\omega_3 \Rightarrow \cos\omega_3 = -\frac{1}{\sqrt{1-q_1}}[\frac{\sqrt{q_2}-\sqrt{q_1}cos\theta}{sin\theta}]$

$\sin\omega_3 = \sqrt{1-\cos\omega_3} = -\frac{\sqrt{sin^2\theta-q_1-q_2+2\sqrt{q_1q_2}cos\theta}}{\sqrt{1-q_1}sin\theta}$

$U_{21} = \sqrt{r_1} = \cos\omega_1\sin\omega_2 = \cos\omega_1\sqrt{1-q_1} \Rightarrow \cos\omega_1 = \sqrt{\frac{r_1}{1-q_1}}$

$\sin\omega_1 = \sqrt{1-\cos\omega_1} = \sqrt{\frac{p_1}{1-q_1}}$

Substituting the coefficients of reflectivity and transmittance the beamsplitters are:

$$M_1 = \begin{pmatrix} \sqrt{\frac{p_1}{1-q_1}} & \sqrt{\frac{r_1}{1-q_1}} & 0 \\ \sqrt{\frac{r_1}{1-q_1}} & -\sqrt{\frac{p_1}{1-q_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} \sqrt{1-q_1} & 0 & \sqrt{q_1} \\ 0 & 1 & 0 \\ \sqrt{q_1} & 0 & -\sqrt{1-q_1} \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sqrt{sin^2\theta-q_1-q_2+2\sqrt{q_1q_2}cos\theta}}{\sqrt{1-q_1}sin\theta} \\ 0 & -\frac{1}{\sqrt{1-q_1}}[\frac{\sqrt{q_2}-\sqrt{q_1}cos\theta}{sin\theta}] \end{pmatrix} -$$

All the coefficients can be expressed in terms of the $FRIO$. Using the optimal relationship between the individual failure rates $\eta_1 q_1 = \eta_2 q_2 = Q/2$,$q_1 = Q/2\eta_1, q_2 = Q/2\eta_2$ and the above expressions of success and error rates.

$\cos\omega_1 = \sqrt{\frac{r_1}{1-Q/2\eta_1}}$, $\sin\omega_1 = \sqrt{\frac{p_1}{1-Q/2\eta_1}}$

$\cos\omega_2 = \sqrt{Q/2\eta_1}$, $\sin\omega_2 = \sqrt{1-Q/2\eta_1}$

$\cos\omega_3 = -\frac{\sqrt{Q/2\eta_2}-Q_o/2\eta_1\sqrt{Q/2\eta_2}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}}$, $\sin\omega_3 = \frac{\sqrt{1-Q_o^2/4\eta_1\eta_2-Q/(2\eta_1\eta_2)+QQ_o/(2\eta_1\eta_2)}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}}$

.

.

$$M_1 = \begin{pmatrix} \sqrt{\frac{p_1}{1-Q/2\eta_1}} & \sqrt{\frac{r_1}{1-Q/2\eta_1}} & 0 \\ \sqrt{\frac{r_1}{1-Q/2\eta_1}} & -\sqrt{\frac{p_1}{1-Q/2\eta_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \sqrt{1-Q/2\eta_1} & 0 & \sqrt{Q/2\eta_1} \\ 0 & 1 & 0 \\ \sqrt{Q/2\eta_1} & 0 & -\sqrt{1-Q/2\eta_1} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{1-Q_o^2/4\eta_1\eta_2-Q/(2\eta_1\eta_2)+QQ_o/(2\eta_1\eta_2)}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}} & -\frac{\sqrt{Q/2\eta_2}-Q_o/2\eta_1\sqrt{Q/2\eta_2}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}} \\ 0 & -\frac{\sqrt{Q/2\eta_2}-Q_o/2\eta_1\sqrt{Q/2\eta_2}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}} & -\frac{\sqrt{1-Q_o^2/4\eta_1\eta_2-Q/(2\eta_1\eta_2)+QQ_o/(2\eta_1\eta_2)}}{\sqrt{(1-Q/2\eta_1)(1-Q_o^2/4\eta_1\eta_2)}} \end{pmatrix}.$$

