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Common Information, Noise Stability, and Their Extensions

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Contents

Common Information, Noise Stability, and Their Extensions

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ABSTRACT

Common information is ubiquitous in information theory and related areas such as theoretical computer science and discrete probability. However, because there are multiple notions of common information, a unified understanding of the deep interconnections between them is lacking. This monograph seeks to fill this gap by leveraging a small set of mathematical techniques that are applicable across seemingly disparate problems.

In Part I, we review the operational tasks and properties associated with Wyner's and Gács–Körner–Witsenhausen's (GKW's) common information. In Part II, we discuss extensions of the former from the perspective of distributed source simulation. This includes the Rényi common information which forms a bridge between Wyner's common information and the exact common information. Via a surprising equivalence between the Rényi common information of order ∞ and the exact common information, we demonstrate the existence of a joint source in which the exact common

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information strictly exceeds Wyner's common information. Other closely related topics discussed in Part II include the channel synthesis problem and the connection of Wyner's and exact common information to the nonnegative rank of matrices.

In Part III, recognizing that GKW's common information is zero for most non-degenerate sources, we examine it with a more refined lens via the Non-Interactive Correlation Distillation (NICD) problem in which we quantify the agreement probability of extracted bits from a bivariate source. We extend this to the noise stability problem which includes as special cases the k-user NICD and q-stability problems. This allows us to seamlessly transition to discussing their connections to various conjectures in information theory and discrete probability, such as the Courtade–Kumar, Li–Médard and Mossell–O'Donnell conjectures. Finally, we consider functional inequalities (e.g., the hypercontractivity and Brascamp–Lieb inequalities), which constitute a further generalization of the noise stability problem in which the Boolean functions therein are replaced by nonnnegative functions. We demonstrate that the key ideas behind the proofs in Part III can be presented in a pedagogically coherent manner and unified via information-theoretic and Fourier-analytic methods.

1

Introduction

1.1 Motivation

Let X be the statistical description of a set of images whose foregrounds and backgrounds are those of an airplane and the blue sky respectively. Let Y, which is correlated to X, be the statistical description of another set of images whose foregrounds are those of a unicorn and the blue sky respectively. It seems natural and intuitive that the common information in X and Y should be the number of bits needed to describe the blue sky, which is the common part of X and Y. Can we make this observation precise and quantitative for *arbitrary* (X, Y) pairs? This monograph is centered on this fundamental question in information and probability theory. In other words, we would like to quantify, via an assortment of well-motivated measures, the *intrinsic similarity* or *common information* between two correlated random variables X and Y. Regardless of what applications there may be, the pursuit of operationally meaningful measures that quantify the common information between two random variables seems to be an extremely worthy academic endeavor. This is especially so for researchers in information and coding theory, theoretical computer science, and cryptography who are seeking to understand the inherent difficulties in generating correlated bits from a single joint

source, or simulating a joint source using a single source of randomness in a distributed manner.

In probability, statistics, and data analysis, there are numerous popular functionals of joint distributions that quantify the amount of correlation or dependence between two random variables X and Y. If these random variables have joint distribution π_{XY} and means μ_X and μ_Y respectively, such paradigmatic examples include the *Pearson correlation coefficient*

$$\rho(X;Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]}{\sqrt{\mathbb{E}[(X-\mu_X)^2]\mathbb{E}[(Y-\mu_Y)^2]}} \tag{1.1}$$

and the *Hirschfeld–Gebelein–Rényi (HGR) maximal correlation*

$$\rho_{\mathrm{m}}(X;Y) := \sup_{f,g} \rho\big(f(X);g(Y)\big), \tag{1.2}$$

where the supremum is taken over all real-valued functions f and g such that $0 < \mathrm{Var}(f(X)), \mathrm{Var}(g(Y)) < \infty$. In addition, an information-theoretic quantity known as the *mutual information*

$$I(X;Y) = \begin{cases} \displaystyle\int_{\mathcal{X}\times\mathcal{Y}} \log\left(\frac{\mathrm{d}\pi_{XY}}{\mathrm{d}(\pi_X\pi_Y)}\right)\mathrm{d}\pi_{XY} & \text{if } \pi_{XY} \ll \pi_X\pi_Y \\ +\infty & \text{otherwise} \end{cases},$$

also serves to quantify the dependence between two random variables. These measures have the property that they are zero if the two random variables are independent, fulfilling a basic requirement of any measure that quantifies the dependence between two random variables. These measures can be regarded as common information quantities between X and Y, jointly distributed as π_{XY}. Indeed, the mutual information $I(X;Y)$ captures the amount of information about X provided by observing Y, as can observed in the celebrated distributed lossless compression theorem of Slepian and Wolf [41], [156]. Are there any other *operationally-motivated* measures that allow us to gain deeper insights on the common information between X and Y given their numerical values?

In information and coding theory, there are two canonical examples of operationally-motivated common information measures that have been widely accepted since their inceptions in the 1970s. The first, which was

introduced in 1973, is *Gács–Körner–Witsenhausen's (GKW's) common information* [60], [178], defined as

$$C_{\mathrm{GKW}}(\pi_{XY}) := \sup_{f,g} H\big(f(X)\big), \tag{1.3}$$

where the supremum is taken over all pairs of deterministic functions (f, g) defined respectively on $\mathcal{X}$ and $\mathcal{Y}$ such that $f(X) = g(Y)$ with π_{XY}-probability one. The second, which was introduced in 1975, is *Wyner's common information* [182], defined as

$$C_{\mathrm{W}}(\pi_{XY}) := \inf_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} I_P(W; XY), \tag{1.4}$$

where the infimum extends over triples of random variables $(W, X, Y) \sim P_{WXY}$ such that $X - W - Y$ forms a Markov chain and $P_{XY} = \pi_{XY}$.

1.2 Overview of the Monograph

Our twin objectives in this monograph are as follows. Firstly, we seek to provide a concise review of these classical notions of common information. Secondly, we endeavor to connect these quantities to new notions of common information in the literature that have gained traction recently. A flowchart of the sections in this monograph is provided in Fig. 1.1.

1.2.1 Part I: Classic Common Information Quantities

We commence in Part I by reviewing the operational tasks associated with the classical common information quantities in (1.3) and (1.4) and describing their salient properties. This part consists of Sections 2 and 3 on Wyner's and GKW's common information respectively.

1.2.2 Part II: Extensions of Wyner's Common Information

We then extend and generalize Wyner's common information in Part II of this monograph, which consists of four sections. In Section 4, we review the *Rényi common information*, originally studied by the present authors [197], [202]. In his seminal paper [182], Wyner used the normalized relative entropy

$$\frac{1}{n} D\left(P_{X^n Y^n} \| \pi_{XY}^n\right) = \frac{1}{n} \sum_{x^n, y^n} P_{X^n Y^n}(x^n, y^n) \log \frac{P_{X^n Y^n}(x^n, y^n)}{\pi_{XY}^n(x^n, y^n)}.$$

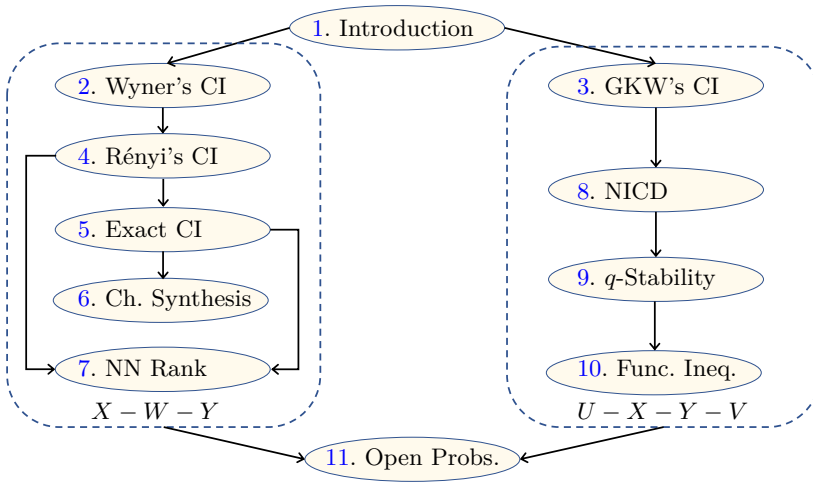


Figure 1.1: Flowchart of the sections in this monograph (CI, NN, and NICD stand for Common Information, Nonnegative, and Non-Interactive Correlation Distillation respectively)

to quantify the discrepancy between the synthesized distribution $P_{X^nY^n}$ and the target distribution π_{XY}^n and sought the minimum rate for distributed source synthesis for which this quantity vanishes as the blocklength n grows. The Rényi common information [197], [202] generalizes this to the case in which the discrepancy measures used belong to the families of normalized and unnormalized Rényi divergences. For Rényi order $1+s \in (0,\infty)\setminus\{1\}$, the unnormalized form can be expressed as

$$D_{1+s}(P_{X^nY^n}\|\pi_{XY}^n) = \frac{1}{s}\log \sum_{x^n,y^n} P_{X^nY^n}(x^n,y^n)\left(\frac{P_{X^nY^n}(x^n,y^n)}{\pi_{XY}^n(x^n,y^n)}\right)^s.$$

We use this family of measures to build a bridge to the topic of discussion in Section 5, namely, the *exact common information*, a quantity first defined and studied by Kumar, Li, and El Gamal [103]; see Definition 5.3 for its precise definition. In contrast to the Rényi common information, the exact version requires that synthesized distribution be *exactly* equal to the target distribution for some blocklength n; however, *variable-length* codes are permitted. Using an unexpected equivalence

between the unnormalized Rényi common information of order ∞ (the limit of D_{1+s} as $s \to \infty$)

$$D_\infty(P_{X^nY^n} \| \pi_{XY}^n) = \log \max_{(x^n,y^n):P_{X^nY^n}(x^n,y^n)>0} \frac{P_{X^nY^n}(x^n,y^n)}{\pi_{XY}^n(x^n,y^n)},$$

and the exact common information, we argue that the latter can be strictly larger than Wyner's common information for some sources, specifically the doubly symmetric binary source (DSBS).

In Section 6, we use the preceding notions to describe the problem of *channel synthesis*. We review this problem in both the approximate and exact settings and show that it produces a continuum of common information measures that interpolate from the mutual information to Wyner's or exact common information.

In Section 7, we describe a seemingly tangential topic in numerical linear algebra, namely the *nonnegative rank* of nonnegative matrices [65], [168]. It turns out that this area of research has intimate connections to the preceding notions of common information, leading to some interesting open problems.

1.2.3 Part III: Extensions of Gács–Körner–Witsenhausen's Common Information

It is known that GKW's common information is zero for most non-pathological sources such as the doubly symmetric binary source and the bivariate Gaussian source. Consequently, in itself, GKW's common information does not provide any tangible quantification of how "similar" two sources are. The goal of Part III, which consists of three sections, is thus to consider several *refinements* of GKW's common information in which new insights can be readily gleaned.

We start in Section 8 by providing an extensive discussion of the 2-*user Non-Interactive Correlation Distillation* (NICD) problem [94], [124]. Given a pair of random vectors $(X^n, Y^n) \sim \pi_{XY}^n$ in which each (X_i, Y_i) is drawn independently from a DSBS, this problem concerns the agreement probability of the random bits that can be extracted from X^n and Y^n individually. In other words, we wish to quantify

$$\max_{f,g} \Pr\left(U = V\right) \quad \text{and} \quad \min_{f,g} \Pr\left(U = V\right),$$

where $U = f(X^n)$ and $V = g(Y^n)$ and f and g are $\{0,1\}$-valued (i.e., Boolean) functions such that the marginals $\Pr(U = 1)$ and $\Pr(V = 1)$ are appropriately constrained. For example, for the maximization version of the NICD problem, we place upper bounds on $\Pr(U = 1)$ and $\Pr(V = 1)$. We quantify these agreement probabilities by studying various geometric structures such as Hamming subcubes and Hamming balls. We discuss their optimality in several asymptotic regimes (such as the central limit or large deviations regimes) using results from concentration of measure and Boolean Fourier analysis, among other techniques.

In Section 9, we extend the NICD problem to the *multi-user* version. For the k-user case, there are k correlated sources $X_1^n, X_2^n, \ldots, X_k^n$ that are generated independently conditioned on another source Y^n such that the joint distribution of X_i^n and Y^n is π_{XY}^n. We are interested in quantifying

$$\max_{f_1, f_2, \ldots, f_k} \Pr\left(U_1 = U_2 = \ldots = U_k\right),$$

where $U_i = f_i(X_i^n), i = 1, 2, \ldots, k$ and the maximum extends over all k-tuples of Boolean functions f_i's whose marginals are also constrained by placing upper bounds on $\Pr(U_i = 1)$. We also discuss the connection of the k-user NICD problem to *q-stability* [52], [110] in which the number of users k is replaced by an arbitrary real number q. This allows us to seamlessly segue into a review of recent advances in contemporary conjectures in information theory and discrete probability. These include the Courtade–Kumar conjecture [40], the Mossel–O'Donnell conjecture [122], and the Li–Médard conjecture [110]. Mathematical tools used here include the analysis of Boolean functions [131] and, in particular, edge-isoperimetric inequalities and the study of the maximal degree-1 Fourier weight.

In Section 10, we connect these notions and results to *functional inequalities* including the hypercontractivity, the logarithmic Sobolev, the Brascamp–Lieb inequalities, as well as their strengthened counterparts. This section generalizes the preceding two sections in that the Boolean functions f_i are replaced by arbitrary nonnnegative functions.

The monograph is concluded in Section 11 in which we summarize open problems in this fascinating area of study.

The common theme in Part II is the Markov chain $X - W - Y$; this corresponds to the constraint that defines Wyner's common information in (1.4). In contrast, in Part III, we focus on the Markov chain $U - X - Y - V$; this corresponds to the Markov chain in the NICD problem in which $U = f(X^n)$ and $V = g(Y^n)$ for some Boolean functions f and g. It is also present in GKW's common information. At first glance, this appears to be different from the constraint in (1.3). However, this constraint is merely a special case of $U - X - Y - V$ by taking U and V to be deterministic functions of X and Y respectively such that they are also constrained to be equal almost surely.

1.3 Notation

To appreciate the material in this monograph, the reader is expected to have some background in information theory at the level of Cover and Thomas [42]. We will also make frequent use of the method of types, for which an excellent exposition can be found in Csiszár and Körner [45].

In this monograph, we generally follow the notation in Cover and Thomas [42], El Gamal and Kim [51], and Csiszár and Körner [45].

1.3.1 Random Variables and Probability Distributions

Random variables and their realizations are denoted by upper case letters (such as X and Y) and lower case letters (such as x and y) respectively. The sets of values that the realizations take on, also called *alphabets*, are denoted by calligraphic letters such as $\mathcal{X}$ and $\mathcal{Y}$. We use $P_X, \tilde{P}_X, Q_X$, and π_X to denote various probability distributions on alphabet $\mathcal{X}$. If a random variable X is distributed according to P_X, we write $X \sim P_X$. As we work with both discrete and continuous random variables in this monograph, we will often have to distinguish between probability mass functions (PMFs) for discrete random variables and probability density functions (PDFs) for continuous random variables. If X is discrete, we use $x \in \mathcal{X} \mapsto P_X(x)$ to denote its PMF. The PDF of a (real-valued) continuous random variable is denoted as $f_X : x \in \mathbb{R} \mapsto (\mathrm{d}P_X/\mathrm{d}\mu)(x)$, where μ is the Lebesgue measure on $\mathbb{R}$. These will also be denoted as P or f when the random variable X is clear from the context. Throughout

the monograph, the notations π_X and π_{XY} are reserved for *target* and *source distributions*.

The set of PMFs on $\mathcal{X}$ is denoted as $\mathcal{P}(\mathcal{X})$ and the set of conditional PMFs on $\mathcal{Y}$ given a variable taking values in $\mathcal{X}$ is denoted as $\mathcal{P}(\mathcal{Y}|\mathcal{X}) = \{P_{Y|X} : P_{Y|X}(\cdot|x) \in \mathcal{P}(\mathcal{Y}), x \in \mathcal{X}\}$. The joint distribution induced by $P_X \in \mathcal{P}(\mathcal{X})$ and $P_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ is denoted as $P_X P_{Y|X} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$. The *support* of a discrete distribution is denoted as $\mathrm{supp}(P) := \{x \in \mathcal{X} : P(x) > 0\}$. Given an input distribution $P_X \in \mathcal{P}(\mathcal{X})$ and a conditional distribution $P_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$, if the induced output distribution is $P_Y(y) = \sum_x P_X(x)P_{Y|X}(y|x)$ (for the discrete case), we write this as $P_X \to P_{Y|X} \to P_Y$. For two distributions P and Q (defined on the same measurable space), we use $P \ll Q$ to denote that P is *absolutely continuous* with respect to Q. In the finite alphabet case, $P \ll Q$ means that for every $x \in \mathcal{X}$ such that $Q(x) = 0$, it holds that $P(x) = 0$.

We say that three random variables X, Y, and Z *form a Markov chain in this order* if X and Z are conditionally independent given Y. In this case, we write $X - Y - Z$. For discrete random variables, $X - Y - Z$ if and only if $P_{XYZ}(x, y, z) = P_Y(y)P_{X|Y}(x|y)P_{Z|Y}(z|y)$ for all $(x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$. As is customary in information theory, for two integers m and n, we write X_m^n to mean the random vector $(X_m, X_{m+1}, \ldots, X_n)$; when $m = 1$, this is abbreviated to X^n. A particular realization of X^n, a deterministic vector, is denoted as $x^n = (x_1, x_2, \ldots, x_n)$. We denote the *n-fold product distribution* of P as P^n, which is defined by the formula $P^n(x^n) = \prod_{i=1}^n P(x_i)$ for all $x^n \in \mathcal{X}^n$.

A *stationary memoryless source*, denoted by $X \sim P_X \in \mathcal{P}(\mathcal{X})$, is a discrete-time stochastic process $\{X_i\}_{i \in \mathbb{N}}$ such that X_i's are independent copies of X. We also denote a source X by its distribution P_X. We use X^n to denote the first n random variables in the stochastic process $\{X_i\}_{i \in \mathbb{N}}$. With a slight abuse of terminology, X^n is also called a *source sequence* of the source X. A *stationary memoryless channel*, denoted by $P_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$, is a random transformation that outputs a length-n random vector $Y^n \sim P_{Y|X}^n(\cdot|x^n)$ if the input is the length-n vector $x^n \in \mathcal{X}^n$. Since we deal almost exclusively with stationary memoryless sources and channels in this monograph, we will omit the term "stationary memoryless" when we mention sources and channels.

We will work mainly with three types of random variables in this monograph. A discrete *uniform* random variable X takes equal probabilities on its support $\mathcal{X}$ and its probability distribution is denoted as $\mathrm{Unif}(\mathcal{X})$. A *Bernoulli* random variable X is one with support $\{0,1\}$. Its probability distribution is abbreviated as $\mathrm{Bern}(a)$ if $\Pr(X=1)=a$. A (d-dimensional) *normal* or *Gaussian* random variable or vector X has a PDF that is denoted by

$$\mathbf{x} \in \mathbb{R}^d \mapsto \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^d \det(\boldsymbol{\Sigma})}} \exp\Big(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\Big),$$

(or simply $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$) where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the mean vector and the covariance matrix respectively.

1.3.2 Types or Empirical Distributions

We will often use the method of types [45] in our calculations, especially for finite alphabets. Given a sequence $x^n \in \mathcal{X}^n$, we use

$$T_{x^n}(a) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{x_i = a\} \quad \text{for all } a \in \mathcal{X}$$

to denote its *type* or *empirical distribution*. The type of a length-n sequence will be denoted by T or $T_X^{(n)}$ depending on the context. The set of all sequences with type T is denoted as $\mathcal{T}_T \subset \mathcal{X}^n$. This is known as the *type class* of T. The set of all types that can be formed from sequences of length n taking values in alphabet $\mathcal{X}^n$ is denoted as $\mathcal{P}_n(\mathcal{X})$, which is a subset of the probability simplex $\mathcal{P}(\mathcal{X})$.

1.3.3 Information Measures

We now recap the necessary information measures used in this monograph. For $X \sim P_X$, we denote its *Shannon entropy* as

$$H(X) = H_P(X) = H(P_X) := -\sum_{x \in \mathrm{supp}(P_X)} P_X(x) \log P_X(x). \tag{1.5}$$

All logarithms are to the base 2 unless otherwise specified. For $(X, Y) \sim P_{XY}$, we denote the *conditional entropy* of X given Y as

$$\begin{aligned} H(X|Y) = H_P(X|Y) &= H(P_{X|Y}|P_Y) \\ &:= -\sum_{y\in\mathcal{Y}} P_Y(x) \sum_{x\in\mathrm{supp}(P_{X|Y}(\cdot|y))} P_{X|Y}(x|y) \log P_{X|Y}(x|y). \end{aligned}$$

The *mutual information* between X and Y where $(X, Y) \sim P_{XY}$ is denoted as

$$I_P(X;Y) = I(P_X, P_{Y|X}) := H_P(X) - H_P(X|Y).$$

The subscripts in H_P and I_P are used to emphasize the distribution of (X, Y) under which these information measures are computed. When the distribution is clear from the context, the subscripts will be omitted. The *relative entropy* or *Kullback–Leibler divergence* between two distributions P_X and Q_X defined on the same (countable) alphabet is[1]

$$D(P_X\|Q_X) := \sum_{x\in\mathrm{supp}(P_X)} P_X(x) \log \frac{P_X(x)}{Q_X(x)}.$$

The *conditional relative entropy* of two conditional distributions $P_{Y|X}$ and $Q_{Y|X}$, given a distribution P_X, is

$$D(P_{Y|X}\|Q_{Y|X}|P_X) := D(P_X P_{Y|X}\|P_X Q_{Y|X}). \tag{1.6}$$

In addition to the Shannon information measures above, we need to recap the family of Rényi information measures [144], [166] as this is central to the majority of our discussion in this monograph. For two distributions $P_X, Q_X \in \mathcal{P}(\mathcal{X})$ on a countable set $\mathcal{X}$, the *Rényi divergence* of order $1 + s \in (0, 1) \cup (1, \infty)$ is

$$D_{1+s}(P_X\|Q_X) := \frac{1}{s} \log \sum_{x\in\mathrm{supp}(P_X)} P_X(x) \left(\frac{P_X(x)}{Q_X(x)}\right)^s.$$

[1] This definition is only applicable when the alphabets are countable. For P_X and Q_X defined on a general probability space, the ratio P_X/Q_X should be replaced with the Radon–Nikodym derivative $\mathrm{d}P_X/\mathrm{d}Q_X$ (if $P_X \ll Q_X$), and the expectation with respect to P_X should be written as a Lebesgue integral over $\mathcal{X}$. If P_X is not absolutely continuous with respect to Q_X, $D(P_X\|Q_X)$ is defined to be $+\infty$. In the following, for simplicity, we only provide definitions of information-theoretic quantities for countable alphabets.

The Rényi divergence is monotonically nondecreasing in its order. Sibson's [155] version of the *conditional Rényi divergence* between two conditional distributions $P_{Y|X}$ and $Q_{Y|X}$ given a distribution P_X is

$$D_{1+s}(P_{Y|X}\|Q_{Y|X}|P_X) := D_{1+s}(P_X P_{Y|X}\|P_X Q_{Y|X}). \tag{1.7}$$

We note that while the conditional relative entropy in (1.6) is the expectation of $D(P_{Y|X}(\cdot|X)\|Q_{Y|X}(\cdot|X))$ over $X \sim P_X$, the conditional Rényi divergence in (1.7) depends on $D_{1+s}(P_{Y|X}(\cdot|X)\|Q_{Y|X}(\cdot|X))$ in a more involved way; indeed, it is a generalized mean of the random variable $D_{1+s}(P_{Y|X}(\cdot|X)\|Q_{Y|X}(\cdot|X))$ evaluated at s. For a more detailed discussion on this point, the reader is referred to Cai and Verdú [32]. We also note that there are other definitions of the conditional Rényi divergence but we will use the definition in (1.7) in this monograph; see [20], [43], [155]. The Rényi divergence and its conditional version in (1.7) can be extended to all orders $1+s \in \{0, 1, \infty\}$ by taking the appropriate limits. In particular, when $s \to 0$, we recover the usual relative entropy. An order of the Rényi divergence that will be of particular interest to us in this monograph is the *Rényi divergence of order* ∞. This is the divergence we obtain when we let $s \to \infty$, i.e.,

$$D_\infty(P_X\|Q_X) := \log \sup_{x \in \mathrm{supp}(P_X)} \frac{P_X(x)}{Q_X(x)}.$$

The *Rényi entropy* of order $1 + s \in (0, 1) \cup (1, \infty)$ of a probability mass function $P_X \in \mathcal{P}(\mathcal{X})$ is defined as

$$H_{1+s}(P_X) = -\frac{1}{s} \log \sum_{x \in \mathrm{supp}(P_X)} \big(P_X(x)\big)^{1+s}. \tag{1.8}$$

It is easy to check that

$$H_{1+s}(P_X) := \log|\mathcal{X}| - D_{1+s}(P_X\|\mathrm{Unif}(\mathcal{X})). \tag{1.9}$$

Similarly to the Rényi divergence, we define $H_0(P_X)$ and $H_\infty(P_X)$ as the limits of $H_{1+s}(P_X)$ as $s \downarrow -1$ and $s \to \infty$ respectively. These are known as the *max-entropy* and *min-entropy* respectively. Of special importance is the case when $s \to 0$, in which case $H_{1+s}(P_X)$ reduces to the Shannon entropy defined in (1.5). Since the relation in (1.9)

holds and the Rényi divergence is nondecreasing in its order, the Rényi entropy is nonincreasing in its order.

We need one additional measure of the discrepancy between two distributions. The *total variation distance* or simply the *TV distance* is defined for two distributions P and Q on a common (countable) alphabet $\mathcal{X}$ as

$$|P - Q| := \frac{1}{2} \sum_{x \in \mathcal{X}} |P(x) - Q(x)|.$$

More generally, $|P - Q| = \sup_{\mathcal{A} \subset \mathcal{X}} |P(\mathcal{A}) - Q(\mathcal{A})|$, where $\mathcal{A}$ runs over all (measurable) subsets of $\mathcal{X}$. Pinsker's inequality yields the following bound on the TV distance in terms of the relative entropy

$$|P - Q|^2 \leq \frac{\ln 2}{2} \cdot D(P\|Q). \tag{1.10}$$

1.3.4 Typical Sets

In our achievability proofs, we will often need to use the notion of *typical sets* [42], [51], [135]. The *ϵ-strongly typical set* with respect to a distribution $P_X \in \mathcal{P}(\mathcal{X})$ is defined as

$$\mathcal{T}_\epsilon^{(n)}(P_X) := \left\{ x^n \in \mathcal{X}^n : |T_{x^n}(x) - P_X(x)| \leq \epsilon P_X(x), \forall\, x \in \mathcal{X} \right\}.$$

This notion of typicality, proposed by Orlitsky and Roche [135], is also commonly known as *robust typicality* and is convenient for coding problems with cost constraints or rate-distortion problems. However, it suffers from the deficiency that it is amenable only to *finite* alphabets. This is mitigated by the availability of the *ϵ-weakly typical set* with respect to a distribution $P_X \in \mathcal{P}(\mathcal{X})$, which is defined as

$$\mathcal{A}_\epsilon^{(n)}(P_X) := \left\{ x^n \in \mathcal{X}^n : \left| \frac{1}{n} \log \frac{1}{P_X^n(x^n)} - H(P_X) \right| < \epsilon \right\}.$$

When X is a continuous random variable, $H(P_X)$ is to be replaced by the *differential entropy* of X [42]. The conditional versions of these sets can be defined in a natural manner, e.g., the *conditionally ϵ-strongly typical set* of Y given a sequence $x^n \in \mathcal{X}^n$ is

$$\mathcal{T}_\epsilon^{(n)}(P_{XY}|x^n) := \left\{ y^n \in \mathcal{Y}^n : (x^n, y^n) \in \mathcal{T}_\epsilon^{(n)}(P_{XY}) \right\}.$$

1.3.5 Asymptotic Notations

Asymptotic notation is used in the usual way [39]. Given two real-valued sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$, we say that $a_n = O(b_n)$ if $\limsup_{n\to\infty} |a_n/b_n| < \infty$, $a_n = \Omega(b_n)$ if $\liminf_{n\to\infty} |a_n/b_n| > 0$, and $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$. Similarly, $a_n = o(b_n)$ if $\lim_{n\to\infty} |a_n/b_n| = 0$. Finally, if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are positive sequences, we write $a_n \doteq b_n$ if these sequences are *equal to first-order in the exponent* [42], i.e., $\lim_{n\to\infty} n^{-1} \log(a_n/b_n) = 0$.

1.3.6 Miscellaneous

For two integers m and n, we write $[m : n] = \{m, m+1, \ldots, n\}$ to denote the discrete interval. When $m = 1$, this is abbreviated as $[n]$. Often, for an $R > 0$, we write $[2^{nR}]$ to refer to the set $\{1, 2, \ldots, 2^{\lfloor nR \rfloor}\}$. Given a number $a \in [0, 1]$, we write $\overline{a} := 1 - a$. Given two numbers $a, b \in [0, 1]$, we write $a * b := a\overline{b} + b\overline{a}$ to denote their binary convolution. We write $[a]^+$ to mean $\max\{a, 0\}$ for $a \in \mathbb{R}$. For two bits $a, b \in \{0, 1\}$, $a \oplus b$ denotes the binary addition (modulo-2 sum) operation, i.e., $a \oplus b = 0$ if $a = b$ and 1 otherwise. Logarithms are always to the base 2 unless otherwise specified. When we write ln, we are referring to the natural logarithm (to base e).

Vectors are interchangeably denoted by boldface lower case font (e.g., $\mathbf{u}$) or, as mentioned in Section 1.3.1, with a lower case letter and with a superscript indicating its length (e.g., $u^n = (u_1, u_2, \ldots, u_n)$). Matrices (e.g., $\mathbf{M}$) are denoted in boldface upper case font. The i^{th} element of a vector $\mathbf{u}$ is denoted interchangeably as u_i or $[\mathbf{u}]_i$. Similarly, the $(i, j)^{\text{th}}$ element of a matrix $\mathbf{M}$ is denoted interchangeably as $M_{i,j}$ or $[\mathbf{M}]_{i,j}$.

1.4 Mathematical Tools

1.4.1 The Method of Types

We summarize a few key property of types which will turn out to be useful in proving both achievability and converse parts of various common information problems, particularly those with finite alphabets.

For an extensive discussion, the reader is referred to the book by Csiszár and Körner [45].

First, the number of types $|\mathcal{P}_n(\mathcal{X})| \leq (n+1)^{|\mathcal{X}|}$ is polynomial in n. Second, for a given type $T \in \mathcal{P}_n(\mathcal{X})$, the size of the type class $(n+1)^{-|\mathcal{X}|}2^{nH(T)} \leq |\mathcal{T}_T| \leq 2^{nH(T)}$ is related to the entropy of the type $H(T)$. Third, the Q^n-probability of a sequence $x^n \in \mathcal{T}_T$ is $Q^n(x^n) = 2^{-n(D(T\|Q)+H(T))}$. Consequently, the Q^n-probability of the type class $\mathcal{T}_T$ is bounded as $(n+1)^{-|\mathcal{X}|}2^{-nD(T\|Q)} \leq Q^n(\mathcal{T}_T) \leq 2^{-nD(T\|Q)}$.

A particularly useful result that we use repeatedly in Part III of the monograph is *Sanov's theorem* [42], [49], [150], so we review it here.

Theorem 1.1 (Sanov's theorem)**.** Let the components of the random vector $X^n = (X_1, X_2, \ldots, X_n)$ be generated in an independently and identically distributed (i.i.d.) manner from a PMF $Q \in \mathcal{P}(\mathcal{X})$. For any $n \in \mathbb{N}$ and any set of distributions $\mathcal{S} \subset \mathcal{P}(\mathcal{X})$,

$$Q^n\big(\{x^n : T_{x^n} \in \mathcal{S}\}\big) \leq (n+1)^{|\mathcal{X}|}2^{-nD(P^*\|Q)},$$

where the *information projection* of Q onto $\mathcal{S}$ is any distribution P^* that satisfies

$$D(P^*\|Q) = \inf_{P\in\mathcal{S}} D(P\|Q).$$

If additionally, $\mathcal{S}$ is equal to the closure of its interior (under the relative topology),[2]

$$\liminf_{n\to\infty} -\frac{1}{n}\log Q^n\big(\{x^n : T_{x^n} \in \mathcal{S}\}\big) \geq D(P^*\|Q),$$

and hence,

$$Q^n\big(\{x^n : T_{x^n} \in \mathcal{S}\}\big) \doteq 2^{-nD(P^*\|Q)}.$$

Sanov's theorem basically says that the exponent of the probability that the type T_{X^n} of a random sequence $X^n \sim Q^n$ belongs to a set $\mathcal{S}$ is dominated by the relative entropy between the information projection of Q onto $\mathcal{S}$ and Q.

[2]This regularity condition will always be satisfied in the sections to follow.

1.4.2 Couplings

In this monograph, we will often encounter the optimization problems over joint distributions for which their marginals are fixed. Such a joint distribution is known as a coupling. More precisely, a *coupling* P_{XY} of two distributions $Q_X \in \mathcal{P}(\mathcal{Y})$ and $Q_Y \in \mathcal{P}(\mathcal{Y})$ is a joint distribution on $\mathcal{X} \times \mathcal{Y}$ whose X- and Y-marginals are respectively Q_X and Q_Y. The set of all couplings with marginals Q_X and Q_Y is denoted as

$$\mathcal{C}(Q_X, Q_Y) := \{P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : P_X = Q_X, P_Y = Q_Y\}.$$

Similarly, a conditional coupling $P_{XY|W}$ is a joint conditional distribution whose X- and Y-marginals agree with given marginals $Q_{X|W}$ and $Q_{Y|W}$ respectively. The set of all conditional couplings with marginals $Q_{X|W}$ and $Q_{Y|W}$ is

$$\begin{aligned}&\mathcal{C}(Q_{X|W}, Q_{Y|W})\\ &\quad := \{P_{XY|W} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}|\mathcal{W}) : P_{X|W} = Q_{X|W}, P_{Y|W} = Q_{Y|W}\}.\end{aligned}$$

Couplings have many beautiful properties, but we will not elaborate on them in this monograph; see Thorisson [163] or Yu and Tan [199] for example. One property that is quite remarkable is the *maximal coupling equality* which says that given two distributions Q_X and Q_Y, the total variation distance between them is equal to the probability that X is not equal to Y minimized over all couplings induced by Q_X and Q_Y, i.e.,

$$\min_{P_{XY} \in \mathcal{C}(Q_X, Q_Y)} \Pr(X \neq Y) = |Q_X - Q_Y|.$$

A generalization of the maximal coupling equality that turns out to be useful in the GKW common information problem (Section 3) is stated as follows. This lemma is due to the present authors [199].

Lemma 1.2 (Maximal guessing coupling equality)**.** Given two distributions Q_X and Q_Y, we have

$$\min_{P_{XY} \in \mathcal{C}(Q_X, Q_Y)} \min_{f: \mathcal{X} \to \mathcal{Y}} \Pr\left(Y \neq f(X)\right) = \min_{f: \mathcal{X} \to \mathcal{Y}} \left|Q_Y - Q_{f(X)}\right|. \quad (1.11)$$

The minimization problem on the left-hand side of (1.11) is termed the *maximal guessing coupling problem* (because we would like to maximize the probability that Y is guessed correctly by f acting on X).

The minimization problem on the right-hand side is a classical problem in information theory which is termed the *distribution approximation* or *random number generation* problem [71, Chapter 2]. Lemma 1.2 implies that the maximal guessing coupling problem is equivalent to the distribution approximation problem.

The concept of coupling is naturally involved when we study a problem involving Markov chains, e.g., Wyner's common information and its extensions. One key step to analyze such problems is to simplify multi-letter expressions that involve optimizations over couplings to single-letter ones. This is conveniently facilitated by the *chain rule* on couplings. Before stating this, we first define the *product coupling set*

$$\prod_{i=1}^{n} \mathcal{C}(Q_{X_i|X^{i-1}W}, P_{Y_i|Y^{i-1}W}) := \Bigg\{ \prod_{i=1}^{n} P_{X_iY_i|X^{i-1}Y^{i-1}W} : P_{X_iY_i|X^{i-1}Y^{i-1}W} \in \mathcal{C}(Q_{X_i|X^{i-1}W}, Q_{Y_i|Y^{i-1}W}), i \in [n] \Bigg\}.$$

Lemma 1.3 (Chain Rule for Coupling Sets)**.** For any pair of conditional distributions $(Q_{X^n|W}, Q_{Y^n|W})$, we have

$$\prod_{i=1}^{n} \mathcal{C}(Q_{X_i|X^{i-1}W}, Q_{Y_i|Y^{i-1}W}) \subset \mathcal{C}(Q_{X^n|W}, Q_{Y^n|W}).$$

This lemma can be interpreted as follows. By the usual chain rule for joint distributions, the conditional distributions $Q_{X^n|W}$ and $Q_{Y^n|W}$ can be factorized as $\prod_{i=1}^{n} Q_{X_i|X^{i-1}W}$ and $\prod_{i=1}^{n} Q_{Y_i|Y^{i-1}W}$ respectively. Let $P_{X_iY_i|X^{i-1}Y^{i-1}W}$ be a coupling of each pair of component conditional distributions $(Q_{X_i|X^{i-1}W}, Q_{Y_i|Y^{i-1}W})$. Then, this lemma says that the product of $P_{X_iY_i|X^{i-1}Y^{i-1}W}$ forms a coupling of the product of $Q_{X_i|X^{i-1}W}$ and the product of $Q_{Y_i|Y^{i-1}W}$.

The proof of this lemma can be found in Yu and Tan [204].

Part I

Classic Common Information Quantities

2

Wyner's Common Information

What constitutes a meaningful notion of the *common information* between two random variables X and Y? As mentioned in the Introduction, there are at least two such notions that have gained traction in the information theory community as well as adjacent communities such as theoretical computer science and cryptography. In this section, we focus on *Wyner's common information* [182]. To motivate this fundamental quantity, let us consider the special case in which X and Y can be written as $X = (\tilde{X}, V)$ and $Y = (\tilde{Y}, V)$ where $\tilde{X}$, $\tilde{Y}$ and V are independent. It seems natural to define the amount of common information between X and Y as the entropy $H(V)$ of the common part they share, namely V. Taking this idea (much) further is the subject of the current and later sections (in Part II).

We review the notion of Wyner's common information from two seemingly disparate information processing tasks. We show that these perspectives are, somewhat surprisingly, equivalent. In Section 2.1, we consider the scenario in which one would like to *simulate* a joint distribution π_{XY} given a single source of common randomness. The minimum amount of common randomness to obtain an asymptotically exact reconstruction of π_{XY} constitutes Wyner's common information between X

and Y. The perspective concerning simulation of random variables is the common thread throughout the monograph. Nevertheless, we find it useful to provide a complementary perspective of Wyner's common information by revisiting the *Gray–Wyner source coding* problem in Section 2.2. In this problem, Wyner's common information is the minimum common rate R_0 such that the sum of the two private rates R_1 and R_2 and the common rate R_0 is constrained to be almost equal to the joint entropy of the source $H(XY)$. We evaluate Wyner's common information for the doubly symmetric binary source (DSBS) and the symmetric binary erasure source (SBES) in Sections 2.3 and 2.4 respectively.

Moving on to more contemporary topics, in Section 2.5, we discuss the subtleties and techniques to extend Wyner's common information to continuous sources, allowing us to evaluate it for jointly Gaussian random variables. Finally, in Section 2.6, we discuss several recent extensions and applications of Wyner's common information.

2.1 Distributed Simulation of a Target Joint Distribution

How much common randomness is needed to simulate a joint source $(X, Y) \sim \pi_{XY}$ in a distributed fashion? This problem, as depicted in Fig. 2.1 and termed *distributed source simulation*, was first studied by Wyner [182] in his celebrated paper on common information. In this problem, there is a *target distribution* π_{XY} and we would like to use a uniform random variable and two distributed *processors* to approximate the product distribution π_{XY}^n to an arbitrary precision as the number of copies n of the target distribution tends to infinity. What is the minimum cardinality (or rate) of the support of the uniform random variable such that this is achievable?

Before turning to formal definitions and results, let us revisit the simple example in which $X = (\tilde{X}, V)$ and $Y = (\tilde{Y}, V)$ for some tuple of independent random variables $\tilde{X}$, $\tilde{Y}$ and V. Clearly, one can use a lossless source code to encode V^n by a binary string of length approximately $nH(V)$. This binary string is then sent through the processors. Shannon's lossless source coding theorem tells us that we can reconstruct V^n almost losslessly as long as n is sufficiently large. Additionally, the processors can themselves generate $\tilde{X}^n$ and $\tilde{Y}^n$ independently. Thus,

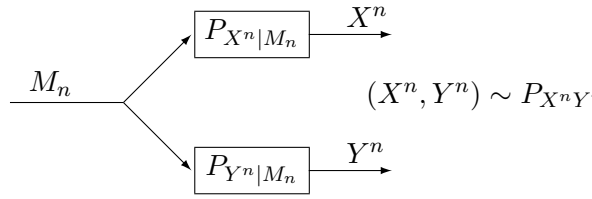


Figure 2.1: The distributed source simulation problem

it is clear that an achievable rate of common randomness is $H(V)$. It is also plausible that any rate strictly below $H(V)$ is not achievable as the common part of X and Y cannot be reliably reconstructed.

We now turn to formal definitions of the problem. Consider the distributed source simulation setup depicted in Fig. 2.1. Each of the two terminals has access to a uniformly distributed random variable M_n, also known as the *common* or *shared randomness*. Given a target distribution π_{XY}, one of terminals uses $M_n \in \mathcal{M}_n$ and its own local randomness to generate a random vector X^n and the other one uses M_n and its own local randomness to generate another random vector Y^n. The terminals' goal is to ensure that the *synthesized distribution*

$$P_{X^nY^n}(x^n, y^n) = \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} P_{X^n|M_n}(x^n|m) P_{Y^n|M_n}(y^n|m) \qquad (2.1)$$

is "close to" the n-fold product of the target distribution π_{XY}^n. We wish to quantify the minimum amount of common randomness—that is the cardinality $|\mathcal{M}_n|$ or its normalized logarithm $\frac{1}{n}\log|\mathcal{M}_n|$—satisfying this requirement. Of course, we have to quantify what we mean by "close to". In Wyner's original paper, this discrepancy between $P_{X^nY^n}$ and π_{XY}^n was quantified via the *normalized relative entropy*

$$\frac{1}{n} D\left(P_{X^nY^n} \| \pi_{XY}^n\right) = \frac{1}{n} \sum_{x^n, y^n} P_{X^nY^n}(x^n, y^n) \log \frac{P_{X^nY^n}(x^n, y^n)}{\pi_{XY}^n(x^n, y^n)}.$$

Definition 2.1. An (n, R)*-fixed-length distributed source simulation code* consists of a pair of random mappings called *processors* $P_{X^n|M_n} \in \mathcal{P}(\mathcal{X}^n|\mathcal{M}_n)$ and $P_{Y^n|M_n} \in \mathcal{P}(\mathcal{Y}^n|\mathcal{M}_n)$ such that $\log|\mathcal{M}_n| \le nR$.

In the above definition, n and R are known respectively as the *blocklength* and the *rate* of the code $(P_{X^n|M_n}, P_{Y^n|M_n})$. We are now

ready to define Wyner's common information from the distributed source simulation perspective.

Definition 2.2. The *minimal distributed simulation rate* $T(\pi_{XY})$ between a pair of random variables $(X, Y) \sim \pi_{XY}$ is the infimum of all rates R such that there exists a sequence of (n, R)-fixed-length distributed source simulation codes $\{(P_{X^n|M_n}, P_{Y^n|M_n})\}_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} D\left(P_{X^nY^n} \| \pi_{XY}^n\right) = 0, \tag{2.2}$$

where $P_{X^nY^n}$ denotes the synthesized distribution in (2.1).

At this point, the reader may wonder whether the minimal distributed simulation rate $T(\pi_{XY})$ as defined in Definition 2.2 is "sensitive" to the choice of the discrepancy measure—namely, that it is the normalized relative entropy in (2.2). We reassure the reader that this will be discussed extensively in the sequel—as a matter of fact, this is a central theme in Part II of the monograph. Just to provide a sneak peek at the results in the subsequent sections, we mention the $T(\pi_{XY})$ remains unchanged if we choose not to normalize by n in (2.2); this results in a *more stringent* criterion. Furthermore, $T(\pi_{XY})$ also remains the same if the normalized relative entropy is replaced by the TV distance $|P_{X^nY^n} - \pi_{XY}^n|$. More importantly, we discuss the ramifications of changing the discrepancy measure to various members of the family of normalized and unnormalized Rényi divergences; these have implications for other notions of common information such as the exact common information.

One of Wyner's key contributions in his seminal paper on common information [182] is the following.

Theorem 2.1. The minimal distributed simulation rate is given by

$$T(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}) = \min_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} I_P(XY; W). \tag{2.3}$$

Thus, the minimal distributed simulation rate is exactly Wyner's common information as defined in (1.4). We reiterate that the minimization in (2.3) is performed over all triples of random variables (W, X, Y) such that $X - W - Y$ forms a Markov chain in this order and the

marginal distribution of $(X, Y) \sim P_{XY}$ is exactly the target distribution π_{XY}. The use of the min (in place of an inf as in (1.4)) in (2.3) is justified by the fact that the cardinality of W can be restricted to be no more than $|\mathcal{X}||\mathcal{Y}|$. This is a consequence of an application of the convex cover method; see El Gamal and Kim [51, Appendix C] for a detailed discussion. Having established the equivalence between the minimal distributed simulation rate and Wyner's common information, in the following, we will no longer distinguish between these two notions.

Example 2.1. Let us do a sanity check of the expression in (2.3) based on our running example in which $X = (\tilde{X}, V)$ and $Y = (\tilde{Y}, V)$ and $\tilde{X}$, $\tilde{Y}$ and V are mutually independent. By taking $W = V$, we see that $T(\pi_{XY}) \leq H(V)$. On the other hand, we have the Markov chain $V - X - W - Y - V$, so V is a deterministic function of W. As a result,

$$I(XY; W) = I(\tilde{X}\tilde{Y}V; W) = I(\tilde{X}\tilde{Y}V; WV) \geq H(V).$$

Since this holds true for all $X - W - Y$, minimizing the left-hand side over all such joint distributions yields $T(\pi_{XY}) \geq H(V)$ as desired. So indeed, the formula in (2.3) coincides with the intuitive expression for the common information of $X = (\tilde{X}, V)$ and $Y = (\tilde{Y}, V)$, namely $H(V)$.

Although we will not provide detailed proofs in this monograph, we briefly mention the main idea to prove the direct (or achievability) part of Theorem 2.1 as it is a prevailing theme in Part II. This is based on the following lemma, which, in today's information theory parlance, is known as *approximation of output statistics* [73], *channel resolvability* [76], [79], or *soft-covering* [48]. We term any subset $\mathcal{C}_n$ of $\mathcal{W}^n$ as a *codebook*. Any codebook $\mathcal{C}_n$ takes the form $\{w^n(m) : m \in \mathcal{M}_n\}$. The elements of $\mathcal{C}_n$, namely $w^n(m)$, are called *codewords*.

Lemma 2.2 (Soft-Covering)**.** Let $(U, W) \sim P_{UW} \in \mathcal{P}(\mathcal{U} \times \mathcal{W})$ be a given pair of random variables with mutual information $I(U; W)$. For any $R > I(U; W)$, there exists a sequence of codebooks $\{\mathcal{C}_n\}_{n=1}^{\infty}$ with

$$\limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{M}_n| \leq R$$

such that the corresponding sequence of synthesized distributions

$$P_{U^n}(u^n) := \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} P_{U|W}^n(u^n | w^n(m)), \quad n \in \mathbb{N} \tag{2.4}$$

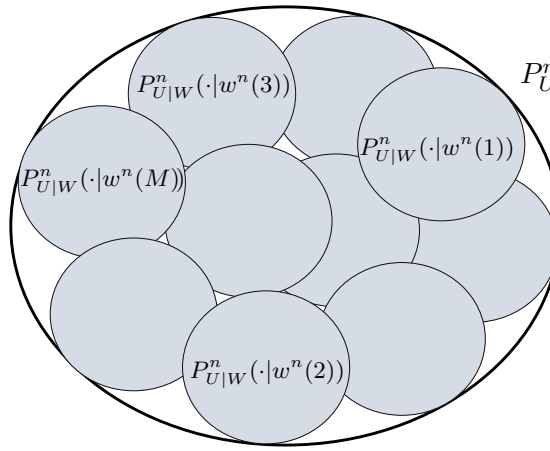


Figure 2.2: Illustration of the soft-covering lemma. If M is large enough (i.e., its exponential rate exceeds $I(U;W)$), the uniform mixture of the conditional distributions $P_{U|W}^n(\cdot|w^n(m))$ approximates the product distribution P_U^n well in the sense of the normalized relative entropy and the total variation distance.

is arbitrarily close in the normalized relative entropy to the product distribution P_U^n, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} D\big(P_{U^n} \big\| P_U^n\big) = 0. \tag{2.5}$$

In addition, the TV distance between P_{U^n} and P_U^n vanishes, i.e.,

$$\lim_{n\to\infty} \big|P_{U^n} - P_U^n\big| = 0. \tag{2.6}$$

We can interpret the soft-covering lemma by considering drawing a codeword w^n from the codebook $\mathcal{C}_n$ uniformly at random. This codeword is then sent through n uses of the *test channel* $P_{U|W}$. Lemma 2.2 says that as long as the cardinality of $\mathcal{C}_n$ is large enough in the sense that its rate exceeds $I(U;W)$, the synthesized distribution P_{U^n} can be made arbitrarily close to P_U^n in the sense of (2.5) or (2.6); see Fig. 2.2. The statement in (2.5) is due to Wyner [182] while that in (2.6) is due to Han and Verdú [73], Hayashi [76] and Cuff [48]. The soft-covering lemma has found numerous applications in information-theoretic security.

