Common Information: Old and New

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National University of Singapore and Nankai University

2021 International Symposium on Information Theory

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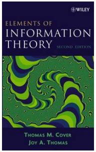
- Prerequisite: Information theory at the level of Cover and Thomas



Outline

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- As information theorists, we like operational interpretations
- Wyner's CI and Gács–Körner–Witsenhausen's CI are the two archetypal notions of information among RVs that admit operational interpretations.

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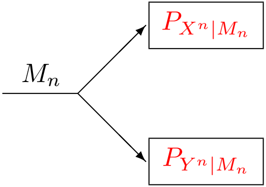

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$$P_{X^n|M_n} : \mathcal{M}_n \to \mathcal{X}^n \quad \text{and} \quad P_{Y^n|M_n} : \mathcal{M}_n \to \mathcal{Y}^n.$$

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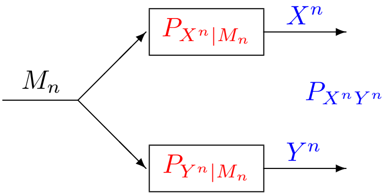

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- The distribution induced by the code $(P_{X^n|M_n}, P_{Y^n|M_n})$ is

$$P_{X^nY^n}(x^n, y^n) := \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} P_{X^n|M_n}(x^n|m) P_{Y^n|M_n}(y^n|m)$$

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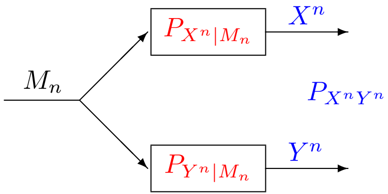

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- Desideratum:

$$P_{X^nY^n} \approx \pi_{XY}^n \qquad \text{(target distribution)}$$

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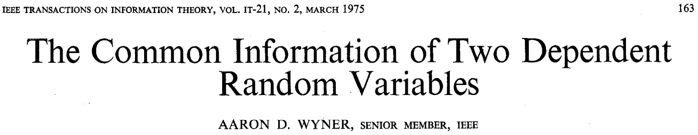

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where $C_{\mathsf{W}}(\pi_{XY})$ is named Wyner's Common Information.

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- So Wyner said that a reasonable notion of common information is

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- Intuitively, we should get $H(V)$ as the common information. Do we?
- Take $W = V$, satisfies $X - W - Y$. Then

$$I(XY; W) = I(XY; V) \le H(V) \qquad \text{so far so good...}$$

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- Minimize over $X - W - Y$ so

$$C_{\mathsf{W}}(\pi_{XY}) \geq H(V)$$

Proof Idea of the Achievability Part

Lemma (Soft-covering lemma [Wyner, 1975] [Cuff, 2012])

Let $(U, W) \sim P_{UW}$ *have mutual information* $I(U;W)$. *For any*

$$R > I(U;W),$$

there exists a sequence of codebooks $\mathcal{C}_n = \{w^n(m) : m \in [2^{nR}]\}$ *such that the synthesized distribution*

$$P_{U^n}(u^n) = \frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} P_{U|W}^n(u^n|w^n(m)) \qquad \forall\, n \in \mathbb{N}$$

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Also known as resolvability [Han and Verdú, 1993] [Hayashi, 2006].

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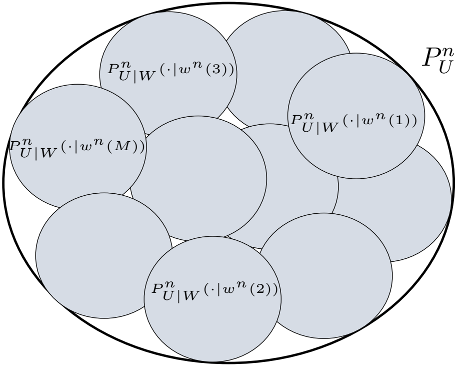

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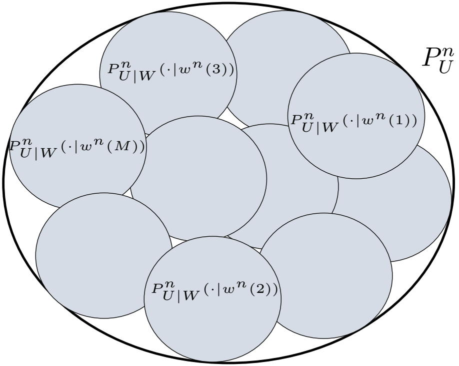
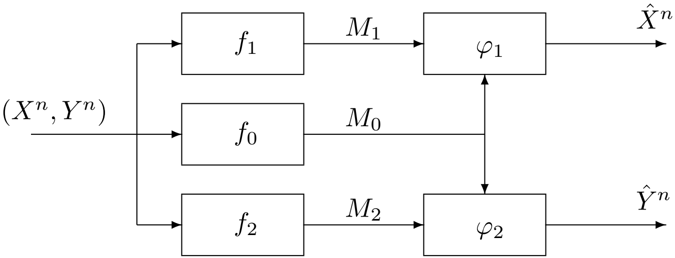


Figure: If $M = 2^{nR}$ and $R > I(U;W)$, then $\frac{1}{n}D(P_{U^n} \| P_U^n) \to 0$.

Now take $U = (X,Y) \sim \pi_{XY}$ and note by Markovity $X - W - Y$ that

$$P_{X^n|M_n}(x^n|m)P_{Y^n|M_n}(y^n|m) = P_{U^n|W^n}(u^n|w^n(m)) \text{ and } I(W;U) = I(W;XY).$$

Alternative Interpretation of Wyner's Common Information



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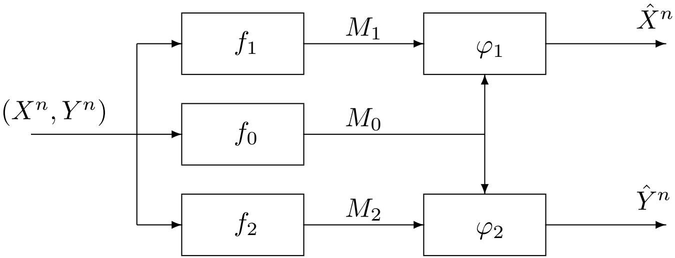

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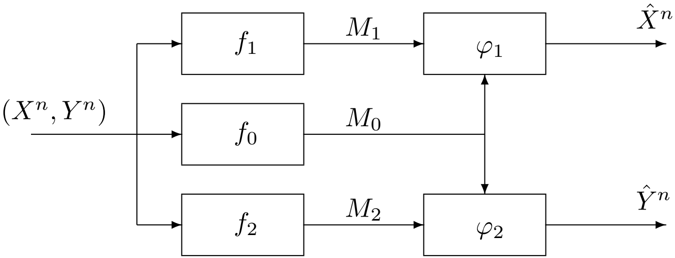

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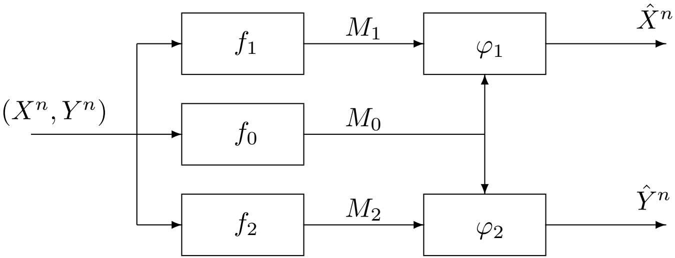

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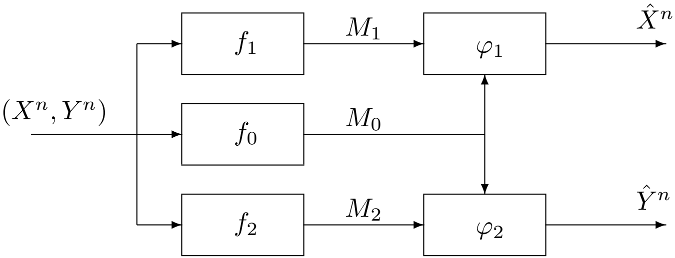

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- The probability of error of the code is

$$\Pr\Big(\big(\varphi_1(M_0, M_1), \varphi_2(M_0, M_2)\big) \neq (X^n, Y^n)\Big).$$

where $M_i = f_i(X^n, Y^n)$ for $i = 0, 1, 2$.

Alternative Interpretation of Wyner's Common Information

Common information based on the Gray-Wyner system $T_{\mathrm{GW}}(\pi_{XY})$ for $(X,Y) \sim \pi_{XY}$

$\Longleftrightarrow$

Smallest common rate R_0 such that for all $\epsilon > 0$, there exists sequence of (n, R_0, R_1, R_2) Gray-Wyner codes $\{(f_{0,n}, f_{1,n}, f_{2,n}, \varphi_{1,n}, \varphi_{2,n})\}_{n=1}^{\infty}$ such that

$$R_0 + R_1 + R_2 \leq H(XY) + \epsilon$$

and the probability of error vanishes.

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Theorem ([Wyner, 1975])

$$T_{\mathrm{GW}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$$

Example: Doubly Symmetric Binary Source (DSBS)

- Consider a DSBS $(X,Y) \in \{0,1\}^2$ which is defined for $p \in (0,1/2)$ by

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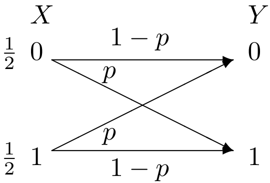

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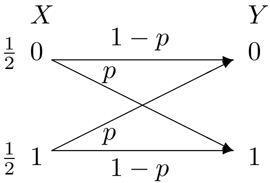

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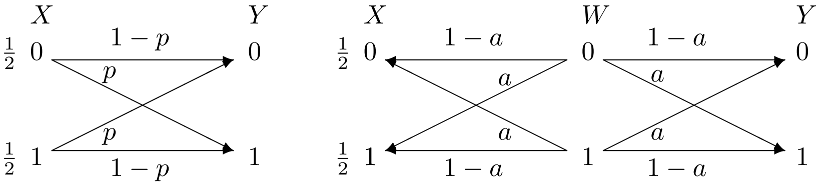

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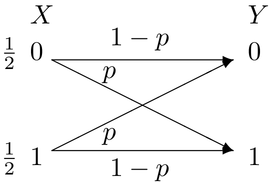
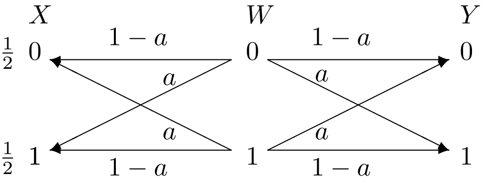
- Interpretation in terms of $X - W - Y$





- Here, $a * a = p$ and

$$a = \frac{1 - \sqrt{1 - 2p}}{2} \in (0, 1/2).$$

Example: DSBS

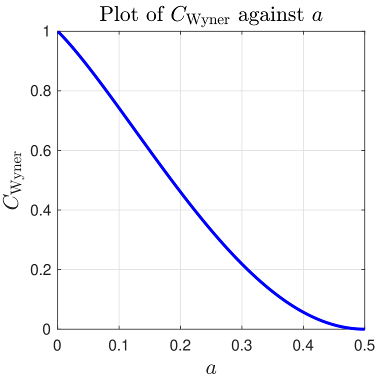


Figure: Plots of Wyner's common information for the DSBS in terms of p and a

Outline

Motivation for Alternative Measures

- Wyner used the normalized relative entropy, i.e.,

$$\inf\left\{R : \lim_{n\to\infty} \frac{D(P_{X^nY^n}\|\pi_{XY}^n)}{n} = 0\right\} = C_{\mathsf{W}}(\pi_{XY}) = \min_{X-W-Y} I(W;XY).$$

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- What if we do not normalize?

$$\tilde{T}(\pi_{XY}) := \inf\left\{R : \lim_{n\to\infty} D(P_{X^nY^n}\|\pi_{XY}^n) = 0\right\} \geq C_{\mathsf{W}}(\pi_{XY}).$$

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- What if we want an even stronger measure of dependence?
- Rényi common information for orders ≥ 1 [Yu and Tan, 2018]!

$$T_{1+s}(\pi_{XY}) := \inf\left\{R : \lim_{n\to\infty} \frac{D_{1+s}(P_{X^nY^n}\|\pi_{XY}^n)}{n} = 0\right\}$$

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Rényi Common Information

- Rényi divergence

$$D_{1+s}(P\|Q) := \frac{1}{s} \log \sum_{x\in\mathrm{supp}(P)} P(x) \left(\frac{P(x)}{Q(x)}\right)^s \quad s \in [-1, \infty)$$

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- And for a fixed order $1+s\in[0,\infty]$,

$$T_{1+s}(\pi_{XY}) \le \tilde{T}_{1+s}(\pi_{XY}).$$

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  for some joint sources π_{XY}.
- But let's soldier on and tackle the Rényi common information for now.

