A Journey in Information Theory: From Second-Order Asymptotics to Common Information

Vincent Y. F. Tan

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National University of Singapore (NUS)

Plenary at ISITA (Taipei)
Nov 2024

Outline

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A Large-Deviation Analysis of the Maximum-Likelihood Learning of Markov Tree Structures

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distributions. In this respect, graphical models [2] provide a significant simplification of joint distribution as the distribution can be factorized according to a graph defined on the set of nodes. Many specialized algorithms [3]–[9] exist for exact and approximate learning of graphical models Markov on sparse graphs.

There are many applications of learning graphical models, including clustering and dimensionality reduction. Suppose we have d genetic variables and we would like to group the ones that are similar together. Then the construction of a graphical model provides a visualization of the relationship between genes. Those genes that have high degree are highly correlated

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parameters and are tractable for learning [5] and statistical inference [1], [4].

The problem of maximum-likelihood (ML) learning of a Markov tree distribution from i.i.d. samples has an elegant solution, proposed by Chow and Liu in [5]. The ML tree structure is given by the maximum-weight spanning tree (MWST) with empirical mutual information quantities as the edge weights. Furthermore, the ML algorithm is *consistent* [6], which implies that the error probability in learning the tree structure decays to zero with the number of samples available for learning.

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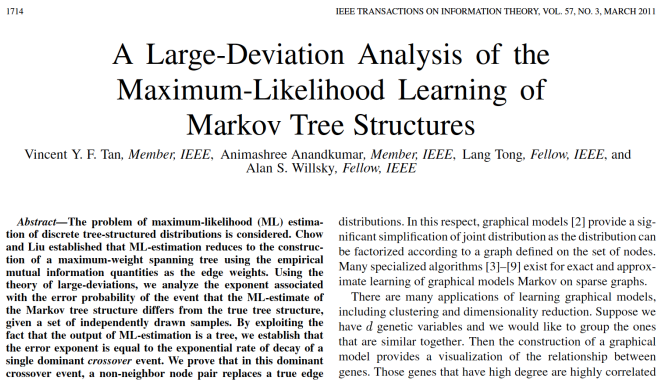

1714 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 57, NO. 3, MARCH 2011

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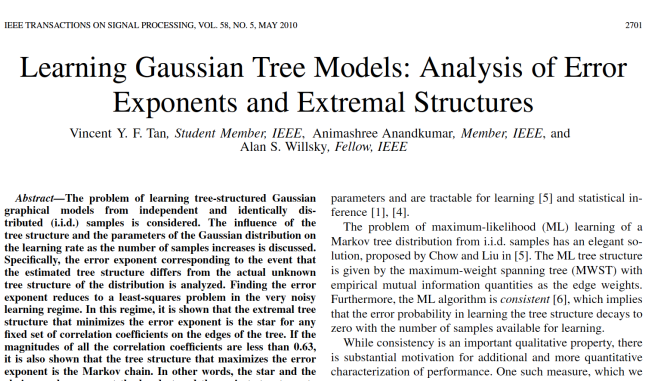

IEEE TRANSACTIONS ON SIGNAL PROCESSING, VOL. 58, NO. 5, MAY 2010 2701

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The Real Start of my Foray into Information Theory

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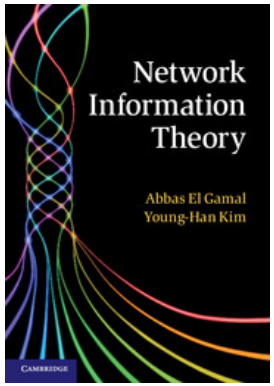

Network Information Theory
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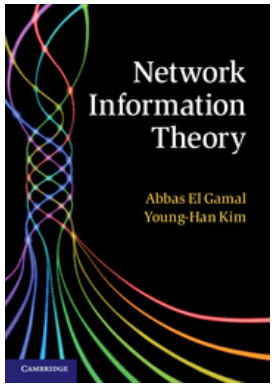

Network Information Theory
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- Came across the paper by Ingber and Kochman (DCC 2011)

The Dispersion of Lossy Source Coding

Amir Ingber
Dept. of EE-Systems, TAU
Tel Aviv 69978, Israel
Email: ingber@eng.tau.ac.il

Yuval Kochman
EECS Dept., MIT
Cambridge, MA 02139, USA
Email: yuvalko@mit.edu

Abstract

In this work we investigate the behavior of the minimal rate needed in order to guarantee a given probability that the distortion exceeds a prescribed threshold, at some fixed finite quantization block length. We show that the excess coding rate above the rate-distortion function is inversely proportional (to the first order) to the square root of the block length. We give an explicit expression for the proportion constant, which is given by the inverse Q-function of the allowed excess distortion probability, times the square root of a constant, termed the *excess distortion dispersion*. This result is the dual of a corresponding channel coding result, where the dispersion above is the dual of the channel dispersion. The work treats discrete memoryless sources, as well as the quadratic-Gaussian case.

$$\frac{\sqrt{n}\big(R(\hat{P}_{X^n}, D) - R(P, D)\big)}{\sqrt{V(P, D)}} \xrightarrow{\text{d}} \mathcal{N}(0, 1)$$

Extension to NIT: The Slepian–Wolf problem

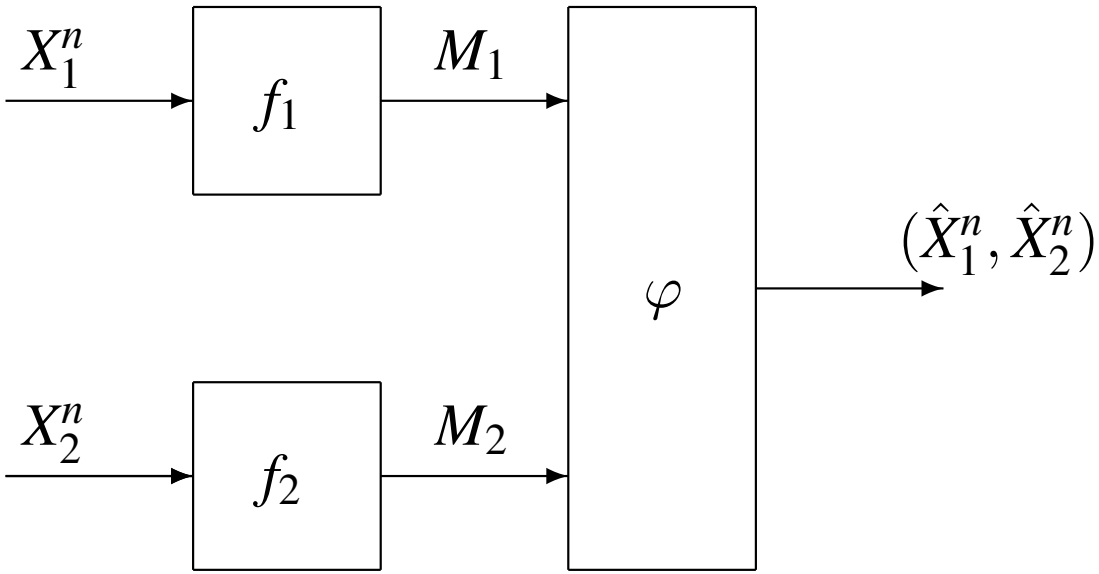

Illustration of the Slepian–Wolf problem.

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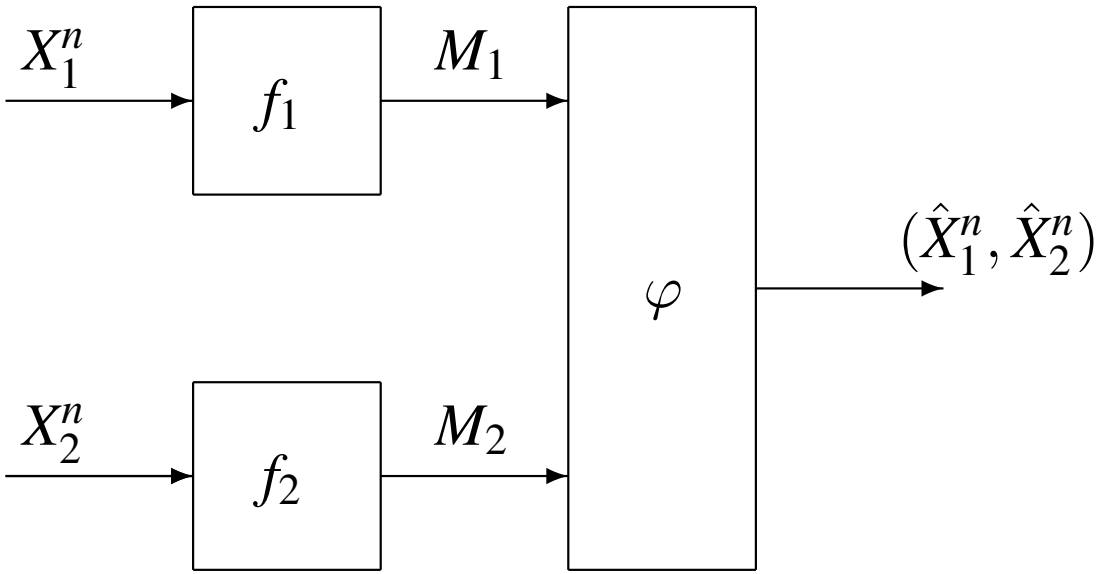

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- Two correlated sources $(X_1^n, X_2^n) \sim \prod_{i=1}^n P_{X_1X_2}(x_{1i}, x_{2i})$.

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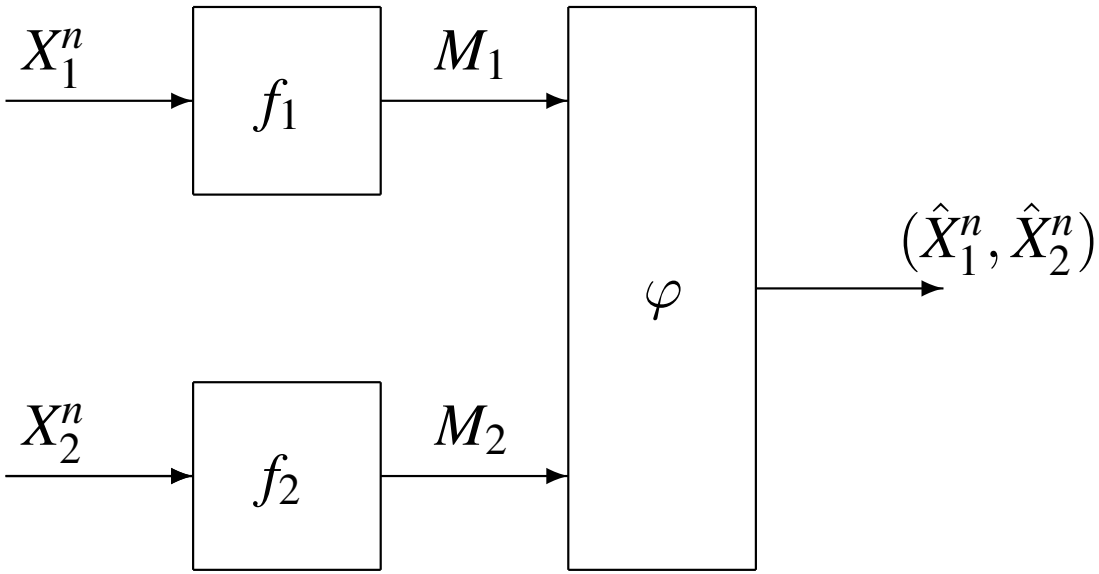

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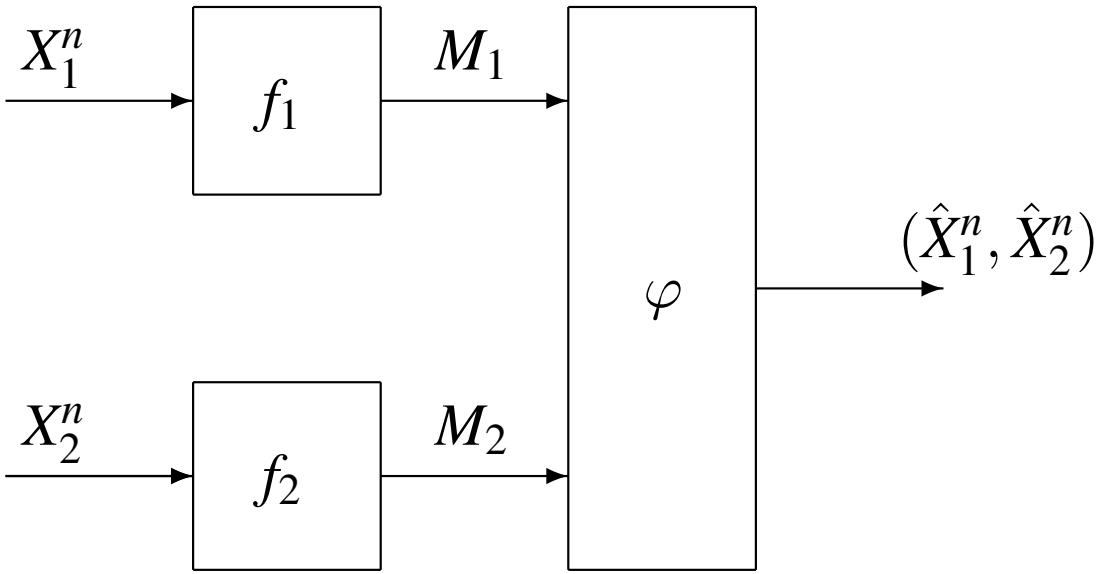

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- Both to be decoded at destination

The Slepian–Wolf theorem

- Sources to be compressed to nR_1 and nR_2 bits respectively.

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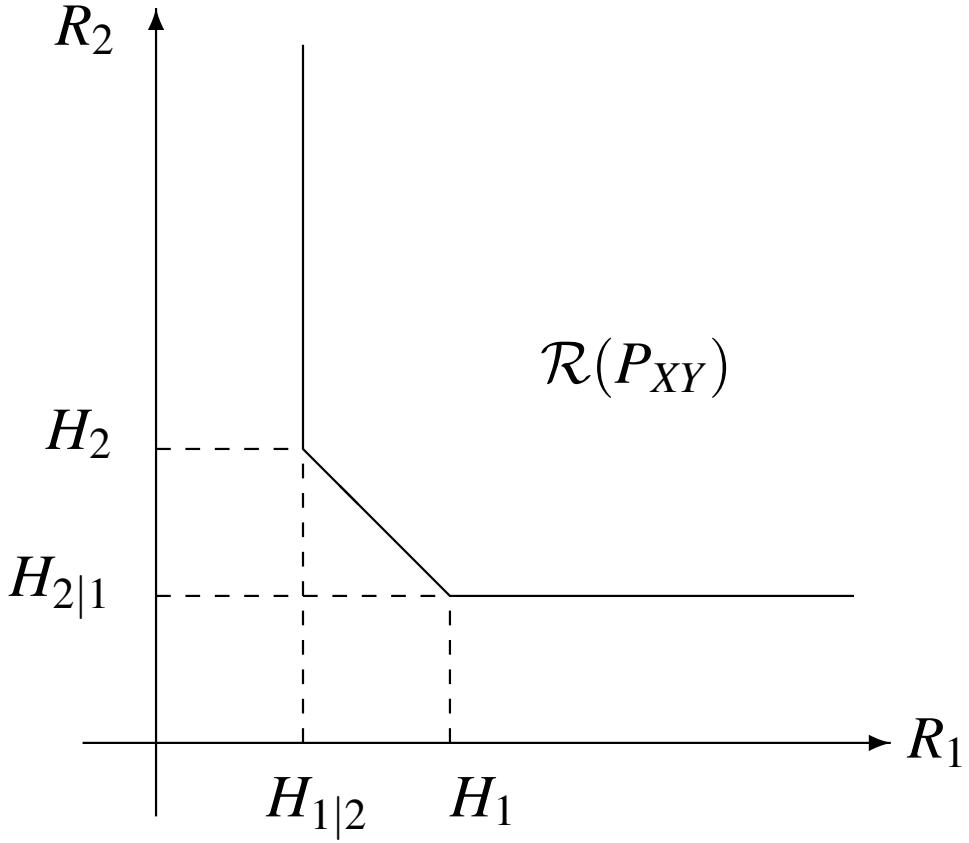
- Slepian and Wolf (1973)

$$\mathcal{R}(P_{XY}) = \{R_1 \geq H(X_1|X_2), R_2 \geq H(X_2|X_1), R_1 + R_2 \geq H(X_1, X_2)\}$$

D. Slepian

J. Wolf

The Slepian-Wolf region



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Second-Order Asymptotics for Slepian–Wolf

Joint work with Oliver Kosut (postdoc at MIT at that time)

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 60, NO. 2, FEBRUARY 2014 881

On the Dispersions of Three Network Information Theory Problems

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***Abstract*— We analyze the dispersions of distributed lossless source coding (the Slepian–Wolf problem), the multiple-access channel, and the asymmetric broadcast channel. For the two-encoder Slepian–Wolf problem, we introduce a quantity known as the entropy dispersion matrix, which is analogous**

While the characterization of capacity regions is a difficult problem in general, there have been positive results for several special classes of networks such as the multiple-access channel [2], [3] and the asymmetric [4] or degraded

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3552 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 68, NO. 6, JUNE 2022

A Second-Order Converse Bound for the Multiple-Access Channel via Wringing Dependence

Oliver Kosut, *Member, IEEE*

Setup for Second-Order Asymptotics for Slepian–Wolf

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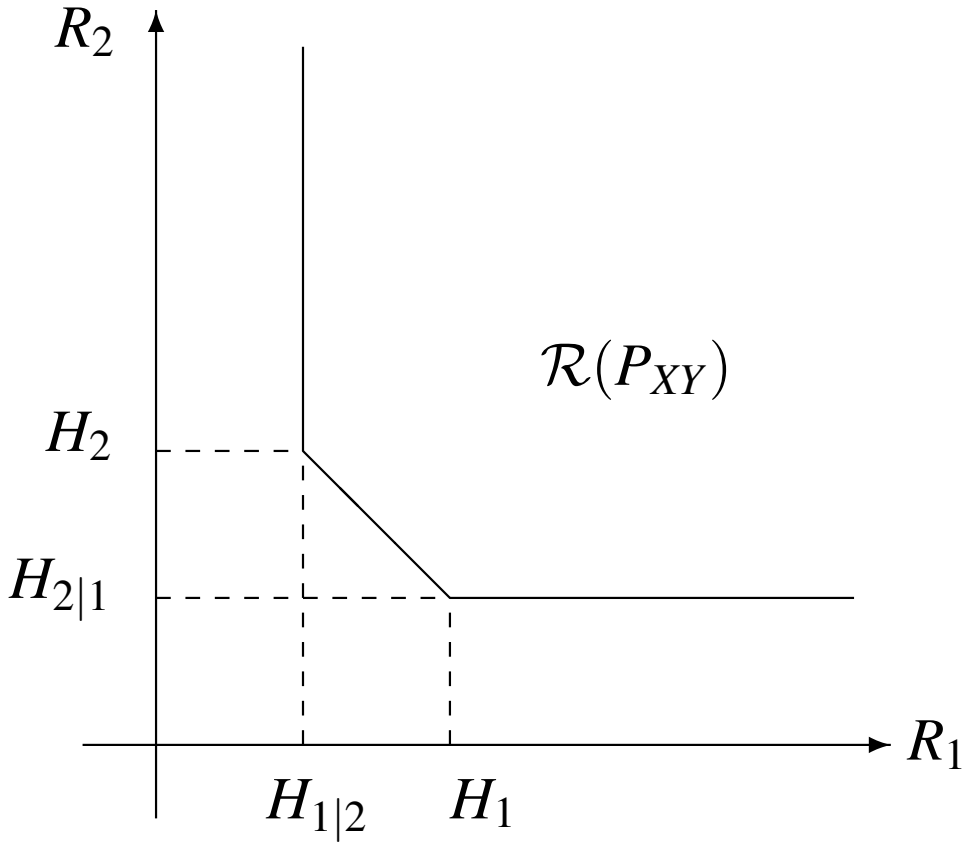

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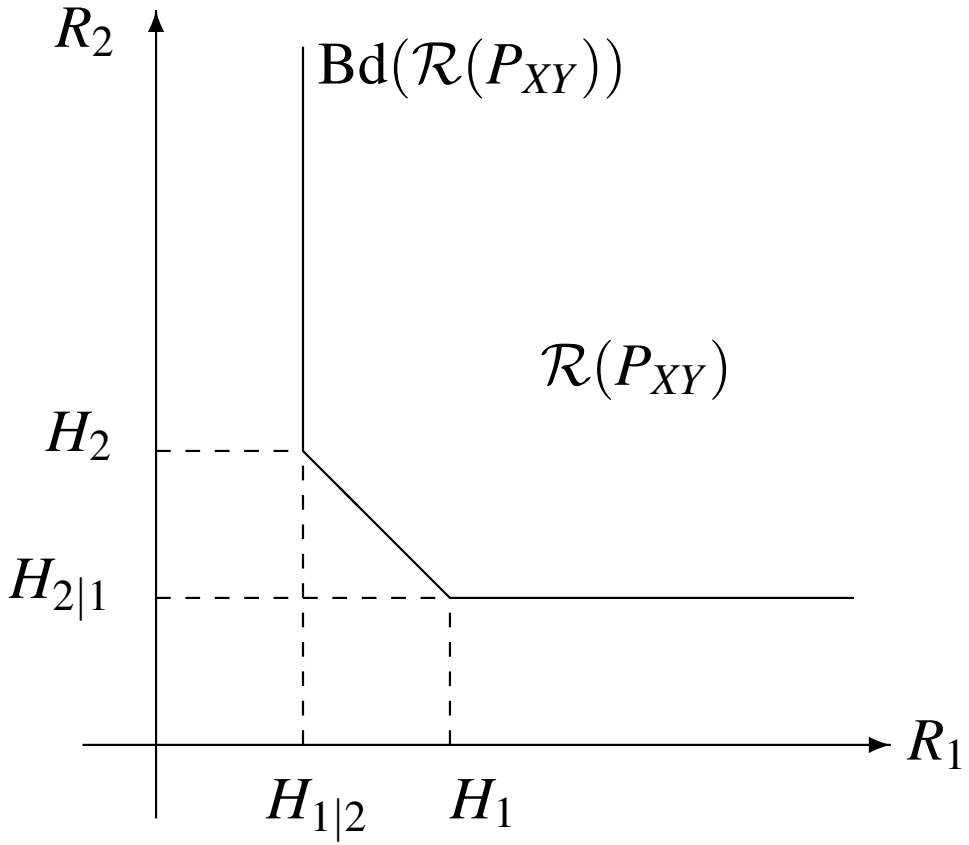

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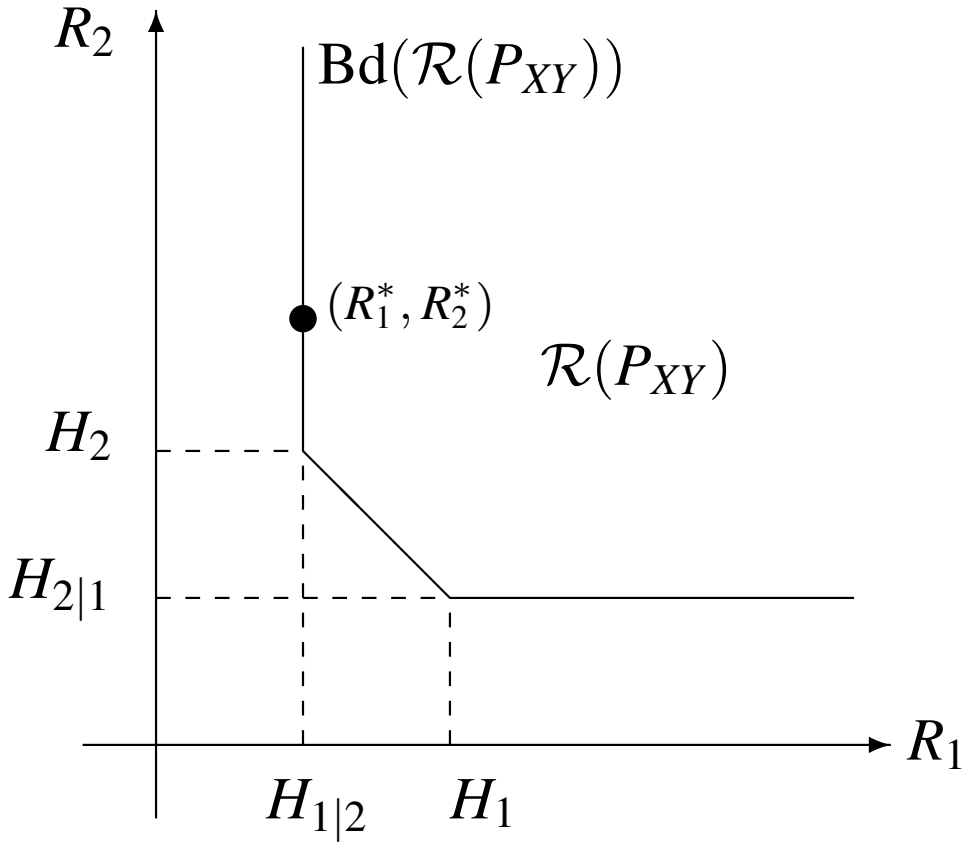

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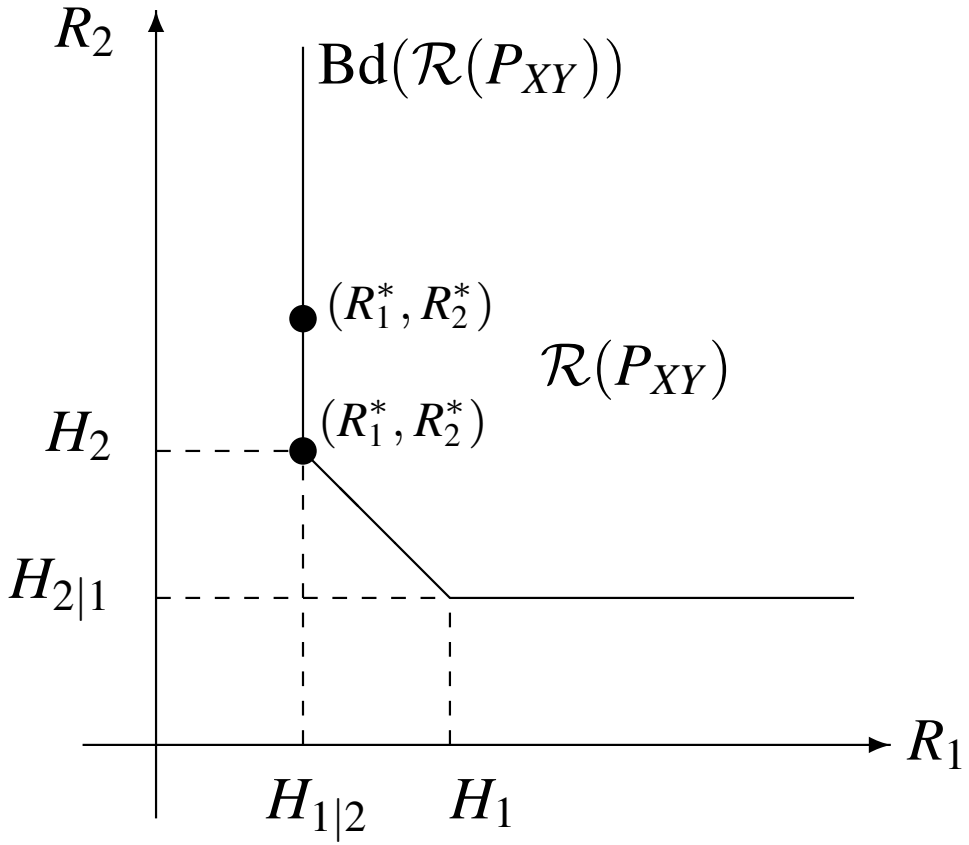