The application of the soft-covering lemma to prove the achievability part of Wyner's common information is now apparent. Particularize

$U \sim P_U$ in Lemma 2.2 to be $(X, Y) \sim \pi_{XY}$ and since $X - W - Y$ forms a Markov chain, the synthesized distribution in (2.4) reduces to that in (2.1) by setting for each $u^n = (x^n, y^n)$ and $m \in \mathcal{M}_n$,

$$P_{X^n|M_n}(x^n|m)P_{Y^n|M_n}(y^n|m) = P_{U^n|W^n}(u^n|w^n(m)).$$

The converse is proved via single-letterization steps that are commonplace in network information theory. We omit them here as we will, in Section 2.5.1 and subsequent sections, sketch proofs that yield stronger and more general results, thus recovering the converse of Theorem 2.1 "for free". See [182, Section 5] for the original converse proof.

Remark 2.1. Wyner's common information can be alternatively written as

$$C_{\mathrm{W}}(\pi_{XY}) = H_\pi(X, Y) + \min \Bigg\{ \sum_{x\in\mathcal{X}} \mathbb{E}_W \left[P_{X|W}(x|W) \log P_{X|W}(x|W)\right] + \sum_{y\in\mathcal{Y}} \mathbb{E}_W \left[P_{Y|W}(y|W) \log P_{Y|W}(y|W)\right] \Bigg\}, \tag{2.7}$$

where the minimum extends over all pairs of collections of random variables $\{P_{X|W}(x|W)\}_{x\in\mathcal{X}}$ and $\{P_{Y|W}(y|W)\}_{y\in\mathcal{Y}}$ satisfying

$$\begin{aligned} P_{X|W}(x|W), P_{Y|W}(y|W) &\geq 0 && \forall\, (x, y) \in \mathcal{X} \times \mathcal{Y} \\ \sum_{x\in\mathcal{X}} P_{X|W}(x|W) = \sum_{y\in\mathcal{Y}} P_{Y|W}(y|W) &= 1 && \text{and} \\ \mathbb{E}_W \left[P_{X|W}(x|W)P_{Y|W}(y|W)\right] &= \pi_{XY}(x, y) && \forall\, (x, y) \in \mathcal{X} \times \mathcal{Y}. \end{aligned} \tag{2.8}$$

In [182, Eqn. (1.16)], Wyner claims that $C_{\mathrm{W}}(\pi_{XY})$ can be expressed as a max-min of a difference of relative entropies. However, the authors have disproved this claim numerically. The problem with Wyner's argument is that one cannot swap the min and max operations because the Lagrangian corresponding to the minimization in (2.7) and constraints in (2.8) is *bilinear* in $\{P_{X|W}(x|W)\}_x$ and $\{P_{Y|W}(y|W)\}_y$ and not (jointly) linear in them.

2.2 The Gray–Wyner System

In addition to the Wyner's common information being interpreted as the minimum rate required to simulate a joint source π_{XY} in a distributed

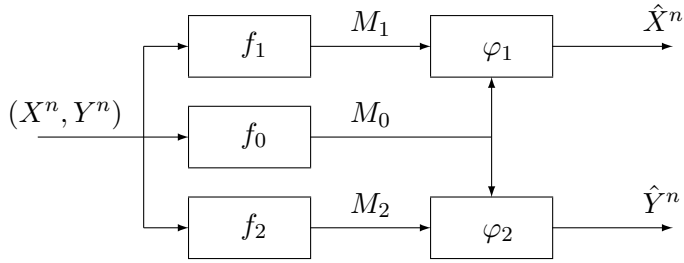


Figure 2.3: The Gray–Wyner source coding problem [68].

manner, there is another natural interpretation in terms of a distributed lossless source coding system—the Gray–Wyner system [68] as depicted in Fig. 2.3. In this problem, there is a joint source $(X, Y) \sim \pi_{XY}$ that is to be reconstructed almost losslessly. This joint source is encoded into three bit strings (M_0, M_1, M_2) of rates (R_0, R_1, R_2) via three encoders that observe n independent copies of (X, Y). There are two decoders. Bit strings M_0 and M_1 are sent to the first decoder, while bit strings M_0 and M_2 are sent to the second decoder. The two decoders generate estimates $\hat{X}^n$ and $\hat{Y}^n$ of X^n and Y^n respectively.

In distributed lossless source coding problems, one is concerned with the tradeoff among the rates; in this case, (R_0, R_1, R_2). If the three encoders are combined into a single entity—equivalently, the common rate is allowed to be arbitrarily large—by Shannon's lossless source coding theorem, we can describe the joint source using roughly $nH(XY)$ bits, or at a rate of $H(XY)$. Clearly, we can do more to reduce the common rate. Using our running example in which $X = (\tilde{X}, V)$ and $Y = (\tilde{Y}, V)$ with $\tilde{X}, \tilde{Y}$, and V being independent, any coding scheme involves compressing the common part of X and Y using encoder f_0. For lossless reconstruction, this requires a rate of roughly $H(V)$. The other encoders f_1 and f_2 are tasked with compressing the private parts of the sources, namely $\tilde{X}$ and $\tilde{Y}$ respectively. These require rates of roughly $H(\tilde{X})$ and $H(\tilde{Y})$. Reconstruction of the sources by the decoders is clearly possible. For example, φ_1 takes the descriptions (M_0, M_1) and reconstructs V^n and $\tilde{X}^n$, which when concatenated, is approximately X^n. Thus, the required sum rate is $H(V) + H(\tilde{X}) + H(\tilde{Y}) = H(XY)$. Motivated by this special case, it seems natural to alternatively define the

common information of the any source $(X, Y) \sim \pi_{XY}$ as the minimum common rate R_0 such that the sum rate $R_0 + R_1 + R_2$ is no larger than the joint entropy $H(XY)$. The set of all (R_0, R_1, R_2) such that $R_0 + R_1 + R_2 = H(XY)$ is known as the *Pangloss plane* of the source. The term "Pangloss plane" was coined by Gray and Wyner [68].

Definition 2.3. An (n, R_0, R_1, R_2)-*Gray–Wyner code* consists of

- Three encoders $f_i : \mathcal{X}^n \times \mathcal{Y}^n \to [2^{nR_i}]$ where $i = 0, 1, 2$;
- Two decoders $\varphi_1 : [2^{nR_0}] \times [2^{nR_1}] \to \mathcal{X}^n$ and $\varphi_2 : [2^{nR_0}] \times [2^{nR_2}] \to \mathcal{Y}^n$.

The *probability of error* of the code is

$$\Pr\left((\varphi_1(M_0, M_1), \varphi_2(M_0, M_2)) \neq (X^n, Y^n)\right), \tag{2.9}$$

where $M_i = f_i(X^n, Y^n)$ for $i = 0, 1, 2$.

Definition 2.4. The *Pangloss-common information based on the Gray–Wyner system* $T_{\mathrm{GW}}(\pi_{XY})$ between two random variables $(X, Y) \sim \pi_{XY}$ is the infimum of all R_0 such that for all $\epsilon > 0$, there exists a sequence of (n, R_0, R_1, R_2)-Gray–Wyner codes $\{(f_{0,n}, f_{1,n}, f_{2,n}, \varphi_{1,n}, \varphi_{2,n})\}_{n=1}^{\infty}$ such that $R_0 + R_1 + R_2 \leq H(XY) + \epsilon$ for all n sufficiently large and the probability of error in (2.9) vanishes as the length of the code n tends to infinity.

The term "Pangloss" is used in the above definition to emphasize that sum rate should be close $H(XY)$; this is to distinguish this definition from an analogous one for the GKW common information (Definition 3.4). We also adopt the somewhat verbose qualifier "based on the Gray–Wyner system" and the subscript GW in $T_{\mathrm{GW}}(\pi_{XY})$ because *a priori*, there is little evidence to suggest that $T_{\mathrm{GW}}(\pi_{XY})$ equals to the quantity in Definition 2.2. The qualifier can, however, be jettisoned in view of the following theorem also due to Wyner [182].

Theorem 2.3. The Pangloss-common information based on the Gray–Wyner system

$$T_{\mathrm{GW}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}). \tag{2.10}$$

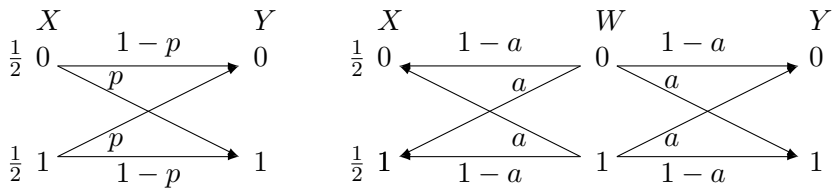


Figure 2.4: Left: DSBS with crossover probability p. Right: Interpretation in terms of the common random variable W.

Thus, both definitions of the common information (in Definitions 2.2 and 2.4) coincide and we can use a single symbol $C_{\mathrm{W}}(\pi_{XY})$ to name the quantity on the right-hand side of (2.10). The subscript W refers to Wyner. The quantity $C_{\mathrm{W}}(\pi_{XY})$ thus has two operational interpretations; one as the minimum rate required to simulate a joint source in a distributed manner and another as the minimum common rate of the Gray–Wyner system keeping the sum rate at the joint entropy of π_{XY}.

2.3 Doubly Symmetric Binary Sources

Due to the optimization over the Markov chain $X - W - Y$, Wyner's common information is difficult to evaluate for most pairs of sources π_{XY}. Two notable exceptions are the doubly symmetric binary source (DSBS) and the symmetric binary erasure source (SBES). We describe the former in this section and the latter in the next.

Consider a DSBS $(X, Y) \in \{0,1\}^2$ which is defined by the joint distribution

$$\pi_{XY} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}, \tag{2.11}$$

where $\alpha = (1-p)/2$, $\beta = p/2$ and $p \in (0, 1/2)$. This is equivalent to $X \sim \mathrm{Bern}(1/2)$ and $Y = X \oplus E$ with $E \sim \mathrm{Bern}(p)$ and independent of X. Here, p represents the crossover probability of a binary symmetric channel (BSC) with X being the input and Y the output. Intuitively, if $p \downarrow 0$, X and Y become highly correlated, and the common information increases. On the other hand, if $p \uparrow 1/2$, X and Y become close to independent and the common information decreases to 0.

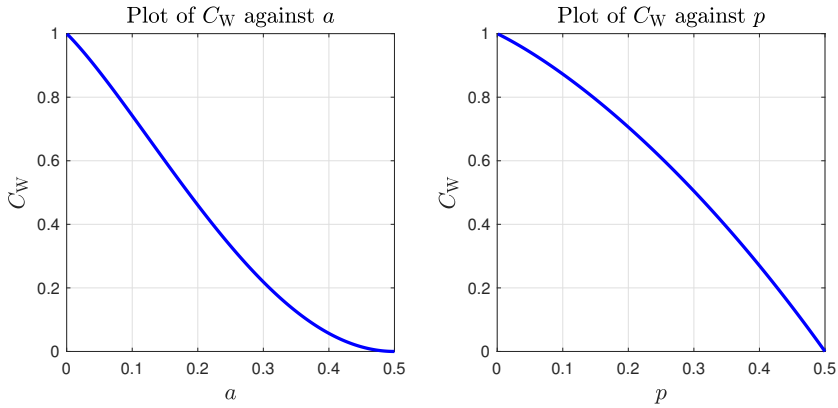


Figure 2.5: Plots of Wyner's common information for the DSBS in terms of p and a

Equivalently, $X = W \oplus A$ and $Y = W \oplus B$ with $W \sim \mathrm{Bern}(1/2)$, $A, B \sim \mathrm{Bern}(a)$ mutually independent with $a := (1 - \sqrt{1-2p})/2 \in (0, 1/2)$ so $a * a = p$. Thus, similarly to p, as a increases, the common information decreases. We can express α and β in (2.11) in terms of a as $\alpha = \frac{1}{2}(a^2 + (1-a)^2)$ and $\beta = a(1-a)$. In this parametrization, W is the common random variable that achieves the minimum in the formula for Wyner's common information in (2.3). The two interpretations of the DSBS are illustrated in Fig. 2.4. Clearly, there is no loss in generality in restricting p (or a) to be in $(0, 1/2)$; if not, replace X by $X \oplus 1$.

Wyner [182] successfully evaluated the common information for the DSBS in closed form.

Proposition 2.1. For the DSBS as described in (2.11), Wyner's common information is

$$C_{\mathrm{W}}(\pi_{XY}) = 1 + h(2a\bar{a}) - 2h(a)$$

where $h(a) := -a \log a - \bar{a} \log \bar{a}$ is the binary entropy function.

This function is plotted in Fig. 2.5 and shows clearly that $C_{\mathrm{W}}(\pi_{XY})$ for a DSBS is decreasing in p and a. We conclude this section by mentioning that Witsenhausen [177] calculated $C_{\mathrm{W}}(\pi_{XY})$ for a variety of other discrete sources.

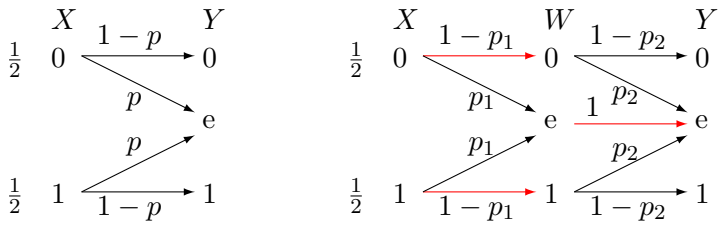


Figure 2.6: Left: SBES with erasure probability p. Right: Interpretation in terms of the common random variable W due to Cuff [48].

2.4 Symmetric Binary Erasure Sources

The SBES is a joint source π_{XY} with binary input $\mathcal{X} = \{0, 1\}$ and ternary output $\mathcal{Y} = \{0, \mathrm{e}, 1\}$. The "output" Y is identical to the "input" X with probability $1 - p$ and takes on the "erasure symbol" e with probability $1 - p$. The input variable is uniformly distributed on $\mathcal{X}$, leading to the joint distribution

$$\pi_{XY} = \begin{bmatrix} (1-p)/2 & p/2 & 0 \\ 0 & p/2 & (1-p)/2 \end{bmatrix}. \tag{2.12}$$

This is illustrated in the left diagram of Fig. 2.6. Cuff [48] proved the following proposition.

Proposition 2.2. For the SBES as described in (2.12), Wyner's common information is

$$C_{\mathrm{W}}(\pi_{XY}) = \begin{cases} 1 & p \leq 0.5 \\ h(p) & p > 0.5 \end{cases}. \tag{2.13}$$

The optimal distribution $P_W P_{X|W} P_{Y|W}$ in Wyner's common information for the SBES is shown in the right diagram of Fig. 2.6 where X is uniform on $\mathcal{X}$ and p_1 and p_2 satisfy $(1 - p_1)(1 - p_2) = 1 - p$. Hence, the channel from X to Y is a concatenation of a binary erasure channel (BEC) with erasure probability p_1 and a BEC-like channel with three inputs $0, \mathrm{e}$, and 1 in which, restricted to the inputs in $\{0, 1\}$, it is a BEC with erasure probability p_2 but e is transmitted noiselessly. Wyner's common information for an SBES is plotted in Fig. 2.7.

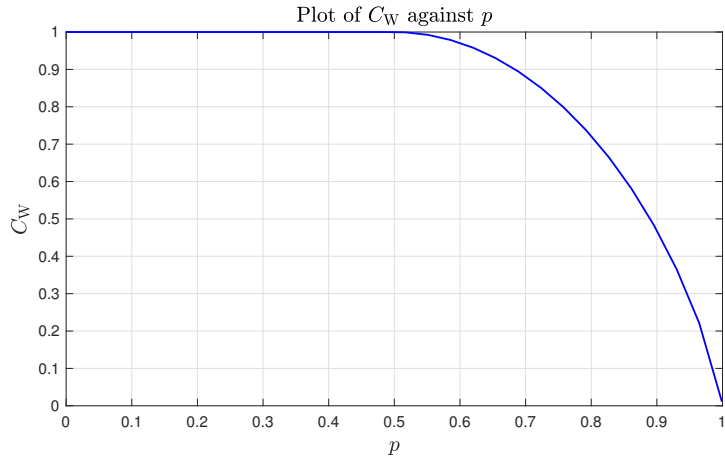


Figure 2.7: Plot of Wyner's common information for the SBES in terms of the erasure probability p

2.5 Continuous and Gaussian Sources

Even though the expression for Wyner's common information in (2.3) remains valid for arbitrary random variables when the min replaced by an inf, i.e.,

$$C_{\mathrm{W}}(\pi_{XY}) = \inf_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} I_P(XY; W), \tag{2.14}$$

the operational stories for Wyner's common information for continuous random variables are more intricate. Indeed, the operational interpretation in terms of minimum common rate in the Gray–Wyner system (fixing the sum rate to be $H(XY)$) is only applicable to discrete random variables. An operational interpretation for continuous random variables in terms of the *lossy* Gray–Wyner system [173] was discovered by Xu, Liu, and Chen [186]. The interpretation in terms of distributed source simulation remains valid, though the result is more subtle [107], [204]. In this section, we first generalize Wyner's common information in this direction then discuss generalizations of the Gray–Wyner system to be amenable to continuous sources. Finally, we justify why these interpretations yield the same result for jointly Gaussian sources.

2.5.1 Distributed Source Simulation

In his seminal paper, Wyner [182] characterized the common information for *finite alphabet* sources from the perspective of distributed source simulation. Here, we extend his results to arbitrary and, in particular, continuous sources in the context of the distributed source simulation problem. The operational quantity $T(\pi_{XY})$ in the following theorem pertains to that in Definition 2.2 (for distributed source simulation).

Theorem 2.4. Let $(X, Y) \sim \pi_{XY}$ be a joint source with distribution defined on the product of two arbitrary alphabets. Then we have[1]

$$\tilde{C}_{\mathrm{W}}(\pi_{XY}) \leq T(\pi_{XY}) \leq \hat{C}_{\mathrm{W}}(\pi_{XY}) \tag{2.15}$$

where

$$\tilde{C}_{\mathrm{W}}(\pi_{XY}) := \lim_{\epsilon \downarrow 0} \inf_{\substack{P_W P_{X|W} P_{Y|W}: \\ D(P_{XY} \| \pi_{XY}) \leq \epsilon}} I(XY; W) \quad \text{and} \tag{2.16}$$

$$\hat{C}_{\mathrm{W}}(\pi_{XY}) := \inf_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY} = \pi_{XY}}} \lim_{s \downarrow 0} D_{1+s}(P_{X|W} P_{Y|W} \| P_{XY} | P_W). \tag{2.17}$$

This result is due to the present authors [204]. An alternative upper bound on the Wyner's common information of a set of continuous random variables in terms of the dual total correlation between them was derived by Li and El Gamal [107]. We remark that when the joint source π_{XY} is finitely supported, both $\tilde{C}_{\mathrm{W}}(\pi_{XY})$ and $\hat{C}_{\mathrm{W}}(\pi_{XY})$ reduce to Wyner's common information $C_{\mathrm{W}}(\pi_{XY})$ as defined in (2.3). In particular, we recall that as $s \downarrow 0$, the conditional Rényi divergence $D_{1+s}(P_{X|W} P_{Y|W} \| P_{XY} | P_W)$ reduces to the conditional relative entropy $D(P_{X|W} P_{Y|W} \| P_{XY} | P_W)$ which in turn equals the mutual information $I(XY; W)$.

[1] Since we consider arbitrary probability spaces here, to be formal, we need to generalize several notions in probability theory, e.g., conditional distributions and conditional independence. We use $P_{X|W}$ to denote a *regular conditional probability distribution* [37]. Random variables X and Y, defined on an arbitrary measurable space, are *conditionally independent* given W, denoted as $X - W - Y$, if $\sigma(X)$ and $\sigma(Y)$ are conditionally independent of $\sigma(W)$, where $\sigma(X)$ denotes the σ-algebra generated by X [37]. When the regular conditional $P_{XY|W}$ exists, it holds that $X - W - Y$ if and only if $P_{XY|W=w}$ is a product distribution (see Definition 4.3(a) for the countable alphabet case) for P_W-almost every $w \in \mathcal{W}$.

We highlight some key ideas of the proof. For the achievability part, we leverage a one-shot (non-asymptotic) soft-covering lemma that can be thought of as a strengthened version of Lemma 2.2. This result first appeared in the work of the present authors [200] en route to proving generalized security theorems for the wiretap channel [44], [183].

Lemma 2.5 (One-Shot Soft-Covering)**.** Let $(U, W) \sim P_{UW} \in \mathcal{P}(\mathcal{U} \times \mathcal{W})$ be a given pair of random variables defined on some arbitrary measurable space. Consider a random codebook $\mathscr{C} = \{W(m) : m \in \mathcal{M}\}$ where $|\mathcal{M}| = 2^{\lfloor R \rfloor}$ for some $R > 0$. For each realization of the codebook $\mathcal{C} = \{w(m) : m \in \mathcal{M}\}$, define the synthesized distribution

$$P_{U|\mathscr{C}}(u|\mathcal{C}) := \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} P_{U|W}(u|w(m)).$$

Let π_U be a target distribution such that for some $s \in (0, 1]$, both $D_{1+s}(P_{U|W} \| \pi_U | P_W)$ and $D_{1+s}(P_U \| \pi_U)$ exist (and hence are finite). Then for any $s \in (0, 1]$, we have

$$\begin{aligned} &\exp\left(sD_{1+s}(P_{U|\mathscr{C}} \| \pi_U | P_{\mathscr{C}})\right) \\ &\quad \le \exp\left(sD_{1+s}(P_{U|W} \| \pi_U | P_W) - sR\right) + \exp\left(sD_{1+s}(P_U \| \pi_U)\right). \end{aligned}$$

By setting $\pi_U \leftarrow \pi_{XY}^n$, $P_{U|W} \leftarrow P_{X|W}^n P_{Y|W}^n$, $P_W \leftarrow P_W^n$ and $R \leftarrow nR$, for some distribution $P_W P_{X|W} P_{Y|W}$ such that its marginal on (X, Y) equals π_{XY}, Lemma 2.5 tells us that if

$$R > D_{1+s}(P_{X|W} P_{Y|W} \| \pi_{XY} | P_W),$$

then $D_{1+s}(P_{X^n Y^n | \mathscr{C}_n} \| \pi_{XY}^n | P_{\mathscr{C}_n}) \to 0$. Thus, we conclude that there exists (at least) one sequence of (deterministic) codebooks $\{\mathcal{C}_n\}_{n=1}^{\infty}$ such that $D(P_{X^n Y^n | \mathscr{C}_n}(\cdot|\mathcal{C}_n) \| \pi_{XY}^n) \le D_{1+s}(P_{X^n Y^n | \mathscr{C}_n}(\cdot|\mathcal{C}_n) \| \pi_{XY}^n) \to 0$. Letting s tend to 0 (from above) and minimizing over all $P_W P_{X|W} P_{Y|W}$ concludes the proof of the achievability part.

Remark 2.2. The reader will observe that what we have proved is stronger than what Definition 2.2 demands of a common information code. The one-shot soft-covering lemma as stated in Lemma 2.5 is strong enough to drive the *unnormalized* relative entropy $D(P_{X^n Y^n} \| \pi_{XY}^n)$ to zero as $n \to \infty$. Compare this to (2.2) in which the *normalized* relative entropy $\frac{1}{n} D(P_{X^n Y^n} \| \pi_{XY}^n)$ is required to vanish. This strengthening will be central to our discussion in Part II.

The converse follows from standard single-letterization steps that we outline here. Fix any code $(P_{X^n|M_n}, P_{Y^n|M_n})$ per Definition 2.1. Observe that

$$\begin{aligned} R &\geq \frac{1}{n} H(M_n) \geq \frac{1}{n} I(X^n Y^n; M_n) \\ &= \frac{1}{n} D(P_{X^n Y^n M_n} \| P_{X^n Y^n} P_{M_m}) \\ &= \frac{1}{n} D(P_{X^n Y^n M_n} \| \pi_{X^n Y^n} P_{M_m}) - \frac{1}{n} D(P_{X^n Y^n} \| \pi_{XY}^n). \end{aligned} \tag{2.18}$$

The first term can be further lower bounded as

$$\begin{aligned} &\frac{1}{n} D(P_{X^n Y^n M_n} \| \pi_{X^n Y^n} P_{M_m}) \\ &= \frac{1}{n} \sum_{i=1}^{n} D(P_{X_i Y_i | M_n X^{i-1} Y^{i-1}} \| \pi_{XY} | P_{M_n X^{i-1} Y^{i-1}}) \qquad (2.19) \\ &\geq \frac{1}{n} \sum_{i=1}^{n} D(P_{X_i Y_i | M_n} \| \pi_{XY} | P_{M_n}) \qquad (2.20) \\ &= D(P_{X_J Y_J | M_n J} \| \pi_{XY} | P_{M_n J}) = D(P_{XY|W} \| \pi_{XY} | P_W), \qquad (2.21) \end{aligned}$$

where (2.19) follows from the chain rule for relative entropy, (2.20) follows from the convexity of the relative entropy, and (2.21) follows from introducing $J \sim \mathrm{Unif}[n]$ independent of (M_n, X^n, Y^n) and by setting $X := X_J$, $Y := Y_J$ and $W := (M_n, J)$. These identifications of the random variables satisfy the Markovity condition $X - W - Y$. Using similar steps, we can show that $D(P_{XY} \| \pi_{XY}) \leq \frac{1}{n} D(P_{X^n Y^n} \| \pi_{XY}^n)$. Since the code requires that the final term in (2.18) to vanish, $D(P_{XY} \| \pi_{XY})$ also vanishes. This establishes the bound $D(P_{XY} \| \pi_{XY}) \leq \epsilon$ for any $\epsilon > 0$ and any $X - W - Y$ satisfying $P_{XY} = \pi_{XY}$. Taking $\epsilon \downarrow 0$ completes the proof of the converse part of Theorem 2.4.

It is natural to wonder when $C_{\mathrm{W}}(\pi_{XY})$, $\tilde{C}_{\mathrm{W}}(\pi_{XY})$ and $\hat{C}_{\mathrm{W}}(\pi_{XY})$, as defined in (2.14), (2.16), and (2.17) respectively coincide, beyond the case in which π_{XY} is finitely supported. This is partially addressed in the following proposition due to the present authors [204].

Proposition 2.3. The following hold:

- If there exists a joint distribution $P_W P_{X|W} P_{Y|W}$ that attains $C_{\mathrm{W}}(\pi_{XY})$ and satisfies $D_{1+s}(P_{X|W} P_{Y|W} \| P_{XY} | P_W) < \infty$ for some $s > 0$, then $C_{\mathrm{W}}(\pi_{XY}) = \hat{C}_{\mathrm{W}}(\pi_{XY})$.

- Assume that π_{XY} is an absolutely continuous distribution on $\mathbb{R}^2$ with PDF f_{XY} such that $C_{\mathrm{W}}(\pi_{XY}) = \hat{C}_{\mathrm{W}}(\pi_{XY})$ (e.g., based on the sufficient condition in the point above), f_{XY} is log-concave,[2] and $I(X;Y) < \infty$. For each $d > 0$, define the constant

$$\kappa_d := \sup_{(x,y)\in[-d,d]^2} \left| \frac{\partial}{\partial x} \log f_{XY}(x,y) \right| + \left| \frac{\partial}{\partial y} \log f_{XY}(x,y) \right|$$

and $\epsilon_d := 1 - \pi_{XY}([-d,d]^2)$. If $\epsilon_d \log(d\kappa_d) \to 0$ as $d \to \infty$, then all inequalities in (2.15) become equalities.

It holds that jointly Gaussian sources satisfy both regularity conditions in Proposition 2.3. This will be discussed in detail in Section 2.5.3.

2.5.2 Lossy Gray–Wyner System

An operational interpretation for continuous random variables in terms of the *lossy* Gray–Wyner system [173] was discovered by Xu, Liu, and Chen [186]. Recall that in the Gray–Wyner problem, one seeks to reconstruct a pair of sources losslessly. Obviously, this is only meaningful if the sources are discrete otherwise they cannot be reliably reconstructed with probability one for all finite rates. However, if one allows for the sources to be reconstructed to within some distortion levels, then it is meaningful to discuss the tradeoff between the rates (R_0, R_1, R_2) and allowable distortions. To this end, we introduce two per-letter distortion measures $d_1 : \mathcal{X} \times \hat{\mathcal{X}} \to [0,\infty)$ and $d_2 : \mathcal{Y} \times \hat{\mathcal{Y}} \to [0,\infty)$ that operate on length-n sequences as follows: $d_1(x^n, \hat{x}^n) = \frac{1}{n}\sum_{i=1}^n d_1(x_i, \hat{x}_i)$ and similarly for d_2. Instead of demanding that the probability of error in (2.9) vanishes, in the lossy case, we only require the reconstructions $(\hat{X}^n, \hat{Y}^n)$ in Fig. 2.3 to satisfy

$$\limsup_{n\to\infty} \mathbb{E}\big[d_1(X^n, \hat{X}^n)\big] \le \Delta_1 \quad \text{and} \quad \limsup_{n\to\infty} \mathbb{E}\big[d_2(Y^n, \hat{Y}^n)\big] \le \Delta_2 \tag{2.22}$$

for some permissible distortions Δ_1 and Δ_2. This is known as the *lossy* Gray–Wyner system [173]. Similarly to Definition 2.4, we define the (Δ_1, Δ_2)-*Pangloss-common information based on the lossy Gray–Wyner system* $T_{\mathrm{GW}}(\pi_{XY}; \Delta_1, \Delta_2)$ to be the infimum of all common rates R_0

[2] This means that $\log f_{XY}$ is concave on $\mathbb{R}^2$.

such that for each $\epsilon > 0$, there exists a sequence of Gray–Wyner codes satisfying the distortion constraints in (2.22) and

$$R_0 + R_1 + R_2 \leq R_{XY}(\Delta_1, \Delta_2) + \epsilon, \tag{2.23}$$

for all sufficiently large n, where the *joint rate-distortion function* is defined as

$$R_{XY}(\Delta_1, \Delta_2) := \inf_{P_{\hat{X}\hat{Y}|XY}: \mathbb{E}[d_1(X,\hat{X})] \leq \Delta_1,\, \mathbb{E}[d_2(Y,\hat{Y})] \leq \Delta_2} I(XY; \hat{X}\hat{Y}).$$

The *Pangloss plane* in this lossy case is given by the set of (R_0, R_1, R_2) such that (2.23) holds with equality. The quantity $T_{\mathrm{GW}}(\pi_{XY}; \Delta_1, \Delta_2)$, in general, depends on (Δ_1, Δ_2). However, Xu, Liu, and Chen [186, Theorem 5] showed that in certain non-degenerate cases, this dependence vanishes.

Theorem 2.6. Let $P_W P_{X|W} P_{Y|W}$ be any distribution that achieves the infimum in the optimization problem in (2.14). Let the reproduction alphabets $\hat{\mathcal{X}} = \mathcal{X}$ and $\hat{\mathcal{Y}} = \mathcal{Y}$ and the two distortion measures d_1 and d_2 satisfy $d_1(x, \hat{x}) > d_1(x, x) = 0$ for all $x \neq \hat{x}$ and $d_2(y, \hat{y}) > d_2(y, y) = 0$ for all $y \neq \hat{y}$. If the following conditions are satisfied

- For any $w \in \mathcal{W}$, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $P_{W|XY}(w|x, y) > 0$,
- There exists $\hat{x} \in \mathcal{X}$ and $\hat{y} \in \mathcal{Y}$ such that

$$\mathbb{E}[d_1(X, \hat{x})] < \infty \quad \text{and} \quad \mathbb{E}[d_2(Y, \hat{y})] < \infty.$$

Then there exists a positive constant γ such that for all $0 \leq \Delta_1, \Delta_2 \leq \gamma$,

$$T_{\mathrm{GW}}(\pi_{XY}; \Delta_1, \Delta_2) = C_{\mathrm{W}}(\pi_{XY}). \tag{2.24}$$

In other words, under relatively mild conditions, for sufficiently small distortion levels, $T_{\mathrm{GW}}(\pi_{XY}; \Delta_1, \Delta_2)$ does not depend on (Δ_1, Δ_2) and additionally, there admits an operational interpretation of the expression on the right-hand side of (2.24), i.e., it is the minimum common rate of the lossy Gray–Wyner system for small distortion levels. Moreover, if the regularity conditions of Proposition 2.3 also hold, then the two operational definitions for the common information for

continuous sources (as presented in Sections 2.5.1 and 2.5.2) coincide. This dovetails nicely with the discrete case.

From now on, we assume that $0 \leq \Delta_1, \Delta_2 \leq \gamma$ so it is permissible to write $T_{\mathrm{GW}}(\pi_{XY}; \Delta_1, \Delta_2)$ interchangeably as $T(\pi_{XY})$ or $C_{\mathrm{W}}(\pi_{XY})$.

2.5.3 Jointly Gaussian Sources

In this section, we consider jointly Gaussian sources. Our discussions up until this point inform us that there are two ways of computing Wyner's common information for such sources. In particular, Xu, Liu, and Chen [186] and Yu and Tan [204] used Theorem 2.6 and Proposition 2.3 respectively to compute $T(\pi_{XY})$ for a jointly Gaussian source.

Let $(X, Y) \sim \pi_{XY}$ be a pair of jointly Gaussian random variables with covariance matrix given by

$$\mathbf{K} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

The constant $\rho \in (-1, 1)$ is known as the *correlation coefficient* of X and Y. Without loss of generality, it suffices for us to consider $\rho \in [0, 1)$. Otherwise, we can replace X by $-X$ and the results go through *mutatis mutandis* with ρ replaced by $-\rho$.[3] We expect that as $\rho \downarrow 0$, the common information $C_{\mathrm{W}}(\pi_{XY})$ should tend to 0 as X and Y tend towards being independent. On the other hand as $\rho \uparrow 1$, $C_{\mathrm{W}}(\pi_{XY})$ should increase as X and Y tend towards being completely dependent. The following proposition is due to Xu, Liu, and Chen [186] and Yu and Tan [204].

Proposition 2.4. For a jointly Gaussian source with correlation coefficient $\rho \in [0, 1)$, Wyner's common information is

$$T(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}) = \frac{1}{2} \log \left(\frac{1+\rho}{1-\rho} \right). \tag{2.25}$$

This function is plotted in Fig. 2.8 and confirms our intuition about the limiting cases $\rho \downarrow 0$ and $\rho \uparrow 1$. Note that for continuous random variables, Wyner's common information $C_{\mathrm{W}}(\pi_{XY})$ can increase

[3] Equivalently, if we do not make the assumption that $\rho \in [0, 1)$, the results for Gaussian sources here and in the following would hold with ρ replaced by $|\rho|$.

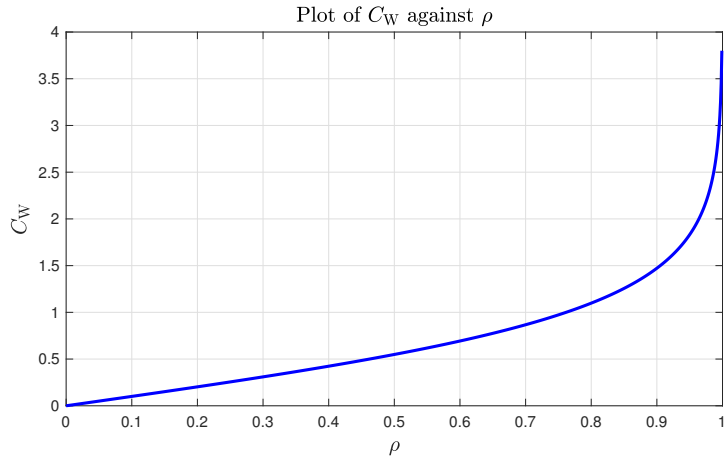


Figure 2.8: Plot of Wyner's common information for the jointly Gaussian source

without bound but for discrete random variables $C_{\mathrm{W}}(\pi_{XY})$ is clearly overbounded (by $\log(|\mathcal{X}||\mathcal{Y}|)$).

The test channels that achieve the infimum in (2.14) for jointly Gaussian sources are also Gaussian. Indeed, the optimum $P_W P_{X|W} P_{Y|W}$ takes the form

$$X = \sqrt{\rho}\, W + \sqrt{1-\rho}\, N_1 \quad \text{and} \quad Y = \sqrt{\rho}\, W + \sqrt{1-\rho}\, N_2,$$

where W, N_1 and N_2 are independent standard Gaussian random variables. This does not come as a surprise in view of the optimum common random variable and test channels for the DSBS; see Fig. 2.4. Note that with this choice of test channels, and $0 \le s \le \sqrt{\frac{1+\rho}{2\rho}}$,

$$D_{1+s}(P_{X|W} P_{Y|W} \| P_{XY} | P_W) = \frac{1}{2} \log \frac{1+\rho}{1-\rho} - \frac{1}{2s} \log \left(1 - \frac{2s^2 \rho}{1+\rho}\right).$$

Hence, the first condition of Proposition 2.3 is satisfied. It is also easy to verify by straightforward, albeit tedious, calculus that the second condition is satisfied, so all inequalities in (2.15) are equalities.

2.6 Generalizations and Applications

We conclude this section by briefly mentioning some extensions of Wyner's common information and its applications that we do not discuss

further in the monograph. This list is by no means exhaustive and serves as a teaser for the reader to explore the many generalizations of this useful quantity.

Liu, Xu, and Chen [114] extended Wyner's common information for two random variables to a quantity representing the common information among N random variables, namely,

$$C_{\mathrm{W}}(\pi_{X_1X_2\ldots X_N}) = \min I(X_1X_2\ldots X_N; W), \tag{2.26}$$

where the minimum is over all joint distributions $P_W \prod_{i=1}^{N} P_{X_i|W}$ such that the marginal $P_{X_1\ldots X_N}$ equals the target distribution $\pi_{X_1\ldots X_N}$. This has the same operational interpretation in terms of distributed simulation of random variables and the Gray–Wyner network with N decoders and $N+1$ encoders. Cuff [48] considered a distributed channel synthesis problem and showed that in the absence of any shared common randomness between the encoder and decoder, the minimum rate required to synthesize a channel is exactly Wyner's common information. At the other extreme, if the amount of shared common randomness is sufficiently large, the rate required is the mutual information. We revisit the channel synthesis problem in Section 6. Recently, motivated by problems in caching, Gastpar and Suha [62] found an operational interpretation of the following *relaxed* version of Wyner's common information

$$C_{\mathrm{W}}^{(\delta)}(\pi_{XY}) = \min_{P_{WXY}: P_{XY}=\pi_{XY}, I(X;Y|W)\leq\delta} I(XY; W), \tag{2.27}$$

which is parametrized by $\delta \geq 0$. Notice that if $\delta = 0$, this quantity particularizes to the usual Wyner's common information as the constraint $I(X;Y|W) \leq \delta$ reduces to the Markovity constraint $X - W - Y$. In another recent work, Graczyk, Lapidoth, and Wigger [67] defined a conditional version of Wyner's common information

$$C_{\mathrm{W}}(\pi_{XY|Z}|\pi_Z) = \min_{P_{WZ}P_{X|WZ}P_{Y|WZ}: P_{XYZ}=\pi_{XYZ}} I(XY; W|Z),$$

which has obvious operational interpretations in terms of the distributed source simulation and Gray–Wyner problems when the terminals have access to correlated side-information $Z^n \sim \pi_Z^n$. The same authors also studied a quantity known as the *relevant common information*.

$$C_{\mathrm{Rel}}(\pi_{XY|S} \to \pi_S) = \min_{\substack{P_{WXYS}: P_{XYS}=\pi_{XYS},\\ X-W-Y,\, S-(X,Y)-W}} I(S; W),$$

where the minimization is over all tuples of random variables (X, Y, S, W) such that the marginal of (X, Y, S) matches the given $\pi_{XY|S}\pi_S$, $X - W - Y$ and $S - (X, Y) - W$. As can be seen from the two Markov chains, $C_{\mathrm{Rel}}(\pi_{XY|S} \to \pi_S)$ represents the common information in (X, Y) that is *relevant* to a correlated random variable S. It has the interesting operational interpretation as the rate of the common randomness required at two terminals to—through their inputs—strongly coordinate the output of a two-user multiple-access channel (MAC) according to a target distribution π_S.

Tyagi [165] introduced the notion of *r-interactive common information*, which is a variant of Wyner's common information. This quantity characterizes the minimum overall rate of interactive communication required to generate a maximum rate secret key in an interactive manner between two parties.

Extending the seminal work of Maddah-Ali and Niesen [115] on the information-theoretic limits of caching, Wang, Lim, and Gastpar [174] formulated another caching problem from an information-theoretic perspective in which users' requests change over time. They cast the problem as a multi-terminal lossless source coding problem with side-information. For the N-user scenario, Wang, Lim, and Gastpar [174] showed that the optimal caching strategy is closely related to $C_{\mathrm{W}}(\pi_{X_1X_2\ldots X_N})$ in (2.26), which represents Wyner's common information for N dependent random variables.

3

Gács–Körner–Witsenhausen's Common Information

As mentioned at the start of Section 2, there are two well-known notions of common information, the first of which—Wyner's common information—has already been discussed in detail in Section 2. In this section, we introduce the other classical notion of common information, namely, Gács–Körner–Witsenhausen's common information. Recall that in the definition of Wyner's common information, a *common* or *shared source of randomness* M_n is used to generate a pair of random vectors X^n and Y^n in a distributed manner such that the joint distribution of (X^n, Y^n) is close to a target product distribution π_{XY}^n. We now consider a counterpart of this problem, illustrated in Fig. 3.1, in which a pair of random vectors $(X^n, Y^n) \sim \pi_{XY}^n$ is given, random variables $U = f(X^n)$ and $V = g(Y^n)$ are to be extracted from X^n and Y^n individually using functions f_n and g_n, and these random variables, called *common randomnesses*, should be *almost identical*. This setting was first considered by Gács and Körner in their celebrated paper [60] in which they defined the common information between X and Y, jointly distributed as π_{XY}, as the maximum information rate of the common randomness U or, equivalently, V. This notion of common information was later coined *Gács–Körner–Witsenhausen's* or *GKW's*

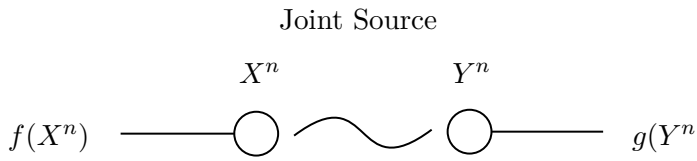


Figure 3.1: The distributed randomness extraction problem

common information. In fact, Gács and Körner [60] were the first to investigate the notion of common information in 1973, prior to Wyner's work [182] in 1975.

In this section, we review GKW's common information. In Section 3.1, we introduce the distributed randomness extraction system, and define GKW's common information in the context of this system. In Section 3.2, we introduce several properties of GKW's common information. We also mention some probability- and graph-theoretic interpretations of GKW's common information. We verify that GKW's common information is zero for the DSBS and also for bivariate Gaussian sources; this observation motivates Part III of the monograph. In Section 3.3, we introduce an operational interpretation of GKW's common information in the context of the Gray–Wyner lossless source coding system [68]. GKW's common information turns out to be the maximum common rate under some conditions on the sums of the private and common rates of the messages. In Section 3.4, we discuss an operational interpretation of GKW's common information due to the present authors that is not too well-known. Specifically, we relate it to the channel capacity in which the input distribution is fixed to be a given product distribution. Finally, we discuss some extensions and applications in Section 3.5.

3.1 Distributed Randomness Extraction

Consider the distributed randomness extraction problem illustrated in Fig. 3.1. For a joint source $(X, Y) \sim \pi_{XY}$, we use a pair of functions f and g, respectively acting on X^n and Y^n, to generate random variables $f(X^n)$ and $g(Y^n)$. Our goal is to ensure that $f(X^n)$ and $g(Y^n)$ are

equal with high probability and, at the same time, to maximize the information rate of $f(X^n)$ or, equivalently, $g(Y^n)$. Formally, we define distributed extraction codes and the ε-common information as follows. These definitions are due to Csiszár and Narayan [46]; we discuss the original formulation by Gács and Körner [60] in Remark 3.1.

Definition 3.1. An *X-sided (n, R)-distributed extraction code* consists of a pair of (deterministic) functions[1] (f, g) defined respectively on $\mathcal{X}^n$ and $\mathcal{Y}^n$ such that

$$\frac{1}{n} H(f(X^n)) \geq R. \tag{3.1}$$

A *Y-sided (n, R)-distributed extraction code* is defined similarly, but with (3.1) replaced by $\frac{1}{n} H(g(Y^n)) \geq R$.

Definition 3.2. Fix $\varepsilon \in (0, 1)$. The *maximal X-sided ε-error extraction rate* $S_\varepsilon^{(X)}(\pi_{XY})$ between a pair of random variables $(X, Y) \sim \pi_{XY}$ is defined as the supremum of all rates R such that there exists a sequence of X-sided (n, R)-distributed extraction codes $\{(f_n, g_n)\}_{n \in \mathbb{N}}$ satisfying

$$\Pr\left(f_n(X^n) \neq g_n(Y^n)\right) \leq \varepsilon, \tag{3.2}$$

for all sufficiently large n, where $(X^n, Y^n) \sim \pi_{XY}^n$. The *maximal Y-sided ε-error extraction rate* $S_\varepsilon^{(Y)}(\pi_{XY})$ between $(X, Y) \sim \pi_{XY}$ is defined analogously.

One can easily verify that the maximal X- and Y-sided ε-error extraction rates do not differ significantly in the limit as $n \to \infty$ and $\varepsilon \downarrow 0$. This is because, by Fano's inequality [51, Section 2.1],

$$1 + \varepsilon \log |\operatorname{supp}(f_n(X^n))| \geq H\big(f_n(X^n) | g_n(Y^n)\big). \tag{3.3}$$

Note that $|\operatorname{supp}(f_n(X^n))| \leq |\mathcal{X}|^n$ since f_n is a deterministic function. Therefore,

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} H\big(f_n(X^n) | g_n(Y^n)\big) = 0.$$

By symmetry, it also holds that

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} H\big(g_n(Y^n) | f_n(X^n)\big) = 0.$$

[1] Without loss of generality, we can set the codomains of f and g to be the set of natural numbers $\mathbb{N}$, i.e., $f : \mathcal{X}^n \to \mathbb{N}$ and $g : \mathcal{Y}^n \to \mathbb{N}$.

Combining these two limits yields that

$$\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\frac{1}{n}\big(H(f_n(X^n))-\frac{1}{n}H(g_n(Y^n))$$
$$=\lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\Big(H\big(f_n(X^n)|g_n(Y^n)\big)-H\big(g_n(X^n)|f_n(Y^n)\big)\Big)=0. \quad (3.4)$$

The exact expressions for the maximal X- and Y-sided ε-extraction rate as $\varepsilon\downarrow 0$ are given by Gács and Körner [60].

Theorem 3.1. For a joint source $(X,Y)\sim\pi_{XY}$, it holds that

$$\lim_{\varepsilon\downarrow 0}S_\varepsilon^{(X)}(\pi_{XY})=\lim_{\varepsilon\downarrow 0}S_\varepsilon^{(Y)}(\pi_{XY})=C_{\mathrm{GKW}}(\pi_{XY}),$$

where

$$C_{\mathrm{GKW}}(\pi_{XY}):=\max_{f,g:f(X)=g(Y)}H\big(f(X)\big), \quad (3.5)$$

and where the maximization is taken over all pairs of deterministic functions (f,g) defined respectively on $\mathcal{X}$ and $\mathcal{Y}$ such that $f(X)=g(Y)$ with π_{XY}-probability one.

In the literature, for example in El Gamal and Kim [51], $C_{\mathrm{GKW}}(\pi_{XY})$ is known as *GKW's common information*. Theorem 3.1 says that the maximal X- and Y-sided ε-error extraction rates are equal to GKW's common information, so in the following, we will use these terminologies interchangeably. It is clear that the objective function in the maximization in (3.5) can be replaced by $H(g(Y))$ since $f(X)$ and $g(Y)$ are constrained to be equal almost surely. Roughly speaking, the quantity $C_{\mathrm{GKW}}(\pi_{XY})$ corresponds to a single-letter version (i.e., $n=1$ version) of the maximal X- or Y-sided 0-error extraction rates (defined formally in Definition 3.3), in the sense that $C_{\mathrm{GKW}}(\pi_{XY})$ is equal to the supremum of all rates R such that $H(f(X))\geq R$ and $\Pr\big(f(X)\neq g(Y)\big)=0$.

Proof of Theorem 3.1. By symmetry, it clearly suffices to prove that $\lim_{\varepsilon\downarrow 0}S_\varepsilon^{(X)}(\pi_{XY})=C_{\mathrm{GKW}}(\pi_{XY})$. We first prove that $\lim_{\varepsilon\downarrow 0}S_\varepsilon^{(X)}(\pi_{XY})\geq C_{\mathrm{GKW}}(\pi_{XY})$, the achievability part. Let $f^*:\mathcal{X}\to\mathcal{U}$ and $g^*:\mathcal{Y}\to\mathcal{V}$ be an optimal pair of functions that attains the maximum in (3.5), where $\mathcal{U}$ and $\mathcal{V}$ are two fixed sets (that can be assumed to be the same). Then, let $f_n:x^n\in\mathcal{X}^n\mapsto(f^*(x_1),f^*(x_2),\ldots,f^*(x_n))\in\mathcal{U}^n$ and

$g_n : y^n \in \mathcal{Y}^n \mapsto (g^*(y_1), g^*(y_2), \dots, g^*(y_n)) \in \mathcal{V}^n$. Then, by the mutual independence of $X_1, X_2, \dots, X_n$,

$$\frac{1}{n} H(f_n(X^n)) = H(f^*(X)) = C_{\mathrm{GKW}}(\pi_{XY})$$

and

$$\Pr(f_n(X^n) = g_n(Y^n)) = 1. \tag{3.6}$$

Therefore, $S_\varepsilon^{(X)}(\pi_{XY}) \geq C_{\mathrm{GKW}}(\pi_{XY})$ for any $\varepsilon \in (0,1)$.

