Recent advances in nonnegative matrix factorization

Part I: Generalities, optimization, regularization

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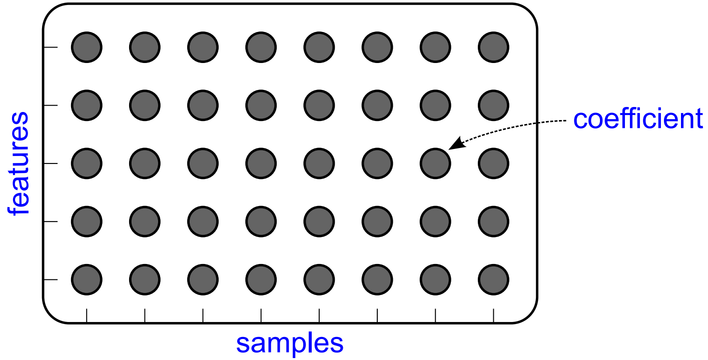
Outline

Matrix factorization models

Data often available in matrix form.



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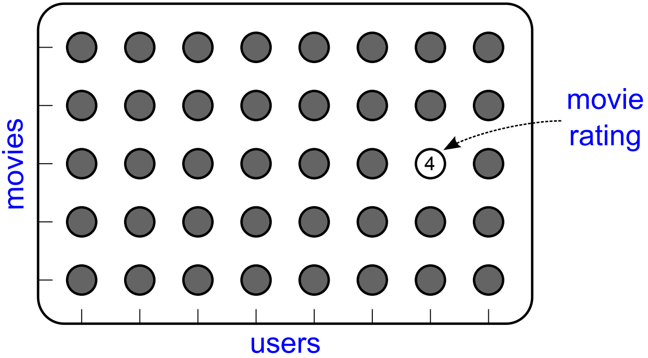

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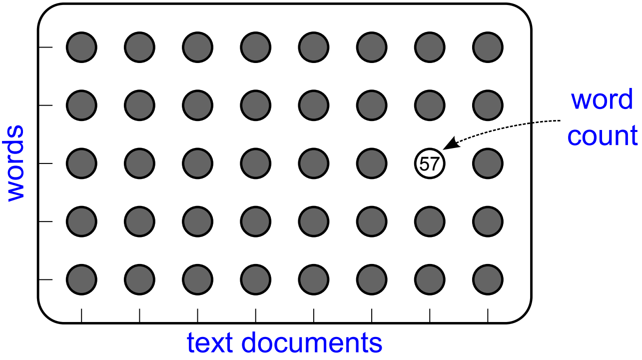

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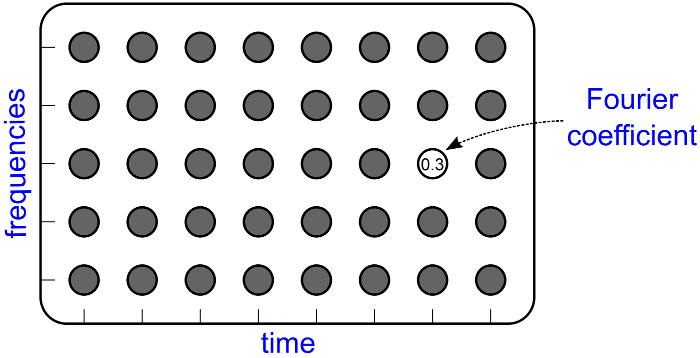
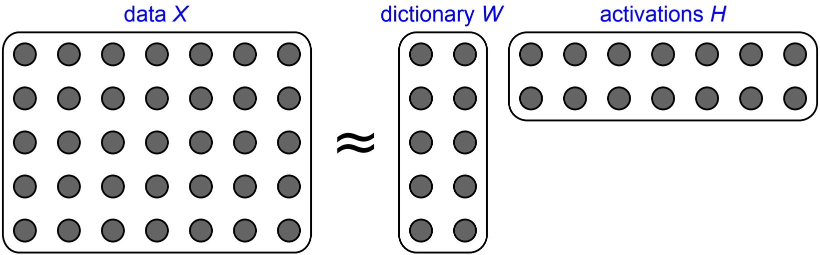


Matrix factorization models

$\approx$ **dictionary learning**
low-rank approximation
factor analysis
latent semantic analysis



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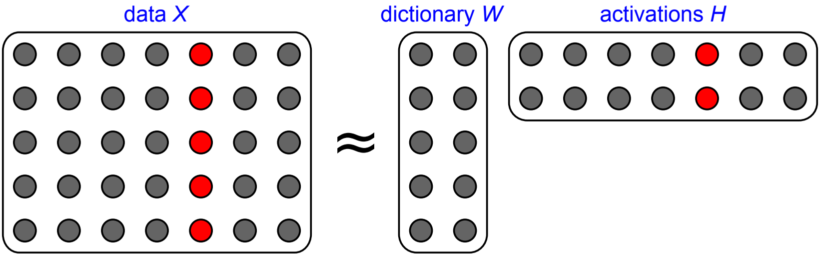
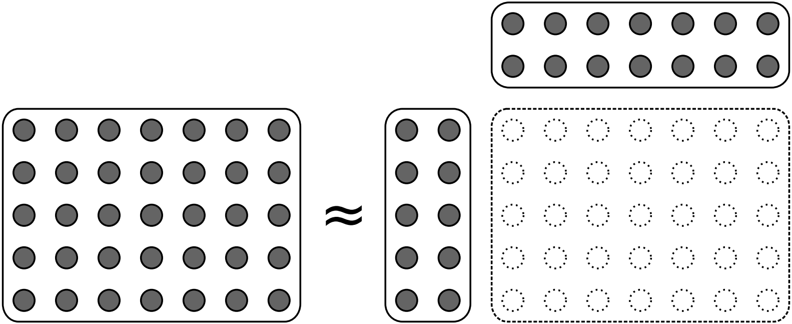


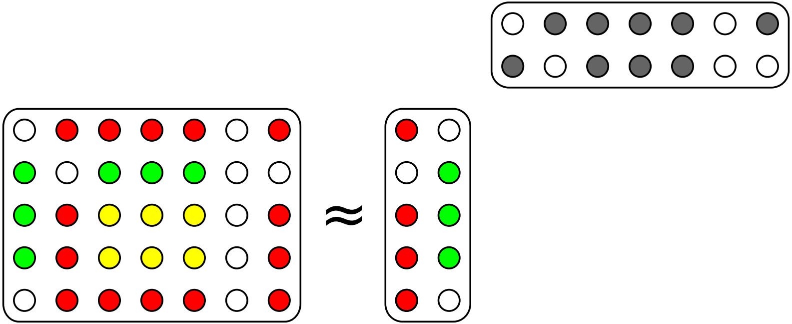
Matrix factorization models

for dimensionality reduction (coding, low-dimensional embedding)



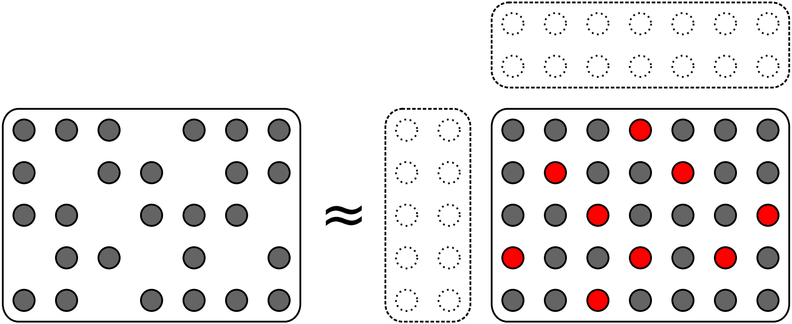
Matrix factorization models

for unmixing (source separation, latent topic discovery)



