1 Proof of stochastical equicontinuity

As we assume the adversary's strategy is $(\tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_1)$, then the true distribution of Y^n is $P_0\tilde{\mathbf{A}}_0$ under H_0 and $P_1\tilde{\mathbf{A}}_1$ under H_1. Thus, under H_0, by the strong law of large numbers, we have that $\hat{Q}_{Y^n} \to P_0\tilde{\mathbf{A}}_0$ a.s. as $n \to \infty$. Consequently, we conclude from the continuity of $D(\cdot\|P_1\mathbf{A}_1)$ on finite alphabet $\mathcal{X}$ that under H_0, $D(\hat{Q}_{Y^{T^*}}\|P_1\mathbf{A}_1) \to D(P_0\tilde{\mathbf{A}}_0\|P_1\mathbf{A}_1)$ and $D(\hat{Q}_{Y^{(T^*-1)}}\|P_1\mathbf{A}_1) \to D(P_0\tilde{\mathbf{A}}_0\|P_1\mathbf{A}_1)$ a.s. as $\alpha \to 0^+$ for each $\mathbf{A}_1 \in \mathcal{A}_1$. Thus, we have the pointwise convergence for each $\mathbf{A}_1 \in \mathcal{A}_1$. Now we want to obtain the uniform almost sure convergence of $\min_{\mathbf{A}_1\in\mathcal{A}_1} D(\hat{Q}_{Y^{(T^*-1)}}\|P_1\mathbf{A}_1)$.

We have assumed that $\mathcal{A}_1$ is a compact set. It also easy to show that there is a unique $P_1\mathbf{A}_1$ that minimizes $D(\hat{Q}_{Y^{T^*}}\|P_1\mathbf{A}_1)$. We also need to show that $\{Q_1 \mapsto D(\hat{Q}_{Y^{T^*}}\|Q_1)\}$ is stochastically equicontinuous. That is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\lim_{\alpha\to 0^+} \Pr\left(\sup_{\|Q_1-Q_1'\|_2\le\delta} |D(\hat{Q}_{Y^{T^*}}\|Q_1) - D(\hat{Q}_{Y^{T^*}}\|Q_1')| > \epsilon\right) = 0.$$

Now we have, for every $\epsilon > 0$, there exists a $0 < \delta < \epsilon \min_{y\in\mathcal{X}} Q_1(y)$ such that

$$\begin{aligned}
&\Pr\left(\sup_{\|Q_1-Q_1'\|_2\le\delta} |D(\hat{Q}_{Y^{T^*}}\|Q_1) - D(\hat{Q}_{Y^{T^*}}\|Q_1')| > \epsilon\right) \\
&\quad = \Pr\left(\sup_{\|Q_1-Q_1'\|_2\le\delta} \left|\sum_{a\in\mathcal{X}} \hat{Q}_{Y^{T^*}}(a) \log\frac{Q_1(a)}{Q_1'(a)}\right| > \epsilon\right) \\
&\quad \le \Pr\left(\sup_{\|Q_1-Q_1'\|_2<\delta} \sum_{a\in\mathcal{X}} \hat{Q}_{Y^{T^*}}(a) \left|\log\frac{Q_1(a)}{Q_1'(a)}\right| > \epsilon\right) \\
&\quad \overset{(a)}{\le} \Pr\left(\sup_{\|Q_1-Q_1'\|_2<\delta} \sum_{a\in\mathcal{X}} \hat{Q}_{Y^{T^*}}(a) \frac{1}{\min_{y\in\mathcal{X}} Q_1(y)} |Q_1(a) - Q_1'(a)| > \epsilon\right) \\
&\quad = \Pr\left(\sup_{\|Q_1-Q_1'\|_2<\delta} \left[\sum_{a\in\mathcal{X}} (\hat{Q}_{Y^{T^*}}(a) - Q_0(a)) \frac{1}{\min_{y\in\mathcal{X}} Q_1(y)} |Q_1(a) - Q_1'(a)|\right.\right. \\
&\qquad \left.\left. + \sum_{a\in\mathcal{X}} Q_0(a) \frac{1}{\min_{y\in\mathcal{X}} Q_1(y)} |Q_1(a) - Q_1'(a)|\right] > \epsilon\right) \\
&\quad \le \Pr\left(\sup_{\|Q_1-Q_1'\|_2<\delta} \sum_{a\in\mathcal{X}} (\hat{Q}_{Y^{T^*}}(a) - Q_0(a)) \frac{1}{\min_{y\in\mathcal{X}} Q_1(y)} |Q_1(a) - Q_1'(a)|\right. \\
&\qquad \left. + \frac{\delta}{\min_{y\in\mathcal{X}} Q_1(y)} > \epsilon\right),
\end{aligned}$$

