# Minimax Optimal Fixed-Budget Best Arm Identification in Linear Bandits


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## Linear Bandits

- An arm set $\mathcal{A} = [K]$, which corresponds to arm vectors
$$\{a(1), a(2), \ldots, a(K)\} \subset \mathbb{R}^d.$$
- At each time $t$, the agent chooses an arm $A_t$ from the arm set $\mathcal{A}$ and then observes a noisy reward
$$X_t = \langle \theta^*, a(A_t) \rangle + \eta_t,$$
where $\theta^* \in \mathbb{R}^d$ is the unknown parameter vector and $\eta_t$ is independent zero-mean 1-subgaussian random noise.

## Best Arm Identification in the Fixed-Budget Setting

- To maximize the probability of identifying the best arm with no more than $T$ arm pulls.
- The agent uses an online algorithm $\pi$ to decide the arm $A_t$ to pull at each time step $t$, and the arm $i_{\text{out}} \in \mathcal{A}$ to output as the identified best arm by time $T$.
- We seek to minimize the error probability
$$\Pr\left[ i_{\text{out}} \neq \arg\max_{j \in \mathcal{A}} \langle \theta^*, a(j) \rangle \right].$$

## Notations

- For any arm $i \in \mathcal{A}$, let $p(i) = \langle \theta^*, a(i) \rangle$ denote the expected reward.
- Assume that $p(1) > p(2) \geq \cdots \geq p(K)$.
- For any suboptimal arm $i$, we denote $\Delta_i = p(1) - p(i)$ as the optimality gap. Also set $\Delta_1 = \Delta_2$.

## Hardness Quantities

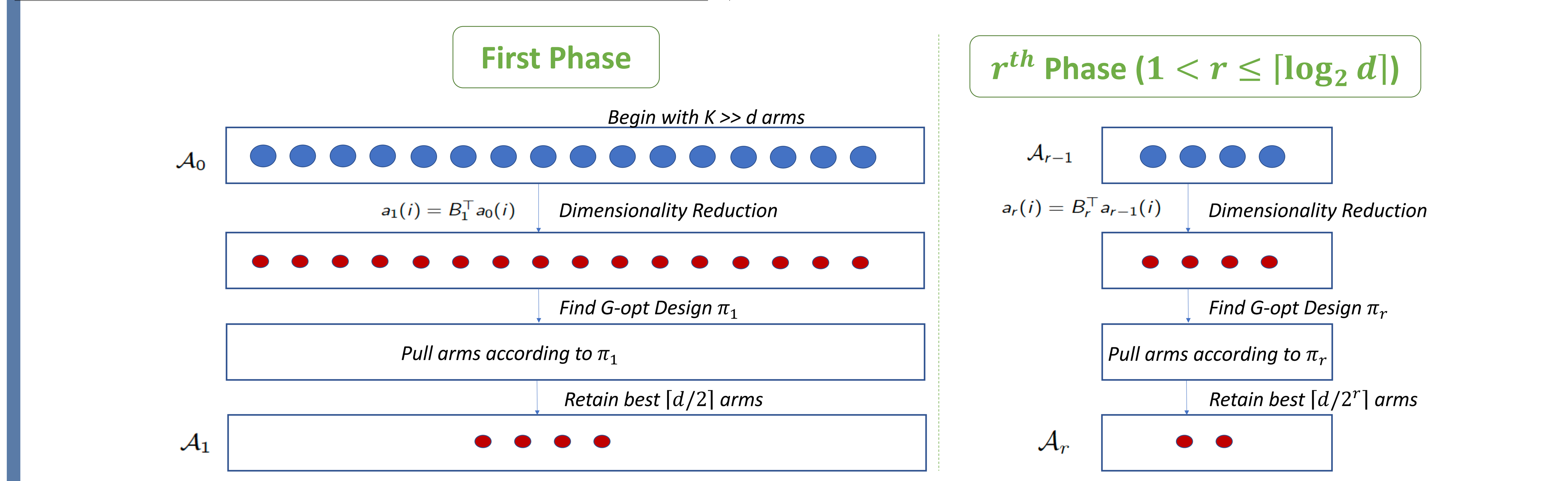
| | | |
|---|---|---|
| $H_1 = \sum_{1 \leq i \leq K} \Delta_i^{-2}$ | $H_2 = \max_{2 \leq i \leq K} \frac{i}{\Delta_i^2}$ | $1 \leq \frac{H_1}{H_2} \leq \log(2K)$ |
| $H_{1,\text{lin}} = \sum_{1 \leq i \leq d} \Delta_i^{-2}$ | $H_{2,\text{lin}} = \max_{2 \leq i \leq d} \frac{i}{\Delta_i^2}$ | $1 \leq \frac{H_{1,\text{lin}}}{H_{2,\text{lin}}} \leq \log(2d)$ |
| $1 \leq \frac{H_1}{H_{1,\text{lin}}} \leq \frac{K}{d}$ | $1 \leq \frac{H_2}{H_{2,\text{lin}}} \leq \frac{K}{d}$ | |