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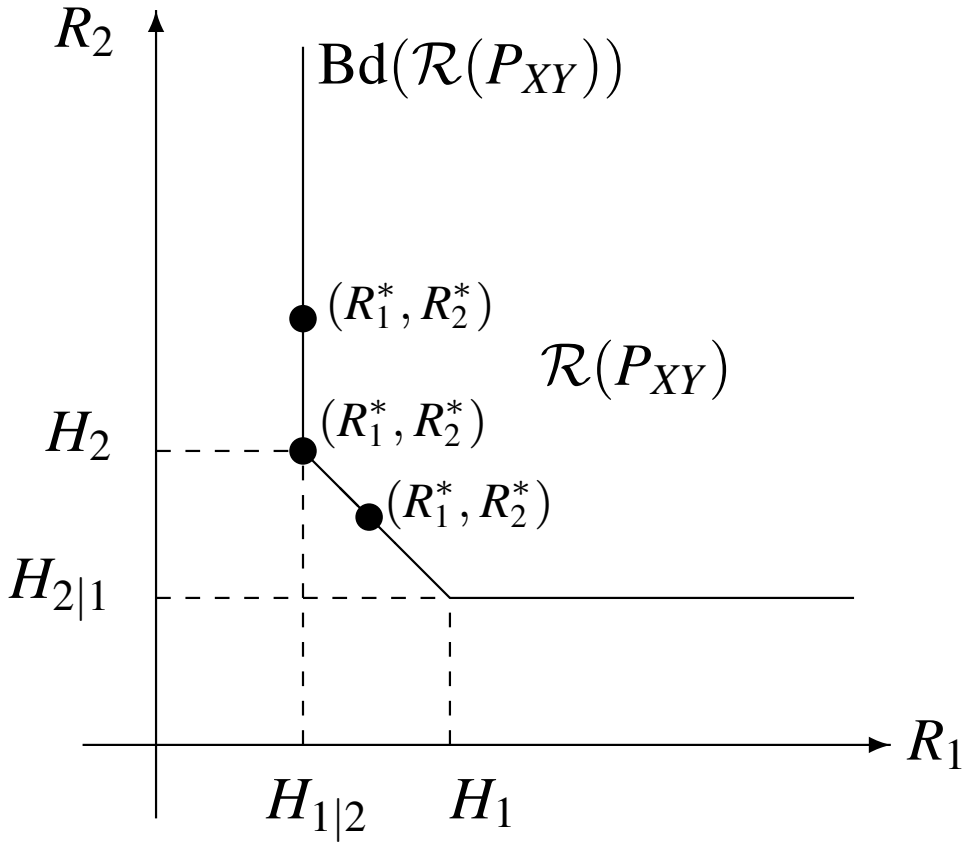

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$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_{1n} - nR_1^*) \le L_1$$

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_{2n} - nR_2^*) \le L_2$$

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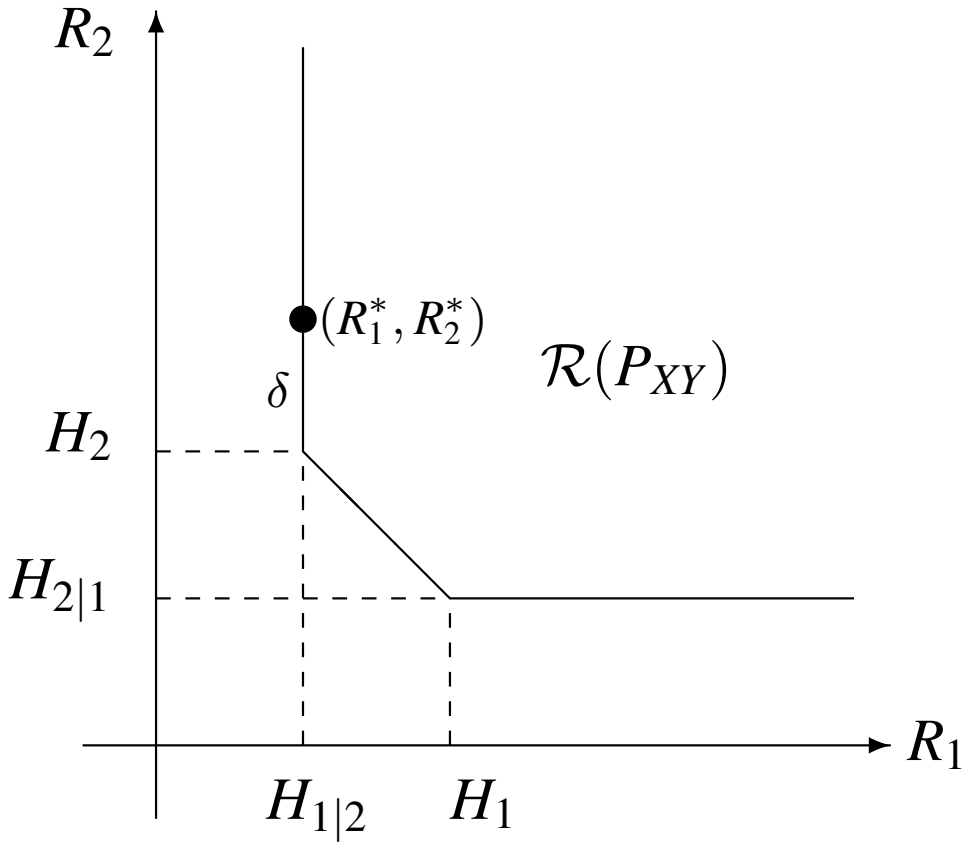
- $\mathcal{L}(\varepsilon; R_1^*, R_2^*)$ is the set of all $(R_1^*, R_2^*, \varepsilon)$-achievable (L_1, L_2) pairs

Case: Non-Corner Point



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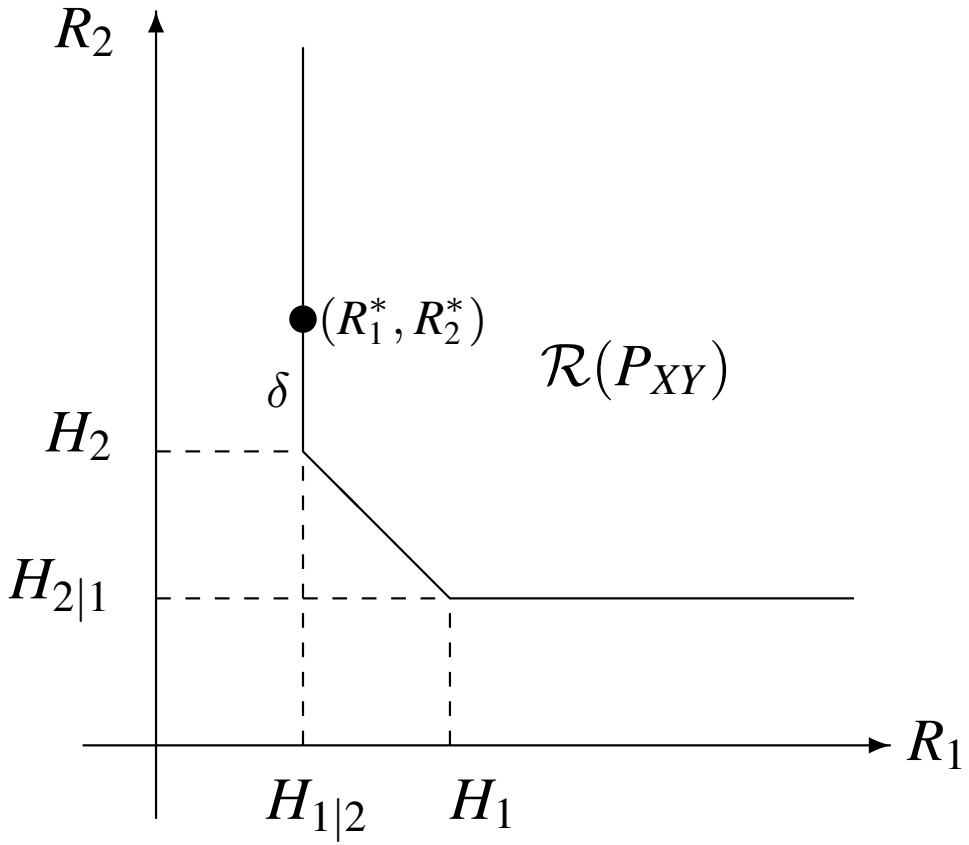

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Suppose $(R_1^*, R_2^*) = (H(X_1|X_2), H(X_2) + \delta)$ for some $\delta > 0$.

Second-Order Asymptotics for Non-Corner Point Case

Theorem (Tan–Kosut (2014), Nomura–Han (2015))

When $(R_1^*, R_2^*) = (H(X_1|X_2), H(X_2) + \delta)$ *for some* $\delta > 0$*, the set*

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*) = \left\{(L_1, L_2) : L_1 \geq \sqrt{V(X_1|X_2)}\Phi^{-1}(1-\varepsilon)\right\}$$

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When $(R_1^*, R_2^*) = (H(X_1|X_2), H(X_2) + \delta)$ *for some* $\delta > 0$*, the set*

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*) = \left\{(L_1, L_2) : L_1 \geq \sqrt{V(X_1|X_2)}\Phi^{-1}(1-\varepsilon)\right\}$$

- Same as lossless source coding with full decoder side information

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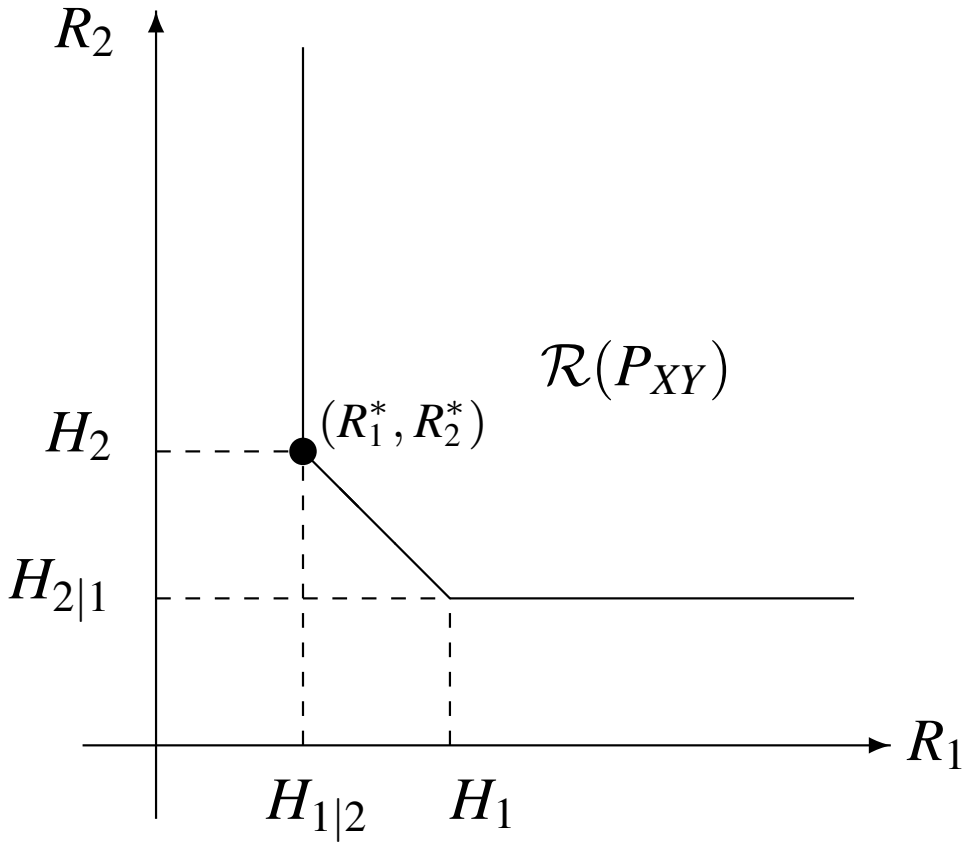
- Same as lossless source coding with full decoder side information
- No constraints on $L_2 \in \mathbb{R}$
- Intuition is that R_2^* is strictly above $H(X_2)$ which means

$$\Pr(\hat{X}_2^n \neq X_2^n) \leq \exp(-nc(\delta))$$

for some $c(\delta) > 0$. Doesn't affect second-order asymptotics.

Corner-Point Case



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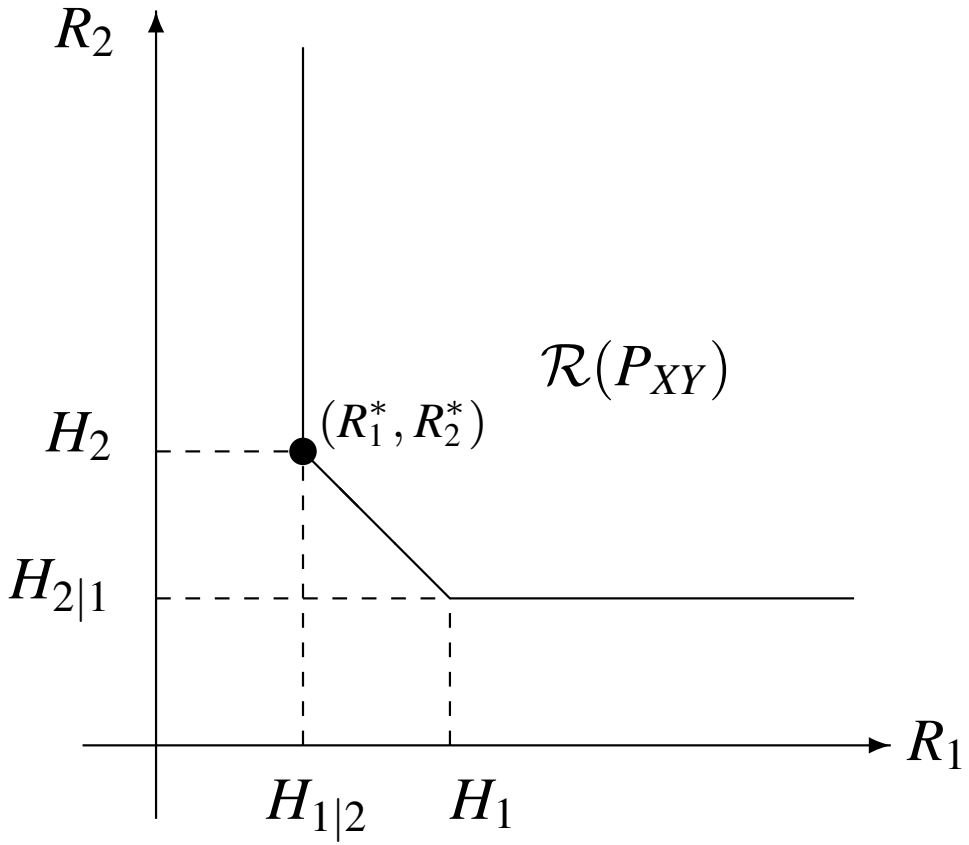

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Suppose $(R_1^*, R_2^*) = (H(X_1|X_2), H(X_2))$, i.e., near the corner point.

Second-Order Asymptotics for Corner-Point Case

Theorem (Tan–Kosut (2014), Nomura–Han (2015))

When $(R_1^*, R_2^*) = (H(X_1|X_2), H(X_2))$,

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*) = \left\{(L_1, L_2) : \Psi(L_1, L_1 + L_2; [\mathbf{V}]_{1,12}) \geq 1 - \varepsilon\right\}$$

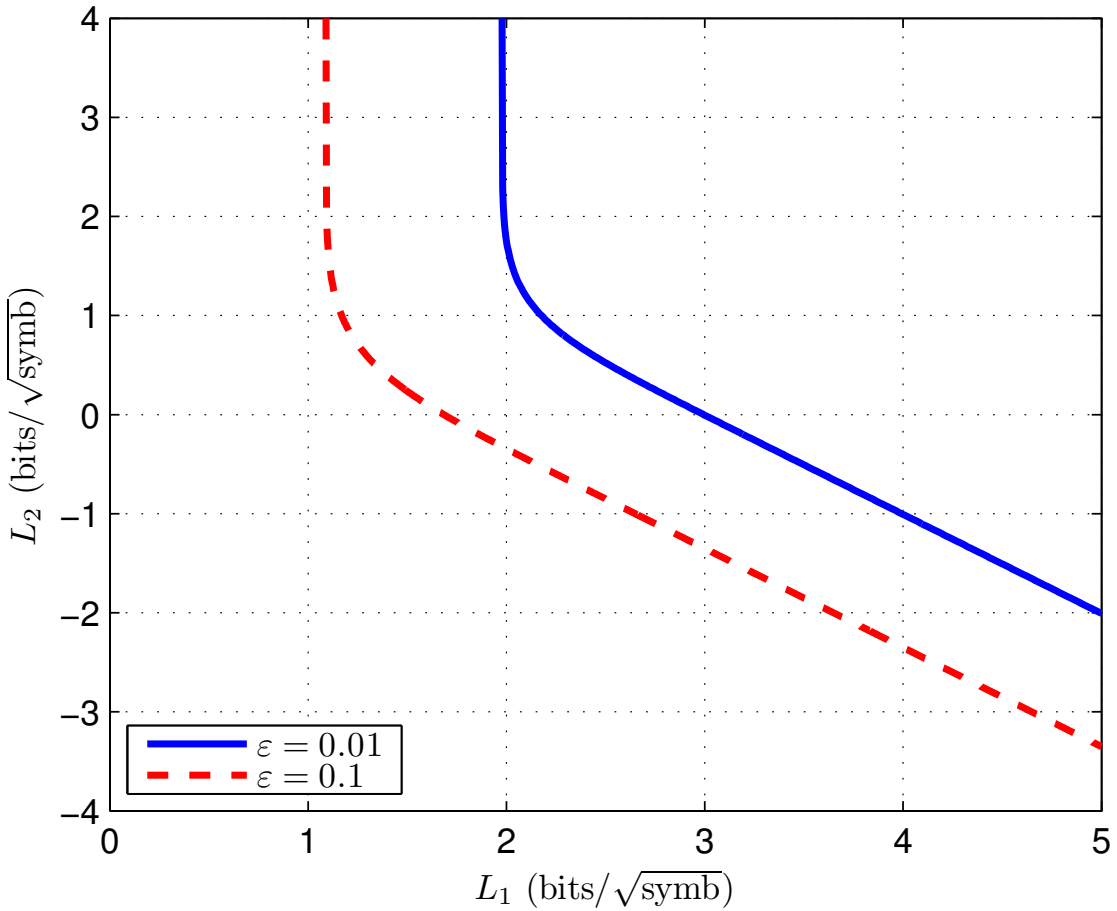
where

$$\Psi(a, b; \mathbf{V}) = \int_{-\infty}^{a} \int_{-\infty}^{b} \mathcal{N}(\mathbf{x}; \mathbf{0}, \mathbf{V})\, d\mathbf{x}.$$

and

$$\mathbf{V} = \mathrm{Cov} \begin{bmatrix} -\log P_{X_1|X_2}(X_1|X_2) \\ -\log P_{X_2|X_1}(X_2|X_1) \\ -\log P_{X_1X_2}(X_1, X_2) \end{bmatrix}.$$

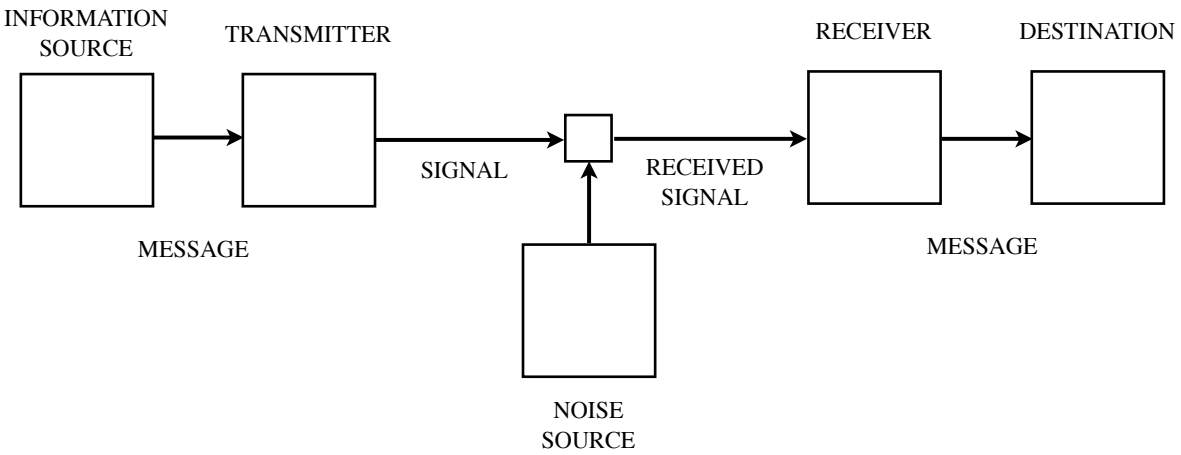
Second-Order Asymptotics for Corner-Point Case

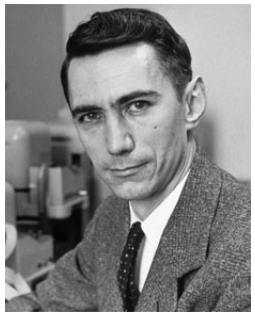


$\mathcal{L}(\varepsilon; R_1^*, R_2^*)$ for the source $\begin{bmatrix} 0.7 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$ with $\varepsilon = 0.01, 0.1$.

Outline

Transmission of Information





Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission

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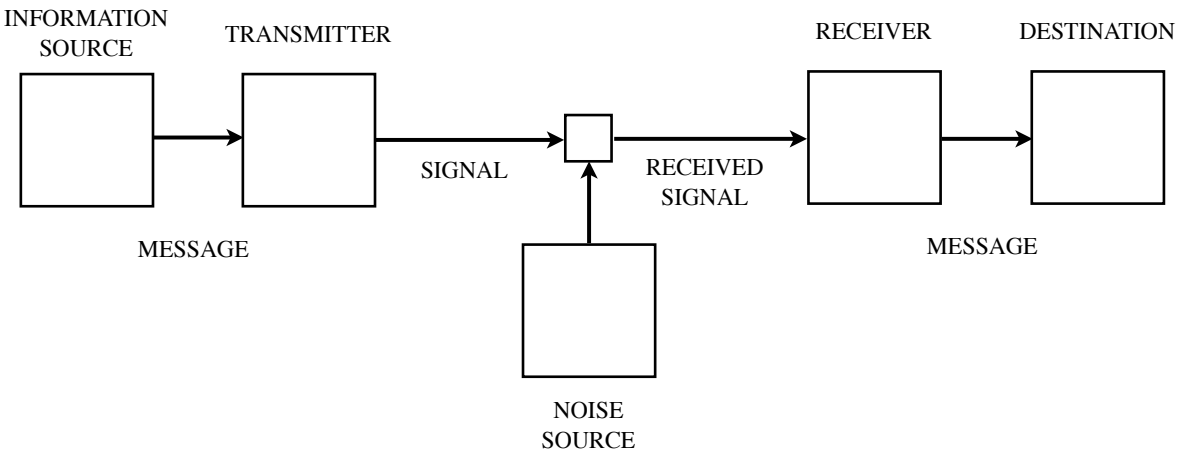
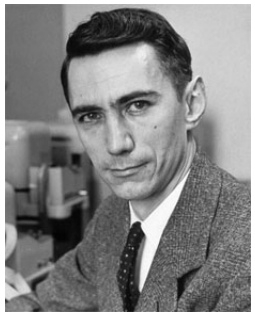
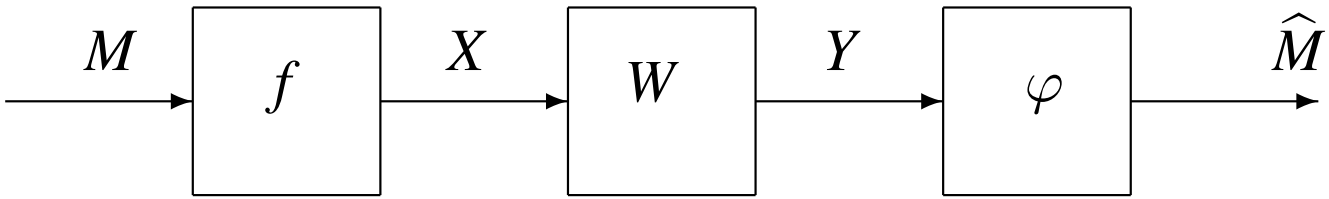




Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission
- Channel coding: Concerned with the maximum rate of communication in bits per channel use

Channel Coding (One-Shot)



- A code is an triple $\mathcal{C} = \{\mathcal{M}, f, \varphi\}$ where $\mathcal{M}$ is the message set
- The average error probability $p_{\text{err}}(\mathcal{C})$ is

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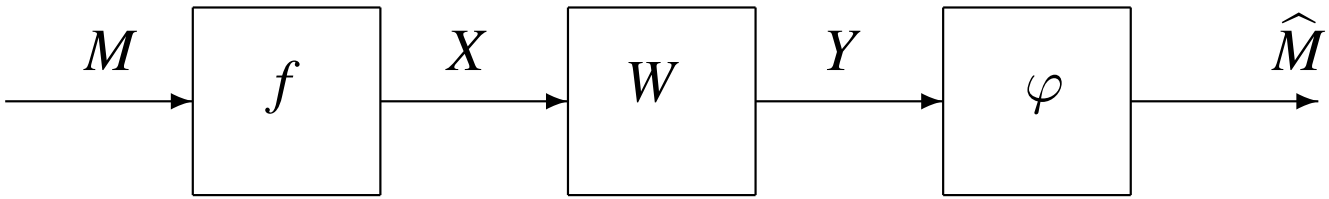

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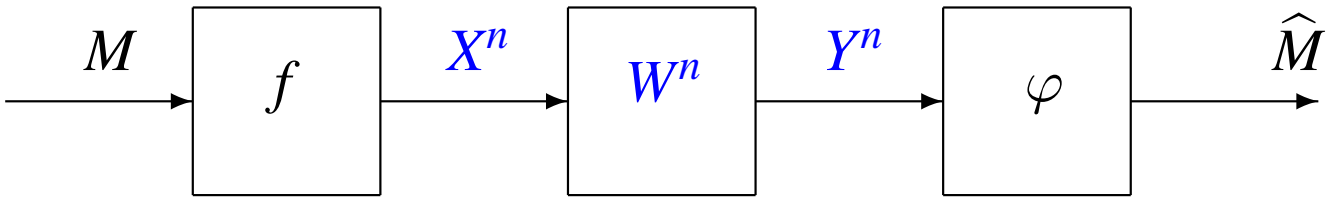
  where M is uniform on $\mathcal{M}$.

- A non-asymptotic fundamental limit can be defined as

$$M^*(W, \varepsilon) := \max\left\{m \in \mathbb{N} \ : \ \exists \mathcal{C} \ \text{ s.t. } \ m = |\mathcal{M}|, \, p_{\text{err}}(\mathcal{C}) \leq \varepsilon\right\}$$

- Central problem in information theory to characterize $M^*(W, \varepsilon)$.