Matrix factorization models

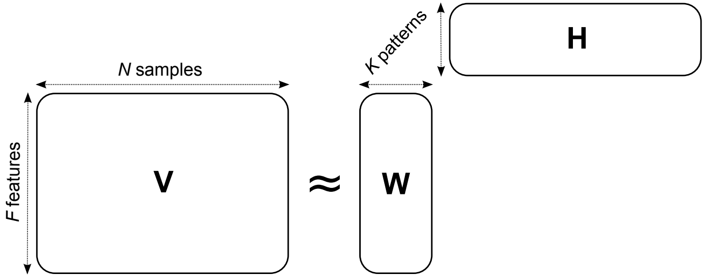
for interpolation (collaborative filtering, image inpainting)



Matrix factorization models

- simple generative & interpretable models, popular in unsupervised settings.
- used in many fields for a long time:
  - Principal component analysis PCA (Pearson, 1901)
  - Factor analysis (Spearman, 1904)
  - Latent semantic analysis LSA (Deerwester et al., 1988)
  - Independent component analysis ICA (Comon, 1994)
  - Nonnegative matrix factorization NMF (Lee & Seung, 1999)
  - Latent Dirichlet allocation LDA (Blei et al., 2003)
  - Sparse dictionary learning, e.g., K-SVD (Aharon et al., 2006)
- active topics:
  - design of nonconvex optimization algorithms with proven convergence
  - landscape analysis, search for global optima
  - conditions for identifiability
  - rank selection
  - probabilistic models & statistical approaches (e.g., integer-valued or binary data)

Nonnegative matrix factorization

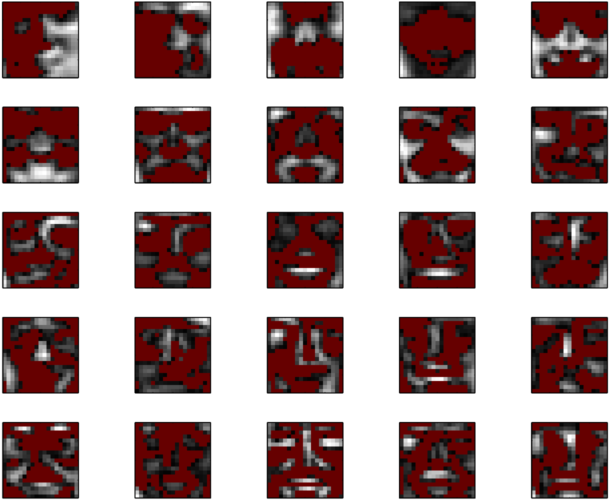


- data $\mathbf{V}$ and factors $\mathbf{W}$, $\mathbf{H}$ have nonnegative entries.
- nonnegativity of $\mathbf{W}$ ensures interpretability of the dictionary, because patterns $\mathbf{w}_k$ and samples $\mathbf{v}_n$ belong to the same space.
- nonnegativity of $\mathbf{H}$ tends to produce part-based representations, because subtractive combinations are forbidden.

Early work by (Paatero and Tapper, 1994), landmark *Nature* paper by (Lee and Seung, 1999)

49 images among 2429 from MIT's CBCL face dataset

PCA dictionary with $K = 25$



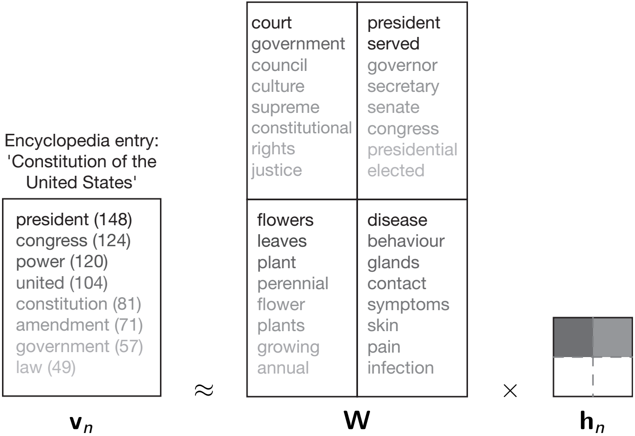
red pixels indicate negative values

NMF dictionary with $K = 25$

experiment reproduced from (Lee and Seung, 1999)

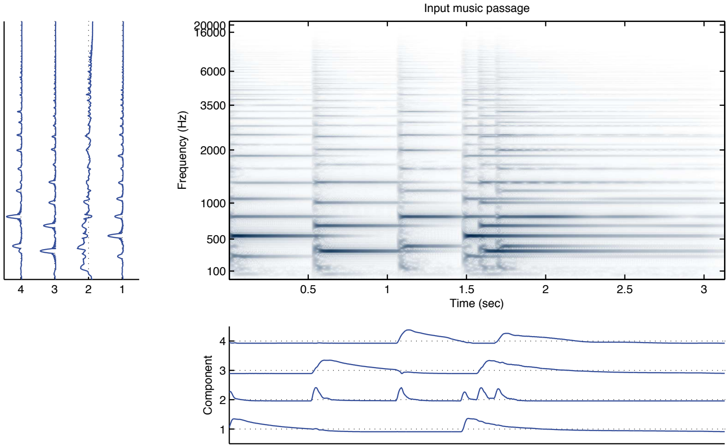
NMF for latent semantic analysis

(Lee and Seung, 1999; Hofmann, 1999)



reproduced from (Lee and Seung, 1999)

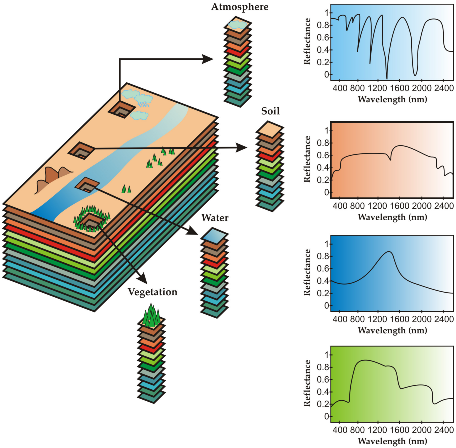
NMF for audio spectral unmixing
(Smaragdis and Brown, 2003)



reproduced from (Smaragdis, 2013)

NMF for hyperspectral unmixing

(Berry, Browne, Langville, Pauca, and Plemmons, 2007)



reproduced from (Bioucas-Dias et al., 2012)

Outline

NMF as a constrained minimization problem

Minimize a measure of fit between $\mathbf{V}$ and $\mathbf{WH}$, subject to nonnegativity:

$$\min_{\mathbf{W},\mathbf{H}\geq\mathbf{0}} D(\mathbf{V}|\mathbf{WH}) = \sum_{fn} d([\mathbf{V}]_{fn}|[\mathbf{WH}]_{fn}),$$

where $d(x|y)$ is a scalar cost function, e.g.,

- squared Euclidean distance (Paatero and Tapper, 1994; Lee and Seung, 2001)
- Kullback-Leibler divergence (Lee and Seung, 1999; Finesso and Spreij, 2006)
- Itakura-Saito divergence (Févotte, Bertin, and Durrieu, 2009)
- α-divergence (Cichocki et al., 2008)
- β-divergence (Cichocki et al., 2006; Févotte and Idier, 2011)
- Bregman divergences (Dhillon and Sra, 2005)
- and more in (Yang and Oja, 2011)

Regularization terms often added to $D(\mathbf{V}|\mathbf{WH})$ for sparsity, smoothness, etc.
Nonconvex problem.