## Algorithm: Optimal Design-based Linear Best Arm Identification (OD-LinBAI)

**Input:** time budget $T$, arm set $\mathcal{A} = [K]$ and arm vectors $\{a(1), a(2), \ldots, a(K)\} \subset \mathbb{R}^d$.

```
Initialize t_0 = 1, A_0 ← A and d_0 = d.
For each arm i ∈ A_0, set a_0(i) = a(i).
Calculate m = Θ(T / log_2 d).
for r = 1 to ⌈log_2 d⌉ do
    Set d_r = dim(span({a_{r-1}(i) : i ∈ A_{r-1}})).
    if d_r = d_{r-1} then
        For each arm i ∈ A_{r-1}, set a_r(i) = a_{r-1}(i).
    else
        Find matrix B_r ∈ R^{d_{r-1} × d_r} whose columns form a orthonormal basis of the subspace spanned by {a_{r-1}(i) : i ∈ A_{r-1}}.
        For each arm i ∈ A_{r-1}, set a_r(i) = B_r^T a_{r-1}(i).
    end if
    if r = 1 then
        Find a G-optimal design π_r : {a_r(i) : i ∈ A_{r-1}} → [0,1] with |Supp(π_r)| ≤ d(d+1)/2.
    else
        Find a G-optimal design π_r : {a_r(i) : i ∈ A_{r-1}} → [0,1].
    end if
    Set
        T_r(i) = ⌈π_r(a_r(i)) · m⌉   and   T_r = Σ_{i ∈ A_{r-1}} T_r(i).
    Choose each arm i ∈ A_{r-1} exactly T_r(i) times.
    Calculate the OLS estimator:
        θ̂_r = V_r^{-1} Σ_{t=t_r}^{t_r+T_r-1} a_r(A_t) X_t  with  V_r = Σ_{i ∈ A_{r-1}} T_r(i) a_r(i) a_r(i)^T.
    For each arm i ∈ A_{r-1}, estimate the expected reward:
        p̂_r(i) = ⟨θ̂_r, a_r(i)⟩.
    Let A_r be the set of ⌈d/2^r⌉ arms in A_{r-1} with the largest estimates of the expected rewards.
    Set t_{r+1} = t_r + T_r.
end for
```

**Output:** the only arm $i_{\text{out}}$ in $\mathcal{A}_{\lceil \log_2 d \rceil}$.




## Minimax Lower Bound

Let $\mathcal{E}(a)$ denote the set of linear bandit instances whose $H_{1,\text{lin}}$ is bounded by $a > 0$, i.e.,
$$\mathcal{E}(a) = \{\nu : H_{1,\text{lin}}(\nu) \leq a\}.$$

If $T \geq a^2 \log(6Td)/900$, then
$$\min_{\pi} \max_{\nu \in \mathcal{E}(a)} \Pr[i_{\text{out}}^{\pi} \neq 1] \geq \frac{1}{6} \exp\left( -\frac{240T}{a} \right).$$

Further if $a \geq 15d^2$, then
$$\min_{\pi} \max_{\nu \in \mathcal{E}(a)} \left( \Pr[i_{\text{out}}^{\pi} \neq 1] \cdot \exp\left( \frac{2700T}{H_{1,\text{lin}}(\nu) \log_2 d} \right) \right) \geq \frac{1}{6}.$$

## Error Probability of OD-LinBAI

For any linear bandit instance $\nu$, OD-LinBAI outputs an arm $i_{\text{out}}$ satisfying
$$\Pr[i_{\text{out}} \neq 1] \leq \left( \frac{4K}{d} + 3 \log_2 d \right) \exp\left( -\frac{m}{32 H_{2,\text{lin}}} \right).$$