We next prove $\lim_{\varepsilon \downarrow 0} S_\varepsilon^{(X)}(\pi_{XY}) \leq C_{\mathrm{GKW}}(\pi_{XY})$, the converse part. The proof is based on the following lemma due to Csiszár and Narayan [46, Lemma 1.1]. As assumed in the achievability part, let (f^*, g^*) be an optimal pair of functions attaining the maximization in (3.5). Let $W = f^*(X) = g^*(Y) \in \mathcal{W}$; this random variable is called the *common part* of $(X, Y) \sim \pi_{XY}$.

Lemma 3.2. For $(X^n, Y^n) \sim \pi_{XY}^n$, let U and V be two random variables such that $U - X^n - Y^n - V$ and

$$\Pr(U \neq V) \leq \varepsilon$$

for $\varepsilon > 0$. Then

$$\min_{h : \mathcal{W}^n \to \mathcal{U}} \Pr\left(U \neq h(W^n)\right) \leq \delta(\varepsilon),$$

where $\delta : (0, \infty) \to (0, \infty)$ is a function that only depends on π_{XY}, is independent of n, and has the property that $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

This lemma is proven by the tensorization property of the *conditional maximal correlation* (the unconditional version of the maximal correlation was defined in (1.2)). It uses some results of Witsenhausen [178], but we will not elaborate on it here; see [46, Lemma 1.1]. Using Lemma 3.2, we know that $\min_{h_n} \Pr\left(f_n(X^n) \neq h_n(W^n)\right) \leq \delta(\varepsilon)$, where the minimization is taken over all functions h_n defined on $\mathcal{W}^n$. Similarly to (3.3), by Fano's inequality [51, Section 2.1], for any function $h_n : \mathcal{W}^n \to \mathcal{U}$,

$$1 + \delta(\varepsilon) \log |\operatorname{supp}(f_n(X^n))| \geq H\big(f_n(X^n) | h_n(W^n)\big).$$

Following an argument similar to the one leading to (3.4), we have that

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \Big(H\big(f_n(X^n)\big) - H\big(h_n(W^n)\big) \Big) = 0. \tag{3.7}$$

By combining (3.7) with the fact that $H\big(h_n(W^n)\big) \le H(W^n) = nH(W)$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \frac{1}{n} H\big(f_n(X^n)\big) \le H(W) = C_{\mathrm{GKW}}(\pi_{XY}),$$

which implies that $\lim_{\varepsilon\downarrow 0} S_\varepsilon^{(X)}(\pi_{XY}) \le C_{\mathrm{GKW}}(\pi_{XY})$. □

From the proof of Theorem 3.1, and in particular (3.6), we know that the constraint on the probability of disagreement $\Pr(f_n(X^n) \neq g_n(Y^n)) \le \varepsilon$ (where $\varepsilon \in (0,1)$) can be strengthened significantly to the zero-error version, i.e.,

$$\Pr\big(f_n(X^n) \neq g_n(Y^n)\big) = 0 \qquad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Definition 3.3. The *maximal X-sided (resp. Y-sided) 0-error extraction rate* $S_\varepsilon^{(X)}(\pi_{XY})$ (resp. $\tilde{S}_0^{(Y)}(\pi_{XY})$) is the supremum of all rates R such that there exists a sequence of (n,R)-distributed extraction codes $\{(f_n,g_n)\}_{n\in\mathbb{N}}$ such that (3.8) holds.

By definition, $\tilde{S}_0^{(X)}(\pi_{XY}) = \tilde{S}_0^{(Y)}(\pi_{XY})$. Moreover, these strengthened definitions are the same as the limiting values of maximal X- and Y-sided ε-error extraction rates as $\varepsilon \downarrow 0$, i.e.,

$$\tilde{S}_0^{(X)}(\pi_{XY}) = \lim_{\varepsilon\downarrow 0} S_\varepsilon^{(X)}(\pi_{XY}) \quad \text{and} \quad \tilde{S}_0^{(Y)}(\pi_{XY}) = \lim_{\varepsilon\downarrow 0} S_\varepsilon^{(Y)}(\pi_{XY}).$$

This is easy to see as, on one hand, according to (3.6), the functions f_n and g_n, defined in the proof of Theorem 3.1, satisfy the zero-error constraint. Hence, $\tilde{S}_0^{(X)}(\pi_{XY}) \ge C_{\mathrm{GKW}}(\pi_{XY})$. On the other hand, observe that the maximal X-sided ε-error extraction rate is no larger than $S_\varepsilon^{(X)}(\pi_{XY})$ for any $\varepsilon \in (0,1)$, since an error is allowed in the latter. Combining this with Theorem 3.1 yields that $\tilde{S}_0^{(X)}(\pi_{XY}) \le C_{\mathrm{GKW}}(\pi_{XY})$. These observations are summarized in the following theorem.

Theorem 3.3. It holds that

$$\tilde{S}_0^{(X)}(\pi_{XY}) = \tilde{S}_0^{(Y)}(\pi_{XY}) = C_{\mathrm{GKW}}(\pi_{XY}).$$

Remark 3.1. The formulation of GKW's common information as presented in Definition 3.2 was introduced by Csiszár and Narayan [46]. This is not the original definition introduced in Gács and Körner [60].

In Gács and Körner's original formulation, instead of the normalized entropy of the common part $W_n \in \mathcal{W}_n$ of X^n and Y^n (cf. (3.1)), the information rate is measured in terms of the *exponent* of its alphabet size $\frac{1}{n}\log|\mathcal{W}_n|$. To ensure that the exponent of the alphabet size is an "effective" measure of the information rate, the distribution of the random variable $f_n(X^n)$ (and $g_n(Y^n)$) is required to be close to the uniform distribution on its alphabet. Hence, lossless source coding is used in Gács and Körner's setting to implement this requirement. Specifically, the juxtaposition of f_n with another function $\tilde{f}_n$ on $\mathcal{X}^n$ is required to be an *almost optimal* fixed-length lossless source code for X^n. A similar constraint was also imposed for the function g_n. These force the outputs of f_n and g_n to be close to uniform on $\mathcal{W}_n$. Indeed, the formulation by Gács and Körner [60] is analogous to fixed-length source coding [153] while the formulation by Csiszár and Narayan [46] is analogous to weak variable-length source coding as studied in Han [72] and Koga and Yamamoto [98] among others.

For their setting, Gács and Körner showed that the maximum asymptotic exponent $\liminf_{n\to\infty}\frac{1}{n}\log|\mathcal{W}_n|$ under the asymptotic probability of disagreement constraint in (3.2) is $C_{\mathrm{GKW}}(\pi_{XY})$ for all $\varepsilon \in (0,1)$. Hence, the strong converse holds for Gács and Körner's formulation, while it does not hold for Csiszár and Narayan's formulation (i.e., Definition 3.2). This observation resembles lossless source coding in that the strong converse holds for the fixed-length version [180] but not the weak variable-length version [72], [98], [100], [146].

3.2 Properties of GKW's Common Information

We next introduce several interesting properties of $C_{\mathrm{GKW}}(\pi_{XY})$ that elucidate more insights on its properties.

3.2.1 Interpretation in terms of Markov chains and bipartite graphs

We first focus on the computation of $C_{\mathrm{GKW}}(\pi_{XY})$, which can be understood using Markov chains and bipartite graphs. Consider a discrete-time Markov chain $X_1 - Y_1 - X_2 - Y_2 - \ldots$ in which $P_{X_1} = \pi_X$ is the initial distribution and the transition probability distributions satisfy

$$P_{Y_i|X_i} = \pi_{Y|X} \quad \text{and} \quad P_{X_{i+1}|Y_i} = \pi_{X|Y}$$

for all $i \in \mathbb{N}$. Then, the subchain $X_1 - X_2 - \ldots$ is a *time-homogeneous* Markov chain with initial distribution π_X and transition probability distribution

$$P_{X_{i+1}|X_i} = \pi_{\hat{X}|X},$$

where

$$\pi_{\hat{X}|X}(x'|x) := \sum_{y \in \mathcal{Y}} \pi_{X|Y}(x'|y)\pi_{Y|X}(y|x) \quad \text{for all } (x, x') \in \mathcal{X}^2.$$

Obviously, π_X is the stationary distribution of $X_1 - X_2 - \ldots$. Moreover, this subchain is *reversible* since the stationary distribution and transition probability distribution satisfy

$$\pi_{\hat{X}|X}(x'|x)\pi_X(x) = \pi_{\hat{X}|X}(x|x')\pi_X(x') \quad \text{for all } (x, x') \in \mathcal{X}^2.$$

Now we recap a few more definitions from Markov chains; see, for example, Gallager [61, Chapter 4]. For two states $x, x' \in \mathcal{X}$, x' is *accessible* from x, abbreviated as $x \to x'$, if $P_{X_N|X_1}(x'|x) > 0$ for some positive integer N. The condition $P_{X_N|X_1}(x'|x) > 0$ is also equivalent to the fact that there exists a sequence of states (also called a *walk*) $(x_1, x_2, \ldots, x_N)$ such that $x_1 = x, x_N = x'$, and $\pi_{\hat{X}|X}(x_i|x_{i-1}) > 0$ for $2 \le i \le N$. If $\mathcal{X}$ is the support of π_X, then the condition $P_{X_N|X_1}(x'|x) > 0$ is also equivalent to $P_{X_1X_N}(x, x') > 0$. Two distinct states x and x' *communicate*, abbreviated as $x \leftrightarrow x'$, if x is accessible from x' and x' is accessible from x. By definition, for a stationary and reversible Markov chain (e.g., the one considered here), the joint distribution $P_{X_iX_j}$ of (X_i, X_j) for $i \neq j$ satisfies $P_{X_iX_j}(x, x') = P_{X_iX_j}(x', x)$ for all $x, x' \in \mathcal{X}$. Hence, $x \to x'$ (or $x' \to x$) is equivalent to $x \leftrightarrow x'$. Obviously, "$\leftrightarrow$" is an equivalence relation, since it satisfies the following three properties:

- Reflexivity: $a \leftrightarrow a$;
- Symmetry: $a \leftrightarrow b$ if and only if $b \leftrightarrow a$;
- Transitivity: If $a \leftrightarrow b$ and $b \leftrightarrow c$ then $a \leftrightarrow c$.

This allows us to define *equivalence classes* for the relation $\leftrightarrow$. In the language of Markov chains, these are known as *communicating classes*, or simply *classes*. A set $\mathcal{A} \subset \mathcal{X}$ is termed a *class* of $\mathcal{X}$ if $\mathcal{A}$ is non-empty

and for all $x \in \mathcal{A}$, each state $x' \in \mathcal{X} \setminus \{x\}$ satisfies $x' \in \mathcal{A}$ if $x \leftrightarrow x'$ and $x' \notin \mathcal{A}$ if $x \not\leftrightarrow x'$. The classes of $\mathcal{X}$, denoted as $\mathcal{X}_i, i \in [r]$, form a partition[2] of $\mathcal{X}$. The classes of $\mathcal{Y}$ are similarly denoted as $\mathcal{Y}_j, j \in [s]$.

Clearly, the Markov chain transitions from a state in $\mathcal{X}_i$ to a state in $\mathcal{Y}_j$ with positive probability in the sense that $\Pr(Y \in \mathcal{Y}_j | X \in \mathcal{X}_i) = \mathbb{1}\{i = j\}$ [61, Theorem 4.2.9] where $(X, Y) \sim \pi_{XY}$, and vice versa. Hence, $r = s$ and

$$\Pr\left(Y \in \mathcal{Y}_j | X \in \mathcal{X}_i\right) = \Pr\left(X \in \mathcal{X}_i | Y \in \mathcal{Y}_j\right) = \mathbb{1}\{i = j\}. \tag{3.9}$$

Such a partition of $\mathcal{X}$ (or $\mathcal{Y}$), termed an *ergodic decomposition* [60], is unique. If we denote $i^*(x)$ as the index i such that $x \in \mathcal{X}_i$, and similarly, $j^*(y)$ as the index j such that $y \in \mathcal{Y}_j$, then by (3.9), $i^*(X) = j^*(Y)$. The pair of functions (i^*, j^*) attains the maximization in (3.5). This is because, on one hand, by definition, $C_{\mathrm{GKW}}(\pi_{XY}) \geq H(i^*(X))$. On the other hand, for (f, g) such that $f(X) = g(Y)$ almost surely,

$$f(X_1) = g(Y_1) = f(X_2) = g(Y_2) = \ldots, \tag{3.10}$$

where $X_1 - Y_1 - X_2 - Y_2 - \ldots$ is the Markov chain as defined at the start of this section. Denote the image of f as $\mathcal{U}$. Then, by (3.10), for each pair of distinct elements (u, u') of $\mathcal{U}$, we have

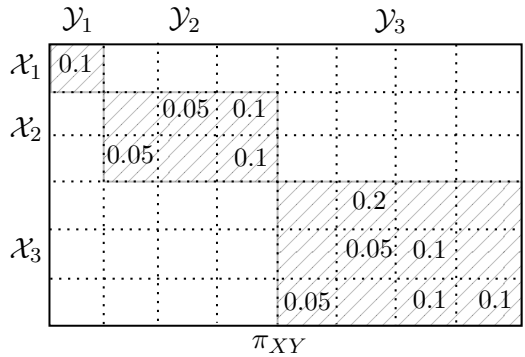
$$\Pr\left(X_m \in f^{-1}(u') | X_1 \in f^{-1}(u)\right) = 0 \qquad \text{for all } m \in \mathbb{N}.$$

Hence, for each $u \in \mathcal{U}$, $f^{-1}(u) \subset \mathcal{X}$ is a class or the union of several classes. This means that $f(X)$ is determined by $i^*(X)$, which in turn implies that $C_{\mathrm{GKW}}(\pi_{XY}) \leq H(i^*(X))$. Hence, (i^*, j^*) is the unique pair of functions (up to a bijection) attaining the maximization in GKW's common information in (3.5).

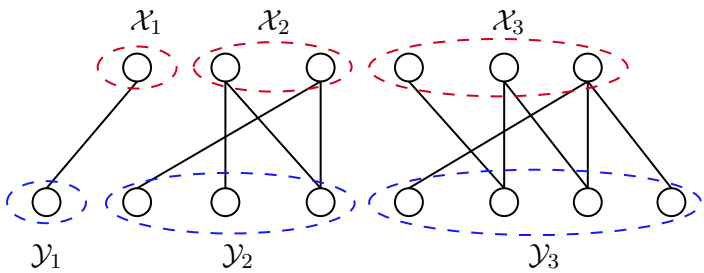
The ergodic decomposition can be also expressed in the language of graph theory. Without loss of generality, we may assume that $\mathcal{X} \cap \mathcal{Y} = \emptyset$. Consider a (undirected) bipartite graph in which the two sets of vertices are represented by $\mathcal{X}$ and $\mathcal{Y}$ and a pair of vertices $(x, y) \in \mathcal{X} \times \mathcal{Y}$ is adjacent if $\pi_{XY}(x, y) > 0$. In an undirected graph, a vertex $v \in \mathcal{X} \cup \mathcal{Y}$ is *reachable* from a vertex $u \in \mathcal{X} \cup \mathcal{Y}$ if there is a path from u to v.

[2] A *partition* of a set $\mathcal{X}$ is a collection of sets $\{\mathcal{X}_\alpha\}_{\alpha \in \mathcal{A}}$ such that $\cup_{\alpha \in \mathcal{A}} \mathcal{X}_\alpha = \mathcal{X}$ and $\mathcal{X}_\alpha \cap \mathcal{X}_{\alpha'} = \emptyset$ for all $\alpha \neq \alpha'$.

Reachability is also an equivalence relation, and the equivalence classes of this equivalence relation are $\mathcal{X}_i \cup \mathcal{Y}_i, i \in [r]$, where $\mathcal{X}_i$ and $\mathcal{Y}_i, i \in [r]$ are the communicating classes. The induced subgraphs formed by these equivalence classes are known as the *connected components* of the graph. The ergodic decomposition corresponds to the decomposition of the graph into connected components. Fig. 3.2 illustrates an example of a joint distribution π_{XY} together with its ergodic decomposition.



(a) An example of π_{XY} with $r = s = 3$ classes each for $\mathcal{X}$ and $\mathcal{Y}$



(b) The ergodic decomposition of the joint distribution π_{XY} given in (a)

Figure 3.2: An example of π_{XY} and its ergodic decomposition

3.2.2 Connections to Other Quantities

We now provide an alternative expression for $C_{\mathrm{GKW}}(\pi_{XY})$, which looks similar to the expression for Wyner's common information in (2.3). This characterization is due to Ahlswede and Körner [2].

Proposition 3.1. It holds that

$$C_{\mathrm{GKW}}(\pi_{XY}) = \max_{\substack{P_{WXY}: P_{XY}=\pi_{XY},\\ W-X-Y,\, W-Y-X}} I(XY;W). \tag{3.11}$$

We remark that the objective function in the maximization above can be replaced by $I(X;W)$ or $I(Y;W)$, since the Markov chains $W-X-Y$ and $W-Y-X$ are assumed.

Proof. Let $U = f^*(X) = g^*(Y)$ be the common part of $(X,Y) \sim \pi_{XY}$ with (f^*, g^*) denoting the optimal pair of functions attaining the maximization in $C_{\mathrm{GKW}}(\pi_{XY})$ in (3.5). By setting $W = U$, we conclude that the right-hand side of (3.11) is at least $C_{\mathrm{GKW}}(\pi_{XY})$.

To prove the opposite inequality, we first state the following lemma.

Lemma 3.4. Every P_{WXY} that satisfies the constraints in (3.11) also satisfies $W - U - X$.

Lemma 3.4 then implies that $I(XY;W) = I(X;W) \le I(X;U) = H(U)$. Hence, the right-hand side of (3.11) is at most $C_{\mathrm{GKW}}(\pi_{XY})$. Combining the two points above yields the equality in (3.11). Hence, it remains to prove Lemma 3.4.

We now prove Lemma 3.4. Since $W - X - Y$ and $W - Y - X$, we have

$$P_{W|XY}(\cdot|x,y) = P_{W|X}(\cdot|x) = P_{W|Y}(\cdot|y) \tag{3.12}$$

for all (x,y) such that $P_{XY}(x,y) > 0$. Using the graph-theoretic interpretation of GKW's common information as described in Section 3.2.1, we assign a distribution $P_W^{(v)} \in \mathcal{P}(\mathcal{W})$ to each vertex $v \in \mathcal{X} \cup \mathcal{Y}$ in the bipartite graph. Here $P_W^{(v)}$ corresponds to $P_{W|X}(\cdot|v)$ if $v \in \mathcal{X}$, or $P_{W|Y}(\cdot|v)$ if $v \in \mathcal{Y}$. From (3.12) and the assumption that $P_{XY} = \pi_{XY}$, these distributions satisfy that $P_W^{(v)} = P_W^{(\hat{v})}$ for any two adjacent vertices $(v, \hat{v})$. As a consequence, these distributions are identical for all vertices in a connected component. As mentioned in Section 3.2.1, f^* is a function indicating which component X belongs to. Hence, if we denote the joint distribution of (W,X,U) as P_{WXU}, then given u, the conditional distribution $P_{W|X}(\cdot|x)$, which is equal to $P_{W|UX}(\cdot|u,x)$, remains the same for all x such that $u = f^*(x)$. That is, given each u,

$$P_{W|UX}(\cdot|u,x) = \sum_{x'} P_{X|U}(x'|u) P_{W|UX}(\cdot|u,x') = P_{W|U}(\cdot|u)$$

for all x such that $P_{UX}(u,x) > 0$. Hence, $W - U - X$ holds, completing the proof of Lemma 3.4. □

We now compare $C_{\mathrm{GKW}}(\pi_{XY})$ with the mutual information $I_\pi(X;Y)$ and Wyner's common information $C_{\mathrm{W}}(\pi_{XY})$.

Proposition 3.2. For any joint source $(X,Y) \sim \pi_{XY}$,

$$C_{\mathrm{GKW}}(\pi_{XY}) \le I_\pi(X;Y) \le C_{\mathrm{W}}(\pi_{XY}). \tag{3.13}$$

Moreover, the two inequalities become equalities if and only if $X - U - Y$ holds, where U is the common part of X and Y.

The leftmost inequality in (3.13) and the corresponding equality conditions were proved by Gács and Körner [60].

Proof. The inequalities in this proposition follow directly by their definitions. We next consider the conditions for equality. Obviously, if the common part U of X and Y satisfies $X - U - Y$, then both inequalities in (3.13) are equalities. On the other hand, if the leftmost inequality in (3.13) is an equality, then $I(X;Y) = H(U)$. Combining this with the fact $I(X;Y) = I(XU;Y) = H(U) + I(X;Y|U)$ yields that $I(X;Y|U) = 0$, i.e., $X - U - Y$.

We next assume that the rightmost inequality in (3.13) is an equality. Let P_{WXY} be a distribution attaining $C_{\mathrm{W}}(\pi_{XY})$. Then, $I(XY;W) \ge I(X;W) \ge I(X;Y)$ due to the Markov chain $X - W - Y$. By assumption, $I(XY;W) = I(X;Y)$, which implies that $I(XY;W) = I(X;W)$, i.e., $W - X - Y$ holds. By symmetry, $W - Y - X$ also holds. Combining these two conditions with Proposition 3.1 yields that $C_{\mathrm{W}}(\pi_{XY}) \le C_{\mathrm{GKW}}(\pi_{XY})$. Since the inequalities in (3.13) imply that $C_{\mathrm{W}}(\pi_{XY})$ cannot be (strictly) smaller than $C_{\mathrm{GKW}}(\pi_{XY})$, we have $C_{\mathrm{W}}(\pi_{XY}) = C_{\mathrm{GKW}}(\pi_{XY})$, which in turn implies that the random variable W under the distribution P_{WXY} is the common part of $(X,Y) \sim \pi_{XY}$. By the choice of W, $X - W - Y$ holds. □

3.2.3 When is GKW's common information positive?

Another interesting property of $C_{\mathrm{GKW}}(\pi_{XY})$ is its intimate connection to the maximal correlation defined in (1.2).

Proposition 3.3. For $(X, Y) \sim \pi_{XY}$, the following are equivalent.

(a) $\rho_{\mathrm{m}}(X; Y) = 1$;

(b) $C_{\mathrm{GKW}}(\pi_{XY}) > 0$;

(c) There exists a pair of nonconstant functions[3] (f, g) such that $f(X) = g(Y)$ almost surely.

Thus, for any source $(X, Y) \sim \pi_{XY}$ with maximal correlation strictly smaller than 1, its GKW's common information is zero. This class of sources includes the DSBS and Gaussian sources with correlation coefficients in $(-1, 1)$. This is essentially why Gács and Körner [60] titled their paper *"Common information is far less than mutual information"*. Further refinements to GKW's common information (when it is equal to zero) that captures other aspects of the sources' correlation will be the main subject of discussion in Part III.

Example 3.1. Let us now revisit Example 2.1 in which $X = (\tilde{X}, V)$ and $Y = (\tilde{Y}, V)$ for mutually independent random variables $\tilde{X} \in \tilde{\mathcal{X}}$, $\tilde{Y} \in \tilde{\mathcal{Y}}$, and $V \in \mathcal{V}$. This example was first presented at the beginning of Section 2 to illustrate that Wyner's common information for this source coincides with the intuitive quantity $H(V)$. Here, we can also easily observe that GKW's common information also coincides with $H(V)$. This is because, the bipartite graph induced by the distribution of (X, Y) is such that given each pair $(v, \hat{v}) \in \mathcal{V}^2$, two vertices $(\tilde{x}, v)$ and $(\tilde{y}, \hat{v})$ with $\tilde{x} \in \tilde{\mathcal{X}}, \tilde{y} \in \tilde{\mathcal{Y}}$ are adjacent if and only if $v = \hat{v}$. Hence, each element in $\mathcal{V}$ identifies a unique connected component of the graph, and *vice versa*. This implies that $C_{\mathrm{GKW}}(\pi_{XY}) = H(V)$ and $i^*(X) = j^*(Y) = V$ (i.e., V is the common part of the joint source $(X, Y) \sim \pi_{XY}$). This example implies that GKW's common information is zero if the sources are independent (i.e., V is constant). However, the converse clearly does not hold. Indeed, for any distribution π_{XY} which is fully supported on $\mathcal{X} \times \mathcal{Y}$ (e.g., the DSBS with $|\rho| \neq 1$), its GKW's common information is identically zero.

[3] A nonconstant function is one whose image contains more than one element.

3.3 The Gray–Wyner System

In Section 3.1, we saw one operational interpretation of GKW's common information. In this and the next section, we present two other operational interpretations; these sections may be omitted at a first reading as further discussions on this topic in Part III depend only on Sections 3.1 and 3.2.

We now relate GKW's common information to the common rate in the Gray–Wyner system, defined in Section 2.2. In the Gray–Wyner system, the common rate is denoted as R_0 and two private rates are denoted as R_1 and R_2. By Shannon's source coding theorem, if there exists a Gray–Wyner code such that the source (X, Y) can be reconstructed almost losslessly by two decoders respectively, then the rate tuple (R_0, R_1, R_2) of this code must satisfy $R_0 + R_1 \geq H(X)$ and $R_0 + R_2 \geq H(Y)$. Obviously, these necessary conditions are not sufficient in general. For example, a tuple (R_0, R_1, R_2) such that $R_1 = R_2 = 0$ and $R_0 = \max\{H(X), H(Y)\}$ satisfies these necessary conditions. However, by Shannon's source coding theorem, the optimal rate for lossless source coding of the joint source (X, Y) is $H(XY)$, which is strictly larger than $\max\{H(X), H(Y)\}$ unless X is a function of Y or Y is a function of X. Hence, in general, there is no Gray–Wyner code with such a rate tuple (R_0, R_1, R_2) such that (X, Y) can be reconstructed almost losslessly by the two decoders. In addition, if $(R_0, R_1, R_2) = (0, H(X), H(Y))$, then coding X and Y separately with rates R_1 and R_2 is clearly feasible. Hence, within the transition between these two extreme cases, there is a maximum common rate R_0 such that $R_0 + R_1 = H(X)$, $R_0 + R_2 = H(Y)$, and the source (X, Y) can be transmitted almost losslessly to the two decoders using a Gray–Wyner code with rate tuple (R_0, R_1, R_2). This maximum common rate R_0 can be regarded as a form of common information of (X, Y). Indeed, if we consider the example $X = (\tilde{X}, V)$ and $Y = (\tilde{Y}, V)$ where $\tilde{X}, \tilde{Y}, V$ are mutually independent (cf. Example 3.1), then the maximum common rate coincides with the intuitive "common information" $H(V)$. Formally, we define the pairwise sum rate-common information based on the Gray–Wyner system as follows (compare to Definition 2.4).

Definition 3.4. The *pairwise sum rate-common information based on the Gray–Wyner system* $S_{\mathrm{GW}}(\pi_{XY})$ between a pair of random variables $(X, Y) \sim \pi_{XY}$ is the supremum of all rates R_0 such that for all $\epsilon > 0$, there exists a sequence of (n, R_0, R_1, R_2) Gray–Wyner codes $\{(f_{0,n}, f_{1,n}, f_{2,n}, \varphi_{1,n}, \varphi_{2,n})\}_{n=1}^{\infty}$ such that $R_0 + R_1 \leq H(X) + \epsilon$ and $R_0 + R_2 \leq H(Y) + \epsilon$ and the probability of error in (2.9) vanishes as the length of the code n tends to infinity.

The common information in Definition 3.4 differs from Gács and Körner's formulation of the common information in [60] in two aspects. Firstly, the functions $f_{0,n}, f_{1,n}, f_{2,n}$ in Definition 3.4 are defined on the set $\mathcal{X}^n \times \mathcal{Y}^n$; while the functions $f_n, \tilde{f}_n$ in [60] are defined on $\mathcal{X}^n$ and similarly, g_n, g'_n are defined on $\mathcal{Y}^n$. Secondly, only *one* function $f_{0,n}$ is used to extract common randomness in Definition 3.4; while in Gács and Körner [60], *two* functions f_n and g_n are employed to extract common randomness in a distributed way. Ahlswede and Körner [2] (and also Kamath and Anantharam [93]) showed that the common information in Definition 3.4 coincides with the one based on distributed randomness extraction in Definition 3.2.

Theorem 3.5. The pairwise sum rate-common information based on the Gray–Wyner system

$$S_{\mathrm{GW}}(\pi_{XY}) = C_{\mathrm{GKW}}(\pi_{XY}).$$

Proof. It was shown by Gray and Wyner [68] that the closure of the set of all rate tuples (R_0, R_1, R_2) such that there exists a sequence of (n, R_0, R_1, R_2) Gray–Wyner codes satisfying that the probability of error vanishes as the length of the code n tends to infinity, is the set of (R_0, R_1, R_2) such that $R_0 \geq I(XY; W), R_1 \geq H(X|W), R_2 \geq H(Y|W)$ for some random variable W. Hence,

$$\begin{aligned} S_{\mathrm{GW}}(\pi_{XY}) &= \max_{\substack{P_{WXY}: P_{XY} = \pi_{XY}, \\ I(XY;W) + H(X|W) = H(X), \\ I(XY;W) + H(Y|W) = H(Y)}} I(XY; W) \\ &= \max_{\substack{P_{WXY}: P_{XY} = \pi_{XY}, \\ W - X - Y,\, W - Y - X}} I(XY; W) = C_{\mathrm{GKW}}(\pi_{XY}), \end{aligned}$$

where the final equality follows from Proposition 3.1. □

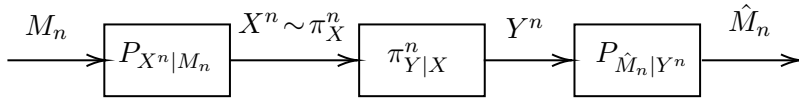


Figure 3.3: The channel coding problem with an input distribution constraint

Similarly to Wyner's common information, $C_{\mathrm{GKW}}(\pi_{XY})$ also has (at least) two operational interpretations; one as the maximum rate of almost identical common randomness that can be extracted from two correlated sources independently, and the other one as the maximum common rate of the Gray–Wyner system keeping the sum of the common rate and the private rate at the entropy of the corresponding source.

We conclude this section by mentioning that it would be interesting to establish an analogue of Theorem 2.4 (due to Xu, Liu, and Chen [186]) for GKW's common information. In particular, how is GKW's common information related to the *lossy* Gray–Wyner system? More specifically, is it true that, like Wyner's common information, under mild conditions and for sufficiently small distortion levels, the lossy version of GKW's common information coincides with its almost lossless counterpart?

3.4 Channel Coding with an Input Distribution Constraint

In this section, we provide a third and final operational interpretation of GKW's common information in the context of the classical problem of channel coding, but with a slight twist. Consider the channel coding problem with an *input distribution constraint* as illustrated in Fig. 3.3. Let $\pi_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ denote the channel, and $M_n \sim \mathrm{Unif}[2^{nR}]$ a uniformly distributed message with rate R.

Definition 3.5. A *stochastic* (n, R)*-code* consists of two stochastic mappings, the stochastic encoder $P_{X^n|M_n} \in \mathcal{P}(\mathcal{X}^n|[2^{nR}])$ and the stochastic decoder $P_{\hat{M}_n|Y^n} \in \mathcal{P}([2^{nR}]|\mathcal{Y}^n)$.

Definition 3.6. The *channel capacity with input distribution* $\pi_X \in \mathcal{P}(\mathcal{X})$, denoted as $C(\pi_X)$, is the supremum of rates R such that there exists a sequence of stochastic (n, R)-codes $\{(P_{X^n|M_n}, P_{\hat{M}_n|Y^n})\}_{n\in\mathbb{N}}$ satisfying

that the distribution of X^n is exactly equal to π_X^n for each $n \in \mathbb{N}$ and the average probability of error $\Pr(\hat{M}_n \neq M_n)$ vanishes as the length of the code n tends to infinity.

This notion is markedly different from the problem of *channel coding with input cost* [51, Section 3.3] in which the input codewords $x^n(m), m \in [2^{nR}]$ are required to satisfy a constraint of the form $\frac{1}{n}\sum_{i=1}^n b(x_i(m)) \leq B$ for some per-letter cost function $b : \mathcal{X} \to [0, \infty)$ and cost constraint $B > 0$. In Definition 3.6, the *distribution* of the channel input X^n, induced by M_n and $P_{X^n|M_n}$, is required to be exactly equal to π_X^n. To satisfy this constraint, a *stochastic* encoder is required. The present authors showed that the channel capacity $C(\pi_X)$ with input distribution π_X is equal to GKW's common information of π_{XY} [199].

Theorem 3.6. For any channel $\pi_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$,

$$C(\pi_X) = C_{\mathrm{GKW}}(\pi_{XY}).$$

From this theorem, we deduce that

$$C(\pi_X) \leq I_\pi(X;Y) \leq C^* := \max_{P_X \in \mathcal{P}(\mathcal{X})} I(P_X, \pi_{Y|X}),$$

where $C^* = C^*(\pi_{Y|X})$ denotes the Shannon capacity of the channel $\pi_{Y|X}$ (i.e., the channel capacity without the input distribution constraint). This channel coding problem can in fact be reinterpreted as a *randomness extraction* problem. Given a bivariate source $(X, Y) \sim \pi_{XY}$, we use two stochastic maps $P_{M_n|X^n} \in \mathcal{P}([2^{nR}]|\mathcal{X}^n)$ and $P_{\hat{M}_n|Y^n} \in \mathcal{P}([2^{nR}]|\mathcal{Y}^n)$ to generate a uniform random variable M_n and an arbitrary random variable $\hat{M}_n$ such that $\Pr(\hat{M}_n \neq M_n) \to 0$ as $n \to \infty$. We aim to maximize the rate R of M_n. This variation of the randomness extraction problem differs from limiting case in which $\varepsilon \downarrow 0$ (vanishing error probability) in Definition 3.2 in Section 3.1 in two aspects. Firstly, the maps in the channel coding with input distribution constraint problem are *stochastic*, while the maps in the distributed randomness extraction problem in Section 3.1 are *deterministic*. Secondly, the output M_n from $P_{M_n|X^n}$ here is a *uniform* random variable, while the output $f(X^n)$ is not necessarily uniform. In spite of these two differences, we observe that the maximum achievable rates of the extracted randomnesses for these two problems coincide, and are both equal to $C_{\mathrm{GKW}}(\pi_{XY})$.

Proof of Theorem 3.6. The inequality $C(\pi_X) \le C_{\mathrm{GKW}}(\pi_{XY})$, which represents the converse, can be proved by combining Lemma 3.2 and Fano's inequality, just as in the proof of Theorem 3.1. We omit the details here and refer the interested reader to Yu and Tan [199, Section VI].

We next prove the more interesting part $C(\pi_X) \ge C_{\mathrm{GKW}}(\pi_{XY})$, which represents the achievability. Observe that the distributions of X^n and M_n are respectively π_X^n and $\mathrm{Unif}[2^{nR}]$, both of which are given. Hence, designing a stochastic map $P_{X^n|M_n} \in \mathcal{P}(\mathcal{X}^n|[2^{nR}])$ (or $P_{M_n|X^n} \in \mathcal{P}([2^{nR}]|\mathcal{X}^n)$) is equivalent to designing a *coupling* (cf. Section 1.4.2) of π_X^n and $\mathrm{Unif}[2^{nR}]$. On the other hand, let U be the common part of $(X, Y) \sim \pi_{XY}$. By definition, $C_{\mathrm{GKW}}(\pi_{XY}) = H(U)$. Moreover, U^n, which corresponds to the common part of X^n and Y^n, can be generated from X^n and Y^n individually. To prove that $C(\pi_X) \ge C_{\mathrm{GKW}}(\pi_{XY})$, it suffices to construct a sequence of couplings $\{P_{U^n M_n}\}_{n\in\mathbb{N}}$ of the distributions of U^n and M_n such that

$$\lim_{n\to\infty} \min_{\varphi:\mathcal{U}^n\to[2^{nR}]} \Pr\left(M_n \ne \varphi(U^n)\right) = 0. \tag{3.14}$$

In other words, we only utilize the common part U^n of X^n and Y^n to transmit the message. To this end, we leverage the maximal guessing coupling equality in Lemma 1.2.

The minimization on the right-hand side of (1.11) is termed the *distribution approximation* or *random number generation* problem [71, Chapter 2], in which a random variable $X \sim P_X$ is used to simulate another random variable $Y \sim P_Y$ using a function $\varphi : \mathcal{X} \to \mathcal{Y}$ such that the TV distance between the distribution $P_{\varphi(X)}$ of the generated random variable $\varphi(X)$ and the target distribution P_Y is minimized. When Y is uniform, this problem reduces to the *intrinsic randomness* problem in which a well-known result due to Vembu and Verdú [171] is the following. For an i.i.d. source sequence $X^n \sim P_X^n$ and the uniform distribution $\mathrm{Unif}[2^{nR}]$, if $R < H(X)$,

$$\lim_{n\to\infty} \min_{\varphi_n:\mathcal{X}^n\to[2^{nR}]} \left|\mathrm{Unif}[2^{nR}] - P_{\varphi_n(X^n)}\right| = 0.$$

Hence, in our setting, with $U^n \sim \pi_U^n$ and $R < H_\pi(U)$,

$$\lim_{n\to\infty} \min_{\varphi_n:\mathcal{U}^n\to[2^{nR}]} \left|\mathrm{Unif}[2^{nR}] - \pi_{\varphi_n(U^n)}\right| = 0.$$

Combining this with Lemma 1.2 yields that if $R < H_\pi(U)$, then

$$\lim_{n\to\infty} \max_{P_{U^n M_n} \in \mathcal{C}(\pi_U^n, \mathrm{Unif}[2^{nR}])} \max_{\varphi_n : \mathcal{U}^n \to [2^{nR}]} \Pr\left(M_n = \varphi_n(U^n)\right) = 1. \quad (3.15)$$

In other words, if $R < H_\pi(U)$, there exists a sequence of couplings $\{P_{U^n M_n}\}_{n\in\mathbb{N}}$ of π_U^n and $\mathrm{Unif}[2^{nR}]$ such that (3.14) holds. Let $\varphi_n^* : \mathcal{U}^n \to [2^{nR}]$ be any function that attains $\max_{\varphi_n} \Pr(M_n = \varphi_n(U^n))$ for $(U^n, M_n) \sim P_{U^n M_n}$. Now define the encoder as

$$P_{X^n|M_n}(x^n|m) := \sum_{u^n \in \mathcal{U}^n} P_{U^n|M_n}(u^n|m) P_{X|U}^n(x^n|u^n), \quad (3.16)$$

and the decoder as

$$P_{\hat{M}_n|Y^n}(m|y^n) := \sum_{u^n \in \mathcal{U}^n} P_{U|Y}^n(u^n|y^n)\ \mathbb{1}\{m = \varphi_n^*(u^n)\}.$$

See the coding scheme in Fig. 3.4. Then, as a result of (3.15)–(3.16), the constraints $X^n \sim \pi_X^n$ and $\Pr(\hat{M}_n \neq M_n) \to 0$ as $n \to \infty$ are respectively satisfied, which implies that $C(\pi_X) \geq H_\pi(U) = C_{\mathrm{GKW}}(\pi_{XY})$. □

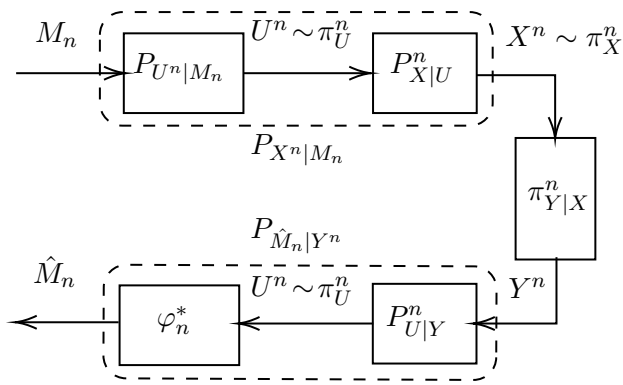


Figure 3.4: A channel coding scheme

3.5 Generalizations and Applications

We conclude this section by introducing several generalizations of GKW's common information. The expression of GKW's common information in (3.11) implies that it can also written as the maximum of

the mutual information $I(X;U)$ over all distributions P_{UXY} such that $P_{XY} = \pi_{XY}$, $U - X - Y$, and $H(U|Y) = 0$. If we relax the constraint $H(U|Y) = 0$ to $H(U|Y) \leq \delta$ for a given $\delta > 0$, then we arrive at the *approximate GKW's common information* of π_{XY}, namely,

$$C_{\mathrm{GKW}}^{(\delta)}(\pi_{XY}) := \max_{P_{UXY}: P_{XY} = \pi_{XY}, U - X - Y, H(U|Y) \leq \delta} I(X;U).$$

This quantity was proposed and used by Salamatian, Cohen, and Médard [147] to characterize an achievability result for zero-error coding in the *distributed lossless compression with helper* [156] problem.

Yu, Li, and Chen [196] introduced another generalization of GKW's common information. Let U be the common part of X and Y. Then, for every $u \in \mathcal{U}$, on the event $\{U = u\}$, the maximal correlation of X and Y, i.e., the maximal correlation of $(X', Y') \sim \pi_{XY|U=u}$ which is denoted as $\rho_{\mathrm{m}}(X;Y|U=u)$, is strictly less than 1. This is because, otherwise, for each $u \in \mathcal{U}$, one can extract a common part V_u of $(X', Y') \sim \pi_{XY|U=u}$ such that $H(V_U|U) > 0$. Note that (U, V_U) also forms a common part of X and Y, and moreover, $H(U, V_U) > H(U)$. This contradicts the assumption that U is the common part of X and Y. This inspires Yu, Li, and Chen [196] to provide another characterization of GKW's common information as follows. For each $\beta \in [0, 1]$, define the *information-correlation function* as

$$C_{\mathrm{IC}}^{(\beta)}(\pi_{XY}) := \max_{P_{UXY}: P_{XY} = \pi_{XY}, \rho_{\mathrm{m}}(X;Y|U) \leq \beta} I(XY;U),$$

where $\rho_{\mathrm{m}}(X;Y|U) := \sup_{u \in \mathcal{U}} \rho_{\mathrm{m}}(X;Y|U=u)$ denotes the *conditional maximal correlation* of X and Y given U. By the support lemma [51], it suffices to consider a variable U with alphabet size $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{Y}| + 1$. Then, GKW's common information can be expressed as

$$C_{\mathrm{GKW}}(\pi_{XY}) = \lim_{\beta \uparrow 1} C_{\mathrm{IC}}^{(\beta)}(\pi_{XY}).$$

If we consider the other end point $\beta = 0$, then the constraint in the definition of the information-correlation function reduces to $\rho_{\mathrm{m}}(X;Y|U) = 0$, which is equivalent to $X - U - Y$. Hence, we recover Wyner's common information from the information-correlation function, i.e.,

$$C_{\mathrm{IC}}^{(0)}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}).$$

Thus the information-correlation function $C_{\mathrm{IC}}^{(\beta)}$ interpolates between GKW's and Wyner's common information as β decreases from 1 to 0. The generalization of Wyner's common information by Gastpar and Suha [62], also given in (2.27), is defined in the same spirit as $C_{\mathrm{IC}}^{(\beta)}$, in which the conditional maximal correlation in the constraint is replaced by the conditional mutual information.

The distributed common randomness extraction problem formulated by Gács and Körner [60] is a type of key agreement or key generation problem, in which interactive communication is not allowed and also secrecy is not considered. These two assumptions are usually not applicable to practical secret key agreement systems. To ameliorate these limitations, Csiszár and Narayan [46] generalize GKW's common information to the setting in which communication is allowed, and moreover, a helper assists the extractors to extract a higher rate of common randomness from the sources. Besides, they also consider another setting in which the communication can be observed by a wiretapper, and the secrecy is measured by certain information-theoretic quantities. The latter setting is known as the *secret key agreement problem*. The regions of achievable rate tuples for these two settings are characterized in terms of certain mutual information quantities, which recover GKW's common information as extreme cases. Further generalizations of GKW's common information to more complicated networks have also been investigated in the literature; see for example the comprehensive surveys by Sudan, Tyagi, and Watanabe [158], Liang, Poor, and Shamai [111], and Bloch and Barros [21]. Other interesting quantities defined based on Gray–Wyner system, such as the Körner graph entropy [99], the privacy funnel [116], and the excess functional information [109], can be found in Li and El Gamal [108]. These quantities can also considered as generalizations of GKW's and Wyner's common information.

Part II

Extensions of Wyner's Common Information

4

Rényi and Total Variation Common Information

In this section, we extend the notion of Wyner's common information by modifying the discrepancy measure used to quantify the distance between the synthesized distribution $P_{X^nY^n}$ and the n-fold product of the target distribution π_{XY}^n. We analyze how the minimum amount of shared randomness in the distributed source simulation problem (the rate of M_n in Fig. 2.1) changes when we employ Rényi divergences of orders $1+s$ where $s \in [-1,1] \cup \{\infty\}$, their normalized versions, and the total variation (TV) distance in place of the normalized relative entropy in (2.2).

The reader might naturally wonder what the value is in going beyond the traditional normalized relative entropy. For one, in security problems, one is usually not content with having the *normalized* amount of leaked information vanish as the length of the code grows; this is known as *weak secrecy*. Systems that satisfy weak secrecy nevertheless allow an unbounded number of bits to be leaked to a potentially malicious party. This is clearly undesirable. In practical systems, we seek to design codes such that the *unnormalized* amount of leaked information vanishes. This is known as *strong secrecy* [22], [117] in which the average number of bits that is leaked vanishes. Analogously, requiring the normalized relative

entropy between $P_{X^nY^n}$ and π_{XY}^n to vanish is usually not a criterion that is sufficiently stringent. Our objective is to design and analyze codes that drive the unnormalized relative entropy to zero. It turns out that there is typically no additional cost to satisfy this more stringent criterion compared to the normalized case.

More importantly, prior to our work that this section is based on [197], [202], the unnormalized relative entropy was the *strongest* or *most stringent* criterion for measuring the discrepancy between $P_{X^nY^n}$ and π_{XY}^n. Are there families of divergences that further strengthen the unnormalized relative entropy? It turns out that the answer is yes. Since the Rényi divergence D_{1+s} is monotonically non-decreasing in its order $1+s$, if we increase s from zero to infinity and mandate that $D_{1+s}(P_{X^nY^n}\|\pi_{XY}^n)$ vanishes, we obtain a family of Wyner-inspired common information measures that strengthens the original Wyner's common information.

En route to proving coding theorems for the common information when the Rényi divergence (in both its normalized and unnormalized forms) is employed, we find it convenient to segue to working with the TV distance. Via Pinsker-like inequalities relating the Rényi divergence to the TV distance, results concerning the TV distance can be rather conveniently translated to those for the Rényi divergence and *vice versa*.

It would be remiss for us to not mention that the study of common information under Rényi divergence measures leads us to one of the main results of this part of the monograph. In particular, we show in Section 5 that the Rényi common information of order ∞ is exactly the same as the so-called exact common information. This allows us to interpret the latter quantity in a whole new different light, thus providing a pathway to computing it and showing that the exact common information can be strictly larger than Wyner's common information for some joint sources.

Finally, it is worth noting that it is quite natural to use various divergences to measure the discrepancy between two distributions. For instance, Hayashi [76], [79] and Yu and Tan [200] respectively used the KL divergence and the Rényi divergence to study the channel resolvability problem. The latter also applied their results to study the capacity of the wiretap channel under the condition that the security requirement is measured by these generalized measures. Special instances

of Rényi entropies and divergences including the relative entropy, the collision entropy, and the min-entropy (corresponding to the Rényi divergence of order ∞) were used to study various problems in probability theory, cryptography, and quantum information recently. See Bobkov, Chistyakov, and Götze [23], Dodis and Yu [50], Iwamoto and Shikata [88], Hayashi and Tan [81], Tan and Hayashi [161], and Beigi and Gohari [14], and references therein for a non-exhaustive list.

This section starts by formally defining some useful quantities and stating some of their properties. These quantities are used to express bounds or exact expressions for the Rényi and ε-TV common informations (to be defined in Definition 4.5) in terms of single-letter quantities, rendering their computations for a variety of joint sources feasible. We evaluate the Rényi common information for the DSBS. We show that for Rényi orders greater than 1 (resp. in $(0,1]$), the Rényi common information generally exceeds (resp. coincides with) Wyner's common information. This section, being technical in nature, also provides glimpses of how various proofs are intertwined and hinge on some basic results introduced in Sections 1 and 2.

4.1 Preliminary Definitions

We commence by stating a couple of definitions that are used extensively to characterize the common information quantities of interest in this and the following sections.