Let us now check the bounds of the general unitary matrix for equal priors to see if it reproduces the unitary in 4.1.36. Indeed, everything checks out and the equal priors unitary matrix is reproduced:

$$U = \begin{pmatrix} \sqrt{p} & \frac{\sqrt{r}-\sqrt{p}Q_o}{\sqrt{1-Q_o^2}} & \sqrt{\frac{Q}{1+Q_o}} \\ \sqrt{r} & \frac{[\sqrt{p}-\sqrt{r}Q_o]}{\sqrt{1-Q_o^2}} & \sqrt{\frac{Q}{1+Q_o}} \\ \sqrt{Q} & \sqrt{\frac{Q(1-Q_o)}{1+Q_o}} & -\frac{\sqrt{p}+\sqrt{r}}{\sqrt{1+Q_o}} \end{pmatrix}, \tag{4.1.42}$$

4.1.8. Minimum Error. Solution to the minimum error problem was derived in Chapter II. Here we use the explicit solution to error and success rates to construct the unitary operator which in turn will be expressed in terms of

beamsplitters. We start with the Neumark setup where a unitary operator entangles the input states with the ancilla. Let the input states be $|\psi_1\rangle_{in} = |1\rangle$ and $|\psi_2\rangle_{in} = \cos\theta|1\rangle + \sin\theta|2\rangle$

$$U|1\rangle_{in}|0\rangle = \sqrt{p_1}|1\rangle + \sqrt{r_1}|2\rangle \tag{4.1.43}$$

$$U(\cos\theta|1\rangle + \sin\theta|2\rangle)|0\rangle = \sqrt{r_2}|1\rangle + \sqrt{p_2}|2\rangle \tag{4.1.44}$$

Where the error and success rates were calculated in chapter two:

$$r_i = \frac{1}{2}[1 - \frac{1-2\eta_j s^2}{\sqrt{1-4\eta_1\eta_2 s^2}}] \tag{4.1.45}$$

$$p_i = \frac{1}{2}[1 - \frac{1-2\eta_j s^2}{\sqrt{1-4\eta_1\eta_2 s^2}}] \tag{4.1.46}$$

To get the unitary elements multiply on the left by $\langle 1|$ and $\langle 2|$

First column is:

$\langle 1|U|1\rangle = U_{11} = \sqrt{p_1}$

$\langle 2|U|1\rangle = U_{21} = \sqrt{r_1}$

Using the elements of the first column the second column can be calculated and is:

$\cos\theta\langle 1|U|1\rangle + \sin\theta\langle 1|U|2\rangle = \cos\theta U_{11} + \sin\theta U_{12} = \cos\theta\sqrt{p_1} + \sin\theta U_{12} = \sqrt{r_2} \Rightarrow U_{12} = \frac{\sqrt{r_2} - \cos\theta\sqrt{p_1}}{\sin\theta}$

$\cos\theta\langle 2|U|1\rangle + \sin\theta\langle 2|U|2\rangle = \cos\theta U_{21} + \sin\theta U_{22} = \cos\theta\sqrt{r_1} + \sin\theta U_{22} = \sqrt{p_2} \Rightarrow U_{22} = \frac{\sqrt{p_2} - \cos\theta\sqrt{r_1}}{\sin\theta}$

The Unitary matrix is:

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{p_1} & \frac{\sqrt{r_2} - \cos\theta\sqrt{p_1}}{\sin\theta} \\ \sqrt{r_1} & \frac{\sqrt{p_2} - \sin\theta\sqrt{r_1}}{\sin\theta} \end{pmatrix} \tag{4.1.47}$$

Only one beam splitter is necessary.

4.1.9. summary. We derived the optimal rate of error for a fixed failure rate when discriminating between two pure states with fixed a-priori probabilities. Along the way we found expressions for the individual error rates. Then we created an experimental implementation of this procedure using the six-rail representation, and found that three beam-splitters are sufficient to perform this experiment.