Rényi Common Information: The Weaker Case

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Let's start with a simple exercise. Consider the case $s \in (-1, 0]$ in which

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Theorem ([Yu and Tan, 2018] [Yu and Tan, 2020a])

For Rényi orders in $(0, 1]$ *(i.e.,* $s \in (-1, 0]$*),*

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Theorem ([Yu and Tan, 2018])

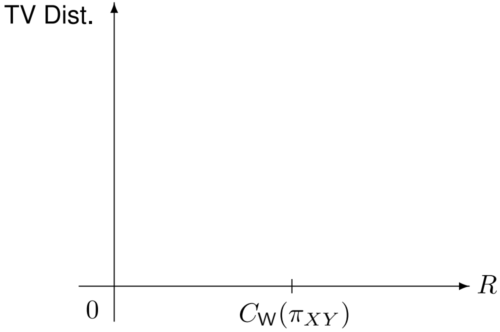
For any $\varepsilon \in [0, 1)$,

$$T_{\varepsilon}^{\mathrm{TV}}(\pi_{XY}) = C_{\mathsf{W}}(\pi_{XY}), \qquad \textit{(Strong converse)}$$

where $T_{\varepsilon}^{\mathrm{TV}}(\pi_{XY})$ *is the minimum simulation rate required to ensure*

$$\limsup_{n\to\infty} |P_{X^nY^n} - \pi_{XY}^n| \leq \varepsilon.$$

Total Variation Common Information



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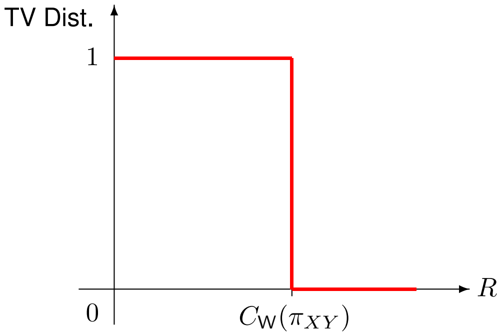

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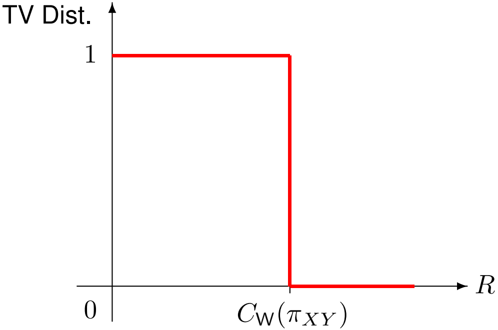

In fact, we have an exponential strong converse, i.e., if $R < C_{\mathsf{W}}(\pi_{XY})$,

$$|P_{X^nY^n} - \pi_{XY}^n| \geq 1 - 2^{-nE} \quad \text{for some} \quad E > 0.$$

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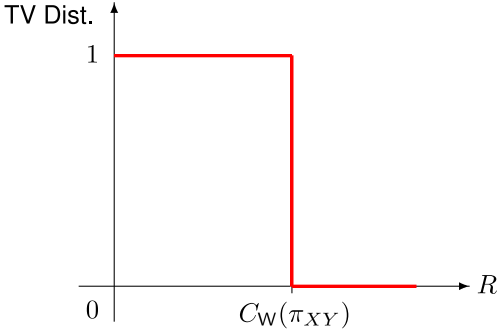

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Amenable to second-order?

Total Variation Common Information

- Achievability part follows from the soft-covering lemma.

If $\quad R > I(XY;W) \quad$ then $\quad \lim_{n\to\infty} |P_{X^nY^n} - \pi_{XY}^n| = 0.$

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- Converse requires a very cool information spectrum, single-letterization idea from [Oohama, 2018].

entropy

MDPI

Article

Exponential Strong Converse for Source Coding with Side Information at the Decoder †

Yasutada Oohama

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† This paper is an extended version of our paper published in 2016 International Symposium on Information Theory and Its Applications, Monterey, CA, USA, 6–9 November 2016; pp. 171–175.

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check for updates

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On the Rényi Divergence, Joint Range of Relative Entropies, and a Channel Coding Theorem

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Lemma

For any $s \in (-1, 0]$*,*

$$\inf_{P_X, Q_X : |P_X - Q_X| \geq \epsilon} D_{1+s}(P_X \| Q_X) = \inf_{q \in [0, 1-\epsilon]} d_{1+s}(q + \epsilon \| q)$$

and

$$\inf_{q \in [0, 1-\epsilon]} d_{1+s}(q + \epsilon \| q) \geq \left[\min\left\{ 1, \frac{1+s}{s} \right\} \log \frac{1}{1-\epsilon} + \frac{1}{s} \log 2 \right]^{+}$$

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- If $R < C_{\mathsf{W}}(\pi_{XY})$, exponential strong converse to TV CI says

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- Thus, if $R < C_{\mathsf{W}}(\pi_{XY})$

$$\frac{1}{n} \inf_{P_X, Q_X : |P_X - Q_X| \geq \epsilon} D_{1+s}(P_X \| Q_X) \geq \frac{1}{n} \left[\min\left\{1, \frac{1+s}{s}\right\} nE + \frac{1}{s} \log 2 \right]^+$$

and the normalized Rényi divergence cannot vanish.

Rényi CI: The Stronger Case $s \in (0,1] \cup \{\infty\}$

- For $s \in (0,1] \cup \{\infty\}$,

$$C_{\mathsf{W}}(\pi_{XY}) \leq T_{1+s}(\pi_{XY}).$$

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Definition

The maximal cross entropy w.r.t. $(X,Y) \sim \pi_{XY}$ over couplings of (P_X, P_Y) is

$$\mathsf{H}_\infty(P_X, P_Y \| \pi_{XY}) := \max_{Q_{XY} \in \mathcal{C}(P_X,P_Y)} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)},$$

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- $\mathsf{H}_\infty(\pi_X, \pi_Y \| \pi_{XY}) \ge H_\pi(X;Y)$ with equality iff $\pi_{XY} = \pi_X \pi_Y$.

Intuition for the Maximal Cross Entropy

- Take a sequence of n-types $T_X^{(n)} \in \mathcal{P}_n(\mathcal{X})$ and $T_Y^{(n)} \in \mathcal{P}_n(\mathcal{Y})$.

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- So $\mathsf{H}_\infty(P_X, P_Y \| \pi_{XY})$ is the exponential decay rate of this probability.

Upper and Lower Pseudo Common Informations

Definition

The upper pseudo-common information is

$$\overline{\Gamma}_\infty(\pi_{XY}) := \min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) + \mathsf{E}_{P_W}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \| \pi_{XY})\right]$$

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Contrast to Wyner's common information

$$C_{\mathsf{W}}(\pi_{XY}) = \min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) + H(XY).$$

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The lower pseudo-common information is

$$\underline{\Gamma}_\infty(\pi_{XY}) := \inf_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) + \inf_{Q_{WW'} \in \mathcal{C}(P_W, P_W)} \mathsf{E}_{Q_{WW'}}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W'} \| \pi_{XY})\right].$$

Rényi Common Information of order ∞

Theorem ([Yu and Tan, 2020a] [Yu and Tan, 2020c])

The order-∞ Rényi common information admits the following single-letter bounds

$$\tilde{T}_\infty(\pi_{XY}) \geq T_\infty(\pi_{XY}) \geq \max\{\underline{\Gamma}_\infty(\pi_{XY}), C_{\mathsf{W}}(\pi_{XY})\}$$

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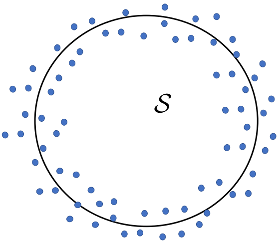

Product distribution

$$P_W^n(w^n) = \prod_{i=1}^{n} P_W(w_i)$$

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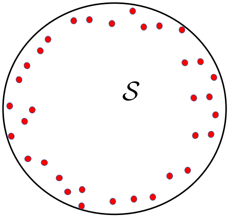

Truncated product distribution

$$P_{W^n}(w^n) \propto \left(\prod_{i=1}^n P_W(w_i)\right) \mathbb{1}\{w^n \in \mathcal{S}\}$$

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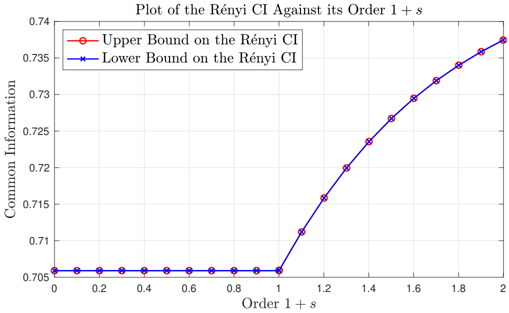

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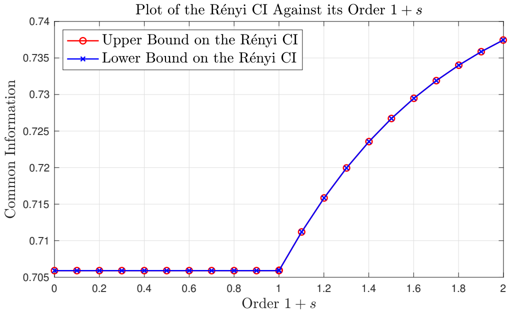

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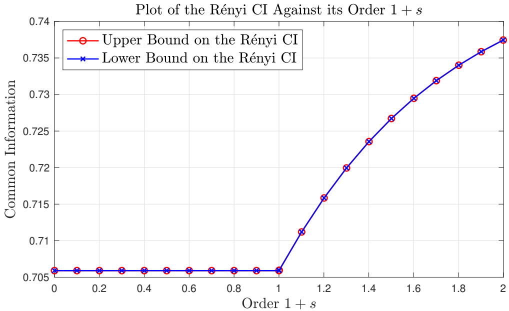

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Does this have more profound implications?

Outline

Exact Common Information?

- In the distributed source simulation problem à la Wyner, we mandated that

$$\frac{1}{n} D(P_{X^n Y^n} \| \pi_{XY}^n) \to 0.$$

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- In come [Kumar et al., 2014], who introduced

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Exact Common Information

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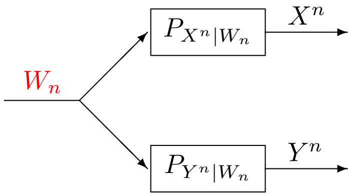

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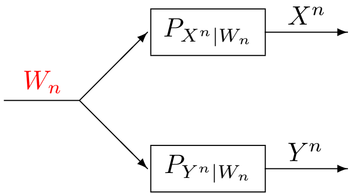

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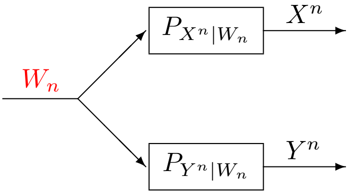

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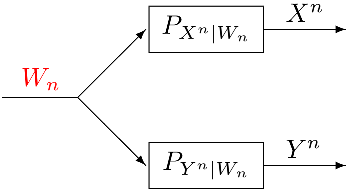

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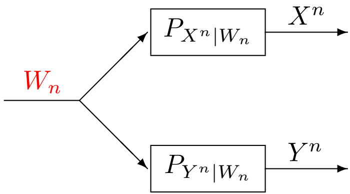

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- Let the length of W_n be $\ell(W_n)$.
- Then, by Shannon's zero-error compression theorem, the optimal expected codeword length $L(W_n) = \mathbb{E}[\ell(W_n)]$ satisfies

$$H(W_n) \leq L(W_n) < H(W_n) + 1$$

which implies that

$$\lim_{n\to\infty} \frac{L(W_n)}{n} = \lim_{n\to\infty} \frac{H(W_n)}{n}.$$

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The exact common information is defined as

$$T_{\mathrm{Ex}}(\pi_{XY}) := \inf\left\{\lim_{n\to\infty}\frac{L(W_n)}{n} : P_{X^nY^n} = \pi_{XY}^n \text{ for some } n \geq 1\right\}$$

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As expected the exact common information rate is greater than or equal to the Wyner common information.

Proposition 3.

$$\overline{G}(X;Y) \geq J(X;Y).$$

In the following section, we show that they are equal for the SBES in Example 1. We do not know if this is the case in general, however.

From [Kumar et al., 2014]

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the exact common information rate. While this multiletter characterization is in general greater than or equal to the Wyner common information, we showed that they are equal for the SBES. The main open question is whether the exact common information rate has a single letter characterization in general. Is it always equal to the Wyner common information? Is there an example 2-DMS for which the exact common information rate is strictly larger than the Wyner common information? It would also be interesting to further explore the application to machine learning.

From [Kumar et al., 2014]

Surprising Equivalence: ∞-Rényi CI and Exact CI

Theorem ([Yu and Tan, 2020c])

For a bivariate source π_{XY} on a finite alphabet,

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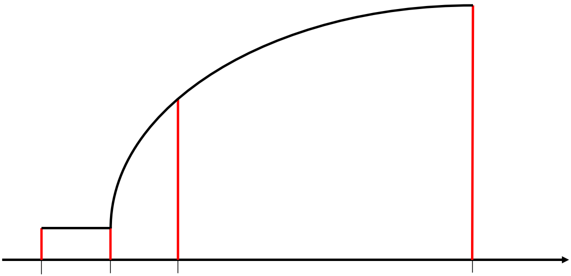

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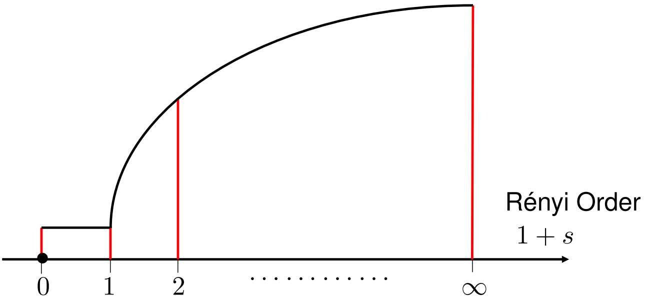

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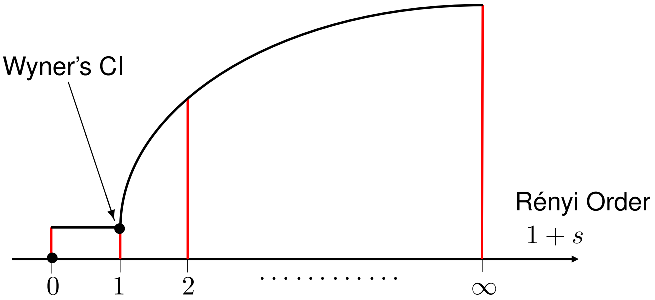

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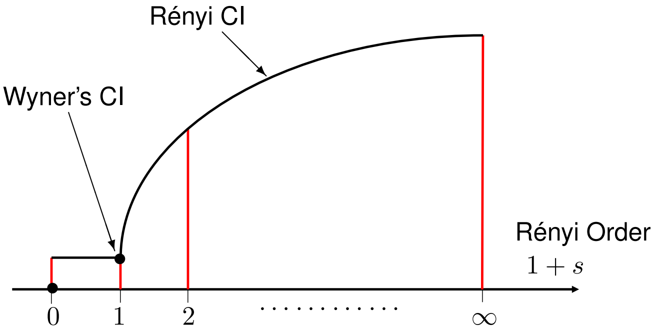

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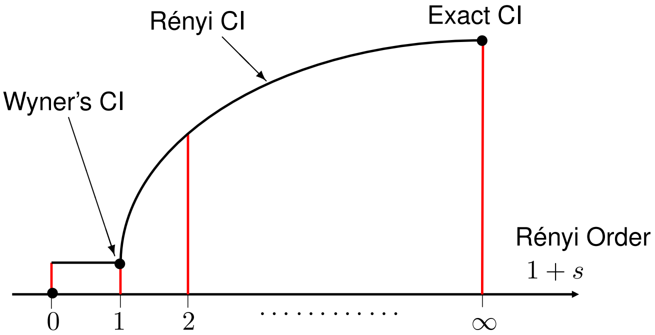

Proof of $\Longrightarrow$ Part of Equivalence Theorem

Lemma ([Kumar et al., 2014], [Vellambi and Kliewer, 2016])

$\exists$ *rate-*R ∞*-Rényi CI code* $\Longrightarrow$ $\exists$ *rate-*R *Exact CI code*

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- $\exists$ rate-R ∞-Rényi CI code

$$D_\infty(P_{X^nY^n} \| \pi_{XY}^n) < \epsilon \quad \Longrightarrow \quad P_{X^nY^n}(x^n, y^n) < 2^\epsilon \pi_{XY}^n(x^n, y^n)$$

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then obviously, $\widehat{P}_{X^nY^n}(x^n, y^n)$ is a valid distribution.