Definition 4.1. For $s > 0$, the *maximal s-mixed cross entropy* with respect to π_{XY} over all couplings of P_X and P_Y is

$$\begin{aligned}&\mathsf{H}_s(P_X, P_Y \| \pi_{XY})\\ &:= \max_{Q_{XY} \in \mathcal{C}(P_X, P_Y)} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)} + \frac{1}{s} H(Q_{XY}).\end{aligned}$$

When $s = \infty$, the above definition reduces to the *maximal cross entropy* with respect to π_{XY} over all couplings of P_X and P_Y, i.e.,

$$\mathsf{H}_\infty(P_X, P_Y \| \pi_{XY}) := \max_{Q_{XY} \in \mathcal{C}(P_X, P_Y)} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)}. \quad (4.1)$$

Some intuition for these quantities can be gleaned by considering the case $s = \infty$ in (4.1). Consider a sequence of pairs of marginal types $\{(T_X^{(n)}, T_Y^{(n)})\}_{n\in\mathbb{N}} \subset \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$ such that $T_X^{(n)} \in \mathcal{P}_n(\mathcal{X})$ converges to P_X and $T_Y^{(n)} \in \mathcal{P}_n(\mathcal{Y})$ converges to P_Y as $n \to \infty$ (in TV distance, for example). The minimum π_{XY}^n-probability of (x^n, y^n) such that the *marginal* types of x^n and y^n are $T_X^{(n)}$ and $T_Y^{(n)}$ respectively is given by

$$\begin{aligned}
&\min_{T_{x^n}=T_X^{(n)}, T_{y^n}=T_Y^{(n)}} \pi_{XY}^n(x^n, y^n) \\
&\quad \doteq \exp\Bigg(-n \max_{Q_{XY}\in\mathcal{C}(P_X,P_Y)} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)} \Bigg) \\
&\quad = \exp\big(-n\mathsf{H}_\infty(P_X, P_Y\|\pi_{XY})\big). \qquad (4.2)
\end{aligned}$$

The intuitive reason why we consider the *minimum* π_{XY}^n-probability leading to *maximal* cross entropy is that, as alluded to in the introduction of this section, we are considering *strengthenings* of Wyner's common information using the discrepancy measures D_{1+s} for $s \in (0, \infty]$. Consequently, the required resolution or common information rate of M_n would, in general, need to be larger than that for Wyner's common information. In fact, it is determined by the *minimum* of the π_{XY}^n-probability of certain type classes. This will be made clear when we discuss the notion of exact common information in Section 5.

We now state a few properties of the maximal cross-entropy.

Lemma 4.1. Let π_{XY} be a joint distribution on a finite alphabet $\mathcal{X} \times \mathcal{Y}$ and with marginals π_X and π_Y.

1. We have
$$\mathsf{H}_\infty(\pi_X, \pi_Y\|\pi_{XY}) \geq H(\pi_{XY})$$
where equality holds if and only if $\pi_{XY} = \pi_X\pi_Y$.

2. Assume that $\mathrm{supp}(\pi_{XY}) = \mathcal{X} \times \mathcal{Y}$. Then for any pair of distributions P_X and P_Y such that $\mathrm{supp}(P_X) = \mathcal{X}$ and $\mathrm{supp}(P_Y) = \mathcal{Y}$, we have
$$\mathsf{H}_\infty(P_X, P_Y\|\pi_{XY}) \geq \sum_{x,y} P_X(x)P_Y(y) \log \frac{1}{\pi_{XY}(x,y)} \qquad (4.3)$$
where equality holds if and only if $\pi_{XY} = \pi_X\pi_Y$.

We now provide a couple of examples to show that H_∞ can be calculated in closed form for some archetypal joint sources.

Example 4.1. Consider the DSBS in Section 2.3. Fix $P_X = \mathrm{Bern}(a)$ and $P_Y = \mathrm{Bern}(b)$ where $a, b \in [0,1]$. Then

$$\begin{aligned}\mathsf{H}_\infty(P_X, P_Y \| \pi_{XY}) &= \log\frac{1}{\alpha} + \big(\min\{a, \bar{b}\} + \min\{\bar{a}, b\}\big)\log\frac{\alpha}{\beta} \\ &= \log\frac{1}{\alpha} + \min\{a+b, \bar{a}+\bar{b}\}\log\frac{\alpha}{\beta},\end{aligned}$$

where $\alpha = (1-p)/2$ and $\beta = p/2$. Furthermore, when P_X and P_Y are particularized to $\pi_X = \mathrm{Bern}(1/2)$ and $\pi_Y = \mathrm{Bern}(1/2)$,

$$\mathsf{H}_\infty(\pi_X, \pi_Y \| \pi_{XY}) = \log\frac{1}{\beta}.$$

In contrast, the joint entropy is

$$H(\pi_{XY}) = 2\alpha\log\frac{1}{\alpha} + 2\beta\log\frac{1}{\beta} \le \mathsf{H}_\infty(\pi_X, \pi_Y \| \pi_{XY})$$

with equality if and only if $p = 1/2$, i.e., $\alpha = \beta = 1/4$.

Example 4.2. Let $(X, Y) \sim \pi_{XY}$ be a jointly Gaussian source with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$, and correlation coefficient ρ. Let $P_X = \mathcal{N}(\mu_X, \sigma_X^2)$ and $P_Y = \mathcal{N}(\mu_Y, \sigma_Y^2)$. Then,

$$\begin{aligned}&\mathsf{H}_\infty(P_X, P_Y \| \pi_{XY}) \\ &= \log\big(2\pi\sqrt{1-\rho^2}\big) + \frac{1 - \rho\big(\min_{P_{XY} \in \mathcal{C}(P_X, P_Y)} \mathbb{E}[XY]\big)}{1-\rho^2}\log\mathrm{e} \\ &= \log\big(2\pi\sqrt{1-\rho^2}\big) + \frac{1 + \rho(\sigma_X\sigma_Y - \mu_X\mu_Y)}{1-\rho^2}\log\mathrm{e},\end{aligned}$$

where the last equality easily follows from the condition for equality in the Cauchy–Schwarz inequality. We note that this step is equivalent to computing the Wasserstein distance of order 2 between X and $-Y$; see [141, Example 3.2.14]. Furthermore, if $P_X = \pi_X = \mathcal{N}(0,1)$ and $P_Y = \pi_Y = \mathcal{N}(0,1)$ (so $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$),

$$\mathsf{H}_\infty(\pi_X, \pi_Y \| \pi_{XY}) = \log\big(2\pi\sqrt{1-\rho^2}\big) + \frac{\log\mathrm{e}}{1-\rho}.$$

In contrast, the joint (differential) entropy of π_{XY} is

$$H(\pi_{XY}) = \log\left(2\pi e\sqrt{1-\rho^2}\right) \leq \mathsf{H}_\infty(\pi_X, \pi_Y \| \pi_{XY}),$$

with equality if and only if $\rho = 0$, i.e., $(X, Y) \sim \pi_{XY}$ is a pair of independent Gaussian random variables.

The quantities in Definition 4.1 are used to characterize the following upper and lower bounds on the Rényi common information.

Definition 4.2. For $s > 0$, define the *upper pseudo-common information of order* $(1+s)$ as

$$\overline{\Psi}_{1+s}(\pi_{XY}) := \min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY} = \pi_{XY}}} -\frac{1+s}{s} H(XY|W) + \mathbb{E}_{P_W}\left[\mathsf{H}_s(P_{X|W}, P_{Y|W} \| \pi_{XY})\right], \tag{4.4}$$

where the expectation on the second line can be explicitly written as $\sum_w P_W(w) \mathsf{H}_s(P_{X|W=w}, P_{Y|W=w} \| \pi_{XY})$. Similarly, define the *lower pseudo-common information of order* $(1+s)$ as

$$\underline{\Psi}_{1+s}(\pi_{XY}) := \min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY} = \pi_{XY}}} -\frac{1+s}{s} H(XY|W) + \min_{\substack{Q_{WW'} \\ \in \mathcal{C}(P_W, P_W)}} \mathbb{E}_{Q_{WW'}}\left[\mathsf{H}_s(P_{X|W}, P_{Y|W'} \| \pi_{XY})\right], \tag{4.5}$$

where the expectation on the second line can be explicitly written as $\sum_{w,w'} Q_{WW'}(w,w') \mathsf{H}_s(P_{X|W=w}, P_{Y|W'=w'} \| \pi_{XY})$.

Also define $\overline{\Psi}_1(\pi_{XY})$, $\underline{\Psi}_1(\pi_{XY})$, $\overline{\Psi}_\infty(\pi_{XY})$, and $\underline{\Psi}_\infty(\pi_{XY})$ to be the limits of $\overline{\Psi}_{1+s}(\pi_{XY})$ and $\underline{\Psi}_{1+s}(\pi_{XY})$ as $s \downarrow 0$ or $s \to \infty$.

Observe that these two definitions are rather similar. Indeed, if the inner infimum in (4.5) is achieved by a coupling $Q_{WW'} \in \mathcal{C}(P_W, P_W)$ such that $Q_{WW'}(w,w') = P_W(w)\mathbb{1}\{w = w'\}$ for all $(w,w') \in \mathcal{W}^2$, then we have the favorable scenario in which $\overline{\Psi}_{1+s}(\pi_{XY}) = \underline{\Psi}_{1+s}(\pi_{XY})$ for all $s > 0$. This coupling is known as the *equality coupling*.

Even though the expression in (4.4) is somewhat involved, for some special classes of distributions, $\overline{\Psi}_{1+s}(\pi_{XY})$ turns out to be equal to Wyner's common information $C_{\mathrm{W}}(\pi_{XY})$ for all $s \in (0, \infty]$. To state the desired result, we now define a hierarchy of product-like distributions.

Definition 4.3. Consider the following hierarchy of joint distributions.

(a) A *product distribution* $\pi_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is one in which there exists functions $\alpha : \mathcal{X} \to [0, \infty)$ and $\beta : \mathcal{Y} \to [0, \infty)$ such that $\pi_{XY}(x, y) = \alpha(x)\beta(y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. In this case, $\alpha(x) = \sum_{y \in \mathcal{Y}} \pi_{XY}(x, y)$ is the marginal of π_{XY} on $\mathcal{X}$ and similarly for $\beta(y)$. The matrix of probabilities corresponding to π_{XY} has rank 1 and X is independent of Y.

(b) A *pseudo-product distribution* $\pi_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is one in which there exists some subset $\mathcal{A} \subset \mathcal{X} \times \mathcal{Y}$ such that

$$\pi_{XY}(x, y) = \begin{cases} \alpha(x)\beta(y) & (x, y) \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

for some functions $\alpha : \mathcal{X} \to [0, \infty)$ and $\beta : \mathcal{Y} \to [0, \infty)$ such that $\sum_{(x,y) \in \mathcal{A}} \alpha(x)\beta(y) = 1$.

(c) A *Wyner-product distribution* $\pi_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is one in which there exists a distribution $P_W P_{X|W} P_{Y|W}$ attaining the infimum in the definition of $C_{\mathrm{W}}(\pi_{XY})$ in (2.14) such that π_{XY} restricted to $\mathcal{A}_w = \mathrm{supp}(P_{X|W=w}) \times \mathrm{supp}(P_{Y|W=w})$ is a product distribution for all $w \in \mathrm{supp}(P_W)$. In other words,

$$\pi_{XY}(x, y|\mathcal{A}_w) := \frac{\pi_{XY}(x, y)}{\pi_{XY}(\mathcal{A}_w)} \mathbb{1}\{(x, y) \in \mathcal{A}_w\} \tag{4.7}$$

is a product distribution for all $w \in \mathrm{supp}(P_W)$.

It can be seen that a pseudo-product distribution is a Wyner-product distribution. This is because $\mathcal{A}_w \subset \mathcal{A}$ for all w, otherwise there is some $(x, y) \in \mathcal{X} \times \mathcal{Y}$ such that $P_{XY}(x, y) > 0$ but $\pi_{XY}(x, y) = 0$. Since π_{XY} has the property in (4.6), so does it on each $\mathcal{A}_w$.

Obviously, a product distribution is a pseudo-product distribution (take $\mathcal{A}$ to be $\mathcal{X} \times \mathcal{Y}$). However, a pseudo-product distribution need not be a *bona fide* product distribution as the next example shows. Nevertheless, if $\mathrm{supp}(\pi_{XY})$ is a *product set* (i.e., a set $\mathcal{A}$ that can be written as the Cartesian product $\mathcal{X}' \times \mathcal{Y}'$ where $\mathcal{X}' \subset \mathcal{X}$ and $\mathcal{Y}' \subset \mathcal{Y}$), then a pseudo-product distribution is a product distribution. A Venn diagram of these classes of distributions is shown in Fig. 4.1.

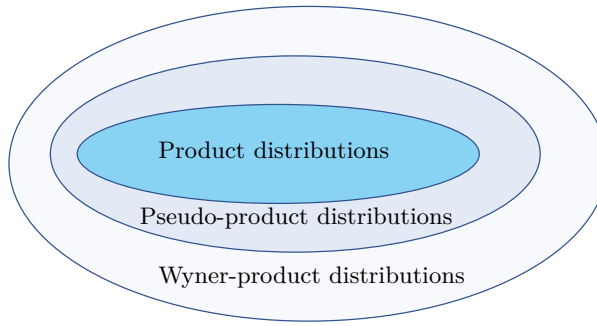


Figure 4.1: Venn diagram of various types of product distributions

Example 4.3. Consider the joint distribution supported on $\{0,1\}^2$ with matrix of probabilities given by

$$\pi_{XY} = \frac{1}{\alpha_0\beta_0 + \alpha_0\beta_1 + \alpha_1\beta_0} \begin{bmatrix} \alpha_0\beta_0 & \alpha_0\beta_1 \\ \alpha_1\beta_0 & 0 \end{bmatrix}, \tag{4.8}$$

where $\alpha_x, \beta_y > 0$ for $x, y \in \{0,1\}$. This is a pseudo-product distribution but not a product distribution. To show the former statement, take the set $\mathcal{A}$ to be $\{0,1\}^2 \setminus \{(1,1)\}$ and functions $\alpha(x) \propto \alpha_x$ and $\beta(y) \propto \beta_y$ for $x, y \in \{0,1\}$. For the latter statement, note that since the rank of the matrix in (4.8) is not one, π_{XY} is not a product distribution.

We now state some useful properties of $\overline{\Psi}_{1+s}$ and $\underline{\Psi}_{1+s}$.

Lemma 4.2. The upper and lower pseudo-common information quantities satisfy the following properties.

(a) For the optimization in (4.4) that defines $\overline{\Psi}_{1+s}(\pi_{XY})$, it suffices to restrict the cardinality $|\mathcal{W}| \leq |\mathcal{X}||\mathcal{Y}|$.

(b) The functions $s \mapsto \overline{\Psi}_{1+s}(\pi_{XY})$ and $s \mapsto \underline{\Psi}_{1+s}(\pi_{XY})$ are non-decreasing in $s > 0$.

(c) As $s \downarrow 1$, the following limiting case holds:

$$\underline{\Psi}_1(\pi_{XY}) \leq \overline{\Psi}_1(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}).$$

(d) As $s \to \infty$, the following limiting cases hold:

$$\begin{aligned}\lim_{s\to\infty} \overline{\Psi}_{1+s}(\pi_{XY}) &= \overline{\Psi}_{\infty}(\pi_{XY}) \\ &= \min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) + \mathbb{E}_{P_W}\left[\mathsf{H}_{\infty}(P_{X|W}, P_{Y|W}\|\pi_{XY})\right],\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\lim_{s\to\infty} \underline{\Psi}_{1+s}(\pi_{XY}) &= \underline{\Psi}_{\infty}(\pi_{XY}) \\ &= \inf_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) \\ &\quad + \inf_{\substack{Q_{WW'} \\ \in \mathcal{C}(P_W, P_W)}} \mathbb{E}_{Q_{WW'}}\left[\mathsf{H}_{\infty}(P_{X|W}, P_{Y|W'}\|\pi_{XY})\right].\end{aligned}$$

(e) For any $s \in (0, \infty]$, $\overline{\Psi}_{1+s}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$ if and only if π_{XY} is a Wyner-product distribution.

Statement (a) says that $\overline{\Psi}_{1+s}(\pi_{XY})$ is efficiently computable as there is a cardinality bound on the auxiliary random variable W. Statement (b) is clear and mirrors that of the operational definition of the Rényi common information we state later. Statements (c) and (d) say that the limit operations (as $s \downarrow 1$ and $s \to \infty$) "commute" with the minimizations. Finally, Statement (e) says that the upper pseudo-common information of any order greater than 1 is the same as Wyner's common information when the target distribution is a Wyner-product distribution so $\overline{\Psi}_{1+s}(\pi_{XY})$ offers another representation of Wyner's common information for this class of distributions.

4.2 Rényi Common Information

In this section, we formally define and state some known results on the Rényi common information. The following definition, which differs from an alternative one in Graczyk and Lapidoth [66], mirrors that of the Wyner's common information from the perspective of the distributed simulation problem (Definition 2.2). In the following, we only consider $s \in (-1, 1] \cup \{\infty\}$.

Definition 4.4. The *normalized Rényi common information*[1] *of order* $1+s$ between a pair of random variables $(X,Y) \sim \pi_{XY}$, denoted as $T_{1+s}(\pi_{XY})$, is the infimum of all rates R such that there exists a sequence of (n,R)-fixed-length distributed source simulation codes $\{(P_{X^n|M_n}, P_{Y^n|M_n})\}_{n\in\mathbb{N}}$ (Definition 2.1) satisfying

$$\lim_{n\to\infty} \frac{1}{n} D_{1+s}\left(P_{X^nY^n} \| \pi_{XY}^n\right) = 0. \tag{4.10}$$

Similarly, the *unnormalized Rényi common information of order* $1+s$, denoted as $\tilde{T}_{1+s}(\pi_{XY})$, is analogous to the normalized version except that the criterion in (4.10) is replaced with the more stringent condition

$$\lim_{n\to\infty} D_{1+s}\left(P_{X^nY^n} \| \pi_{XY}^n\right) = 0. \tag{4.11}$$

A few remarks on Definition 4.4 are in order. First, note that $T_1(\pi_{XY}) = T(\pi_{XY})$ is exactly Wyner's common information, as defined in Definition 2.2. Second, since the unnormalized criterion in (4.11) is more stringent than that of the normalized one in (4.10), we have

$$T_{1+s}(\pi_{XY}) \le \tilde{T}_{1+s}(\pi_{XY}) \quad \text{for all } s \ge -1.$$

Furthermore, by the monotonically non-decreasing nature of the Rényi divergence in its parameter, we see that T_{1+s} and $\tilde{T}_{1+s}$ are also monotonically non-decreasing in their parameter, i.e.,

$$T_{1+s}(\pi_{XY}) \le T_{1+s'}(\pi_{XY}) \quad \text{and} \quad \tilde{T}_{1+s}(\pi_{XY}) \le \tilde{T}_{1+s'}(\pi_{XY})$$

for all $-1 \le s \le s' \le \infty$. Finally, for the special case in which $s = 0$, we obtain

$$T(\pi_{XY}) = T_1(\pi_{XY}) = \tilde{T}_1(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}),$$

where the statement for the unnormalized case (the final equality) comes from the one-shot soft covering lemma as discussed in Remark 2.2.

We are now ready to state the main result of this section; this result is due to the present authors [197], [202].

[1] To be analogous to Definition 2.2, we should term $T_{1+s}(\pi_{XY})$ as the *minimal normalized Rényi distributed simulation rate of order* $1+s$. However, since we have established that the mimimal distributed simulation rate is Wyner's common information in Theorem 2.1, henceforth, to avoid having too many different terminologies, we refer to such fundamental limits (operational definitions) as common information quantities. In other words, we define common information quantities operationally.

Theorem 4.3 (Bounds on Rényi common information)**.** The following hold:

(a) For $s = -1$,
$$\tilde{T}_0(\pi_{XY}) = T_0(\pi_{XY}) = 0.$$

(b) For $s \in (-1, 0]$,
$$\tilde{T}_{1+s}(\pi_{XY}) = T_{1+s}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}).$$

(c) For $s \in (0, 1] \cup \{\infty\}$,
$$\tilde{T}_{1+s}(\pi_{XY}) \geq T_{1+s}(\pi_{XY}) \geq \max\left\{\underline{\Psi}_{1+s}(\pi_{XY}), C_{\mathrm{W}}(\pi_{XY})\right\}, \quad (4.12)$$
and
$$T_{1+s}(\pi_{XY}) \leq \tilde{T}_{1+s}(\pi_{XY}) \leq \overline{\Psi}_{1+s}(\pi_{XY}). \quad (4.13)$$

For $s \in [-1, 0]$, we have tight characterizations of the normalized and unnormalized Rényi common information. For $s \in (0, 1] \cup \{\infty\}$, we only have bounds in general. Despite only having bounds for this case, combining (4.12) and Lemma 4.2(e) yields the following corollary.

Corollary 4.4 (Sufficient condition for equality of Rényi and Wyner's common information)**.** Let $s \in (-1, 1] \cup \{\infty\}$. For any Wyner-product distribution π_{XY},
$$T_{1+s}(\pi_{XY}) = \tilde{T}_{1+s}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}). \quad (4.14)$$

By the inclusions shown in Fig. 4.1, the equality in (4.14) also holds for pseudo-product distributions.

4.3 TV Common Information and Its Strong Converse

Interestingly, the converse part of the proof of Part (b) of Theorem 4.3 requires an auxiliary result concerning the so-called ε-TV common information. We formally define this quantity in the following.

Definition 4.5. For $0 \leq \varepsilon < 1$, the *ε-TV common information* $T_{\varepsilon}^{\mathrm{TV}}(\pi_{XY})$ between a pair random variables $(X, Y) \sim \pi_{XY}$ is the infimum of all

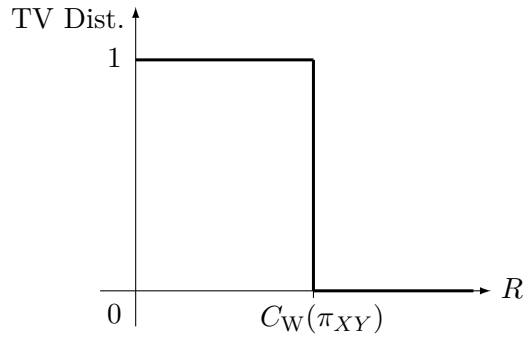


Figure 4.2: Plot of the asymptotic TV distance $\lim_{n\to\infty} |P_{X^nY^n} - \pi_{XY}^n|$ against the rate R. Observe the sharp phase transition at $C_{\mathrm{W}}(\pi_{XY})$.

rates R such that there exists a sequence of (n, R)-fixed-length distributed source simulation codes $\{(P_{X^n|M_n}, P_{Y^n|M_n})\}_{n\in\mathbb{N}}$ (Definition 2.1) satisfying

$$\limsup_{n\to\infty} |P_{X^nY^n} - \pi_{XY}^n| \leq \varepsilon. \tag{4.15}$$

We abbreviate the 0-TV common information as the *TV common information*. If $T_\varepsilon^{\mathrm{TV}}(\pi_{XY})$ does not depend on $0 \leq \varepsilon < 1$, we say that *the strong converse property* holds.

If the strong converse property [71], [179] holds, there is a sharp phase transition in rates such that the TV distance between the synthesized and target distributions can be made arbitrarily small and those rates such that the TV distance necessarily tends to one as the blocklength grows. This is usually a very pleasing phenomenon in Shannon theory because in this case, there is no tradeoff between a permissible error and the rate, at least in the first-order sense; see Fig. 4.2. The tradeoff between the error probability and rate can be seen in the *second-order coding rate* [77], [78], [138], [160].

Unlike the Rényi common information with orders $(1, 2] \cup \{\infty\}$, a full characterization of $T_\varepsilon^{\mathrm{TV}}(\pi_{XY})$ is available. The strong converse part is due to the present authors [197] leveraging an ingenious information spectrum-based [71], [172], single-letterization technique by Oohama [132], while the achievability part can be obtained using arguments in Hayashi [76] or Cuff [48, Lemma IV.1].

Theorem 4.5 (ε-TV Common Information)**.** The following hold:

(a) For any $\varepsilon \in [0,1)$,

$$T_\varepsilon^{\mathrm{TV}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}).$$

(b) Let $R > C_{\mathrm{W}}(\pi_{XY})$. Then, there exists a sequence of rate-R codes such that $|P_{X^nY^n} - \pi_{XY}^n|$ converges to 0 exponentially fast (i.e., $\limsup_{n\to\infty} \frac{1}{n} \log |P_{X^nY^n} - \pi_{XY}^n| < 0$).

(c) Let $R < C_{\mathrm{W}}(\pi_{XY})$. Then, all sequences of rate-R codes result in $|P_{X^nY^n} - \pi_{XY}^n|$ converging to 1 exponentially fast (i.e., $\limsup_{n\to\infty} \frac{1}{n} \log(1 - |P_{X^nY^n} - \pi_{XY}^n|) < 0$).

This theorem is illustrated in an alternative way in Fig. 4.2. In fact, Parts (b) and (c) say that not only do we have matching achievability and strong converse results, these results are also *exponentially* strong in the sense that the TV distance converges to 0 and 1 exponentially fast. This rate of convergence, for the exponentially strong converse part, has implications for the converse proof of Part (b) of Theorem 4.3. We discuss this in its proof sketch in Section 4.5.4.

4.4 Doubly Symmetric Binary Sources

We now consider the DSBS with crossover probability p as depicted in Fig. 2.4. Since the Rényi common information for the case $s \in (-1, 0]$ is exactly Wyner's common information, we can see how it depends on p from Fig. 2.5. Thus, we will only be concerned with the case $s \in (0,1] \cup \{\infty\}$. Here we show that for the DSBS, we have strong numerical evidence that the upper bound on the Rényi common information coincides with the lower bound. Recall the definitions of a, α and β from Section 2.3.

Proposition 4.1. If π_{XY} is a DSBS with crossover probability p and $s \in (0,1]$, the Rényi common information can be upper bounded as

$$\begin{aligned} T_{1+s}(\pi_{XY}) &\le \tilde{T}_{1+s}(\pi_{XY}) \\ &\le -\frac{1+s}{s} \cdot 2h(a) + \frac{1}{s}\Big[h_4(q^*, a-q^*, a-q^*, 1+q^*-2a) \\ &\qquad - 2s(a-q^*)\log\beta\Big], \end{aligned} \tag{4.16}$$

where $h_4(a_1, a_2, a_3, a_4) = -\sum_{i=1}^4 a_i \log a_i$ is the *quaternary entropy* and

$$q^* := \frac{\sqrt{\kappa(\bar{a}-a)^2 + 4\kappa a\bar{a}} - (\kappa(\bar{a}-a)+2a)}{2(\kappa-1)} \tag{4.17}$$

where $\kappa := (\alpha/\beta)^{2s}$. For $s = \infty$,

$$T_\infty(\pi_{XY}) = \tilde{T}_\infty(\pi_{XY}) \tag{4.18}$$

$$= -2h(a) - (1-2a)\log\left(\frac{a^2+\bar{a}^2}{2}\right) - 2a\log(a\bar{a}). \tag{4.19}$$

The idea of the proof of (4.16) in Proposition 4.1 is straightforward but tedious. It involves considering the Markov chain as shown on the right plot of Fig. 2.4 and noticing for the random variables (X, W, Y) (such that $X - W - Y$ forms a Markov chain), the coupling set is

$$\mathcal{C}(P_{X|W=w}, P_{Y|W=w}) = \left\{ \begin{bmatrix} a & a-q \\ a-q & 1+q-2a \end{bmatrix} : 0 \le q \le a \right\}.$$

By noticing this, we can then evaluate the maximal s-mixed cross entropy $\mathsf{H}_s(P_{X|W=w}, P_{Y|W=w} \| \pi_{XY})$ by optimizing over the scalar parameter $0 \le q \le a$ to yield q^* in (4.17). This shows (4.16). We defer the discussion and justification of the Rényi common information of order ∞ in (4.18)–(4.19) to the next section.

Upper and lower bounds for the Rényi common information of order $1 + s = 2$, as well as Wyner's common information for the DSBS are illustrated in Figs. 4.3 and 4.4. Unlike the upper bound, we do not have a closed-form expression for the lower bound so we resort to numerical optimization to evaluate (4.12). To do so, we gradually increase the alphabet size of W from 2 to 10 and we notice for the DSBS that this does not change the resulting curve and in fact it appears to coincide with the upper bound. Hence it is natural to conjecture the upper bound in (4.16) in Proposition 4.1 for the DSBS is tight. Finally, we note that the Rényi common information of orders larger than 1 for the DSBS are strictly larger than Wyner's common information.

4.5 Proof Sketches

In this section, which may be skipped at a first reading, we provide proof sketches of Theorems 4.3 and 4.5. See [197] and [202] for details.

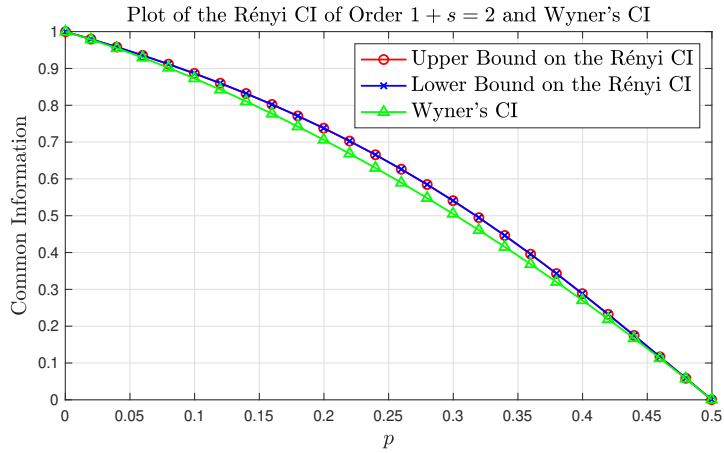


Figure 4.3: Plots of the upper bound in (4.16) and lower bound in (4.12) of the Rényi common information with order $1 + s = 2$, and Wyner's common information

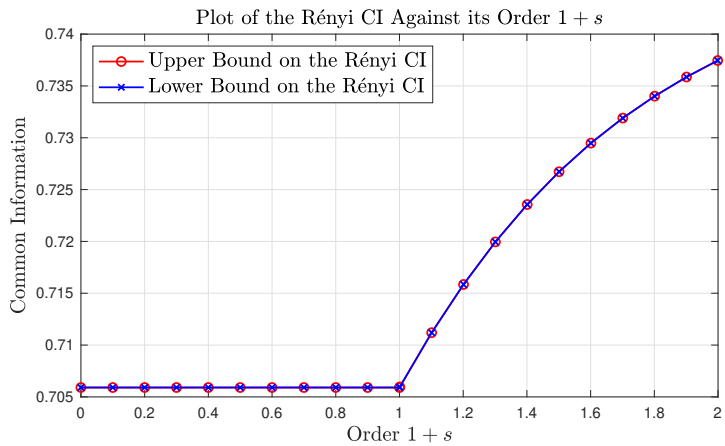


Figure 4.4: Illustrations of the upper bound in (4.16) and lower bound in (4.12) of the Rényi common information as functions of $s \in (-1, 1]$ (or order $1 + s \in (0, 2]$) for the DSBS with crossover probability $p = 0.2$. Notice that for $s \in (-1, 0]$, the Rényi common information $T_{1+s}(\pi_{XY})$ is Wyner's common information $T(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$ so the curve is constant in this range.

High-level ideas and interconnections among the proofs are presented in Table 4.1.

Table 4.1: Summary of proof ideas for the various cases of Theorems 4.3 and 4.5. RCI, WCI, norm. and unnorm. respectively stand for Rényi and Wyner's common information, normalized and unnormalized.

	Achievability	Converse
TV CI	Soft-covering [48], [76] or implied by unnorm. WCI	Information spectrum method [132]
Unnorm. RCI $s \in (-1, 0]$	Implied by unnorm. WCI	Implied by norm. RCI $s \in (-1, 0]$
Norm. RCI $s \in (-1, 0]$	Implied by unnorm. RCI $s \in (-1, 0]$	TV exp. strong converse & Pinsker's inequality (1.10)
Unnorm. RCI $s \in (0, 1] \cup \{\infty\}$	Soft-covering & truncated product dist.	Implied by norm. RCI $s \in (0, 1] \cup \{\infty\}$
Norm. RCI $s \in (0, 1] \cup \{\infty\}$	Implied by unnorm. RCI $s \in (0, 1] \cup \{\infty\}$	Lem 4.7 & Chain rule for coupling sets (Lem. 1.3)

4.5.1 Sketch of the Achievability of Theorem 4.5

First, we note from Theorem 2.4 and Remark 2.2 that if $R > C_{\mathrm{W}}(\pi_{XY})$, there exists a sequence of codes $\{(P_{X^n|M_n}, P_{Y^n|M_n})\}_{n\in\mathbb{N}}$ such that the unnormalized relative entropy $D(P_{X^nY^n}\|\pi_{XY}^n)$ vanishes. By Pinsker's inequality in (1.10) (which says that the relative entropy dominates the TV distance), we see that the TV distance also vanishes.

Alternatively, one can directly leverage the soft-covering lemma for the TV distance (as stated in (2.6) in Lemma 2.2) to show that if $R > C_{\mathrm{W}}(\pi_{XY})$, there exists a sequence of rate-R distributed source simulation codes such that TV distance between $P_{X^nY^n}$ and π_{XY}^n vanishes.

4.5.2 Sketch of the Exponential Strong Converse of Theorem 4.5

The proof of the exponential strong converse requires a careful application of the information spectrum method [172] due to Oohama [132], who used this technique to provide the first proof of the strong converse for the Wyner-Ziv problem [185]. The idea is to express the TV distance $|P_{X^nY^n} - \pi_{XY}^n|$ in terms of the probability of some "error events". Roughly speaking, for any synthesis code with

$$\frac{1}{n}\log|\mathcal{M}_n| \le R, \tag{4.20}$$

we can lower bound the TV distance as

$$|P_{X^nY^n} - \pi_{XY}^n| \geq 1 - \Pr\left(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \mid \mathcal{S}\right) - 3 \cdot 2^{-n\eta}, \tag{4.21}$$

where for any $\eta > 0$ and $Q_{X^nY^n}$ and $Q_{X^nY^n|M_n}$, the sets above are defined as

$$\mathcal{A}_1 := \left\{(x^n, y^n) : \frac{1}{n} \log \frac{\pi_{XY}^n(x^n, y^n)}{Q_{X^nY^n}(x^n, y^n)} \geq -\eta\right\} \times \mathcal{M}_n$$

$$\mathcal{A}_2 := \left\{(x^n, y^n, m) : \frac{1}{n} \log \frac{P_{X^n|M_n}(x^n|m) P_{Y^n|M_n}(y^n|m)}{Q_{X^nY^n|M_n}(x^n, y^n|m)} \geq -\eta\right\}$$

$$\mathcal{A}_3 := \left\{(x^n, y^n, m) : \frac{1}{n} \log \frac{Q_{X^nY^n|M_n}(x^n, y^n|m)}{\pi_{XY}^n(x^n, y^n)} \leq R + \eta\right\},$$

and $\mathcal{S} := \left(\operatorname{supp}(\pi_{XY}^n) \times \mathcal{M}_n\right) \cap \operatorname{supp}\left(P_{X^nY^nM_n}\right)$. The rest of the proof follows from choosing $Q_{X^nY^n}$ and $Q_{X^nY^n|M_n}$ appropriately; the freedom to allow us to do so in converse proofs was first noticed by Hayashi and Nagaoka [80]. One then applies a Chernoff bound to the probability in (4.21) and single-letterizes the resultant exponent. All in all, we obtain that under (4.20),

$$|P_{X^nY^n} - \pi_{XY}^n| \geq 1 - 2^{-nF(R)},$$

where $F(R)$ is an exponent function that is strictly positive when $R < C_{\mathrm{W}}(\pi_{XY})$ and equal to 0 otherwise. This completes the proof of the exponential strong converse.

4.5.3 Sketch of the Achievability of Theorem 4.3(b)

For any $s \in (-1, 0]$, the fact that $\tilde{T}_{1+s}(\pi_{XY}) \leq C_{\mathrm{W}}(\pi_{XY})$ is obvious due to monotonically non-decreasing nature of $s \mapsto \tilde{T}_{1+s}(\pi_{XY})$ and the fact that $\tilde{T}_1(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$.

4.5.4 Sketch of the Converse of Theorem 4.3(b)

The converse part for the case $s \in (-1, 0]$, i.e., that $T_{1+s}(\pi_{XY}) \geq C_{\mathrm{W}}(\pi_{XY})$ is more interesting and leverages a Pinsker-like relationship between the TV distance and the Rényi divergence due to Sason [151], which we restate here as it may be of independent interest.

Lemma 4.6 (Pinsker-like inequality for Rényi divergence)**.** For any $s > -1$,

$$\inf_{P_X, Q_X : |P_X - Q_X| \geq \epsilon} D_{1+s}(P_X \| Q_X) = \inf_{q \in [0, 1-\epsilon]} d_{1+s}(q + \epsilon \| q),$$

and for any $s \in (-1, 0)$,

$$\inf_{q \in [0, 1-\epsilon]} d_{1+s}(q + \epsilon \| q) \geq \left[\min \left\{ 1, \frac{1+s}{s} \right\} \log \frac{1}{1-\epsilon} + \frac{1}{s} \log 2 \right]^+,$$

where

$$d_{1+s}(p \| q) := \begin{cases} \frac{1}{s} \log \left(p^{1+s} q^{-s} + \bar{p}^{1+s} \bar{q}^{-s} \right), & s \geq -1, s \neq 0 \\ p \log \frac{p}{q} + \bar{p} \log \frac{\bar{p}}{\bar{q}}, & s = 0 \end{cases}$$

denotes the *binary Rényi divergence of order* $1 + s$.

Remark 4.1. Sason [151] showed that

$$\inf_{q \in [0, 1-\epsilon]} d_{1/2}(q + \epsilon \| q) = \log \frac{1}{1 - \epsilon^2}.$$

Remark 4.2. Gilardoni [64] showed for $\alpha = 1 + s \in (0, 1)$ that

$$\inf_{P_X, Q_X : |P_X - Q_X| \geq \epsilon} D_\alpha(P_X \| Q_X) \geq \frac{1}{2} \alpha \epsilon^2 + \frac{1}{9} \alpha (1 + 5\alpha - 5\alpha^2) \epsilon^4.$$

These two remarks imply that the minimal Rényi divergence of order less than 1 subject to the TV distance between the two distributions having TV distance ϵ behaves quadratically in ϵ. Thus, these can be considered as Pinsker-type inequalities for the Rényi divergence.

Using Lemma 4.6, the converse part for Theorem 4.3(b) is obvious. If $R < C_{\mathrm{W}}(\pi_{XY})$, the TV distance converges to 1 exponentially fast. In other words,

$$|P_{X^n Y^n} - \pi_{XY}^n| \geq 1 - 2^{-n\delta_n}$$

for some sequence $\{\delta_n\}_{n \in \mathbb{N}} \subset [0, \infty)$ satisfying $\liminf_{n \to \infty} \delta_n > 0$. Thus using Lemma 4.6 (with $1 - 2^{-n\delta_n}$ in place of ϵ),

$$\begin{aligned} \liminf_{n \to \infty} \frac{1}{n} D_{1+s}(P_{X^n Y^n} \| \pi_{XY}^n) &\geq \liminf_{n \to \infty} \left\{ \min \left\{ 1, \frac{1+s}{s} \right\} \delta_n + \frac{\log 2}{ns} \right\} \\ &> 0, \end{aligned}$$

showing that if the rate R is strictly smaller than Wyner's common information $C_{\mathrm{W}}(\pi_{XY})$, the normalized Rényi divergence cannot converge to zero. Thus, $T_{1+s}(\pi_{XY}) \geq C_{\mathrm{W}}(\pi_{XY})$ for any $s \in (-1, 0)$. The case $s = 0$ follows from the converse for Wyner's common information.

4.5.5 Sketch of the Achievability of Theorem 4.3(c)

We only consider $s \in (0, 1]$ since the proof ideas for $s = \infty$ are similar to those for $s \in (0, 1]$. The achievability of Theorem 4.3(c) follows by carefully evaluating the one-shot soft-covering result in Lemma 2.5. We set $\pi_U, P_{U|W}, P_W$, and R to be π_{XY}^n, $P_{X^nY^n|W^n} = P_{X^n|W^n} P_{Y^n|W^n}$, P_{W^n} and nR respectively. Note that if there exists a sequence of distributions $\{P_{W^n} P_{X^n|W^n} P_{Y^n|W^n}\}_{n\in\mathbb{N}}$ such that $D_{1+s}(P_{X^nY^n} \| \pi_{XY}^n) \to 0$ and

$$R > \limsup_{n\to\infty} \frac{1}{n} D_{1+s}(P_{X^nY^n|W^n} \| \pi_{XY}^n | P_{W^n}), \tag{4.22}$$

then from Lemma 2.5, there exists a sequence of distributed source simulation codes $\{(P_{X^n|M_n}, P_{Y^n|M_n})\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty} D_{1+s}(P_{X^nY^n|M_n} \| \pi_{X^nY^n} | P_{M_n}) \\
&\leq \limsup_{n\to\infty} \frac{1}{s} \log \Big[\exp\left(sD_{1+s}(P_{X^nY^n|W^n} \| \pi_{XY}^n | P_{W^n}) - nsR\right) \\
&\qquad + \exp\left(sD_{1+s}(P_{X^nY^n} \| \pi_{XY}^n)\right) \Big] \\
&\leq \limsup_{n\to\infty} \frac{1}{s} \log \Big[\exp\big(sD_{1+s}(P_{X^nY^n|W^n} \| \pi_{XY}^n | P_{W^n}) - nsR\big) + 1 \Big] \\
&= 0,
\end{aligned}$$

where the last equality follows from (4.22). Thus, for $s \in (0, 1]$,

$$\tilde{T}_{1+s}(\pi_{XY}) \leq \inf \limsup_{n\to\infty} \frac{1}{n} D_{1+s}(P_{X^nY^n|W^n} \| \pi_{XY}^n | P_{W^n}), \tag{4.23}$$

where the infimum is over all sequences $\{P_{W^n} P_{X^n|W^n} P_{Y^n|W^n}\}_{n\in\mathbb{N}}$ such that $D_{1+s}(P_{X^nY^n} \| \pi_{XY}^n) \to 0$.

As a result, the achievability proof reduces to finding a tractable joint distribution $P_{W^n} P_{X^n|W^n} P_{Y^n|W^n}$ such that the conditional Rényi diver-

gence in (4.23) can be single-letterized. We first choose a distribution $Q_{WXY} \in \mathcal{P}(\mathcal{W} \times \mathcal{X} \times \mathcal{Y})$ such that $Q_{XY} = \pi_{XY}$ and

$$\begin{aligned} P_{W^n}(w^n) &\propto Q_W^n(w^n) \mathbb{1}\left\{w^n \in \mathcal{T}_{\epsilon'}(Q_W)\right\}, \\ P_{X^n|W^n}(x^n|w^n) &\propto Q_{X|W}^n(x^n|w^n) \mathbb{1}\left\{x^n \in \mathcal{T}_{\epsilon}(Q_{WX}|w^n)\right\}, \\ P_{Y^n|W^n}(x^n|w^n) &\propto Q_{Y|W}^n(y^n|w^n) \mathbb{1}\left\{y^n \in \mathcal{T}_{\epsilon}(Q_{WY}|w^n)\right\}, \end{aligned}$$

where $0 < \epsilon' < \epsilon \leq 1$. This triple is known as a *truncated product distribution*, also used by Vellambi and Kliewer [169] and has two desirable features. First, it behaves like a *bona fide* product distribution. Indeed,

$$P_{X^nY^n}(x^n, y^n) \leq \frac{\pi_{XY}^n(x^n, y^n)}{1 - \gamma_n} \quad \text{for all } (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n,$$

where, roughly speaking, $\gamma_n = o(1)$ represents the probability of atypical sets. This property ensures that the constraint $D_{1+s}(P_{X^nY^n} \| \pi_{XY}^n) \to 0$ is satisfied and the single-letterization of the conditional Rényi divergence in (4.23) is tractable. Secondly, any triple of sequences (w^n, x^n, y^n) generated from the truncated product distribution has marginal types T_{w^n,x^n} and T_{w^n,y^n} that are close to Q_{WX} and Q_{WY} respectively, so necessary approximations of types by distributions can be done to yield that the right-hand side of (4.23) is not larger than $\overline{\Psi}_{1+s}(\pi_{XY})$, which in turn implies that $\tilde{T}_{1+s}(\pi_{XY}) \leq \overline{\Psi}_{1+s}(\pi_{XY})$ for $s \in (0, 1]$.

Truncated product distributions will also be used extensively in the next section on exact common information; see Section 5.3.1.

4.5.6 Sketch of the Converse of Theorem 4.3(c)

Note that for $s \in (0, 1]$, $T_{1+s}(\pi_{XY}) \geq C_{\mathrm{W}}(\pi_{XY})$ is obvious in view of the monotonically non-decreasing nature of $s \mapsto T_{1+s}$. Hence, we only have to show that $T_{1+s}(\pi_{XY}) \geq \underline{\Psi}_{1+s}(\pi_{XY})$ for $s \in (0, 1]$. This proceeds in a few steps and we highlight the key ideas.

First, we derive a lower bound for $T_{1+s}(\pi_{XY})$ in terms of a multi-letter expression. This hinges on the following non-asymptotic converse lemma which is due to the present authors [200].

Lemma 4.7. Let M be a uniform random variable on the set $[L]$ and let $P_{X|M}$ be an arbitrary stochastic map, whence $P_{XM}(x, m) =$

$L^{-1}P_{X|M}(x|m)$ for all $(x,m) \in \mathcal{X} \times [L]$. Then for $s \in [0,\infty]$ and any distribution π_X, we have

$$D_{1+s}(P_X\|\pi_X) \geq \max\left\{D_{1+s}(P_{MX}\|P_M\pi_X) - \log L, D_{1+s}(P_X\|\pi_X)\right\}.$$

By particularizing π_X, $P_{X|M}$, P_M, and $\log L$ to be π_{XY}^n, $P_{X^n|M_n}$, $P_{Y^n|M_n}$, P_{M_n} and nR respectively, Lemma 4.7 implies that

$$T_{1+s}(\pi_{XY}) \geq \inf \limsup_{n\to\infty} \frac{1}{n} D_{1+s}(P_{M_nX^nY^n}\|P_{M_n}\pi_{XY}^n), \tag{4.24}$$

where the infimum runs over all distributed source simulation codes $\{(P_{X^n|M_n}, P_{Y^n|M_n})\}_{n\in\mathbb{N}}$ such that $\frac{1}{n}D_{1+s}(P_{X^nY^n}\|\pi_{XY}^n) \to 0$.

Now, it is easy to check by elementary calculus (see, for example, Shayevitz [154] and Anantharam [4]) that the Rényi divergence admits a variational representation of the form

$$D_{1+s}(P\|Q) = \max_{\tilde{Q}\in\mathcal{P}(\mathcal{X})} \frac{1}{s}\left\{\sum_{x\in\mathcal{X}} \tilde{Q}(x)\log\left(P^{1+s}(x)Q^{-s}(x)\right) + H(\tilde{Q})\right\}.$$

By particularizing P, Q, and $\tilde{Q}$ above to $P_{M_nX^nY^n}$, $P_{M_n}\pi_{XY}^n$, and $\tilde{Q} = P_{M_n}Q_{X^nY^n|M_n}$ respectively, and performing some algebraic manipulations, (4.24) yields

$$\begin{aligned} T_{1+s}(\pi_{XY}) \geq \inf \limsup_{n\to\infty} & -\frac{1+s}{s} H(X^nY^n|M_n) \\ & + \frac{1}{s} \max_{\substack{Q_{X^nY^n|M_n}\in \\ \mathcal{C}(P_{X^n|M_n}, P_{Y^n|M_n})}} \mathbb{E}_Q\left[\log \frac{1}{(\pi^n(X^n,Y^n))^s Q(X^n,Y^n|M_n)}\right], \end{aligned} \tag{4.25}$$

where the infimum runs over the same sequence of distributions under the same constraints as in (4.24).

The multi-letter expression in (4.25) consists of two parts. The entropy term can be single-letterized using standard techniques in network information theory. In particular,

$$\begin{aligned} \frac{1}{n}H(X^nY^n|M_n) &= \frac{1}{n}\sum_{i=1}^n H(X_i|X^{i-1}M_n) + \frac{1}{n}\sum_{i=1}^n H(Y_i|Y^{i-1}M_n) \\ &= H(X_J|X^{J-1}M_nJ) + H(Y_J|Y^{J-1}M_nJ), \end{aligned}$$

where we introduced the random variable $J \sim \mathrm{Unif}[n]$ which is independent of (M_n, X^n, Y^n). The second term is more involved but the main

ingredient for simplifying it is the chain rule for couplings (Lemma 1.3) which implies that for any function $f : \mathcal{P}(\mathcal{X}^n \times \mathcal{Y}^n) \to \mathbb{R}$,

$$\begin{aligned}
&\max_{Q_{X^nY^n|M_n} \in \mathcal{C}(P_{X^n|M_n}, P_{Y^n|M_n})} f(Q_{X^nY^n|M_n}) \\
&\quad \geq \max_{\substack{Q_{X^nY^n|M_n} \in \\ \prod_{i=1}^n \mathcal{C}(P_{X_i|X^{i-1}M_n}, P_{Y_i|Y^{i-1}M_n})}} f\Big(\prod_{i=1}^n Q_{X_iY_i|X^{i-1}Y^{i-1}W}\Big). \quad (4.26)
\end{aligned}$$

Observe that for a fixed $m \in \mathcal{M}_n$ and $Q_{X^nY^n|M_n=m}$, we have

$$\begin{aligned}
&\mathbb{E}_{Q_{X^nY^n|M_n=m}}\left[\log \frac{1}{(\pi^n(X^n,Y^n))^s Q(X^n,Y^n|M_n)}\right] \\
&= \sum_{i=1}^n \sum_{x_i,y_i} \sum_{x^{i-1},y^{i-1}} Q(x^{i-1},y^{i-1}|m) Q(x_i,y_i|x^{i-1},y^{i-1},m) \\
&\qquad \times \log \frac{1}{\pi(x_i,y_i)^s Q(x_i,y_i|x^{i-1},y^{i-1},m)} \\
&\geq \sum_{i=1}^n \min_{\substack{\tilde{Q}_{X^{i-1}Y^{i-1}|M_n} \in \\ \mathcal{C}(P_{X^{i-1}|M_n}, P_{Y^{i-1}|M_n})}} \sum_{x_i,y_i} \sum_{x^{i-1},y^{i-1}} \tilde{Q}_{X^{i-1}Y^{i-1}|M_n}(x^{i-1},y^{i-1}|m) \\
&\qquad \times Q(x_i,y_i|x^{i-1},y^{i-1},m) \log \frac{1}{\pi(x_i,y_i)^s Q(x_i,y_i|x^{i-1},y^{i-1},m)}. \quad (4.27)
\end{aligned}$$

Now, the main idea from here onwards is to use the consequence of the chain rule for couplings in (4.26) and then to justify swapping the maximization in (4.25) and the minimization in (4.27). Then we see that we will be left with an inner maximization over couplings $Q_{XY|UVW} \in \mathcal{C}(P_{X|UW}, P_{Y|VW})$ where $U := X^{J-1}$, $V = Y^{J-1}$, $X := X_J$, $Y := Y_J$ and $W := (M_n, J)$. These ideas, together with a few additional approximation arguments, gives rise to the maximization over couplings $Q_{XY} \in \mathcal{C}(P_{X|W=w}, P_{Y|W=w'})$ in the definition of H_s and minimization over couplings $Q_{WW'} \in \mathcal{C}(P_W, P_W)$ in (4.5). This completes our sketch of the proof of the lower bound (converse) $T_{1+s}(\pi_{XY}) \geq \Psi_{1+s}(\pi_{XY})$ for the case $s \in (0, 1]$.