Channel Coding (n-Shot)



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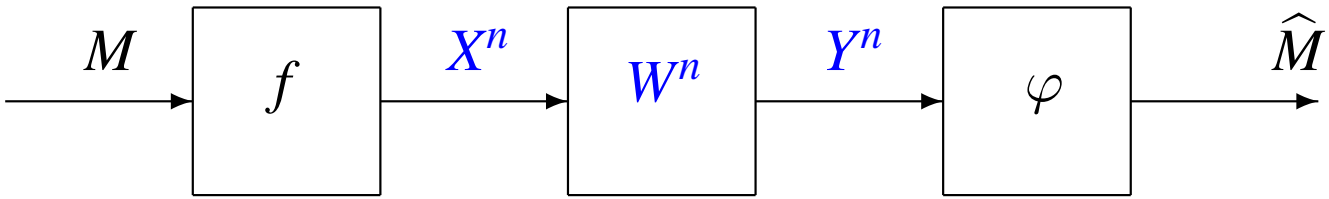

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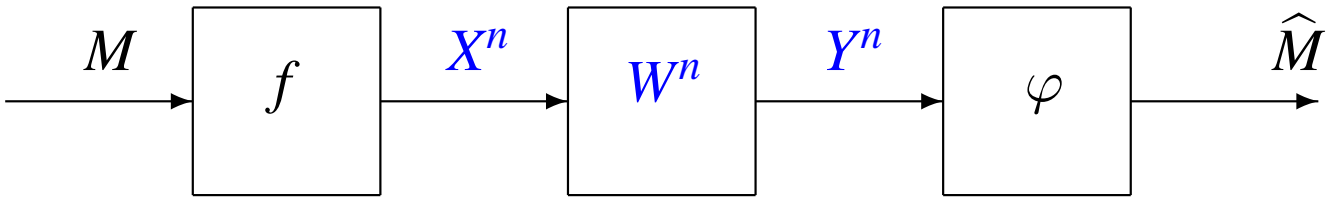

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$$\log M^*(W^n, \varepsilon)$$

Second-Order Asymptotics

Theorem (Strassen '62, Hayashi '09, Polyanskiy–Poor–Verdú '10)

For all "well-behaved" DMCs and Gaussian channels,

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M. Hayashi

Polyanskiy–Poor–Verdú

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Theorem (Tomamichel–Tan (2013))

If W is a DMC with positive ε-dispersion,

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IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 59, NO. 11, NOVEMBER 2013 7041

A Tight Upper Bound for the Third-Order Asymptotics for Most Discrete Memoryless Channels

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***Abstract*—This paper shows that the logarithm of the ε-error capacity (average error probability) for n uses of a discrete memoryless channel (DMC) is upper bounded by the normal approximation plus a third-order term that does not exceed $\frac{1}{2}\log n + O(1)$ if the ε-dispersion of the channel is positive. This matches a lower bound by Y. Polyanskiy (2010) for DMCs with positive reverse dispersion. If the ε-dispersion vanishes, the logarithm of the ε-error capacity is upper bounded by n times the capacity plus a constant term except for a small class of DMCs and $\varepsilon \geq \frac{1}{2}$.**

where $\rho_n = O(\log n)$, V_ε is the *ε-channel dispersion* [4], [5], and $\Phi(\cdot)$ is the Gaussian cumulative distribution function.[1] These quantities will be defined precisely in Section II-A. In fact, this asymptotic expansion also holds for $M^*(W^n, \varepsilon)$ [4, (284)–(286)] and implies that if an error probability of ε is tolerable, the backoff from channel capacity C at finite blocklength n is roughly $\sqrt{V_\varepsilon/n}\,\Phi^{-1}(\varepsilon)$. There have been several recent refinements to and extensions of Strassen's normal

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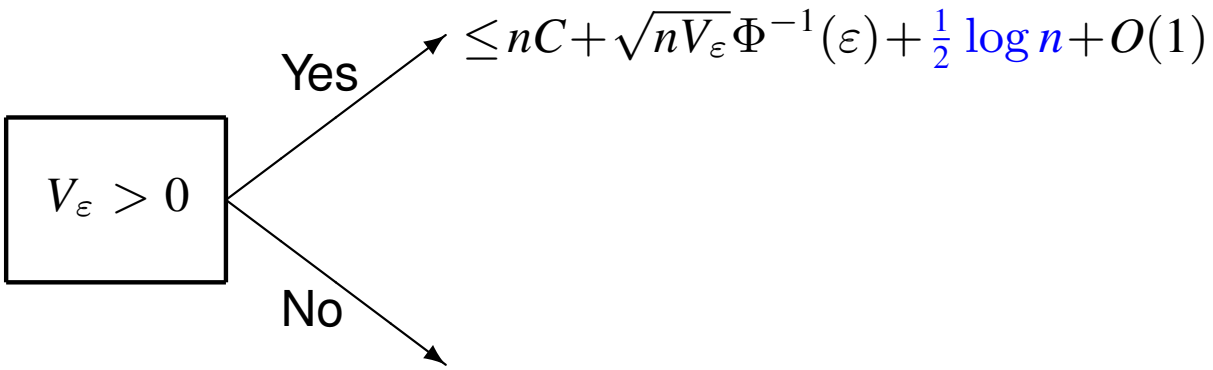
$\rho_n \geq \frac{1}{2}\log n + O(1)$ for non-singular DMCs (e.g., BSCs) [Polyanskiy '10].

Main Result: Tight Third-Order Term

All cases are covered



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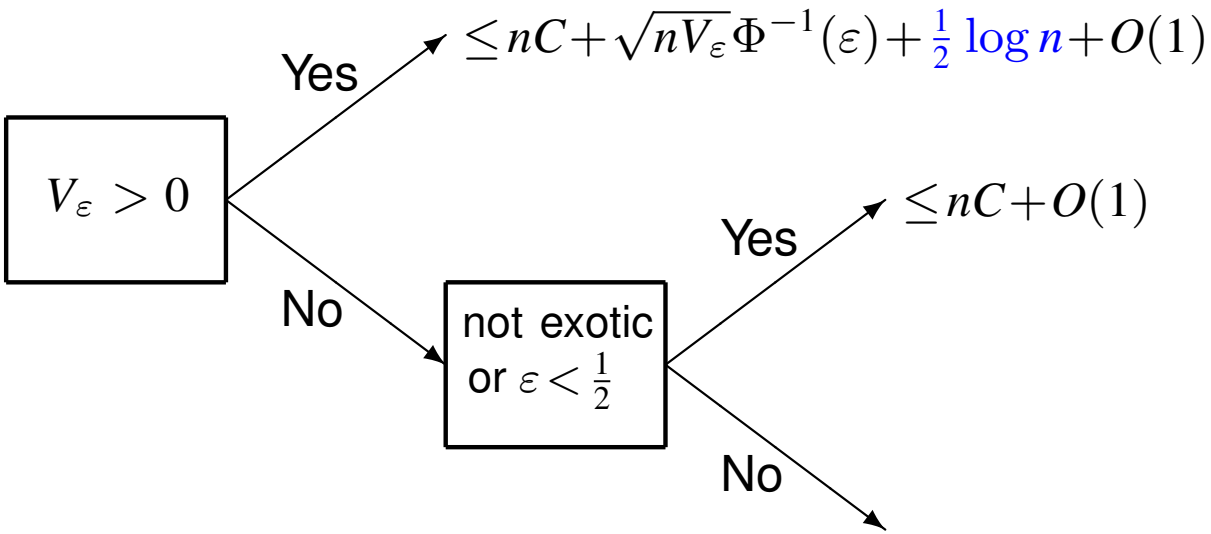

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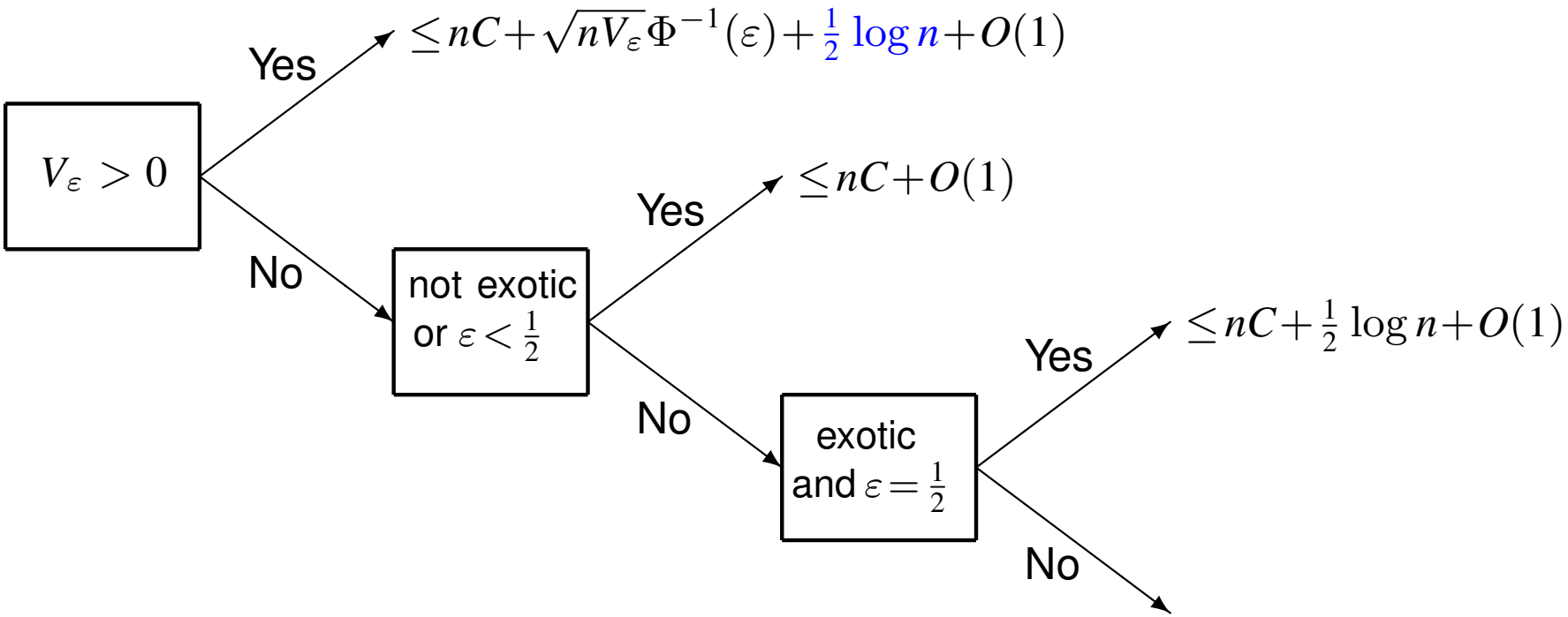

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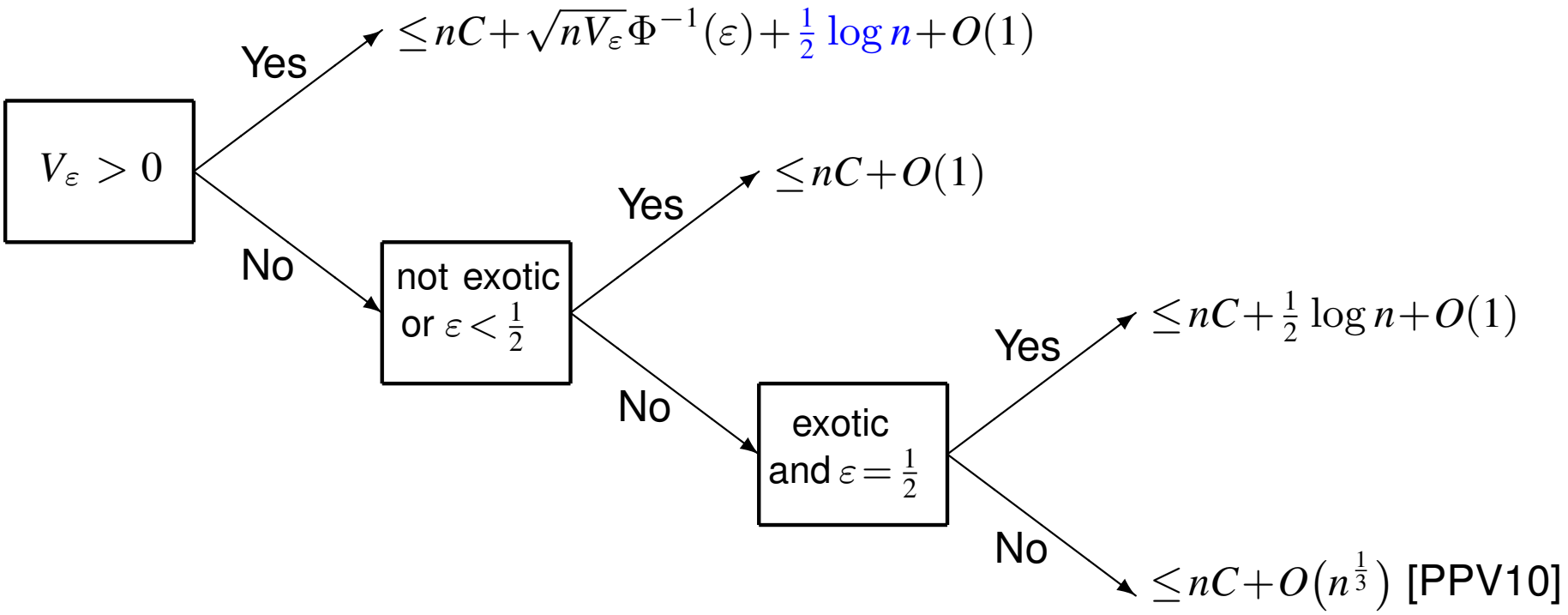

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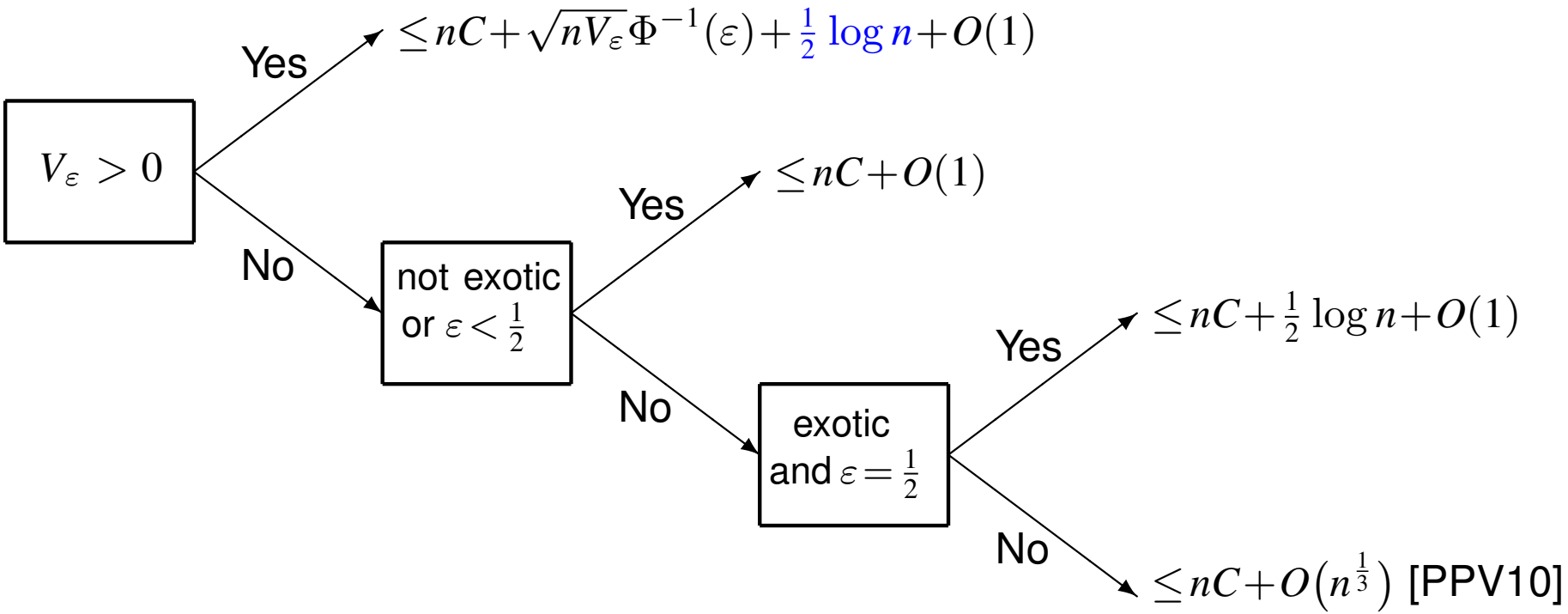

W is exotic if $V_{\max}(W) = 0$ and $\exists\, x_0 \in \mathcal{X}$ such that

$$D(W(\cdot|x_0) \,\|\, Q^*) = C, \quad \text{and} \quad V(W(\cdot|x_0) \,\|\, Q^*) > 0.$$

Proof Technique for Tight Third-Order Term

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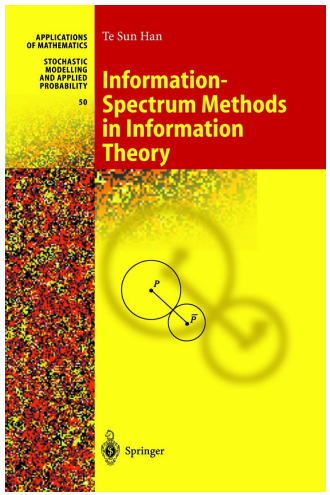
- Need a convenient converse bound for general DMCs.
- Information spectrum divergence

$$D_s^{\varepsilon}(P \,\|\, Q) := \sup\left\{R \geq 0 \,:\, P\left\{x : \log \frac{P(x)}{Q(x)} \leq R\right\} \leq \varepsilon\right\}$$



Te Sun Han

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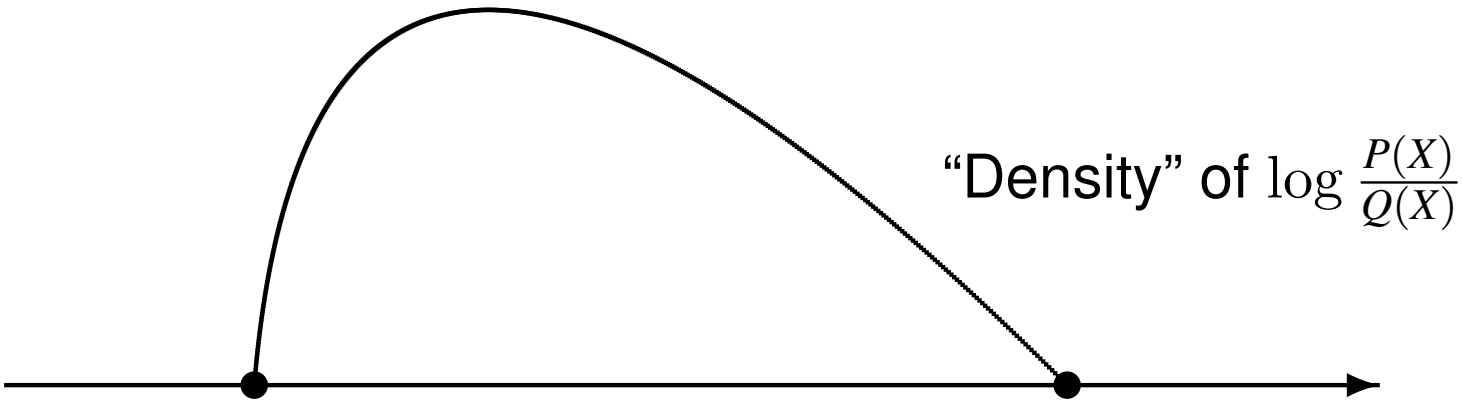

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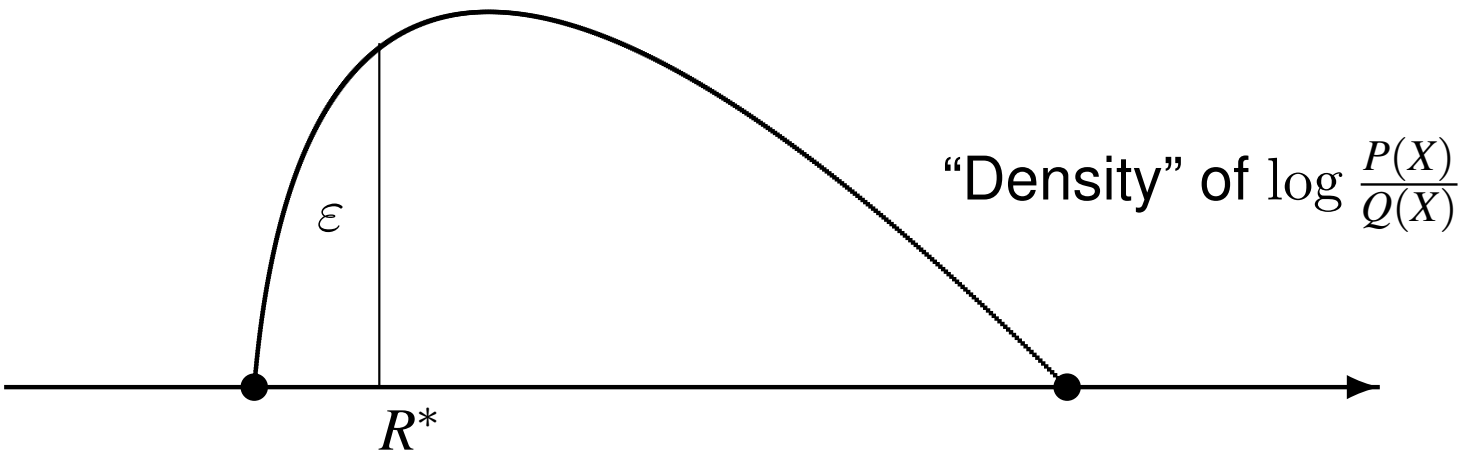

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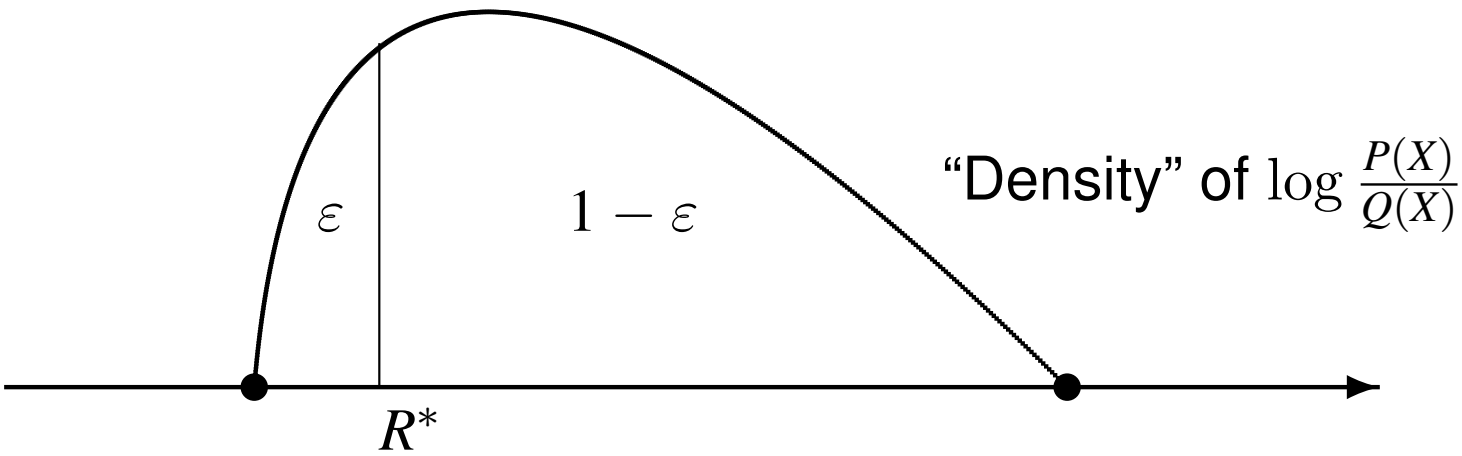

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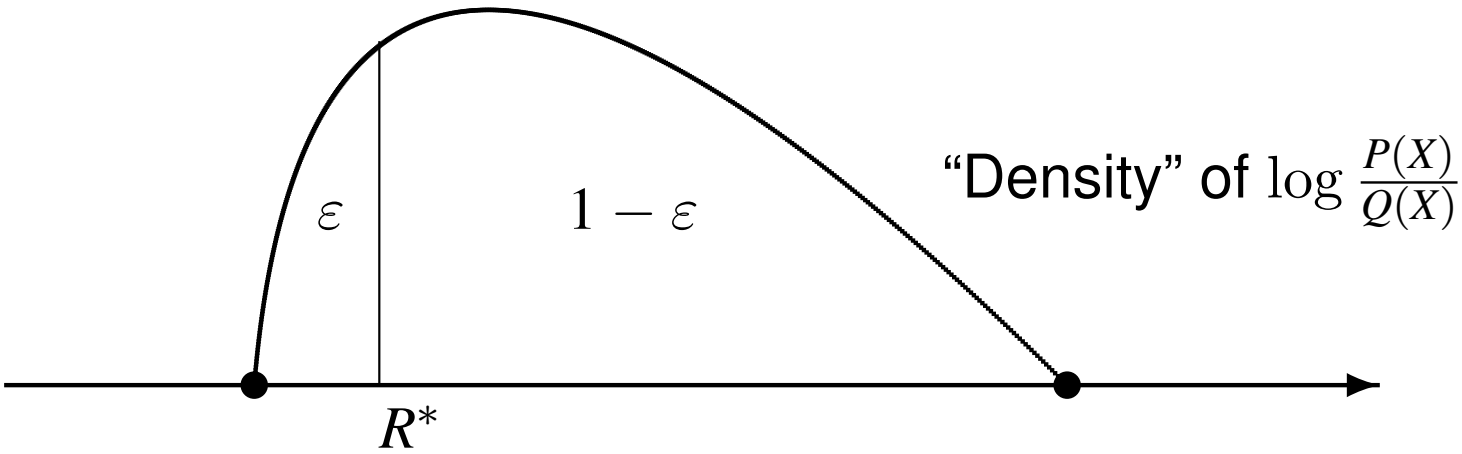

If X^n is i.i.d. P, the Berry–Esseen theorem yields

$$D_s^{\varepsilon}(P^n \,\|\, Q^n) = nD(P \,\|\, Q) + \sqrt{nV(P \,\|\, Q)}\Phi^{-1}(\varepsilon) + O(1)$$

Proof Technique: Symbol-Wise Converse Bound

Lemma (Tomamichel–Tan (2013))

For every channel W, every $\varepsilon \in (0,1)$ and $\delta \in (0, 1-\varepsilon)$, we have

$$\log M^*(W,\varepsilon) \leq \min_{Q \in \mathcal{P}(\mathcal{Y})} \max_{x \in \mathcal{X}} D_s^{\varepsilon+\delta}(W(\cdot|x) \,\|\, Q) + \log \frac{1}{\delta}$$

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- Choose $\delta = n^{-\frac{1}{2}}$ so $\log \frac{1}{\delta} = \frac{1}{2} \log n$
- Since all $\mathbf{x}$ within a type class result in the same $D_s^{\varepsilon+\delta}$ (if $Q^{(n)}$ is permutation invariant), it's really a $\max$ over types $P_{\mathbf{x}} \in \mathcal{P}_n(\mathcal{X})$

Proof Technique: Choice of Output Distribution

$$\log M^*(W^n, \varepsilon) \leq \max_{\mathbf{x} \in \mathcal{X}^n} D_s^{\varepsilon+\delta}(W^n(\cdot|\mathbf{x}) \,\|\, Q^{(n)}) + \log \frac{1}{\delta}, \qquad \forall\, Q^{(n)} \in \mathcal{P}(\mathcal{Y}^n)$$

- $Q^{(n)}(\mathbf{y})$: invariant to permutations of the n channel uses

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- First term: $Q_{\mathbf{k}}$'s and $\lambda(\mathbf{k})$'s designed to form an $n^{-\frac{1}{2}}$-cover of $\mathcal{P}(\mathcal{Y})$:

$$\forall\, Q \in \mathcal{P}(\mathcal{Y}), \quad \exists\, \mathbf{k} \in \mathcal{K} \quad \text{s.t.} \quad \|Q - Q_{\mathbf{k}}\|_2 \leq n^{-\frac{1}{2}}.$$

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- Second term: Uniform mixture over output distributions induced by input types [Hayashi (2009)]
- Take cares of "bad input types" (i.e., types $P \in \mathcal{P}_n(\mathcal{X})$ such that PW is far from Q^*)

Proof Technique: Novel Choice of Output Distribution

- First term is

$$\sum_{\mathbf{k}\in\mathcal{K}} \lambda(\mathbf{k}) Q_{\mathbf{k}}^n(\mathbf{y}), \qquad \lambda(\mathbf{k}) = \frac{\exp\left(-\gamma\|\mathbf{k}\|_2^2\right)}{Z}, \qquad \sum_{\mathbf{k}\in\mathcal{K}} \lambda(\mathbf{k}) = 1,$$

and $\mathbf{k} = \{k_y\}_{y\in\mathcal{Y}} \in \mathbb{Z}^{|\mathcal{Y}|}$ indexes distance to the capacity-achieving output distribution (CAOD). Note $Z < \infty$.