Lagrange Multipliers

In our works we have relied quite heavily in the Lagrange multipliers when optimizing a function which was under the restriction of a constraint. We now show why it works. The method can be applied to a function of any number of variable but it can be more clearly explained in two variables. Suppose that we need to find the stationary points of a function $f(x, y)$, where x and y are the two variables, subject to the constraint $g(x, y) = 0$. If the constraint is simple then we can solve for x in terms of y, plug it into the function then solve $\partial f/\partial y = 0$. However for slightly more complicated constraint this can easily lead to a very high order equation with cannot be solved analytically. In the case of exact cloning, doing just so leads to a sixth order equation which is of little use.

To find the stationary points of a function of two variable such as $f(x, y)$, we could just take the total differentia df and set it to zero

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \tag{4.1.48}$$

which leads to two conditions:

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \tag{4.1.49}$$

However there is a constraint which means that the differentials dx and dy are not independent, they are related to the total differential of g by:

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \tag{4.1.50}$$

Multiplying 4.1.50 by the Lagrange parameter λ and adding it to 4.1.48 we get

$$d(f+\lambda g) = (\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x})dx + (\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y})dy \tag{4.1.51}$$

This equation can be satisfied by choosing the Lagrange multiplier λ such that the following two conditions are satisfied:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \tag{4.1.52}$$

and:

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \tag{4.1.53}$$

To get the stationary points of $f(x, y)$ follow this procedure:

- Solve the two equations:4.1.52 and 4.1.53 in terms of λ, $x(\lambda)$ and $y(\lambda)$
- Plug $x(\lambda)$ and $y(\lambda)$ into the constraint $g(x, y)$
- Solve for λ
- Plug the value of λ into $x(\lambda)$ and $y(\lambda)$
- Plug $x(\lambda)$ and $y(\lambda)$ into the function which was to be optimized $f(x, y)$

Now that we have seen how the Lagrange multipliers method works, we can simplify the procedure by optimizing the function following function:

$$F(x, y) = f(x, y) + \lambda g(x, y) \tag{4.1.54}$$

with respect to the the independent variable x and y. Differentiating 4.1.54 with respect to x and y, we obtain equations 4.1.52 and 4.1.53. The rest of the procedure is the same. This is the exact procedure we used for our works, for example in optimizing the error rate $P_E(r_1, r_2)$ with one constraint $s(r_1, r_2)$.

A more general procedure involving a function of n variables, $f(x_1, x_2, ..., x_n)$, and j constrains goes in similar lines. We start by setting up 4.1.54

$$F(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) + \sum_{\lambda_j=1}^{m} \lambda_j g_j(x_1, x_2, ..., x_n). \tag{4.1.55}$$

Optimizing the function $f(x_1, x_2, ..., x_n)$ we differentiate with respect to all the independent variables $(x_1, x_2, ..., x_n)$ and follow the procedure defined above.

Reck-Zeilinger Algorithm

In their letter **[5]** prove that any discrete finite-dimensional unitary operator can be constructed using optical devices only. Then they provide a general algorithm which decomposes any N X N unitary matrix into a product of two-dimensional $U(2)$ transformations which can be expressed as beam splitters, phase shifters and mirrors. This optical multi-port can act upon various fields such as electrons, neutrons, atoms, photons etc. The authors decide to work with photons purely for convenience and widespread availability of high power lasers. It is this very proof which allows us to implement our various works in state discrimination and cloning. In addition the proof has greatly simplified the experimental realizations of many quantum computation, quantum information and quantum cryptography schemes. Besides these very practical applications it has also answered a long standing question: Does an experiment measuring the variables corresponding to any arbitrary Hermitian operator exists? They show that indeed an experimental realization does exist for an arbitrary operator in a finite dimensional Hilbert space.

It has long been known that a lossless beam splitter and a phase shifter can be implement any $U(2)$ transformation: a beam splitter and a phase shifter at one output port transforms the input operators into output operators as

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} e^{i\phi}\sin\omega & e^{i\phi}\cos\omega \\ \cos\omega & -\sin\omega \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{4.1.56}$$

where, ϕ is the phase shifter which can be realized as an external phase shifter after the beam splitter, ω represents the transmittance and reflectivity coefficient, $\sqrt{T} = \cos\omega$, $\sqrt{R} = \sin\omega$. In their Letter Reck, *et al.* considered the use of a Mach-Zehner interferometer to simulate the effect of a beam splitter which splits

the incoming beam according to the given parameters of transmittance and reflectivity. For an actual two by two beam splitter the coefficients of transmittance and reflectivity should be $\sqrt{T} = \sin\omega$ and $\sqrt{R} = \sin\theta$.