5

Exact Common Information

In this section, we depart from two key assumptions that we employed in the previous sections on Wyner's and Rényi common information. These assumptions are that *fixed-length* codes are used and *approximate* generation of the target distribution π_{XY}^n is desired. By *fixed-length*, we mean that the shared or common randomness M_n in Fig. 2.1 takes on values in the set $\mathcal{M}_n$, which contains no more than 2^{nR} elements. Equivalently the bit string that corresponds to M_n has length no larger than nR. By *approximate* generation, we mean that we only demand that some notion of the "discrepancy" between $P_{X^nY^n}$ and π_{XY}^n converges to zero as the length of the code grows. The metrics that govern the discrepancy include the (normalized and unnormalized) Rényi divergence and the TV distance.

In this section, we consider the distributed source simulation problem under the assumptions that the codes used are allowed to be *variable-length* and we demand that the reconstruction $P_{X^nY^n}$ of the target distribution π_{XY}^n be *exact* for some blocklength $n \in \mathbb{N}$; this formulation is due to Kumar, Li, and El Gamal [103]. These distinctions are analogous to the problem of lossless source coding [42] in which there are also two formulations. The first, which mirrors our discussion in Section 4, is of

fixed-length lossless source coding with approximate reconstruction of the source. Shannon [153] showed using ideas from what is now known as the *asymptotic equipartition property* (AEP) [42, Ch. 3] that the minimum rate of compression is the entropy of the source. The second formulation, analogous to the current section, is *variable-length source coding* in which each source symbol is allowed to be encoded to bit strings of varying lengths and the *minimal average codeword length* is sought under the constraint of zero-error reconstruction. In this case, the asymptotic minimal average per symbol codeword length is also the entropy of the source. This can be achieved via a variety of schemes including the Shannon–Fano–Elias code [42, Ch. 5] or the Huffman code [86].

One of the key benefits of variable-length coding in data compression is the ability to obtain *exact* reconstructions. In contrast, if we are constrained to use fixed-length codes for the distributed source simulation problem, then we would require a much higher rate to obtain an exact reconstruction, namely, $\min\{\log|\mathcal{X}|, \log|\mathcal{Y}|\}$ in the worst case. Since the fundamental limits for the lossless and zero-error source compression problems are the same—i.e., the Shannon entropy $H(X)$—it is natural to wonder whether the same is true for the distributed source simulation problem. This was an open problem posed at the 2014 International Symposium on Information Theory by Kumar, Li, and El Gamal [103].

In this section, we answer this question in the negative. The way we do so is to show a surprising equivalence between the unnormalized Rényi common information of order ∞ and the exact common information. This is done by relating both problems at the operational level. To wit, we show that if there exists a rate-R exact common information code, this code can be suitably modified to be a rate-R order-∞ Rényi common information code and *vice versa*. Thus, the family of Rényi common information provides a *bridge* between Wyner's common information and the exact common information; see Fig. 5.1. We recall that the Rényi common information is monotonically non-decreasing in its order and as we have seen from Section 4.4 for the DSBS, it can be *strictly* increasing. Thus, exact generation of a joint source requires strictly larger rate compared to approximate generation in general, answering the open problem posed by Kumar, Li, and El Gamal [103]. We identify

classes of sources for which the exact common information is equal to Wyner's common information and provide intuition for why no extra rate is needed for exact generation of these sources [169]. We extend our discussion to sources with continuous alphabets and provide bounds on the exact common information for the bivariate Gaussian source.

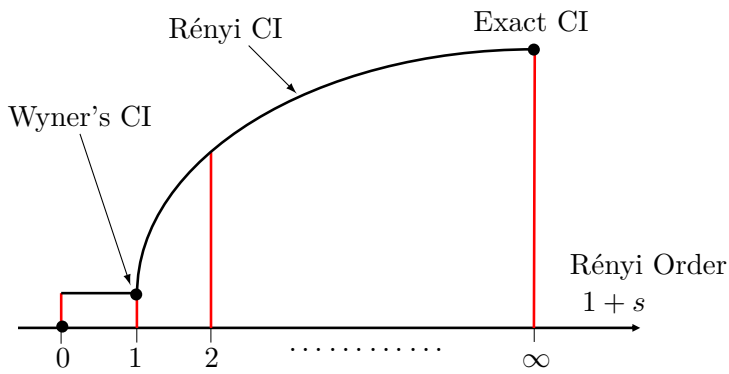


Figure 5.1: A schematic showing that the Rényi common information provides a bridge between Wyner's common information and the exact common information

5.1 Preliminary Definitions

Define $\{0,1\}^* = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ to be the set of all finite-length binary strings. Denote the alphabet of the common random variable W_n as the countable set $\mathcal{W}_n$. We assume, without loss of generality, that $\mathcal{W}_n \subset \mathbb{N}$. Recall that a *prefix-free code* is a source code in which no codeword is a prefix of another. Consider any prefix-free code $f : \mathcal{W}_n \to \{0,1\}^*$ which yields the codebook $\mathcal{C} = \{f(w) : w \in \mathcal{W}_n\}$ whose elements are known as *codewords*. Then for each symbol $w \in \mathcal{W}_n$ and the code f, let $\ell_f(w)$ be the *length* of the codeword $f(w)$.

Example 5.1. Let $\mathcal{W}_n = \{1,2,3,4\}$. Consider the prefix-free code

$$f(1) = 0, \quad f(2) = 10, \quad f(3) = 110, \quad \text{and} \quad f(4) = 111.$$

This code has corresponding lengths

$$\ell_f(1) = 1, \quad \ell_f(2) = 2, \quad \ell_f(3) = 3, \quad \text{and} \quad \ell_f(4) = 3.$$

Definition 5.1. The *expected codeword length* of a code $f : \mathcal{W}_n \to \{0,1\}^*$ for compressing the source $W_n \sim P_{W_n}$ is

$$L_f(W_n) = \mathbb{E}[\ell_f(W_n)] = \sum_{w \in \mathcal{W}_n} P_{W_n}(w)\ell_f(w).$$

Definition 5.2. An (n, R)*-variable-length distributed source simulation code* $(P_{W_n}, f, P_{X^n|W_n}, P_{Y^n|W_n})$ consists of

- A distribution P_{W_n} supported on a countable set $\mathcal{W}_n \subset \mathbb{N}$;
- A prefix-free source code $f : \mathcal{W}_n \to \{0,1\}^*$;
- A pair of random mappings called *processors* $P_{X^n|W_n} \in \mathcal{P}(\mathcal{X}^n|\mathcal{W}_n)$ and $P_{Y^n|W_n} \in \mathcal{P}(\mathcal{Y}^n|\mathcal{W}_n)$;

such that the per-symbol expected codeword length

$$\frac{L_f(W_n)}{n} \leq R. \tag{5.1}$$

As usual, n and R are known as the *blocklength* and *rate* respectively. Observe that if f in Definition 5.2 is constrained to output codewords whose lengths do not exceed nR and W_n is constrained to be uniform on $\mathcal{W}_n$, the expected length constraint in (5.1) is automatically satisfied and the definition reverts to that for a fixed-length distributed source simulation code (cf. Definition 2.1).

Using a variable-length code, we assume that the common random variable W_n is transmitted in an error-free manner to the two processors which then generate the *synthesized distribution*

$$P_{X^nY^n}(x^n, y^n) = \sum_{w \in \mathcal{W}_n} P_{W_n}(w)P_{X^n|W_n}(x^n|w)P_{Y^n|W_n}(y^n|w). \tag{5.2}$$

Definition 5.3. The *exact common information* $T_{\mathrm{Ex}}(\pi_{XY})$ between a pair of random variables $(X, Y) \sim \pi_{XY}$ is the infimum of all rates R such that there exists an (n, R)-variable-length distributed source simulation code $(P_{W_n}, f_n, P_{X^n|W_n}, P_{Y^n|W_n})$ satisfying

$$P_{X^nY^n} = \pi_{XY}^n \quad \text{for some } n \in \mathbb{N},$$

where $P_{X^nY^n}$ denotes the synthesized distribution in (5.2).

Remark 5.1. Note that since we assume that f_n is a *prefix-free* code, we can synthesize target distributions of arbitrarily long lengths by concatenating codewords $f_n(W_n)$ and decoding them *uniquely*.

It is well known [42, Sec. 5.4] that the minimal per-letter expected codeword length $L_{f_n}(W_n)$ for a prefix-free code f_n satisfies

$$H(W_n) \le L_{f_n}(W_n) < H(W_n) + 1. \tag{5.3}$$

The lower bound follows from Kraft's inequality [101] while the upper bound follows from Shannon's code assignment $\ell_{f_n}(w) = \lceil -\log P_{W_n}(w) \rceil$. The bounds in (5.3) are colloquially known as *Shannon's zero-error compression theorem*.

Define the *common entropy* of the joint source $(X, Y) \sim \pi_{XY}$ as

$$G(\pi_{XY}) := \min_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} H(W). \tag{5.4}$$

It can be shown that $\lim_{n\to\infty} \frac{1}{n} G(\pi_{XY}^n) = \inf_{n\in\mathbb{N}} \frac{1}{n} G(\pi_{XY}^n)$ so we can define the *common entropy rate* of the source $(X, Y) \sim \pi_{XY}$ as

$$\overline{G}(\pi_{XY}) := \lim_{n\to\infty} \frac{G(\pi_{XY}^n)}{n} = \inf_{n\in\mathbb{N}} \frac{G(\pi_{XY}^n)}{n}. \tag{5.5}$$

Kumar, Li, and El Gamal [103] showed the following proposition.

Proposition 5.1. The exact common information

$$T_{\mathrm{Ex}}(\pi_{XY}) = \overline{G}(\pi_{XY}).$$

Since the proof, due to Kumar, Li, and El Gamal [103], is brief and insightful, we reproduce it here.

Proof. For the achievability, fix any $R > \overline{G}(\pi_{XY})$. From (5.5), we see that for sufficiently large n, $\overline{G}(\pi_{XY}) \ge \frac{1}{n}(G(\pi_{XY}^n) + 1)$. By the upper bound in Shannon's zero-error compression theorem in (5.3), we see that it is possible to exactly generate $(X^n, Y^n) \sim \pi_{XY}^n$ with rate at most $\frac{1}{n}(G(\pi_{XY}^n) + 1)$. Hence, R is achievable.

For the converse part, assume R is achievable. Then there exists a simulation code $(P_{W_n}, f_n, P_{X^n|W_n}, P_{Y^n|W_n})$ with large enough blocklength n that exactly generates $(X^n, Y^n) \sim \pi_{XY}^n$. Therefore, by the lower bound in Shannon's zero-error compression theorem, $R \ge \frac{1}{n} G(\pi_{XY}^n)$ for some n. Thus, by (5.5), $R \ge \overline{G}(\pi_{XY})$. □

In view of Proposition 5.1, a variable-length synthesis code can be represented by the triple $(P_{W_n}, P_{X^n|W_n}, P_{Y^n|W_n})$ and the dependence on the prefix-free source code f_n can be omitted.

A particularly important property of the common entropy rate is stated in the following lemma [103].

Lemma 5.1. The common entropy rate is an upper bound on Wyner's common information, i.e.,

$$\overline{G}(\pi_{XY}) \geq C_{\mathrm{W}}(\pi_{XY}). \tag{5.6}$$

This result can be shown by first defining W_n^* to be the common random variable achieving the common entropy of the product source $G(\pi_{XY}^n)$. Then it follows that

$$\begin{aligned}
\overline{G}(\pi_{XY}) &= \lim_{n\to\infty} \frac{1}{n} H(W_n^*) \\
&\geq \lim_{n\to\infty} \frac{1}{n} I(W_n^*; X^n Y^n) \\
&\geq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} I(W_n^*; X_i Y_i) \qquad (5.7) \\
&\geq \min_{P_W P_{X|W} P_{Y|W}: P_{XY} = \pi_{XY}} I(W; XY) \\
&= C_{\mathrm{W}}(\pi_{XY}), \qquad (5.8)
\end{aligned}$$

where (5.7) follows because the source $\{(X_i, Y_i)\}_{i=1}^{\infty}$ is memoryless. The central question of this section is whether the inequality in (5.6) is strict for some sources π_{XY}. We answer this in the affirmative in Section 5.6.

5.2 Equivalence

We now establish a somewhat surprising equivalence between the exact and unnormalized Rényi common information of order ∞ and characterize them via an alternative multi-letter expression. This equivalence was noticed by the present authors [204].

Theorem 5.2 (Equivalence of exact and ∞-Rényi common information)**.** For a source with distribution π_{XY} defined on a finite alphabet $\mathcal{X} \times \mathcal{Y}$,

$$T_{\mathrm{Ex}}(\pi_{XY}) = \tilde{T}_{\infty}(\pi_{XY}). \tag{5.9}$$

Furthermore, the quantities in (5.9) are equal to

$$\lim_{n\to\infty} \frac{\overline{\Psi}(\pi_{XY}^n)}{n}, \tag{5.10}$$

where $\overline{\Psi}$ is the upper pseudo-common information of order ∞, i.e.,

$$\begin{aligned}\overline{\Psi}(\pi_{XY}) := \overline{\Psi}_\infty(\pi_{XY}) \overset{(4.9)}{=} \min_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} & -H(XY|W) \\ & + \mathbb{E}_{P_W}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W}\|\pi_{XY})\right]\end{aligned} \tag{5.11}$$

and H_∞ is the maximal cross-entropy defined in (4.1).

Similarly to the common entropy, the function $\overline{\Psi}(\pi_{XY}^n)$ can also be shown to be subadditive; hence, the limit in (5.10) exists due to Fekete's lemma [57]. We also remark that to compute the minimization in (5.11), we can restrict the cardinality of W to be no more than $|\mathcal{X}||\mathcal{Y}|$.

Theorem 5.2 says that for any joint source defined on a finite alphabet, the exact common information is equal to the unnormalized Rényi common information of order ∞. The former is defined in Definition 5.3. The latter, on the other hand, involves the seemingly stringent condition $D_\infty(P_{X^nY^n}\|\pi_{XY}^n) \to 0$ which is equivalent to

$$\max_{(x^n,y^n)\in\mathrm{supp}(P_{X^nY^n})} \frac{P_{X^nY^n}(x^n,y^n)}{\pi_{XY}^n(x^n,y^n)} = 1 + o(1). \tag{5.12}$$

This is surprising as two aspects of the definition have changed, yet they serendipitously resulted in common information quantities that coincide. The theorem also presents an alternative multi-letter expression for the exact common information in (5.10). This comes about due to the evaluation of $\tilde{T}_\infty(\pi_{XY})$ instead of $T_{\mathrm{Ex}}(\pi_{XY})$ and is more useful than the common entropy rate in (5.5) for the purposes of single-letterization.

5.2.1 Sketch of the Proof of $T_{\mathrm{Ex}}(\pi_{XY}) = \tilde{T}_\infty(\pi_{XY})$

Because the equality in (5.9) is particularly important, we sketch its proof in this subsection. For the impatient reader, this subsection can be omitted at a first reading.

We first show that $T_{\mathrm{Ex}}(\pi_{XY}) \le \tilde{T}_\infty(\pi_{XY})$. To so, we let R be achievable rate for a fixed-length distributed source simulation code

for which $D_\infty(P_{X^nY^n}\|\pi_{XY}^n) \to 0$. This means that for every $\epsilon > 0$, for all sufficiently large n, there exists a fixed-length simulation code $(P_{X^n|M_n}, P_{Y^n|M_n})$ with rate R such that $D_\infty(P_{X^nY^n}\|\pi_{XY}^n) \le \epsilon$, where the synthesized distribution $P_{X^nY^n}$ is defined in (2.1). We show that R is also achievable for exact reconstruction using a variable-length code. The idea is to consider a "mixing" scheme in which with high probability, we use the given fixed-length code, and with low probability, we use a completely lossless code.

By the definition of D_∞, we have $P_{X^nY^n}(x^n, y^n) \le 2^\epsilon \pi_{XY}^n(x^n, y^n)$ for all $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$. Define

$$P_{\hat{X}^n\hat{Y}^n}(x^n, y^n) := \frac{2^\epsilon \pi_{XY}^n(x^n, y^n) - P_{X^nY^n}(x^n, y^n)}{2^\epsilon - 1},$$

which is a valid distribution (as it is non-negative and sums to one). Note now that π_{XY}^n is a *mixture distribution* that can be written as a convex combination of $P_{X^nY^n}$ and $P_{\hat{X}^n\hat{Y}^n}$ as follows

$$\pi_{XY}^n(x^n, y^n) = 2^{-\epsilon} P_{X^nY^n}(x^n, y^n) + (1 - 2^{-\epsilon}) P_{\hat{X}^n\hat{Y}^n}(x^n, y^n). \quad (5.13)$$

The variable-length code first generates a Bernoulli random variable $U \sim \text{Bern}(2^{-\epsilon})$ which can be described by 1 bit. It then transmits U to the two processors. If $U = 1$, the encoder also generates $M_n \sim \text{Unif}[2^{nR}]$ and uses the given fixed-length code $(P_{X^n|M_n}, P_{Y^n|M_n})$ with rate R to generate $P_{X^nY^n}$. Otherwise (if $U = 0$), the encoder generates $(\hat{X}^n, \hat{Y}^n) \sim P_{\hat{X}^n\hat{Y}^n}$ using $\log(|\mathcal{X}||\mathcal{Y}|)$ bits per source symbol. By the law of total probability, the distribution generated is *exactly* π_{XY}^n in (5.13). Since $\Pr(U = 1) = 2^{-\epsilon}$, the average codeword length required is

$$\frac{1}{n} + 2^{-\epsilon} R + (1 - 2^{-\epsilon}) \log(|\mathcal{X}||\mathcal{Y}|). \quad (5.14)$$

Taking $n \to \infty$ and then $\epsilon \downarrow 0$ yields the conclusion that R is an achievable rate for the exact synthesis of π_{XY}^n. Because we use a mixture distribution in (5.13), this technique is known as the *mixture decomposition technique* and will also be used in Section 6.

We next argue that $\tilde{T}_\infty(\pi_{XY}) \le T_{\text{Ex}}(\pi_{XY})$. For this purpose, assume that there exists a (k, R)-variable-length distributed source simulation code $(P_{W_k}, P_{X^k|W_k}, P_{Y^k|W_k})$ that exactly generates π_{XY}^k, i.e.,

$$\pi_{XY}^k(x^k, y^k) = \sum_{w_k} P_{W_k}(w_k) P_{X^k|W_k}(x^k|w_k) P_{Y^k|W_k}(y^k|w_k). \quad (5.15)$$

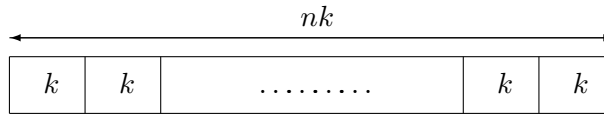


Figure 5.2: Code construction for the proof that $\tilde{T}_\infty(\pi_{XY}) \le T_{\mathrm{Ex}}(\pi_{XY})$.

For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that R can come arbitrarily close to $\frac{1}{k}H(W_k)$. In particular, we can assume

$$R \le \frac{H(W_k)}{k}(1+2\epsilon).$$

Using the above variable-length code, we construct a fixed-length super-code which is the concatenation of n independent length-k blocks with rate

$$R' := \frac{H(W_k)}{k}(1+\epsilon). \tag{5.16}$$

See Fig. 5.2. We will now verify that this super-code has the desired property in (5.12). The common random variable $W_k^n = (W_{k1}, \ldots, W_{kn}) \sim P_{W_k}^n$ and $(P_{X_k|W_k}^n, P_{Y_k|W_k}^n)$ is the pair of processors. The main idea of the proof is to suitably "shape" a uniform random variable M_n into the non-uniform W_k^n so that the variable-length code $(P_{W_k}^n, P_{X_k|W_k}^n, P_{Y_k|W_k}^n)$ can be used subsequently.

We now design a function f such that given a uniform random variable $M_n \sim \mathrm{Unif}(\mathcal{M}_n)$ where $\mathcal{M}_n = [2^{nkR'}]$, the function f applied to M_n simulates $P_{W_k}^n$ in the sense that the Rényi divergence of order ∞ from $P_{f(M_n)}$ to $P_{W_k}^n$ vanishes. This function is constructed as follows [201, Theorem 7]. It maps multiple elements of $\mathcal{M}_n$ to each sequence in the weakly typical set $\mathcal{A}_\epsilon^{(n)}(P_{W_k})$. For each sequence w_k^n, we control the number of elements that are mapped to w_k^n to be directly proportional to $P_{W_k}^n(w_k^n)$. By the asymptotic equipartition property [42], $W_k^n \sim P_{W_k}^n$ is distributed almost uniformly on $\mathcal{A}_\epsilon^{(n)}(P_{W_k})$. According to the theory of (Rényi) source resolvability [73], [157], [201], since $\frac{1}{n}\log|\mathcal{M}_n| = kR' \ge (1+\epsilon)H(W_k)$ (cf. (5.16)),

$$\lim_{n\to\infty} D_\infty(P_{f(M_n)} \| P_{W_k}^n) = 0.$$

This essentially follows because the $P_{W_k}^n$-probability of the weakly typical set $\mathcal{A}_\epsilon^{(n)}(P_{W_k})$ converges to one exponentially fast as $n \to \infty$. Now, we

consider the concatenation scheme in Fig. 5.3. From (5.15) and the constructed fixed-length code, we have

$$P_{W_k}^n \to P_{X^k|W_k}^n P_{Y^k|W_k}^n \to \pi_{XY}^{kn} \quad \text{and}$$
$$P_{f(M_n)} \to P_{X^k|W_k}^n P_{Y^k|W_k}^n \to P_{X^{kn}Y^{kn}},$$

where $P_X \to V_{Y|X} \to P_Y$ means that P_Y is the induced output distribution when the input distribution is P_X and the stochastic kernel (channel) is $V_{Y|X}$. Thus, by the data-processing inequality for the Rényi divergence,

$$D_\infty(P_{X^{kn}Y^{kn}} \| \pi_{XY}^{kn}) \le D_\infty(P_{f(M_n)} \| P_{W_k}^n) \to 0 \quad \text{as } n \to \infty.$$

This concludes the proof that the Rényi divergence of order ∞ converges to zero along blocklengths n that are integers multiples of k. For other n's, a standard approximation argument suffices. Thus, R is an achievable rate for the approximate synthesis problem under the Rényi divergence of order ∞, which in turn implies $R \ge \tilde{T}_\infty(\pi_{XY})$.

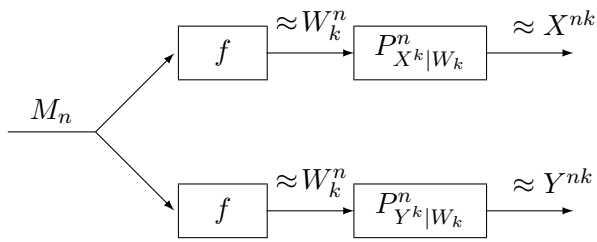


Figure 5.3: Concatenation scheme for the proof that $\tilde{T}_\infty(\pi_{XY}) \le T_{\mathrm{Ex}}(\pi_{XY})$.

5.3 Single-Letter Bounds for Exact Common Information

In anticipation of evaluating $T_{\mathrm{Ex}}(\pi_{XY}) = \tilde{T}_\infty(\pi_{XY})$ for various sources π_{XY}, we provide single-letter bounds on these common information quantities. Recall the definition of the upper pseudo-common informa-

tion of order ∞, namely $\overline{\Psi}(\pi_{XY}) = \overline{\Psi}_\infty(\pi_{XY})$ in (5.11). Additionally, we rename the lower pseudo-common information of order ∞ as

$$\begin{aligned}\underline{\Psi}(\pi_{XY}) &:= \underline{\Psi}_\infty(\pi_{XY}) \\ &= \inf_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} -H(XY|W) \\ &\qquad + \inf_{\substack{Q_{WW'} \\ \in \mathcal{C}(P_W, P_W)}} \mathbb{E}_{Q_{WW'}}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W'} \| \pi_{XY})\right], \quad (5.17)\end{aligned}$$

where the last equality is the same as (4.2). Note that the only difference between $\underline{\Psi}(\pi_{XY})$ and $\overline{\Psi}(\pi_{XY})$ is the inner sum; the former is an optimization over all couplings $Q_{WW'} \in \mathcal{C}(P_W, P_W)$ while latter replaces this optimization with P_W.

We are now ready to state single-letter bounds on the (unnormalized) Rényi common information of order ∞ which is the same as the exact common information (cf. Theorem 5.2).

Theorem 5.3 (Bounds on exact common information)**.** For a source with distribution π_{XY} defined on a finite alphabet $\mathcal{X} \times \mathcal{Y}$,

$$\begin{aligned}\max\left\{\underline{\Psi}(\pi_{XY}), C_{\mathrm{W}}(\pi_{XY})\right\} \le T_\infty(\pi_{XY}) &\le \tilde{T}_\infty(\pi_{XY}) \\ &= T_{\mathrm{Ex}}(\pi_{XY}) \le \overline{\Psi}(\pi_{XY}). \quad (5.18)\end{aligned}$$

We note that this theorem is just a combination of Theorem 4.3(c) (concerning bounds on the Rényi common information of orders in $(1, 2] \cup \{\infty\}$) and Theorem 5.2.

5.3.1 Coding Scheme and Type Overflow Phenomenon

We now comment on the coding scheme used to achieve the upper bound $\tilde{T}_\infty(\pi_{XY}) \le \overline{\Psi}(\pi_{XY})$ in (5.18). It shares many similarities to the achievability of the Rényi common information for orders in $(1, 2]$ as outlined in Section 4.5.5. We use truncated product distributions. Sequences generated from these distributions are useful in upper bounding T_∞ and Wyner's common information. This is because under both scenarios, $X^n - W_n - Y^n$ forms a Markov chain. Hence given $W_n = w$, the support of $P_{X^n|W_n}(\cdot|w)P_{Y^n|W_n}(\cdot|w)$ is a *product set*, i.e., $\mathcal{A} \times \mathcal{B}$ where $\mathcal{A} \subset \mathcal{X}^n$ and $\mathcal{B} \subset \mathcal{Y}^n$. Thus the support of $P_{X^nY^n}$ is the *union of product sets*.

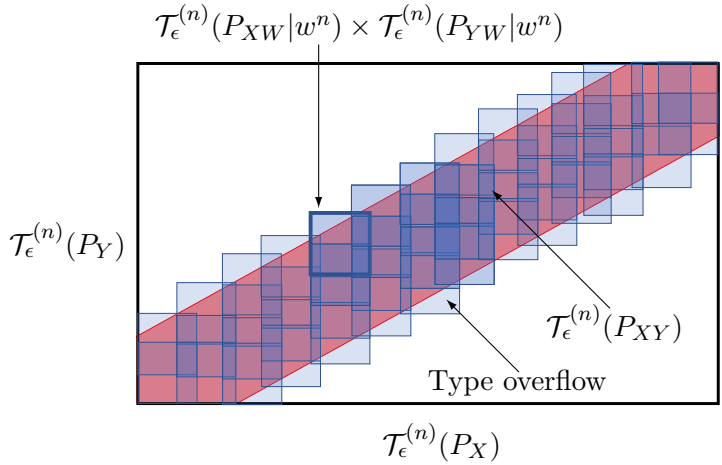


Figure 5.4: Illustration of the type overflow phenomenon. The rectangle represents the Cartesian product of the marginal typical sets $\mathcal{T}_\epsilon^{(n)}(P_X) \times \mathcal{T}_\epsilon^{(n)}(P_Y)$. The jointly typical set is the shaded diagonal area $\mathcal{T}_\epsilon^{(n)}(P_{XY})$. Each small square is the Cartesian product of conditionally typical sets $\mathcal{T}_\epsilon^{(n)}(P_{XW}|w^n) \times \mathcal{T}_\epsilon^{(n)}(P_{YW}|w^n)$ indexed by the codewords w^n in $\mathcal{C} = \{w^n(m) : m \in [2^{nR}]\}$. As can be seem from this schematic, the union of small squares can be a strict superset of the jointly typical set.

This union consists of not only the jointly typical set $\mathcal{T}_\epsilon^{(n)}(P_{XY})$ but also *other* joint type classes. This is what we term as the *type overflow phenomenon*. See Fig. 5.4 for a schematic. Designing a synthesis code that achieves Wyner's common information (under the relative entropy measure) only requires sequences in the jointly typical set $\mathcal{T}_\epsilon^{(n)}(P_{XY})$ to be well-simulated. However, ∞-Rényi approximate synthesis requires *all* the sequences in the support of $P_{X^nY^n}$ to be well-simulated; see (5.12). Hence, the type overflow phenomenon does not affect Wyner's synthesis asymptotically, but plays a critical role in determining the optimal rate for ∞-Rényi approximate synthesis (or equivalently, exact synthesis). Truncated i.i.d. coding turns out to be a convenient approach to control all possible types of the output sequence of a code to mitigate the effects of type overflow.

5.3.2 Intuition for the upper bound (Achievability)

Let us provide some intuition for the upper bound in (5.18). Exact synthesis requires that $P_{X^nY^n}$ multiplicatively approximates π_{XY}^n pointwise for all $(x^n, y^n) \in \mathrm{supp}(P_{X^nY^n})$; see (5.12). By using the truncated i.i.d. coding technique, we can essentially restrict our attention to random variables $(W^n, X^n) \in \mathcal{T}_\epsilon^{(n)}(P_{WX})$ and $(W^n, Y^n) \in \mathcal{T}_\epsilon^{(n)}(P_{WY})$. Let $M_n \in \mathcal{M}_n$ be the common randomness for approximate synthesis based on the Rényi divergence of order ∞. Then, for sufficiently large n,

$$\begin{aligned} &P_{X^nY^n}(x^n, y^n) \\ &\quad\approx \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} P_{X^n|W^n}(x^n|w^n(m)) P_{Y^n|W^n}(y^n|w^n(m)) \\ &\quad\approx \exp(-nR) N(x^n, y^n) \exp\Big(- n\big(H(X|W) + H(Y|W)\big)\Big), \end{aligned} \tag{5.19}$$

where $N(x^n, y^n)$ is the number of $w^n(m)$ sequences in the codebook $\mathcal{C}$ that are jointly typical with x^n and jointly typical with y^n (individually). On the other hand, by a similar intuition for the maximal cross-entropy in (4.2), we have

$$\begin{aligned} &\min_{(x^n,y^n) \in \mathrm{supp}(P_{X^nY^n})} \pi_{XY}^n(x^n, y^n) \\ &\quad\approx \min_{\substack{(w^n,x^n,y^n):\\ T_{w^nx^n} \approx P_{WX},\, T_{w^ny^n} \approx P_{WY}}} \pi_{XY}^n(x^n, y^n) \\ &\quad\approx \exp\Big(- n\mathbb{E}_W\big[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \| \pi_{XY})\big]\Big). \end{aligned} \tag{5.20}$$

Since $N(x^n, y^n) \geq 1$ for $(x^n, y^n) \in \mathrm{supp}(P_{X^nY^n})$ and $H(X|W) + H(Y|W) = H(XY|W)$, combining (5.12), (5.19) and (5.20) yields that any rate R satisfying

$$R \geq -H(XY|W) + \mathbb{E}_W\big[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \| \pi_{XY})\big]$$

is achievable. Taking the minimum of the right-hand side over all joint distributions $P_W P_{X|W} P_{Y|W}$ such that $P_{XY} = \pi_{XY}$ and noticing that the resultant expression is $\overline{\Psi}(\pi_{XY})$ (defined in (5.11)) completes the proof that $\tilde{T}_\infty(\pi_{XY}) \leq \overline{\Psi}(\pi_{XY})$.

5.4 Equality of Exact and Wyner's Common Information

As we have seen from Section 4.4, the Rényi common information for orders larger than 1 can be strictly larger than Wyner's common information. We now discuss various conditions under which Wyner's common information $C_{\mathrm{W}}(\pi_{XY})$ is equal to the exact common information $T_{\mathrm{Ex}}(\pi_{XY})$. Under these conditions, in view of the monotonicity of $\tilde{T}_{1+s}(\pi_{XY})$ for $s \geq -1$, the entire family of Rényi common information for all positive orders is equal to $C_{\mathrm{W}}(\pi_{XY})$.

Theorem 5.4. For every Wyner-product distribution $\pi_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ (see Definition 4.3),

$$T_{\mathrm{Ex}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}). \tag{5.21}$$

This theorem, due to the present authors [204], follows easily by combining Lemma 4.2(e) and Theorem 5.2. The former for the case $s = \infty$ is restated here for ease of reference.

Lemma 5.5. The equality $\overline{\Psi}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$ holds if and only if π_{XY} is a Wyner-product distribution.

Since every pseudo-product distribution is a Wyner-product distribution (cf. Fig. 4.1), the equality in (5.21) also applies to pseudo-product distributions. The fact that pseudo-product distributions result in the equality $T_{\mathrm{Ex}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$ was also realized by Vellambi and Kliewer [170], albeit via a different consideration.

We now provide a brief justification of Lemma 5.5.

Proof Sketch of Lemma 5.5. If π_{XY} is a Wyner-product distribution, by the second part of Lemma 4.1,

$$\mathsf{H}_\infty(P_{X|W=w}, P_{Y|W=w} \| \pi_{XY}) = \sum_{x,y} P(x|w)P(y|w) \log \frac{1}{\pi(x,y)}, \tag{5.22}$$

where $P_W P_{X|W} P_{Y|W}$ is a joint distribution that attains the infimum in $C_{\mathrm{W}}(\pi_{XY})$. Taking the expectation with respect to P_W, and noticing that $P_{XY} = \pi_{XY}$, we obtain

$$\mathbb{E}\big[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \| \pi_{XY})\big] = H(XY). \tag{5.23}$$

Substituting $P_W P_{X|W} P_{Y|W}$ into the definition of $\overline{\Psi}(\pi_{XY})$ in (5.11), we obtain $\overline{\Psi}(\pi_{XY}) \le C_{\mathrm{W}}(\pi_{XY})$. Obviously (see Theorem 5.3), the reverse inequality holds and so $\overline{\Psi}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$.

Now suppose that $\overline{\Psi}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$. Let $P_W P_{X|W} P_{Y|W}$ attain the infimum in the upper pseudo-common information of order ∞, namely $\overline{\Psi}(\pi_{XY})$. Then for every $w \in \mathrm{supp}(P_W)$, $\mathrm{supp}(P_{X|W=w}) \times \mathrm{supp}(P_{Y|W=w}) \subset \mathrm{supp}(\pi_{XY})$. Otherwise, $\mathbb{E}[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \| \pi_{XY})] = \infty$, contradicting the optimality of $P_W P_{X|W} P_{Y|W}$. At the same time,

$$\begin{aligned}\underline{\Psi}(\pi_{XY}) &= -H(XY|W) + \mathbb{E}\big[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \| \pi_{XY})\big] \\ &\ge -H(XY|W) + H(XY) \ge C_{\mathrm{W}}(\pi_{XY}),\end{aligned}$$

where the first inequality follows from (4.3). Thus, all inequalities above are equalities. In particular, $P_W P_{X|W} P_{Y|W}$ also attains the infimum in $C_{\mathrm{W}}(\pi_{XY})$ and (5.23) holds. This implies that (5.22) holds for all $w \in \mathrm{supp}(P_W)$. By the second part of Lemma 4.1, for all $w \in \mathrm{supp}(P_W)$, π_{XY} is a product distribution on $\mathcal{A}_w = \mathrm{supp}(P_{X|W=w}) \times \mathrm{supp}(P_{Y|W=w})$. Hence π_{XY} is a Wyner-product distribution. □

We now state a couple of other easy-to-verify sufficient conditions for Wyner's common information to be equal to the exact common information. These conditions are due to Vellambi and Kliewer [169].

Corollary 5.6. Let π_{XY} be a distribution defined on a finite alphabet. Let $P_W P_{X|W} P_{Y|W}$ achieve the infimum in $C_{\mathrm{W}}(\pi_{XY})$. If either

$$H(W|XY) = 0 \quad \text{or} \tag{5.24}$$

$$\sum_{w \in \mathcal{W}} H(X|W=w) H(Y|W=w) = 0, \tag{5.25}$$

then $T_{\mathrm{Ex}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$.

If either of these conditions hold, it is easy to see that π_{XY} is a Wyner-product distribution; thus Theorem 5.4 generalizes these sufficient conditions. Indeed, if $H(W|XY) = 0$ (i.e., (5.24) holds), $\mathcal{X} \times \mathcal{Y}$ can be partitioned into a collection of subsets $\{\mathcal{A}_w : w \in \mathcal{W}\}$. For each w, $P_{XY|W=w}$ is the restriction of π_{XY} to $\mathcal{A}_w$, defined in (4.7). Since by assumption, $X - W - Y$ holds, we have $P_{XY|W=w} = P_{X|W=w} P_{Y|W=w}$.

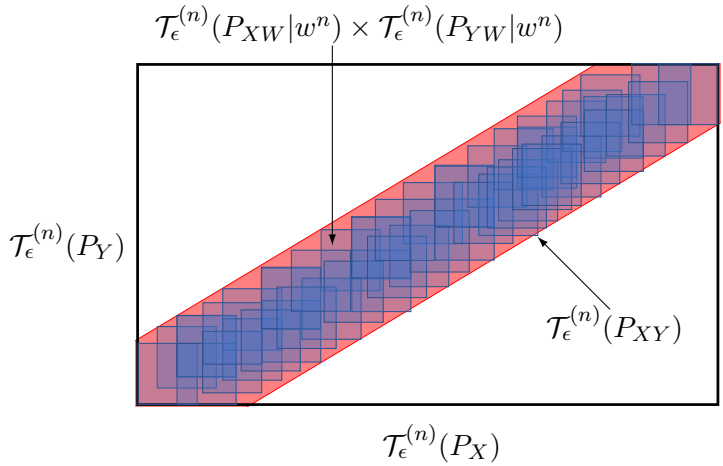


Figure 5.5: Illustration of no type overflow. If the condition in (5.25) holds, there is no type overflow. Indeed, the jointly typical set $\mathcal{T}_\epsilon^{(n)}(P_{XY})$ is approximately the union of the Cartesian product of conditional typical sets as written in (5.26).

This implies the restriction of π_{XY} to each $\mathcal{A}_w$ can be written as a product distribution, i.e., π_{XY} is a Wyner-product distribution.

On the other hand, if (5.25) holds, either the support of $P_{X|W=w}$ or the support of $P_{Y|W=w}$ (or both) is a singleton. Hence, the restriction of any joint distribution to $\mathrm{supp}(P_{X|W=w}) \times \mathrm{supp}(P_{Y|W=w})$ can be written as $P_X(x)\mathbb{1}\{y = y_0\}$ or $P_Y(y)\mathbb{1}\{x = x_0\}$ for some $(P_X, P_Y) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$ and some $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$, i.e., π_{XY} is a Wyner-product distribution. Another way of seeing this, and as illustrated in Fig. 5.5, is that if $H(X|W = w)H(Y|W = w) = 0$ for each w, then the coupling set $\mathcal{C}(P_{X|W}, P_{Y|W})$ is a singleton consisting solely of the distribution $P_{X|W}P_{Y|W}$. In other words, the jointly typical set is approximately the union of Cartesian products of conditionally typical sets, i.e.,

$$\mathcal{T}_\epsilon^{(n)}(P_{XY}) \approx \bigcup_{w^n \in \mathcal{C}} (\mathcal{T}_\epsilon^{(n)}(P_{XW}|w^n) \times \mathcal{T}_\epsilon^{(n)}(P_{YW}|w^n)). \tag{5.26}$$

Hence, the jointly typical set $\mathcal{T}_\epsilon^{(n)}(P_{XY})$ is approximately $\mathrm{supp}(P_{X^nY^n})$, nullifying the type overflow phenomenon as discussed in Section 5.3.1. Thus, the equality $T_{\mathrm{Ex}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$ holds.

In the remaining sections, we turn to examples to illustrate the exact common information for various joint sources.

5.5 Symmetric Binary Erasure Sources

Recall the SBES introduced in Section 2.4. For this source in which its Wyner's common information is stated in Proposition 2.2, observe the following important feature from Fig. 2.6. If $W = 0$, then we know for sure that $X = 0$. Similarly if $W = 1$, we also know that $X = 1$. The final possibility is that $W = \mathrm{e}$, in which case $Y = \mathrm{e}$. That is to say, for all $w \in \mathcal{W} = \{0, 1, \mathrm{e}\}$, either $H(X|W = w) = 0$ or $H(Y|W = w) = 0$ (indicated by the red arrows in Fig. 2.6). Thus, by the sufficient condition in (5.25) in Corollary 5.6, we know that $T_{\mathrm{Ex}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$. This is summarized in the following proposition, which was originally proved from first principles (i.e., without using Corollary 5.6) by Kumar, Li, and El Gamal [103].

Proposition 5.2. The exact common information for the SBES with erasure probability p is

$$T_{\mathrm{Ex}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}) = \begin{cases} 1 & p \leq 0.5 \\ h(p) & p > 0.5 \end{cases}.$$

This function is illustrated in Fig. 2.7.

5.6 Doubly Symmetric Binary Sources

As we have just seen, in the case of the SBES, the exact common information can be computed in closed form and is equal to Wyner's common information. This begs the following two questions. Are there any other sources for which the exact common information $T_{\mathrm{Ex}}(\pi_{XY})$ can be computed in closed form? From what we have gathered up to this point, in general, $T_{\mathrm{Ex}}(\pi_{XY})$ can only be expressed via a *multi-letter* form (in Proposition 5.1) or via single-letter *bounds* (in Theorem 5.3). In addition, are there sources for which the exact common information is strictly larger than Wyner's common information? The latter is the content of an open question posed by Kumar, Li, and El Gamal [103].

In this section, we consider the DSBS with crossover probability $p \in (0, 1/2)$ as described in Section 2.3. Surprisingly, the exact common information can also be evaluated in closed form. Recall from Section 2.3

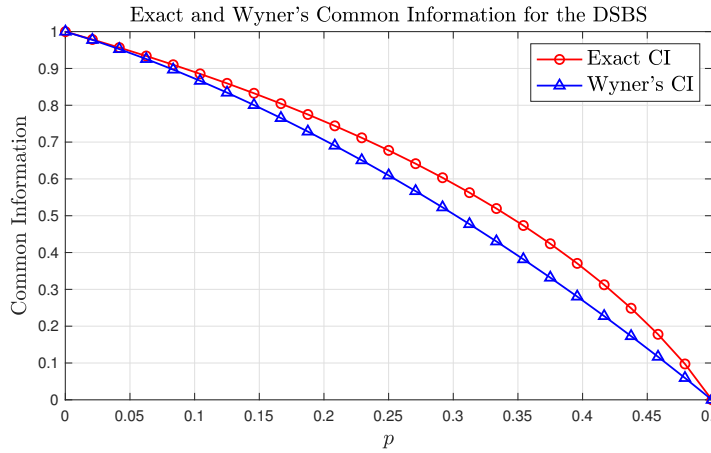


Figure 5.6: Plot of the exact and Wyner's common information for the DSBS

that $a = a(p) \in (0, 1/2)$ is defined as the unique number satisfying $a * a = p$.

Proposition 5.3. The exact common information of the DSBS with crossover probability p is

$$T_{\mathrm{Ex}}(\pi_{XY}) = -2h(a) - (1-2a)\log\left(\frac{1}{2}(a^2+\bar{a}^2)\right) - 2a\log(a\bar{a}).$$

This result follows by Theorem 5.2 and (4.18)–(4.19) in Proposition 4.1 where we stated T_∞ and $\tilde{T}_\infty$ for the DSBS. From Proposition 5.3 and Proposition 2.1, we see that the difference between the exact and Wyner's common information is

$$T_{\mathrm{Ex}}(\pi_{XY}) - C_{\mathrm{W}}(\pi_{XY}) = 2a^2 \log\left(\frac{a^2+\bar{a}^2}{2a\bar{a}}\right) > 0.$$

This difference is positive for all $a \in (0, 1/2)$; equivalently, $p \in (0, 1/2)$. This answers the open problem posed by Kumar, Li, and El Gamal [103]. We conclude that there exists sources (namely the DSBS with $p \in (0, 1/2)$) for which the exact common information strictly exceeds Wyner's common information. Note that the DSBS does not satisfy any of the sufficient conditions in Section 5.4. The two common information quantities and their gap are illustrated in Fig. 5.6.

Proof Sketch of Proposition 5.3. Because $T_{\mathrm{Ex}}(\pi_{XY}) = \tilde{T}_{\infty}(\pi_{XY})$ (Theorem 5.2), it suffices to prove (4.18)–(4.19). The crux in the evaluation of both bounds is in the understanding of the maximal cross-entropy terms in $\overline{\Psi}(\pi_{XY})$ and $\underline{\Psi}(\pi_{XY})$.

In view of (4.13), we first evaluate the upper bound $\overline{\Psi}(\pi_{XY})$ for the DSBS. We set $P_W P_{X|W} P_{Y|W}$ as the distribution that achieves the minimum in Wyner's common information. Hence $W \sim \mathrm{Bern}(1/2)$ and $X = W \oplus A$ and $Y = W \oplus B$ where A and B are mutually independent $\mathrm{Bern}(a)$ random variables. The key terms in $\overline{\Psi}(\pi_{XY})$ are thus the maximal cross-entropies for each w. For a fixed w, this can be simplified as follows

$$\begin{aligned}
&\mathsf{H}_{\infty}(P_{X|W=w}, P_{Y|W=w} \| \pi_{XY}) \\
&= \max_{Q_{XY} \in \mathcal{C}(P_{X|W=w}, P_{Y|W=w})} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)} \\
&= \log \frac{1}{\alpha} + 2 \min\{a, \bar{a}\} \log \frac{\alpha}{\beta} = \log \frac{1}{\alpha} + 2a \log \frac{\alpha}{\beta}.
\end{aligned}$$

See Example 4.1 for details of this calculation. Hence, we have

$$\overline{\Psi}(\pi_{XY}) \le -2h(a) + \log \frac{1}{\alpha} + 2a \log \frac{\alpha}{\beta}.$$

Recalling that $\alpha = \frac{1}{2}(a^2 + \bar{a}^2)$ and $\beta = a\bar{a}$ completes the proof of the upper bound.

The evaluation of the lower bound in (5.17) is more involved but is essentially inspired by Wyner [182] in his evaluation of Wyner's common information for the DSBS. Let $\alpha_w := \Pr(X = 0 | W = w)$ and $\beta_w = \Pr(Y = 0 | W = w)$. The condition that $P_{XY} = \pi_{XY}$ implies that $\mathbb{E}[\alpha_W] = \mathbb{E}[\beta_W] = \Pr(X = 0) = \Pr(Y = 0) = 1/2$ and $\mathbb{E}[\alpha_W \beta_W] = \Pr(X = 0, Y = 0) = \alpha$. In view of these equalities, we lower bound the maximal cross-entropy for each w as follows

$$\begin{aligned}
&\mathsf{H}_{\infty}(P_{X|W=w}, P_{Y|W=w'} \| \pi_{XY}) \\
&= \max_{Q_{XY} \in \mathcal{C}(P_{X|W=w}, P_{Y|W=w'})} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)} \\
&= \log \frac{1}{\alpha} + \Big(\min\{\alpha_w, \overline{\beta_{w'}}\} + \min\{\overline{\alpha_w}, \beta_{w'}\} \Big) \log \frac{\alpha}{\beta}
\end{aligned}$$

$$\geq \log \frac{1}{\alpha} + \Big(\min\{\alpha_w, \overline{\alpha_w}\} + \min\{\beta_{w'}, \overline{\beta_{w'}}\} \Big) \log \frac{\alpha}{\beta}.$$

Now, we plug this lower bound into the definition of $\underline{\Psi}(\pi_{XY})$ in (5.17). We conclude by leveraging ideas from Wyner [182]; these ideas include the concavity of the functions $x \in (0, 1/2) \mapsto h(x)$ and $x \in \mathbb{R}_+ \mapsto \sqrt{x}$, to solve the optimization problem in (5.17). See [204] for details. □

5.7 Jointly Gaussian Sources

In this final section, we briefly discuss the generalization of the concept of exact common information to continuous sources and, specifically, the important family of jointly Gaussian sources. Per the theme of this section, we aim to establish that the unnormalized Rényi common information of order ∞ is equal to the exact common information. However, this is not true in general for arbitrary continuous sources. Nevertheless, the proof that $T_{\mathrm{Ex}}(\pi_{XY}) \geq \tilde{T}_\infty(\pi_{XY})$ (in the second half of Section 5.2.1) goes through verbatim as the weakly typical set and its properties, which are applicable to arbitrary sources, are exploited therein. It also holds that $T_{\mathrm{Ex}}(\pi_{XY}) = \tilde{T}_\infty(\pi_{XY})$ for sources with countable alphabets; this follows from another typicality and truncation argument. Hence, it remains to establish some mild regularity conditions such that $T_{\mathrm{Ex}}(\pi_{XY}) \geq \tilde{T}_\infty(\pi_{XY})$ holds for sources with uncountable alphabets.