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$$Q_{\mathbf{k}}(y) := Q^*(y) + \frac{k_y}{\sqrt{n\zeta}},$$

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- Construction ensures that

$$\forall\, Q \in \mathcal{P}(\mathcal{Y}), \quad \exists\, \mathbf{k} \in \mathcal{K}, \quad \text{s.t.} \quad \|Q - Q_{\mathbf{k}}\|_2 \leq \frac{1}{\sqrt{n}}.$$

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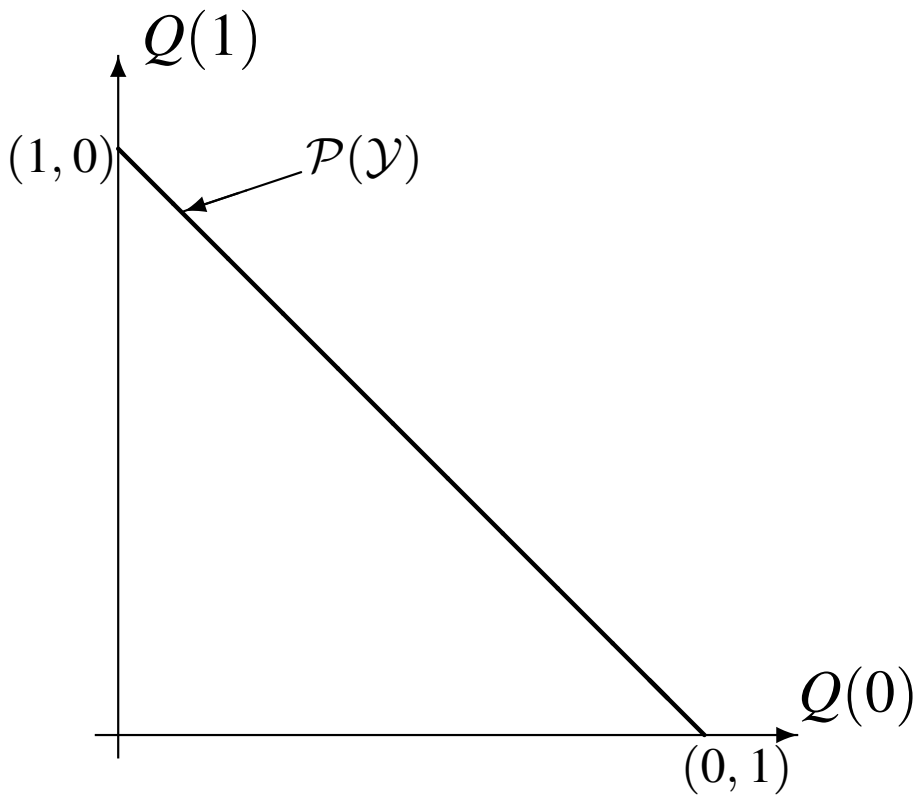
$$Q^{(n)}(\mathbf{y}) := \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{K}} \lambda(\mathbf{k}) Q_{\mathbf{k}}^{n}(\mathbf{y}) + \frac{1}{2}\sum_{P\in\mathcal{P}_n(\mathcal{X})} \frac{1}{|\mathcal{P}_n(\mathcal{X})|}(PW)^n(\mathbf{y})$$



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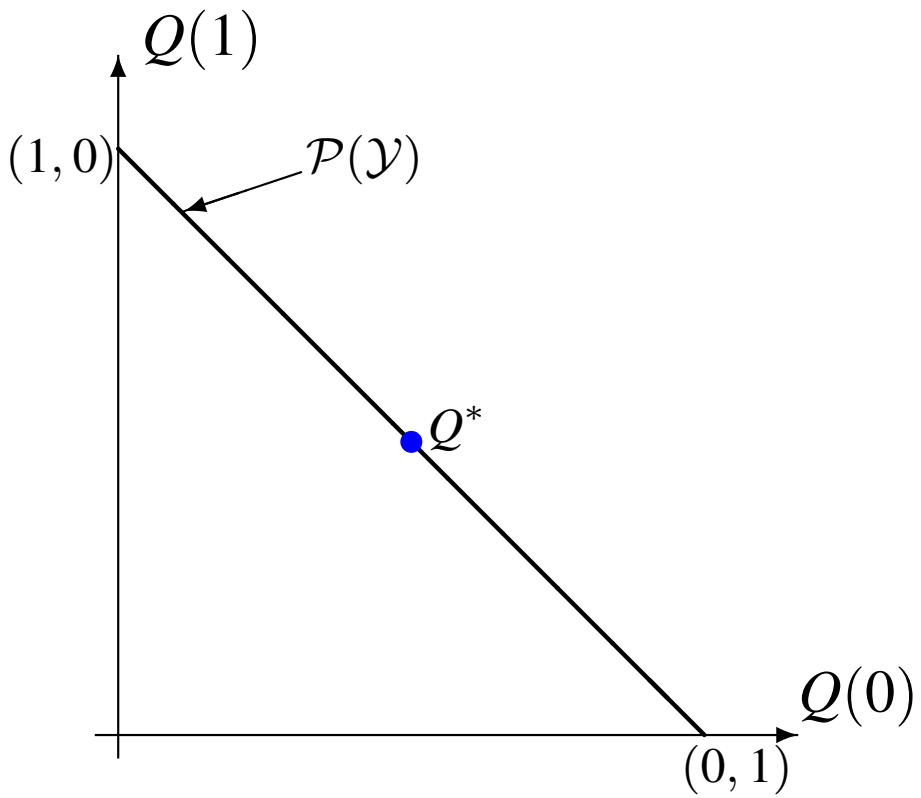

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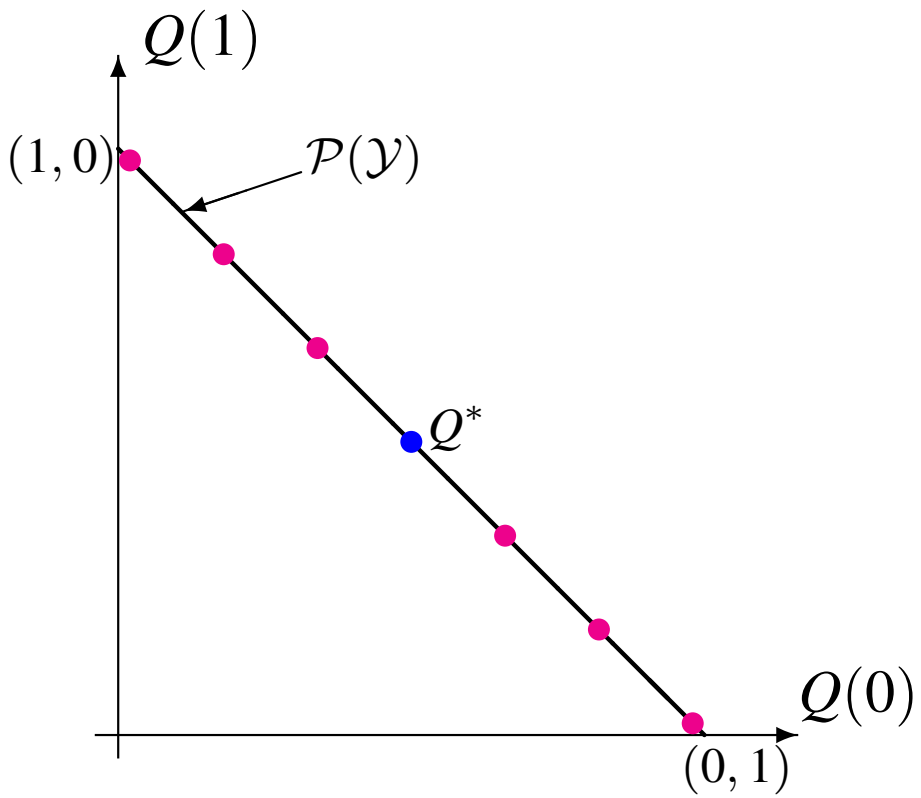

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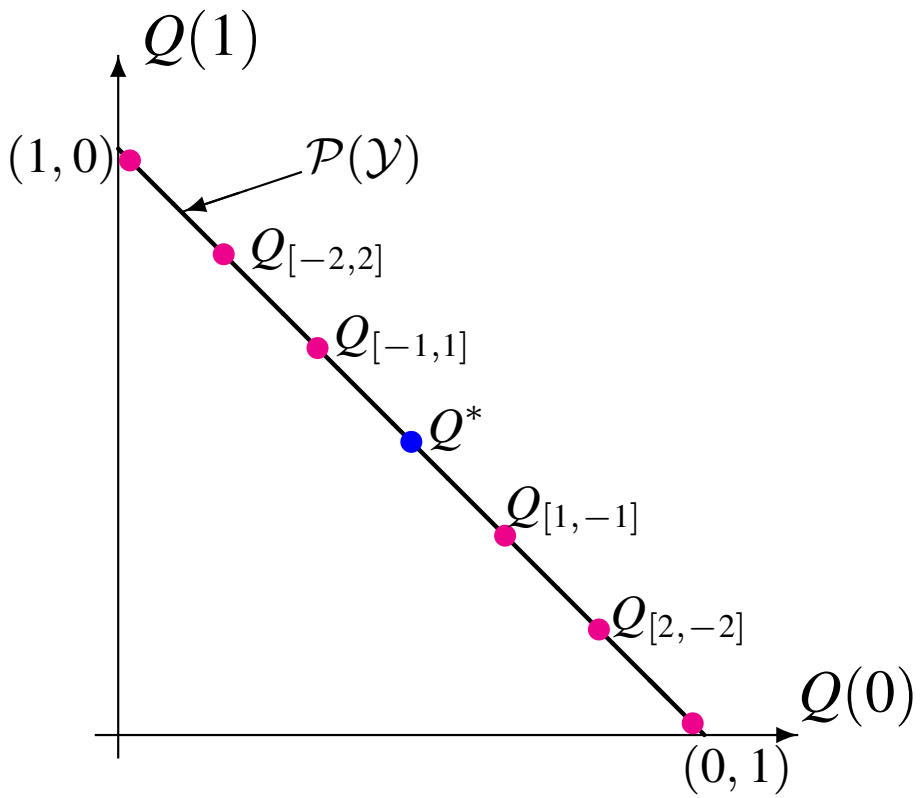

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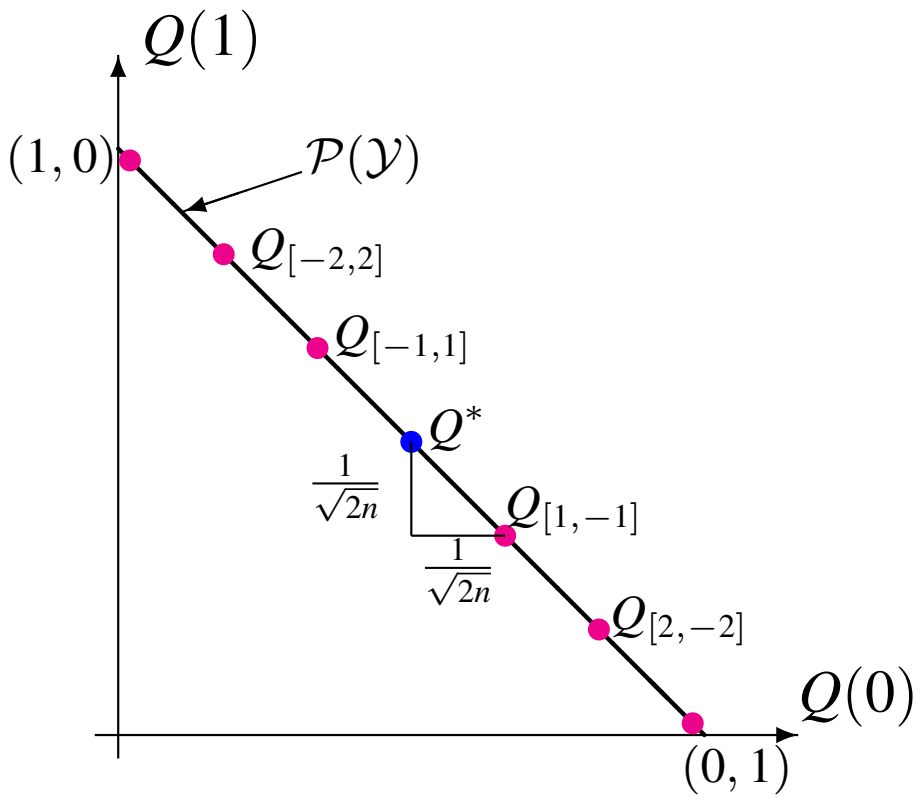

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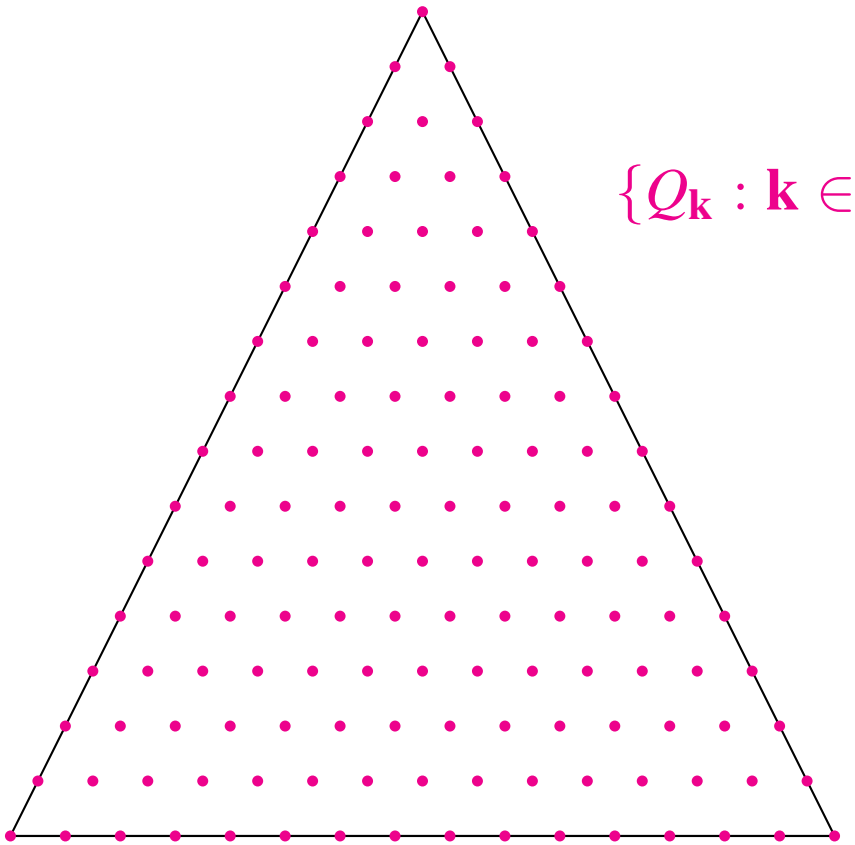
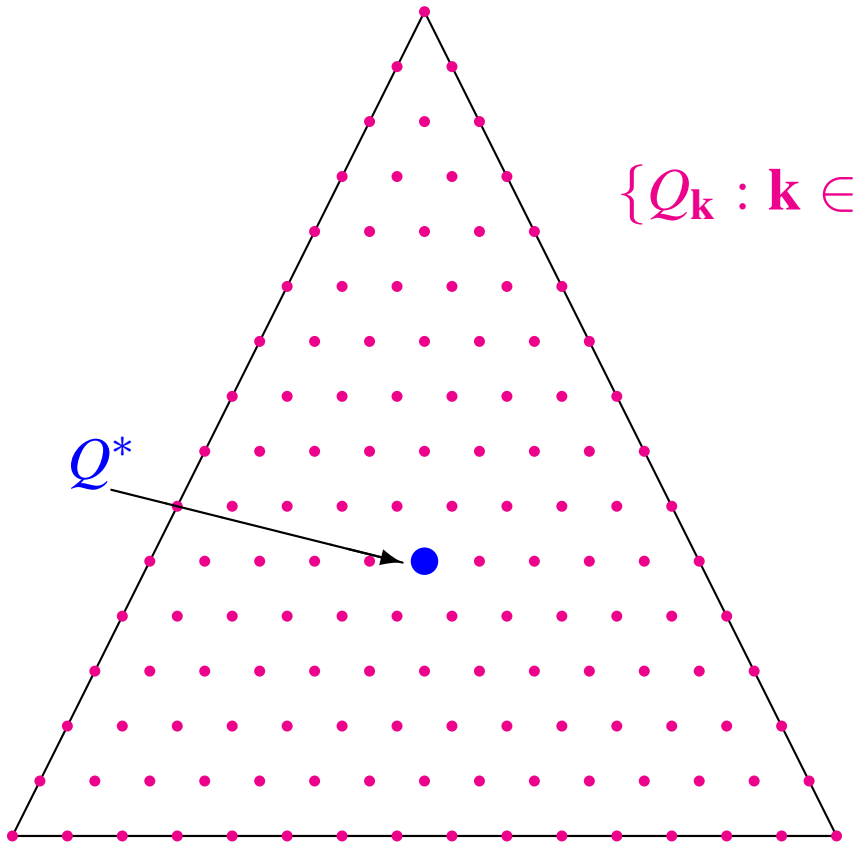


$$\{Q_{\mathbf{k}} : \mathbf{k} \in \mathcal{K}\} \subset \mathcal{P}(\mathcal{Y})$$

Proof Technique: Novel Choice of Output Distribution



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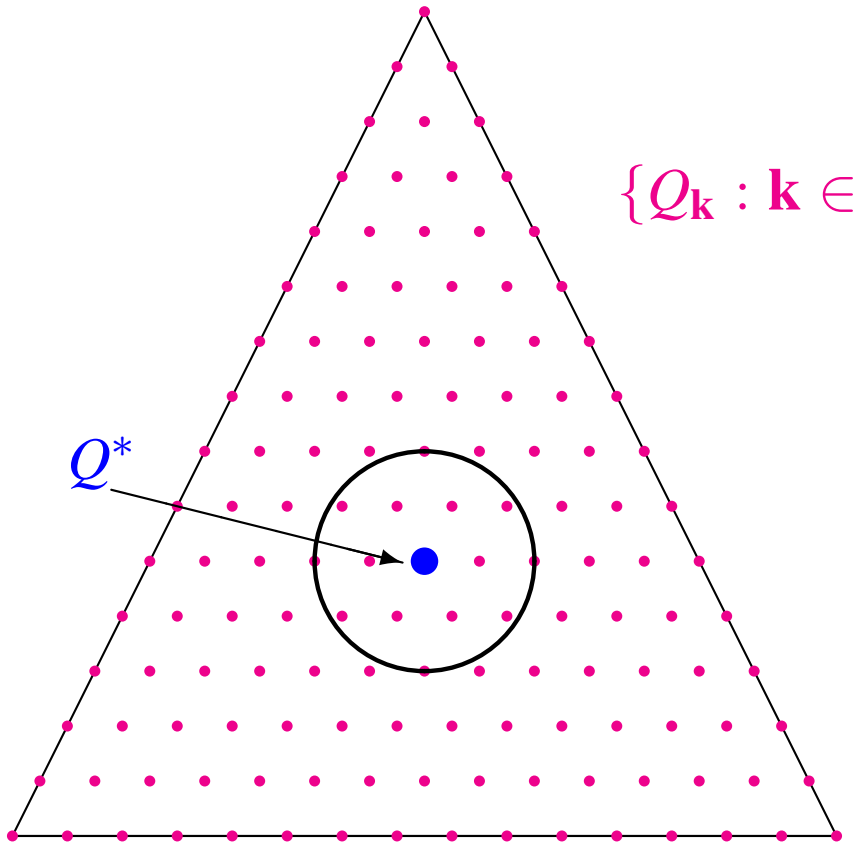

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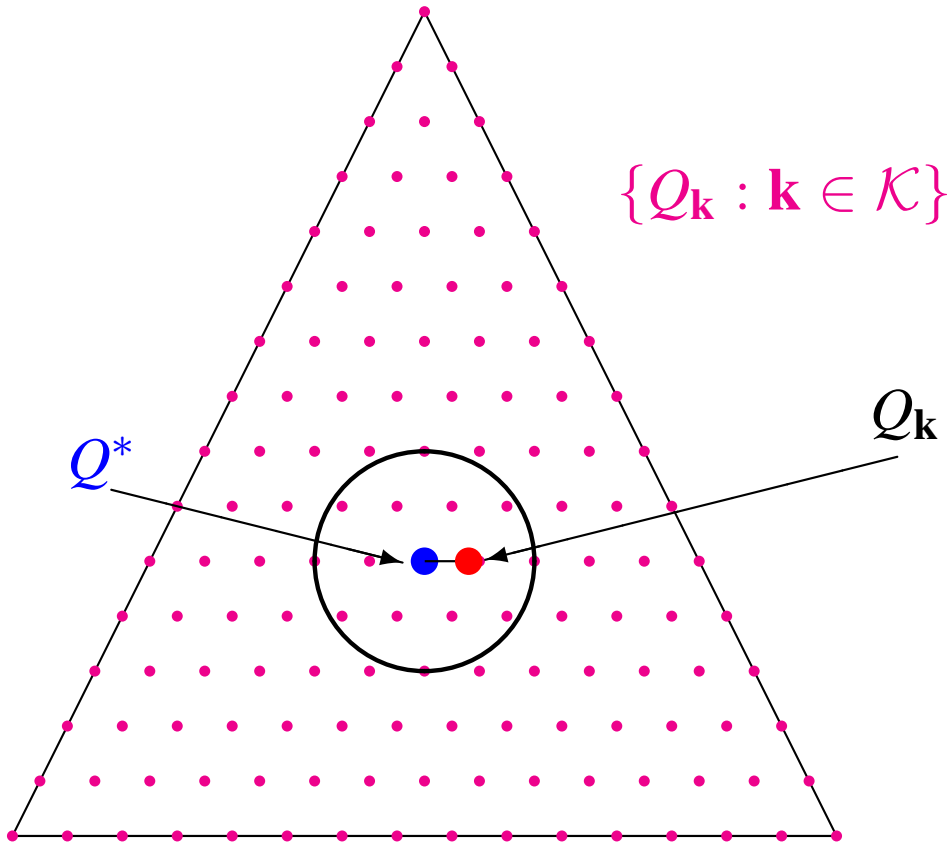

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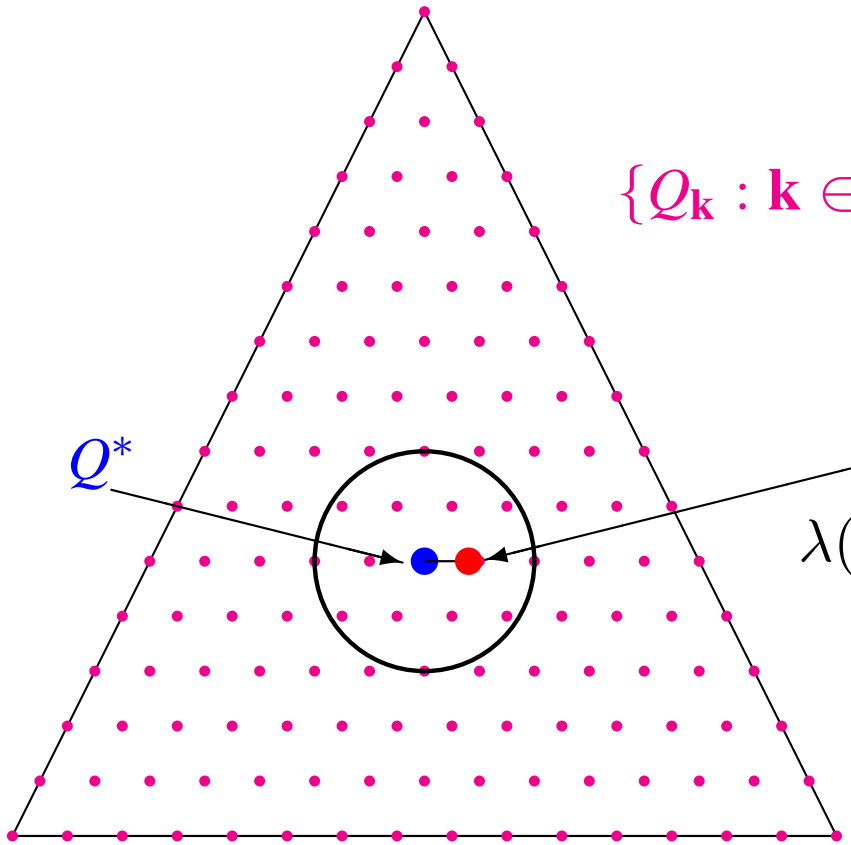

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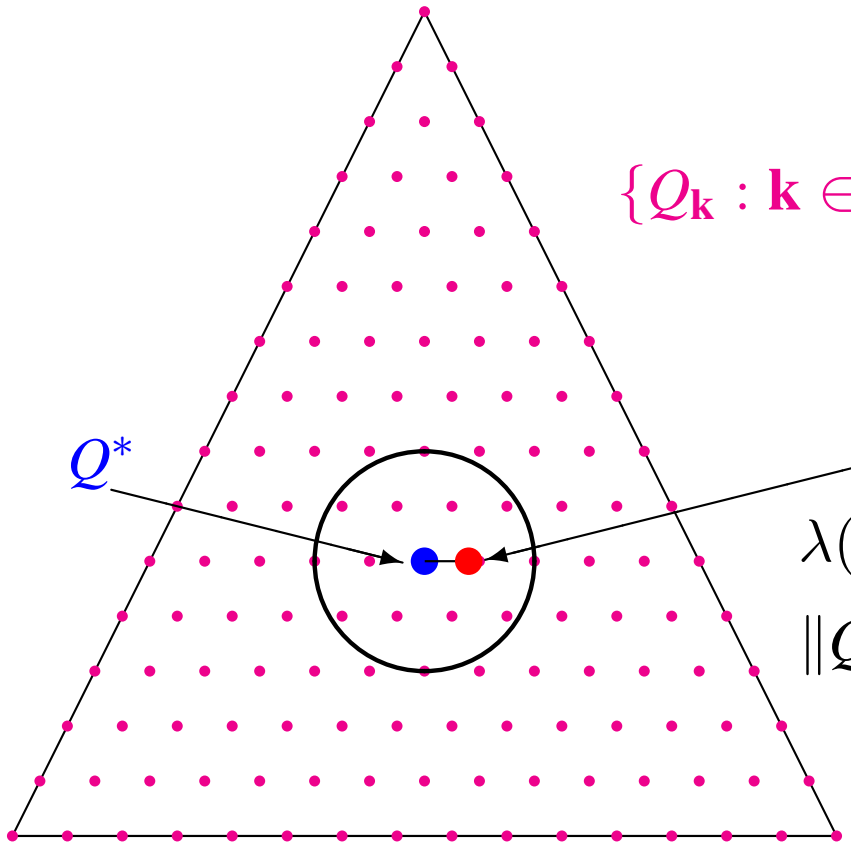