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- Hence π_{XY}^n can be written as a mixture distribution

$$\pi_{XY}^n(x^n, y^n) = 2^{-\epsilon} P_{X^nY^n}(x^n, y^n) + \left(1 - 2^{-\epsilon}\right) \widehat{P}_{X^nY^n}(x^n, y^n)$$

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- A time-sharing variable-length scheme:
  - The encoder first generates $U \sim \mathrm{Bern}(2^{-\epsilon})$, and transmits it to two generators using 1 bit
  - If $U = 1$, then the encoder and two generators use the rate-R ∞-Rényi CI code to generate $P_{X^nY^n}$
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- The induced distribution is π_{XY}^n exactly
- The total code rate

$$\leq \frac{1}{n} + 2^{-\epsilon} R + \left(1 - 2^{-\epsilon}\right) \log(|\mathcal{X}||\mathcal{Y}|) \to R$$

as $n \to \infty, \epsilon \to 0$

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Lemma

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$$\lim_{k\to\infty} \frac{1}{k} H(P_{W_k}) = R$$

but W_k is not uniform. 

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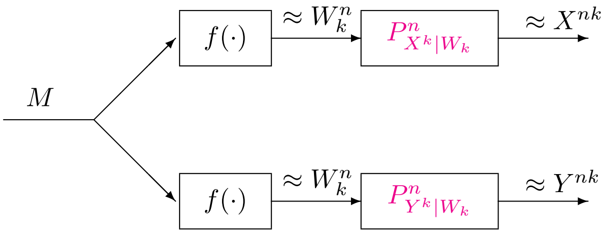
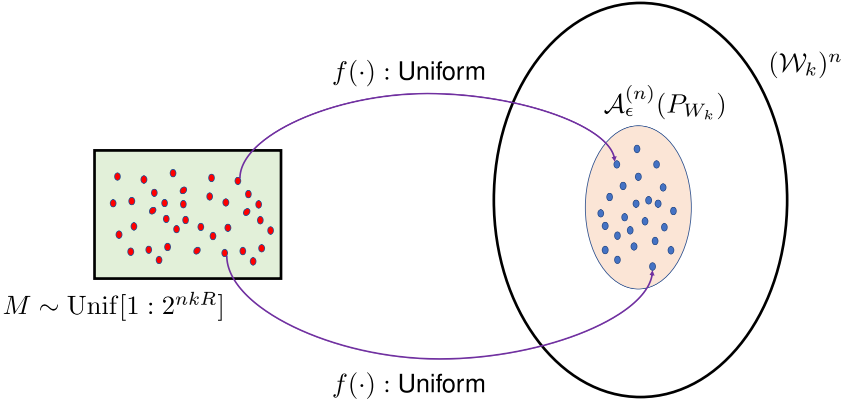
  but W_k is not uniform. 
- Simulate W_k^n using two Rényi source resolvability codes!



Proof of $\Longleftarrow$ Part of Equivalence Theorem



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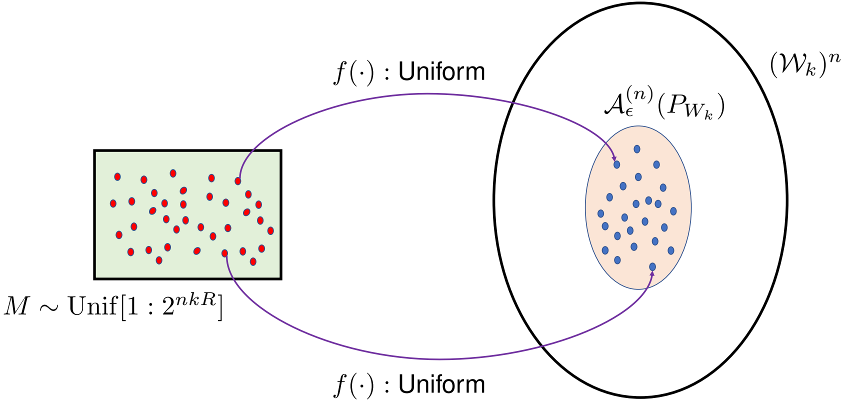

Succeed in the sense of $D_\infty(P_{f(M)} \| P_{W_k}^n) \to 0$ if [Yu and Tan, 2019]

$$R > \frac{1}{k} H(P_{W_k})$$

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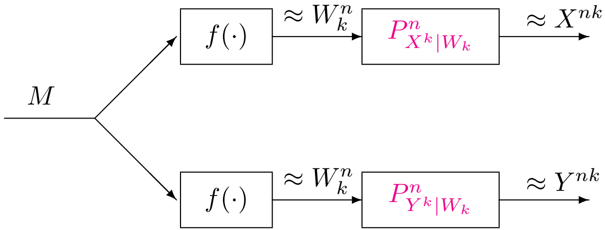

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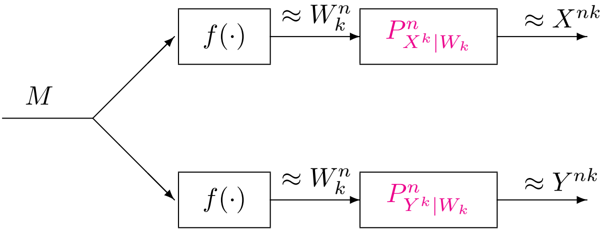

- For the given stochastic kernel (channel) $P^n_{X^k|W_k} P^n_{Y^k|W_k}$,

$$P_W^n \longrightarrow P^n_{X^k|W_k} P^n_{Y^k|W_k} \longrightarrow \pi^{kn}_{XY}$$

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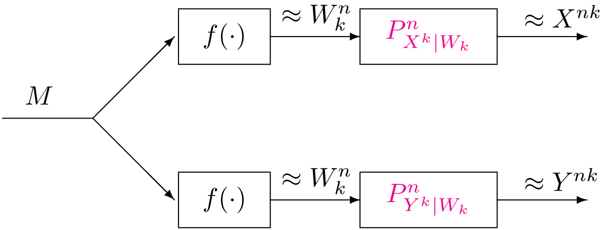

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- By the data processing inequality (DPI) for Rényi divergence,

$$D_\infty(P_{X^{kn}Y^{kn}} \| \pi_{XY}^{kn}) \le D_\infty(P_{f(M)} \| P_{W_k}^n) \xrightarrow{n\to\infty} 0$$

Combining with Single-Letter Bounds from Rényi CI

Theorem ([Yu and Tan, 2020c])

For $(X, Y) \sim \pi_{XY}$ *on a finite alphabet,*

$$\underline{\Gamma}_\infty(\pi_{XY}) \leq T_{\mathrm{Ex}}(\pi_{XY}) = \tilde{T}_\infty(\pi_{XY}) \leq \overline{\Gamma}_\infty(\pi_{XY}).$$

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- Gone from a multi-letter expression by [Kumar et al., 2014]

$$\lim_{n\to\infty} \frac{1}{n} \min_{\substack{P_{W_n} P_{X^n|W_n} P_{Y^n|W_n}: \\ P_{X^n Y^n} = \pi_{XY}^n}} H(W_n)$$

to single-letter bounds.

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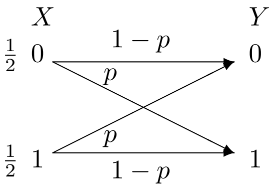
to single-letter bounds.

- Presumably the bounds are more amenable to numerical evaluation?

Revisiting the DBSS



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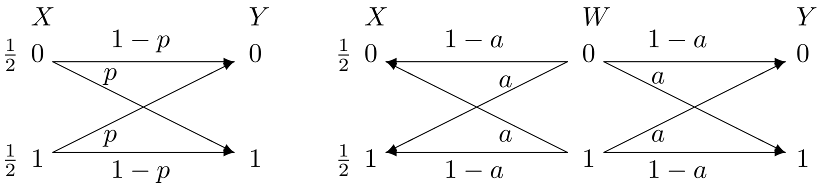

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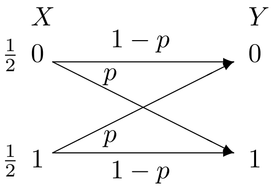
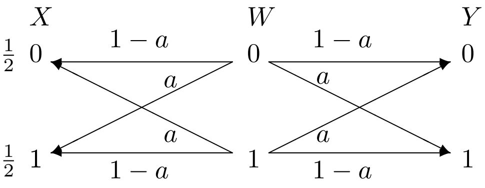




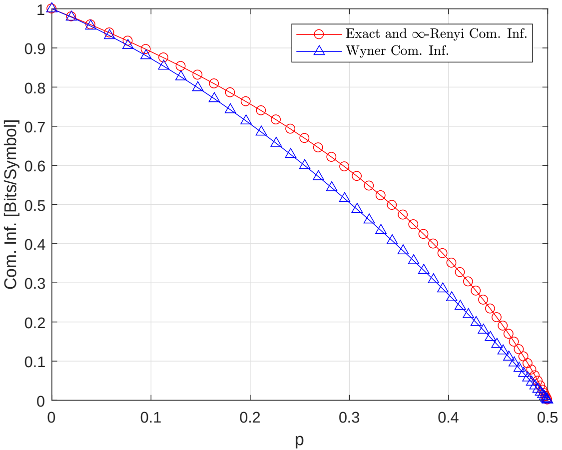
Theorem (Evaluation of Upper and Lower Bounds for DSBS(p))

For a DSBS $(X, Y) \sim \mathrm{DSBS}(p)$ *with crossover probability* $p \in (0, 1/2)$*,*

$$\begin{aligned}\tilde{T}_\infty(\pi_{XY}) &= T_{\mathrm{Ex}}(\pi_{XY}) \\ &= -2h(a) - (1-2a)\log\left[\frac{1}{2}\left(a^2 + (1-a)^2\right)\right] - 2a\log\left[a(1-a)\right],\end{aligned}$$

where $a := \frac{1-\sqrt{1-2p}}{2} \in (0, \frac{1}{2})$ *and* $h(a) := -a\log a - (1-a)\log(1-a)$*.*

Numerical Results — DSBS



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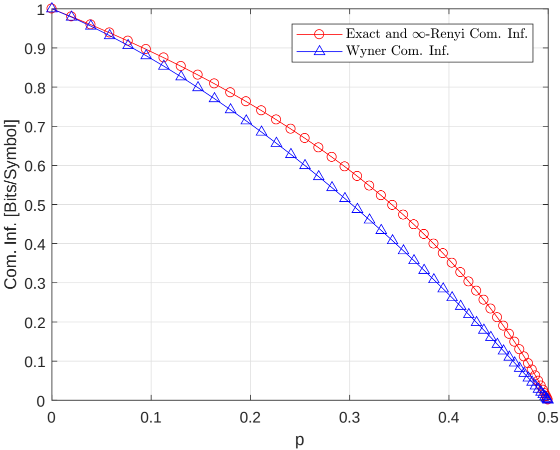

$$T_{\mathrm{Ex}}(\mathrm{DSBS}(p)) > C_{\mathrm{W}}(\mathrm{DSBS}(p)) \quad \forall\, p \in (0, 1/2).$$

Answers the open question in [Kumar et al., 2014].