## Minimax Optimality

As the time budget $T$ tends to infinity,

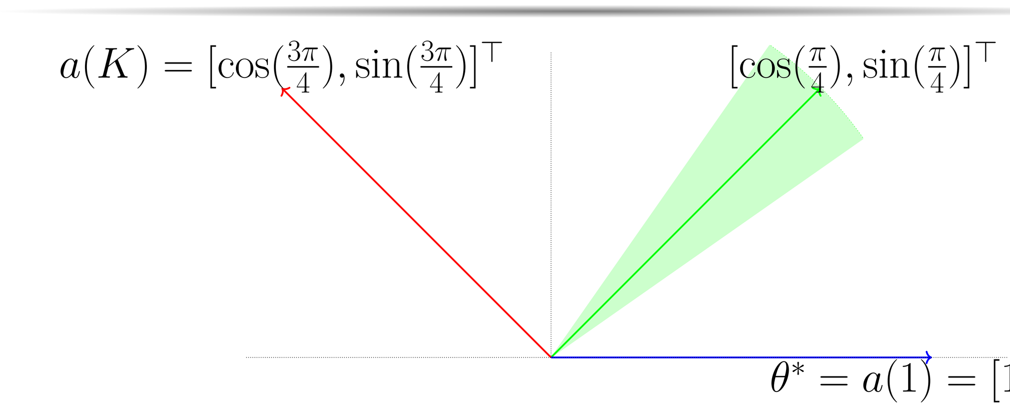
- Upper bound: $\exp\left( -\Omega\left( \frac{T}{H_{2,\text{lin}} \log_2 d} \right) \right)$
- Lower bound: $\exp\left( -O\left( \frac{T}{H_{1,\text{lin}} \log_2 d} \right) \right)$

$H_{2,\text{lin}} \leq H_{1,\text{lin}}$

## Comparison to Existing Art

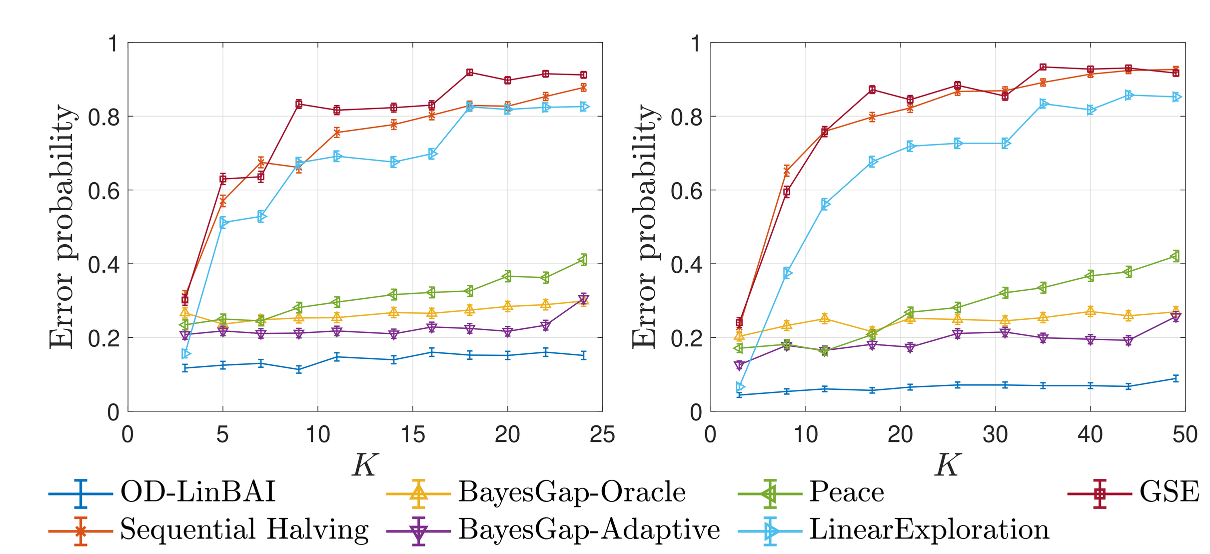
- BayesGap [1]:
  - Not parameter-free (require the knowledge of the instance)
  - Error probability: $\exp\left( -\Omega\left( \frac{T}{H_1} \right) \right)$
- Peace [2]:
  - Not parameter-free (require the knowledge of the instance)
  - Not minimax optimal
- LinearExploration [3]:
  - Error probability: $\exp\left( -\Omega\left( \frac{T}{\tilde{H}_2 \log_2 K} \right) \right)$
- GSE [4]:
  - Error probability: $\exp\left( -\Omega\left( \frac{T \Delta_1^2}{d \log_2 K} \right) \right)$

## Numerical Experiments




- One best arm, one worst arm and $K-2$ almost second best arms.
- $a(i) = [\cos(\pi/4 + \phi_i), \sin(\pi/4 + \phi_i)]^\top$ with $\phi_i \sim \mathcal{N}(0, 0.09^2)$ for $i = 2, 3, \ldots, K-1$

**Figure 1:** Results for different numbers of arms $K$ with $T = 25, 50$.




## References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**Full Paper is Available at:**

https://arxiv.org/abs/2105.13017