where (a) is because for any $x, y \ge \beta$, $|\log x - \log y| \le \frac{1}{\beta}|x - y|$. Thus, because $\hat{Q}_{Y^{T^*}} \overset{a.s.}{\to} Q_0$ as $\alpha \to 0^+$, we have

$$\lim_{\alpha\to 0^+} \Pr\left(\sup_{\|Q_1-Q_1'\|_2\le\delta} |D(\hat{Q}_{Y^{T^*}}\|Q_1) - D(\hat{Q}_{Y^{T^*}}\|Q_1')| > \epsilon\right) = 0,$$

which shows that $D(\hat{Q}_{Y^{T^*}}\|P_1\mathbf{A}_1)$ is stochastically equicontinuous.

2 Proof of (15)

To go from convergence almost sure to convergence in mean, it now suffices to prove that a family of random variables $\left\{\frac{T^*}{\log(1/\alpha)}\right\}_{\alpha>0}$ is uniformly integrable. That is, there exists $\epsilon_0 > 0$ such that for all $\alpha \in (0, \epsilon_0]$,

$$\lim_{\eta\to\infty} \mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\mathbb{I}_{\{T^*/\log(1/\alpha)\geq\eta\}}\right] = 0.$$

Here we choose an ϵ_0 such that $x\log(1/x)$ is increasing on $[0, \epsilon_0]$ and $\frac{\log(1/\epsilon_0)}{1/\epsilon_0} \leq 1$. Then we have

$$\begin{aligned}
&\mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\mathbb{I}_{\{T^*/\log(1/\alpha)\geq\eta\}}\right] \\
&\quad\leq \mathbb{E}_0\left[\frac{T^* - \lfloor\eta\log(1/\alpha)\rfloor + \eta\log(1/\alpha)}{\log(1/\alpha)}\mathbb{I}_{\{T^*\geq\lfloor\eta\log(1/\alpha)\rfloor\}}\right] \\
&\quad\leq \frac{1}{\log(1/\alpha)}\mathbb{E}_0\left[(T^* - \lfloor\eta\log(1/\alpha)\rfloor)\mathbb{I}_{\{T^*\geq\lfloor\eta\log(1/\alpha)\rfloor\}}\right] + \eta P_0[T^* \geq \lfloor\eta\log(1/\alpha)\rfloor] \\
&\quad= \frac{1}{\log(1/\alpha)}\sum_{l=1}^{\infty} P_0[T^* \geq \lfloor\eta\log(1/\alpha)\rfloor + l] + \eta P_0[T^* \geq \lfloor\eta\log(1/\alpha)\rfloor] \\
&\quad\leq \frac{1/\alpha}{\log(1/\alpha)}\sum_{l=1}^{\infty}(\lfloor\eta\log(1/\alpha)\rfloor + l)^{2|\mathcal{X}|}e^{-(\eta\log(1/\alpha)+l-2)2B^*+(\eta\log(1/\alpha)+l)^{2/3}} \\
&\qquad + \eta\frac{1}{\alpha}e^{-(\eta\log(1/\alpha)-2)2B^*+(\eta\log(1/\alpha))^{2/3}}(\lfloor\eta\log(1/\alpha)\rfloor)^{2|\mathcal{X}|} \\
&\quad\leq \frac{1/\alpha}{\log(1/\alpha)}\sum_{l=1}^{\infty}(\lfloor\eta\log(1/\alpha)\rfloor + l)^{2|\mathcal{X}|}e^{-(\eta\log(1/\alpha)+l-4)B^*} + \eta\frac{1}{\alpha}e^{-(\eta\log(1/\alpha)-4)B^*}(\lfloor\eta\log(1/\alpha)\rfloor)^{2|\mathcal{X}|} \\
&\quad\overset{(a)}{\leq} \frac{1/\alpha}{\log(1/\alpha)}\sum_{l=1}^{\infty}2^{2|\mathcal{X}|-1}(\lfloor\eta\log(1/\alpha)\rfloor^{2|\mathcal{X}|} + l^{2|\mathcal{X}|})e^{-(\eta\log(1/\alpha)+l-4)B^*} \\
&\qquad + \eta\frac{1}{\alpha}e^{-(\eta\log(1/\alpha)-4)B^*}(\lfloor\eta\log(1/\alpha)\rfloor)^{2|\mathcal{X}|} \\
&\quad= \frac{1/\alpha}{\log(1/\alpha)}2^{2|\mathcal{X}|-1}e^{4B^*}e^{-\eta\log(1/\alpha)B^*}\sum_{l=1}^{\infty}l^{2|\mathcal{X}|}e^{-lB^*} + \eta\frac{1}{\alpha}e^{-\eta\log(1/\alpha)B^*}e^{4B^*}(\lfloor\eta\log(1/\alpha)\rfloor)^{2|\mathcal{X}|} \\
&\qquad + \frac{1/\alpha}{\log(1/\alpha)}2^{2|\mathcal{X}|-1}\lfloor\eta\log(1/\alpha)\rfloor^{2|\mathcal{X}|}e^{4B^*}e^{-\eta\log(1/\alpha)B^*}\sum_{l=1}^{\infty}e^{-lB^*} \\
&\quad\leq C_1\left(\frac{1}{\alpha}\right)^{1-\eta B^*}\frac{1}{\log 1/\alpha} + C_2\left(\frac{1}{\alpha}\right)^{1-\eta B^*}\eta^{2|\mathcal{X}|+1}\left(\log\frac{1}{\alpha}\right)^{2|\mathcal{X}|} + C_3\eta^{2|\mathcal{X}|}\left(\log\frac{1}{\alpha}\right)^{2|\mathcal{X}|-1}\left(\frac{1}{\alpha}\right)^{1-\eta B^*},
\end{aligned}$$