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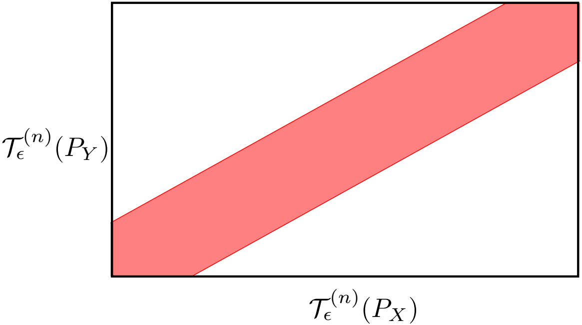

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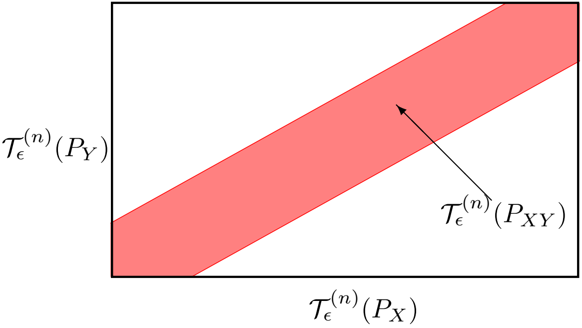

Wyner's common information requires

$$\frac{P_{X^nY^n}(x^n, y^n)}{\pi_{XY}^n(x^n, y^n)} = 1 + o(1) \quad \text{for almost all} \quad (x^n, y^n) \in \mathcal{T}_\epsilon^{(n)}(\pi_{XY})$$

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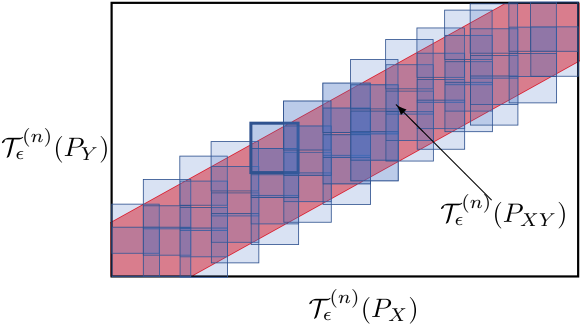

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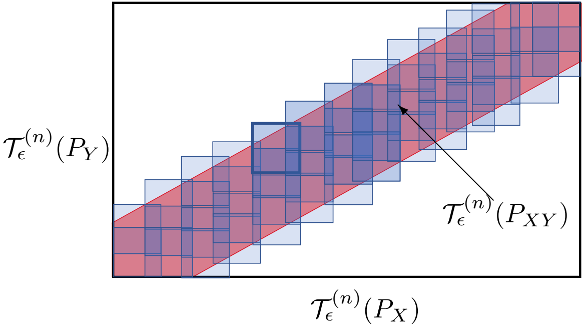

Rényi CI of order ∞ or Exact CI requires

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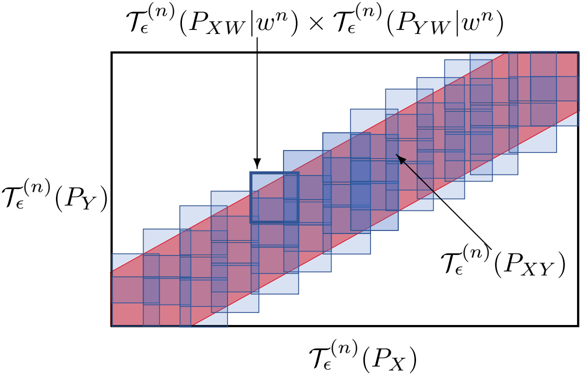

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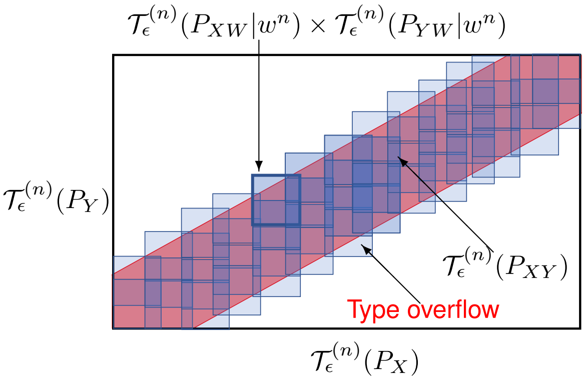

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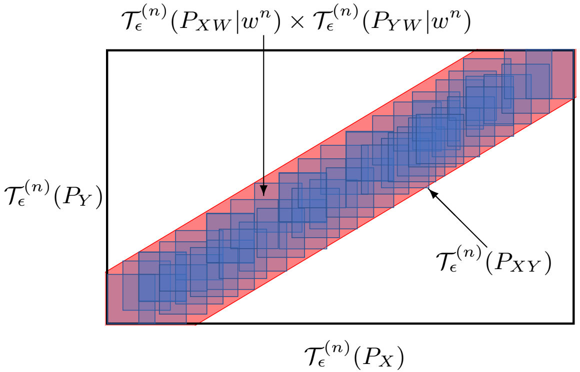

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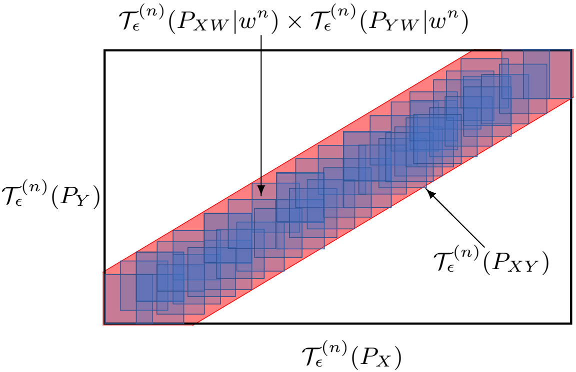

Sufficient Condition [Vellambi and Kliewer, 2016]

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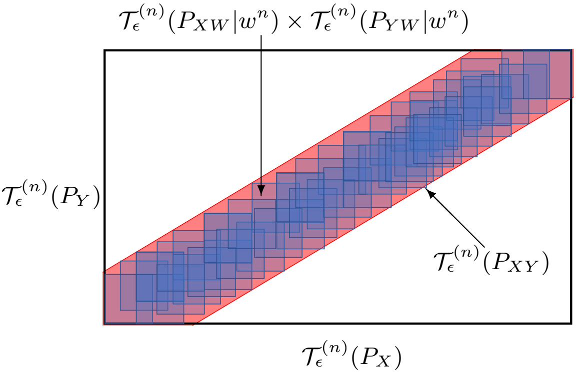

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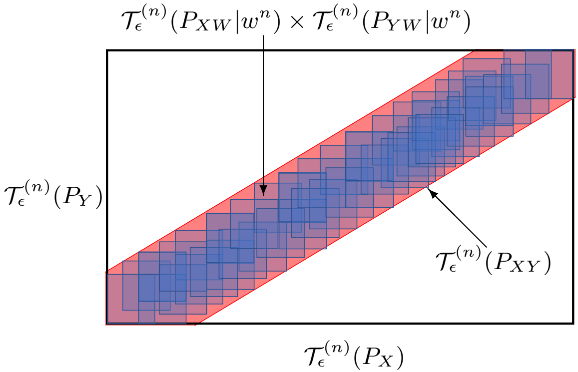

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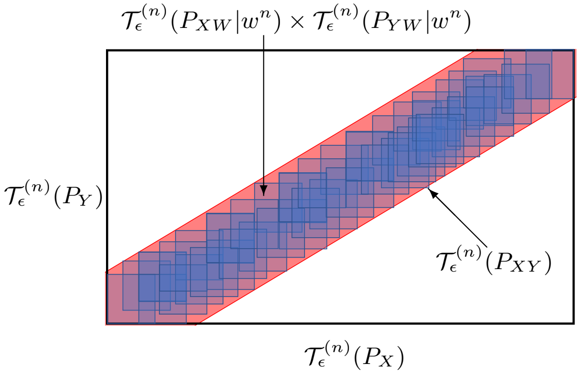

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$$\iff \mathcal{T}_\epsilon^{(n)}(P_{XY}) \approx \bigcup_{w^n\in\mathcal{C}} \Big(\mathcal{T}_\epsilon^{(n)}(P_{XW}|w^n) \times \mathcal{T}_\epsilon^{(n)}(P_{YW}|w^n)\Big)$$

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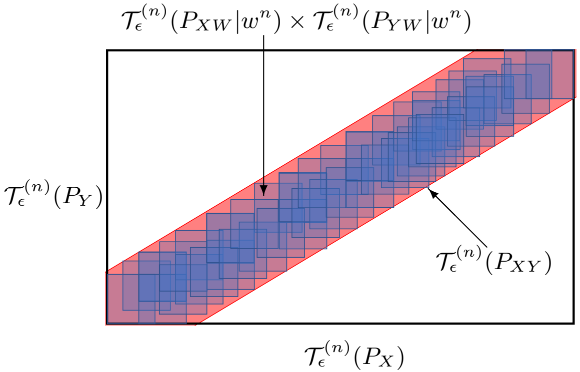

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$$\iff \mathcal{T}_\epsilon^{(n)}(P_{XY}) \approx \mathrm{supp}(P_{X^nY^n}) \qquad \text{(No type overflow)}$$

When Exact CI (or ∞-Rényi CI) $=$ Wyner's CI

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$$H(X|W=w)H(Y|W=w)=0 \qquad \forall\, w$$

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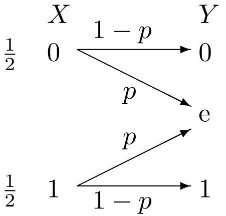
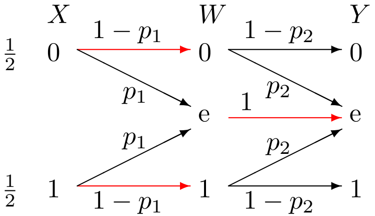
- Symmetric Binary Erasure Source (SBES)





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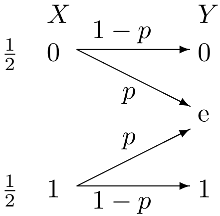

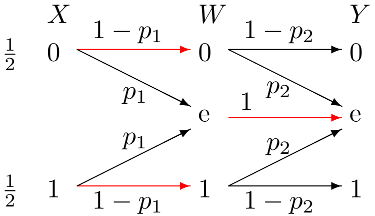

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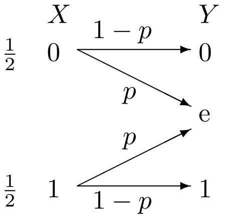
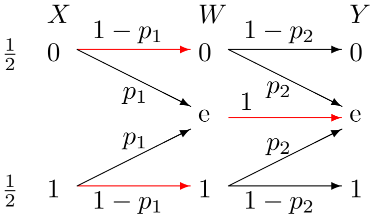




- $(1-p_1)(1-p_2)=1-p$.

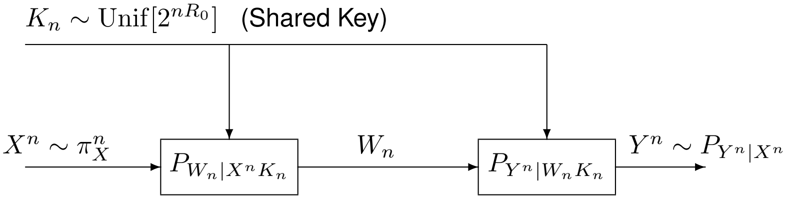
- The Exact CI is equal to Wyner's CI and

$$\tilde{T}_\infty(\pi_{XY}) = T_{\mathrm{Exact}}(\pi_{XY}) = C_{\mathsf{Wyner}}(\pi_{XY}) = \begin{cases} 1 & p \le \frac{1}{2} \\ H(p) & p > \frac{1}{2} \end{cases}.$$

Outline

Channel Synthesis

- Given $\pi_{XY} = \pi_X \pi_{Y|X}$ consider the following task:



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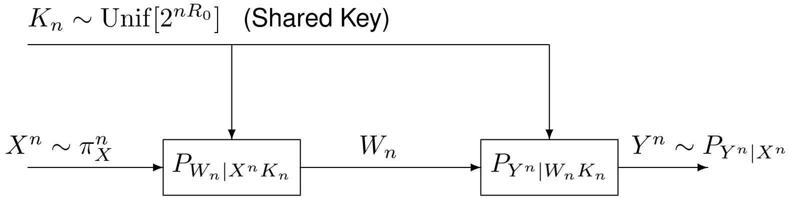

- Goal: Ensure that

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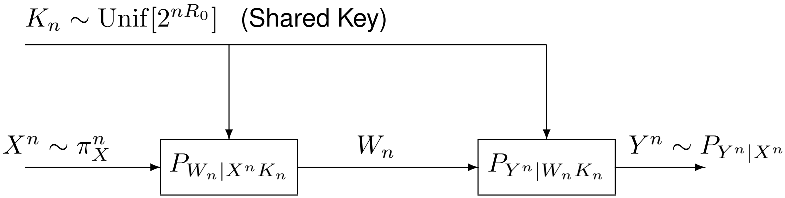

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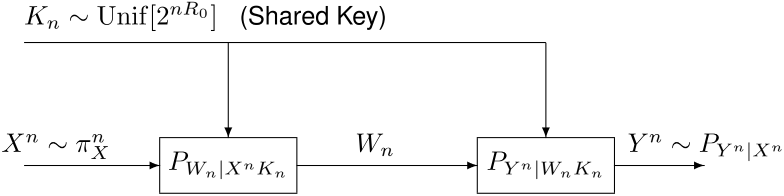

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- Known as channel synthesis [Cuff, 2012].