Probabilistic models

- Let $\mathbf{V} \sim p(\mathbf{V}|\mathbf{WH})$ such that
  - $\mathrm{E}[\mathbf{V}|\mathbf{WH}] = \mathbf{WH}$
  - $p(\mathbf{V}|\mathbf{WH}) = \prod_{fn} p(v_{fn}|[\mathbf{WH}]_{fn})$
- then the following correspondences apply with

$$D(\mathbf{V}|\mathbf{WH}) = -\log p(\mathbf{V}|\mathbf{WH}) + \text{cst}$$

data support	distribution/noise	divergence	examples
real-valued	additive Gaussian	quadratic loss	many
integer	multinomial*	weighted KL	word counts
integer	Poisson	generalized KL	photon counts
nonnegative	multiplicative Gamma	Itakura-Saito	spectrogram
generally nonnegative	Tweedie	β-divergence	generalizes above models

*conditional independence over f does not apply

The β-divergence

A popular measure of fit in NMF (Basu et al., 1998; Cichocki and Amari, 2010)

$$d_\beta(x|y) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{\beta\,(\beta-1)} \left(x^\beta + (\beta-1)\, y^\beta - \beta\, x\, y^{\beta-1}\right) & \beta \in \mathbb{R}\backslash\{0,1\} \\ x \log \frac{x}{y} + (y-x) & \beta = 1 \\ \frac{x}{y} - \log \frac{x}{y} - 1 & \beta = 0 \end{cases}$$

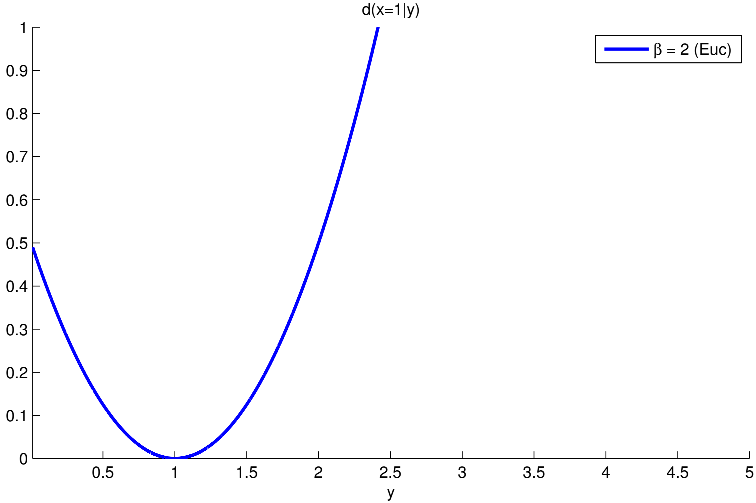
Special cases:

- squared Euclidean distance / quadratic loss ($\beta = 2$)
- generalized Kullback-Leibler (KL) divergence ($\beta = 1$)
- Itakura-Saito (IS) divergence ($\beta = 0$)

Properties:

- Homogeneity: $d_\beta(\lambda x|\lambda y) = \lambda^\beta d_\beta(x|y)$
- $d_\beta(x|y)$ is a convex function of y for $1 \leq \beta \leq 2$
- Bregman divergence

The β-divergence



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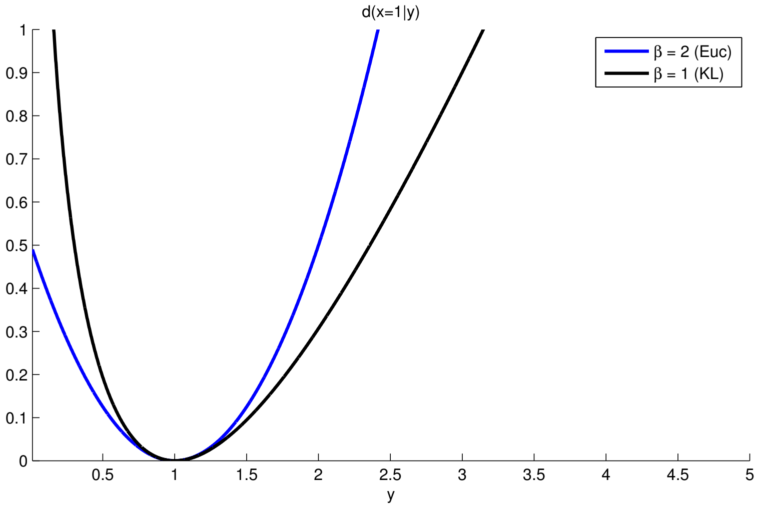

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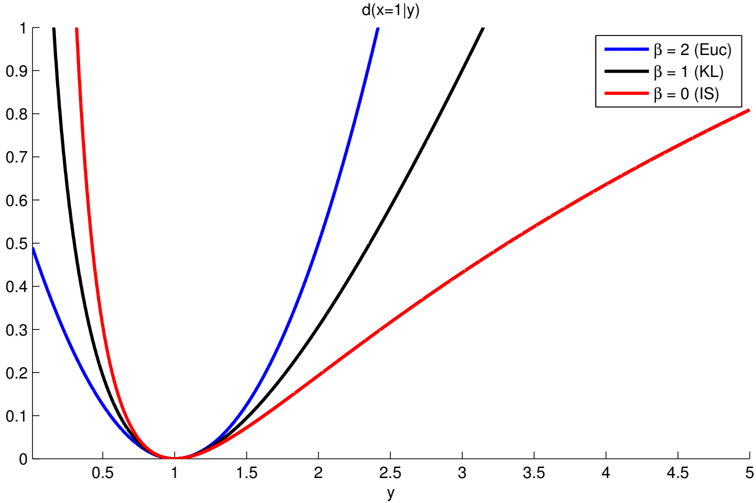

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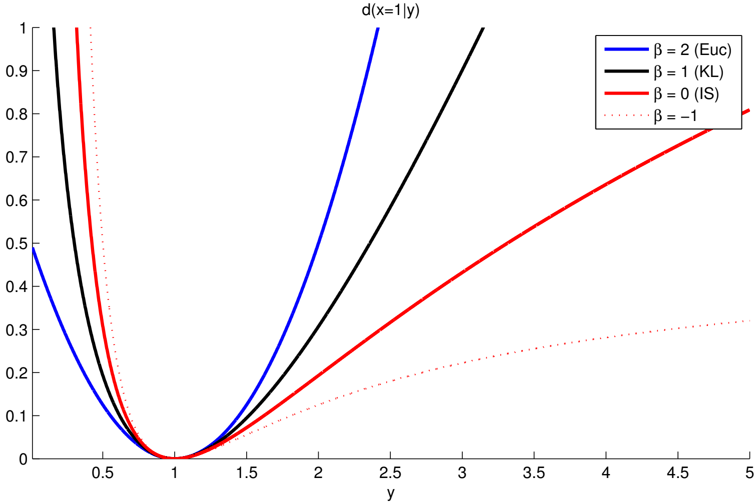