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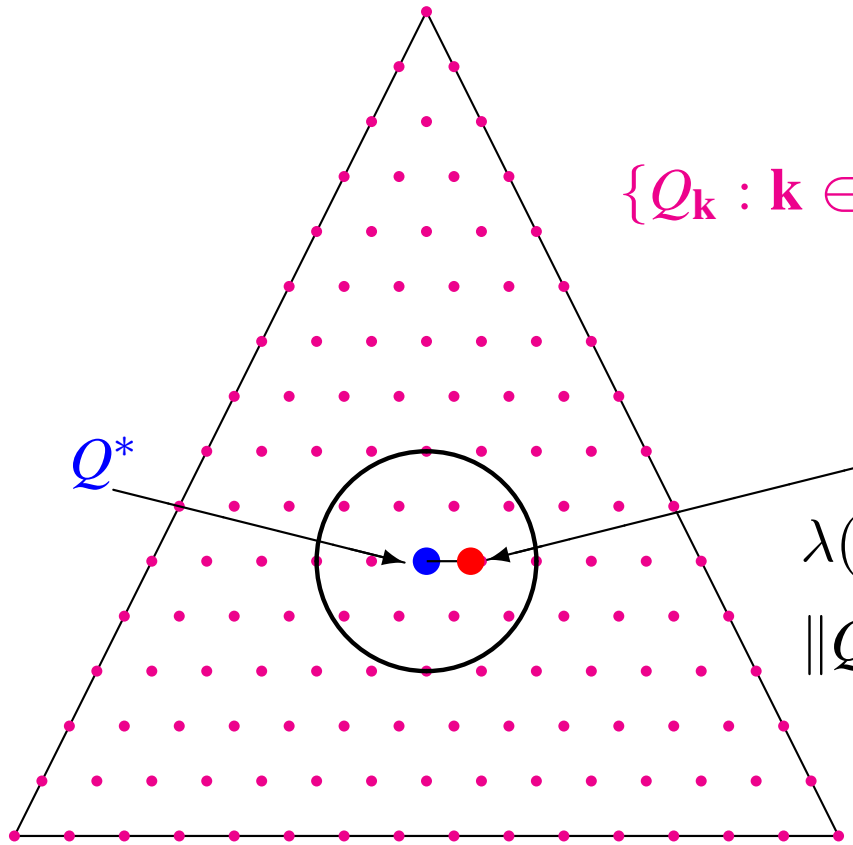
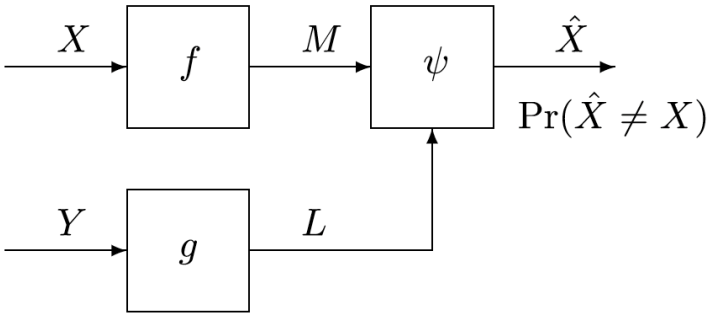


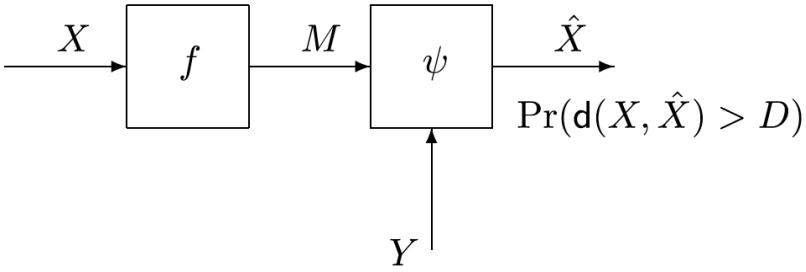
For most DMCs,

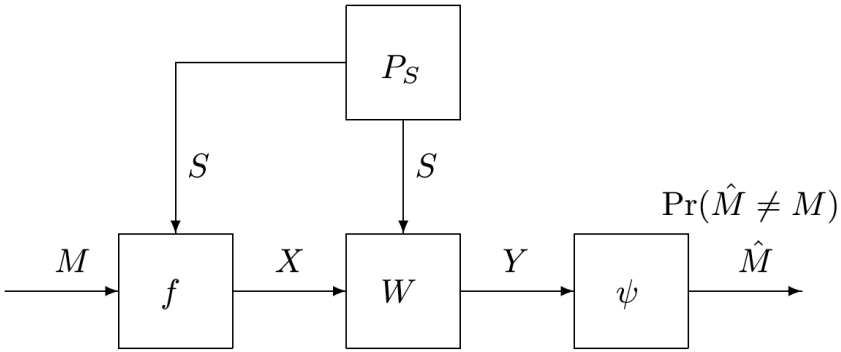
$$\log M^*(W^n, \varepsilon) - \left[nC + \sqrt{nV}\Phi^{-1}(\varepsilon)\right] = \frac{1}{2}\log n + O(1).$$

Other Contributions to Second- and Third-Order

Side Information with S. Watanabe and S. Kuzuoka







Nonasymptotic and Second-Order Achievability Bounds for Coding With Side-Information

Shun Watanabe, *Member, IEEE*, Shigeaki Kuzuoka, *Member, IEEE*, and Vincent Y. F. Tan, *Member, IEEE*

***Abstract*— We present a novel nonasymptotic or finite blocklength achievability bounds for three side-information problems in network information theory. These include: 1) the Wyner-Ahlswede-Körner (WAK) problem of almost-lossless source coding with rate-limited side-information; 2) the Wyner–Ziv (WZ) problem of lossy source coding with side-information at the decoder; and 3) the Gel'fand–Pinsker (GP) problem of channel coding with noncausal state information available at the encoder. The bounds are proved using ideas from channel simulation and channel resolvability. Our bounds for all three problems improve on all previous nonasymptotic**

problems whose asymptotic rate characterizations are well known. These include

- The *Wyner-Ahlswede-Körner* (WAK) problem of almost-lossless source coding with rate-limited (aka coded) side-information [2], [3],
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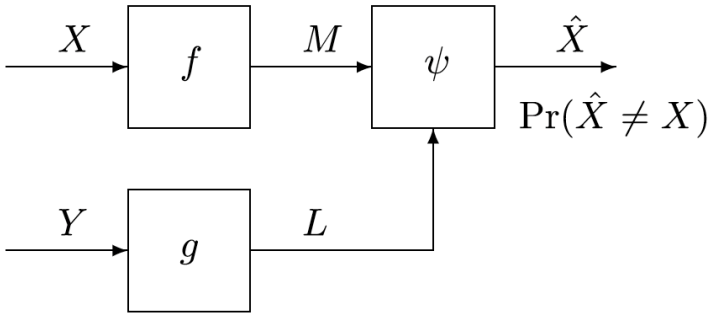

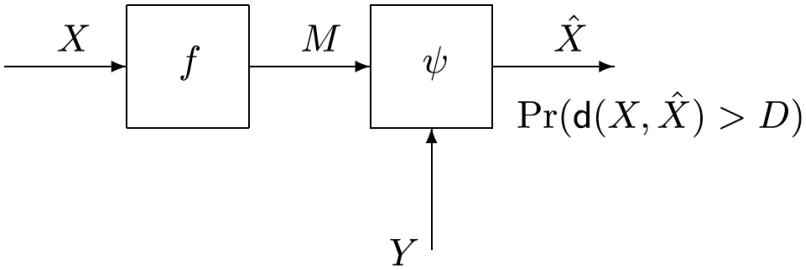

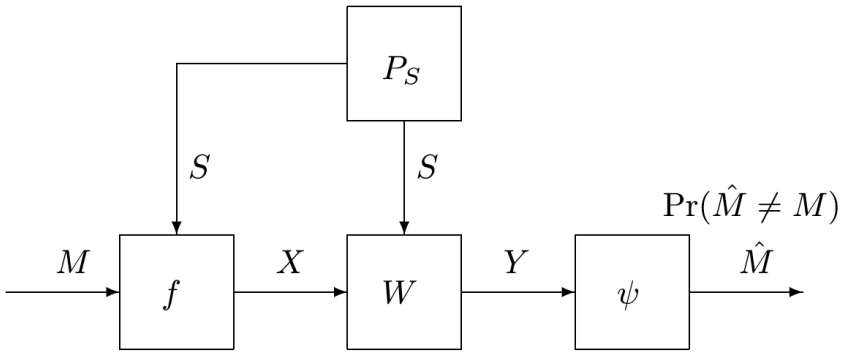

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$$\varepsilon \lessapprox \Pr\Big(\text{covering error} \cup \text{packing error}\Big)$$

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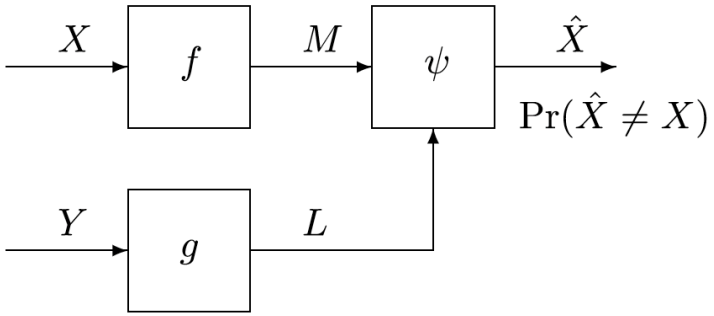

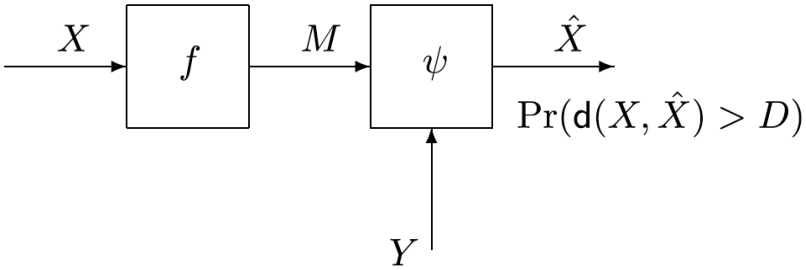

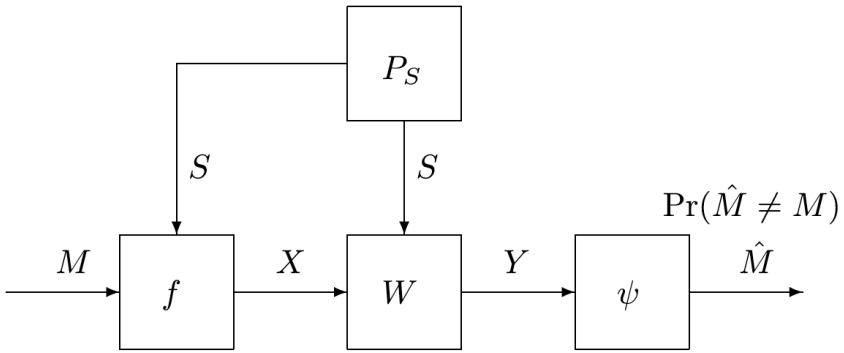

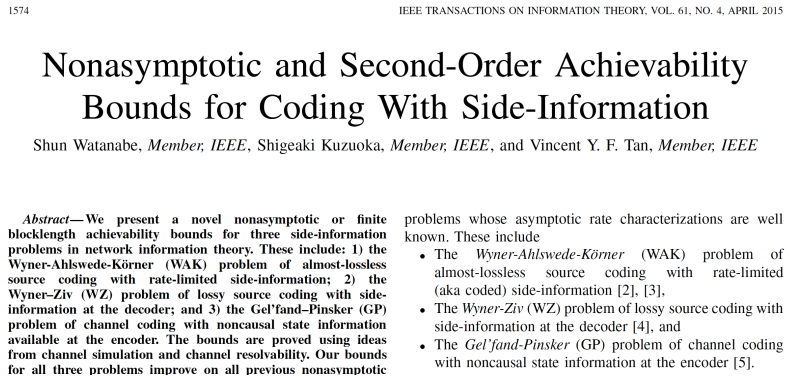

1574 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 61, NO. 4, APRIL 2015

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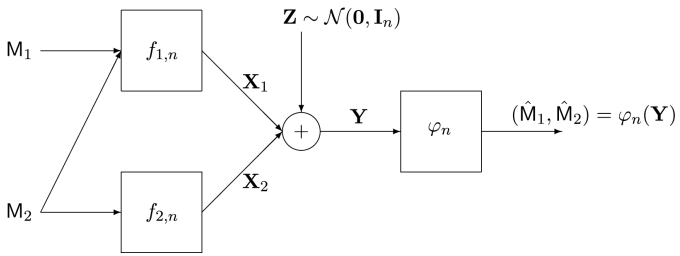
$$\varepsilon \lessapprox \Pr\Big(\text{covering error} \cup \text{packing error}\Big)$$

Also see Jingbo Liu's work for a converse for WAK (IEEE T-IT '20)

Other Contributions to Second- and Third-Order

Gaussian MAC with degraded message sets with J. Scarlett



6700 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 61, NO. 12, DECEMBER 2015

Second-Order Asymptotics for the Gaussian MAC With Degraded Message Sets

Jonathan Scarlett, *Member, IEEE*, and Vincent Y. F. Tan, *Senior Member, IEEE*

***Abstract*— This paper studies the second-order asymptotics of the Gaussian multiple-access channel with degraded message sets. For a fixed average error probability $\varepsilon \in (0, 1)$ and an arbitrary point on the boundary of the capacity region, we characterize the speed of convergence of rate pairs that converge to that boundary point for codes that have asymptotic error probability no larger than ε. As a stepping stone to this local notion of the second-order asymptotics, we study a global notion, and establish relationships between the two. We provide a numerical example**

given by the set of rate pairs (R_1, R_2) satisfying

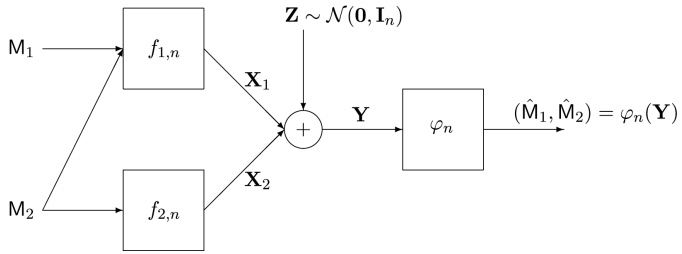
$$R_1 \le \mathsf{C}\big((1-\rho^2)S_1\big) \quad (1)$$

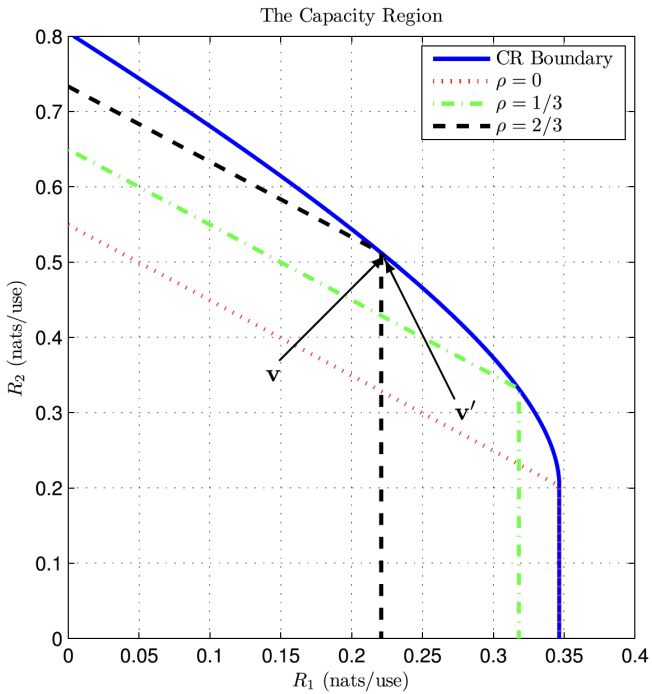
$$R_1 + R_2 \le \mathsf{C}\big(S_1 + S_2 + 2\rho\sqrt{S_1 S_2}\big) \quad (2)$$

for some $\rho \in [0, 1]$, where S_1 and S_2 are the admissible transmit powers, and $\mathsf{C}(x) := \frac{1}{2}\log(1+x)$ is the Gaussian capacity function. The capacity region $\mathcal{C}$ does not depend on

Other Contributions to Second- and Third-Order

Gaussian MAC with degraded message sets with J. Scarlett





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$$R_1 \le \mathsf{C}\big((1-\rho^2)S_1\big) \tag{1}$$

$$R_1 + R_2 \le \mathsf{C}\big(S_1 + S_2 + 2\rho\sqrt{S_1 S_2}\big) \tag{2}$$

for some $\rho \in [0, 1]$, where S_1 and S_2 are the admissible transmit powers, and $\mathsf{C}(x) := \frac{1}{2}\log(1+x)$ is the Gaussian capacity function. The capacity region $\mathcal{C}$ does not depend on

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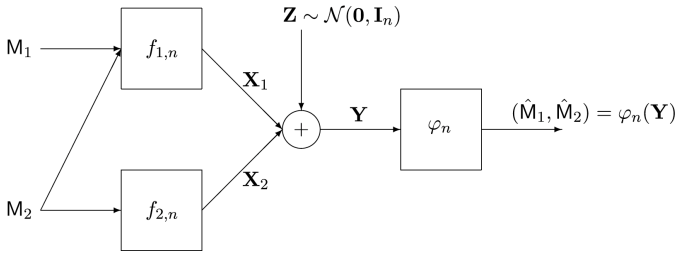

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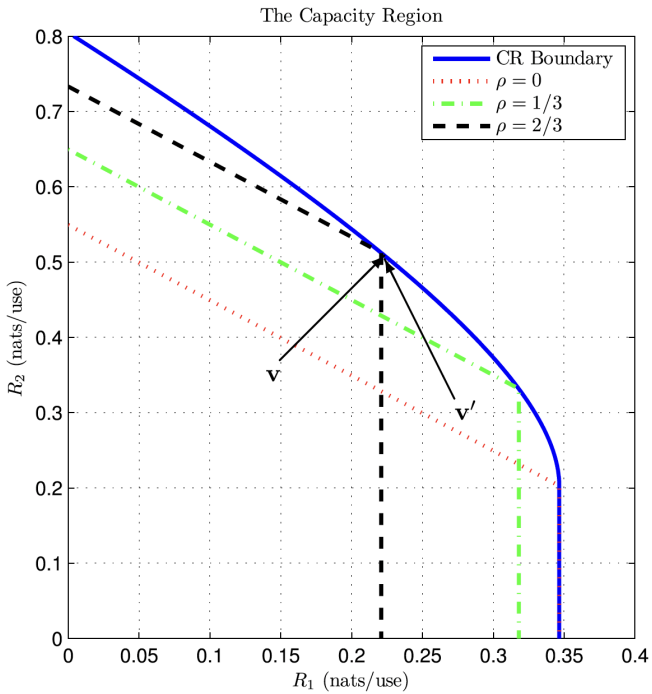
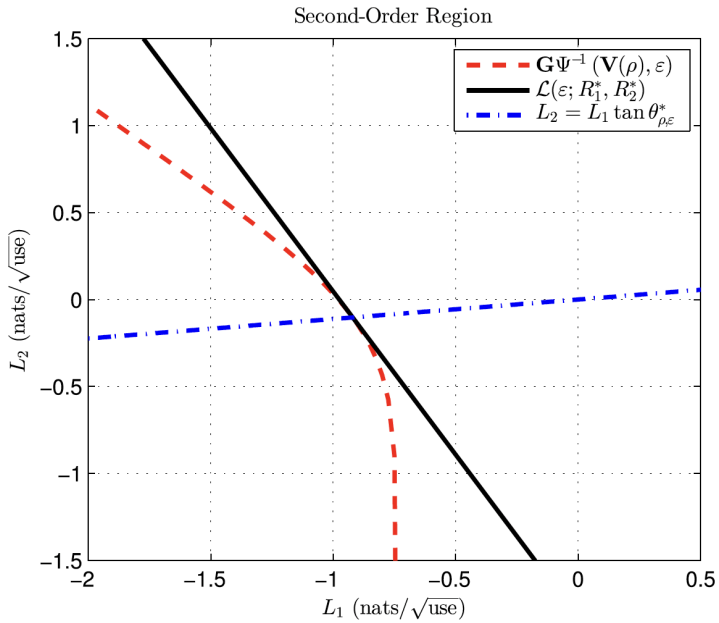
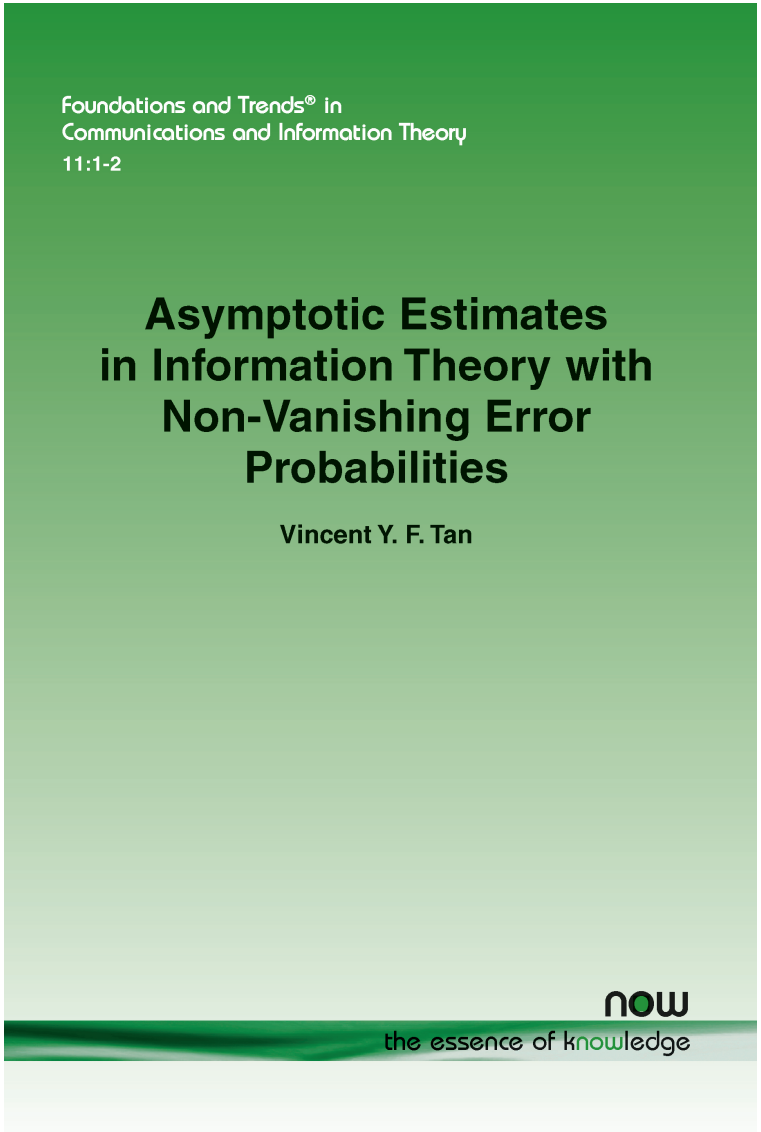




Summarized in First Monograph



Outline

Transitioning...

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- (2014 to 2018) Collaborated with M. Hayashi on equivocations for hash functions under Rényi information measures.

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 63, NO. 2, FEBRUARY 2017 975

Equivocations, Exponents, and Second-Order Coding Rates Under Various Rényi Information Measures

Masahito Hayashi, *Fellow, IEEE*, and Vincent Y. F. Tan, *Senior Member, IEEE*

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source (A^n, E^n). One of the central tasks in information-theoretic security is to understand the effect of applying a hash function [1] (binning operator) f on A^n. This hash function is used to ensure that the compressed source $f(A^n)$ is almost uniform on its alphabet and also almost independent of another discrete memoryless source E^n. Mathematically, we want to understand the deviation of $f(A^n) \in \{1, \ldots, \lceil e^{nR} \rceil\}$

3734 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 64, NO. 5, MAY 2018

Analysis of Remaining Uncertainties and Exponents Under Various Conditional Rényi Entropies

Vincent Y. F. Tan, *Senior Member, IEEE*, and Masahito Hayashi, *Fellow, IEEE*

***Abstract*— We analyze the asymptotics of the normalized remaining uncertainty of a source when a compressed or hashed version of it and correlated side information is observed. For this system, commonly known as Slepian–Wolf source coding, we establish the optimal (minimum) rate of compression of the source to ensure that the remaining uncertainties vanish. We also study the exponential rate of decay of the remaining uncertainty to zero when the rate is above the optimal rate of compression. In this paper, we consider various classes of random universal hash functions. Instead of measuring remaining uncertainties using traditional Shannon information measures, we do so using two forms of the conditional Rényi entropy.**

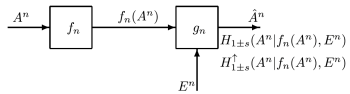

Fig. 1. The Slepian-Wolf [4] source coding problem. We are interesting in quantifying the asymptotic behaviors of the remaining uncertainty of A^n given $(f_n(A^n), E^n)$ measured according to the conditional Rényi entropies $H_{1\pm s}$ and $H^{\uparrow}_{1\pm s}$ defined in (10) and (12).

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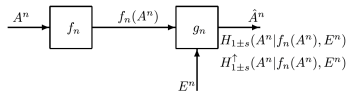

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1862 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 65, NO. 3, MARCH 2019

Rényi Resolvability and Its Applications to the Wiretap Channel

Lei Yu and Vincent Y. F. Tan, *Senior Member, IEEE*

***Abstract*— The conventional channel resolvability problem refers to the determination of the minimum rate required for an input process so that the output distribution approximates a target distribution in either the total variation distance or the relative entropy. In contrast to previous works, in this paper, we use the (normalized or unnormalized) Rényi divergence (with the Rényi parameter in $[0, 2] \cup \{\infty\}$) to measure the level of approximation. We also provide asymptotic expressions for normalized Rényi divergence when the Rényi parameter is larger than or equal to 1 as well as (lower and upper)**

a target output distribution? This is the so-called *channel resolvability problem*, studied by Han and Verdú [2]. In [2], the total variation (TV) distance and the normalized relative entropy (Kullback-Leibler divergence) were used to measure the level of approximation. The resolvability problem with the *unnormalized* relative entropy was studied by Hayashi [3], [4]. In [2]–[4] it was shown that in the memoryless case the minimum rates of randomness needed for simulating a channel output under the TV, normalized relative entropy, or unnor-

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- Given two random variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ with joint distribution π_{XY}, how common are they?