In this section, we use f_{XY} to denote the PDF of the distribution π_{XY}, which is assumed to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. To state the results succinctly, for each $\epsilon > 0$ and $n \in \mathbb{N}$, we define

$$\kappa_{\epsilon,n} := \sup_{(x,y) \in \mathcal{I}_{\epsilon,n}^2} \left\{ \left| \frac{\partial}{\partial x} \log f_{XY}(x,y) \right| + \left| \frac{\partial}{\partial y} \log f_{XY}(x,y) \right| \right\},$$

where $\mathcal{I}_{\epsilon,n}$ is the interval $[-\sqrt{n(1+\epsilon)}, \sqrt{n(1+\epsilon)}]$. The following lemma and Proposition 5.4 to follow are due to the present authors [204].

Lemma 5.7. Assume that the joint source π_{XY} satisfies the following three assumptions.

(A1) π_{XY} is absolutely continuous on $\mathbb{R}^2$ with $\mathbb{E}[X^2], \mathbb{E}[Y^2] < \infty$;

(A2) The PDF f_{XY} is log-concave and continuously differentiable and that $I(X;Y)$ exists (and thus is finite);

(A3) $\log \kappa_{\epsilon,n}$ is sub-exponential in n (i.e., $\frac{1}{n}\log\log \kappa_{\epsilon,n} \to 0$ as $n \to \infty$).

If there exists a sequence of fixed-length distributed source simulation codes with rate R (Definition 2.1) that generates $P_{X^nY^n}$ (defined in (2.1)) such that

$$D_\infty(P_{X^nY^n} \| \pi_{XY}^n) = o\left(\frac{1}{n + \log \kappa_{\epsilon,n}}\right), \tag{5.27}$$

then there exists a sequence of variable-length distributed source simulation codes with rate R (Definition 5.2) that generates π_{XY}^n exactly. In other words, $T_{\mathrm{Ex}}(\pi_{XY}) \le \tilde{T}_\infty^{\mathrm{cts}}(\pi_{XY})$ where $\tilde{T}_\infty^{\mathrm{cts}}(\pi_{XY})$ is the infimum of all rates R such that (5.27) holds for all $\epsilon > 0$.

In short, if the continuous source $\pi_{XY} \in \mathcal{P}(\mathbb{R}^2)$ satisfies Assumptions (A1)–(A3) and the Rényi divergence of order ∞ vanishes sufficiently rapidly relative to the smoothness of the source density (captured by $\kappa_{\epsilon,n}$), we are able to relate T_{Ex} to $\tilde{T}_\infty^{\mathrm{cts}}$, a proxy of $\tilde{T}_\infty$.

One important example satisfying the conditions in Lemma 5.7 is the class of jointly Gaussian sources as described in Section 2.5.3. Consider two jointly Gaussian random variables X and Y that have zero means and unit variances, and the pair (X, Y) has correlation coefficient $\rho \in (0, 1)$.[1] In this case, it is easy to check that

$$\kappa_{\epsilon,n} = \sup_{(x,y)\in\mathcal{I}_{\epsilon,n}^2} \left|\frac{x - \rho y}{1-\rho^2}\right| + \left|\frac{y - \rho x}{1-\rho^2}\right| = \frac{2\sqrt{n(1+\epsilon)}}{1-\rho}.$$

Hence, for every fixed $\epsilon > 0$ and $\rho \in (0, 1)$, $\log \kappa_{\epsilon,n} = O(\log n)$ is clearly sub-exponential in n. Furthermore, $(n + \log \kappa_{\epsilon,n})^{-1} = \Theta(1/n)$. Hence, by Lemma 5.7, if there exists a sequence of fixed-length codes of rate R such that $D_\infty(P_{X^nY^n} \| \pi_{XY}^n) = o(1/n)$, then there also exists a sequence of rate-R variable-length codes that exactly generates π_{XY}^n.

Using Lemma 5.7, we are able to provide bounds for the exact common information of jointly Gaussian sources.

[1] The results also hold for negative correlation coefficients in which case ρ should be replaced by $|\rho|$.

Proposition 5.4. For a jointly Gaussian source with correlation coefficient $\rho \in (0,1)$,

$$\begin{aligned}\frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right) &= C_{\mathrm{W}}(\pi_{XY}) \le T_{\infty}(\pi_{XY}) \le \tilde{T}_{\infty}(\pi_{XY}) \\ &= T_{\mathrm{Ex}}(\pi_{XY}) \le \frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right) + \frac{\rho \log \mathrm{e}}{1+\rho}. \end{aligned} \tag{5.28}$$

Thus, the upper and lower bounds differ by $\rho/(1+\rho)$. These bounds are illustrated in Fig. 5.7.

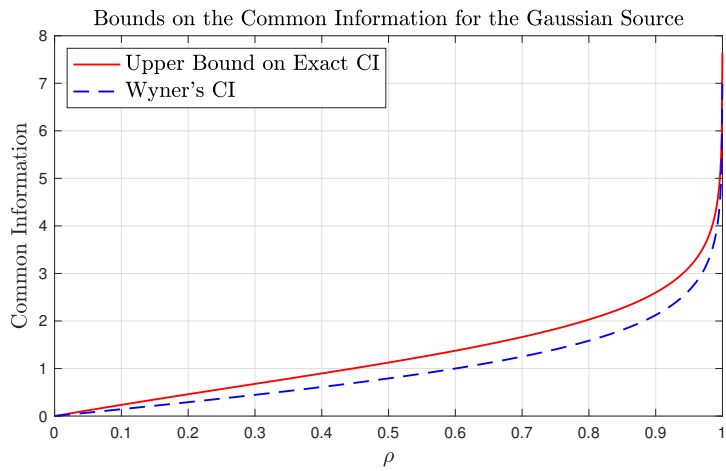


Figure 5.7: Plot of the upper bound on exact common information (5.28) and Wyner's common information (lower bound on the exact common information) for the jointly Gaussian source as stated in (2.25). The exact common information lies between these two curves and Conjecture 5.1 says the upper bound on the exact common information is tight.

Remark 5.2. Li and El Gamal [107] showed using a dyadic decomposition scheme that

$$T_{\mathrm{Ex}}(\pi_{XY}) \le G(\pi_{XY}) \le I(X;Y) + 24 = \frac{1}{2}\log\left(\frac{1}{1-\rho^2}\right) + 24.$$

This bound by Li and El Gamal [107] is based on a *one-shot* scheme and hinges on upper bounding the common entropy $G(\pi_{XY})$, defined in (5.4). The coding scheme involved in proving Proposition 5.4, however, utilizes

multiple copies of the source and hence, naturally results in a better upper bound. In fact, simple algebra yields that for all $\rho \in (0,1)$

$$\left[\frac{1}{2}\log\left(\frac{1}{1-\rho^2}\right)+24\right] - \left[\frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right) + \frac{\rho\log \mathrm{e}}{1+\rho}\right] \geq 22.28 \text{ bits/symb.}$$

The strategy to achieve the upper bound in (5.28), which we will not describe in detail here, is a combination of Li and El Gamal's dyadic decomposition scheme [107] and the construction of a sequence of fixed-length codes that yields $\{P_{X^nY^n}\}_{n\in\mathbb{N}}$ satisfying $D_\infty(P_{X^nY^n}\|\pi_{XY}^n) = o(1/n)$ (as hinted by Lemma 5.7).

We remark that for the DSBS, the upper bound in Proposition 5.3 is tight. It is thus natural to conjecture that the upper bound in (5.28) is also tight which implies that the gap between Wyner's common information and the exact common information for the bivariate Gaussian source is exactly $(\rho\log \mathrm{e})/(1+\rho)$. We state this as a conjecture.

Conjecture 5.1 (Exact common information for a jointly Gaussian source)**.** The exact common information for a jointly Gaussian source with correlation coefficient $\rho \in (0,1)$ is

$$T_{\mathrm{Ex}}(\pi_{XY}) \stackrel{?}{=} \frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right) + \frac{\rho\log \mathrm{e}}{1+\rho}.$$

6

Approximate and Exact Channel Synthesis

How much information is required to create correlation *remotely*? How much interaction is necessary to create such correlation? These questions form the basis of this section. This setup is depicted in Fig. 6.1. It shows that an observer or encoder observes a sequence of i.i.d. random variables $X^n \sim \pi_X^n$ and describes it using a bit string with a certain rate R to the decoder which itself produces another sequence Y^n. It is the hope that even though the encoder and decoder are remotely located, they can leverage a source of shared randomness K_n to reduce the rate of jointly synthesizing a random process $(X^n, Y^n) \sim \pi_X^n P_{Y^n|X^n}$ such that its joint distribution $\pi_X^n P_{Y^n|X^n}$ is close to (or exactly equal) to a target distribution $\pi_{XY}^n = \pi_X^n \pi_{Y|X}^n$. Since the X-marginals of π_{XY}^n and $\pi_X^n P_{Y^n|X^n}$ are identical, the spotlight is then shone on the generated conditional distribution $P_{Y^n|X^n}$ that is mandated to be close (or exactly equal) to the target conditional distribution or *channel* $\pi_{Y|X}^n$. For this reason, this problem is termed as the *distributed channel synthesis* or *communication complexity of correlation* problem and has been studied in [17], [18], [48], [75], [175] among others.

Aiding the reconstruction of the channel is a source of *shared* or *common randomness* which we denote by K_n in Fig. 6.1. This random

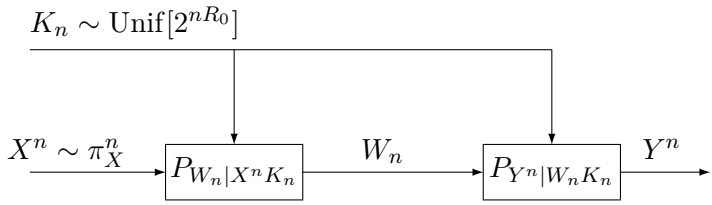


Figure 6.1: The channel synthesis problem. The goal is to ensure that $P_{X^nY^n}$ is either approximately or exactly equal to π_{XY}^n.

variable is uniformly distributed on the index set $\mathcal{K}_n = [2^{nR_0}]$; equivalently it has rate R_0. It can be seen that there is a tradeoff between R_0 and R. Indeed, generally the larger the amount of shared randomness R_0, the more resources the encoder and decoder jointly have, and consequently, the rate of communication R required for synthesizing $\pi_{Y|X}^n$ (exactly or approximately) is usually smaller. The purpose of this section is to quantify this tradeoff precisely.

In the spirit of the previous sections, we study the problems of *approximately* and *exactly* synthesizing the (n-fold product of the) target channel $\pi_{Y|X}^n$. The approximate version consists in quantifying the tradeoff between R and R_0 such that the TV distance between $P_{X^nY^n}$ and π_{XY}^n converges to zero as the blocklength n increases without bound. This problem was studied by Bennett *et al.* [18], Winter [175], Cuff [48], and Bennett *et al.* [17] among others. In particular, Cuff [48] showed that if $R_0 = 0$, then the minimum amount of communication rate required for TV-approximate synthesis is $R = C_{\mathrm{W}}(\pi_{XY})$. In essence, when there is no common randomness, the problem of channel synthesis reduces to the distributed source simulation problem (Section 2.1). On the other hand, if $R_0 = \infty$, the corresponding minimum amount of rate is $R = I(X;Y)$. See Table 6.1. Thus by varying R_0, one traces out a tradeoff curve that interpolates between two familiar notions of correlation, namely Wyner's common information and the mutual information. We elaborate on this in Section 6.1.

We are also concerned with synthesizing the channel $\pi_{Y|X}$ *exactly* using *variable-length* codes. This problem was also studied in several works, including by Bennett *et al.* [18], Harsha *et al.* [75] and Li and El Gamal [109]. Bennett *et al.* [18] showed that when there is unlimited

Table 6.1: Summary of results for the minimum communication rate for the extreme cases of the common randomness (CR) rate $R_0 = 0$ and $R_0 = \infty$

Syntheses \ CR Rate	$R_0 = \infty$	$R_0 = 0$
TV Approx. Synthesis	$I(X;Y)$ [18], [48], [175]	$C_{\mathrm{W}}(\pi_{XY})$ [48]
Exact Synthesis	$I(X;Y)$ [18]	$T_{\mathrm{Ex}}(\pi_{XY})$ [103]

shared randomness, the minimum rate of communication is $I(X;Y)$. At the other extreme, if there is no shared randomness, the problem of exact channel synthesis reduces to the exact common information problem. From Section 5, we saw that for the DSBS, exact channel synthesis (with a uniform source X) requires a strictly larger communication rate $\tilde{T}_\infty(\pi_{XY}) = T_{\mathrm{Ex}}(\pi_{XY})$ compared to that required for the TV-approximate version $\tilde{T}_1(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$ (Theorem 4.5). These results are also summarized in Table 6.1.

In this section, we are concerned with *refinements* to these extreme cases. Some results in the literature are worth highlighting. Harsha *et al.* [75] used a rejection sampling scheme to study the one-shot version of exact simulation for the discrete source (X, Y). The authors showed that the number of bits of the shared randomness can be limited to $O(\log\log|\mathcal{X}| + \log|\mathcal{Y}|)$ if the expected description length of X is increased by $O(\log(I(X;Y)+1) + \log\log|\mathcal{Y}|)$ bits from the mutual information lower bound $I(X;Y)$. Li and El Gamal [109] showed that if the expected description length is increased by $\log(I(X;Y)+1)+5$ bits from $I(X;Y)$, then the number of bits of shared randomness can be upper bounded by $\log(|\mathcal{X}|(|\mathcal{Y}|-1)+2)$. This section is concerned with the fundamental limits of the amount of shared randomness when the sequence of communication rates is required to approach the minimum rate $I(X;Y)$ *only asymptotically* as $n \to \infty$. In this case, what is the minimum amount of shared randomness required to realize exact synthesis? Bennett *et al.* [17] conjectured that an exponential number of bits (and hence an infinite rate) of shared randomness is necessary. This was disproved by Harsha *et al.* [75] and Li and El Gamal [109] where finite bounds on the rate were established. This section, and in particular Section 6.4, surveys advances on this question and provides

the best known bounds on the minimum amount of shared randomness in Section 6.4.2. We supplement our discussions with numerical examples using the DSBS and the bivariate Gaussian source.

Besides the works surveyed above, local TV-approximate simulation of a channel was studied by Steinberg and Verdú [157]. TV-approximate simulation of a "bidirectional" channel via interactive communication was studied by Yassaee, Gohari, and Aref [190]. Both the exact and TV-approximate versions of the simulation of a channel over another noisy channel were studied by Haddadpour *et al.* [70]. In particular, [70] addressed the case of exact simulation of a binary symmetric channel over a binary erasure channel. The relationship between the problem of exact channel simulation over another channel and the problem of zero-error capacity was studied by Cubitt *et al.* [47].

6.1 Approximate Channel Synthesis

In this section, we set the stage by describing the problem of approximate channel synthesis. The problem is depicted in Fig. 6.1 in which the encoder provides a description of the source sequence $X^n \sim \pi_X^n$ at a certain rate R. The rate-R description, also known as the *message*, is denoted as W_n. A rate-R_0 random variable K_n, uniformly distributed on $\mathcal{K}_n$, represents *common randomness* available to *both* the encoder and decoder. The decoder generates a sequence Y^n based on the message W_n and the common randomness K_n.

The following definition is parallel to Definition 2.1 for fixed-length distributed source simulation codes.

Definition 6.1. An (n, R, R_0)-*fixed-length channel synthesis code* consists of a pair of random mappings $P_{W_n|X^nK_n} \in \mathcal{P}(\mathcal{W}_n|\mathcal{X}^n \times \mathcal{K}_n)$ and $P_{Y^n|W_nK_n} \in \mathcal{P}(\mathcal{Y}^n|\mathcal{W}_n \times \mathcal{K}_n)$ such that

$$\frac{1}{n}\log|\mathcal{W}_n| \le R \quad \text{and} \quad \frac{1}{n}\log|\mathcal{K}_n| \le R_0.$$

These two mappings are known as the *encoder* and *decoder* respectively.

Given a code $(P_{W_n|X^nK_n}, P_{Y^n|W_nK_n})$, the joint distribution of the message W_n and output Y^n given (X^n, K_n) is

$$P_{Y^nW_n|X^nK_n} = P_{Y^n|W_nK_n} P_{W_n|X^nK_n}. \tag{6.1}$$

The joint distribution of all the random variables (X^n, Y^n, W_n, K_n) is

$$P_{X^nY^nW_nK_n} = P_{Y^nW_n|X^nK_n} P_{X^nK_n},$$

where, by definition,

$$P_{X^nK_n}(x^n, k) = \frac{\pi_X^n(x^n)}{|\mathcal{K}_n|} \quad \text{for all } (x^n, k) \in \mathcal{X}^n \times \mathcal{K}_n.$$

Given a code, the *synthesized distribution* is

$$P_{X^nY^n}(x^n, y^n) := \sum_{(w,k)\in\mathcal{W}_n\times\mathcal{K}_n} P_{X^nY^nW_nK_n}(x^n, y^n, w, k). \tag{6.2}$$

Definition 6.2. The pair $(R, R_0) \in \mathbb{R}_+^2$ is said to be *achievable for synthesizing the channel $\pi_{Y|X}$ with input π_X* if there exists a sequence of (n, R, R_0)-fixed-length channel synthesis codes such that the TV distance between the synthesized distribution in (6.2) and the target distribution π_{XY}^n vanishes, i.e.,

$$\lim_{n\to\infty} |P_{X^nY^n} - \pi_{XY}^n| = 0.$$

Define the *optimal rate region* $\mathcal{T}(\pi_{XY}) \subset \mathbb{R}_+^2$ to be the closure of the set of achievable rate pairs (R, R_0) for synthesizing $\pi_{Y|X}$ with input π_X.

We remark that this definition is generally more stringent than the analogous one for distributed source synthesis in Definition 4.5 as we only require that the TV distance vanishes. In contrast, in Definition 4.5, the TV distance is only required to be asymptotically bounded by $\varepsilon \in [0, 1)$. To state the next result succinctly, let us define the following set:

$$\mathcal{C}_{\mathrm{W}}(\pi_{XY}) := \bigcup_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} \left\{ (R, R_0) : \begin{array}{c} R \geq I(X;W) \\ R + R_0 \geq I(XY;W) \end{array} \right\}. \tag{6.3}$$

Here, just like in (2.3) for Wyner's common information, the union runs over all triples of random variables (X, W, Y) such that $X - W - Y$ forms a Markov chain in this order and $P_{XY} = \pi_{XY}$. To exhaust the rate region, it suffices to take $|\mathcal{W}| \leq |\mathcal{X}||\mathcal{Y}| + 1$. Cuff [48] proved the following fundamental result.

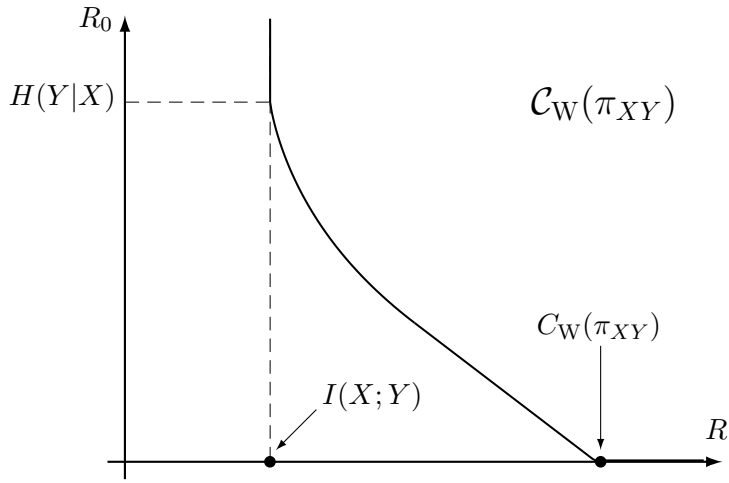


Figure 6.2: A schematic of the region $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$ defined in (6.3)

Theorem 6.1. For any joint distribution π_{XY} defined on a finite alphabet $\mathcal{X} \times \mathcal{Y}$,

$$\mathcal{T}(\pi_{XY}) = \mathcal{C}_{\mathrm{W}}(\pi_{XY}).$$

Let us examine the extreme points of the region $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$. When there is no common randomness, i.e., $R_0 = 0$, the second inequality in (6.3) dominates and Theorem 6.1 says that rate of communication needs to be at least $I(XY;W)$ where the joint distribution $P_W P_{X|W} P_{Y|W}$ satisfies $P_{XY} = \pi_{XY}$. This rate is precisely Wyner's common information $C_{\mathrm{W}}(\pi_{XY})$. On the other hand, when $R_0 = \infty$, the second inequality is inactive and we can easily see that the minimum communication rate is $R = I(X;Y)$. This can be rigorously justified as follows. By the data processing inequality for the mutual information and the Markov chain $X - W - Y$, we have $I(X;W) \geq I(X;Y)$. This inequality can be met with equality by choosing $W = Y$. Furthermore, this choice implies that

$$\begin{aligned} R_0 &= I(XY;W) - R = I(XY;Y) - R \\ &= H(Y) - I(X;Y) = H(Y|X) \end{aligned} \tag{6.4}$$

is a sufficient common randomness rate for achieving $R = I(X;Y)$. A schematic of the region $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$ is shown in Fig. 6.2. Hence, we see that the approximate channel synthesis problem provides us with a

tuning knob R_0 to obtain a continuum of common information measures that interpolate from the mutual information to Wyner's common information.

We now devote the final paragraphs of this section to sketch the achievability proof of Theorem 6.1. The main idea is to invoke the TV distance version of the soft-covering lemma (cf. (2.6) in Lemma 2.2 and Section 4.5.1) multiple times, together with some properties of the TV distance.

We proceed by a random selection (random coding) argument. Fix any distribution $P_W P_{X|W} P_{Y|W}$ such that $P_{XY} = \pi_{XY}$. Randomly and independently generate a codebook $\mathscr{C}_n = \{W^n(m,k) : m \in \mathcal{W}_n, k \in \mathcal{K}_n\}$ where $\log|\mathcal{W}_n| = nR$ and $\log|\mathcal{K}_n| = nR_0$ and where each codeword $W^n(m,k)$ is generated independently from the n-fold product distribution P_W^n. Using $\mathscr{C}_n$, define the (random) distribution

$$Q_{X^nY^nW_nK_n}(x^n,y^n,m,k) = \frac{P^n_{X|W}(x^n|W^n(m,k))P^n_{Y|W}(y^n|W^n(m,k))}{2^{n(R+R_0)}}.$$

Based on $Q_{X^nY^nW_nK_n}$, define the *synthesized distribution* as

$$P_{X^nY^nW_nK_n}(x^n,y^n,m,k) = \frac{\pi_X^n(x^n)Q_{Y^nW_n|X^nK_n}(y^n,m|x^n,k)}{2^{nR_0}}. \quad (6.5)$$

This distribution satisfies all the properties in (6.1)–(6.2).

By the soft-covering lemma for the TV distance, if

$$R + R_0 > I_P(XY;W), \quad (6.6)$$

then the expectation of the TV distance between $Q_{X^nY^n}$ and π^n_{XY} vanishes, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\Big[\big|Q_{X^nY^n} - \pi^n_{XY}\big|\Big] = 0. \quad (6.7)$$

Unfortunately, $Q_{X^nY^n}$ is not the synthesized distribution $P_{X^nY^n}$ so we must do a little more. Applying the soft-covering lemma for the TV distance again, we see that if

$$R > I_P(X;W), \quad (6.8)$$

then for all $k \in \mathcal{K}_n$,

$$\lim_{n\to\infty} \mathbb{E}\Big[\big|Q_{X^n|K_n=k} - \pi^n_X\big|\Big] = 0.$$

Consequently, by invoking the definition of the TV distance,

$$\lim_{n\to\infty} \mathbb{E}\Big[\big|Q_{X^nK_n} - \pi_X^n Q_{K_n}\big|\Big] = 0, \tag{6.9}$$

where $Q_{K_n} = \text{Unif}[2^{nR_0}]$. Now, we compare the synthesized distribution to the target distribution as follows

$$\begin{aligned}
&\big|P_{X^nY^n} - \pi_{XY}^n\big| \\
&\le \big|P_{X^nY^n} - Q_{X^nY^n}\big| + \big|Q_{X^nY^n} - \pi_{XY}^n\big| && (6.10)\\
&\le \big|P_{X^nY^nW_nK_n} - Q_{X^nY^nW_nK_n}\big| + \big|Q_{X^nY^n} - \pi_{XY}^n\big| && (6.11)\\
&= \big|P_{X^nK_n} - Q_{X^nK_n}\big| + \big|Q_{X^nY^n} - \pi_{XY}^n\big| && (6.12)\\
&= \big|\pi_X^n Q_{K_n} - Q_{X^nK_n}\big| + \big|Q_{X^nY^n} - \pi_{XY}^n\big|, && (6.13)
\end{aligned}$$

where (6.10) follows from the triangle inequality for the TV distance, (6.11) follows from the fact that the TV distance between joint distributions is at least as large as the TV distance between marginal distributions, (6.12) follows from the fact that $P_{Y^nW_n|X^nK_n} = Q_{Y^nW_n|X^nK_n}$ by the construction in (6.5) and finally, (6.13) follows from the definition of $P_{X^nY^nW_nK_n}$ in (6.5). We can now take expectations on both sides of the above chain of inequalities. The expectations of both terms in (6.13) vanish due to (6.7) and (6.9), which means that the synthesized distribution $P_{X^nY^n}$ is arbitrarily close in TV distance to the target distribution π_{XY}^n as $n \to \infty$ if (6.6) and (6.8) are satisfied.

6.2 Exact Channel Synthesis

In this section, we consider an exact synthesis counterpart to that considered in Section 6.1. That is, we require that the decoder in Fig. 6.2 outputs a sequence of random variables Y^n whose joint distribution with the source sequence X^n is *exactly* π_{XY}^n. Just as we discussed in Section 5 on exact common information, to ensure exact reconstruction, one has to be given the freedom to use *variable-length* codes. In this channel synthesis setting, the notion of variable-length codes is parallel to that in Section 5 albeit slightly more involved due to the presence of the common randomness K_n. Our objective is to compare and contrast the optimal rate regions for approximate and exact reconstructions of the channel $\pi_{Y|X}$.

Formally, let the alphabet of the common randomness K_n be $\mathcal{K}_n = [2^{nR_0}]$. In other words, K_n can be represented by nR_0 bits and this length is kept *fixed*. The length that is allowed to vary is that of $W_n \sim P_{W_n|X^nK_n}(\cdot|x^n, k)$ whose alphabet we denote by $\mathcal{W}_n$. This alphabet, without loss of generality, can be regarded as a subset of $\mathbb{N}$. We now consider a set of source codes $\mathbf{f} = \{f_k : k \in \mathcal{K}_n\}$, where each element of $\mathbf{f}$ is a prefix-free code [42] $f_k : \mathcal{W}_n \to \{0,1\}^*$ indexed by $k \in \mathcal{K}_n$. Then for each message-common randomness pair $(w, k) \in \mathcal{W}_n \times \mathcal{K}_n$, and the set of codes $\mathbf{f}$, let $\ell_{\mathbf{f}}(w|k)$ denote the *length* of the codeword $f_k(w)$ (see Example 5.1) where f_k is the k^{th} component of $\mathbf{f}$.

Definition 6.3. The *expected codeword length* $L_{\mathbf{f}}$ of a code $\mathbf{f} = \{f_k : k \in \mathcal{K}_n\}$ for compressing the source W_n given K_n is

$$L_{\mathbf{f}}(W_n|K_n) = \mathbb{E}[\ell_{\mathbf{f}}(W_n|K_n)] = \sum_{(w,k)\in\mathcal{W}_n\times\mathcal{K}_n} P_{W_nK_n}(w,k)\ell_{\mathbf{f}}(w|k),$$

where the joint distribution between the message and uniformly distributed common randomness (i.e., $P_{K_n}(k) = |\mathcal{K}_n|^{-1}$ for all $k \in \mathcal{K}_n$) is

$$P_{W_nK_n}(w,k) = \sum_{x^n\in\mathcal{X}^n} \frac{1}{|\mathcal{K}_n|}\pi_X^n(x^n)P_{W_n|X^nK_n}(w|x^n,k).$$

Note that if $\mathcal{K}_n = \emptyset$, this definition reduces to that in Definition 5.1 for the exact common information problem.

Definition 6.4. An (n, R, R_0)-*variable-length channel simulation code* $(\mathbf{f}, P_{W_n|X^nK_n}, P_{Y^n|W_nK_n})$ consists of

- A set of prefix-free source codes $\mathbf{f} = \{f_k : \mathcal{W}_n \to \{0,1\}^*\}_{k\in\mathcal{K}_n}$;
- A pair of random mappings $P_{W_n|X^nK_n} \in \mathcal{P}(\mathcal{W}_n|\mathcal{X}^n \times \mathcal{K}_n)$ and $P_{Y^n|W_nK_n} \in \mathcal{P}(\mathcal{Y}^n|\mathcal{W}_n \times \mathcal{K}_n)$ called the *encoder* and *decoder* respectively;

such that the per-symbol expected codeword length

$$\frac{1}{n}L_{\mathbf{f}}(W_n|K_n) \le R,$$

and the rate of the common randomness $|\mathcal{K}_n|$ satisfies

$$\frac{1}{n}\log|\mathcal{K}_n| \le R_0.$$

By the variable-length nature of the code, W_n can be transmitted to the decoder error-free. The *synthesized channel* is then given by

$$P_{Y^n|X^n}(y^n|x^n) = \sum_{(w,k)\in\mathcal{W}_n\times\mathcal{K}_n} \frac{P_{W_n|X^nK_n}(w|x^n,k)P_{Y_n|W_nK_n}(y^n|w,k)}{|\mathcal{K}_n|}. \tag{6.14}$$

In the exact channel synthesis problem we consider in this section, $P_{Y^n|X^n}$ is required to be *exactly* equal to $\pi_{Y|X}^n$ for some large enough n. It is worth noting that under the assumption that $K_n \sim \mathrm{Unif}(\mathcal{K}_n)$, the synthesized channel depends only on the code $(P_{W_n|X^nK_n}, P_{Y_n|W_nK_n})$ and not the distribution π_X^n. However, the code rate R induced by a given channel simulation code (Definition 6.4) depends on π_X.

Definition 6.5. The pair $(R, R_0) \in \mathbb{R}_+^2$ is said to be *achievable for exactly synthesizing the channel* $\pi_{Y|X}$ *with input* π_X if there exists an (n, R, R_0)-variable-length channel synthesis code such that the synthesized distribution in (6.14) and the target distribution $\pi_{Y|X}^n$ are equal, i.e.,

$$P_{Y^n|X^n} = \pi_{Y|X}^n \quad \text{for some } n \in \mathbb{N}.$$

Define the *optimal rate region* $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subset \mathbb{R}_+^2$ to be the closure of the set of achievable rate pairs (R, R_0) for exactly synthesizing $\pi_{Y|X}$ with input π_X.

The central goal of this section is to characterize $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ for various sources $(X, Y) \sim \pi_{XY} = \pi_X\pi_{Y|X}$.

We first perform a simple observation that is parallel to that of Lemma 5.1 for the exact common information problem. Observe from the law of total expectation that $L_{\mathbf{f}}(W_n|K_n) = \mathbb{E}[\mathbb{E}[\ell_{\mathbf{f}}(W_n|K_n)|K_n]]$. Hence, to minimize the expected codeword length $L_{\mathbf{f}}(W_n|K_n)$, it suffices to minimize $\mathbb{E}[\ell_{\mathbf{f}}(W_n|K_n)|K_n = k]$ for each $k \in \mathcal{K}_n$. By applying Shannon's zero-error compression theorem for every k, we have the bounds

$$H(W_n|K_n = k) \le \mathbb{E}[\ell_{\mathbf{f}}(W_n|K_n)|K_n = k] < H(W_n|K_n = k) + 1.$$

Hence, by taking the expectation over K_n, for a set of optimal prefix-free codes $\mathbf{f}^* = \{f_k^* : k \in \mathcal{K}_n\}$, one has

$$H(W_n|K_n) \le L_{\mathbf{f}^*}(W_n|K_n) < H(W_n|K_n) + 1.$$

Consequently,

$$\lim_{n\to\infty} \frac{L_{\mathbf{f}^*}(W_n|K_n)}{n} = \lim_{n\to\infty} \frac{H(W_n|K_n)}{n}. \tag{6.15}$$

Hence, completely analogous to Lemma 5.1, we have the following multi-letter characterization of $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$; this is due to the present authors [203].

Lemma 6.2. The optimal rate region for the exact channel synthesis problem is

$$\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) = \mathrm{Cl}\left(\bigcup_{n\in\mathbb{N}} \left\{ (R, R_0) : \begin{array}{c} \exists\, (P_{W_n|X^nK_n}, P_{Y^n|W_nK_n}) \\ P_{Y^n|X^n} = \pi_{Y|X}^n \\ R \geq \frac{1}{n} H(W_n|K_n) \end{array} \right\}\right).$$

Because of (6.15), the multi-letter expression presented in Lemma 6.2 does not depend on the set of prefix-free codes $\mathbf{f}$ and thus $\mathbf{f}$ may be omitted from Definition 6.4 in our consideration of the optimal rate region (per Definition 6.5). We notice that the limit of $H(W_n|K_n)/n$ can be interpreted as the *conditional common entropy rate* of the process $\{W_n\}_{n\in\mathbb{N}}$ given another process $\{K_n\}_{n\in\mathbb{N}}$. While this lemma presents a characterization of $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$, it is far from explicit and intractable to calculate given a π_{XY}. In the following, we present alternative characterizations of and bounds on $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ that are more explicit.

6.3 Multi-Letter Characterization for Exact Channel Synthesis

In this section, we present an alternative multi-letter characterization in terms of the maximal cross-entropy defined (see (4.1) in Definition 4.1), which as we have seen from Sections 4 and 5, plays a crucial role in the characterization of fundamental limits of common information problems when *exact* reconstruction is required. To do so, we define $\underline{\mathcal{R}}(\pi_{XY})$ to be the set of rate pairs $(R, R_0) \in \mathbb{R}_+^2$ such that there exists a joint distribution $P_W P_{X|W} P_{Y|W}$ with $P_{XY} = \pi_{XY}$ and

$$R \geq I(W;X) \tag{6.16}$$

$$R_0 + R \geq -H(XY;W) + \mathbb{E}_{P_W}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W}\|\pi_{XY})\right]. \tag{6.17}$$

To exhaust the region $\underline{\mathcal{R}}(\pi_{XY})$, it suffices to take $|\mathcal{W}| \leq |\mathcal{X}||\mathcal{Y}|+1$. The bound in (6.17) is analogous to the upper pseudo-common information of order ∞ (Definition 4.2). The following theorem is also parallel to (5.10) in Theorem 5.2 and is due to the present authors [203].

Theorem 6.3. For a source with distribution π_{XY} defined on a finite alphabet $\mathcal{X} \times \mathcal{Y}$,

$$\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) = \mathrm{Cl}\left(\bigcup_{n \in \mathbb{N}} \frac{1}{n} \underline{\mathcal{R}}(\pi_{XY}^n) \right).$$

The intuition for the achievability part of this result (i.e., that $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \supset \frac{1}{n}\underline{\mathcal{R}}(\pi_{XY}^n)$ for any $n \in \mathbb{N}$) is similar to that sketched in Section 5.3.2 for the exact common information problem. In particular, the constraint in (6.17) results from the fact that we can use the *pair* (W_n, K_n) as the common randomness for the exact synthesis of π_{XY}^n. Recall that the exact common information is equal to the Rényi common information of order ∞ (Theorem 5.2). Hence, we require $D_\infty(P_{X^nY^n} \| \pi_{XY}^n)$ to vanish. This is equivalent to

$$\max_{(x^n,y^n) \in \mathrm{supp}(P_{X^nY^n})} \frac{P_{X^nY^n}(x^n, y^n)}{\pi_{XY}^n(x^n, y^n)} = 1 + o(1).$$

According to the discussion in Section 5.3.2, for this condition to hold using truncated i.i.d. codes within a mixture decomposition framework, we need the total rate of the available common randomness $R_0 + R$ to satisfy (6.17). The constraint that $R \geq I(W;X)$ in (6.16) is similar, albeit simpler. It is required to ensure that π_X^n is close to P_{X^n} in the sense that

$$\max_{x^n \in \mathcal{T}_\epsilon^{(n)}(\pi_X)} \frac{P_{X^n}(x^n)}{\pi_X^n(x^n)} = 1 + o(1).$$

Putting these ideas together yields the fact that $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \supset \underline{\mathcal{R}}(\pi_{XY})$. Using the above coding scheme and following steps similar to the approximate synthesis case (i.e., the proof of Theorem 6.1) on source blocks of length n yields that $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \supset \frac{1}{n}\underline{\mathcal{R}}(\pi_{XY}^n)$ for all $n \in \mathbb{N}$, which is the achievability part of Theorem 6.3.

6.4 Single-Letter Bounds for Exact Channel Synthesis

In this section, we present single-letter inner and outer bounds on the optimal rate region for exact channel synthesis $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$.

To state the bounds succinctly, we present a definition that is analogous to $\underline{\mathcal{R}}(\pi_{XY})$ in (6.16)–(6.17). Let $\overline{\mathcal{R}}(\pi_{XY})$ be the set of rate pairs $(R, R_0) \in \mathbb{R}_+^2$ such that there exists a joint distribution $P_W P_{X|W} P_{Y|W}$ with $P_{XY} = \pi_{XY}$ satisfying (6.16) and additionally,

$$R_0 + R \geq -H(XY;W) + \inf_{\substack{Q_{WW'} \\ \in \mathcal{C}(P_W, P_W)}} \mathbb{E}_{Q_{WW'}}[\mathsf{H}_\infty(P_{X|W}, P_{Y|W'} \| \pi_{XY})]. \quad (6.18)$$

This bound is analogous to the lower pseudo-common information of order ∞ (Definition 4.2). The difference between $\overline{\mathcal{R}}(\pi_{XY})$ and $\underline{\mathcal{R}}(\pi_{XY})$ is also similar to the difference between $\underline{\Psi}(\pi_{XY})$ and $\overline{\Psi}(\pi_{XY})$ defined in (5.17) and (5.11) respectively. Clearly, $\underline{\mathcal{R}}(\pi_{XY}) \subset \overline{\mathcal{R}}(\pi_{XY})$ and equality is achieved, for example, when every point on the boundary of $\underline{\mathcal{R}}(\pi_{XY})$ induces an optimal coupling in (6.18) that is the equality coupling, i.e., $Q_{WW'}(w, w') = P_W(w)\mathbb{1}\{w = w'\}$ for all $(w, w') \in \mathcal{W}^2$.

The following theorem, due to the present authors [203], is analogous to Theorem 5.3 for the exact common information problem.

Theorem 6.4 (Bounds on exact channel synthesis region)**.** For a source with distribution π_{XY} defined on a finite alphabet $\mathcal{X} \times \mathcal{Y}$,

$$\underline{\mathcal{R}}(\pi_{XY}) \subset \mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subset \overline{\mathcal{R}}(\pi_{XY}) \cap \mathcal{C}_{\mathrm{W}}(\pi_{XY}).$$

Remark 6.1. To alleviate any possible confusion, we remark that in Theorem 5.3 in which the optimal rate (exact common information) $T_{\mathrm{Ex}}(\pi_{XY})$ is sought, the *achievability* part (resp. converse part) corresponds to the *upper* bound (resp. lower bound) on $T_{\mathrm{Ex}}(\pi_{XY})$. In contrast, in Theorem 6.4 in which the rate region $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ is sought, the *achievability* part (resp. converse part) corresponds to the *inner* bound (resp. outer bound) on $T_{\mathrm{Ex}}(\pi_{XY})$.

The difference between the inner bound $\underline{\mathcal{R}}(\pi_{XY})$ and $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$ is the bound on the sum rate. In the former, the sum rate is lower bounded

by $-H(XY;W) + \mathbb{E}_{P_W}[\mathsf{H}_\infty(P_{X|W}, P_{Y|W}\|\pi_{XY})]$ (see (6.17)) while for the latter (see (6.3)), the sum rate is lower bounded by $I(XY;W)$. It can easily be seen that the inner bound is indeed a subset of $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$. This is because

$$\begin{aligned}
&\sum_{w\in\mathcal{W}} P_W(w)\mathsf{H}_\infty(P_{X|W=w}, P_{Y|W=w}\|\pi_{XY}) \\
&\geq \sum_{w\in\mathcal{W}} P_W(w) \sum_{x,y} P_{X|W}(x|w)P_{Y|W}(y|w)\log\frac{1}{\pi_{XY}(x,y)} \\
&= \sum_{x,y} P_{XY}(x,y)\log\frac{1}{\pi_{XY}(x,y)} = H_\pi(XY), \qquad (6.19)
\end{aligned}$$

where the final equality follows from the fact that $P_{XY} = \pi_{XY}$. As a result,

$$\begin{aligned}
I(XY;W) &= H(XY) - H(XY;W) \\
&\leq \mathbb{E}_{P_W}[\mathsf{H}_\infty(P_{X|W}, P_{Y|W}\|\pi_{XY})] - H(XY;W).
\end{aligned}$$

Thus the lower bound on the sum rate in $\underline{\mathcal{R}}(\pi_{XY})$ is at least as large as that on the sum rate in $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$, which implies that $\underline{\mathcal{R}}(\pi_{XY}) \subset \mathcal{C}_{\mathrm{W}}(\pi_{XY})$.

6.4.1 Ideas for the Proof of Theorem 6.4

In this section, we provide brief sketches of the set inclusions in Theorem 6.4; this section can be omitted at a first reading. In Section 6.3, we have already provided a sketch of the proof that $\underline{\mathcal{R}}(\pi_{XY}) \subset \mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ (the achievability part) so we proceed to show the other inclusions.

Let us now reason that $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subset \mathcal{C}_{\mathrm{W}}(\pi_{XY})$. By the chain of inequalities leading to (6.19), we see that

$$\sum_{w\in\mathcal{W}} P_W(w)\mathsf{H}_\infty(P_{X^n|W=w}, P_{Y^n|W=w}\|\pi_{XY}^n) \geq nH_\pi(XY)$$

and so

$$I(X^nY^n;W) \leq -H(X^nY^n|W) + \mathbb{E}_{P_W}[\mathsf{H}_\infty(P_{X^n|W}, P_{Y^n|W}\|\pi_{XY}^n)].$$

Hence, $\underline{\mathcal{R}}(\pi_{XY}^n) \subset \mathcal{C}_{\mathrm{W}}(\pi_{XY}^n)$. Furthermore, Cuff [48] showed that the set $\mathcal{C}_{\mathrm{W}}(\pi_{XY}^n)$ *tensorizes*, i.e., $\frac{1}{n}\mathcal{C}_{\mathrm{W}}(\pi_{XY}^n) = \mathcal{C}_{\mathrm{W}}(\pi_{XY})$. Thus, by the multi-letter characterization of $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ in terms of the union of the sets

$\frac{1}{n}\underline{\mathcal{R}}(\pi_{XY}^n)$ for $n \in \mathbb{N}$ in Theorem 6.3, we see that $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subset \mathcal{C}_{\mathrm{W}}(\pi_{XY})$. This bound is completely analogous to the fact that the exact common information is at least as large as Wyner's common information; recall the derivation of this in (5.8).

Thus, it remains to prove the alternative outer bound $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subset \overline{\mathcal{R}}(\pi_{XY})$ (converse). This requires a single-letterization result in [204, Theorem 2] which is restated here for the reader's convenience.

Lemma 6.5. For a triple of random variables $(X^n, Y^n, Z) \in \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}$ such that $(X^n, Y^n) \sim \pi_{XY}^n$ and $X^n - Z - Y^n$, we have

$$\begin{aligned} &-\frac{1}{n}H(X^nY^n|Z) + \frac{1}{n}\sum_{z\in\mathcal{Z}} P_Z(z)\mathsf{H}_\infty(P_{X^n|Z=z}, P_{Y^n|Z=z}\|\pi_{XY}^n) \\ &\geq -H(XY|W) + \inf_{\substack{Q_{WW'} \\ \in\mathcal{C}(P_W,P_W)}} \mathbb{E}_{Q_{WW'}}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W'}\|\pi_{XY})\right], \quad (6.20) \end{aligned}$$

where $W := (Z, J, X^{J-1}, Y^{J-1})$, $X := X_J$, $Y := Y_J$, and $J \sim \mathrm{Unif}[n]$ denotes a random index independent of (X^n, Y^n, Z).

This lemma says that the n-letter expression whose expectation is over a single distribution P_Z can be lower bounded by a single-letter expression at the additional "cost" of an optimization over couplings $Q_{WW'} \in \mathcal{C}(P_W, P_W)$. From this lemma, we see that the multi-letter expression of the sum rate in $\underline{\mathcal{R}}(\pi_{XY}^n)$ can be lower bounded by the single-letter expression in (6.20). On the other hand, observe that

$$\begin{aligned} P_{W_nK_nX^iY^{i-1}} &= P_{W_nK_n}P_{X^i|W_nK_n}P_{Y^{i-1}|W_nK_n} \\ &= P_{W_nK_n}P_{X_i|W_nK_n}P_{X^{i-1}|W_nK_n}P_{Y^{i-1}|W_nK_n}, \end{aligned}$$

so that $X_i - (W_n, K_n, X^{i-1}) - Y^{i-1}$ forms a Markov chain for all $i \in [n]$. As in Lemma 6.5, let $J \sim \mathrm{Unif}[n]$ be a random index independent of the random variables in (X^n, Y^n, W_n, K_n) and let $X := X_J$, $Y := Y_J$, and $W := (W_n, K_n, X^{J-1}, Y^{J-1}, J)$. Hence,

$$\begin{aligned} nR &\geq H(W_n|K_n) \\ &\geq I(X^n; W_n|K_n) \\ &= I(X^n; W_n, K_n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} I(X_i; W_n K_n | X^{i-1}) \\
&= \sum_{i=1}^{n} I(X_i; W_n K_n X^{i-1}) && (6.21) \\
&= \sum_{i=1}^{n} I(X_i; W_n K_n X^{i-1} Y^{i-1}) && (6.22) \\
&= nI(X_J; W_n K_n, X^{J-1} Y^{J-1} | J) \\
&= nI(X_J; W_n K_n X^{J-1} Y^{J-1} J) \\
&= nI(X; W),
\end{aligned}
$$

where (6.21) follows from the fact that $\{X_i\}_{i=1}^n$ is a memoryless process and (6.22) follows because $X_i - (W_n, K_n, X^{i-1}) - Y^{i-1}$ forms a Markov chain for all $i \in [n]$. This completes the proof that $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subset \overline{\mathcal{R}}(\pi_{XY})$.

6.4.2 Tradeoff Between the Communication and Common Randomness Rates

We now examine the tradeoff between the communication rate R and the common randomness rate R_0 in the optimal region for exact channel synthesis $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$. For this purpose, define the two optimal rates

$$
\begin{aligned}
T^*(R_0) &:= \inf \{R \in \mathbb{R}_+ : (R, R_0) \in \mathcal{T}_{\mathrm{Ex}}(\pi_{XY})\} \quad \text{and} \quad (6.23) \\
T_0^*(R) &:= \inf \{R_0 \in \mathbb{R}_+ : (R, R_0) \in \mathcal{T}_{\mathrm{Ex}}(\pi_{XY})\}.
\end{aligned}
$$

From the inner and outer bounds in Theorem 6.4, we see that

$$
T^*(\infty) = I_\pi(X; Y), \tag{6.24}
$$

where I_π denotes the mutual information computed with respect to the target distribution π_{XY}. This is because when $R_0 = \infty$, the sum rate bound is inactive. Eqn. (6.24) is consistent with the observation in Bennett *et al.* [18]. Namely, when there is unlimited shared or common randomness at the encoder and the decoder, the target channel $\pi_{Y|X}$ can be successfully synthesized by some protocol if and only if the minimum asymptotic communication rate is at least the mutual information $I_\pi(X; Y)$ (refer to Table 6.1). This is the same as approximate channel synthesis in the TV metric (refer to Fig. 6.2). More

interestingly, Bennett *et al.* [18] showed that an *exponential* number of bits of common randomness *suffices* for (6.24) to hold. This condition is rather different from what we have seen from (6.4) in the context of approximate channel synthesis in which a shared randomness rate of $H_\pi(Y|X)$ is needed for us to ensure that the communication rate is $I_\pi(X;Y)$. Bennett *et al.* [18] also conjectured that an exponential number of bits of common randomness (which implies that $R_0 = \infty$) is, in fact, *necessary* for (6.24) to hold.

Harsha *et al.* [75] and Li and El Gamal [109] disproved this conjecture for $(X,Y) \sim \pi_{XY}$ with finite alphabets. These authors showed that shared randomness with rate $\log|\mathcal{Y}|$ is sufficient to realize (6.24), i.e.,

$$T^*(\log|\mathcal{Y}|) \leq I_\pi(X;Y).$$

The result in Theorem 6.4, in fact, yields a better bound. Consider,

$$\begin{aligned} T_0^*(I_\pi(X;Y)) &= \inf\{R_0 : (I_\pi(X;Y), R_0) \in \mathcal{T}_{\mathrm{Ex}}(\pi_{XY})\} \\ &\leq \min_{P_{W|Y}: X-W-Y} -H(X) - H(Y|W) \\ &\qquad + \mathbb{E}_{P_W}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W}\|\pi_{XY})\right] \qquad (6.25) \\ &\leq H_\pi(Y|X), \qquad (6.26) \end{aligned}$$

where (6.25) results from the inner bound in Theorem 6.4 (setting the communication rate as $R = I(X;Y)$) and (6.26) follows from setting $W = Y$ (so that $\mathbb{E}_{P_W}[\mathsf{H}_\infty(P_{X|W}, P_{Y|W}\|\pi_{XY})] = H(XY)$). We will see from Section 6.6 that the bound in (6.26) is tight for the DSBS.