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- Consider approximate channel synthesis under TV criterion, i.e.,

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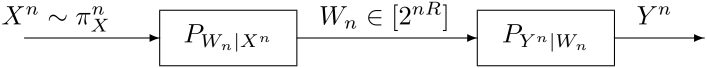

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$$X^n \sim \pi_X^n \longrightarrow \boxed{P_{W_n|X^n}} \xrightarrow{W_n \in [2^{nR}]} \boxed{P_{Y^n|W_n}} \xrightarrow{Y^n}$$

so the minimum compression rate is Wyner's common information

$$R^*(R_0 = 0|\pi_{XY}) = C_{\mathsf{W}}(\pi_{XY})$$

- When $R_0 = \infty$,

$$R^*(R_0 = \infty|\pi_{XY}) = I_\pi(X;Y)$$

Approximate Channel Synthesis

It was shown in [Cuff, 2012] that

$$\mathcal{R}_{\mathsf{W}}(\pi_{XY}) := \bigcup_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} \left\{ (R, R_0) \; : \; \begin{array}{c} R \geq I(X;W) \\ R + R_0 \geq I(XY;W) \end{array} \right\}.$$

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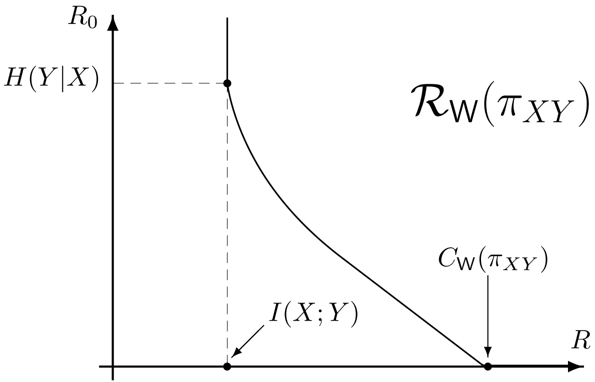
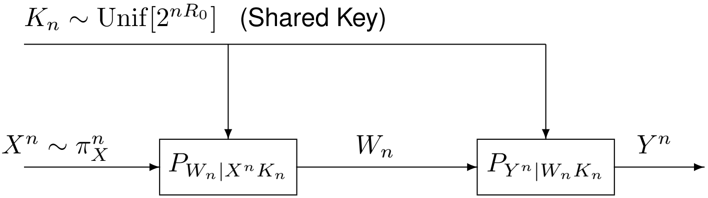


Exact Channel Synthesis



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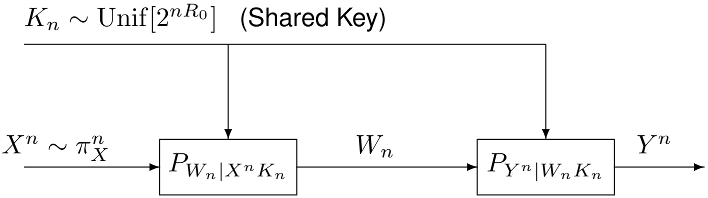

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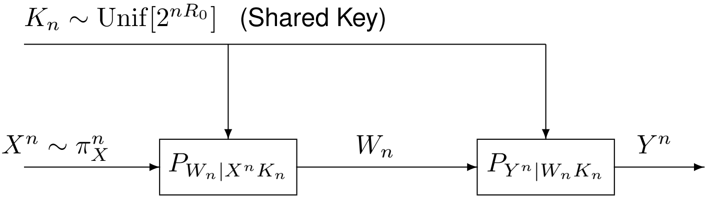

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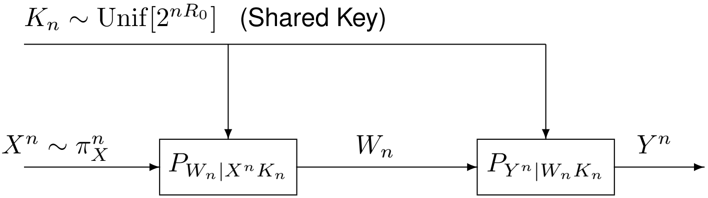

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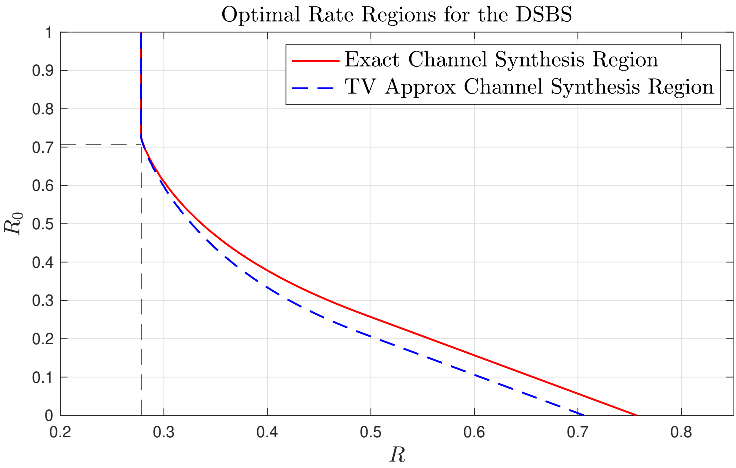
- Best tradeoff between R and R_0 in the non-extremal cases considered by [Yu and Tan, 2020b].

Doubly Binary Symmetric Sources



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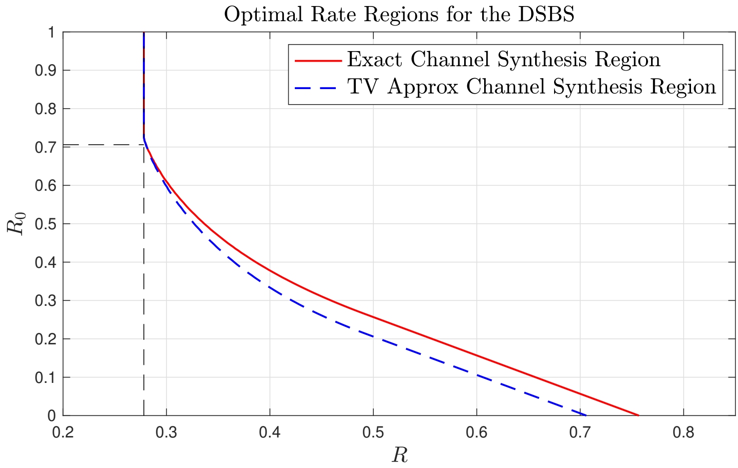

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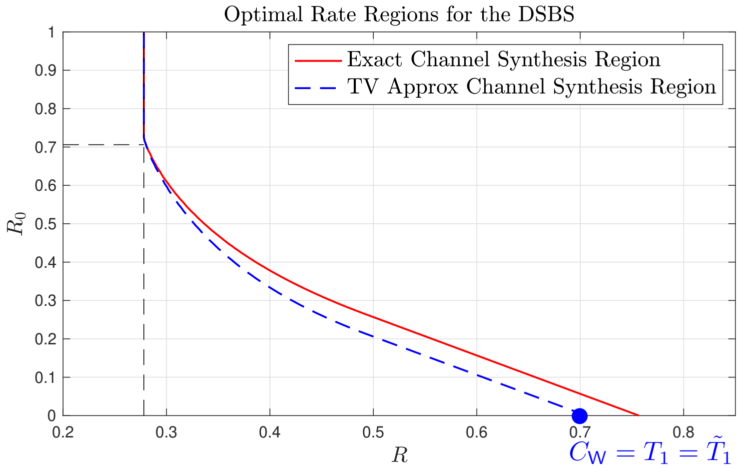

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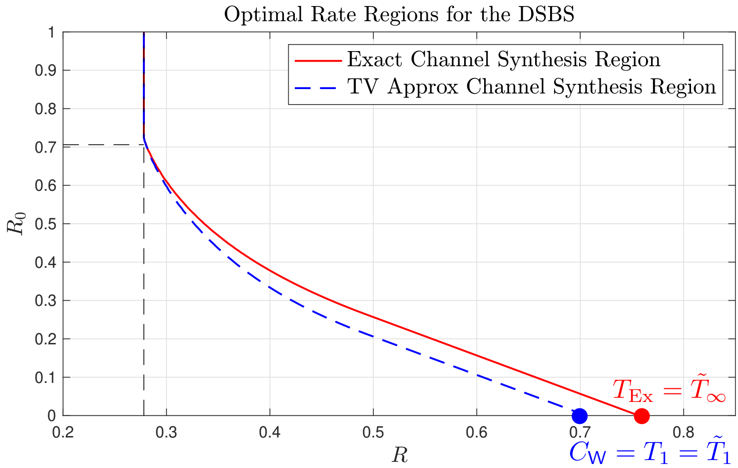

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Outline

Nonnegative Matrix Factorization

- Given a matrix $\mathbf{M} \in \mathbb{R}_+^{m\times k}$, find $\mathbf{U} \in \mathbb{R}_+^{m\times r}$ and $\mathbf{V} \in \mathbb{R}_+^{r\times k}$ such that

$$\mathbf{M} \approx \mathbf{U}\mathbf{V} \qquad \text{or} \qquad \mathbf{M} = \mathbf{U}\mathbf{V}.$$

  Many applications. See [Cichocki et al., 2009] or [Gillis, 2020].

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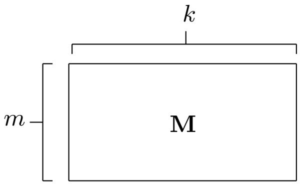

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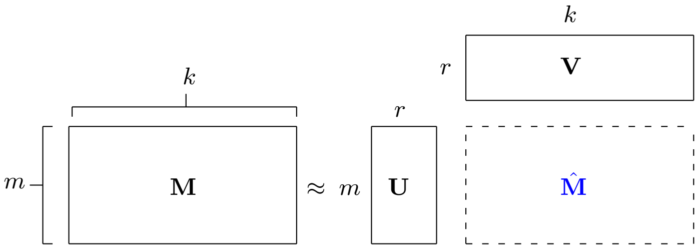

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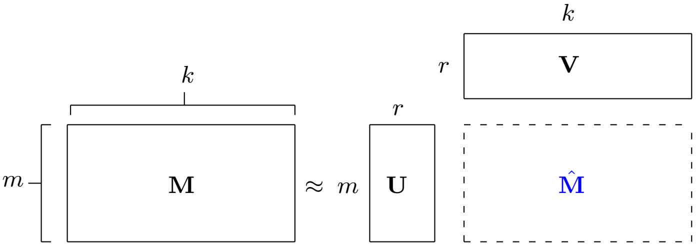

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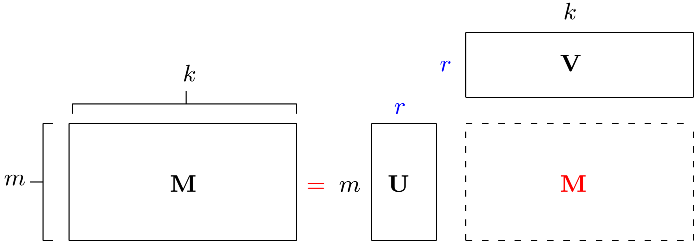

- Only interested in exact factorization.
- What is the minimum r to achieve exact factorization? Is this connected to information theory?

Nonnegative Rank

Definition

The nonnegative rank of $\mathbf{M} \in \mathbb{R}_+^{m \times k}$, denoted as $\mathrm{rank}_+(\mathbf{M})$, is the smallest integer r such that

$$\mathbf{M} = \sum_{w=1}^{r} \mathbf{u}_w \mathbf{v}_w^\top$$

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$$\mathbf{M} = \begin{bmatrix} 0 & (a_1 - a_2)^2 & (a_1 - a_3)^2 & \ldots & (a_1 - a_m)^2 \\ (a_2 - a_1)^2 & 0 & (a_2 - a_3)^2 & \ldots & (a_2 - a_m)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (a_m - a_1)^2 & (a_m - a_2)^2 & (a_m - a_3)^2 & \ldots & 0 \end{bmatrix}.$$

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- $\mathrm{rank}(\mathbf{M}) \le 3$. [Beasley and Laffey, 2009] showed $\mathrm{rank}_+(\mathbf{M}) = \Omega(\log m)$.

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- Wyner's common information for $\mathbf{M}$ is

$$C_{\mathsf{W}}(\mathbf{M}) := C_{\mathsf{W}}(\pi_{XY}).$$

Playing With Definitions

Theorem ([Jain et al., 2013], [Braun and Pokutta, 2013])

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$$P_{W|XY}(w|x,y) = \begin{cases} \dfrac{[\mathbf{u}_w]_x [\mathbf{v}_w]_y}{M_{x,y}} & M_{x,y} > 0 \\ \text{arbitrary} & M_{x,y} = 0 \end{cases}.$$

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By Bayes rule,

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So, $X - W - Y$ and

$$C_{\mathsf{W}}(\mathbf{M}) \leq I_P(XY;W) \leq H(W) \leq \log |\mathcal{W}| = \log \operatorname{rank}_+(\mathbf{M}).$$

□

Gap Between $C_{\mathsf{W}}(\mathbf{M})$ and $\log \mathrm{rank}_+(\mathbf{M})$?

- Consider the diagonal matrix

$$\mathbf{M} = \frac{1}{\sum_{j=1}^m 2^j} \begin{bmatrix} 2^1 & 0 & 0 & \dots & 0 \\ 0 & 2^2 & 0 & \dots & 0 \\ 0 & 0 & 2^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2^m \end{bmatrix}.$$

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- Gap can be arbitrarily large.
- Is the relation between $C_{\mathsf{W}}(\mathbf{M})$ and $\log \operatorname{rank}_+(\mathbf{M})$ fundamental?