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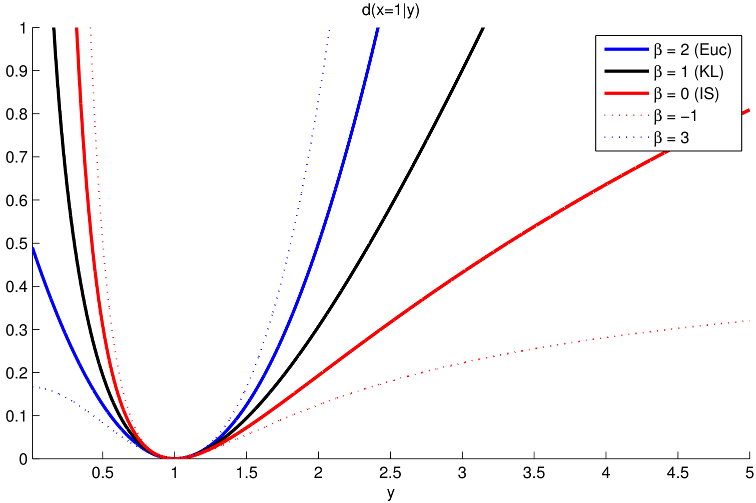


A common NMF algorithm design: alternating methods

- Block-coordinate update of $\mathbf{H}$ given $\mathbf{W}^{(i-1)}$ and $\mathbf{W}$ given $\mathbf{H}^{(i)}$.
- Updates of $\mathbf{W}$ and $\mathbf{H}$ equivalent by transposition:

$$\mathbf{V} \approx \mathbf{W}\mathbf{H} \Leftrightarrow \mathbf{V}^T \approx \mathbf{H}^T\mathbf{W}^T$$

- Objective function separable in the columns of $\mathbf{H}$ or the rows of $\mathbf{W}$:

$$D(\mathbf{V}|\mathbf{W}\mathbf{H}) = \sum_n D(\mathbf{v}_n|\mathbf{W}\mathbf{h}_n)$$

- Essentially left with nonnegative linear regression:

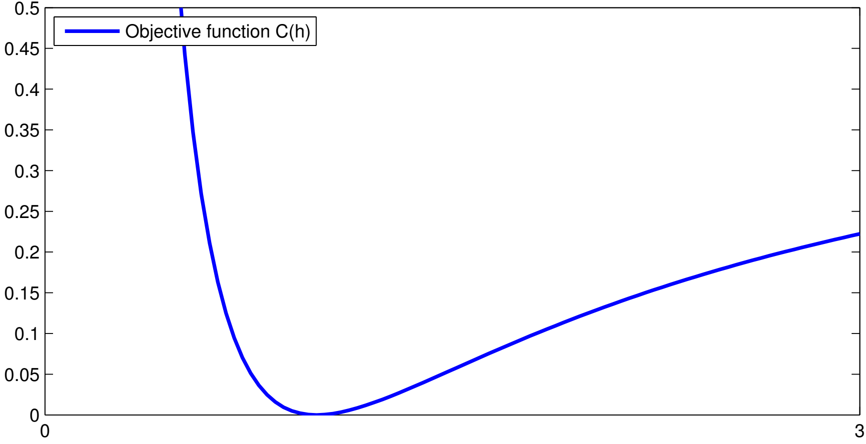
$$\min_{\mathbf{h}\geq\mathbf{0}} C(\mathbf{h}) \stackrel{\text{def}}{=} D(\mathbf{v}|\mathbf{W}\mathbf{h})$$

  Numerous references in the image restoration literature, e.g., (Richardson, 1972; Lucy, 1974; Daube-Witherspoon and Muehllehner, 1986; De Pierro, 1993)

Block-descent algorithm, nonconvex problem, initialization is an issue.

Majorization-minimization (MM)

Build $G(\mathbf{h}|\tilde{\mathbf{h}})$ such that $G(\mathbf{h}|\tilde{\mathbf{h}}) \geq C(\mathbf{h})$ and $G(\tilde{\mathbf{h}}|\tilde{\mathbf{h}}) = C(\tilde{\mathbf{h}})$.
Optimize (iteratively) $G(\mathbf{h}|\tilde{\mathbf{h}})$ instead of $C(\mathbf{h})$.



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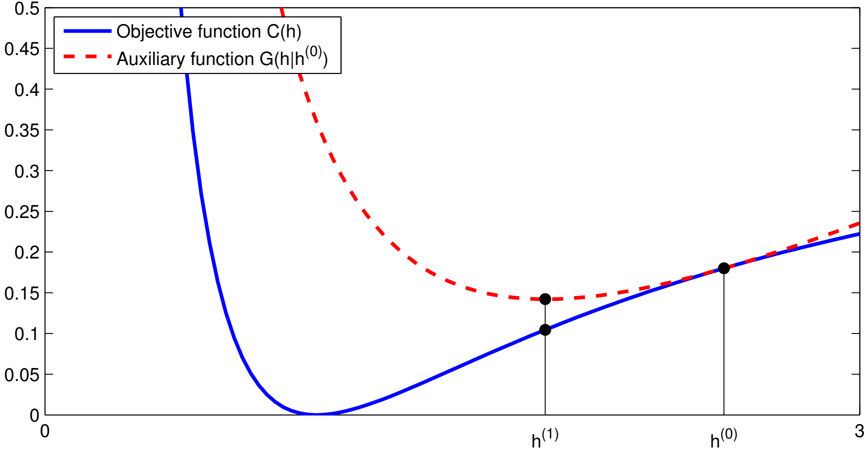

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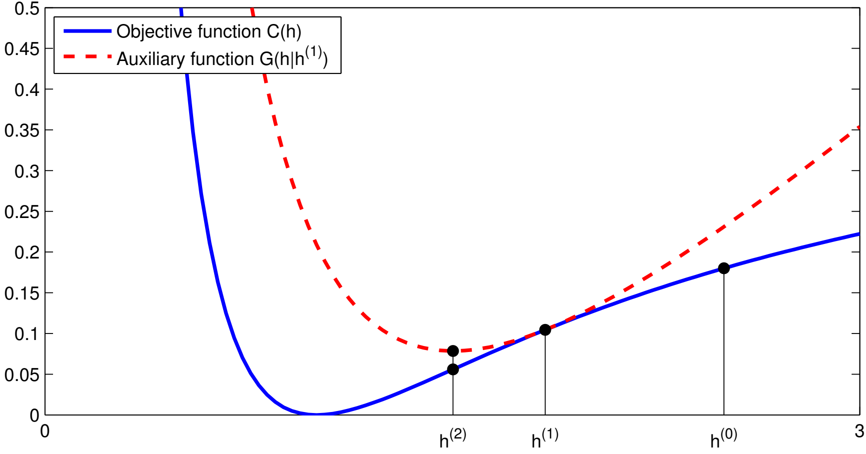

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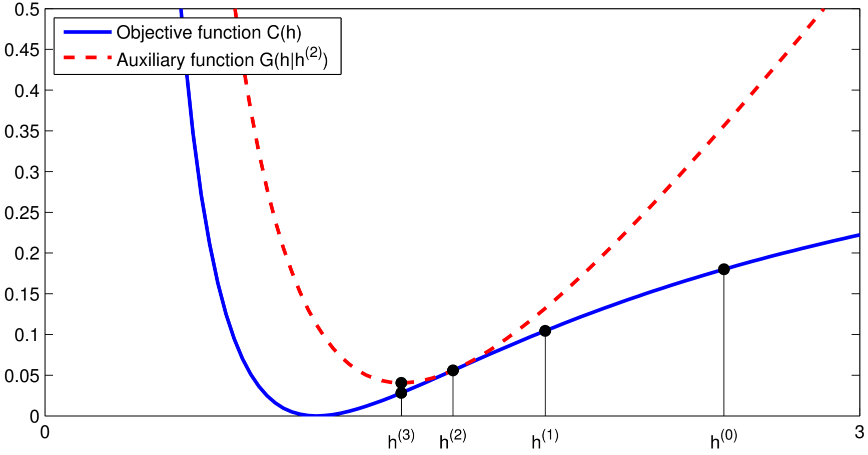