where (a) is based on Minkowski inequality and $C_1 = 2^{2|\mathcal{X}|-1}e^{4B^*}\sum_{l=1}^{\infty}l^{2|\mathcal{X}|}e^{-lB^*}$, $C_2 = e^{4B^*}$ and $C_3 = 2^{2|\mathcal{X}|-1}e^{4B^*}\sum_{l=1}^{\infty}e^{-lB^*}$. We now choose η such that $\eta B^* \geq 2|\mathcal{X}| + 2$. Then for any $0 < \alpha \leq \epsilon_0$, we have

$$
\begin{aligned}
&\mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\mathbb{I}_{\{T^*/\log(1/\alpha)\geq\eta\}}\right]\\
&\leq C_1\left(\frac{1}{\alpha}\right)^{1-\eta B^*}\frac{1}{\log 1/\alpha}+C_2\left(\frac{1}{\alpha}\right)^{1-\eta B^*+2|\mathcal{X}|}\eta^{2|\mathcal{X}|+1}\left(\frac{\log(1/\alpha)}{1/\alpha}\right)^{2|\mathcal{X}|}\\
&\quad+C_3\eta^{2|\mathcal{X}|}\left(\frac{\log(1/\alpha)}{1/\alpha}\right)^{2|\mathcal{X}|-1}\left(\frac{1}{\alpha}\right)^{1-\eta B^*+2|\mathcal{X}|-1}\\
&\leq C_1\left(\frac{1}{\epsilon_0}\right)^{1-\eta B^*}\frac{1}{\log 1/\epsilon_0}+C_2\left(\frac{1}{\epsilon_0}\right)^{1-\eta B^*+2|\mathcal{X}|}\eta^{2|\mathcal{X}|+1}+C_3\eta^{2|\mathcal{X}|}\left(\frac{1}{\epsilon_0}\right)^{1-\eta B^*+2|\mathcal{X}|-1} \qquad (1)
\end{aligned}
$$

where (1) goes to 0 as $\eta\to\infty$. Thus, we have shown the uniform integrability of $\left\{\frac{T^*}{\log(1/\alpha)}:0<\alpha\leq\epsilon_0\right\}$. Then, under H_0, we have

$$\lim_{\alpha\to0^+}\frac{\mathbb{E}_0[T^*]}{\log(1/\alpha)}=\frac{1}{\min_{\mathbf{A}_1\in\mathcal{A}_1}D(P_0\tilde{\mathbf{A}}_0\|P_1\mathbf{A}_1)},$$