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- As information theorists, we like operational interpretations

Wyner's Common Information (Wyner, 1975)

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M_n

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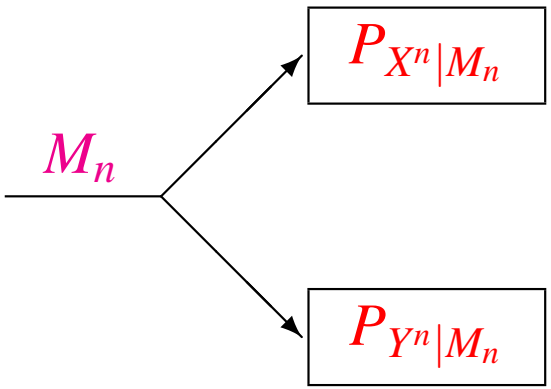

- M_n is uniformly distributed over $\mathcal{M}_n := \{1, \ldots, 2^{nR}\}$
- An (n, R)-synthesis code consists of

$$P_{X^n|M_n} : \mathcal{M}_n \to \mathcal{X}^n \quad \text{and} \quad P_{Y^n|M_n} : \mathcal{M}_n \to \mathcal{Y}^n.$$

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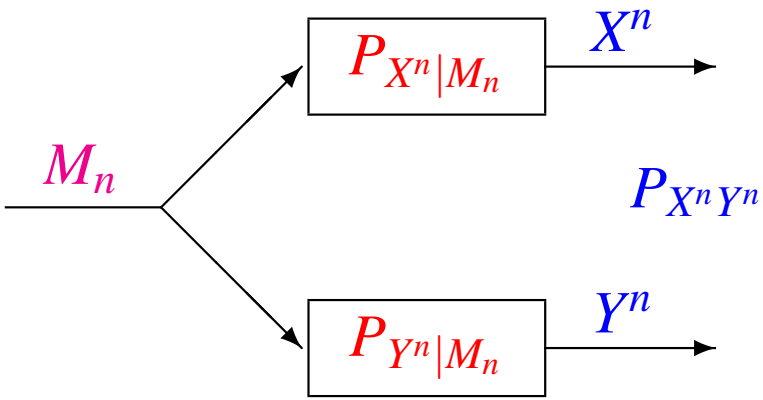

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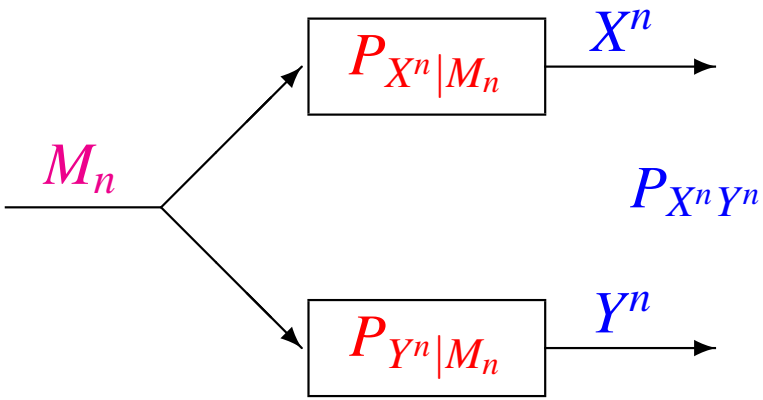

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- Desideratum:

$$P_{X^nY^n} \approx \pi_{XY}^n \quad \text{(target distribution)}$$

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IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-21, NO. 2, MARCH 1975

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where $C_{\mathrm{W}}(\pi_{XY})$ is named Wyner's Common Information.

Example: Doubly Symmetric Binary Source (DSBS)

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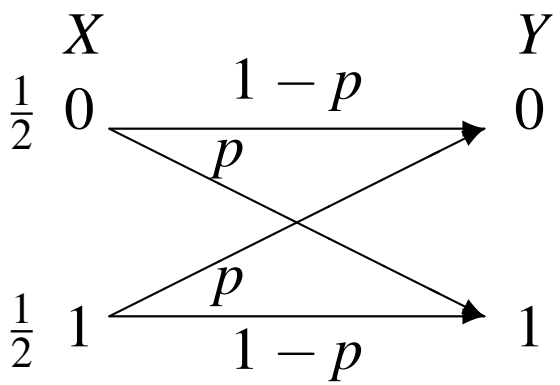

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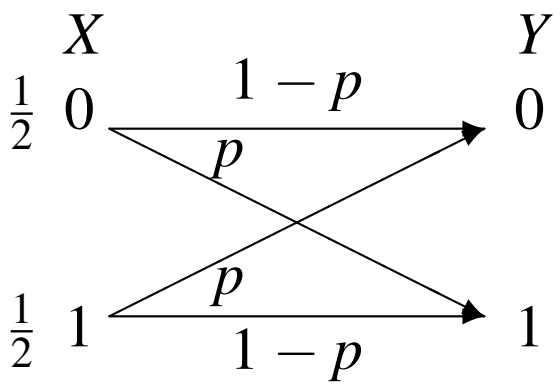

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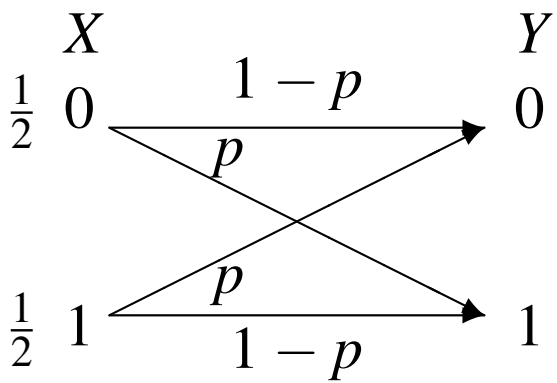

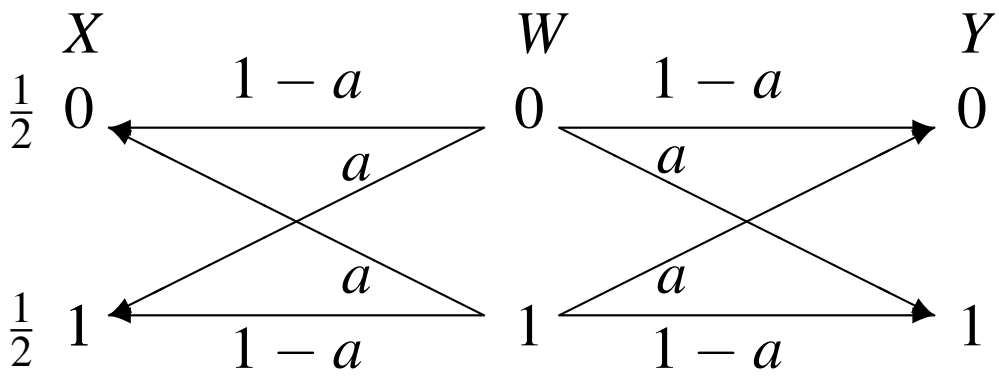

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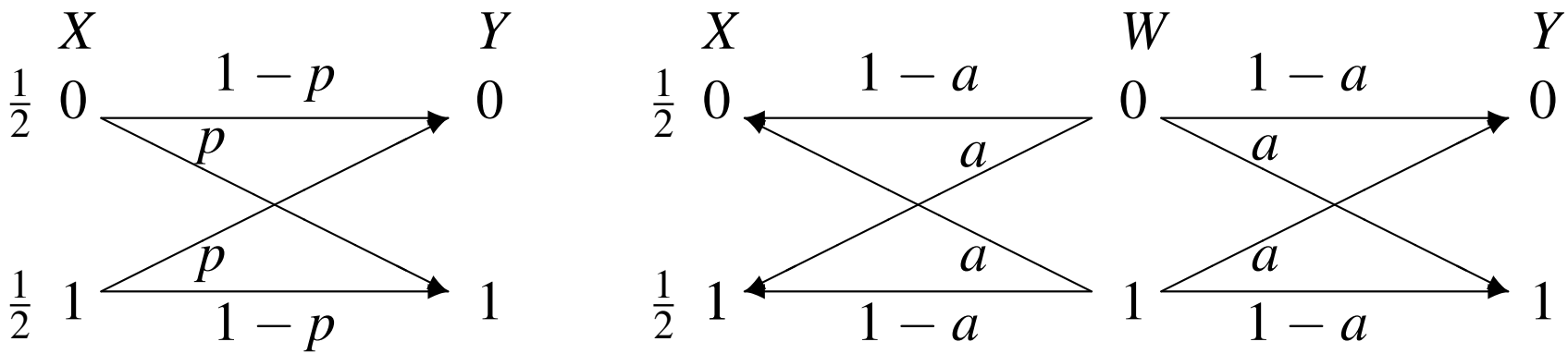
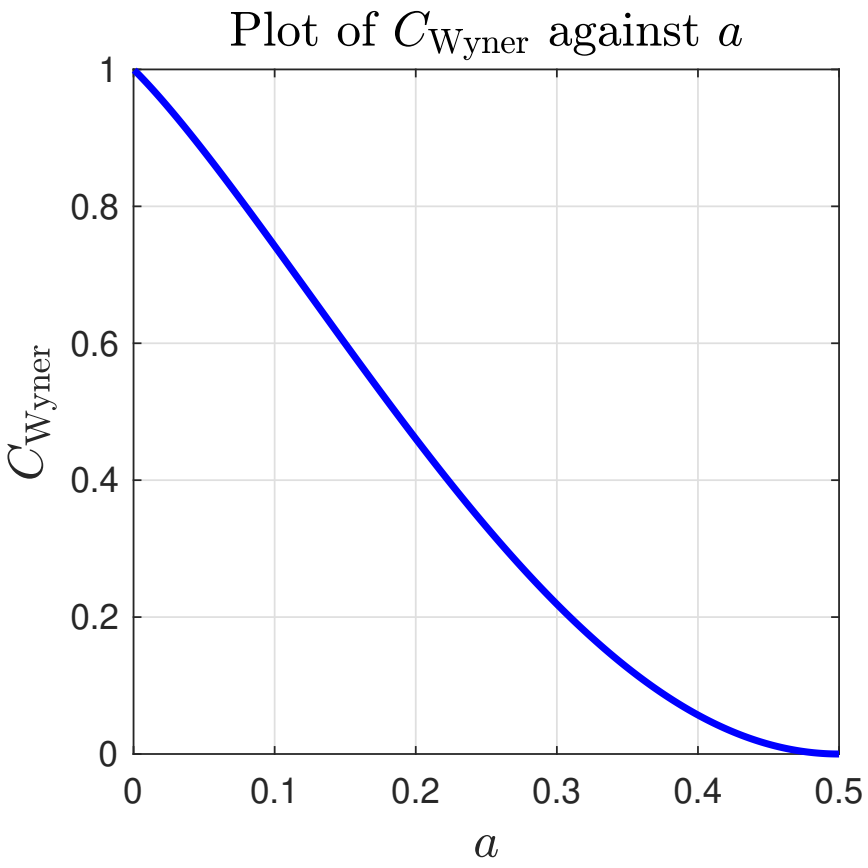
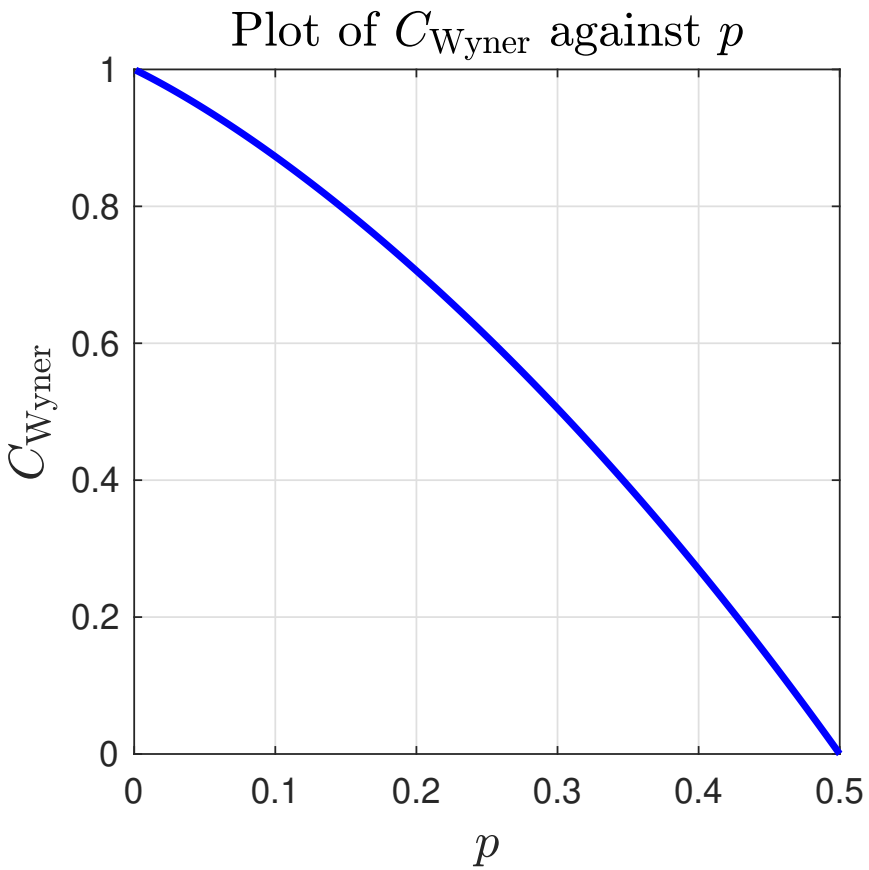


- Here, $a * a = p$ and

$$a = \frac{1 - \sqrt{1-2p}}{2} \in (0, 1/2).$$

Example: DSBS





Plots of Wyner's common information for the DSBS in terms of p and a

Motivation for Alternative Measures

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- What if we do not normalize?

$$\tilde{T}(\pi_{XY}) := \inf\left\{R : \lim_{n\to\infty} D(P_{X^nY^n} \,\|\, \pi_{XY}^n) = 0\right\} \geq C_{\mathrm{W}}(\pi_{XY}).$$

We get a stronger measure of dependence.

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- Wyner used the normalized relative entropy, i.e.,

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- Rényi common information for orders ≥ 1 (Yu and Tan, 2018)!

$$T_{1+s}(\pi_{XY}) := \inf\left\{R : \lim_{n\to\infty} \frac{D_{1+s}(P_{X^nY^n} \,\|\, \pi_{XY}^n)}{n} = 0\right\}$$

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Definition

Maximal cross entropy w.r.t. $(X, Y) \sim \pi_{XY}$ over couplings of (P_X, P_Y) is

$$\mathsf{H}_\infty(P_X, P_Y \,\|\, \pi_{XY}) := \max_{Q_{XY} \in \mathcal{C}(P_X, P_Y)} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)},$$

where $\mathcal{C}(P_X, P_Y) := \{Q_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : Q_X = P_X, Q_Y = P_Y\}$.

Upper and Lower Pseudo Common Informations

Definition

The upper pseudo-common information is

$$\overline{\Gamma}_\infty(\pi_{XY}) := \min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) + \mathbb{E}_{P_W}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \,\|\, \pi_{XY})\right].$$

The lower pseudo-common information is

$$\begin{aligned}\underline{\Gamma}_\infty(\pi_{XY}) := &\min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) \\ &+ \inf_{Q_{WW'} \in \mathcal{C}(P_W, P_W)} \mathbb{E}_{Q_{WW'}}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W'} \,\|\, \pi_{XY})\right].\end{aligned}$$

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cf. $$C_{\mathrm{W}}(\pi_{XY}) = \min_{P_W P_{X|W} P_{Y|W}: P_{XY}=\pi_{XY}} -H(XY|W) + H(XY).$$

Rényi Common Information of order ∞

Theorem (Yu and Tan (2020))

$$\max\left\{\underline{\Gamma}_\infty(\pi_{XY}), C_{\mathrm{W}}(\pi_{XY})\right\} \le T_\infty(\pi_{XY}) \le \tilde{T}_\infty(\pi_{XY}) \le \overline{\Gamma}_\infty(\pi_{XY}).$$

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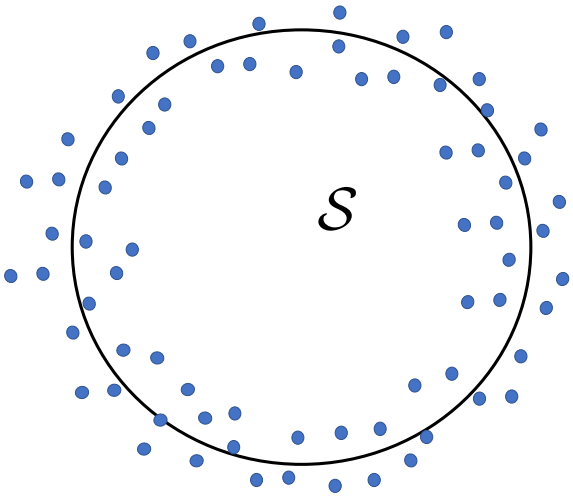

Product distribution

$$P_W^n(w^n) = \prod_{i=1}^n P_W(w_i)$$

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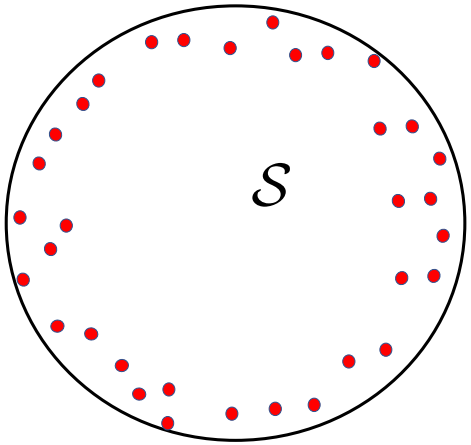

Truncated product distribution

$$P_{W^n}(w^n) \propto \left(\prod_{i=1}^{n} P_W(w_i)\right) \mathbb{1}\{w^n \in \mathcal{S}\}$$

Exact Common Information

- In Wyner's distributed source simulation problem,

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- Kumar, Li, and El Gamal (2014) introduced

2014 IEEE International Symposium on Information Theory

Exact Common Information

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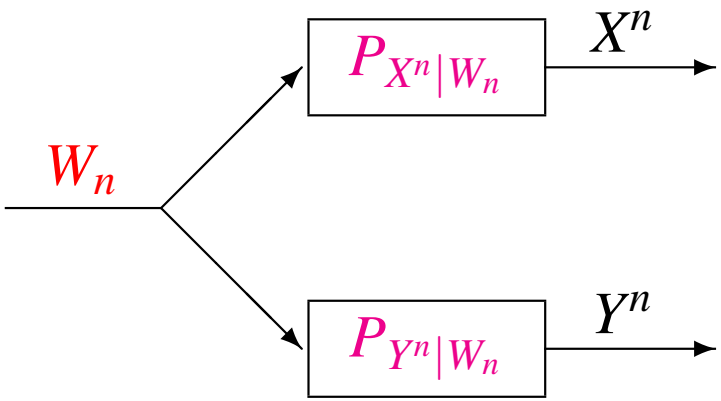

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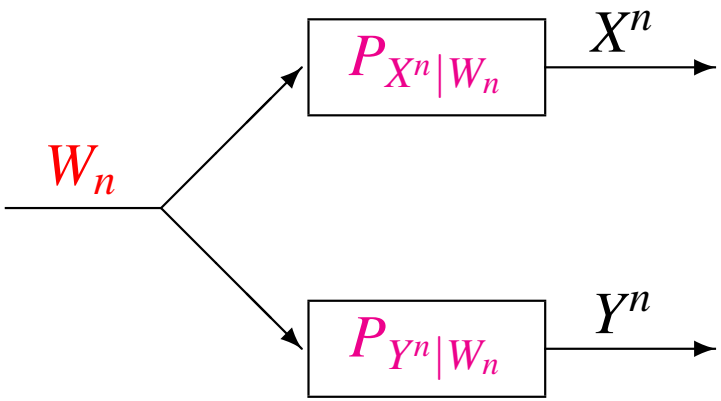

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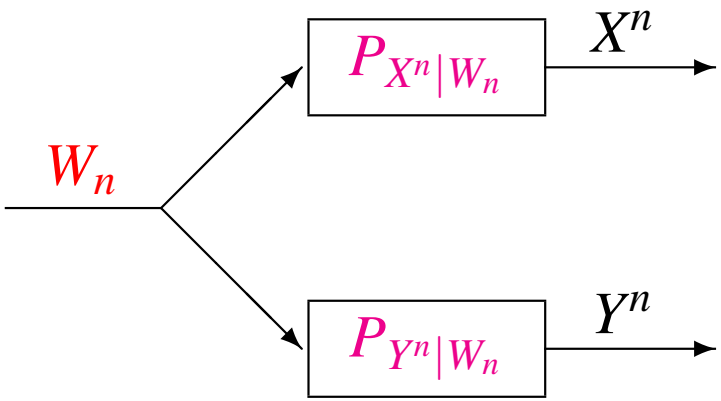

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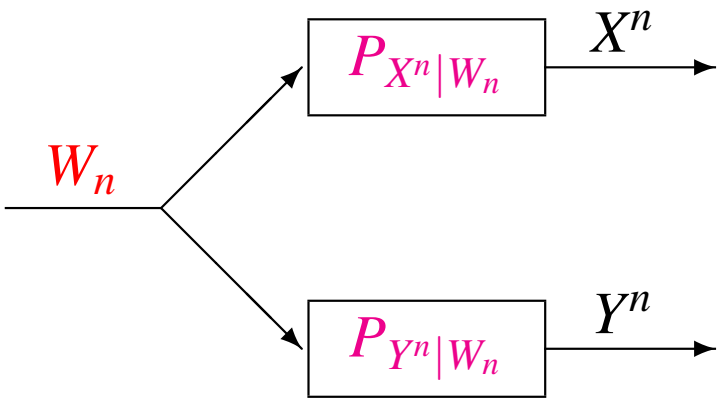

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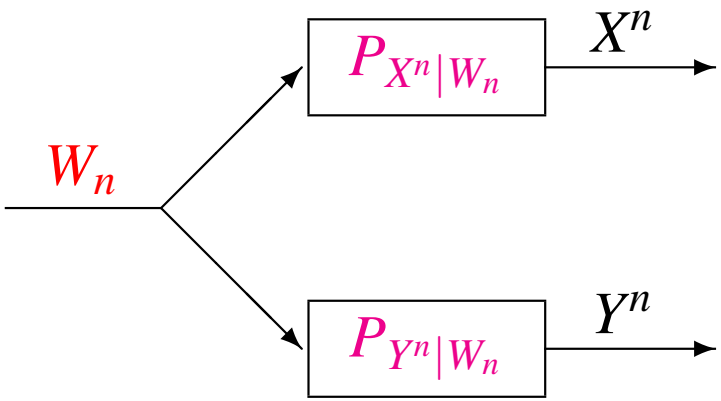

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$$H(W_n) \le L(W_n) < H(W_n) + 1$$

which implies that

$$\lim_{n\to\infty} \frac{L(W_n)}{n} = \lim_{n\to\infty} \frac{H(W_n)}{n}.$$

Exact Common Information

Definition

Exact common information between $(X, Y) \sim \pi_{XY}$

$$T_{\mathrm{Ex}}(\pi_{XY}) := \inf \left\{ \lim_{n\to\infty} \frac{L(W_n)}{n} : P_{X^nY^n} = \pi_{XY}^n \text{ for some } n \geq 1 \right\}$$

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Theorem (Kumar, Li, and El Gamal (2014))

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As expected the exact common information rate is greater than or equal to the Wyner common information.

Proposition 3.

$$\overline{G}(X;Y) \geq J(X;Y).$$

In the following section, we show that they are equal for the SBES in Example 1. We do not know if this is the case in general, however.

Surprising Equivalence: ∞-Rényi CI and Exact CI

Theorem (Yu and Tan (2020))

For a source π_{XY} on a finite alphabet,

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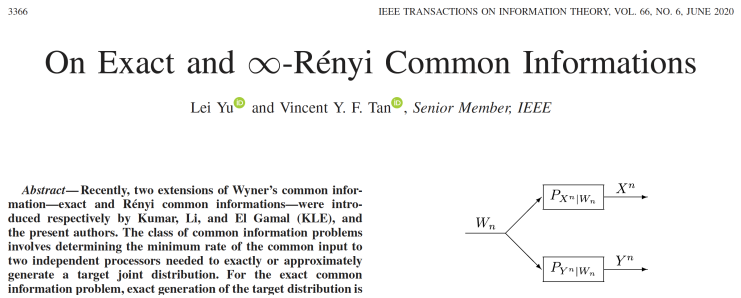

On Exact and ∞-Rényi Common Informations

Lei Yu and Vincent Y. F. Tan, *Senior Member, IEEE*

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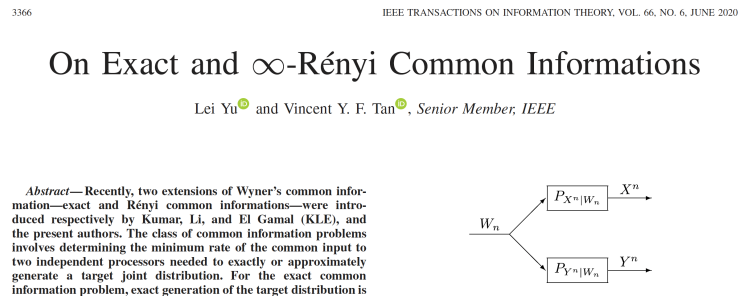

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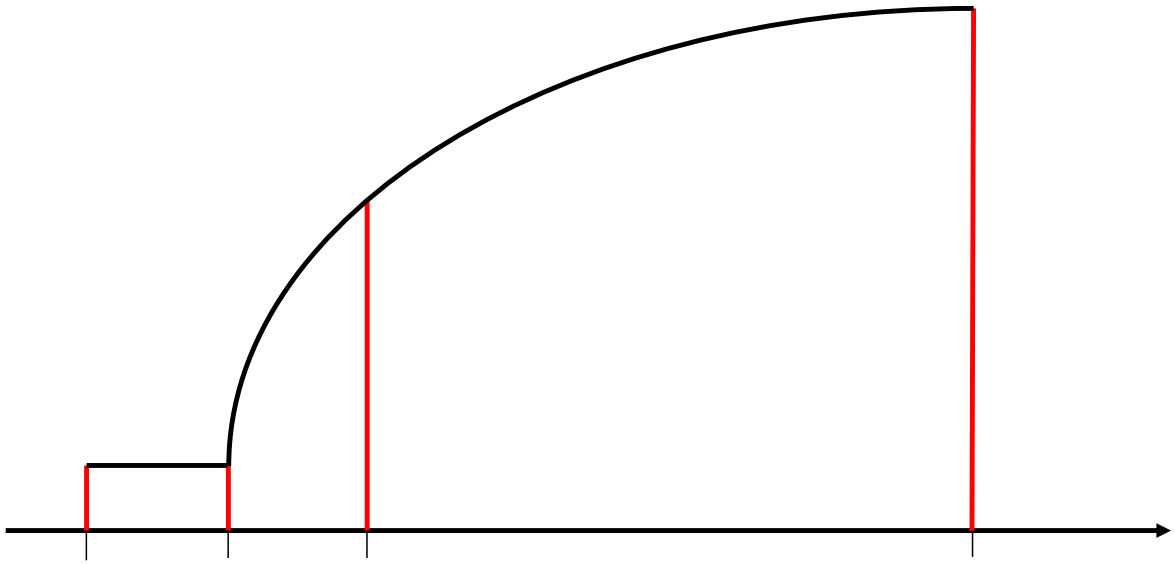

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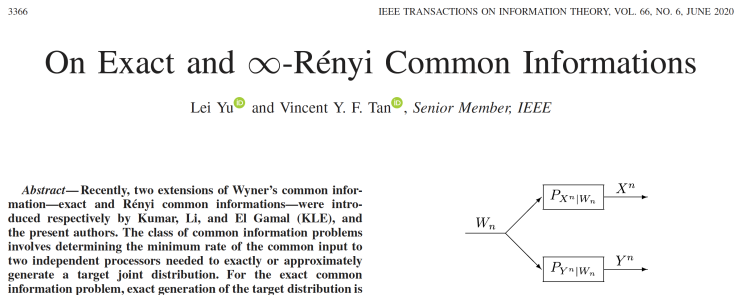