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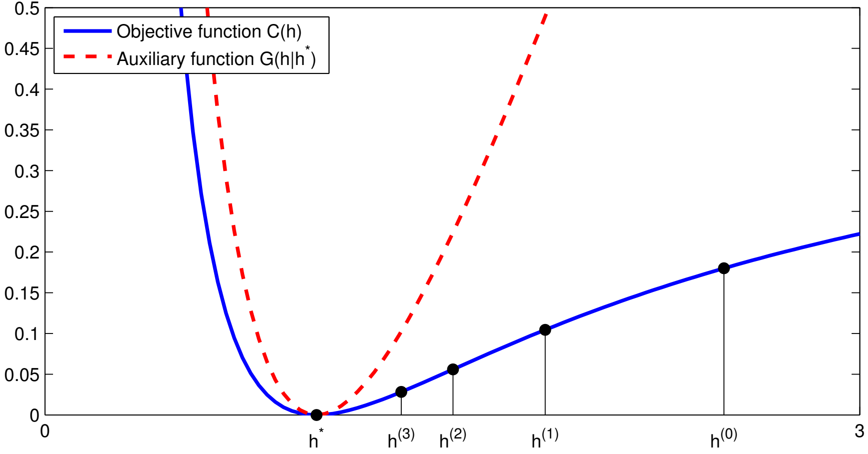

Majorization-minimization (MM)

- Finding a good & workable local majorization is the crucial point.
- Treating convex and concave terms separately with Jensen and tangent inequalities usually works. E.g.:

$$C_{\mathrm{IS}}(\mathbf{h}) = \left[\sum_f \frac{v_f}{\sum_k w_{fk} h_k}\right] + \left[\sum_f \log\left(\sum_k w_{fk} h_k\right)\right] + cst$$

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- In most cases, leads to nonnegativity-preserving multiplicative algorithms:

$$h_k = \tilde{h}_k \left(\frac{\nabla^-_{h_k} C(\tilde{\mathbf{h}})}{\nabla^+_{h_k} C(\tilde{\mathbf{h}})}\right)^\gamma$$

  - $\nabla_{h_k} C(\mathbf{h}) = \nabla^+_{h_k} C(\mathbf{h}) - \nabla^-_{h_k} C(\mathbf{h})$ and the two summands are nonnegative.
  - if $\nabla_{h_k} C(\tilde{\mathbf{h}}) > 0$, ratio of summands < 1 and h_k decreases.
  - γ is a divergence-specific scalar exponent.
- Details in (Nakano et al., 2010; Févotte and Idier, 2011; Yang and Oja, 2011)

Example: derivation for the Itakura-Saito divergence

- IS divergence ($\beta = 0$)

$$d_{IS}(x|y) = \frac{x}{y} - \log\frac{x}{y} - 1$$

- Nonnegative linear regression with the IS divergence

$$\begin{aligned}\min_{\mathbf{h}\geq 0} C_{IS}(\mathbf{h}) &= \sum_f d_{IS}(v_f|[\mathbf{W}\mathbf{h}]_f) \\ &= \underbrace{\left[\sum_f \frac{v_f}{\sum_k w_{fk} h_k}\right]}_{C_1(\mathbf{h})\ (\text{convex})} + \underbrace{\left[\sum_f \log\left(\sum_k w_{fk} h_k\right)\right]}_{C_2(\mathbf{h})\ (\text{concave})} + cst\end{aligned}$$

Example: derivation for the Itakura-Saito divergence

- Majorization of $C_1(\mathbf{h})$ with Jensen's inequality.
  Let $f(x)$ be a convex function and $\boldsymbol{\lambda} \in \mathbb{R}_+^K$ with $\sum_k \lambda_k = 1$. Then:

$$f\left(\sum_k \lambda_k h_k\right) \leq \sum_k \lambda_k f(h_k).$$

- Let $\tilde{\mathbf{h}} \in \mathbb{R}_+^K$ be the current estimate, $\tilde{\mathbf{v}} = \mathbf{W}\tilde{\mathbf{h}}$ be the current approximation and

$$\lambda_{fk} = \frac{w_{fk}\tilde{h}_k}{\tilde{v}_f} = \frac{w_{fk}\tilde{h}_k}{\sum_j w_{fj}\tilde{h}_j} \quad \left(\text{note that } \sum_k \lambda_{fk} = 1\right).$$

- Then, by convexity of $f(x) = x^{-1}$, we may write:

$$C_{\mathrm{IS}}(\mathbf{h}) = \sum_f v_f \left(\sum_k w_{fk} h_k\right)^{-1} = \sum_f v_f \left(\sum_k \lambda_{fk} \frac{w_{fk} h_k}{\lambda_{fk}}\right)^{-1}$$

$$\leq \sum_{fk} v_f \frac{\lambda_{fk}^2}{w_{fk} h_k} = \sum_{fk} w_{fk} \frac{v_f}{\tilde{v}_f^2} \frac{\tilde{h}_k^2}{h_k} = G_1(\mathbf{h}|\tilde{\mathbf{h}}).$$

Example: derivation for the Itakura-Saito divergence

- Majorization of $C_2(\mathbf{h})$ with the tangent inequality.
  Let $g(\mathbf{h})$ be a concave function then:

$$g(\mathbf{h}) \leq g(\tilde{\mathbf{h}}) + \nabla g(\tilde{\mathbf{h}})^\top (\mathbf{h} - \tilde{\mathbf{h}}) = \sum_k [\nabla g(\tilde{\mathbf{h}})]_k h_k + cst.$$

- Given $C_2(\mathbf{h}) = \sum_f \log \left(\sum_k w_{fk} h_k \right)$, we have:

$$[\nabla C_2(\tilde{\mathbf{h}})]_k = \nabla_{h_k} C_2(\tilde{\mathbf{h}}) = \sum_f \frac{w_{fk}}{\tilde{v}_f}.$$

- Finally, we may majorize $C_2(\mathbf{h})$ with:

$$G_2(\mathbf{h}|\tilde{\mathbf{h}}) = \sum_{fk} \frac{w_{fk}}{\tilde{v}_f} h_k + cst.$$

Example: derivation for the Itakura-Saito divergence

- In the end, we may majorize $C_{\mathrm{IS}}(\mathbf{h})$ with:

$$
\begin{aligned}
G(\mathbf{h}|\tilde{\mathbf{h}}) &= G_1(\mathbf{h}|\tilde{\mathbf{h}}) + G_2(\mathbf{h}|\tilde{\mathbf{h}}) + cst \\
&= \sum_{fk} w_{fk} \left[\frac{v_f}{\tilde{v}_f^2} \frac{\tilde{h}_k^2}{h_k} + \frac{1}{\tilde{v}_f} h_k \right] + cst.
\end{aligned}
$$

- Smooth, convex and separable majorizer. Easily minimized by cancelling its gradient, leading to the MM-based multiplicative update

$$
h_k = \tilde{h}_k \left(\frac{\sum_f w_{fk} v_f [\mathbf{W}\tilde{\mathbf{h}}]_f^{-2}}{\sum_f w_{fk} [\mathbf{W}\tilde{\mathbf{h}}]_f^{-1}} \right)^{\frac{1}{2}}.
$$

- Algorithm known from (Cao et al., 1999). The $\frac{1}{2}$ exponent can be dropped using majorization-equalization (Févotte and Idier, 2011).