The authors show that starting with an $N \times N$ unitary matrix $U(N)$, it can be expressed into a succession of two-dimensional matrices which correspond to beam splitters and phase shifters. Hence the $U(N)$ unitary matrix can be realized in the full N dimensional Hilbert space through a succession of two-dimensional $U(2)$ matrices.

The order in which the matrices are multiplied correspond to the sequence in which the beamsplitters are set up. The task of realizing the experimental setup of an arbitrary unitary matrix becomes that of factorizing the matrix in terms of two dimensional beam splitter matrices with phase shifters which can be realized after the beam splitters.

Define an $N-$dimensional identity matrix T_{pq} which multiplies the N dimensional unitary matrix from the right to reduce the dimensionality to $N-1$. In the identity matrix T_{pq} the elements $I_{pq},\ I_{pp},\ I_{qp},\ I_{qq}$ are replaced by the corresponding beam splitter matrix elements $(\cos\omega, \sin\omega)$. Thus:

$$U(N) \times T_{N,N-1} \times T_{N,N-2} \times ...T_{N,1} = \begin{pmatrix} U(N-1) & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{4.1.57}$$

This reduces the dimensionality of $U(N)$ to $U(N-1)$. The process is repeated again until all the off diagonal elements of the original unitary matrix are zero.

$$U(N)\cdot T_{N,N-1}\cdot T_{N,N-2}\cdot ...T_{N,1}\cdot T_{N-1,N-2}\cdot T_{N-2,N-2}\cdot ...T_{2,1}\cdot ...T_{2,1} = \begin{pmatrix} e^{i\alpha_1} & 0 & .. & 0 \\ \vdots & e^{i\alpha_2} & & \\ & & \ddots & \\ 0 & \cdots & & e^{i\alpha N} \end{pmatrix} \tag{4.1.58}$$

Let:

$$D = \begin{pmatrix} e^{-i\alpha_1} & 0 & .. & 0 \\ \vdots & e^{-i\alpha_2} & & \\ & & \ddots & \\ 0 & \cdots & & e^{-i\alpha N} \end{pmatrix} \tag{4.1.59}$$

we have

$$U(N) \cdot T_{N,N-1} \cdot T_{N,N-2} \cdot ...T_{N,1} \cdot T_{N-1,N-2} \cdot T_{N-2,N-2} \cdot ...T_{2,1} \cdot D = I \tag{4.1.60}$$

The unitary matrix can be expressed in terms of Tp, q and D :

$$U(N) = D^{-1} \cdot T_{2,1}.... \cdot T^{-1}_{N-2,N-2} \cdot T^{-1}_{N-1,N-2} \cdot T^{-1}_{N,1}.... \cdot T^{-1}_{N,N-2} \cdot T^{-1}_{N,N-1} \tag{4.1.61}$$

Since the product of matrices represents the order of which the beam splitters are set up, then 4.1.61 is all one needs to implements a finite dimensional unitary matrix. Since this algorithm is recursive, it can factorize any finite dimensional unitary operator. For example a 3×3 unitary matrix, three beam splitters are needed T_{21}, T_{31}, T_{32}, a 4×4 unitary matrix requires six beamsplitters $T_{4,3}, T_{4,2}, T_{4,1}, T_{32}, T_{31}, T_{21}$ in reversed order. In general the maximum number of beam splitters required for any N dimensional unitary operator is $\begin{pmatrix} N \\ 2 \end{pmatrix} = \frac{N(N-1)}{2}$. In practice this method involves a triangular array of beamsplitters, with each diagonal row effectively reducing the dimension of the Hilbert space by one.