3366 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 66, NO. 6, JUNE 2020

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Lei Yu and Vincent Y. F. Tan, *Senior Member, IEEE*

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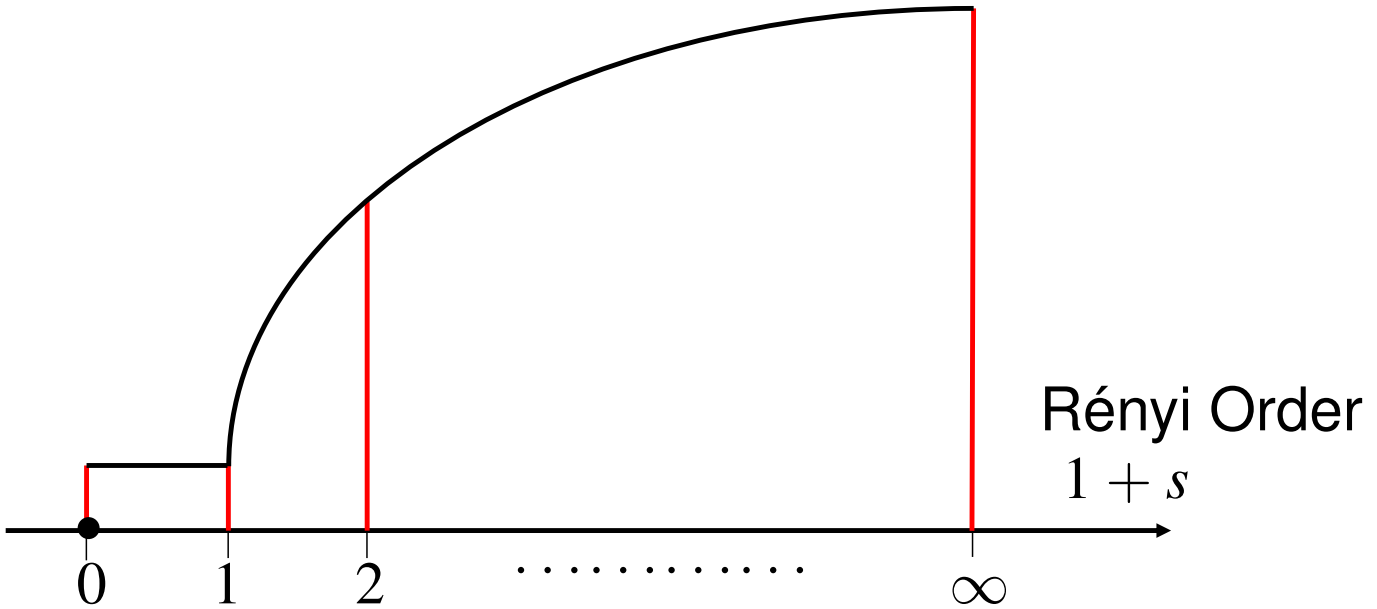

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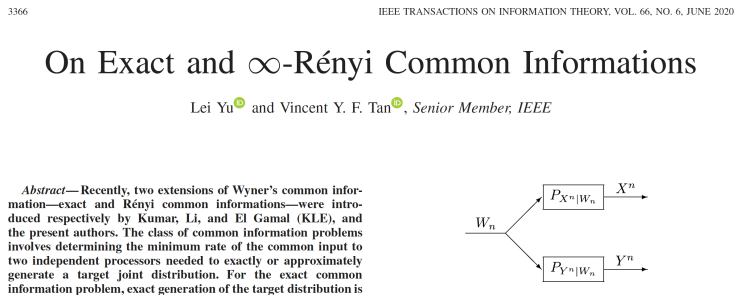

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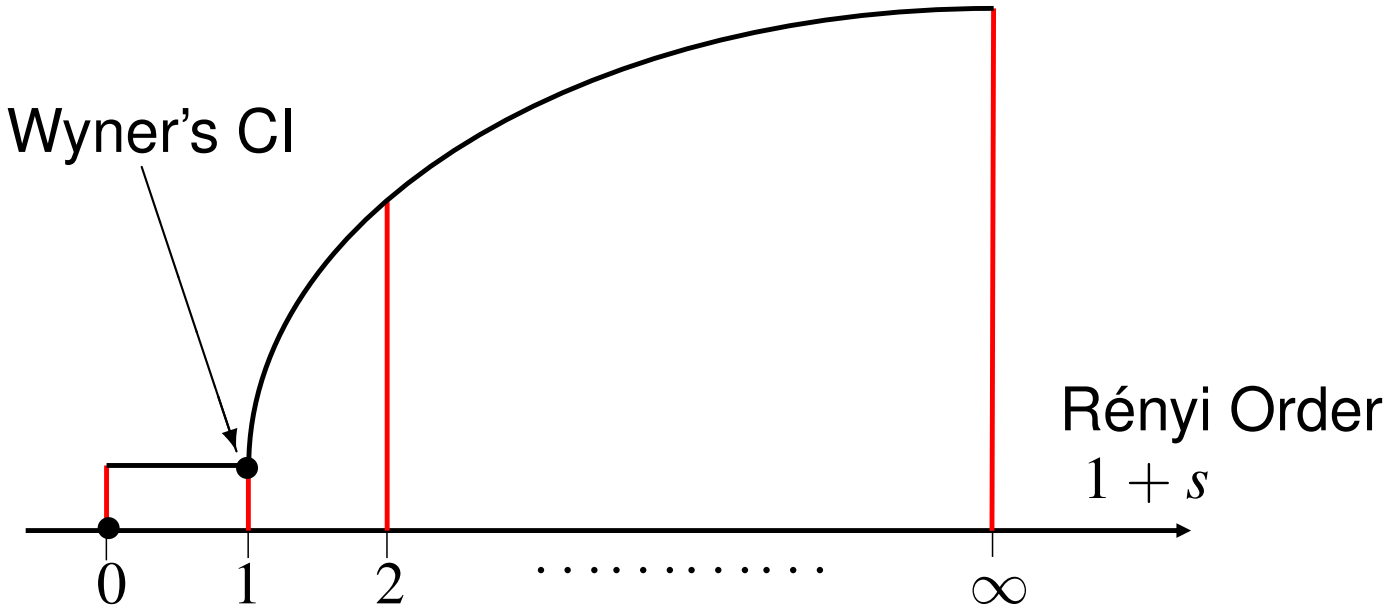

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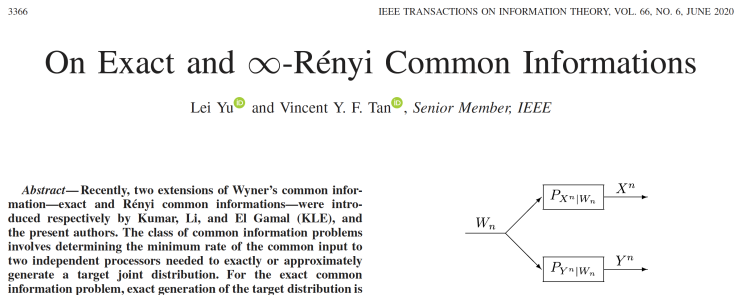

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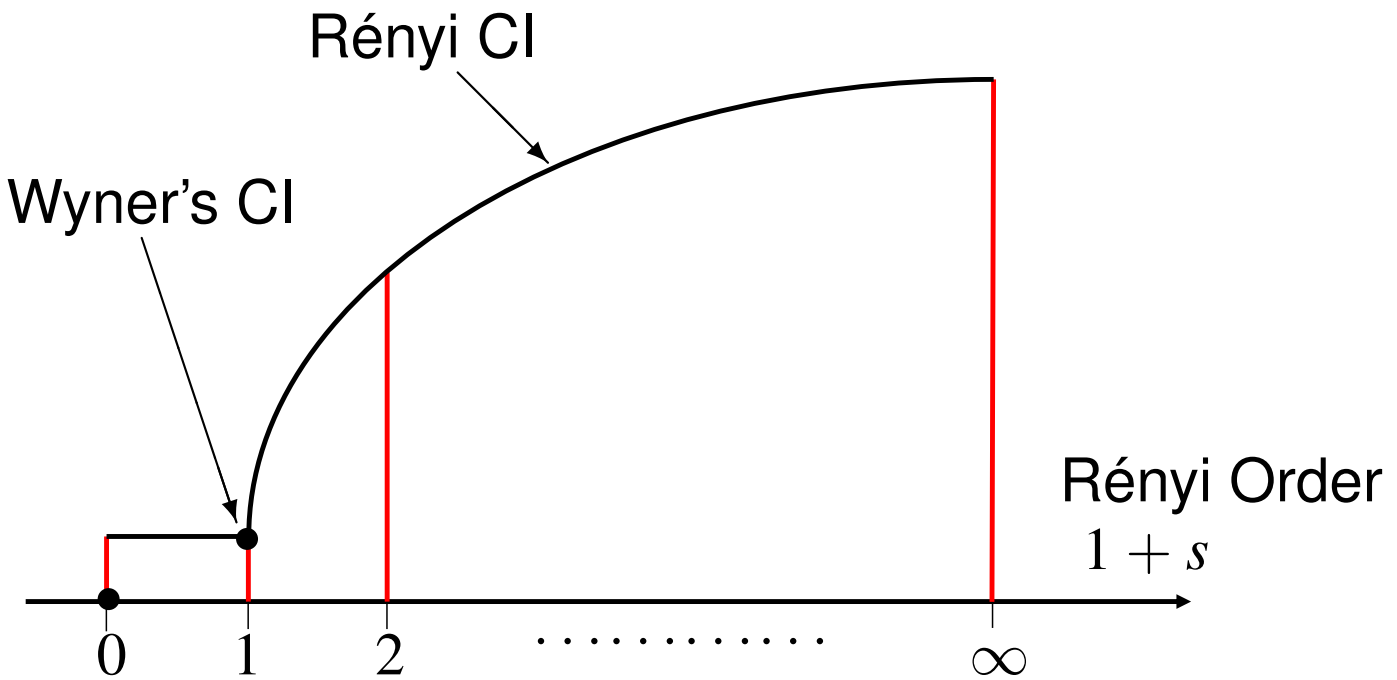

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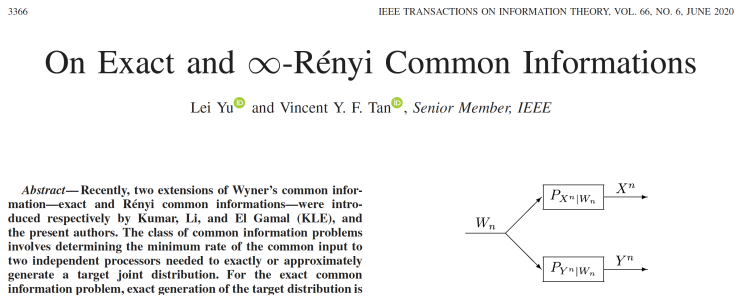

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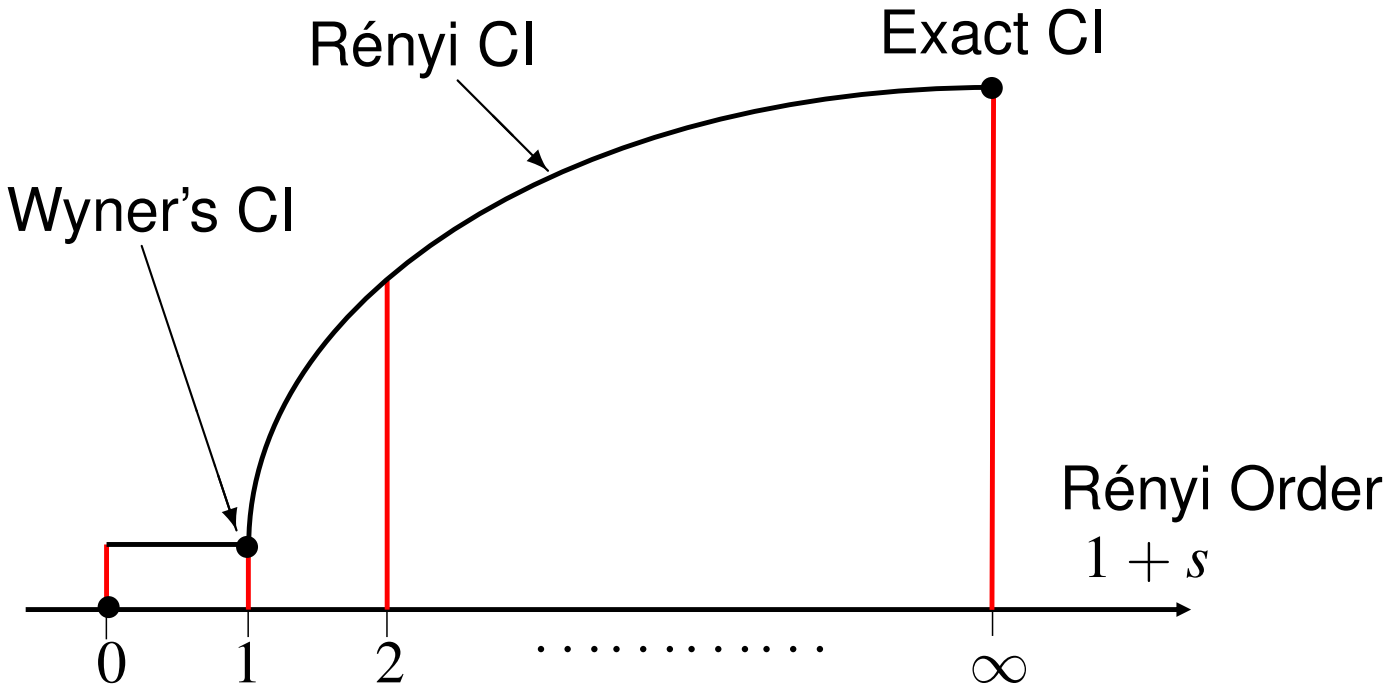

Proof of $\Longrightarrow$ Part of Equivalence Theorem

Lemma (Kumar, Li, and El Gamal '14 & Vellambi and Kliewer '16)

$\exists$ *rate-*R ∞*-Rényi CI code* $\Longrightarrow$ $\exists$ *rate-*R *Exact CI code*

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- Hence π^n_{XY} can be written as a mixture distribution

$$\pi^n_{XY}(x^n, y^n) = 2^{-\epsilon} P_{X^nY^n}(x^n, y^n) + \left(1 - 2^{-\epsilon}\right) \widehat{P}_{X^nY^n}(x^n, y^n)$$

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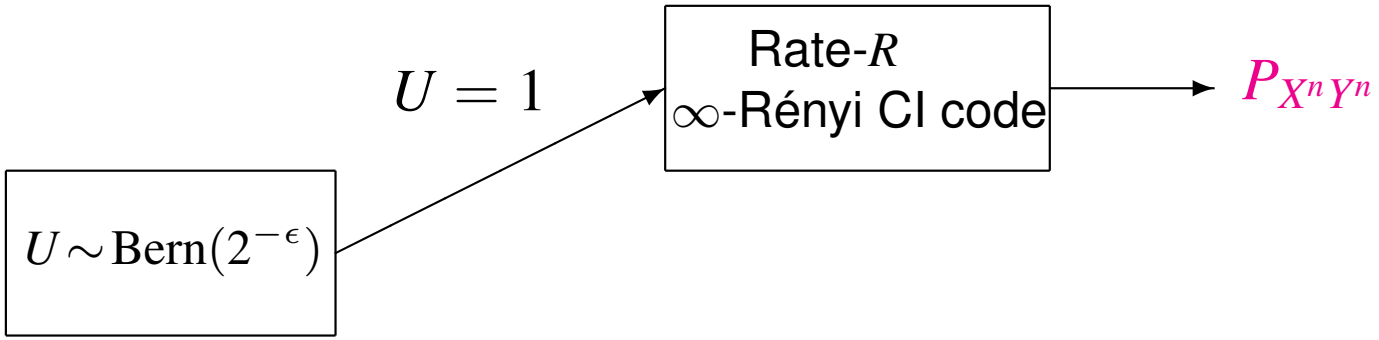

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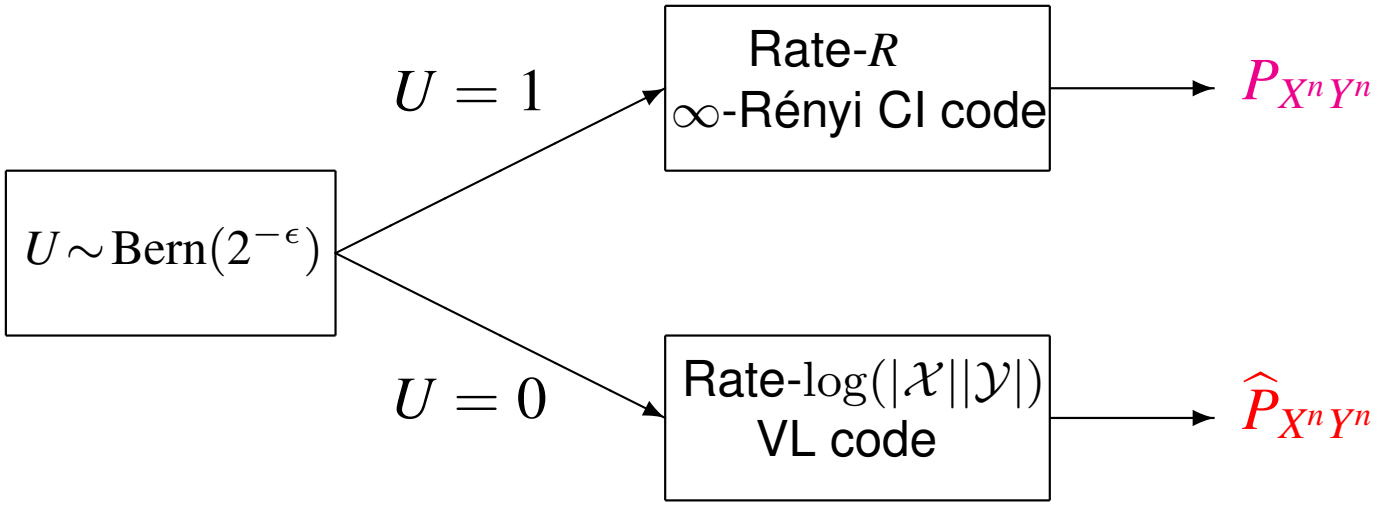

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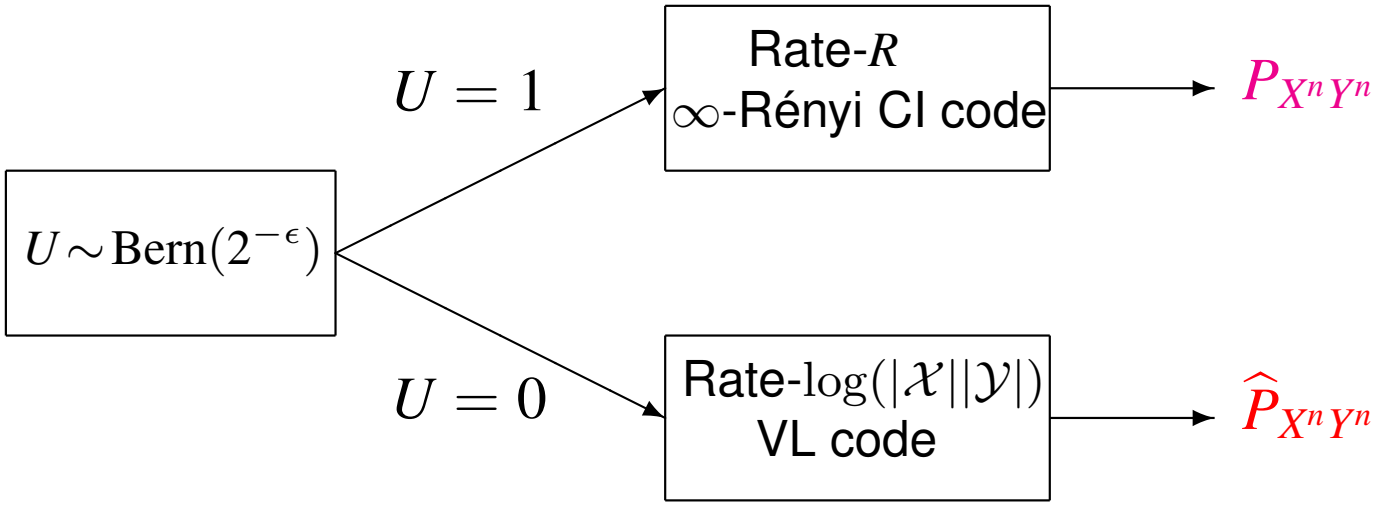

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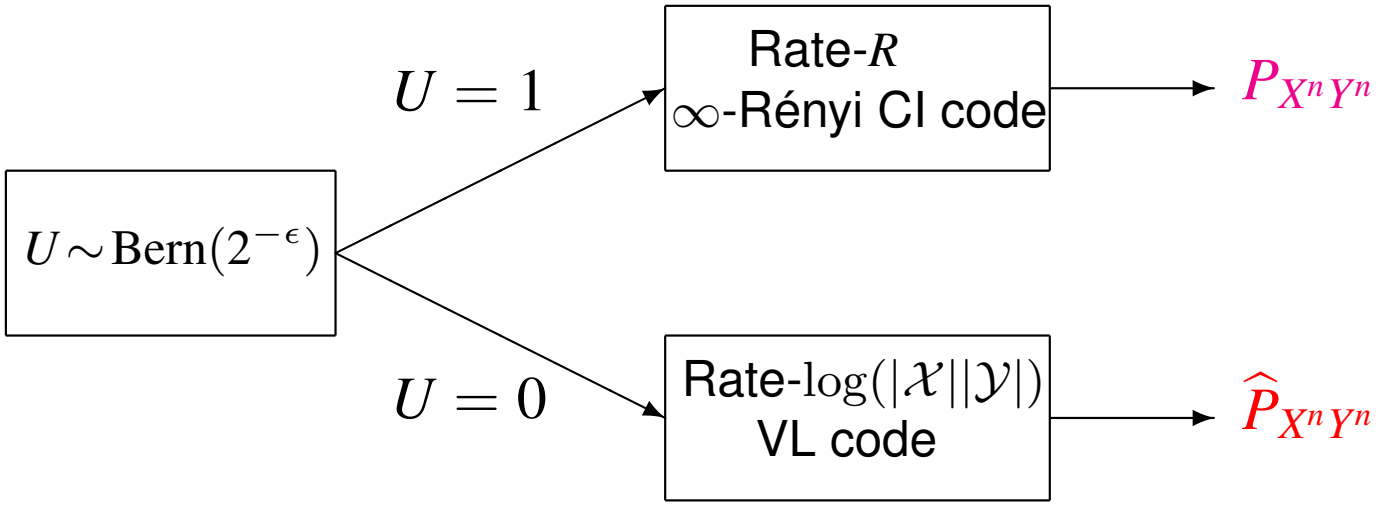

- Induced distribution is exactly π_{XY}^n.
- Expected rate/normalized length of code

$$\leq \frac{1}{n} + 2^{-\epsilon} R + \left(1 - 2^{-\epsilon}\right) \log(|\mathcal{X}||\mathcal{Y}|) \ \to R$$

as $n \to \infty$ then as $\epsilon \to 0$.

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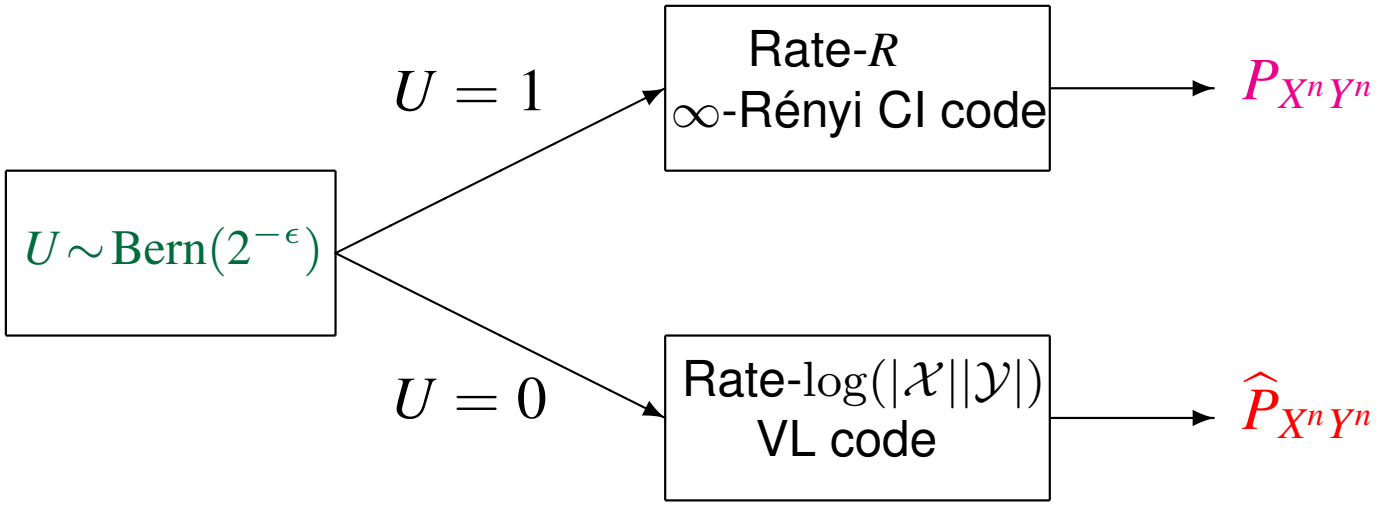

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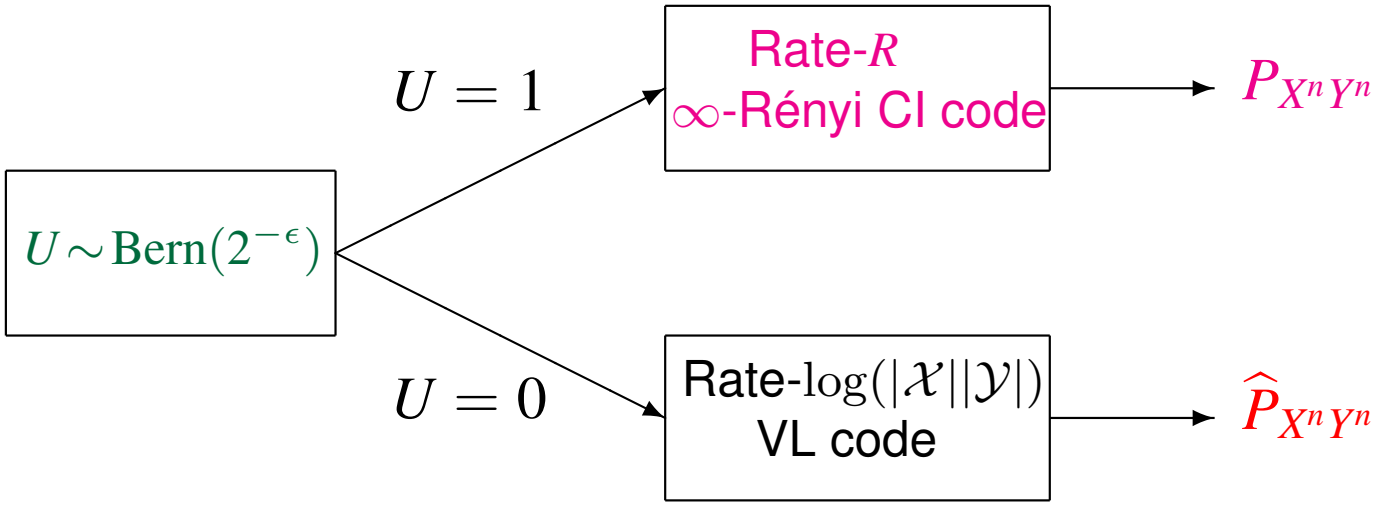