The multiplicative updates (MU) for NMF with β-divergence

- Alternating updates of $\mathbf{W}$ and $\mathbf{H}$.
- In standard practice, only one MM update applied to $\mathbf{W}$ and $\mathbf{H}$, rather than fully solving subproblems $\min_{\mathbf{W}\geq 0} D(\mathbf{V}|\mathbf{WH})$ and $\min_{\mathbf{H}} D(\mathbf{V}|\mathbf{WH})$.
- Leads to a valid descent algorithm with multiplicative updates given by:

$$
\begin{aligned}
\mathbf{H} &\leftarrow \mathbf{H}. \left(\frac{\mathbf{W}^T \left[(\mathbf{WH})^{\cdot(\beta-2)} . \mathbf{V} \right]}{\mathbf{W}^T \left[\mathbf{WH} \right]^{\cdot(\beta-1)}} \right)^{\gamma(\beta)} \\
\mathbf{W} &\leftarrow \mathbf{W}. \left(\frac{\left[(\mathbf{WH})^{\cdot(\beta-2)} . \mathbf{V} \right] \mathbf{H}^T}{\left[\mathbf{WH} \right]^{\cdot(\beta-1)} \mathbf{H}^T} \right)^{\gamma(\beta)}
\end{aligned}
$$

- Very straightforward implementation, no hyperparameters!
- Nonnegativity is automatically preserved given positive initializations.
- Linear complexity per iteration.
- In practice, minimizing $D(\mathbf{V} + \epsilon|\mathbf{WH} + \epsilon)$ prevents from numerical issues.

Convergence of the iterates

- By design, we have convergence of the objective values $C(\mathbf{W}, \mathbf{H}) = D(\mathbf{V}|\mathbf{WH})$.
- What about the iterates ? Only partial answers so far.
- A theoretical challenge arises from the lack of coercivity of the objective: $\|\mathbf{W}\|$ or $\|\mathbf{H}\| \to \infty \not\Rightarrow C(\mathbf{W}, \mathbf{H}) \to \infty$.
- Due to the scale indeterminacy: $C(\mathbf{W}\boldsymbol{\Lambda}^{-1}, \boldsymbol{\Lambda}\mathbf{H}) = C(\mathbf{W}, \mathbf{H})$, with $\boldsymbol{\Lambda} \to 0$.

Possible remedies (modified problems)

1) Impose $\mathbf{W} \geq \epsilon$, $\mathbf{H} \geq \epsilon$ (Takahashi et al., 2018; Hien and Gillis, 2021).
2) Slightly change the objective function to ensure coercivity (Zhao and Tan, 2018):

$$C(\mathbf{W}, \mathbf{H}) = D(\mathbf{V}|\mathbf{WH}) + \epsilon\|\mathbf{W}\|_1 + \epsilon\|\mathbf{H}\|_1$$

MM results in adding ϵ at the denominator of the multiplicative updates.

Selecting β by matrix completion

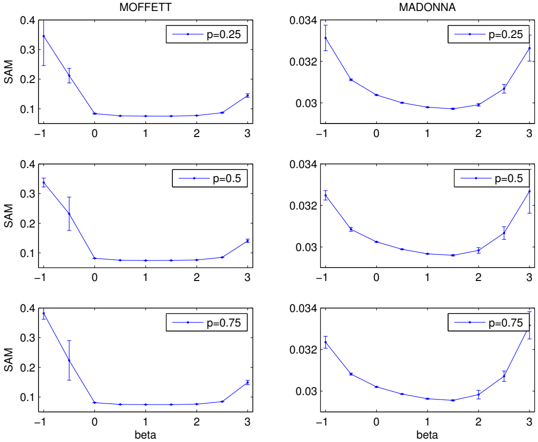
(Févotte and Dobigeon, 2015)

- **Data**: two unfolded hyperspectral cubes, $F \sim 150$, $N = 50 \times 50$
  - Aviris instrument over Moffett Field (CA), lake, soil & vegetation.
  - Hyspex/Madonna instrument over Villelongue (FR), forested area.
- a percentage of the pixels is randomly removed.
- **W** and **H** estimated from observed pixels (simple modification of MU).
- missing pixels are reconstructed from $\hat{\mathbf{V}} = \mathbf{WH}$.
- $K = 3$ ($\sim$ ground truth) and various values of β.
- evaluation using the average spectral angle mapper (aSAM):

$$\mathrm{aSAM}(\mathbf{V}) = \frac{1}{N} \sum_{n=1}^{N} \mathrm{acos}\left(\frac{\langle \mathbf{v}_n, \hat{\mathbf{v}}_n \rangle}{\|\mathbf{v}_n\| \|\hat{\mathbf{v}}_n\|} \right)$$

Selecting β by matrix completion

(Févotte and Dobigeon, 2015)



Recommended value $\beta \approx 1.5$ for these datasets
(compromise between Poisson and additive Gaussian noise).

Other alternating optimization methods

- MM-based multiplicative updates are a simple and competitive choice for many divergences (beyond β-divergences).
- More efficient options have been proposed for specific measures of fit, see books by Cichocki et al. (2009); Gillis (2020)

Quadratic loss (selection)

- Active-set methods (Kim and Park, 2011)
- Hierarchical alternating LS (Cichocki et al., 2007; Gillis and Glineur, 2012)
- Proximal gradient descent (Lin, 2007; Guan et al., 2012; Bolte et al., 2014)
- ADMM (Sun and Févotte, 2014; Huang et al., 2016)

Kullback-Leibler divergence (selection)

- Second-order coordinate descent methods (Hsieh and Dhillon, 2011)
- Hybrid Newton-type algorithms with line search and MU (Hien and Gillis, 2021)

Non-alternating methods (joint optimization)

- Optimize $C(\mathbf{W}, \mathbf{H}) = D(\mathbf{V}|\mathbf{W}, \mathbf{H})$ jointly in $\mathbf{W}$ and $\mathbf{H}$.
- Exciting line of research, driven by recent results in non-convex optimization. Possibly better optima and lower complexity.

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1) Proximal gradient algorithms with global smoothness constant ($\sim$Lipschitz) for the quadratic loss (Rakotomamonjy, 2013; Mukkamala and Ochs, 2019).

Non-alternating methods (joint optimization)

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2) Joint MM algorithm for the β-divergence (Marmin, Goulart, and Févotte, 2021):
   - Global majorizer constructed using Jensen and tangent inequalities:

$$C(\mathbf{W},\mathbf{H}) \leq G(\mathbf{W},\mathbf{H}|\tilde{\mathbf{W}},\tilde{\mathbf{H}})$$
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   - Global minimizer of G not available in closed form. G non-convex.
   - Alternate minimization of G leads to closed-form updates and new multiplicative rules. Important computational savings for some values of β (see paper).

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3) Second-order method for β-NMF based on efficient Hessian approximations and tricks to maintain semidefinite positivity (Vandecappelle et al., 2020).