Why is the amount of common randomness yielded by Theorem 6.4 smaller than those in the works [18], [75], [109] prior to that of the present authors? In the coding scheme of [18], the shared randomness is used to generate a codebook. However, as described in the sketch of the coding scheme for Theorem 6.3, we use the so-called mixture decomposition technique (cf. Section 5.2.1) to construct a variable-length exact synthesis code. This is a mixture of a fixed-length approximate synthesis code that ensures the Rényi divergence of order ∞ vanishes and a completely lossless code of rate $\log(|\mathcal{X}||\mathcal{Y}|)$ (see (5.14)). The performance of this scheme is dominated by that of the approximate synthesis code which requires a much lower rate of shared randomness

compared to the scheme in [18]. Furthermore, the codes in [75] and [109] are such that Y^n is required to be a *deterministic* function of W_n and K_n. In contrast, we allow Y^n to be a *stochastic* function of (W_n, K_n) (cf. Definition 6.4). Hence, naturally, the rate of common randomness is reduced.

Finally, we mention that $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$, in general, is a strict subset of $\mathcal{T}(\pi_{XY}) = \mathcal{C}_{\mathrm{W}}(\pi_{XY})$. This is because, from the operational definitions,

$$T^*(0) = T_{\mathrm{Ex}}(\pi_{XY}),$$

and as we have seen from Section 5.6 for the DSBS with crossover probability $p \in (0, 1/2)$,

$$T_{\mathrm{Ex}}(\pi_{XY}) > C_{\mathrm{W}}(\pi_{XY}).$$

We evaluate the region $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ for the DSBS in Section 6.6 and compare it to $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$.

6.5 Symmetric Binary Erasure Sources

In this section, we revisit the SBES as discussed in Section 2.4. Recall that this is a source with uniform $X \in \{0, 1\}$ and that Y is connected to X via a binary erasure channel with erasure probability p. The joint distribution is given in (2.12). We saw that the exact common information of the SBES is equal to its Wyner's common information because Condition (5.25) in Corollary 5.6, namely that $\sum_{w \in \mathcal{W}} H(X|W = w)H(Y|W = w) = 0$, is satisfied.

For the SBES, Cuff [48] evaluated the optimal rate region for TV approximate synthesis (cf. Definition 6.2). Unsurprisingly, the region is the same as that for exact channel synthesis (cf. Definition 6.5).

Proposition 6.1. For the SBES with erasure probability p, we have

$$\begin{aligned}\mathcal{T}(\pi_{XY}) &= \mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) = \mathcal{C}_{\mathrm{W}}(\pi_{XY}) \\ &= \bigcup_{1-p \le r \le r^*} \left\{ (R, R_0) : \begin{array}{c} R \ge r \\ R + R_0 \ge h(p) + r\left(1 - h\left(\frac{1-p}{r}\right)\right) \end{array} \right\}, \quad (6.27)\end{aligned}$$

where $r^* = \min\{2(p-1), 1\}$ and $h(\cdot)$ is the binary entropy function.

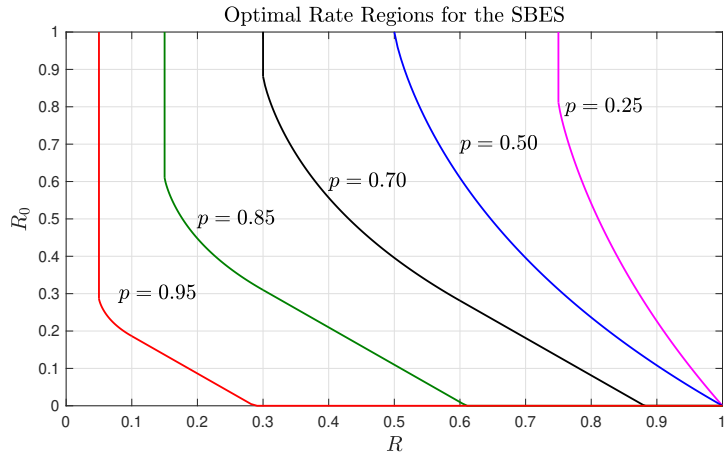


Figure 6.3: Plot of the optimal rate regions (top right hand regions of the boundaries) for approximate and exact channel synthesis for the SBES

Recall that for the SBES, the optimal distribution attaining Wyner's common information is a concatenation of a BEC with erasure probabilities p_1 and a BEC-like channel with erasure probability p_2 such that $(1-p_1)(1-p_2) = 1-p$. The parameter r in (6.27) represents the term $1-p_1$. The various terms in (6.27) are simply the evaluations of $I(W;X)$ and $I(W;XY)$ in the description of $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$ with this parametrization.

The regions for various $p \in [0,1]$ are illustrated in Fig. 6.3. The boundaries cross the horizontal axis at $C_{\mathrm{W}}(\pi_{XY})$, which is equal to 1 for all $0 \leq p \leq 0.5$ (see (2.13) and Fig. 2.7).

6.6 Doubly Symmetric Binary Sources

In this section, we consider the DSBS, a prototypical example in which exact synthesis requires larger rate than approximate synthesis. This is a source in which X is uniform on $\{0,1\}$ and Y is connected to X via a binary symmetric channel with crossover probability p. In Section 2.3, we mentioned that an alternative representation of this source is in terms of the optimal distribution $P_W P_{X|W} P_{Y|W}$ that attains Wyner's common information. This takes the form $W \sim \mathrm{Bern}(1/2)$,

$X = W \oplus A$ (or equivalently, $W = X \oplus A$), and $Y = W \oplus B$ where A and B are independent $\mathrm{Bern}(a)$ random variables such that $a * a = p$. This representation is useful but in this section, as X and Y are not treated symmetrically in the context of distributed channel synthesis, we find it convenient to reparameterize the representation as $X = W \oplus A$ and $Y = W \oplus B$ where $A \sim \mathrm{Bern}(a)$ and $B \sim \mathrm{Bern}(b)$ such that $a * b = p$ and $a, b \in (0, p)$ are not necessarily equal. For a given $a \in (0, p)$, the corresponding b is

$$b = \frac{p - a}{1 - 2a}.$$

Since the exact common information is strictly larger than that of Wyner's common information for the DSBS with crossover probability $p \in (0, 1/2)$ (Proposition 5.3), the optimal rate regions in the context of distributed channel synthesis are also different; in particular $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subsetneq \mathcal{C}_{\mathrm{W}}(\pi_{XY})$ for all $p \in (0, 1/2)$.

Proposition 6.2. For the DSBS with crossover probability p, the optimal rate region for TV approximate channel synthesis (Definition 6.2)

$$\mathcal{C}_{\mathrm{W}}(\pi_{XY}) = \bigcup_{0 \le a \le p} \left\{ (R, R_0) : \begin{array}{c} R \ge 1 - h(a) \\ R + R_0 \ge 1 + h(p) - h(a) - h(b) \end{array} \right\}.$$

The optimal rate region for exact synthesis (Definition 6.5)

$$\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) = \bigcup_{0 \le a \le p} \left\{ (R, R_0) : \begin{array}{c} R \ge 1 - h(a) \\ R + R_0 \ge \log \frac{2}{1-p} + (a+b) \log \frac{1-p}{p} \\ \qquad - h(a) - h(b) \end{array} \right\}.$$

These regions are illustrated in Fig. 6.4. We computed $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ by fixing a point $R_0 \in [0, 1]$ on the ordinate. Then we compute $T^*(R_0)$, defined in (6.23), as

$$T^*(R_0) = \min_{0 \le a \le p} \max \left\{ 1 - h(a),\ \log \frac{2}{1-p} + (a+b) \log \frac{1-p}{p} - h(a) - h(b) - R_0 \right\}.$$

The same can be done for $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$. It can be seen that for $p \in (0, 1/2)$, $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subsetneq \mathcal{C}_{\mathrm{W}}(\pi_{XY})$. Intuitively, this strict inclusion is a

consequence of the type overflow phenomenon described in Section 5.3.1. This observation also confirms that type overflow does not affect the fundamental limits of TV approximate channel synthesis, but it does affect the fundamental limits of exact channel synthesis in the sense of strictly increasing the optimal communication rate for a fixed common randomness rate $R_0 \in [0, H_\pi(Y|X))$.

Finally, if we let $R = I_\pi(X;Y) = 1 - h(p)$ in $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$, we get that $a = p$ and $b = 0$. Hence, the rate of the common randomness is lower bounded as $R \geq h(p)$. Combining this with (6.26) shows that

$$T_0^*(1 - h(p)) = h(p) = H_\pi(Y|X).$$

Thus, for the DSBS, we have identified the optimal rate of the common randomness when the communication rate approaches its optimal value $I_\pi(X;Y) = 1 - h(p)$. In other words, for the DSBS, (6.26) is tight. In fact, one can see that this critical rate $H_\pi(Y|X)$ is the same as that for *approximate* channel synthesis (see Fig. 6.2 and (6.4)).

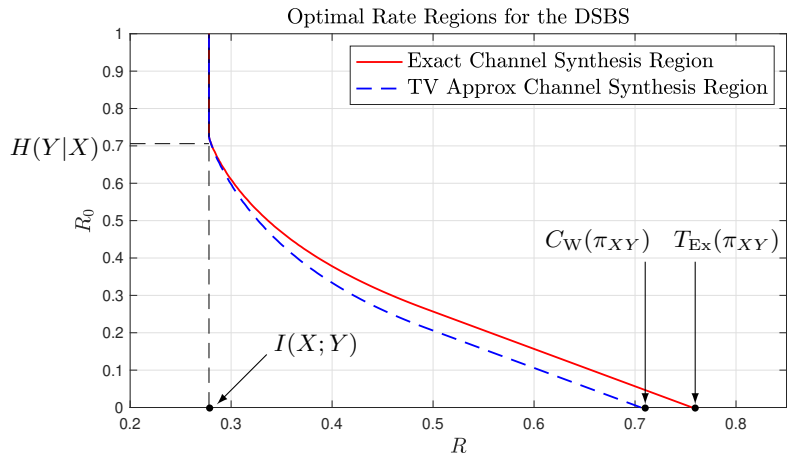


Figure 6.4: Plot of the optimal rate regions (top right hand regions of the boundaries) for TV approximate and exact channel synthesis for the DSBS with crossover probability $p = 0.2$

6.7 Jointly Gaussian Sources

We conclude this section by revisiting the jointly Gaussian source π_{XY} with correlation coefficient $\rho \in (0,1)$. The full description of the source is available in Section 2.5.3 so we will not repeat the details here apart to remind the reader that the source is assumed to have zero mean and correlation coefficient ρ. We saw from Section 2.5.1 that to evaluate Wyner's common information for sources with uncountable alphabets, it should satisfy some regularity conditions (cf. Proposition 2.3). Fortunately, the ubiquitous and canonical jointly Gaussian source satisfies these regularity conditions. As a result, Wyner's common information in terms of the information expression in (2.14) can be evaluated and interpreted as the minimum rate of the description so that π_{XY} can be simulated in a distributed fashion.

For the distributed channel synthesis problem, a similar set of regularity conditions [203, Corollary 2] has to be verified to ensure that optimal rate region under which the TV distance between the synthesized distribution $P_{X^nY^n}$ and the target distribution π_{XY}^n vanishes is equal to $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$ in (6.3). Jointly Gaussian distributions do indeed satisfy these regularity conditions, yielding the following proposition.

Proposition 6.3. For the jointly Gaussian source with correlation coefficient $\rho \in (0,1)$, the optimal rate region for TV approximate channel synthesis $\mathcal{T}(\pi_{XY}) = \mathcal{C}_{\mathrm{W}}(\pi_{XY})$ (Definition 6.2) is the set of rate pairs $(R, R_0) \in \mathbb{R}_+^2$ satisfying

$$R \geq \frac{1}{2} \log \left(\frac{1}{1-\alpha^2} \right) \qquad \text{and} \tag{6.28}$$

$$R + R_0 \geq \frac{1}{2} \log \left(\frac{1-\rho^2}{(1-\alpha^2)(1-\beta^2)} \right) \tag{6.29}$$

for some $\alpha \in [\rho, 1]$ and $\beta = \rho/\alpha$.

Recall that for Wyner's common information, the optimal distribution $P_W P_{X|W} P_{Y|W}$ takes the form that W is standard Gaussian and X and Y are connected to W as

$$X = \sqrt{\rho}\, W + \sqrt{1-\rho}\, N_1 \quad \text{and} \quad Y = \sqrt{\rho}\, W + \sqrt{1-\rho}\, N_2,$$

where N_1 and N_2 are independent standard Gaussian random variables. For the distributed channel synthesis problem, X and Y are not treated symmetrically and so we have to consider a different and more general parametrization of $P_W P_{X|W} P_{Y|W}$. Similarly to the DSBS in Section 6.6, the channels from W to X and from W to Y are respectively

$$X = \alpha\, W + \sqrt{1-\alpha^2}\, N_1 \quad \text{and} \quad Y = \beta\, W + \sqrt{1-\beta^2}\, N_2,$$

where $\alpha\beta = \rho$. Note that X and Y have zero mean and unit variance. By considering this parametrization and evaluating the mutual information terms $I(W;X)$ and $I(W;XY)$, we obtain the expressions in the lower bounds in (6.28) and (6.29).

Similarly to the case for the exact common information, we do not yet have a complete characterization of the optimal rate region for exact channel synthesis for jointly Gaussian sources. It is clearly the case that $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$ constitutes an outer bound to $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$. To derive the inner bound, we have to verify a set of regularity conditions similar to those in Lemma 5.7 and to construct codes such that the conditional Rényi divergence of order ∞ satisfies

$$D_\infty\big(P_{Y^n|X^n} \big\| \pi^n_{Y|X} \big| \tilde{\pi}_{X^n}\big) = o\Big(\frac{1}{n}\Big),$$

where the *truncated* distribution $\tilde{\pi}_{X^n}$ on the X-marginal has PDF

$$\tilde{f}_{X^n}(x^n) \propto \Big(\prod_{i=1}^n f_X(x_i)\Big) \mathbb{1}\{x^n \in \mathcal{A}_\epsilon^{(n)}\},$$

and where f_X and $\mathcal{A}_\epsilon^{(n)}$ are the PDF and the ϵ-weakly typical set of π_X respectively. By using source synthesis codes, one can construct a reliable sequence of approximate channel synthesis codes in the sense that their Rényi divergences of order ∞ vanish (sufficiently rapidly); this translates to a sequence of codes that guarantees exact channel synthesis.

Proposition 6.4. For the jointly Gaussian source with correlation coefficient $\rho \in (0,1)$, the optimal rate region for exact channel synthesis (Definition 6.5) satisfies

$$\mathcal{T}^{(\mathrm{in})}_{\mathrm{Ex}}(\pi_{XY}) \subset \mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subset \mathcal{C}_{\mathrm{W}}(\pi_{XY}),$$

where $\mathcal{T}_{\mathrm{Ex}}^{(\mathrm{in})}(\pi_{XY})$ is the set of rate pairs $(R, R_0) \in \mathbb{R}_+^2$ satisfying

$$R \geq \frac{1}{2} \log \left(\frac{1}{1-\alpha^2} \right) \qquad \text{and}$$
$$R + R_0 \geq \frac{1}{2} \log \left(\frac{1-\rho^2}{(1-\alpha^2)(1-\beta^2)} \right) + \frac{\rho\sqrt{(1-\alpha^2)(1-\beta^2)}}{1-\rho^2} \log \mathrm{e} \quad (6.30)$$

for some $\alpha \in [\rho, 1]$ and $\beta = \rho/\alpha$.

The additional term in the inequality in (6.30) (over the one in (6.29)) is analogous to the additional term of $(\rho \log \mathrm{e})/(1+\rho)$ of the upper bound on the exact common information for jointly Gaussian sources in Proposition 5.4. By the intuition gleaned from the DSBS in Proposition 6.2 (in which $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ was characterized exactly) and the type overflow phenomenon, we conjecture that the inner bound $\mathcal{T}_{\mathrm{Ex}}^{(\mathrm{in})}(\pi_{XY})$ is tight and there is a gap between $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ and $\mathcal{C}_{\mathrm{W}}(\pi_{XY})$, i.e., that $\mathcal{T}_{\mathrm{Ex}}^{(\mathrm{in})}(\pi_{XY}) = \mathcal{T}_{\mathrm{Ex}}(\pi_{XY}) \subsetneq \mathcal{C}_{\mathrm{W}}(\pi_{XY})$. The inner and outer bounds on $\mathcal{T}_{\mathrm{Ex}}(\pi_{XY})$ for a jointly Gaussian source with $\rho = 0.5$ is shown in Fig. 6.5.

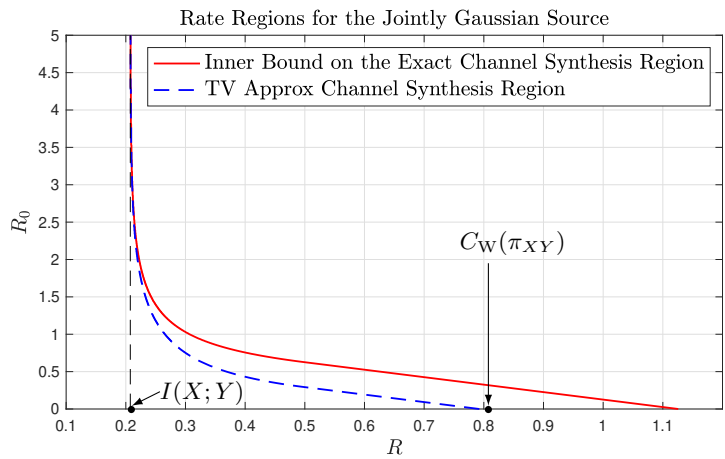


Figure 6.5: Plots of the TV approximate synthesis rate region and the inner bound to the exact channel synthesis region for the jointly Gaussian source with correlation coefficient $\rho = 0.5$. The boundary of the optimal rate region for exact channel synthesis lies between lines. We conjecture that it coincides with the inner bound.

Finally, for a jointly Gaussian source $(X, Y) \sim \pi_{XY}$ with correlation coefficient $\rho \in [0, 1)$, the mutual information between X and Y is

$$I_\pi(X; Y) = \frac{1}{2} \log \frac{1}{1 - \rho^2}.$$

Hence, in this case, under the condition that the sequence of communication rates approaches $I_\pi(X; Y)$ asymptotically, the minimum rate of shared randomness for exact channel synthesis

$$T_0^*(I_\pi(X; Y)) \geq \inf \{R_0 : (I_\pi(X; Y), R_0) \in \mathcal{C}_{\mathrm{W}}(\pi_{XY})\} = \infty.$$

This is attained when $\alpha \downarrow \rho$ and $\beta \uparrow 1$ in (6.29). The same is true for TV approximate channel synthesis. Hence, for a Gaussian source, if the rate of shared randomness R_0 is finite, it is impossible to realize either exact or approximate channel synthesis unless the asymptotic communication rate R is strictly larger than $I_\pi(X; Y)$. This can also be seen from the vertical asymptote in Fig. 6.5.

7

Common Information and Nonnegative Rank

This section completes our discussion of the extensions and generalizations of Wyner's common information. Instead of focusing on coding-inspired operational interpretations of various common information measures, we describe somewhat surprising connections between these measures and a fundamental problem in numerical linear algebra, signal processing, and machine learning, known as *nonnegative matrix factorization* or NMF. The NMF problem was popularized in a landmark paper by Lee and Seung [106] and has received significant attention since its inception. It has numerous applications to audio signal processing, hyperspectral imaging, bioinformatics, and text clustering, among others. See the excellent books by Gillis [65] and Cichocki *et al.* [36] for overviews.

Simply put, in NMF, one is given a nonnegative matrix $\mathbf{M} \in \mathbb{R}_+^{m \times k}$ and one is required to find a factorization of $\mathbf{M}$ into two nonnegative matrices $\mathbf{U} \in \mathbb{R}_+^{m \times r}$ and $\mathbf{V} \in \mathbb{R}_+^{r \times k}$ such that $\mathbf{M}$ is *exactly* or *approximately* equal to the product $\mathbf{UV}$, i.e.,

$$\mathbf{M} = \mathbf{UV} \quad \text{or} \quad \mathbf{M} \approx \mathbf{UV}. \tag{7.1}$$

The matrices $\mathbf{U}$ and $\mathbf{V}$ are usually referred to as the *dictionary* and *coefficient* matrices respectively and the minimum r such that there

exists $\mathbf{U}$ and $\mathbf{V}$ such that $\mathbf{M} = \mathbf{UV}$ is known as the *nonnegative rank* of $\mathbf{M}$. The study of NMF in machine learning and signal processing usually consists in developing algorithms to find $\mathbf{U}$ and $\mathbf{V}$ efficiently and accurately. En route, one typically uses heuristic (e.g., Bayesian) methods [162] to find good approximations of the nonnegative rank. In this section, however, we are concerned with *exact* factorizations and we discuss information-theoretic interpretations of the nonnegative rank. We will see that the nonnegative rank is intimately connected to several common information quantities that we encountered in the previous sections (such as Wyner's common information and the exact common information).

7.1 Nonnegative Rank

We now formally define the nonnegative rank based on the minimal number of rank one factors that sum to the given matrix $\mathbf{M}$.

Definition 7.1. The *nonnegative rank* of a nonnegative matrix $\mathbf{M} \in \mathbb{R}_+^{m \times k}$, denoted as $\mathrm{rank}_+(\mathbf{M})$, is the smallest integer r such that $\mathbf{M}$ can be represented as

$$\mathbf{M} = \sum_{w=1}^{r} \mathbf{u}_w \mathbf{v}_w^\top \tag{7.2}$$

for some nonnegative vectors $\mathbf{u}_w \in \mathbb{R}_+^m$ and $\mathbf{v}_w \in \mathbb{R}_+^k$. Here $\mathbf{u}_w$ and $\mathbf{v}_w^\top$ respectively represent the w^{th} column of $\mathbf{U}$ and w^{th} row of $\mathbf{V}$ where $\mathbf{U}$ and $\mathbf{V}$ are the dictionary and coefficient matrices in (7.1).

As shown by Vavasis [168], the computation of the nonnegative rank is NP-hard. See Moitra [118] for some positive results. For example, checking whether the nonnegative rank is equal to a fixed value r (not part of the input) can be done in polynomial time in the dimensions of the input matrix, namely in time $O((mk)^{r^2})$. The nonnegative rank is of tremendous significance in computational complexity and combinatorial optimization. Of particular importance is the fundamental *factorization theorem* of Yannakakis [189] which states that the nonnegative rank of the slack matrix of a polytope equals its extension complexity (i.e., the minimum number of facets in a higher-dimensional polytope from which

the original one can be obtained as a linear projection). We will not delve into such issues in this section as we focus on various other interesting information-theoretic interpretations of the nonnegative rank.

We note that the usual (linear) rank is a trivial lower bound to the nonnegative rank as the vectors $\mathbf{u}_w$ and $\mathbf{v}_w$ are no longer constrained to be nonnegative, i.e.,

$$\operatorname{rank}(\mathbf{M}) \leq \operatorname{rank}_+(\mathbf{M}).$$

It is known from Cohen and Rothblum [38, Theorem 4.1] that if $\mathbf{M}$ has rank at most two, this inequality is tight. However, this inequality is not tight in general as the following example from [38] shows.

Example 7.1. Let

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

A direct computation shows that $\operatorname{rank}(\mathbf{M}) = 3$. We now claim that $\operatorname{rank}_+(\mathbf{M}) = 4$. On the one hand, the nonnegative rank of any matrix of size m by k cannot exceed $\min\{m, k\}$, which is 4 in this case. To justify the lower bound, we call a pair of entries $M_{a,b}$ and $M_{c,d}$ *pairwise independent* if $M_{a,b}M_{c,d} > 0$ and $M_{a,d}M_{c,b} = 0$. Then by (7.2), we see that if $\mathbf{M}$ contains a set of q pairwise independent entries, then $\operatorname{rank}_+(\mathbf{M}) \geq q$; see Gillis [65, Section 3.4.1] for a detailed justification of this fact. In this example, the entries $M_{1,1}$, $M_{2,3}$, $M_{3,2}$, and $M_{4,4}$ are pairwise independent, so $\operatorname{rank}_+(\mathbf{M}) \geq 4$. Hence, $\operatorname{rank}_+(\mathbf{M}) = 4$, which is strictly larger than the rank of $\mathbf{M}$.

In fact, it is known [13] that the nonnegative rank can be arbitrarily larger than the rank. A canonical example is the family of *distance matrices*.

Example 7.2. For a set of real numbers $\{a_1, \ldots, a_m\} \subset \mathbb{R}$, let $\mathbf{M} = \mathbf{M}(a_1, \ldots, a_m)$ be the $m \times m$ symmetric matrix

$$\mathbf{M} = \begin{bmatrix} 0 & (a_1 - a_2)^2 & (a_1 - a_3)^2 & \ldots & (a_1 - a_m)^2 \\ (a_2 - a_1)^2 & 0 & (a_2 - a_3)^2 & \ldots & (a_2 - a_m)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (a_m - a_1)^2 & (a_m - a_2)^2 & (a_m - a_3)^2 & \ldots & 0 \end{bmatrix}.$$

Thus, the $(i,j)^{\text{th}}$ entry of $\mathbf{M}$ is the square of the distance between a_i and a_j. For obvious reasons, $\mathbf{M}$ is known as a *distance matrix*. Let

$$\mathbf{B} := \begin{bmatrix} a_1^2 & 1 & -2a_1 \\ a_2^2 & 1 & -2a_2 \\ \vdots & \ddots & \vdots \\ a_m^2 & 1 & -2a_m \end{bmatrix} \quad \text{and} \quad \mathbf{C} := \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_1^2 & a_2^2 & \dots & a_m^2 \\ a_1 & a_2 & \dots & a_m \end{bmatrix}.$$

Then $\mathbf{M} = \mathbf{BC}$ so the rank of $\mathbf{M}$ is at most 3. If $|\{a_1, a_2, \dots, a_m\}| \geq 3$, then the rank of $\mathbf{M}$ is exactly 3. However, Beasley and Laffey [13] showed that $\operatorname{rank}_+(\mathbf{M}) = \Omega(\log m)$, so the gap between $\operatorname{rank}(\mathbf{M})$ and $\operatorname{rank}_+(\mathbf{M})$ is arbitrarily large as $m \to \infty$. Also see the work of Hrubeš [85] who showed that $\operatorname{rank}_+(\mathbf{M}) \leq 2 \log m + 2$. We mention in passing that there is yet another notion of rank known as the *positive semidefinite (PSD) rank*. The PSD rank of distance matrices is 2 [55].

Most of the existing lower bounds on the nonnegative rank are based only on the *support* of the matrix, i.e., the sparsity pattern of the entries as in Example 7.1. See Braun *et al.* [30] and Gillis [65, Chapter 3] for reviews and Fawzi and Parrilo [56] for an interesting exception using norm-based methods. The sole utilization of the support has obvious shortcomings as the values of the elements of $\mathbf{M}$ are ignored. In the rest of this section, we take a deeper look at the nonnegative rank from a common information-theoretic perspective.

7.2 Wyner's Common Information as Amortized Nonnegative Rank

In this section, we describe a connection between Wyner's common information and the nonnegative rank of nonnegative matrices. This connection was discovered by Braun *et al.* [30], Braun and Pokutta [31], and Jain *et al.* [89].

To make this connection, for a nonnegative matrix $\mathbf{M} \in \mathbb{R}_+^{m \times k}$, we define its *induced distribution* as

$$\pi_{XY}(x,y) := \frac{M_{x,y}}{\|\mathbf{M}\|_1} \quad \text{for all } (x,y) \in [m] \times [k], \tag{7.3}$$

where $\|\mathbf{M}\|_1 := \sum_{x,y} M_{x,y}$ denotes the ℓ_1 norm or the sum of the (absolute values of the) elements of $\mathbf{M}$. Note that the distribution π_{XY}

defined in (7.3) is valid because $\mathbf{M}$ is nonnegative and, with the normalization by $\|\mathbf{M}\|_1$, π_{XY} sums to unity. Furthermore, we deliberately use the symbol π_{XY} to draw an analogue to the target distribution discussed in Sections 2 and 4–6. In this section, we write $\mathcal{X} = [m]$ and $\mathcal{Y} = [k]$ to denote the finite alphabets of X and Y respectively.

A discrete random variable W with support (or alphabet) $\mathcal{W}$ is said to be a *seed* for the pair of random variables $(X, Y) \sim \pi_{XY}$, or equivalently the matrix $\mathbf{M}$, if X and Y are conditionally independent given W. Given $\mathbf{M}$ and a seed W for $\mathbf{M}$, define the collection of matrices $\{\mathbf{M}_w : w \in \mathcal{W}\}$, each with elements

$$[\mathbf{M}_w]_{x,y} := \Pr(X = x, Y = y, W = w)\|\mathbf{M}\|_1.$$

Every seed W for (X, Y) induces an NMF by writing $\mathbf{M} = \sum_w \mathbf{M}_w$ since $\mathbf{M}_w$ is rank one (a consequence of the Markov chain $X - W - Y$). In Section 2, we referred to W, the seed, as the *common random variable* in the definition of Wyner's common information.

We also note that every NMF of $\mathbf{M} = \sum_w \mathbf{M}_w = \sum_w \mathbf{u}_w \mathbf{v}_w^\top$ induces a seed W for $\mathbf{M}$ by extending the induced distribution π_{XY} via

$$\Pr(X = x, Y = y, W = w) := \frac{[\mathbf{M}_w]_{x,y}}{\|\mathbf{M}\|_1}.$$

By virtue of the fact that $\mathbf{M}_w = \mathbf{u}_w \mathbf{v}_w^\top$ for every w, we see that X and Y are conditionally independent given W. Due to the connection between a nonnegative matrix $\mathbf{M}$ and its induced distribution π_{XY} in (7.3), we can define *Wyner's common information for* $\mathbf{M}$ as

$$C_{\mathrm{W}}(\mathbf{M}) := C_{\mathrm{W}}(\pi_{XY}).$$

We start with a simple observation due to Jain *et al.* [89] and Braun and Pokutta [31] which reinforces the definitions above.

Proposition 7.1. Wyner's common information of $\mathbf{M} \in \mathbb{R}_+^{m \times k}$ is upper bounded by the logarithm of the nonnegative rank of $\mathbf{M}$, i.e.,

$$C_{\mathrm{W}}(\mathbf{M}) \leq \log \operatorname{rank}_+(\mathbf{M}). \tag{7.4}$$

Proof. Let $\mathbf{M}$ have an NMF given by

$$\mathbf{M} = \sum_{w \in \mathcal{W}} \mathbf{u}_w \mathbf{v}_w^\top. \tag{7.5}$$

Define the seed or common random variable W with conditional distribution $P_{W|XY}$ as

$$P_{W|XY}(w|x,y) = \begin{cases} \dfrac{[\mathbf{u}_w]_x[\mathbf{v}_w]_y}{M_{x,y}} & M_{x,y} > 0 \\ \text{arbitrary} & M_{x,y} = 0 \end{cases}. \tag{7.6}$$

This is a valid conditional distribution because for every (x,y) such that $M_{x,y} > 0$,

$$\sum_{w\in\mathcal{W}} P_{W|XY}(w|x,y) = \sum_{w\in\mathcal{W}} \frac{[\mathbf{u}_w]_x[\mathbf{v}_w]_y}{M_{x,y}} = \frac{1}{M_{x,y}} \sum_{w\in\mathcal{W}} [\mathbf{u}_w]_x[\mathbf{v}_w]_y = 1,$$

where the last equality follows from (7.5). Define the joint distribution $P_{WXY} := P_{W|XY}\pi_{XY}$, where π_{XY} is the induced distribution of $\mathbf{M}$. By construction, $(W,X,Y) \sim P_{WXY}$ satisfies $P_{XY} = \pi_{XY}$. Furthermore, by combining (7.3) and (7.6), we obtain

$$P_{XY|W}(x,y|w) = \frac{[\mathbf{u}_w]_x[\mathbf{v}_w]_y}{\sum_{x',y'}[\mathbf{u}_w]_{x'}[\mathbf{v}_w]_{y'}},$$

which, for every fixed w, is clearly a product distribution (cf. Definition 4.3(a)). Hence, $X - W - Y$ forms a Markov chain. To complete the proof, recall that

$$C_{\mathrm{W}}(\mathbf{M}) = C_{\mathrm{W}}(\pi_{XY}) = \min_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} I_P(XY;W).$$

Hence, by choosing a minimal factorization of $\mathbf{M}$ in (7.5) (i.e., one that has the smallest $|\mathcal{W}|$), one has

$$C_{\mathrm{W}}(\mathbf{M}) \le I_P(XY;W) \le H(W) \le \log|\mathcal{W}| = \log \operatorname{rank}_+(\mathbf{M}),$$

completing the proof of (7.4). □

One natural question arising from Proposition 7.1 concerns the tightness of the bound in (7.4). This bound can be arbitrarily loose as the following example from Braun *et al.* [30] demonstrates. Our justification of the upper bound on $C_{\mathrm{W}}(\mathbf{M})$ differs from [30] and, in particular, does not use require the notion of *rectangle covering* [189].

Example 7.3. For a fixed natural number m, let $\mathbf{M} \in \mathbb{R}_+^{m \times m}$ be the diagonal matrix with diagonal elements $M_{i,i} = 2^i / \sum_{j \in [m]} 2^j$ for $i \in [m]$. Then, it is clear that $\operatorname{rank}_+(\mathbf{M}) = m$. However, Wyner's common information for this matrix $\mathbf{M}$, which is normalized, can be bounded as

$$\begin{aligned} C_{\mathrm{W}}(\mathbf{M}) &\le H_\pi(XY) && (7.7)\\ &= H(\pi_X) && (7.8)\\ &= H\left(\frac{2}{\sum_{j\in[m]} 2^j}, \frac{2^2}{\sum_{j\in[m]} 2^j}, \ldots, \frac{2^m}{\sum_{j\in[m]} 2^j}\right)\\ &= -\sum_{i\in[m]} \frac{2^i}{\sum_{j\in[m]} 2^j} \log\left(\frac{2^i}{\sum_{j\in[m]} 2^j}\right), && (7.9) \end{aligned}$$

where (7.7) follows because $I(XY;W) \le H_\pi(XY)$ and (7.8) follows because $X = Y$ in the joint distribution π_{XY} induced by $\mathbf{M}$. The final expression in (7.9) can be shown to be no larger than 2 for all m (and in fact converges to 2 as $m \to \infty$). Thus, $C_{\mathrm{W}}(\mathbf{M}) \le 2$ for all $m \in \mathbb{N}$ and the gap between $C_{\mathrm{W}}(\mathbf{M})$ and $\log \operatorname{rank}_+(\mathbf{M}) = \log m$ in (7.4) can be made arbitrarily large as m tends to infinity.

This somewhat pathological phenomenon can, however, be remedied by considering small ℓ_1 perturbations of the n-fold Kronecker power of the given matrix $\mathbf{M}$. In this case, the limit of the normalized logarithm of the nonnegative rank of the perturbed matrix can be shown to be upper bounded by Wyner's common information of $\mathbf{M}$. This fundamental result is due to Braun *et al.* [30] who used the term *amortization* to describe the perturbation and limiting operations.

Theorem 7.1 (Amortized nonnegative rank and Wyner's common information)**.** Let $\mathbf{M} \in \mathbb{R}_+^{m \times k}$ be a matrix with $\|\mathbf{M}\|_1 = \sum_{x,y} M_{x,y} = \ell$. Then for any $\epsilon > 0$ and $\delta \in (0,1)$, for every

$$n \ge \max\left\{\Omega\Big(\frac{\log^2(mk)}{\epsilon^2 C_{\mathrm{W}}(\mathbf{M})^2} \log\Big(\frac{1}{\delta}\Big)\Big), \Omega\Big(\frac{\delta}{\epsilon}\Big)\right\},$$

there exists a nonnegative matrix $\mathbf{M}_{\epsilon,\delta,n} \in \mathbb{R}_+^{m^n \times k^n}$ with

$$\frac{1}{n} \log \operatorname{rank}_+(\mathbf{M}_{\epsilon,\delta,n}) \le (1+\epsilon)\, C_{\mathrm{W}}(\mathbf{M}) + O\Big(\delta^3 \log \frac{1}{\delta}\Big)\, \frac{\log n}{n}$$

and

$$\left\|\mathbf{M}^{\otimes n} - \mathbf{M}_{\epsilon,\delta,n}\right\|_1 \leq \delta \ell^n. \tag{7.10}$$

In particular, for every $\delta \in (0,1)$, one has

$$\lim_{\epsilon \downarrow 0} \lim_{n\to\infty} \frac{1}{n} \log \operatorname{rank}_+(\mathbf{M}_{\epsilon,\delta,n}) = C_{\mathrm{W}}(\mathbf{M}). \tag{7.11}$$

Thus, Wyner's common information of $\mathbf{M}$ (or of $(X,Y) \sim \pi_{XY}$) admits yet another operational interpretation, namely normalized logarithm of the nonnegative rank of an ℓ_1-perturbed version $\mathbf{M}^{\otimes n}$. This is in addition to its two more familiar operational interpretations in terms of (i) the minimum rate of common randomness of the Gray-Wyner system when the sum rate is constrained to be no larger than the joint entropy and (ii) the minimum amount of common randomness to simulate a joint source in a distributed manner (cf. Section 2).

The reader will notice that Theorem 7.1 is analogous to the fact that the *TV common information* (introduced in Definition 4.5) is equal to *Wyner's common information* (see Cuff [48] and Section 4.3). Indeed, the ℓ_1-relaxation of $\mathbf{M}^{\otimes n}$ to $\mathbf{M}_{\epsilon,\delta,n}$ in (7.10) is analogous to the discrepancy between the target distribution π_{XY}^n and the synthesized distribution $P_{X^nY^n}$ in (4.15). So, in some sense, we have "come full circle" in this part of the monograph.

The proof of Theorem 7.1 involves approximating π_{XY}^n, the n-fold product of the induced distribution given $\mathbf{M}$, by a collection of "better behaving" distributions so that we can bound the log-likelihood ratio $\log P_{XY|W}(X,Y|W) - \log \pi_{XY}(X,Y)$ whose expectation under $(X,Y,W) \sim \pi_{XY} P_{W|XY}$ yields $I(XY;W)$ in the expression for Wyner's common information $C_{\mathrm{W}}(\pi_{XY}) = \min_{X-W-Y} I(XY;W)$. For this purpose, several concentration bounds, such as Chernoff bounds, are used to show that certain well-behaved distributions exist with high probability. As the details are rather involved and delicate, we refer the reader to Braun *et al.* [30].

7.3 Exact Rényi Common Information as Nonnegative Rank

In Section 7.2, we related the nonnegative rank of a matrix $\mathbf{M}$ to its Wyner's common information. Given our discussion of the *exact*

common information in Section 5, it is natural to wonder whether the nonnegative rank has any relation to the exact common information. The purpose of this section is to elaborate on this.

Recall from Proposition 5.1 that the exact common information admits the multi-letter characterization in terms of the *common entropy rate* (previously defined in (5.5)) as follows

$$T_{\mathrm{Ex}}(\pi_{XY}) = \lim_{n\to\infty} \frac{G(\pi_{XY}^n)}{n}, \tag{7.12}$$

where the *common entropy* of $(X, Y) \sim \pi_{XY}$ (previously defined in (5.4)) is

$$G(\pi_{XY}) = \min_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} H(W). \tag{7.13}$$

We can define the *common Rényi entropy of order* $\alpha \in [0, \infty]$ as

$$G_\alpha(\pi_{XY}) := \min_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} H_\alpha(W), \tag{7.14}$$

where H_α is the Rényi entropy of order α (defined in (1.9) and (1.8)). Then the exact common information can be generalized to the *exact Rényi common information of order* α, similarly to (7.12), as follows

$$T_{\mathrm{Ex}}^{(\alpha)}(\pi_{XY}) := \lim_{n\to\infty} \frac{G_\alpha(\pi_{XY}^n)}{n}. \tag{7.15}$$

The existence of the limit in (7.15) follows by the subadditivity of the sequence $\{G_\alpha(\pi_{XY}^n)/n\}_{n\in\mathbb{N}}$ and Fekete's lemma [57]. Note, by definition, that $T_{\mathrm{Ex}}^{(1)}(\pi_{XY}) = T_{\mathrm{Ex}}(\pi_{XY})$. Since $\alpha \mapsto H_\alpha(\pi_{XY})$ is non-increasing, $T_{\mathrm{Ex}}^{(\alpha)}(\pi_{XY})$ is also non-increasing in $\alpha \in [0, \infty]$. We will be concerned with $T_{\mathrm{Ex}}^{(\alpha)}(\pi_{XY})$ for values of $\alpha \in \{0, 1, \infty\}$. The following proposition was shown by the present authors in [204].

Proposition 7.2. We have

$$G_\alpha(\pi_{XY}) = \begin{cases} \log \mathrm{rank}_+(\pi_{XY}) & \alpha = 0 \\ G(\pi_{XY}) & \alpha = 1 \\ \min\limits_{Q_X, Q_Y} D_\infty(Q_X Q_Y \| \pi_{XY}) & \alpha = \infty \end{cases}.$$

The first statement ($\alpha = 0$) in Proposition 7.2 follows by first noticing that $H_0(W) = \log |\mathcal{W}|$, where the support of W is $\mathcal{W}$. Hence,

we see that the exact Rényi common information of order 0 corresponds to the minimum common randomness rate for exact generation of the target distribution in which the common randomness is only allowed to be compressed by *fixed-length* codes. In contrast to Theorem 7.1, this is a *one-shot* characterization of the nonnegative rank and no perturbation or limiting operations are needed. The second statement ($\alpha = 1$) follows by comparing the definitions of $G(\pi_{XY})$ and $G_\alpha(\pi_{XY})$ in (7.13) and (7.14) respectively.

The final statement ($\alpha = \infty$) requires a short calculation which we sketch here. Since $H_\infty(W) = -\log \max_w P_W(w)$,

$$G_\infty(\pi_{XY}) = -\log \max_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} \max_w P_W(w).$$

Swapping the maximization operations,

$$\begin{aligned} G_\infty(\pi_{XY}) &= -\log \max_w \max_{P_W P_{X|W} P_{Y|W} : P_{XY} = \pi_{XY}} P_W(w) \\ &\geq -\log \max_w \max_{\substack{P_{X|W} P_{Y|W}: \\ P_W(w) P_{X|W}(x|w) P_{Y|W}(y|w) \\ \leq \pi_{XY}(x,y)\, \forall (x,y)}} P_W(w) \\ &\geq \min_w \min_{P_{X|W=w}, P_{Y|W=w}} D_\infty\big(P_{X|W=w} P_{Y|W=w} \big\| \pi_{XY}\big) \\ &\geq \min_{Q_X, Q_Y} D_\infty\big(Q_X Q_Y \big\| \pi_{XY}\big). \end{aligned}$$

In the other direction, we let (Q_X^*, Q_Y^*) achieve the minimization in the optimization problem defining $G_\infty(\pi_{XY})$. Let $\epsilon := D_\infty(Q_X^* Q_Y^* \| \pi_{XY})$. Then by the mixture decomposition technique (as described in Section 5.2.1), we see that

$$\pi_{XY} = 2^{-\epsilon} Q_X^* Q_Y^* + (1 - 2^{-\epsilon}) P_{\hat{X}\hat{Y}},$$

where $P_{\hat{X}\hat{Y}} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is a joint distribution defined as

$$P_{\hat{X}\hat{Y}} := \begin{cases} \text{arbitrary} & \epsilon = 0 \\ \dfrac{\pi_{XY} - 2^{-\epsilon} Q_X^* Q_Y^*}{1 - 2^{-\epsilon}} & \epsilon \in (0, \infty) \\ \pi_{XY} & \epsilon = \infty \end{cases}.$$

Now, we choose the common random variable W having alphabet $\mathcal{W} = (\mathcal{X} \times \mathcal{Y}) \cup \{w_0\}$ where $w_0 \notin \mathcal{X} \times \mathcal{Y}$ and W has distribution

$$P_W(w) = \begin{cases} 2^{-\epsilon} & w = w_0 \\ (1 - 2^{-\epsilon}) P_{\hat{X}\hat{Y}}(\hat{x}, \hat{y}) & w = (\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \end{cases}.$$

We construct $P_{X|W}$ and $P_{Y|W}$ as

$$P_{X|W}(x|w) = \begin{cases} Q_X^*(x) & w = w_0 \\ \mathbb{1}\{x = \hat{x}\} & w = (\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \end{cases} \quad \text{and}$$

$$P_{Y|W}(y|w) = \begin{cases} Q_Y^*(y) & w = w_0 \\ \mathbb{1}\{y = \hat{y}\} & w = (\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \end{cases}.$$

By construction, the joint distribution $P_W P_{X|W} P_{Y|W}$ satisfies

$$P_{XY} = \pi_{XY} \quad \text{and} \quad H_\infty(W) \le \epsilon.$$

Thus, $G_\infty(\pi_{XY}) \le \epsilon = D_\infty(Q_X^* Q_Y^* \| \pi_{XY})$ as desired.

When we consider the n-fold product distribution π_{XY}^n (or equivalently the n-fold Kronecker product $\pi_{XY}^{\otimes n}$ of the matrix of joint probabilities π_{XY}), we obtain the following corollary.

Corollary 7.2 (Exact Rényi common information)**.** We have

$$T_{\mathrm{Ex}}^{(\alpha)}(\pi_{XY}) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{n} \log \operatorname{rank}_+ \left(\pi_{XY}^{\otimes n}\right) & \alpha = 0 \\ T_{\mathrm{Ex}}(\pi_{XY}) & \alpha = 1 \\ \min\limits_{Q_X, Q_Y} D_\infty(Q_X Q_Y \| \pi_{XY}) & \alpha = \infty \end{cases}.$$

We note that the first statement ($\alpha = 0$) requires a limiting operation because unlike the linear rank, it is, in general, not true that $\operatorname{rank}_+(\mathbf{M}^{\otimes n}) = \left(\operatorname{rank}_+(\mathbf{M})\right)^n$; see Vandaele *et al.* [167] for a discussion and related conjectures. The second statement ($\alpha = 1$) comes from the multi-letter characterization of the exact common information given in Kumar, Li, and El Gamal [103] (Proposition 5.1) while the last ($\alpha = \infty$) requires some single-letterization steps; see Yu and Tan [204]. Corollary 7.2, illustrated in Fig. 7.1, implies that the exact Rényi common information of α interpolates between the nonnegative rank (when $\alpha = 0$), the exact common information (when $\alpha = 1$),

and $\min_{Q_X,Q_Y} D_\infty(Q_XQ_Y\|\pi_{XY})$ (when $\alpha = \infty$). This is somewhat analogous to the fact that the Rényi common information forms a bridge between Wyner's common information and the exact common information (see Fig. 5.1).

It is important to note a key distinction between Corollary 7.2 and Theorem 7.1. The former tells us that the asymptotic exponent of the nonnegative rank of $\pi_{XY}^{\otimes n}$ can be interpreted as the exact Rényi common information of order 0. The latter, on the other hand, tells us that the asymptotic exponent of the nonnegative rank of *an* ℓ_1 *perturbed version* of $\pi_{XY}^{\otimes n}$ is Wyner's common information; see (7.11).

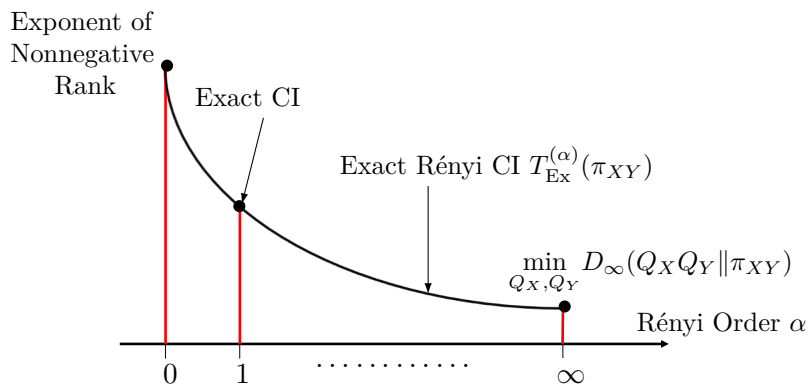


Figure 7.1: A schematic showing that the exact Rényi common information forms a bridge between the nonnegative rank, the exact common information, and $\min_{Q_X,Q_Y} D_\infty(Q_XQ_Y\|\pi_{XY})$

7.4 Nonnegative α-Rank

We conclude this section by briefly mentioning a common information-theoretic generalization of the nonnegative rank. Recall from Proposition 7.2 that the logarithm of the nonnegative rank is the exact Rényi common information of order 0. Inspired by this relationship, we can generalize the notion of the nonnegative rank to the nonnegative α-rank as follows. For a nonnegative matrix (but non-zero matrix) $\mathbf{M}$ and

$\alpha \in [-\infty, \infty]$, we define the *nonnegative α-rank* of $\mathbf{M}$ as

$$\operatorname{rank}_+^{(\alpha)}(\mathbf{M}) := 2^{G_\alpha(\pi_{XY})}, \tag{7.16}$$

where π_{XY} is the induced distribution of $\mathbf{M}$, defined in (7.3). The normalization by $\|\mathbf{M}\|_1$ is required as any reasonable notion of rank should be invariant to the scale of the matrix. This definition reduces to that of $\operatorname{rank}_+(\mathbf{M})$ when $\alpha = 0$ by Proposition 7.2.