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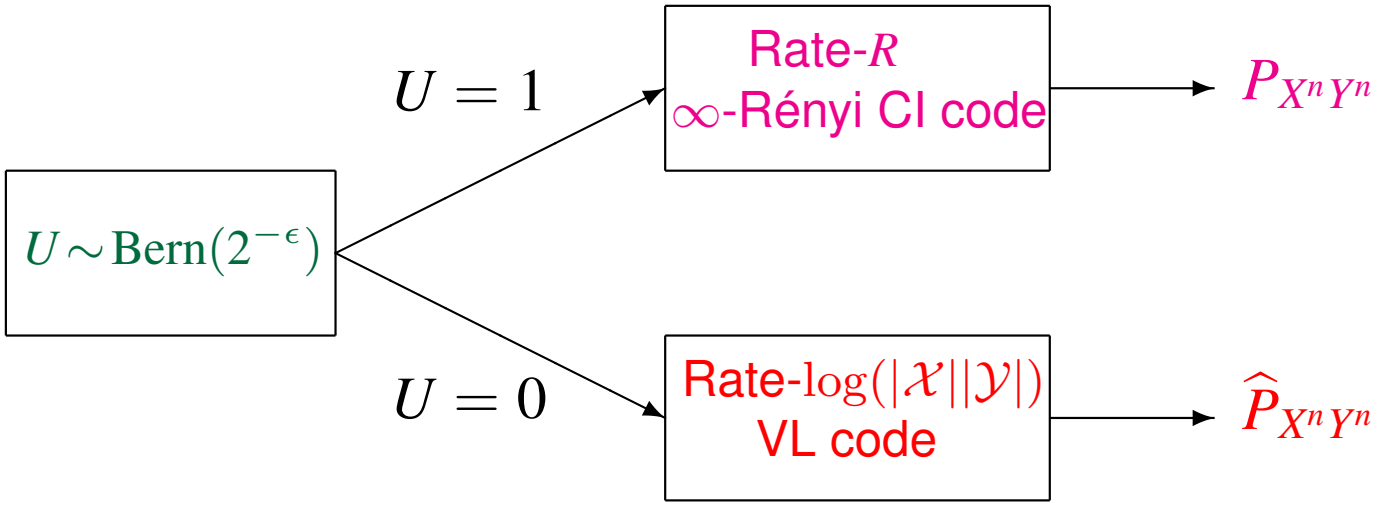

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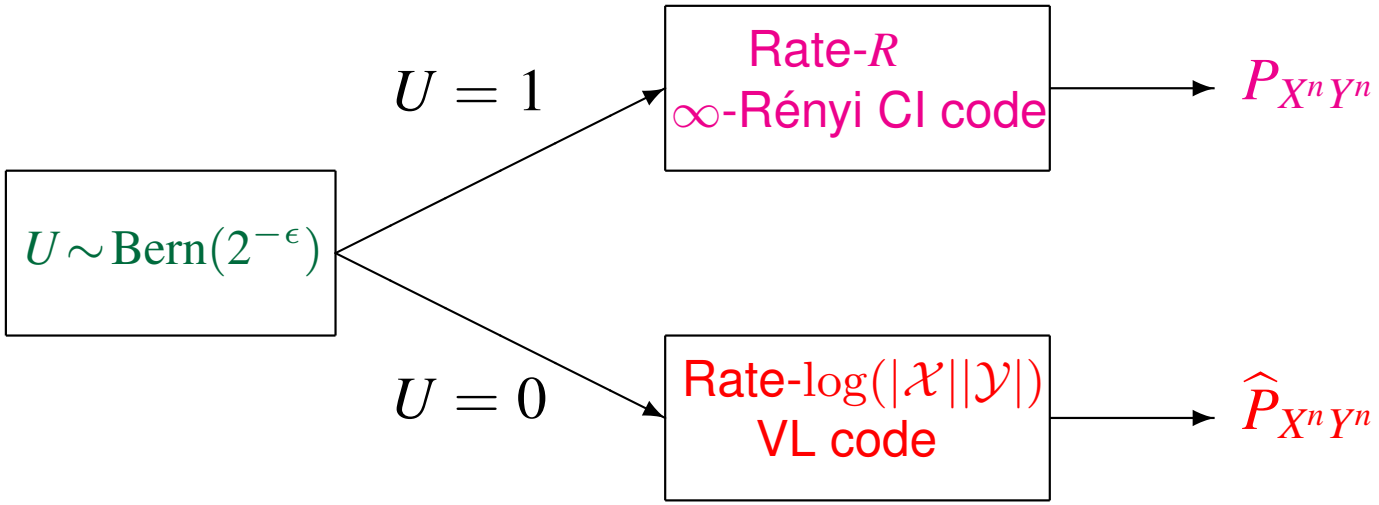

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Combining with Single-Letter Bounds from Rényi CI

Theorem (Yu and Tan (2020))

For $(X, Y) \sim \pi_{XY}$ on a finite alphabet,

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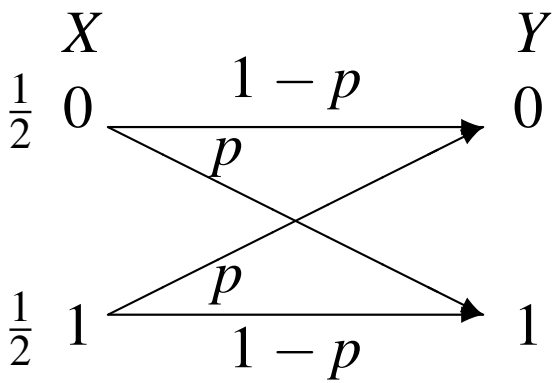
- Bounds are more amenable to numerical evaluation.

Revisiting the DBSS



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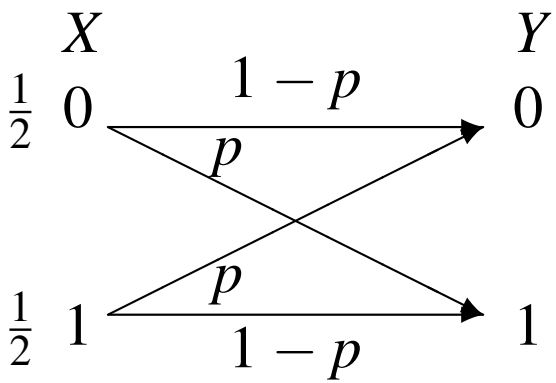

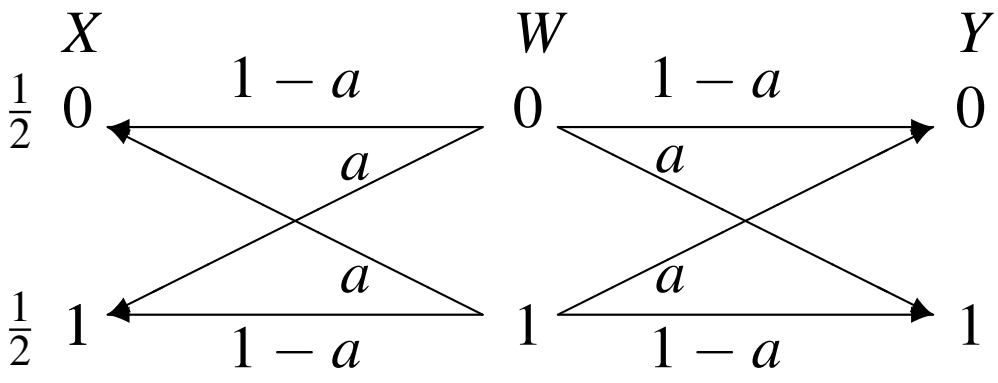

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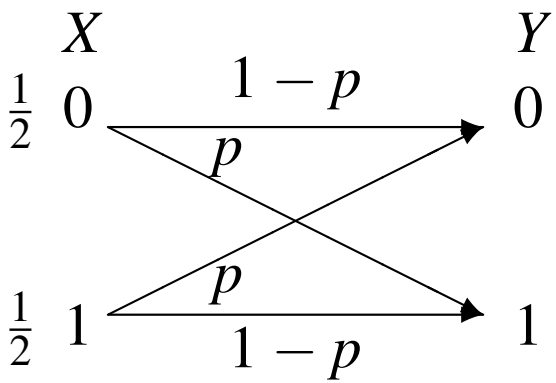
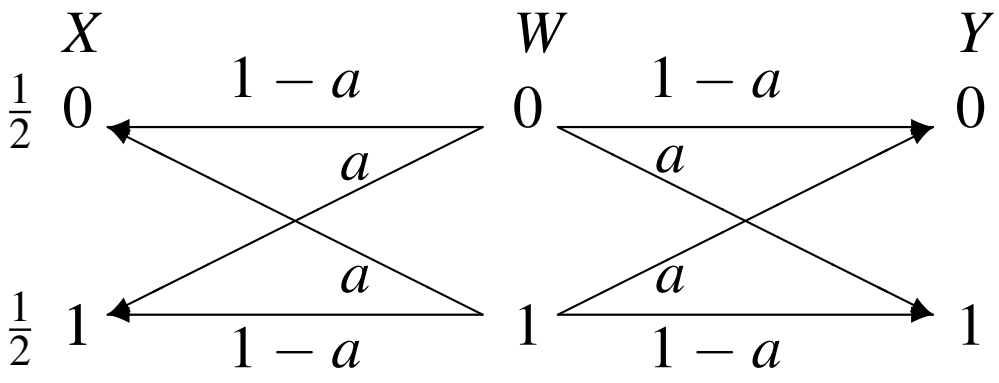




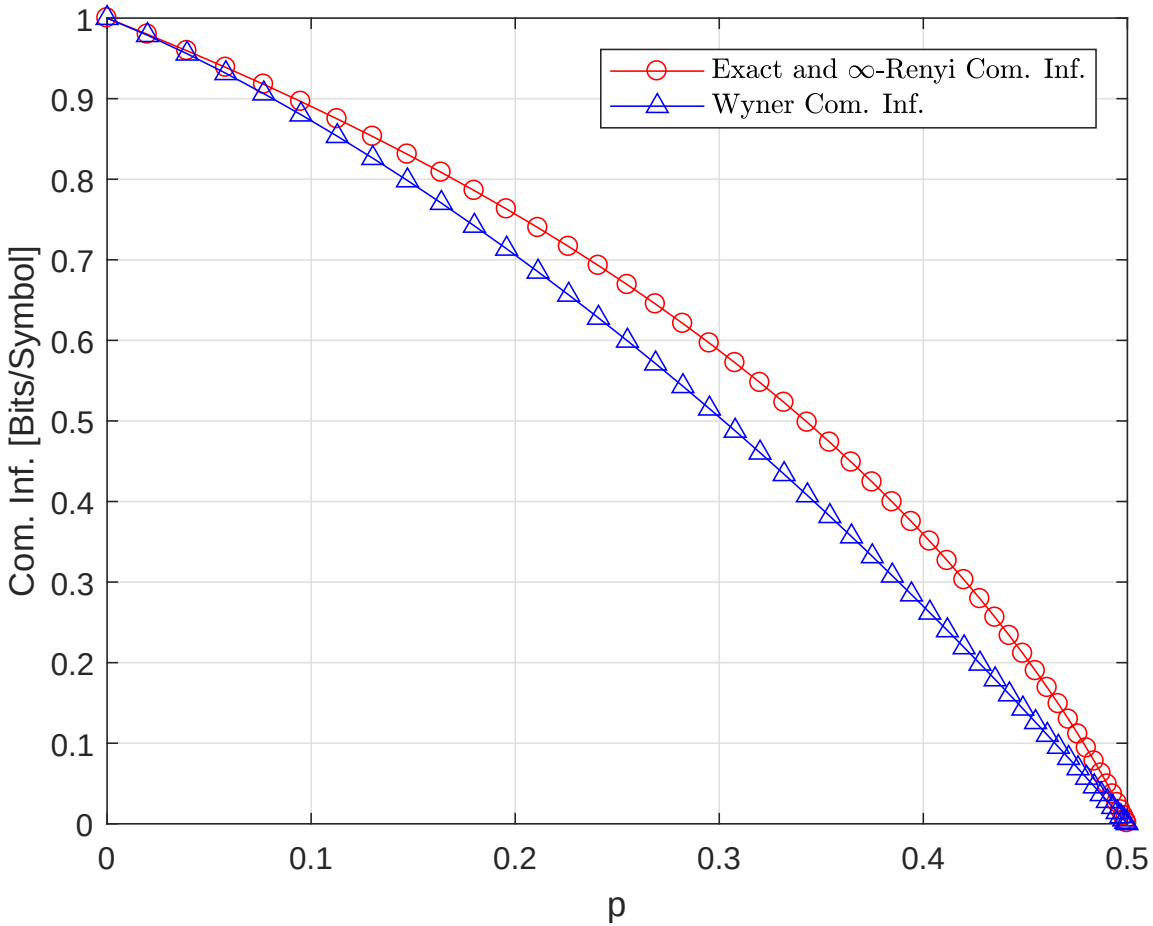
Theorem (Evaluation of Upper and Lower Bounds for DSBS(p))

For a DSBS $(X, Y) \sim \mathrm{DSBS}(p)$ *with crossover probability* $p \in (0, 1/2)$*,*

$$\begin{aligned}\tilde{T}_\infty(\pi_{XY}) &= T_{\mathrm{Ex}}(\pi_{XY}) \\ &= -2h(a) - (1-2a)\log\left[\frac{1}{2}\left(a^2 + (1-a)^2\right)\right] - 2a\log\left[a(1-a)\right],\end{aligned}$$

where $a := \frac{1-\sqrt{1-2p}}{2} \in (0, \frac{1}{2})$ *and* $h(a) := -a\log a - (1-a)\log(1-a)$*.*

Numerical Results — DSBS



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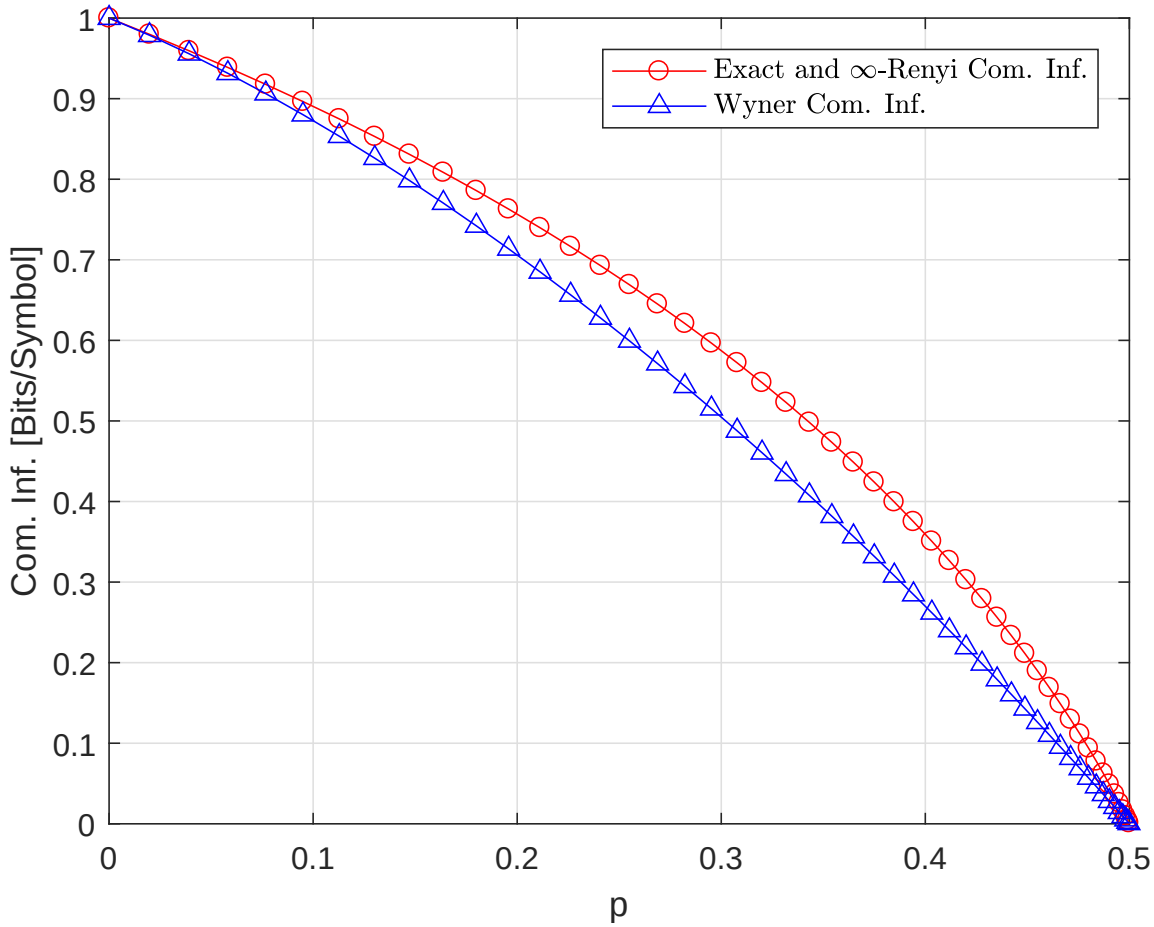

$$T_{\mathrm{Ex}}(\mathrm{DSBS}(p)) > C_{\mathrm{W}}(\mathrm{DSBS}(p)) \quad \forall\, p \in (0, 1/2).$$

Answers the open question in Kumar, Li, and El Gamal (2014).

Other Extensions of Common Information (with Lei Yu)

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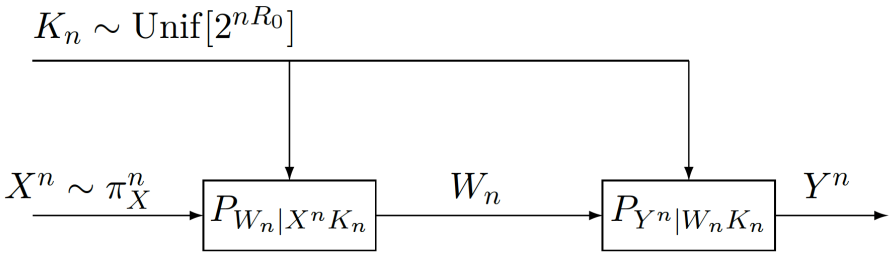
- Exact channel synthesis



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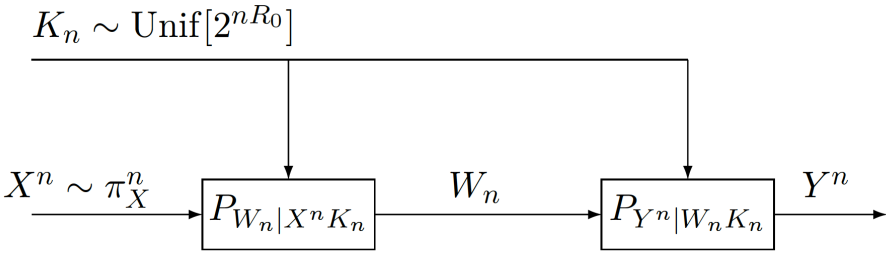

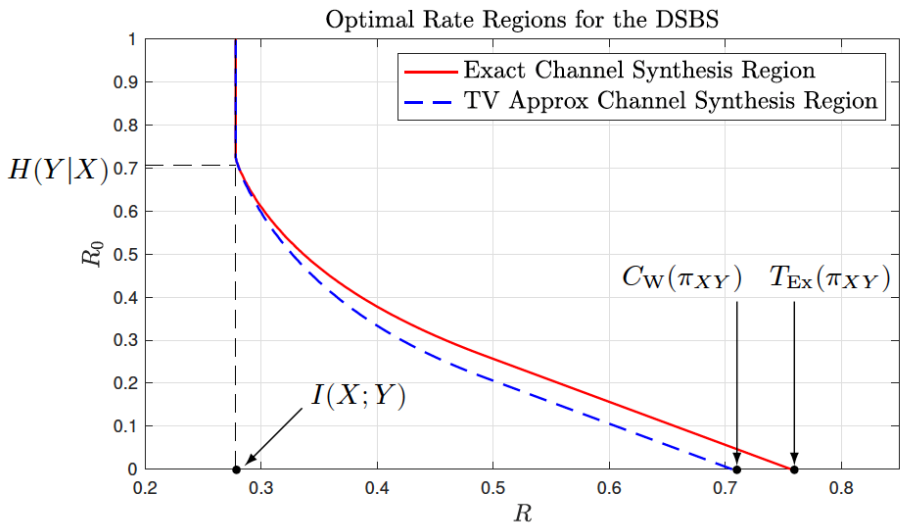

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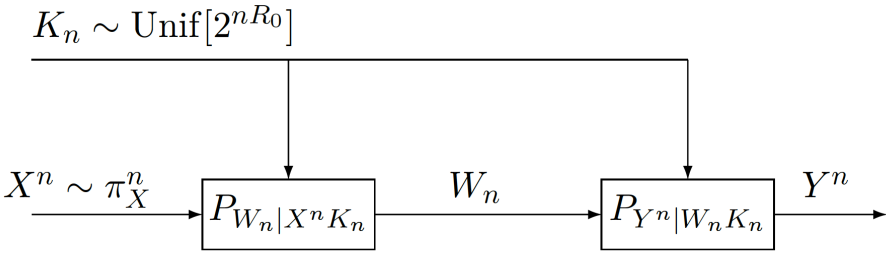
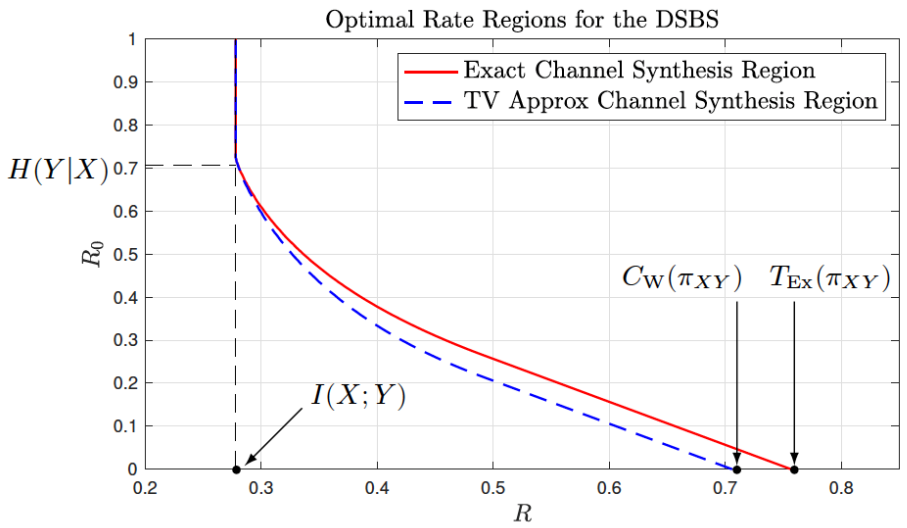
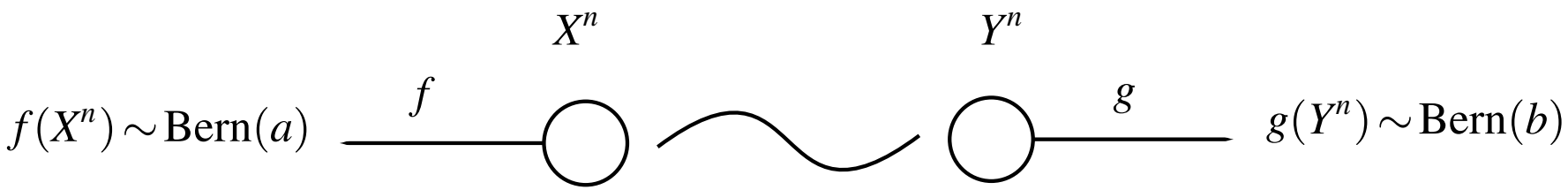




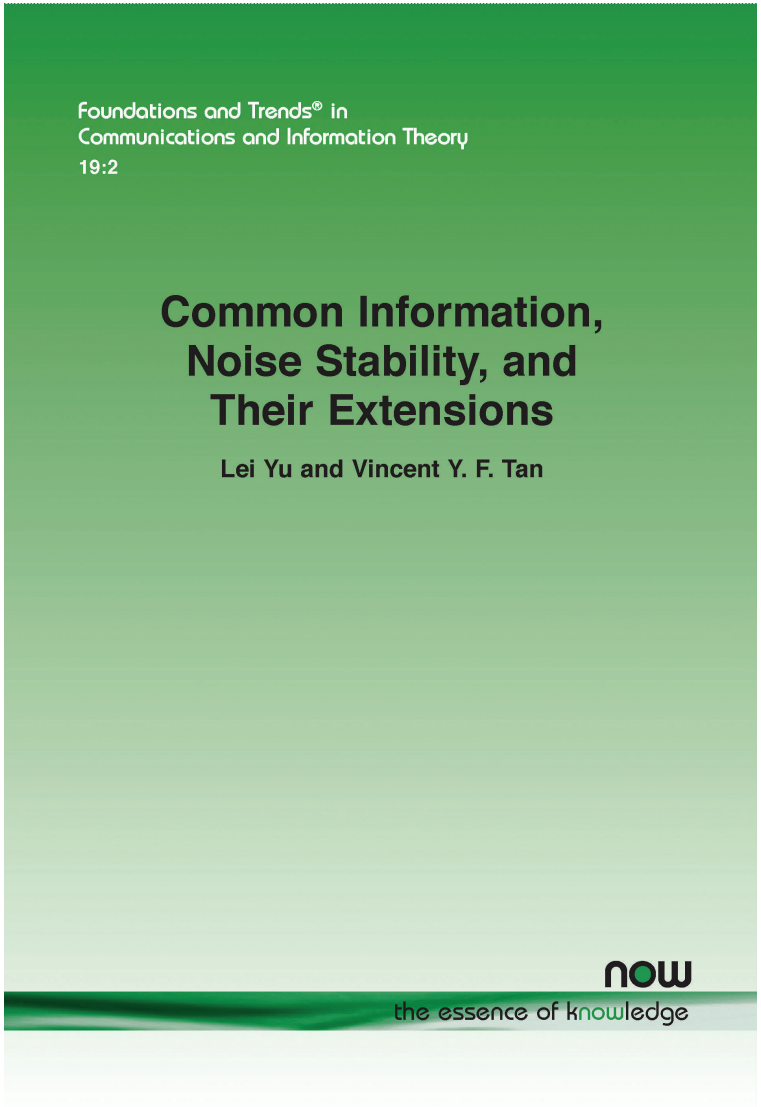
- Non-interactive correlation distillation (cf. Gács–Körner–Witsenhausen CI)

DSBS with correlation coefficient ρ



$$\max / \min \Pr\left(f(X^n) = g(Y^n)\right)$$

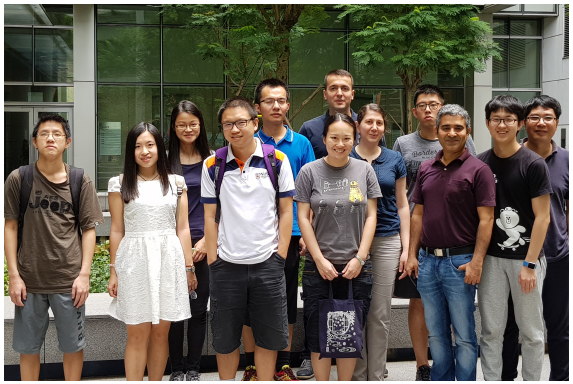
Summarized in Second Monograph (with Lei Yu)



Acknowledgements

Collaborators, students and postdocs!

2017



2018

2021

2022

2023

2024