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Theorem ([Braun et al., 2017])

Let $\mathbf{M} \in \mathbb{R}_+^{m\times k}$ *be such that* $\|\mathbf{M}\|_1 = \sum_{x,y} M_{x,y} = 1$. *For any* $\epsilon, \delta > 0$, *if* $n \geq n_0(\epsilon, \delta, m, k, C_{\mathsf{W}}(\mathbf{M}))$ *is sufficiently large, there exists* $\mathbf{M}_{\epsilon,\delta,n} \in \mathbb{R}_+^{m^n \times k^n}$ *with*

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FullCircle

Outline

References I

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Common Information: Old and New
Part II: Gács-Körner-Witsenhausen's CI and Extensions

Vincent Y. F. Tan and Lei Yu

National University of Singapore and Nankai University

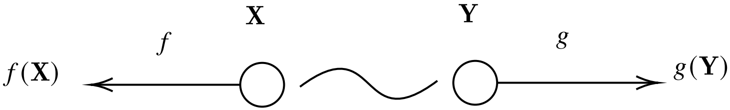
2021 International Symposium on Information Theory

Outline

1. Gács–Körner–Witsenhausen's CI
2. NICD with 2 Users
   - Formulation of NICD
   - Achievability Part: Subcubes, Balls, and Spheres
   - Converse Part: CL Regime with Large a, b
   - Converse Part: MD Regime and CL Regime with Small a, b
   - Converse Part: LD Regime
   - Connection to Hypercontractivity
   - Extension to Gaussian Sources
3. NICD with Multiple Users and q-Stability
   - Formulation
   - Balanced Case
   - MD and LD Regimes

Outline

Gács–Körner–Witsenhausen's System



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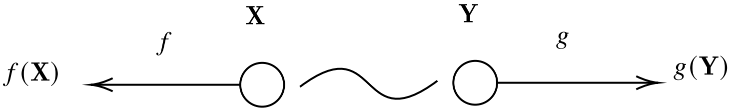

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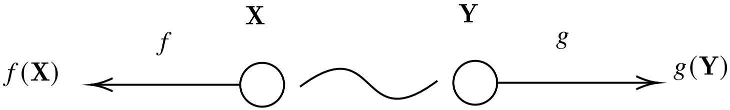

- $(\mathbf{X}, \mathbf{Y}) \sim P_{XY}^n$: a pair of correlated sources
- Define one-sided ϵ-common information:

$$T_X(\epsilon) := \liminf_{n\to\infty} \max_{f,g:\mathbb{P}[f(\mathbf{X})\neq g(\mathbf{Y})]\leq\epsilon} \frac{1}{n} H(f(\mathbf{X}))$$

$$T_Y(\epsilon) := \liminf_{n\to\infty} \max_{f,g:\mathbb{P}[f(\mathbf{X})\neq g(\mathbf{Y})]\leq\epsilon} \frac{1}{n} H(g(\mathbf{Y}))$$

Gács–Körner–Witsenhausen's CI

Theorem ([Gács and Körner, 1973])

$$\lim_{\epsilon \downarrow 0} T_X(\epsilon) = \lim_{\epsilon \downarrow 0} T_Y(\epsilon) = C_{\mathrm{GKW}}(X;Y),$$

where

$$C_{\mathrm{GKW}}(X;Y) := \max_{f,g: f(X)=g(Y)} H(f(X))$$

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- $C_{\mathrm{GKW}}(X;Y)$ called Gács–Körner–Witsenhausen's (GKW's) CI
- Notice: This is a brief version of GKW's system introduced in [Csiszár and Narayan, 2000]

Undesired Properties of GKW's CI

- Fact: GKW's CI $= 0$ for Gaussian sources and doubly symmetric binary sources (DSBSes)
  - More unfortunately, we cannot extract even one pair of identical bits from $(\mathbf{X}, \mathbf{Y})$, if $(\mathbf{X}, \mathbf{Y})$ is jointly Gaussian or if $(\mathbf{X}, \mathbf{Y})$ is a DSBS.

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A Variant of CI: What is the maximal possible correlation of a pair of bits that can be extracted from X, Y individually?

- Coined the binary decision problem [Witsenhausen, 1975],
  the noninteractive correlation distillation (NICD) problem [Mossel et al., 2006],
  the noninteractive binary simulation problem [Kamath and Anantharam, 2016],
  an isoperimetric problem,
  ...

Outline

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Doubly Symmetric Binary Source (DSBS)

Throughout this talk, we mainly consider the DSBS, Unless Otherwise Specified!!!

$$P_{XY} = \begin{array}{ccc} X\backslash Y & 0 & 1 \\ 0 & \frac{1+\rho}{4} & \frac{1-\rho}{4} \\ 1 & \frac{1-\rho}{4} & \frac{1+\rho}{4} \end{array}$$

with correlation $\rho \in (0,1)$, and

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with correlation $\rho \in (0, 1)$, and

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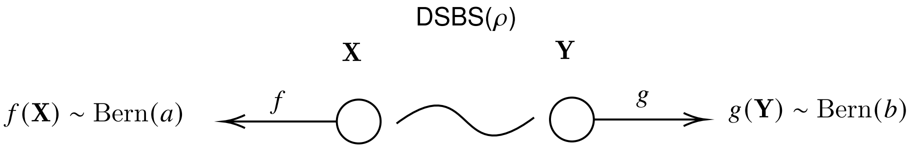
- If you are interested in other sources, please refer to [Ahlswede and Gács, 1976, Carlen and Cordero-Erausquin, 2009, Beigi and Nair, 2016, Yu et al., 2021, Yu, 2021c]...

Noninteractive Correlation Distillation



$\max \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}))$ or equivalently, $\max \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1)$

Noninteractive Correlation Distillation

- Formally, for $a, b \in [0,1]$, define the Forward Joint Probability as

$$\overline{\Gamma}^{(n)}(a,b) := \max_{\substack{f,g:\{0,1\}^n \to \{0,1\}: \mathbb{P}(f(\mathbf{X})=1) \le a, \\ \mathbb{P}(g(\mathbf{Y})=1) \le b}} \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1)$$

$$= \max_{\substack{A,B \subseteq \{0,1\}^n : P_X^n(A) \le a, \\ P_Y^n(B) \le b}} P_{XY}^n(A \times B), \qquad (f = 1_A, g = 1_B)$$

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- Define the Reverse Joint Probability as

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- For $a = \frac{M}{2^n}, b = \frac{N}{2^n}$ (with integers M, N), the "inequalities" in the constraints can be replaced by "equalities"
- Equivalence:

$$\overline{\Gamma}^{(\infty)}(1-a,b) = b - \underline{\Gamma}^{(\infty)}(a,b),$$

where $\overline{\Gamma}^{(\infty)}, \underline{\Gamma}^{(\infty)}$ denote the pointwise limits of $\overline{\Gamma}^{(n)}, \underline{\Gamma}^{(n)}$ as $n \to \infty$.

Interesting Cases

Limit cases as $n \to \infty$

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  - interpolates between the CL and LD regimes

- Notice: In all the above cases, $\alpha, \beta > 0$ are fixed.

Definitions of Several Exponents

- (Forward and Reverse) CL Exponents: For $\alpha, \beta \in (0, \infty)$,

$$\underline{\Theta}_{\mathrm{CL}}^{(n)}(\alpha,\beta) := -\log \overline{\Gamma}^{(n)}\left(e^{-\alpha}, e^{-\beta}\right), \qquad \overline{\Theta}_{\mathrm{CL}}^{(n)}(\alpha,\beta) := -\log \underline{\Gamma}^{(n)}\left(e^{-\alpha}, e^{-\beta}\right)$$

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- Denote $\underline{\Theta}_{\mathrm{CL}}^{(\infty)}, \overline{\Theta}_{\mathrm{CL}}^{(\infty)}, \underline{\Theta}_{\mathrm{LD}}^{(\infty)}, \overline{\Theta}_{\mathrm{LD}}^{(\infty)}, \underline{\Theta}_{\mathrm{MD}}^{(\infty)}, \overline{\Theta}_{\mathrm{MD}}^{(\infty)}$ as the pointwise limits of the above as $n \to \infty$.

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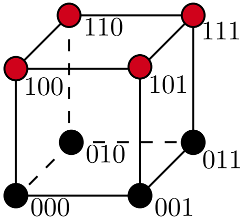
- Equivalence: For finite n, $\underline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha,\beta) = \frac{1}{n}\underline{\Theta}_{\mathrm{CL}}^{(n)}(n\alpha, n\beta)$ and $\underline{\Theta}_{\mathrm{MD}}^{(n)}(\alpha,\beta) = \frac{1}{\theta_n}\underline{\Theta}_{\mathrm{CL}}^{(n)}(\theta_n\alpha, \theta_n\beta)$; Similar formulas hold for the reverse part.

Outline

Achievability: Hamming Subcubes



- An $(n-k)$-subcube C_{n-k} is a set of $\mathbf{x}$ with k components fixed (e.g., $\{\mathbf{1}_k\} \times \{0,1\}^{n-k}$ whose indicator $\mathbf{x} \mapsto \prod_{i=1}^{k} x_i$)

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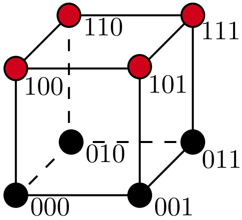

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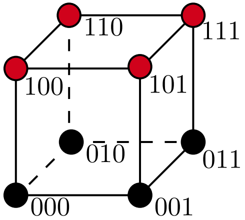

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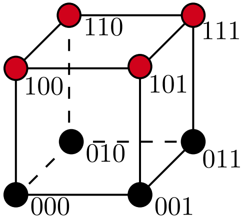
- Case of $a = b = 2^{-k}$: $A = B = C_{n-k}$ (identical) $\Longrightarrow P_{XY}^n(A \times B) = P_{XY}(1,1)^k = \left(\frac{1+\rho}{4}\right)^k$;

  $A = \mathbf{1} - B = C_{n-k}$ (anti-symmetric) $\Longrightarrow P_{XY}^n(A \times B) = P_{XY}(1,0)^k = \left(\frac{1-\rho}{4}\right)^k$

Achievability: Hamming Subcubes



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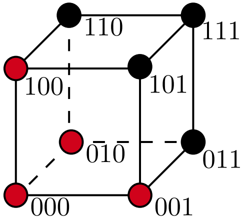
  $A = \mathbf{1} - B = C_{n-k}$ (anti-symmetric) $\Longrightarrow P_{XY}^n(A \times B) = P_{XY}(1,0)^k = \left(\frac{1-\rho}{4}\right)^k$
- Case of $a = 2^{-k_1}, b = 2^{-k_2}, k_1 \le k_2$: $P_{XY}^n(A \times B) = \left(\frac{1}{2}\right)^{k_2-k_1} \left(\frac{1+\rho}{4}\right)^{k_1}$ (resp. $\left(\frac{1}{2}\right)^{k_2-k_1} \left(\frac{1-\rho}{4}\right)^{k_1}$) for "almost identical" (resp. "almost anti-symmetric") subcubes A, B

Achievability: Hamming Balls (CL Regime)



- Hamming Ball: For $r \in [0, n]$, $\mathbb{B}_r(\mathbf{0}) := \{\mathbf{x} : d_{\mathrm{H}}(\mathbf{x}, \mathbf{0}) \le r\} \iff \left\{\mathbf{x} : \sum_{i=1}^n x_i \le r\right\}$

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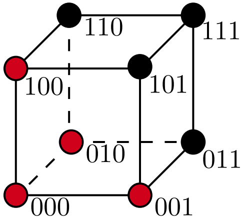

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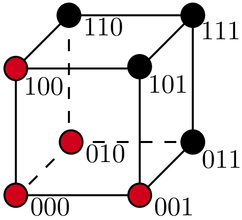

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- By the univariate and multivariate CL theorems,

$$P_X^n(A) \to \Phi(\lambda), \qquad P_Y^n(B) \to \Phi(\mu), \qquad P_{XY}^n(A \times B) \to \Phi_\rho(\lambda, \mu)$$

where Φ is the CDF of the standard Gaussian, and $\Phi_\rho(\cdot, \cdot)$ is the CDF of the zero-mean bivariate Gaussian with covariance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

Achievability: Hamming Balls (CL Regime)

- Achievable CL probabilities:

$$\overline{\Gamma}^{(\infty)}(a,b) \geq \Lambda_{\rho}(a,b) \quad \text{(by concentric balls)}$$

  - Bivariate normal copula (or Gaussian quadrant probability function):

$$\Lambda_\rho(a,b) := \Phi_\rho\left(\Phi^{-1}(a), \Phi^{-1}(b)\right)$$

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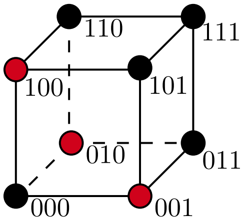
- $\Longleftrightarrow$

$$\underline{\Theta}_{\mathrm{CL}}^{(\infty)}(\alpha,\beta) \leq \underline{\Theta}_{\mathrm{CL}}(\alpha,\beta), \qquad \overline{\Theta}_{\mathrm{CL}}^{(\infty)}(\alpha,\beta) \geq \overline{\Theta}_{\mathrm{CL}}(\alpha,\beta)$$

  - Exponents of Λ_{ρ} and $\Lambda_{-\rho}$:

$$\underline{\Theta}_{\mathrm{CL}}(\alpha,\beta) := -\log \Lambda_{\rho}\left(e^{-\alpha}, e^{-\beta}\right), \qquad \overline{\Theta}_{\mathrm{CL}}(\alpha,\beta) := -\log \Lambda_{-\rho}\left(e^{-\alpha}, e^{-\beta}\right)$$

Achievability: Hamming Spheres (LD Regime)



- Hamming Sphere: For $r \in [0:n]$, $\mathbb{S}_r(\mathbf{0}) := \{\mathbf{x} : d_{\mathrm{H}}(\mathbf{x}, \mathbf{0}) = r\} \Longleftrightarrow \left\{\mathbf{x} : \sum_{i=1}^n x_i = r\right\}$

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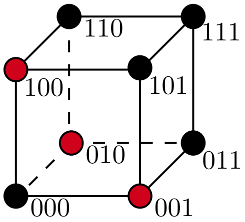

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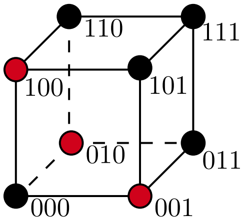

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- LD regime: Choose $A = \mathbb{S}_{r_n}\left(\mathbf{0}\right), B = \mathbb{S}_{s_n}\left(\mathbf{0}\right)$ with $r_n = \lambda n, s_n = \mu n$ where $\lambda, \mu \in [0, 1]$

Achievability: Hamming Spheres (LD Regime)