Large-scale NMF

Online NMF

- Large number of samples $N >> F$.
- Update $\mathbf{W}$ as samples $\mathbf{v}_n$ become available.
- Vectors $\mathbf{h}_n$ act as latent variables, minimize

$$C(\mathbf{W}) = \sum_{n=1}^{N} \min_{\mathbf{h}_n \geq 0} D(\mathbf{v}_n | \mathbf{W}\mathbf{h}_n)$$

- Solved with online MM (Lefèvre et al., 2011b; Mairal, 2015; Zhao et al., 2017)

Stochastic NMF

- Large F and N.
- Online NMF with stochastic subsampling:

$$\min_{\mathbf{h}_n \geq 0} D(\mathbf{v}_n[\mathcal{I}] | \mathbf{W}[\mathcal{I}, :]\mathbf{h}_n)$$

where $\mathcal{I}$ is a random set of indices (Mensch et al., 2018).

Outline

Regularized NMF

- Induce prior information or desired structure on $\mathbf{H}$ (or $\mathbf{W}$) using penalty terms:

$$C(\mathbf{W}, \mathbf{H}) = D(\mathbf{V}|\mathbf{WH}) + S(\mathbf{H})$$

- MM algorithms are easily adapted to that setting:

$$D(\mathbf{V}|\mathbf{WH}) \qquad \leq G(\mathbf{H}|\tilde{\mathbf{H}}, \mathbf{W})$$

- Only the minimization step is changed.
- May however become intractable; sometimes $S(\mathbf{H})$ needs to be majorized itself.
- Similar to adjusting the proximal operator in proximal gradient descent.

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Sparsity

- Promote zeros in $\mathbf{H}$ (or $\mathbf{W}$), e.g,

$$S(\mathbf{H}) = \|\mathbf{H}\|_1 = \sum_{kn} h_{kn}, \quad S(\mathbf{H}) = \sum_{kn} \log(h_{kn} + \epsilon)$$

- Possibly with some group structure, e.g., cancel some rows of $\mathbf{H}$ (see Part II).
- Vast literature! Seminal paper by Hoyer (2004).
- Need to control $\|\mathbf{W}\|$ to avoid degenerate solutions $\|\mathbf{W}\| \to \infty$, $\|\mathbf{H}\| \to 0$.
- Because $C(\mathbf{W}\mathbf{\Lambda}^{-1}, \mathbf{\Lambda}\mathbf{H}) = D(\mathbf{V}|\mathbf{W}\mathbf{H}) + S(\mathbf{\Lambda}\mathbf{H})$, $S(\cdot)$ can be made arbitrary small.
- A common approach:

$$\min_{\mathbf{W},\mathbf{H}\geq 0} C(\mathbf{W}, \mathbf{H}) \quad \text{s.t.} \quad \forall k, \|\mathbf{w}_k\| = 1$$

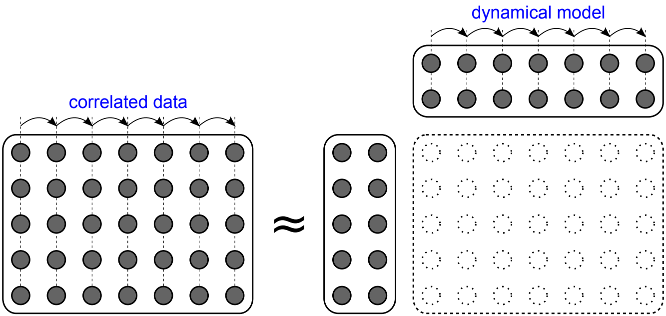
  - Change of variable (Eggert and Körner, 2004; Lefèvre et al., 2011a; Le Roux et al., 2015)
  - Lagrangian method (Leplat et al., 2021)

Smoothness

Impose temporal or spatial regularization, e.g.,

$$S(\mathbf{H}) = \sum_{kn} d(h_{kn} | h_{k(n-1)})$$

- Least squares penalization (Virtanen, 2007; Essid and Févotte, 2013)
- Gamma Markov chains (Smaragdis et al., 2014; Filstroff et al., 2021)



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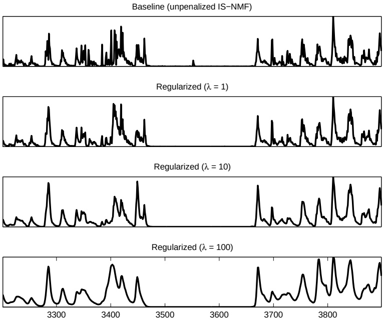


One row of **H** with increasing smoothness (Févotte, 2011)

Other common regularizers

- Orthogonal NMF: $\mathbf{H}\mathbf{H}^T = \mathbf{I}$.
  Essentially nonnegative clustering (Ding et al., 2006).
- Projective NMF: $\mathbf{H} = \mathbf{W}^T\mathbf{V}$.
  Essentially nonnegative PCA (Yang and Oja, 2010).
- Symmetric NMF: $\mathbf{H} = \mathbf{W}^T$.
  Popular in graph clustering (Kuang et al., 2012; Huang et al., 2013).
- Separable NMF: $\mathbf{W}$ is a subset of columns of $\mathbf{V}$.
  Very active research topic! (Donoho and Stodden, 2004; Arora et al., 2016)
- Archetypal NMF: $\mathbf{W}$ belongs to the column-range of $\mathbf{V}$.
  A relaxation of separable NMF (Ding et al., 2010; Chen et al., 2014).
- Minimum-volume NMF: penalize the aperture of $\mathbf{W}$.
  Very active research topic! (Miao and Qi, 2007; Chan et al., 2009)

Robust NMF for nonlinear hyperspectral unmixing

(Févotte and Dobigeon, 2015)

- Variants of the linear mixing model account for "non-linear" effects:

$$\mathbf{v}_n \approx \mathbf{W}\mathbf{h}_n + \mathbf{r}_n$$

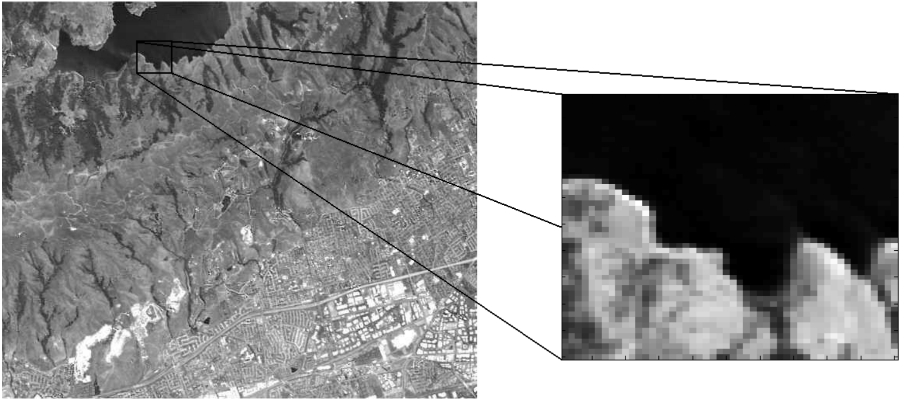
- Often, $\mathbf{r}_n$ has a parametric form, e.g., linear combination of quadratic components $\{\mathbf{w}_k \odot \mathbf{w}_j\}_{kj}$ (Nascimento and Bioucas-Dias, 2009; Fan et al., 2009; Altmann et al., 2012)
- Nonlinear effects usually affect few pixels only.
- We treat them as non-parametric sparse outliers.

$$\min_{\mathbf{W},\mathbf{H},\mathbf{R}\geq 0} D_\beta(\mathbf{V}|\mathbf{W}\mathbf{H} + \mathbf{R}) + \lambda\|\mathbf{R}\|_{2,1}$$

  where $\|\mathbf{R}\|_{2,1} = \sum_{n=1}^{N} \|\mathbf{r}_n\|_2$ induces sparsity at group level.
- A form of robust NMF (Candès et al., 2009)
- Optimized with majorization-minimization.