Similarly, we can also prove that under H_1,

$$\lim_{\alpha\to0^+}\frac{\mathbb{E}_1[T^*]}{\log(1/\alpha)}=\frac{1}{\min_{\mathbf{A}_0\in\mathcal{A}_0}D(P_1\tilde{\mathbf{A}}_1\|P_0\mathbf{A}_0)}.$$

3 Proof of Lemma 5

Let ε be an arbitrary fixed positive number. From the definition of stopping time T^*, we have

$$T^*=T^*_\alpha=\inf\left\{n\geq1:\min_{\mathbf{A}_1\in\mathcal{A}_1}D(\hat{Q}_{Y^n}\|P_1\mathbf{A}_1)\geq\gamma_n\text{ or }\min_{\mathbf{A}_0\in\mathcal{A}_0}D(\hat{Q}_{Y^n}\|P_0\mathbf{A}_0)\geq\gamma_n\right\}.$$

We denote

$$T_1=\inf\left\{n\geq1:\min_{\mathbf{A}_1\in\mathcal{A}_1}D(\hat{Q}_{Y^n}\|P_1\mathbf{A}_1)\geq\gamma_n\right\}.$$

Then we can see that $T_1\geq T^*$. Similar to the proof of equation (16), we also can prove that

$$\lim_{\alpha\to0^+}\frac{\mathbb{E}_0[T_1]}{\log(1/\alpha)}=\frac{1}{\min_{\mathbf{A}_1\in\mathcal{A}_1}D(P_0\tilde{\mathbf{A}}_0\|P_1\mathbf{A}_1)}. \qquad (2)$$

Now we want to show that the convergence in (2) is uniform on $\mathcal{A}_0$, which allows us to establish (22).

According to the definition of T_1, we have

$$
\begin{aligned}
\min_{\mathbf{A}_1\in\mathcal{A}_1}D(\hat{Q}_{Y^{T_1}}\|P_1\mathbf{A}_1)&\geq\frac{\log\left(\frac{1}{\alpha}\right)}{T_1}+(T_1)^{-1/3}+\frac{|\mathcal{X}|\log(T_1+1)}{T_1},\\
\min_{\mathbf{A}_1\in\mathcal{A}_1}D(\hat{Q}_{Y^{T_1-1}}\|P_1\mathbf{A}_1)&\leq\frac{\log\left(\frac{1}{\alpha}\right)}{T_1-1}+(T_1-1)^{-1/3}+\frac{|\mathcal{X}|\log(T_1)}{T_1-1}.
\end{aligned}
$$

Then, we have that

$$\left|\frac{\log(1/\alpha)}{T_1} - \min_{\mathbf{A}_1\in\mathcal{A}_1} D(\hat{Q}_{Y^{T_1}}\|P_1\mathbf{A}_1)\right| \le c_0(T_1)^{-1/3},$$

where c_0 does not depend on $\mathbf{A}_0$. Then, we define $D_{T_1} := \min_{\mathbf{A}_1\in\mathcal{A}_1} D(\hat{Q}_{Y^{T_1}}\|P_1\mathbf{A}_1)$ and $D_0 := \min_{\mathbf{A}_1\in\mathcal{A}_1} D(P_0\tilde{\mathbf{A}}_0\|P_1\mathbf{A}_1)$. We have

$$\begin{aligned}
\left|\mathbb{E}_0\left[\frac{\log(1/\alpha)}{T_1} - D_0\right]\right| &= \left|\mathbb{E}_0\left[\frac{\log(1/\alpha)}{T_1} - D_{T_1} + D_{T_1} - D_0\right]\right| \\
&\le \mathbb{E}_0\left[\left|\frac{\log(1/\alpha)}{T_1} - D_{T_1}\right|\right] + \mathbb{E}_0\left[|D_{T_1} - D_0|\right] \\
&\le c_0\mathbb{E}_0\left[(T_1)^{-1/3}\right] + \mathbb{E}_0\left[|D_{T_1} - D_0|\right]
\end{aligned}$$

