A Survey of Risk-Aware Multi-Armed Bandits

Vincent Tan | **Prashanth L. A.** | **Krishna Jagannathan**

NUS | IIT Madras | IIT Madras



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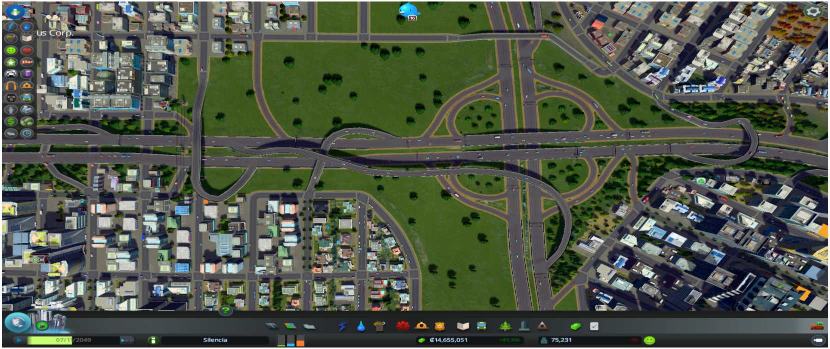
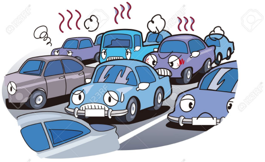
Going to office: Bandit style



On every day

1. Pick a route to office
2. Reach office and record (suffered) delay

Why Consider Risk?



$\mathbb{E}[\text{time}] = 10 \text{ mins}, \Pr(\text{jam}) = 0.1$ $\mathbb{E}[\text{time}] = 11 \text{ mins}, \Pr(\text{jam}) = 0$

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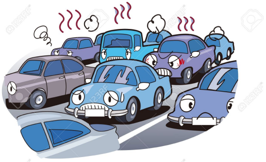

$\mathbb{E}[\text{time}] = 10 \text{ mins}, \Pr(\text{jam}) = 0.1$ $\quad$ $\mathbb{E}[\text{time}] = 11 \text{ mins}, \Pr(\text{jam}) = 0$

- Delays are stochastic.
- In choosing between routes, we need not necessarily minimize expected delay.
- Two route scenario: Average delay of Route 1 slightly below that of Route 2.
- Route 1 has a small chance of very high delay, e.g., jams.
- I might prefer Route 2.

Preliminary Definitions I

Definition

Given i.i.d. random samples $\{X_i\}_{i=1}^n$ from the distribution of a random variable, the empirical distribution function is

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_i \leq x\} \quad \text{for any} \quad x \in \mathbb{R}.$$

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Definition

Random variable X is σ^2-sub-Gaussian if its cumulant generating function

$$\log \mathbb{E}[\exp(rX)] \le \frac{r^2\sigma^2}{2} \quad \text{for all} \quad r \in \mathbb{R}.$$

See Wainwright (2019, Theorem 2.1) for equivalent characterizations.

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The Wasserstein distance between two cumulative distribution functions (CDFs) F_1 and F_2 on $\mathbb{R}$ is

$$W_1(F_1, F_2) := \inf_{F \in \Gamma(F_1, F_2)} \int_{\mathbb{R}^2} |x - y| \, \mathrm{d}F(x, y)$$

where $\Gamma(F_1, F_2)$ is the set of couplings of F_1 and F_2.

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Alternative expressions:

$$\begin{aligned} W_1(F_1, F_2) &= \sup |\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]| \\ &= \int_{-\infty}^{\infty} |F_1(s) - F_2(s)| \mathrm{d}s = \int_0^1 |F_1^{-1}(\beta) - F_2^{-1}(\beta)| \mathrm{d}\beta, \end{aligned}$$

where the supremum is over all 1-Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$.

Concentration of Mean-Variance

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Lemma (Concentration bound for MV (simplified))

For any $\epsilon > 0$:

$$\Pr\left[|\widehat{\mathrm{MV}}_n - \mathrm{MV}| > \epsilon\right] \le 2\exp\left[-\frac{n\epsilon^2}{8\gamma^2\sigma^2}\right] + 2\exp\left(-\frac{n}{16}\min\left[\frac{\epsilon^2}{2\sigma^4}, \frac{\epsilon}{\sigma^2}\right]\right)$$

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- Prashanth and Bhat (2020) use the Wasserstein distance as the underlying norm.

Definition

A risk measure $\rho(\cdot)$ is L-Lipschitz if for all cumulative distribution functions (F, G),

$$|\rho(F) - \rho(G)| \leq L\, W_1(F, G).$$

Idea: Use $\rho_n = \rho(F_n)$ as an estimate of $\rho(F) = \rho(X)$ ($X \sim F$), where

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \leq x\} \quad \text{for any } \; x \in \mathbb{R}.$$

Concentration of Lipschitz-continuous Risk Measures

Theorem

Let X be a sub-Gaussian r.v. with parameter σ^2. Suppose ρ an L-Lipschitz risk measure. Then, for every ϵ satisfying

$$\frac{256\sqrt{2}\sigma}{\sqrt{n}} < \frac{\epsilon}{L} < \frac{256\sqrt{2}\sigma}{\sqrt{n}} + 16\sigma\sqrt{2\mathrm{e}}, \quad \textit{i.e.,} \quad \epsilon = \Omega\Big(\frac{1}{\sqrt{n}}\Big)$$

we have

$$\Pr\left[|\rho_n - \rho(X)| > \epsilon\right] \le \exp\left(-\frac{n}{256\sigma^2\mathrm{e}}\Big(\frac{\epsilon}{L} - \frac{256\sqrt{2}\sigma}{\sqrt{n}}\Big)^2\right).$$