- By LD theory (or Sanov's theorem),

$$-\frac{1}{n}\log P_X^n\left(A\right) \to D\left(\left(\lambda,\bar{\lambda}\right)\|P_X\right) = 1 - H_2\left(\lambda\right)$$

$$-\frac{1}{n}\log P_Y^n\left(B\right) \to D\left(\left(\mu,\bar{\mu}\right)\|P_Y\right) = 1 - H_2\left(\mu\right)$$

$$-\frac{1}{n}\log P_{XY}^n\left(A\times B\right) \to \mathbb{D}\left(\left(\lambda,\bar{\lambda}\right),\left(\mu,\bar{\mu}\right)\|P_{XY}\right)$$

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$$-\frac{1}{n}\log P_{XY}^n(A\times B) \to \mathbb{D}\left(\left(\lambda,\bar{\lambda}\right),(\mu,\bar{\mu})\|P_{XY}\right)$$

  - relative entropy: $D(Q\|P) := \sum_x Q(x)\log\frac{Q(x)}{P(x)}$
  - binary entropy function: $H_2 : t\in[0,1] \mapsto -t\log_2 t - (1-t)\log_2(1-t)$
  - minimum-relative-entropy over couplings of (Q_X, Q_Y):

$$\mathbb{D}(Q_X, Q_Y\|P_{XY}) := \min_{Q_{XY}\in C(Q_X,Q_Y)} D(Q_{XY}\|P_{XY})$$

with $C(Q_X,Q_Y) := \{Q_{XY} \text{ with marginals } Q_X, Q_Y\}$ denoting the coupling set

Achievability: Hamming Spheres (LD Regime)

- Optimizing $\mathbb{D}\left(Q_X, Q_Y \| P_{XY}\right)$ over feasible $Q_X := \left(\lambda, \bar{\lambda}\right), Q_Y := (\mu, \bar{\mu}) \Longrightarrow$

$$\underline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta) \le \underline{\Theta}_{\mathrm{LD}}(\alpha,\beta) := \min_{\substack{Q_X, Q_Y : D(Q_X \| P_X) \ge \alpha, \\ D(Q_Y \| P_Y) \ge \beta}} \mathbb{D}\left(Q_X, Q_Y \| P_{XY}\right),$$

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  - attained by concentric and anti-concentric spheres
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Conjecture (Ordentlich–Polyanskiy–Shayevitz's (OPS's) Conjecture (2019))

For the DSBS and $\alpha, \beta \in (0,1)$,

$$\underline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta) = \underline{\Theta}_{\mathrm{LD}}(\alpha,\beta), \qquad \overline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha,\beta) = \overline{\Theta}_{\mathrm{LD}}(\alpha,\beta).$$

Achievability: Hamming Spheres (MD Regime)

- MD regime: Choose $A = \mathbb{S}_{r_n}(\mathbf{0})$, $B = \mathbb{S}_{s_n}(\mathbf{0})$ with $r_n = \frac{n}{2} + \lambda\sqrt{n\theta_n}$, $s_n = \frac{n}{2} + \mu\sqrt{n\theta_n}$ where $\theta_n \to \infty$, $\frac{\theta_n}{n} \to 0$, and $\lambda, \mu \in \mathbb{R}$

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- By MD theory,

$$-\frac{1}{\theta_n}\log P_X^n(A) \to \frac{1}{2}\chi^2\left((\lambda,\bar{\lambda})\,\|P_X\right), \qquad -\frac{1}{\theta_n}\log P_Y^n(B) \to \frac{1}{2}\chi^2\left((\mu,\bar{\mu})\,\|P_Y\right),$$

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  - χ^2-divergence: $\chi^2(Q\|P) := \sum_x P(x)\left(\frac{Q(x)}{P(x)} - 1\right)^2$
  - minimum-χ^2-divergence over couplings of (Q_X, Q_Y):

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- Same as the LD case but with $D(\cdot\|\cdot)$ replaced by $\frac{1}{2}\chi^2(\cdot\|\cdot)$

Achievability: Hamming Spheres (MD Regime)

- Optimizing $\mathbb{X}^2\left(Q_X, Q_Y \| P_{XY}\right)$ over feasible $Q_X := (r, \bar{r}), Q_Y := (s, \bar{s}) \Longrightarrow$

$$\underline{\Theta}_{\mathrm{MD}}^{(\infty)}(\alpha,\beta) \le \underline{\Theta}_{\mathrm{MD}}(\alpha,\beta) := \min_{\substack{Q_X, Q_Y : \chi^2(Q_X\|P_X)\ge\alpha,\\ \chi^2(Q_Y\|P_Y)\ge\beta}} \mathbb{X}^2\left(Q_X, Q_Y \| P_{XY}\right) = \begin{cases} \frac{\alpha+\beta-2\rho\sqrt{\alpha\beta}}{1-\rho^2}, & \rho^2\alpha \le \beta \le \frac{\alpha}{\rho^2} \\ \alpha, & \beta < \rho^2\alpha \\ \beta, & \alpha < \rho^2\beta \end{cases}$$

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  - attained by concentric and anti-concentric spheres
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- Notice: All $\frac{1}{2}$'s have been removed since the resulting exponents $\frac{1}{k}\underline{\Theta}_{\mathrm{MD}}(k\alpha, k\beta), \frac{1}{k}\overline{\Theta}_{\mathrm{MD}}(k\alpha, k\beta)$ with $k = 2$ are respectively equal to $\underline{\Theta}_{\mathrm{MD}}(\alpha,\beta), \overline{\Theta}_{\mathrm{MD}}(\alpha,\beta)$

Both LD and CL imply MD

- LD implies MD:
  - For MD, $Q_X \to P_X, Q_Y \to P_Y$.
  - Fact: $D(Q\|P) = \frac{1}{2}\chi^2(Q\|P) + o\left(\chi^2(Q\|P)\right)$ (by Taylor's expansion) and $\mathbb{D}(Q_X, Q_Y\|P_{XY}) = \frac{1}{2}\mathbb{X}^2(Q_X, Q_Y\|P_{XY}) + o\left(\mathbb{X}^2(Q_X, Q_Y\|P_{XY})\right)$

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$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\underline{\Theta}_{\mathrm{LD}}(\epsilon\alpha, \epsilon\beta) = \underline{\Theta}_{\mathrm{MD}}(\alpha, \beta), \qquad \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\overline{\Theta}_{\mathrm{LD}}(\epsilon\alpha, \epsilon\beta) = \overline{\Theta}_{\mathrm{MD}}(\alpha, \beta)$$

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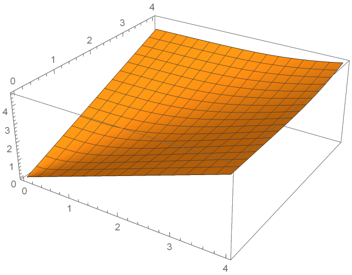
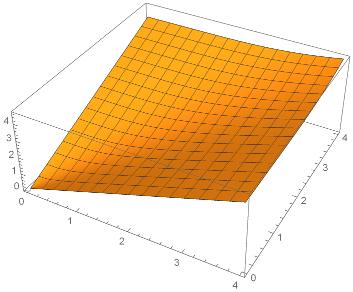
- CL implies MD [O'Donnell, 2014]:

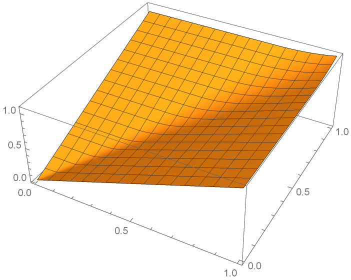
$$\lim_{\theta\to\infty}\frac{1}{\theta}\underline{\Theta}_{\mathrm{CL}}(\theta\alpha, \theta\beta) = \underline{\Theta}_{\mathrm{MD}}(\alpha, \beta), \qquad \lim_{\theta\to\infty}\frac{1}{\theta}\overline{\Theta}_{\mathrm{CL}}(\theta\alpha, \theta\beta) = \overline{\Theta}_{\mathrm{MD}}(\alpha, \beta)$$

  - Recall: $\underline{\Theta}_{\mathrm{CL}}(\alpha, \beta) := -\log\Lambda_\rho\left(e^{-\alpha}, e^{-\beta}\right)$ with $\Lambda_\rho(a, b) := \Phi_\rho\left(\Phi^{-1}(a), \Phi^{-1}(b)\right)$ (and $\overline{\Theta}_{\mathrm{CL}}(\alpha, \beta)$ is defined similarly with $\rho \leftarrow -\rho$)

Exponents induced by Hamming Balls/Spheres for $\rho = 0.9$







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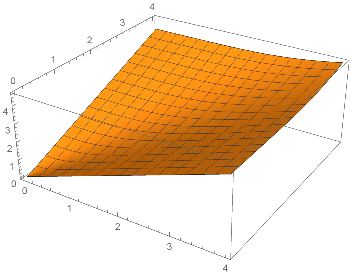
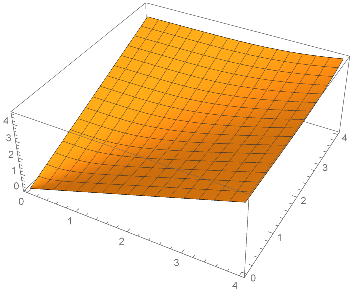
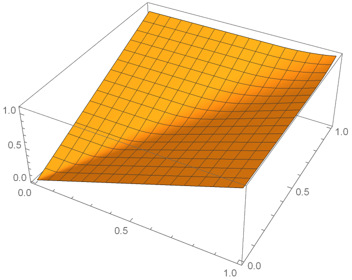
Exponents induced by Hamming Balls/Spheres for $\rho = 0.9$







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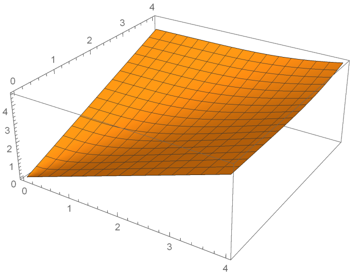

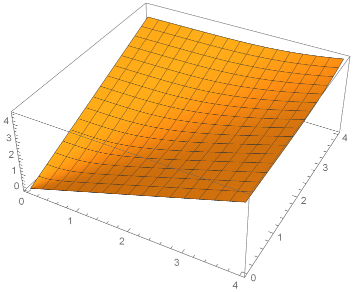

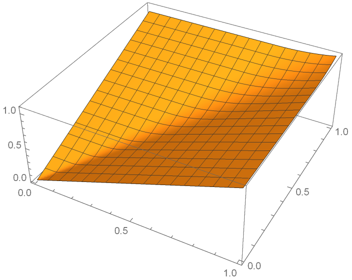

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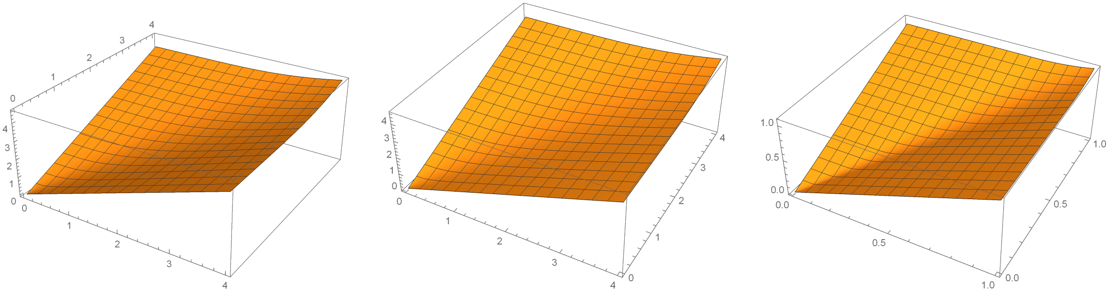

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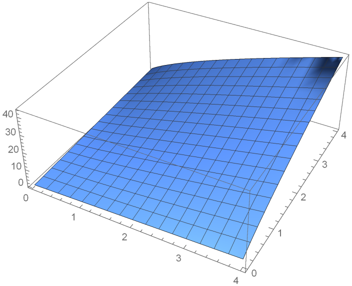

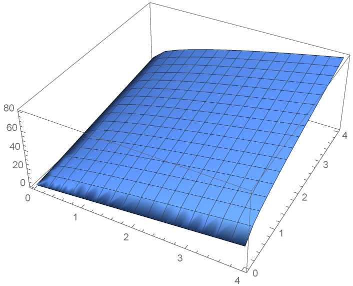

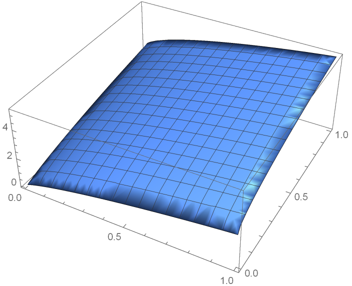

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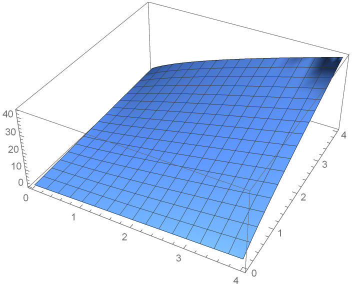

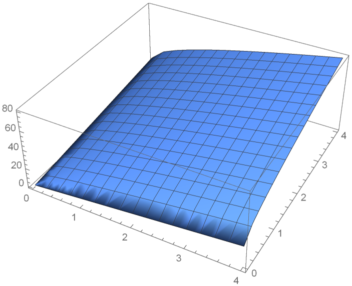

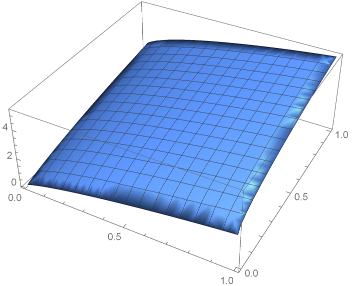

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Comparison: Hamming Subcubes vs. Hamming Balls/Spheres

Regime	Central Limit		Moderate Deviation	Large Deviation
a, b	fixed and large a, b	fixed but small a, b	subexp. vanishing a, b	exp. vanishing a, b
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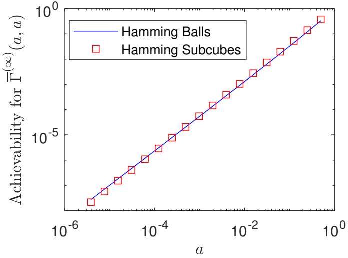

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- We answer these questions in the following.