Let us now give an example to see explicitly how this algorithm works. For a three dimensional unitary operator $U(3)$, the algorithm in 4.1.61 gives:

$$U(3) = D^{-1} \times T^{-1}_{2,1} \times T^{-1}_{3,1} \times T^{-1}_{3,2} \tag{4.1.62}$$

where:

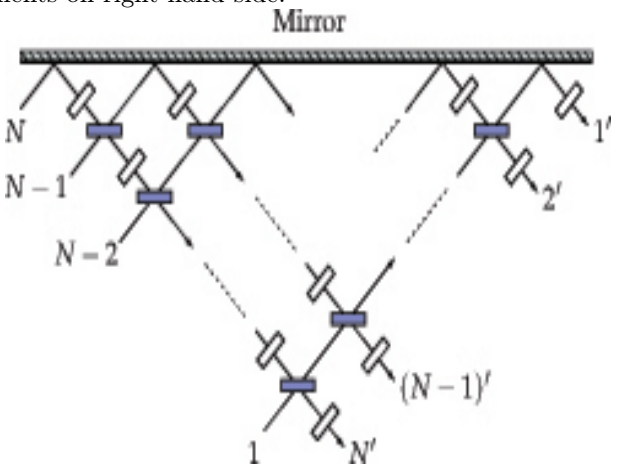
$$D^{-1} = \begin{pmatrix} e^{i\alpha_1} & 0 & .. & 0 \\ \vdots & e^{i\alpha_2} & & \\ & & \ddots & \\ 0 & \cdots & & e^{i\alpha N} \end{pmatrix}, \; T_{21} = \begin{pmatrix} \sin\omega_1 & \cos\omega_1 & 0 \\ \cos\omega_1 & -\sin\omega_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T_{31} = \begin{pmatrix} \sin\omega_2 & 0 & \cos\omega_2 \\ 0 & 1 & 0 \\ \cos\omega_2 & 0 & -\sin\omega_2 \end{pmatrix}, T_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\omega_3 & \cos\omega_3 \\ 0 & \cos\omega_3 & -\sin\omega_3 \end{pmatrix}.$$

Now plug this 2×2 matrices into 4.1.62.

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} = \begin{pmatrix} e^{i\alpha_1} & 0 & .. & 0 \\ \vdots & e^{i\alpha_2} & & \\ & & \ddots & \\ 0 & \cdots & & e^{i\alpha N} \end{pmatrix} \times \begin{pmatrix} sin\omega_1 sin\omega_2 & cos\omega_1 sin\omega_3 + sin\omega_1 cos\omega_2 cos\omega_3 & cos\omega_1 cos\omega \\ cos\omega_1 sin\omega_2 & -sin\omega_1 sin\omega_3 + cos\omega_1 cos\omega_2 cos\omega_3 & -sin\omega_1 cos \\ cos\omega_2 & -sin\omega_2 cos\omega_3 & \end{pmatrix} \tag{4.1.63}$$

To get the elements of the we match the corresponding entries U_{ij} with the elements on right hand side.



Quantum Measurements

In the dissertation we used various quantum measurements schemes to read out information out of a quantum system. For a more thorough understanding of quantum theory of measurements we go along the lines of the review paper by J.A Bergou [**6**]. Starting with the standard quantum measurement theory due essentially to von Neumann the generalized measurements (Positive Operator Valued Measures, POVMs) are introduced as more useful measurements measurement schemes in optimization problems. Using Neumark's theorem the POVMs can be realized experimentally.

4.1.10. Standard Quantum Measurements. .

.

.

Bibliography

[1] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned, 1982.

[2] Anthony Chefles and Stephen Barnett. Strategies and networks for state-dependent quantum cloning. *Physical Review A*, 60(1):136–144, July 1999.

[3] Lu-Ming Duan and Guang-Can Guo. Probabilistic Cloning and Identification of Linearly Independent Quantum States. *Physical Review Letters*, 80(22):4999–5002, June 1998.

[4] E. Bagan, R. Muñoz Tapia, G. a. Olivares-Rentería, and J. a. Bergou. Optimal discrimination of quantum states with a fixed rate of inconclusive outcomes. *Physical Review A - Atomic, Molecular, and Optical Physics*, 86(4):1–5, 2012.

[5] Michael Reck and Anton Zeilinger. Experimental realization of any discrete unitary operator, 1994.

[6] Janos A. Bergou. Discrimination of Quantum States with Selected Applications. In *Conference on Coherence and Quantum Optics*, page CMF4, Washington, D.C., 2007. OSA.