Concentration of CVaR

Definition

The Conditional Value-at-Risk (CVaR) at level $\alpha \in (0,1)$ for a r.v. X is

$$\mathrm{CVaR}_\alpha(X) := \inf_{\xi \in \mathbb{R}} \left\{ \xi + \frac{1}{(1-\alpha)} \mathbb{E}\left[(X-\xi)^+ \right] \right\}.$$

- Empirical CVaR given $\{X_i\}_{i=1}^n$:

$$c_{n,\alpha} = \inf_{\xi \in \mathbb{R}} \left\{ \xi + \frac{1}{n(1-\alpha)} \sum_{i=1}^n (X_i - \xi)^+ \right\}.$$

- But CVaR at level α is $\frac{1}{1-\alpha}$-Lipschitz

$$|\mathrm{CVaR}_\alpha(X) - \mathrm{CVaR}_\alpha(Y)| \le \frac{1}{1-\alpha} W_1(F_X, F_Y).$$

so we can use the preceding concentration bound.

Concentration of Spectral Risk Measures

Definition

Given a risk spectrum $\phi : [0,1] \to [0,\infty)$, the Spectral Risk Measure (SRM) M_ϕ associated with ϕ is defined by Acerbi (2002) as

$$M_\phi(X) = \int_0^1 \phi(\beta) F_X^{-1}(\beta)\, \mathrm{d}\beta.$$

- Suppose $\phi(u) \le K$ for all $u \in [0,1]$, then

$$|M_\phi(X) - M_\phi(Y)| \le K\, W_1(F_X, F_Y).$$

- Use the general concentration result for Lipschitz risk functionals and the estimator

$$m_{n,\phi} = \int_0^1 \phi(\beta) F_n^{-1}(\beta)\, \mathrm{d}\beta.$$

Application to UCB-type Bandit Algorithms

- K-armed bandit with unknown distributions $\nu = (\nu_1, \nu_2, \ldots, \nu_K)$.

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- Seek to minimize the cumulative regret:

$$R_n^\rho(\nu, \pi) := \mathbb{E}\left[n \max_{1 \le i \le K} \rho(\nu_i) - \sum_{t=1}^{n} \rho(\nu_{A_t})\right],$$

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- Play all arms once, then

$$A_t = \arg\min_{1 \le i \le K} \mathrm{LCB}_t(i) \quad \text{where}$$

$$\mathrm{LCB}_t(i) = \rho_{i,T_i(t-1)} - w_{i,T_i(t-1)}$$

and $\rho_{i,T_i(t-1)}$ is the estimate of $\rho(\nu_i)$ with $T_i(t-1)$ samples.

Application to UCB-type Bandit Algorithms

Using the previous bounds for Lipschitz risk measures, we can obtain.

Theorem

The expected regret R_n^ρ of Risk-LCB satisfies the following bound:

$$R_n^\rho \leq \sum_{i:\Delta_i>0} \frac{4L^2\sigma^2[32\sqrt{\mathrm{e}\log n}+256\sqrt{2}]^2}{\Delta_i} + 5K\Delta_i$$

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Bound mimics that of risk-neutral UCB except that Δ_i's depend on ρ.

Thompson Sampling-type Bandit Algorithms

For Gaussian bandits, Zhu and Tan (2020) considered MVTS with the following sampling and update strategy:

1. Sample precision $\tau_{i,t}$ from $\mathrm{Gamma}(\alpha_{i,t-1}, \beta_{i,t-1})$;
2. Sample $\theta_{i,t}$ from $\mathcal{N}(\hat{\mu}_{i,t-1}, 1/T_{i,t-1})$;
3. Play $A_t = \arg\max_{i\in[K]} \gamma\theta_{i,t} - 1/\tau_{i,t}$ and observe X_{t,A_t};
4. $\mathrm{Update}(\hat{\mu}_{A_t,t-1}, T_{A_t,t-1}, \alpha_{A_t,t-1}, \beta_{A_t,t-1})$ using Bayes rule.

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Theorem (Zhu and Tan (2020))

The expected regret of MVTS is

$$\limsup_{n\to\infty} \frac{R_n^\rho}{\log n} \le \sum_{i=2}^{K} \max\left\{\frac{2}{\Gamma_{1,i}^2}, \frac{1}{h(\sigma_i^2/\sigma_1^2)}\right\}(\Delta_i + 2\overline{\Gamma}_i^2),$$

where $\Gamma_{1,j} := \mu_1 - \mu_j$, $\overline{\Gamma}_i^2 := \max_{j\in[K]}(\mu_i - \mu_i)^2$*,* $\Delta_i := \mathrm{MV}_{i^*} - \mathrm{MV}_i$*, and* $h(x) := \frac{1}{2}(x-1-\log x)$*. Bound is asymptotically optimal as* $\gamma \to \{0, \infty\}$*.*

Conclusion and Future Work

- Follow up work by Baudry et al. (2021) and Chang and Tan (2022) on Thompson sampling for CVaR and continuous risk measures

$$\limsup_{n\to\infty} \frac{R_n^\rho}{\log n} \le \sum_{i=2}^{K} \frac{\Delta_k^\rho}{K_{\inf}^\rho(\nu_k, r_1^\rho)} \quad \text{where} \quad K_{\inf}^\rho(\nu, r) = \inf_{\mu:\rho(\mu)\ge r} \mathrm{KL}(\mu, \nu).$$

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- Best arm identification (pure exploration) problems under risk constraints
  - Fixed budget (Kagrecha et al., 2019; Prashanth et al., 2020; Zhang and Ong, 2021)
  - Fixed confidence (David and Shimkin, 2016; David et al., 2018; Hou et al., 2022; Szorenyi et al., 2015)

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