Define $c_1 := -\log\min_{\tilde{\mathbf{A}}_1\in\mathcal{A}_1}\min_{y\in\mathcal{X}}\tilde{Q}_1(y)$. For the first term, because

$$P_0\left(T_1 < \frac{\log(1/\alpha)}{c_1}\right) = 0,$$

we have

$$T_1 \ge \frac{\log(1/\alpha)}{c_1}, \quad \text{a.s.}.$$

Thus,

$$\mathbb{E}_0\left[(T_1)^{-1/3}\right] \le \left(\frac{\log(1/\alpha)}{c_1}\right)^{-1/3}, \tag{3}$$

where c_1 does not depend on $\tilde{\mathbf{A}}_0$. For the second term, we define $c_2 := -\log\min_{\tilde{\mathbf{A}}_0\in\mathcal{A}_0}\min_{y\in\mathcal{X}}\tilde{Q}_0(y)$. Then we have that

$$\begin{aligned}
&\mathbb{E}_0\left[|D_{T_1} - D_0|\right] \\
&= \mathbb{E}_0\left[D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) + \min_{\mathbf{A}_1\in\mathcal{A}_1}\sum_{a\in\mathcal{X}}\left|\hat{Q}_{Y^{T_1}}(a) - \tilde{Q}_0(a)\log\frac{\tilde{Q}_0(a)}{Q_1(a)}\right|\right] \\
&\le \mathbb{E}_0\left[D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0)\right] + c_1|\mathcal{X}|\mathbb{E}_0\left[\sum_{a\in\mathcal{X}}|\hat{Q}_{Y^{T_1}}(a) - \tilde{Q}_0(a)|\right] \\
&\overset{(a)}{\le} \mathbb{E}_0\left[D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0)\right] + c_3\mathbb{E}_0\left[\sqrt{D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0)}\right] \\
&= \mathbb{E}_0\left[D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0)\middle| D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) \ge \epsilon\right] P_0\left(D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) \ge \epsilon\right) \\
&\quad + \mathbb{E}_0\left[D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0)\middle| D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) < \epsilon\right] P_0\left(D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) < \epsilon\right) \\
&\quad + c_3\mathbb{E}_0\left[\sqrt{D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0)}\middle| D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) \ge \epsilon\right] P_0\left(D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) \ge \epsilon\right) \\
&\quad + c_3\mathbb{E}_0\left[\sqrt{D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0)}\middle| D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) < \epsilon\right] P_0\left(D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) < \epsilon\right) \\
&\le \epsilon + \left(c_2|\mathcal{X}| + \sqrt{c_2|\mathcal{X}|}\right)P_0\left(D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) \ge \epsilon\right) + c_3\sqrt{\epsilon},
\end{aligned}$$

where (a) is based on Pinsker's inequality and c_2, c_3 do not depend on $\tilde{\mathbf{A}}_0$. We also have

$$\begin{aligned} P_0\left(D(\hat{Q}_{Y^{T_1}}\|P_0\tilde{\mathbf{A}}_0) \geq \epsilon\right) &\leq \sum_{k\geq \log(1/\alpha)/c_1} P_0\left(D(\hat{Q}_{Y^k}\|P_0\tilde{\mathbf{A}}_0) \geq \epsilon\right) \\ &\leq \sum_{k\geq \log(1/\alpha)/c_1} c_4 e^{-k\epsilon} \\ &\leq c_5 e^{-\frac{\log(1/\alpha)}{c_1}\epsilon}, \end{aligned}$$

where c_5 only depends on $|\mathcal{X}|$. Thus,

$$\mathbb{E}_0\left[|D_{T_1} - D_0|\right] \leq \epsilon + \left(c_2|\mathcal{X}| + \sqrt{c_2|\mathcal{X}|}\right)c_5 e^{-\frac{\log(1/\alpha)}{c_1}\epsilon} + c_3\sqrt{\epsilon}. \tag{4}$$

Therefore, combining (3) and (4), we have

$$\left|\mathbb{E}_0\left[\frac{\log(1/\alpha)}{T_1} - D_0\right]\right| \leq \epsilon + \left(c_2|\mathcal{X}| + \sqrt{c_2|\mathcal{X}|}\right)c_5 e^{-\frac{\log(1/\alpha)}{c_1}\epsilon} + c_3\sqrt{\epsilon} + c_0\left(\frac{\log(1/\alpha)}{c_1}\right)^{-1/3}.$$