Robust NMF for nonlinear hyperspectral unmixing
(Févotte and Dobigeon, 2015)

Moffett Field data

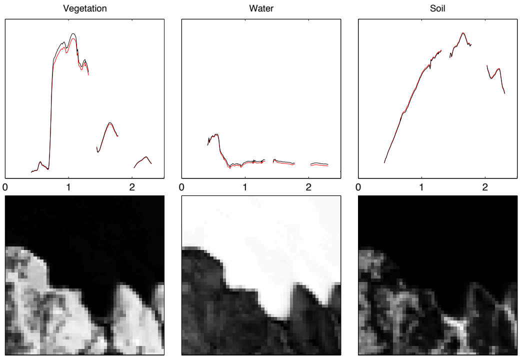


reproduced from (Dobigeon, 2007)

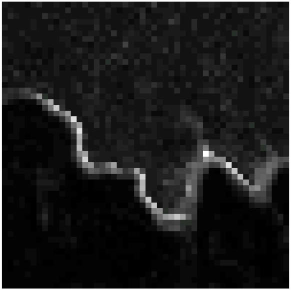
Robust NMF for nonlinear hyperspectral unmixing
(Févotte and Dobigeon, 2015)

Unmixing results

spectral endmembers & activation maps
(**red:** $\beta = 1$, **black:** $\beta = 2$)



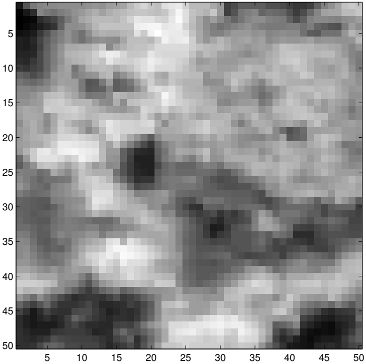
outlier energy $\{\|\mathbf{r}_n\|\}_n$
($\beta = 1$)



Outlier term captures specific water/soil interactions.

Robust NMF for nonlinear hyperspectral unmixing
(Févotte and Dobigeon, 2015)

Villelongue/Madonna data (forested area)



Robust NMF for nonlinear hyperspectral unmixing

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Unmixing results

spectral endmembers & activation maps
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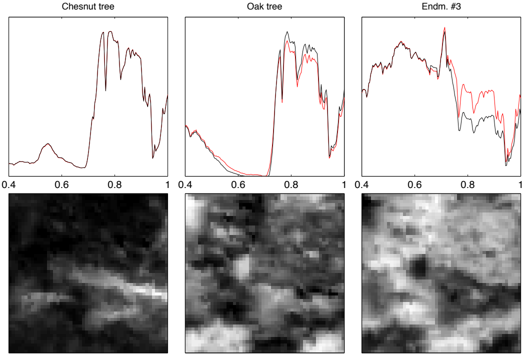

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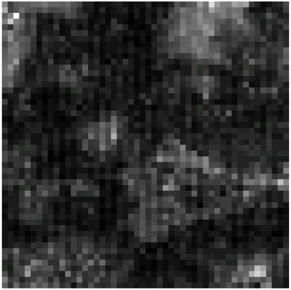
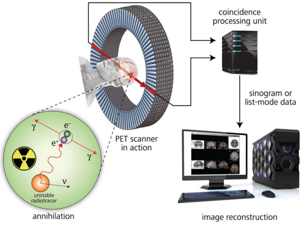


Outlier term seems to capture patterns due to sensor miscalibration.

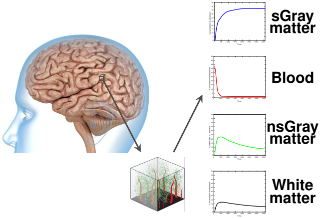
Factor analysis in dynamical PET

(Cavalcanti, Oberlin, Dobigeon, Févotte, Stute, Ribeiro, and Tauber, 2019)

- 3D functional imaging
- Observe the temporal evolution of the brain activity after injecting a radiotracer (biomarker of a specific compound).
- $\mathbf{v}_n$ is the time-activity curve (TAC) in voxel n.
- Neuroimaging: mixed contributions of 4 TAC signatures in each voxel.



Dynamic positron emission tomography



PET voxel decomposition

reproduced from (Cavalcanti, 2018)

Factor analysis in dynamical PET

(Cavalcanti, Oberlin, Dobigeon, Févotte, Stute, Ribeiro, and Tauber, 2019)

Mixing model

- the specific-binding TAC signature varies in space:

$$\begin{aligned}\mathbf{v}_n &\approx [\mathbf{w}_1 + \boldsymbol{\delta}_n] h_{1n} + \sum_{k=2}^{K} \mathbf{w}_k h_{kn} \\ &\approx [\mathbf{w}_1 + \mathbf{D}\mathbf{b}_n] h_{1n} + \sum_{k=2}^{K} \mathbf{w}_k h_{kn} \\ &\approx \mathbf{W}\mathbf{h}_n + h_{1n}\, \mathbf{D}\mathbf{b}_n\end{aligned}$$

- $\mathbf{D}$ is fixed and pre-trained using labeled or simulated data.

Estimation

$$\min_{\mathbf{W},\mathbf{H},\mathbf{B} \geq 0} D_\beta(\mathbf{V} | \mathbf{W}\mathbf{H} + \mathbf{1}\, \underline{\mathbf{h}}_1 \odot \mathbf{D}\mathbf{B}) + \lambda \|\mathbf{B}\|_{2,1}$$

Optimized with majorization-minimization.

Factor analysis in dynamical PET

(Cavalcanti, Oberlin, Dobigeon, Févotte, Stute, Ribeiro, and Tauber, 2019)

Unmixing results

- real dynamic PET image of a stroke subject injected with a tracer for neuroinflammation.
- MRI ground-truth region of the stroke.

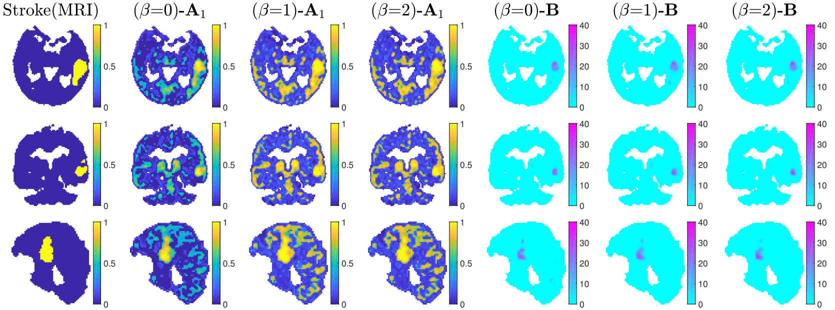


Fig.: Specific-binding activation (h_{1n}) and variability maps ($\|\mathbf{b}_n\|_{2,1}$) in three different planes and for three values of β

End of Part I

Half-time conclusions

- NMF has become a popular data processing tool over the last 20 years.
- Very suited to unmixing problems in unsupervised settings.
- Exciting non-convex optimization problem with non-Euclidean measures of fit.
- MM is a versatile algorithmic framework for NMF.
  - Simple multiplicative algorithms for the β-divergence and beyond.
  - Can be adapted to regularized NMF and variants.
  - More efficient algorithms exist for the quadratic loss.

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