Outline

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Theorem ([Witsenhausen, 1975])

For any A, B *with* $P_X^n(A) = a, P_Y^n(B) = b$,

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  - Note: This point also can be proven by hypercontractivity and Fourier analysis

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where $|\mathbf{y}|$ denotes the Hamming weight of $\mathbf{y}$.

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- However, $\underline{\Gamma}^{(n)}\left(\frac{1}{4}, \frac{1}{4}\right)$ is still open!

Outline

Converse for MD and CL with Small a, b: Small-Set Expansion (SSE) Theorem

Theorem (Small-Set Expansion [Ahlswede and Gács, 1976, Kahn et al., 1988, Mossel et al., 2006, O'Donnell, 2014])

For any $n \geq 1$ and $\alpha, \beta > 0$,

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Balls/spheres are optimal in MD regime!

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  - Substituting $f \leftarrow 1_A, g \leftarrow 1_B$ and optimizing p, q, SSE Theorem follows.

Outline

Converse for LD: Strong SSE Theorem

Theorem (Strong SSE [Yu et al., 2021, Yu, 2021c])

For any $n \geq 1$ *and* $\alpha, \beta \in (0, 1]$,

$$\underline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha, \beta) \geq \mathfrak{K}\left[\underline{\Theta}_{\mathrm{LD}}\right](\alpha, \beta),$$
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- Recall: $\underline{\Theta}_{\mathrm{LD}}(\alpha,\beta), \overline{\Theta}_{\mathrm{LD}}(\alpha,\beta)$ can be achieved by spheres/balls
- Consequence: Time-sharing certain spheres/balls is optimal in LD regime! —— A weaker version of OPS's conjecture

Converse for LD: Strong SSE Theorem

Lemma ([Yu, 2021a])

$\underline{\Theta}_{\mathrm{LD}}$ *is convex, and* $\overline{\Theta}_{\mathrm{LD}}$ *is concave.*

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$$\boxed{\text{OPS's conjecture is true: Balls/spheres are optimal in LD regime!}}$$

- Notice:
  - The limiting cases as $\rho \to 0$ or 1 were previously proven in [Ordentlich et al., 2020].
  - The special case with $\alpha = \beta$ was previously proven in [Kirshner and Samorodnitsky, 2021].

Summary of Converse Parts

Bounds	Central Limit		Moderate Deviation	Large Deviation
	fixed and large a, b	fixed but small a, b	subexp. vanishing a, b	exp. vanishing a, b
Maximal Correlation	Sharp for $a = b = 1/2$ (Subcubes)			
Fourier Analysis	Sharp for $a = b = 1/2$ and $a = b = 1/4$ (Subcubes)			
SSE		Almost sharp (Balls)	Sharp (Balls/Spheres)	
Strong SSE				Sharp (Balls/Spheres)

Outline

Hypercontractivity (HC) $\Longrightarrow$ SSE

- Forward HC: For any nonnegative f, g,

$$\mathbb{E}\left[f(\mathbf{X})\, g(\mathbf{Y})\right] \le \|f(\mathbf{X})\|_p \, \|g(\mathbf{Y})\|_q$$

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$$f \approx \sum_{i=1}^{M} c_i 1_{A_i}, \qquad g \approx \sum_{j=1}^{N} d_j 1_{B_j}$$

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- Strong SSE + "level-partition" technique $\implies$ Strong HC [Kirshner and Samorodnitsky, 2021, Yu et al., 2021]
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$$\boxed{\text{(Strong) SSE} \Longleftrightarrow \text{(Strong) HC (in some sense)!}}$$

Outline

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- Parallel half-spaces are optimal (e.g., $f = 1_{\{x_1 \le r\}}, g = 1_{\{y_1 \le s\}}$)

Outline

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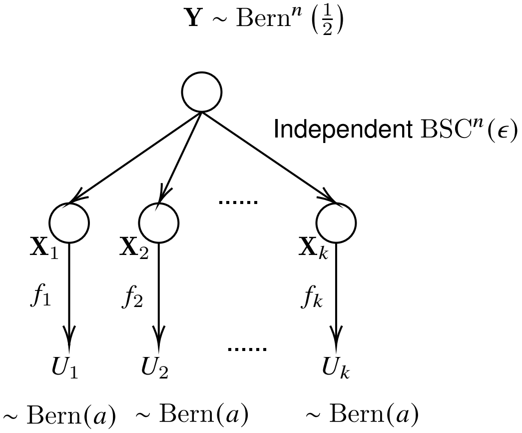
NICD with k Users



Asymmetric Version: $\max \mathbb{P}(U_1 = U_2 = \ldots = U_k = 1)$

Symmetric Version: $\max \mathbb{P}(U_1 = U_2 = \ldots = U_k)$

k-User NICD and q-Stability: Asymmetric Version

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- Asymmetric NICD: For $a = \frac{M}{2^n}$ (with integer M), define (asymmetric) max k-stability (or forward joint probability) as

$$\overline{\Gamma}_k(a) := \max_{\text{Boolean } f_i, 1 \le i \le k : \mathbb{E}[f_i(\mathbf{X}_i)] = a} \mathbb{P}(f_1(\mathbf{X}_1) = f_2(\mathbf{X}_2) = \ldots = f_k(\mathbf{X}_k) = 1)$$

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k-User NICD and q-Stability: Asymmetric Version

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\begin{aligned}
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&= \max_{\text{Boolean } f : \mathbb{E} f(\mathbf{X}) = a} \mathbb{E}_{\mathbf{Y}} \left[\left(T_\rho f(\mathbf{Y}) \right)^k \right] \\
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- Generalization to (asymmetric) max q-stability (or max α-stability) [Li and Médard, 2021]: For $q \in [1, \infty)$,

$$
\overline{\Gamma}_q(a) := \max_{\text{Boolean } f : \mathbb{E} f(\mathbf{X}) = a} \mathbb{E}_{\mathbf{Y}} \left[T_\rho f(\mathbf{Y})^q \right] = \max_{A : P_X^n(A) = a} \mathbb{E}_{\mathbf{Y}} \left[P_{X|Y}^n (A|\mathbf{Y})^q \right]
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k-User NICD and q-Stability: Symmetric Version

- Symmetric NICD: Define symmetric max k-stability as (removing "$= 1$")

$$\overline{\Gamma}_k^{\mathrm{sym}}(a) := \max_{\text{Boolean } f_i, 1\le i\le k: \mathbb{P}(f_i(\mathbf{X}_i)=1)=a} \mathbb{P}(f_1(\mathbf{X}_1) = f_2(\mathbf{X}_2) = \ldots = f_k(\mathbf{X}_k))$$

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- Better definitions of asymmetric and symmetric max q-stability?

Variants of max q-Stability

- Define

$$\Phi_q(t) := \begin{cases} \frac{t^q - t}{q-1} & q > 1 \\ t \log t & q = 1 \end{cases} \quad \text{and} \quad \Phi_q^{\mathrm{sym}}(t) := \Phi_q(t) + \Phi_q(1-t) = \begin{cases} \frac{t^q + (1-t)^q - 1}{q-1} & q > 1 \\ t \log t + (1-t) \log (1-t) & q = 1 \end{cases}$$

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  - Attained by dictators for $a = 1/2$? —— Courtade–Kumar Conjecture (Balanced Function Version) (2013)

Max q-Stability: Interesting Cases

- Balanced case (CL regime): $a = 1/2$
- LD regime: $a = 2^{-n\alpha}$ is exponentially small
- MD regime: $a = 2^{-\theta_n \alpha}$ is “subexponentially” small with $\theta_n \to \infty$, $\frac{\theta_n}{n} \to 0$

Outline

Related Conjectures for the Balanced Case

q	Are dictators optimal for $a = 1/2$?
$q = 1$	Courtade–Kumar Conjecture (2013) (balanced version)
$1 < q < 2$	Li–Médard Conjecture (2019)
$q = 2$ (2-User NICD)	True [Witsenhausen, 1975]
$q > 2$	Mossel–O'Donnell Conjecture for $2 < q \leq 9$ (2006)

q-Stability: Balanced Case

Lemma ([Barnes and Özgür, 2020])

For $a = 1/2$, there are two thresholds $1 \leq q_{\min} < 2 < q_{\max}$ s.t. dictators are optimal if and only if $q \in [q_{\min}, q_{\max}]$.

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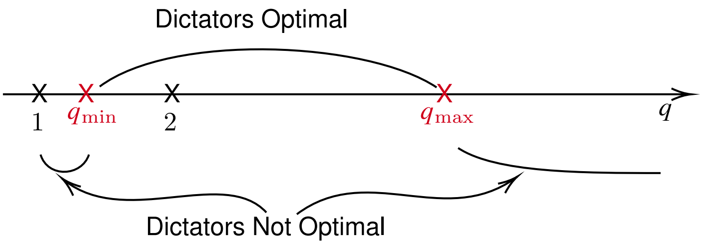

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  - $q_{\max} < 10$ [Mossel and O'Donnell, 2005]

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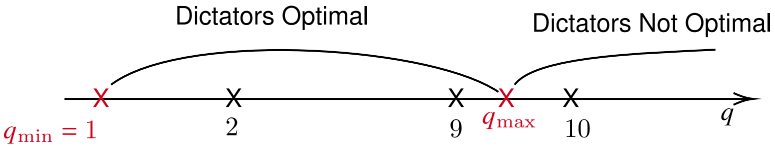

Progress on Courtade–Kumar Conjecture (Li–Médard Conjecture)

Related Works	Upper Bounds on $\max_{\text{Boolean } f:\mathbb{E}f=1/2} I\left(f\left(\mathbf{X}\right);\mathbf{Y}\right)$	Tools
Courtade–Kumar Conjecture	$\mathbf{1}-\boldsymbol{H_2}\left(\frac{\mathbf{1}-\boldsymbol{\rho}}{\mathbf{2}}\right)$	
[Witsenhausen and Wyner, 1975]	ρ^2	Mrs. Gerber's lemma (or HC)
[Ordentlich et al., 2016]	$\frac{\log_2 e}{2}\rho^2+9\left(1-\frac{\log_2 e}{2}\right)\rho^4$ for $0\le\rho\le\frac{1}{\sqrt{3}}$ (asymptotically sharp as $\rho\to 0$)	Fourier analysis + HC
[Samorodnitsky, 2016]	Sharp bound for $\rho\in[0,\rho_0]$ with some $0<\rho_0<1$	Fourier analysis + Random restrictions + ...
[Yu, 2021b]	Sharp bound for $\rho\in[0,\rho_1]$ with $\rho_1\approx 0.46$ (explicitly given)	Fourier analysis + KKT conditions
[Pichler et al., 2018]	A weaker version: $\max_{\text{Boolean } f,g} I\left(f\left(\mathbf{X}\right);g\left(\mathbf{Y}\right)\right)= 1-H_2\left(\frac{1-\rho}{2}\right)$	Fourier analysis + Partition technique

Progress on Mossel–O'Donnell Conjecture

Related Works	Dictators are optimal for ...	Tools
Mossel–O'Donnell Conjecture	**$2 < q \leq 9$ (both symmetric and asymmetric max q-stability)**	
[Mossel and O'Donnell, 2005]	$2 < q \leq 3$ (symmetric)	Reducing $q = 3$ to $q = 2$
[Witsenhausen, 1975]	$q = 2$ (asymmetric)	Maximal correlation
[Yu, 2021b]	$2 < q \leq 5$ (symmetric); $2 < q \leq 3$ (asymmetric)	Fourier analysis + KKT conditions
[Mossel and O'Donnell, 2005, Li and Médard, 2021]	$\rho \to 0$ or 1 (symmetric and asymmetric)	Fourier analysis

Outline

MD and LD Regimes

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Theorem (q-Stability Theorem)

Hamming balls/spheres are optimal in attaining max q-stability for $q \in [1, \infty)$ and min q-stability for $q \in (-\infty, 1) \backslash \{0\}$ in the MD regime.

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- Substituting $f \leftarrow 1_A$ to the Single-Function Version of HC $\Longrightarrow$ q-Stability Theorem

LD Regime

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Theorem (Strong q-Stability Theorem [Yu, 2021c, Yu, 2021a])

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Bounds	Moderate Deviation	Large Deviation
	subexp. vanishing a	exp. vanishing a
q-Stability Theorem	Sharp (Balls/Spheres Optimal)	
Strong q-Stability Theorem [Yu, 2021c]		Sharp (Balls/Spheres Optimal)

Summary of Tools used in NICD and q-Stability

<table>
<tr><th colspan="2">Methods</th><th colspan="2">Central Limit</th><th>Moderate Deviation</th><th>Large Deviation</th></tr>
<tr><td colspan="2"></td><td>fixed and large a (and b)</td><td>fixed but small a (and b)</td><td>subexp. vanishing a (and b)</td><td>exp. vanishing a (and b)</td></tr>
<tr><td rowspan="3">Information-Theoretic Methods</td><td>Maximal Correlation</td><td rowspan="3">Sharp for 2-user NICD with $a = b = 1/2$</td><td></td><td></td><td></td></tr>
<tr><td>HC/SSE (stronger than MC)</td><td rowspan="2">Almost sharp</td><td rowspan="2">Sharp</td><td></td></tr>
<tr><td>Strong HC/SSE (stronger than HC/SSE)</td><td>Sharp</td></tr>
<tr><td>Fourier Analysis</td><td>Combined with Optimization Theory (LP or KKT)</td><td>Sharp for 2-user NICD with $a = b = 1/2$, $1/4$; Sharp for q-Stability with $a = 1/2$ and certain (q, ρ)</td><td></td><td></td><td></td></tr>
</table>

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