As c_0, c_1, c_2, c_3, c_5 do not depend on $\tilde{\mathbf{A}}_0$, the convergence in (2) is uniform over $\mathcal{A}_0$. Now we show that the uniform convergence over $\mathcal{A}_0$ also holds for $\left\{\frac{\mathbb{E}_0[T^*]}{\log(1/\alpha)}\right\}_{0<\alpha\leq 1}$. Denote

$$\begin{aligned} B_0 &= \{\min_{\mathbf{A}_0\in\mathcal{A}_0} D(\hat{Q}_{Y^{T^*}}\|P_0\mathbf{A}_0) \geq \gamma_{T^*}\} \\ B_1 &= \{\min_{\mathbf{A}_1\in\mathcal{A}_1} D(\hat{Q}_{Y^{T^*}}\|P_1\mathbf{A}_1) \geq \gamma_{T^*}\}. \end{aligned}$$

On B_0, we have $T^* < T_1$ and on B_1, we have $T^* = T_1$. Then

$$\begin{aligned} \mathbb{E}_0[T^*] &= \mathbb{E}_0[T_1\mathbb{I}_{\{B_1\}}] + \mathbb{E}_0[T^*\mathbb{I}_{\{B_0\}}] \\ &= \mathbb{E}_0[T_1] + \mathbb{E}_0[(T^* - T_1)\mathbb{I}_{\{B_0\}}] \\ &\geq \mathbb{E}_0[T_1] - \mathbb{E}_0[T^*\mathbb{I}_{\{B_0\}}]. \end{aligned} \tag{5}$$

From equation (15) in the proof of uniform integrability, it follows that for the given $\varepsilon > 0$, there exists a constant K which does not depend on $\tilde{\mathbf{A}}_0$ such that for any $0 < \alpha \leq \alpha_0$ and any $(\tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_1)$,

$$\mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\mathbb{I}_{\{T^*(\alpha)/\log(1/\alpha)\geq K\}}\right] \leq \varepsilon.$$

Therefore, we have that

$$\begin{aligned} &\mathbb{E}_0[T^*\mathbb{I}_{\{B_0\}}] \\ &\quad= \mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\mathbb{I}_{\{T^*(\alpha)/\log(1/\alpha)\geq K\}}\mathbb{I}_{\{B_0\}}\right]\log\left(\frac{1}{\alpha}\right) + \mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\mathbb{I}_{\{T^*(\alpha)/\log(1/\alpha)\leq K\}}\mathbb{I}_{\{B_0\}}\right]\log\left(\frac{1}{\alpha}\right) \\ &\quad\leq \varepsilon\log\left(\frac{1}{\alpha}\right) + KP_0(B_0)\log\left(\frac{1}{\alpha}\right) \\ &\quad\overset{(a)}{\leq} \varepsilon\log\left(\frac{1}{\alpha}\right) + K\alpha\log\left(\frac{1}{\alpha}\right), \end{aligned}$$

where (a) is because $P_0(B_0)$ is the type-I error and it is upper bounded by α. From (5), we have

$$\mathbb{E}_0\left[\frac{T_1}{\log(1/\alpha)}\right] - \mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\right] \leq \mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\mathbb{I}_{\{B_0\}}\right]$$
$$\leq \varepsilon + K\alpha.$$

Therefore,

$$\lim_{\alpha\to 0^+} \sup_{\mathbf{A}_0\in\mathcal{A}_0} \left(\mathbb{E}_0\left[\frac{T_1}{\log(1/\alpha)}\right] - \mathbb{E}_0\left[\frac{T^*}{\log(1/\alpha)}\right]\right) = 0,$$

which together with the uniform convergence of $\mathbb{E}_0\left[\frac{T_1}{\log(1/\alpha)}\right]$ over $\mathcal{A}_0$, implies that

$$\lim_{\alpha\to 0^+} \left\{ \sup_{\tilde{\mathbf{A}}_0\in\mathcal{A}_0} \left(\frac{\mathbb{E}_0[T^*]}{\log(1/\alpha)} - \frac{1}{\min_{\mathbf{A}_1\in\mathcal{A}_1} D(P_0\tilde{\mathbf{A}}_0 \| P_1\mathbf{A}_1)} \right) \right\} = 0,$$

as desired. Similarly, we can also prove that

$$\lim_{\alpha\to 0^+} \left\{ \sup_{\tilde{\mathbf{A}}_1\in\mathcal{A}_1} \left(\frac{\mathbb{E}_1[T^*]}{\log(1/\alpha)} - \frac{1}{\min_{\mathbf{A}_0\in\mathcal{A}_0} D(P_1\tilde{\mathbf{A}}_1 \| P_0\mathbf{A}_0)} \right) \right\} = 0.$$