For a diagonal matrix $\mathbf{D}$, let $\|\mathbf{D}\|_\alpha$ be the α-norm of its diagonal, i.e., $\|\mathbf{D}\|_\alpha = \left(\sum_i D_{i,i}^\alpha\right)^{1/\alpha}$. By appealing to the definition of the Rényi entropy, we see that the nonnegative α-rank of $\mathbf{M} \in \mathbb{R}_+^{m \times k}$ can be equivalently expressed as

$$\operatorname{rank}_+^{(\alpha)}(\mathbf{M}) = \min_{\mathbf{U},\mathbf{D},\mathbf{V}} \|\mathbf{D}\|_\alpha^{\frac{\alpha}{1-\alpha}}, \tag{7.17}$$

where the minimization runs over all triples of matrices $\mathbf{U} \in \mathbb{R}_+^{m \times r}$, $\mathbf{V} \in \mathbb{R}_+^{k \times r}$ and diagonal $\mathbf{D} \in \mathbb{R}_+^{r \times r}$ for some $r \in \mathbb{N}$ such that

$$\sum_{x=1}^{m} [\mathbf{U}]_{xw} = 1 \quad \text{for all } w \in [r], \tag{7.18}$$

$$\sum_{y=1}^{k} [\mathbf{V}]_{yw} = 1 \quad \text{for all } w \in [r], \quad \text{and} \tag{7.19}$$

$$\mathbf{U}\mathbf{D}\mathbf{V}^\top = \frac{\mathbf{M}}{\|\mathbf{M}\|_1}.$$

The equality conditions in (7.18) and (7.19) are to ensure that each column of $\mathbf{U}$ and each column of $\mathbf{V}$ is a PMF. The alternative definition of $\operatorname{rank}_+^{(\alpha)}(\mathbf{M})$ in (7.17) can be seen to be a generalization of the usual nonnegative rank as defined in Definition 7.1. Indeed,

$$\lim_{\alpha \downarrow 0} \operatorname{rank}_+^{(\alpha)}(\mathbf{M}) = \operatorname{rank}_+(\mathbf{M}).$$

The properties of $\operatorname{rank}_+^{(\alpha)}(\mathbf{M})$ as defined in (7.16) or (7.17) are not well understood and constitute a fertile avenue for further investigations.

Part III

Extensions of Gács–Körner–Witsenhausen's Common Information

8

Non-Interactive Correlation Distillation

In this section, we consider an extension of GKW's common information, termed *Non-Interactive Correlation Distillation*. We recall that GKW's common information measures the amount of "almost identical" randomnesses that can be extracted individually from a pair of correlated sources. By Gács and Körner's theorem [60] (also recall Proposition 3.3), the GKW's common information of a joint source (X, Y) is positive if and only if there exists a pair of non-constant functions (f, g) such that $f(X) = g(Y)$ almost surely. Unfortunately, GKW's common information is zero for many common pairs of sources, such as jointly Gaussian sources and doubly symmetric binary sources (DSBS) with correlation coefficients $\rho \in (-1, 1)$. For these joint sources, even if we wish to extract *a single pair* of identical bits from these sources individually, this innocuous task still turns out to be infeasible.

This observation begs the following natural question: *How can we refine the quantification of common information for these and other sources such that it resembles the GKW's common information and yet is non-zero?* Even though any randomnesses extracted from these sources individually cannot agree almost surely, the extracted randomnesses can indeed agree with a certain probability, which, in this section, we quantify

via various probability limit theorems such as the central limit and large deviations theorems. In other words, the extracted randomnesses can be correlated. It is thus natural to quantify the "common information" by the maximal correlation of a pair of random bits that can be extracted from the sources individually. In the literature, determining this maximal correlation is coined the *Noise Stability Problem* (two-set version), the *Non-Interactive Correlation Distillation* or NICD problem. Other names include the *Non-Interactive Binary Simulation Problem* and the *Binary Decision Problem*. This problem was studied by Kamath and Anantharam [94], Yang [188], Mossel *et al.* [124] and Witsenhausen [178] among others.

In this section, we focus mainly on the doubly symmetric binary source (DSBS) parametrized by its correlation coefficient $\rho \in (-1, 1)$. Even though this source is simple, the NICD problem for this source is nontrivial and insights can be drawn from it. In Section 8.1, we define the 2-user NICD problem for the DSBS. Based on the means of the extracted random bits, we define several asymptotic regimes of interest, including the central limit, moderate, and large deviations regimes. In Section 8.2, we discuss various achievability schemes for the NICD problem based on certain geometric structures in Hamming space; these include subcubes and Hamming spheres. These geometric structures are useful to prove existence results in the above-mentioned asymptotic regimes. In Sections 8.3, 8.4, and 8.5 we discuss the optimality of these schemes. Finally, in Section 8.6 we discuss known results in the NICD problem for other sources such as bivariate Gaussians.

8.1 Non-Interactive Correlation Distillation with 2 Users

Consider a doubly symmetric binary distribution π_{XY} on the alphabet $\mathcal{X} \times \mathcal{Y} = \{0, 1\}^2$ with correlation coefficient $\rho \in (0, 1)$, i.e.,

$$\pi_{XY}(x, y) = \begin{cases} \dfrac{1+\rho}{4} & x = y \\ \dfrac{1-\rho}{4} & x \neq y \end{cases}. \tag{8.1}$$

With this parametrization, the *correlation coefficient* of (X, Y), defined in (1.1), is indeed ρ. The pair of random variables $(X, Y) \sim \pi_{XY}$

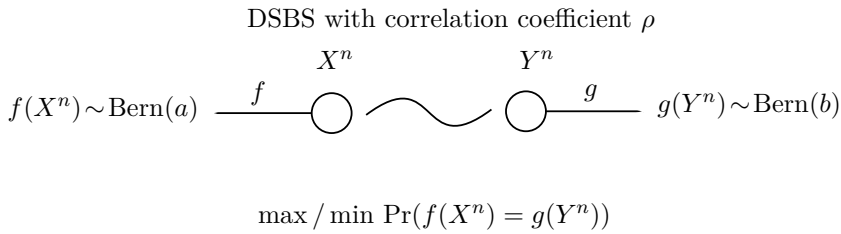


Figure 8.1: The Non-Interactive Correlation Distillation problem with 2 users

corresponds to the DSBS as described in Section 2.3 with *crossover probability* $p = (1-\rho)/2 \in (0, 1/2)$. In this section, we find it convenient to parametrize the DSBS by its correlation coefficient ρ instead of its crossover probability p. It suffices to consider positive ρ as the results carry over to the case for negative ρ by replacing X with $1 - X$. Throughout this section except for Section 8.6, we let (X^n, Y^n) be distributed as the n-fold product distribution π_{XY}^n.

We now introduce the NICD problem with 2 users. This problem is illustrated in Fig. 8.1, in which a source sequence (X^n, Y^n) generated by a DSBS is given, and two random bits $f(X^n)$ and $g(Y^n)$ are generated in a distributed manner using a pair of Boolean functions $f, g : \{0,1\}^n \to \{0,1\}$. The objective of the NICD problem is to maximize or minimize the *agreement probability* of $f(X^n)$ and $g(Y^n)$, i.e., $\Pr(f(X^n) = g(Y^n))$, under the condition that the means of $f(Y^n)$ and $g(Y^n)$ are bounded.

Definition 8.1. Given $a, b \in [0,1]$, the *forward joint probability* is

$$\overline{\Gamma}^{(n)}(a,b) := \max_{\substack{f,g:\{0,1\}^n \to \{0,1\}: \Pr(f(X^n)=1) \le a, \\ \Pr(g(Y^n)=1) \le b}} \Pr\left(f(X^n) = g(Y^n) = 1\right). \quad (8.2)$$

Similarly, define the *reverse joint probability* as

$$\underline{\Gamma}^{(n)}(a,b) := \min_{\substack{f,g:\{0,1\}^n \to \{0,1\}: \Pr(f(X^n)=1) \ge a, \\ \Pr(g(Y^n)=1) \ge b}} \Pr\left(f(X^n) = g(Y^n) = 1\right). \quad (8.3)$$

In Definition 8.1, we maximize or minimize the probability that both generated bits are equal to one, i.e., $\Pr(f(X^n) = g(Y^n) = 1)$, rather than $\Pr(f(X^n) = g(Y^n))$, since by noting that the marginal probabilities $\Pr(f(X^n) = 1)$ and $\Pr(g(Y^n) = 1)$ are constrained in (8.2) and (8.3), determining the former is equivalent to that of the latter.

8.1.1 Optimizing over Supports of Boolean Functions

Instead of optimizing over the Boolean functions f and g, in the following, we find it convenient for the sake of exploiting the properties of geometric structures (such as Hamming balls and spheres) to optimize over their *supports*. The *support* of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is defined as the set $\mathcal{A} := \{x^n \in \{0,1\}^n : f(x^n) = 1\}$.

If we denote the supports of f and g as $\mathcal{A}$ and $\mathcal{B}$ respectively, then one can rewrite (8.2) and (8.3) respectively as

$$\overline{\Gamma}^{(n)}(a,b) = \max_{\mathcal{A},\mathcal{B}\subset\{0,1\}^n \,:\, \pi_X^n(\mathcal{A})\le a,\, \pi_Y^n(\mathcal{B})\le b} \pi_{XY}^n(\mathcal{A}\times\mathcal{B}), \tag{8.4}$$

and

$$\underline{\Gamma}^{(n)}(a,b) = \min_{\mathcal{A},\mathcal{B}\subset\{0,1\}^n \,:\, \pi_X^n(\mathcal{A})\ge a,\, \pi_Y^n(\mathcal{B})\ge b} \pi_{XY}^n(\mathcal{A}\times\mathcal{B}). \tag{8.5}$$

Let $\overline{\Gamma}^{(\infty)}$ and $\underline{\Gamma}^{(\infty)}$ respectively denote the pointwise limits of $\overline{\Gamma}^{(n)}$ and $\underline{\Gamma}^{(n)}$ as $n \to \infty$, i.e.,

$$\overline{\Gamma}^{(\infty)}(a,b) := \lim_{n\to\infty} \overline{\Gamma}^{(n)}(a,b) \quad \text{and} \quad \underline{\Gamma}^{(\infty)}(a,b) := \lim_{n\to\infty} \underline{\Gamma}^{(n)}(a,b). \tag{8.6}$$

These are respectively known as the *asymptotic forward* and *asymptotic reverse joint probabilities*.

By definition, the forward and reverse joint probabilities are non-decreasing in each of the parameters when the other is fixed. This implies that there exists an optimal pair of sets $\mathcal{A}, \mathcal{B} \subset \{0,1\}^n$ (or Boolean functions (f,g)) attaining the forward joint probability such that

$$\pi_X^n(\mathcal{A}) = \frac{\lfloor a \cdot 2^n \rfloor}{2^n} \quad \text{and} \quad \pi_Y^n(\mathcal{B}) = \frac{\lfloor b \cdot 2^n \rfloor}{2^n}.$$

Indeed, if either of these statements were not true, we can enlarge $\mathcal{A}$ (resp. $\mathcal{B}$) to make its π_X^n-probability (resp. π_Y^n-probability) closer to a (resp. b). Similarly, there exists an optimal pair $(\mathcal{A},\mathcal{B})$ (or Boolean functions (f,g)) attaining the reverse joint probability such that

$$\pi_X^n(\mathcal{A}) = \frac{\lceil a \cdot 2^n \rceil}{2^n} \quad \text{and} \quad \pi_Y^n(\mathcal{B}) = \frac{\lceil b \cdot 2^n \rceil}{2^n}.$$

As a consequence, for dyadic rationals a and b (i.e., $a = M/2^n, b = N/2^n$ with integers $M, N \in \{0,1,\ldots,2^n\}$), the inequalities in the constraints

in the definitions of forward and reverse probabilities (i.e., $\overline{\Gamma}^{(n)}(a,b)$ and $\underline{\Gamma}^{(n)}(a,b)$) can be replaced by equalities, without affecting their values. These observations also allow us to conclude that

$$\overline{\Gamma}^{(n)}(1-a,b) = b - \underline{\Gamma}^{(n)}(a,b) \quad \text{for all dyadic rationals } a,b.$$

When we consider the asymptotic case in which $n \to \infty$, i.e., the quantities in (8.6), the requirement that a and b are dyadic rationals can be removed. This implies that for any $a,b \in [0,1]$,

$$\overline{\Gamma}^{(\infty)}(1-a,b) = b - \underline{\Gamma}^{(\infty)}(a,b). \tag{8.7}$$

Hence, for all $(a,b) \in [0,1]^2$, determining the asymptotic forward joint probability in (8.6) is equivalent to determining the asymptotic reverse joint probability and vice versa.

8.1.2 Asymptotic Regimes and Exponents of Interest

The identification of the optimal pairs $(\mathcal{A},\mathcal{B})$ that attain the forward or reverse joint probabilities in (8.4) and (8.5) constitutes a combinatorial problem and is thus difficult in general. Hence, we focus on the limiting cases as $n \to \infty$ as this simplifies the problem, and the resultant problems are also information-theoretic in nature. Specifically, the following three asymptotic regimes will be considered.

1. Central limit (CL) regime: We set a and b to be constants. We write $a = 2^{-\alpha}$ and $b = 2^{-\beta}$ for a pair of constants $(\alpha,\beta) \in [0,\infty)^2$.

2. Large deviations (LD) regime: We set a and b to be sequences that vanish exponentially fast as $n \to \infty$. In particular, we write $a = 2^{-n\alpha}$ and $b = 2^{-n\beta}$ for a pair of constants $(\alpha,\beta) \in [0,1]^2$.

3. Moderate deviations (MD) regime: We set a and b to be sequences that vanish subexponentially fast as $n \to \infty$. More precisely, $a = 2^{-\theta_n\alpha}, b = 2^{-\theta_n\beta}$ for a pair of constants $(\alpha,\beta) \in [0,\infty)^2$, where $\{\theta_n\}_{n\in\mathbb{N}}$ is a positive sequence satisfying $\theta_n \to \infty$ and $\theta_n/n \to 0$, henceforth called an *MD sequence*.

The MD regime straddles between the CL and LD regimes. It is usually the case if one solves a certain information-theoretic problem in

the CL or the LD regimes, a result for the MD regime can be derived as a corollary, for example, by appealing to Taylor's theorem; see Altuğ and Wagner [3], Polyanskiy and Verdú [140], and Tan [159] for example. We will see that this is also the case for the NICD problem.

In the following section, we will set $\mathcal{A}$ and $\mathcal{B}$ to be subcubes, Hamming balls, and Hamming spheres. These are prototypical subsets in the Hamming space that are amenable to analyses. We will then apply various probabilistic limit theorems—such as the central limit theorem and large and moderate deviations theorems—to derive the "performances" of these subsets in attaining the forward and reverse joint probabilities. We formally define several exponents of interest.

Definition 8.2. Consider the following exponents:

1. Forward and reverse CL exponents: For $\alpha, \beta \in [0, \infty)$,

$$\underline{\Upsilon}^{(n)}_{\mathrm{CL}}(\alpha, \beta) := -\log \overline{\Gamma}^{(n)}(2^{-\alpha}, 2^{-\beta}) \quad \text{and} \tag{8.8}$$
$$\overline{\Upsilon}^{(n)}_{\mathrm{CL}}(\alpha, \beta) := -\log \underline{\Gamma}^{(n)}(2^{-\alpha}, 2^{-\beta}).$$

2. Forward and reverse LD exponents: For $\alpha, \beta \in [0, 1]$,

$$\underline{\Upsilon}^{(n)}_{\mathrm{LD}}(\alpha, \beta) := -\frac{1}{n} \log \overline{\Gamma}^{(n)}(2^{-n\alpha}, 2^{-n\beta}) \quad \text{and} \tag{8.9}$$
$$\overline{\Upsilon}^{(n)}_{\mathrm{LD}}(\alpha, \beta) := -\frac{1}{n} \log \underline{\Gamma}^{(n)}(2^{-n\alpha}, 2^{-n\beta}). \tag{8.10}$$

3. Forward and reverse MD exponents: Given an MD sequence $\{\theta_n\}$, and for $\alpha, \beta \in [0, \infty)$,

$$\underline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta) := -\frac{1}{\theta_n} \log \overline{\Gamma}^{(n)}(2^{-\theta_n \alpha}, 2^{-\theta_n \beta}) \quad \text{and} \tag{8.11}$$
$$\overline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta) := -\frac{1}{\theta_n} \log \underline{\Gamma}^{(n)}(2^{-\theta_n \alpha}, 2^{-\theta_n \beta}). \tag{8.12}$$

4. Define $\underline{\Upsilon}^{(\infty)}_{\mathrm{CL}}$, $\overline{\Upsilon}^{(\infty)}_{\mathrm{CL}}$, $\underline{\Upsilon}^{(\infty)}_{\mathrm{LD}}$, $\overline{\Upsilon}^{(\infty)}_{\mathrm{LD}}$, $\underline{\Upsilon}^{(\infty)}_{\mathrm{MD}}$, and $\overline{\Upsilon}^{(\infty)}_{\mathrm{MD}}$ as the pointwise limits of the above exponents as $n \to \infty$.

The reader may notice that the definitions in (8.8)–(8.12) appear to be redundant, since each of the forward (resp. reverse) exponents is

equivalent to the forward (resp. reverse) joint probability in the sense that if the forward (resp. reverse) joint probability has been determined, then each of the forward (resp. reverse) exponents has also been determined. This also means the forward (resp. reverse) exponents are also "equivalent". For example, for each $n \in \mathbb{N}$, $\underline{\Upsilon}_{\mathrm{LD}}^{(n)}(\alpha, \beta) = \frac{1}{n}\underline{\Upsilon}_{\mathrm{CL}}^{(n)}(n\alpha, n\beta)$ and $\underline{\Upsilon}_{\mathrm{MD}}^{(n)}(\alpha, \beta) = \frac{1}{\theta_n}\underline{\Upsilon}_{\mathrm{CL}}^{(n)}(\theta_n\alpha, \theta_n\beta)$. We introduce these notations because in the sequel, we will introduce several *dimension-free* bounds (e.g., Theorem 8.9) that can be conveniently expressed in terms of the exponents defined in (8.8)–(8.12). Here, a dimension-free bound is one that is independent of the dimension (or blocklength) n, but is valid for all dimensions n.

In the following, we introduce bounds on the NICD exponents in (8.8)–(8.12). As is conventional in information theory, there are two parts to this endeavor. In the achievability part that will be discussed in Section 8.2, we construct subsets $\mathcal{A}$ and $\mathcal{B}$ that upper bound the forward exponents and lower bound the reverse exponents. In the converse parts that will be discussed in Section 8.3–8.5, we demonstrate impossibility results, i.e., lower bounds on the forward exponents and upper bounds on the reverse exponents. The achievability and converse bounds match in some special cases.

8.2 Achievability: Subcubes, Hamming Balls, and Spheres

We now consider the achievability parts, i.e., deriving lower bounds for the forward joint probability and upper bounds for the reverse joint probability. For these parts, we consider three canonical types of subsets in Hamming space—subcubes, Hamming balls, and Hamming spheres.

8.2.1 Subcubes

An $(n-k)$-*subcube* $\mathbb{C}_{n-k}$ is a set of vectors $x^n \in \{0,1\}^n$ with k components held fixed. For example, if we fix the first k components to 1, then we get the $(n-k)$-subcube $\{1^k\} \times \{0,1\}^{n-k}$, where 1^k denotes the length-k all-ones vector. For any set $\mathcal{A} \subset \{0,1\}^n$, we say that its *indicator*, denoted as $\mathbb{1}_{\mathcal{A}}$, is the function $f : \{0,1\}^n \to \{0,1\}$ such that $f(x^n) = 1$ for all $x^n \in \mathcal{A}$ and $f(x^n) = 0$ for all $x^n \notin \mathcal{A}$. The indicator of

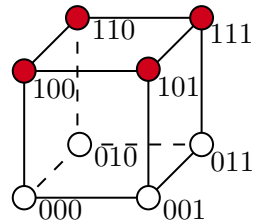


Figure 8.2: A subcube (shaded) in $\{0,1\}^3$ with the first component fixed to 1

the subcube $\{1^k\} \times \{0,1\}^{n-k}$ is $x^n \in \{0,1\}^n \mapsto \prod_{i=1}^k x_i$. An important class of subcubes is the class of $(n-1)$-subcubes, e.g., $\{1\} \times \{0,1\}^{n-1}$. An $(n-1)$-subcube with $n=3$ is illustrated in Fig. 8.2. The indicators of $(n-1)$-subcubes are the functions $x^n \mapsto x_i$ or $x^n \mapsto 1-x_i$ for $i \in [n]$. Such functions are known as *dictator* functions.

We now return to the NICD problem. For $a = b = 2^{-k}$ for a positive integer k, we choose $\mathcal{A}$ and $\mathcal{B}$ as a pair of identical $(n-k)$-subcubes. By referring to the joint distribution in (8.1), we see that the joint probability induced by $(\mathcal{A}, \mathcal{B})$ is

$$\pi_{XY}^n(\mathcal{A} \times \mathcal{B}) = \pi_{XY}(1,1)^k = \Big(\frac{1+\rho}{4}\Big)^k. \tag{8.13}$$

On the other hand, if we choose $\mathcal{A}$ and $\mathcal{B}$ as a pair of anti-symmetric $(n-k)$-subcubes, i.e., $\mathcal{A} = 1^n - \mathcal{B} = \mathcal{C}_{n-k}$, then the induced joint probability is

$$\pi_{XY}^n(\mathcal{A} \times \mathcal{B}) = \pi_{XY}(1,0)^k = \Big(\frac{1-\rho}{4}\Big)^k. \tag{8.14}$$

For the more general case in which $a = 2^{-k_1}$ and $b = 2^{-k_2}$ for integers $0 \le k_1 \le k_2$, if we choose $(\mathcal{A}, \mathcal{B})$ as a pair of "nested" subcubes, i.e., $\mathcal{A} = \{1^{k_1}\} \times \{0,1\}^{n-k_1}$ and $\mathcal{B} = \{1^{k_2}\} \times \{0,1\}^{n-k_2}$, then the induced joint probability

$$\pi_{XY}^n(\mathcal{A} \times \mathcal{B}) = \Big(\frac{1}{2}\Big)^{k_2-k_1}\Big(\frac{1+\rho}{4}\Big)^{k_1}.$$

For the same case, if we choose $(\mathcal{A}, \mathcal{B})$ as a pair of "anti-nested" subcubes, i.e., $\mathcal{A} = \{1^{k_1}\} \times \{0,1\}^{n-k_1}$ and $\mathcal{B} = \{0^{k_2}\} \times \{0,1\}^{n-k_2}$, then

$$\pi_{XY}^n(\mathcal{A} \times \mathcal{B}) = \Big(\frac{1}{2}\Big)^{k_2-k_1}\Big(\frac{1-\rho}{4}\Big)^{k_1}.$$

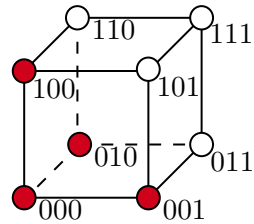


Figure 8.3: A Hamming ball (shaded) in $\{0,1\}^3$ centered at $(0,0,0)$ with radius 1

We now discuss the case in which a and b are dyadic rationals (i.e., $a = M/2^n, b = N/2^n$ for some integers M, N). Observe that if a dyadic rational a is not equal to 2^{-k} for some integer k, then there is no subcube with π_X^n-probability *exactly* equal to a. Hence, to achieve better performances, a generalization of subcubes $\{0^k\} \times \{0,1\}^{n-k}$ and $\{1^k\} \times \{0,1\}^{n-k}$, called *lexicographic sets*, turns out to be useful. A subset of $\{0,1\}^n$ is called *lexicographic* if the elements are selected as the first sequences in some lexicographic order (either ascending or descending). A Boolean function is called *lexicographic* if its support is a lexicographic set. By setting $\mathcal{A}$ and $\mathcal{B}$ to be two lexicographic sets both in ascending (or descending) order, we can obtain a relatively large joint probability $\pi_{XY}^n(\mathcal{A} \times \mathcal{B})$. On the other hand, if we set $\mathcal{A}$ and $\mathcal{B}$ to be two lexicographic sets such that one is chosen in ascending order and the other in descending order, we can obtain a relatively small joint probability $\pi_{XY}^n(\mathcal{A} \times \mathcal{B})$. The explicit expressions for these two joint probabilities are complicated, and thus we omit them. A lexicographic set chosen in ascending order can then be written as $\{x^n \in \{0,1\}^n : \sum_{i=1}^n 2^{i-1} x_i \leq r\}$ for some r. This is a special case of so-called *linear threshold functions*, which is discussed in detail in [131].

8.2.2 Hamming Balls

A *Hamming ball* centered at $y^n \in \{0,1\}^n$ with radius $r \in \{0,1,\ldots,n\}$ takes the form $\mathbb{B}_r(y^n) := \{x^n \in \{0,1\}^n : d_{\mathrm{H}}(x^n, y^n) \leq r\}$, where $d_{\mathrm{H}}(x^n, y^n) := \sum_{i=1}^n \mathbb{1}\{x_i \neq y_i\}$ denotes the *Hamming distance* between vectors x^n and y^n. An example of a Hamming ball with radius 1 is illustrated in Fig. 8.3. In the following, we only consider Hamming balls that are centered at $0^n = (0,0,\ldots,0)$ or $1^n = (1,1,\ldots,1)$. For these

Hamming balls (with radius r), we can rewrite them as $\{x^n \in \{0,1\}^n : \sum_{i=1}^n x_i \leq r\}$ and $\{x^n \in \{0,1\}^n : \sum_{i=1}^n x_i \geq n - r\}$ respectively.

We now set $\mathcal{A}$ and $\mathcal{B}$ in the NICD problem to be Hamming balls. We first consider the CL regime in which we choose $\mathcal{A}$ and $\mathcal{B}$ to be a pair of *concentric* Hamming balls. More specifically, $\mathcal{A}_n := \mathbb{B}_{r_n}(0^n)$ and $\mathcal{B}_n = \mathbb{B}_{s_n}(0^n)$ for some sequences $\{r_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$. We append the subscript n to $\mathcal{A}$ and $\mathcal{B}$, to indicate that these two sets depend on n. We can rewrite $\mathcal{A}_n$ as $\{x^n : \sum_{i=1}^n x_i \leq r_n\}$. Hence, the marginal probability $\pi_X^n(\mathcal{A}_n)$ can be written as $\Pr(\sum_{i=1}^n X_i \leq r_n)$ where $\{X_i\}_{i=1}^n$ are i.i.d. with each $X_i \sim \mathrm{Bern}(1/2)$. To calculate the limiting value of this probability as $n \to \infty$, one may apply several well-known concentration of measure theorems, including the central limit theorem or various large deviations theorems. Since we focus on the CL regime here, we require that $\pi_X^n(\mathcal{A}_n)$ tends to a non-vanishing constant. Hence, we set the radius $r_n = \frac{n}{2} + \frac{\lambda\sqrt{n}}{2}$ for some $\lambda \in \mathbb{R}$. Then, the (univariate) central limit theorem yields

$$\lim_{n\to\infty} \pi_X^n(\mathcal{A}_n) = \Phi(\lambda), \tag{8.15}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard univariate Gaussian distribution. Similarly, if we set the radius $s_n = \frac{n}{2} + \frac{\mu\sqrt{n}}{2}$ for some $\mu \in \mathbb{R}$, we obtain

$$\lim_{n\to\infty} \pi_Y^n(\mathcal{B}_n) = \Phi(\mu).$$

We now estimate the asymptotic value of the joint probability $\pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n)$ where $\mathcal{A}_n$ and $\mathcal{B}_n$ are concentric spheres with radii r_n and s_n respectively. Note that this probability can be restated as $\Pr(\sum_{i=1}^n X_i \leq r_n, \sum_{i=1}^n Y_i \leq s_n)$ where $(X^n, Y^n) = \{(X_i, Y_i)\}_{i=1}^n$ is a source sequence generated by a DSBS with correlation coefficient ρ. The multivariate central limit theorem then yields

$$\lim_{n\to\infty} \pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n) = \Phi_\rho(\lambda, \mu), \tag{8.16}$$

where $\Phi_\rho(\cdot,\cdot)$ is the joint CDF of the zero-mean bivariate Gaussian distribution with covariance matrix

$$\mathbf{K} := \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{8.17}$$

Based on the asymptotic results in (8.15)–(8.16), one can obtain a lower bound on the forward joint probability in the NICD problem [131, Ex. 9.24 and 10.5].

Proposition 8.1. For $a, b \in (0,1)$,

$$\overline{\Gamma}^{(\infty)}(a,b) \geq \Lambda_\rho(a,b), \tag{8.18}$$

where

$$\Lambda_\rho(a,b) := \Phi_\rho(\Phi^{-1}(a), \Phi^{-1}(b)). \tag{8.19}$$

Here $\Lambda_\rho(\cdot,\cdot)$ is known as the *bivariate normal copula* or the *Gaussian quadrant probability function*. Thanks to the equivalence between the forward and reverse joint probabilities as stated in (8.7), (8.18) can alternatively be expressed in terms of the reverse joint probability as

$$\underline{\Gamma}^{(\infty)}(a,b) \leq \Lambda_{-\rho}(a,b). \tag{8.20}$$

The upper bound $\Lambda_{-\rho}(a,b)$ is achieved by a sequence of pairs of anti-concentric balls $\mathcal{A}_n = \mathbb{B}_{r_n}(0^n)$ and $\mathcal{B}_n = \mathbb{B}_{s_n}(1^n)$.

Considering the exponents of the probabilities in (8.18) and (8.20),

$$\underline{\Upsilon}_{\mathrm{CL}}^{(\infty)}(\alpha,\beta) \leq \underline{\Upsilon}_{\mathrm{CL}}(\alpha,\beta) := -\log \Lambda_\rho(2^{-\alpha}, 2^{-\beta}) \quad \text{and} \tag{8.21}$$

$$\overline{\Upsilon}_{\mathrm{CL}}^{(\infty)}(\alpha,\beta) \geq \overline{\Upsilon}_{\mathrm{CL}}(\alpha,\beta) := -\log \Lambda_{-\rho}(2^{-\alpha}, 2^{-\beta}). \tag{8.22}$$

We next consider the LD and MD regimes. Although it is certainly possible to set $\mathcal{A}_n$ and $\mathcal{B}_n$ to be Hamming balls to obtain achievability results for these two regimes, we prefer not to do so here. This is because, it is much easier to derive the same results by using Hamming *spheres* or *spherical shells*. Therefore, we consider the LD and MD regimes in the following subsection after we introduce Hamming spheres.

8.2.3 Hamming Spheres

A *Hamming sphere* centered at $y^n \in \{0,1\}^n$ with radius $r \in \{0,1,\ldots,n\}$ takes the form $\mathbb{S}_r(y^n) := \{x^n \in \{0,1\}^n : d_{\mathrm{H}}(x^n, y^n) = r\}$. See Fig. 8.4 for an illustration. The definition of Hamming spheres differs from that for Hamming balls in the condition $d_{\mathrm{H}}(x^n, y^n) = r$ in which *equality* is mandated. Similarly to the previous subsection, here we

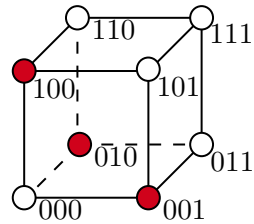


Figure 8.4: A Hamming sphere (shaded) in $\{0,1\}^3$ centered at $(0,0,0)$ with radius 1

also only consider Hamming spheres centered at either 0^n or 1^n, for which we can rewrite them respectively as $\{x^n : \sum_{i=1}^n x_i = r\}$ and $\{x^n : \sum_{i=1}^n x_i = n - r\}$. These Hamming spheres can be regarded as *type classes* with types $(\bar{\lambda}, \lambda)$ and $(\lambda, \bar{\lambda})$ respectively in Hamming space, where $\lambda := \frac{r}{n}$ and $\bar{\lambda} := 1 - \lambda$. Observe that $\mathbb{S}_r(0^n)$ is the same as $\mathbb{S}_{n-r}(1^n)$. Notwithstanding this equivalence, we term a pair of spheres $\mathbb{S}_{r_1}(0^n)$ and $\mathbb{S}_{r_2}(0^n)$ as a pair of *concentric* spheres if $r_1, r_2 \leq n/2$ or $r_1, r_2 \geq n/2$, and as a pair of *anti-concentric* spheres if $r_1 \leq n/2 \leq r_2$ or $r_2 \leq n/2 \leq r_1$.

For the LD regime, we choose $\mathcal{A}_n$ and $\mathcal{B}_n$ to be a pair of concentric or anti-concentric Hamming spheres, i.e., $\mathcal{A}_n = \mathbb{S}_{r_n}(0^n)$ and $\mathcal{B}_n = \mathbb{S}_{s_n}(0^n)$ with $r_n = \lfloor \lambda n \rfloor$ or $\lceil \lambda n \rceil$ and $s_n = \lfloor \mu n \rfloor$ or $\lceil \mu n \rceil$, where $\lambda, \mu \in [0,1]$. By Sanov's theorem [49] (stated in Theorem 1.1),

$$\lim_{n\to\infty} -\frac{1}{n} \log \pi_X^n(\mathcal{A}_n) = D((\bar{\lambda}, \lambda) \| \pi_X) \quad \text{and}$$
$$\lim_{n\to\infty} -\frac{1}{n} \log \pi_Y^n(\mathcal{B}_n) = D((\bar{\mu}, \mu) \| \pi_Y).$$

Since X is uniform on $\{0,1\}$, we can write $D((\bar{\lambda}, \lambda) \| \pi_X) = 1 - h(\lambda)$.

For the joint probability, observe that the set $\mathcal{A}_n \times \mathcal{B}_n$ is a union of joint type classes with types T_{XY} satisfying the condition that its marginals T_X and T_Y are equal to $(\bar{\lambda}, \lambda)$ and $(\bar{\mu}, \mu)$ respectively. Hence, by Sanov's theorem, the joint probability satisfies

$$\lim_{n\to\infty} -\frac{1}{n} \log \pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n) = \mathsf{D}((\bar{\lambda}, \lambda), (\bar{\mu}, \mu) \| \pi_{XY}),$$

where, in analogy to Definition 4.1, the *minimal relative entropy* with respect to π_{XY} over all couplings of Q_X and Q_Y is defined as

$$\mathsf{D}(Q_X, Q_Y \| \pi_{XY}) := \min_{Q_{XY} \in \mathcal{C}(Q_X, Q_Y)} D(Q_{XY} \| \pi_{XY}). \tag{8.23}$$

Optimizing the exponent $\mathsf{D}((\bar{\lambda}, \lambda), (\bar{\mu}, \mu) \| \pi_{XY})$ over all feasible pairs of (λ, μ), yields the following achievability result.

Proposition 8.2. For all $\alpha, \beta \in (0, 1)$,

$$\begin{aligned} \underline{\Upsilon}_{\mathrm{LD}}^{(\infty)}(\alpha, \beta) &\leq \underline{\Upsilon}_{\mathrm{LD}}(\alpha, \beta) \\ &:= \min_{\substack{Q_X, Q_Y: \\ D(Q_X \| \pi_X) \geq \alpha,\, D(Q_Y \| \pi_Y) \geq \beta}} \mathsf{D}(Q_X, Q_Y \| \pi_{XY}), \end{aligned} \tag{8.24}$$

and

$$\begin{aligned} \overline{\Upsilon}_{\mathrm{LD}}^{(\infty)}(\alpha, \beta) &\geq \overline{\Upsilon}_{\mathrm{LD}}(\alpha, \beta) \\ &:= \min_{\substack{Q_X, Q_Y: \\ D(Q_X \| \pi_X) \leq \alpha,\, D(Q_Y \| \pi_Y) \leq \beta}} \mathsf{D}(Q_X, Q_Y \| \pi_{XY}). \end{aligned} \tag{8.25}$$

The bounds in (8.24) and (8.25) are attained by sequences of concentric and anti-concentric Hamming spheres respectively. By the method of types, it is easy to observe that they also can be respectively attained by sequences of concentric and anti-concentric *balls* (since a Hamming ball consists of several spheres and there is one sphere that dominates the others in the sense of the exponent). The above inequalities were conjectured to be tight by Ordentlich, Polyanskiy, and Shayevitz [133]. We refer to this as the *OPS conjecture* in the sequel.

Conjecture 8.1 (OPS Conjecture)**.** For the DSBS and $\alpha, \beta \in (0, 1)$,

$$\underline{\Upsilon}_{\mathrm{LD}}^{(\infty)}(\alpha, \beta) \stackrel{?}{=} \underline{\Upsilon}_{\mathrm{LD}}(\alpha, \beta) \quad \text{and} \quad \overline{\Upsilon}_{\mathrm{LD}}^{(\infty)}(\alpha, \beta) \stackrel{?}{=} \overline{\Upsilon}_{\mathrm{LD}}(\alpha, \beta).$$

In Section 8.5, we discuss the optimality of Hamming spheres in the LD regime, leading to the proof this conjecture. However, before doing this, we first focus on achievability results by Hamming spherical shells in the MD regime.

For the MD regime, we choose the sets in the NICD problem to be two spherical shells (annuli), with thickness in the order of $\sqrt{n\theta_n}$.

Specifically, for a fixed and small $\epsilon > 0$, we choose

$$\mathcal{A}_n = \bigcup_{r \in n/2 + [\lambda, \lambda+\epsilon]\sqrt{n\theta_n}} \mathbb{S}_r(0^n) \quad \text{and} \quad \mathcal{B}_n = \bigcup_{s \in n/2 + [\mu, \mu+\epsilon]\sqrt{n\theta_n}} \mathbb{S}_s(0^n),$$

where $\{\theta_n\}$ is an MD sequence, and $\lambda, \mu \in \mathbb{R}$. In other words, we choose $\mathcal{A}_n$ and $\mathcal{B}_n$ to be unions of type classes induced by types $Q_X = \pi_X + \sqrt{\theta_n/n}\,\eta_X$ and $Q_Y = \pi_Y + \sqrt{\theta_n/n}\,\eta_Y$ respectively, where η_X and η_Y are functions such that $\sum_{x\in\{0,1\}} \eta_X(x) = 0$ and $\sum_{y\in\{0,1\}} \eta_Y(y) = 0$ and $\eta_X(1) \in [\lambda, \lambda+\epsilon]$ and $\eta_Y(1) \in [\mu, \mu+\epsilon]$. Let

$$\hat{\chi}^2(\eta\|\pi) := \sum_{x\in\{0,1\}} \frac{\eta(x)^2}{\pi(x)}.$$

and notice that $\hat{\chi}^2(Q - \pi\|\pi)$ is the chi-squared divergence from Q to π. In analogy to the minimal relative entropy in (8.23), we define

$$\hat{\mathsf{X}}^2(\eta_X, \eta_Y\|\pi_{XY}) := \inf_{\eta_{XY}\in\overline{\mathcal{C}}(\eta_X,\eta_Y)} \hat{\chi}^2(\eta_{XY}\|\pi_{XY}),$$

where $\overline{\mathcal{C}}(\eta_X, \eta_Y)$ is the set of all bivariate functions $\eta_{XY} : \{0,1\}^2 \to \mathbb{R}$ such that their X- and Y-marginals are equal to η_X and η_Y respectively and $\sum_{x,y} \eta_{XY}(x,y) = 0$. Then, letting $\theta_n \to \infty$ and then $\epsilon \downarrow 0$, by the moderate deviations theorem [49], [181],

$$\lim_{n\to\infty} -\frac{1}{\theta_n} \log \pi_X^n(\mathcal{A}_n) = \frac{1}{2}\hat{\chi}^2(\eta_X\|\pi_X), \tag{8.26}$$

$$\lim_{n\to\infty} -\frac{1}{\theta_n} \log \pi_Y^n(\mathcal{B}_n) = \frac{1}{2}\hat{\chi}^2(\eta_Y\|\pi_Y), \quad \text{and}$$

$$\lim_{n\to\infty} -\frac{1}{\theta_n} \log \pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n) = \frac{1}{2}\hat{\mathsf{X}}^2(\eta_X, \eta_Y\|\pi_{XY}). \tag{8.27}$$

In fact, (8.27) requires the continuity of $(\eta_X, \eta_Y) \mapsto \hat{\mathsf{X}}^2(\eta_X, \eta_Y\|\pi_{XY})$; this follows from the following lemma.

Lemma 8.1. For $\eta_X = (-\lambda, \lambda)$ and $\eta_Y = (-\mu, \mu)$, we have

$$\hat{\mathsf{X}}^2(\eta_X, \eta_Y\|\pi_{XY}) = \frac{2(\lambda+\mu)^2}{1+\rho} + \frac{2(\lambda-\mu)^2}{1-\rho}. \tag{8.28}$$

Proof. One can calculate that the optimal η_{XY} attaining the maximum in the definition of $\hat{\mathsf{X}}^2(\eta_X, \eta_Y\|\pi_{XY})$ is

$$\eta_{XY} = \begin{bmatrix} p-\lambda-\mu & \mu-p \\ \lambda-p & p \end{bmatrix},$$

where $p = (\lambda+\mu)/2$. Hence, (8.28) follows. □

Optimizing the exponent $\frac{1}{2}\hat{\mathsf{X}}^2(\eta_X, \eta_Y \| \pi_{XY})$ over all feasible $\eta_X = (-\lambda, \lambda)$ and $\eta_Y = (-\mu, \mu)$ yields the following proposition.

Proposition 8.3. For $\alpha, \beta > 0$,

$$\underline{\Upsilon}^{(\infty)}_{\mathrm{MD}}(\alpha, \beta) \le \underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) := \inf \hat{\mathsf{X}}^2(\eta_X, \eta_Y \| \pi_{XY}) \quad \text{and} \tag{8.29}$$

$$\overline{\Upsilon}^{(\infty)}_{\mathrm{MD}}(\alpha, \beta) \ge \overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) := \sup \hat{\mathsf{X}}^2(\eta_X, \eta_Y \| \pi_{XY}). \tag{8.30}$$

where the inf in (8.29) is over the set of functions $\eta_X, \eta_Y : \{0, 1\} \to \mathbb{R}$ such that $\sum_x \eta_X(x) = \sum_y \eta_Y(y) = 0$ and

$$\hat{\chi}^2(\eta_X \| \pi_X) \ge \alpha \quad \text{and} \quad \hat{\chi}^2(\eta_Y \| \pi_Y) \ge \beta, \tag{8.31}$$

and the sup in (8.30) is over the same set of functions (η_X, η_Y) but with the directions of the inequalities in (8.31) reversed.

The bounds in (8.29) and (8.30) are respectively attained by sequences of concentric and anti-concentric Hamming spheres or balls. The reader may have noticed that the constant 1/2 in (8.26)–(8.27) has been removed in (8.29) and (8.30). This is because, by definition, $\underline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{MD}}$ are *homogeneous* (of degree 1), i.e., for any $\gamma > 0$,

$$\underline{\Upsilon}_{\mathrm{MD}}(\gamma\alpha, \gamma\beta) = \gamma \underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) \quad \text{and} \tag{8.32}$$

$$\overline{\Upsilon}_{\mathrm{MD}}(\gamma\alpha, \gamma\beta) = \gamma \overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta). \tag{8.33}$$

The bounds in (8.29) and (8.30) can be further simplified as follows.

Lemma 8.2. For $\alpha, \beta > 0$,

$$\underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) = \begin{cases} \dfrac{\alpha + \beta - 2\rho\sqrt{\alpha\beta}}{1 - \rho^2} & \rho^2\alpha \le \beta \le \frac{\alpha}{\rho^2} \\ \alpha & \beta < \rho^2\alpha \\ \beta & \alpha < \rho^2\beta \end{cases} \quad \text{and} \tag{8.34}$$

$$\overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) = \frac{\alpha + \beta + 2\rho\sqrt{\alpha\beta}}{1 - \rho^2}. \tag{8.35}$$

Proof. Observe by the uniformity of π_X and π_Y that $\hat{\chi}^2(\eta_X \| \pi_X) = 4\lambda^2$ and $\hat{\chi}^2(\eta_Y \| \pi_Y) = 4\mu^2$. Combining these with Lemma 8.1 yields that

$$\underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) = \min_{\lambda, \mu : 4\lambda^2 \ge \alpha, 4\mu^2 \ge \beta} \frac{2(\lambda + \mu)^2}{1 + \rho} + \frac{2(\lambda - \mu)^2}{1 - \rho} \quad \text{and}$$

$$\overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) = \max_{\lambda, \mu : 4\lambda^2 \le \alpha, 4\mu^2 \le \beta} \frac{2(\lambda + \mu)^2}{1 + \rho} + \frac{2(\lambda - \mu)^2}{1 - \rho}.$$

By the rearrangement inequality and by symmetry, it suffices to consider $\lambda, \mu \geq 0$ for $\underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta)$ and $\lambda \leq 0 \leq \mu$ for $\overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta)$. This results in

$$\underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) = \min_{\lambda \geq \frac{\sqrt{\alpha}}{2}, \mu \geq \frac{\sqrt{\beta}}{2}} \frac{2(\lambda+\mu)^2}{1+\rho} + \frac{2(\lambda-\mu)^2}{1-\rho} \quad \text{and} \tag{8.36}$$

$$\overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) = \max_{-\frac{\sqrt{\alpha}}{2} \leq \lambda \leq 0 \leq \mu \leq \frac{\sqrt{\beta}}{2}} \frac{2(\lambda+\mu)^2}{1+\rho} + \frac{2(\lambda-\mu)^2}{1-\rho}. \tag{8.37}$$

By calculus, one can verify that the right-hand sides of (8.36) and (8.37) are respectively equal to the right-hand sides of (8.34) and (8.35). □

We conclude this section by discussing the relationships between the MD and CL exponents as well as the MD and LD exponents. We can recover the MD exponents from the CL or LD exponents if the MD sequence $\{\theta_n\}$ additionally satisfies $(\log n)/\theta_n \to 0$ as $n \to \infty$. Roughly speaking, in the MD regime, we chose the radii r_n and s_n of Hamming spheres such that $\frac{r_n}{n} \approx \frac{1}{2} + \lambda\sqrt{\epsilon}$ and $\frac{s_n}{n} \approx \frac{1}{2} + \mu\sqrt{\epsilon}$, where $\epsilon := \frac{\theta_n}{n} \to 0$ as $n \to \infty$. This implies that the types corresponding to the spheres are $Q_X \approx \pi_X + \sqrt{\epsilon}\,\eta_X$ and $Q_Y \approx \pi_Y + \sqrt{\epsilon}\,\eta_Y$ as $\epsilon \downarrow 0$. Note that in Sanov's theorem, the LD exponent of the probability of a Hamming sphere with type Q_X is $D(Q_X\|\pi_X) + O(\frac{\log n}{n})$. Hence, if the MD sequence $\{\theta_n\}$ additionally satisfies $(\log n)/\theta_n \to 0$ as $n \to \infty$, this LD exponent is dominated by the term $D(Q_X\|\pi_X)$, which allows us to omit the $O(\frac{\log n}{n})$ term. Moreover, by Taylor's theorem,

$$D(Q_X\|\pi_X) = \frac{\epsilon}{2}\hat{\chi}^2(\eta_X\|\pi_X) + o(\epsilon),$$
$$D(Q_Y\|\pi_Y) = \frac{\epsilon}{2}\hat{\chi}^2(\eta_Y\|\pi_Y) + o(\epsilon),$$

and similarly,

$$\mathsf{D}(Q_X, Q_Y\|\pi_{XY}) = \frac{\epsilon}{2}\hat{\mathsf{X}}^2(\eta_X, \eta_Y\|\pi_{XY}) + o(\epsilon) \quad \text{as } \epsilon \downarrow 0.$$

We obtain the MD exponents by replacing D and D in the LD exponents with $\frac{\epsilon}{2}\hat{\chi}^2$ and $\frac{\epsilon}{2}\hat{\mathsf{X}}^2$ respectively. Formally,

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\underline{\Upsilon}_{\mathrm{LD}}(\epsilon\alpha, \epsilon\beta) = \underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) \quad \text{and} \tag{8.38}$$

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\overline{\Upsilon}_{\mathrm{LD}}(\epsilon\alpha, \epsilon\beta) = \overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta). \tag{8.39}$$

Furthermore, the MD exponents can be also recovered from the CL exponents. By the Berry–Esseen theorem [19], [54], under the condition that the MD sequence $\{\theta_n\}$ satisfies $(\log n)/\theta_n \to 0$ as $n \to \infty$, the probability of a Hamming ball is dominated by the term involving the Gaussian cumulative distribution function $\Phi(\cdot)$. In other words, the additive error term in the Berry–Esseen theorem, which scales as $O(\frac{1}{\sqrt{n}})$, is negligible asymptotically. On the other hand, O'Donnell [131, Ex. 9.24 and 10.5] shows that

$$\lim_{\theta\to\infty} \frac{1}{\theta} \underline{\Upsilon}_{\mathrm{CL}}(\theta\alpha, \theta\beta) = \underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) \quad \text{and} \tag{8.40}$$

$$\lim_{\theta\to\infty} \frac{1}{\theta} \overline{\Upsilon}_{\mathrm{CL}}(\theta\alpha, \theta\beta) = \overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta), \tag{8.41}$$

where $\underline{\Upsilon}_{\mathrm{CL}}$ and $\overline{\Upsilon}_{\mathrm{CL}}$ are defined in (8.21) and (8.22) respectively.

8.2.4 Numerical Results and Comparisons

We now evaluate the various exponents for the DSBS with correlation coefficient ρ. Define $\kappa := (\frac{1+\rho}{1-\rho})^2$,

$$D_{a,b}(p) := D\left(\begin{bmatrix} 1+p-a-b & b-p \\ a-p & p \end{bmatrix} \middle\| \pi_{XY}\right) \quad \text{and}$$

$$\mathsf{D}(a,b) := \min_{\max\{0,a+b-1\}\le p\le \min\{a,b\}} D_{a,b}(p) = D_{a,b}(p^*_{a,b}),$$

where $h(\cdot)$ is the binary entropy function, and

$$p^*_{a,b} := \frac{(\kappa-1)(a+b)+1-\sqrt{((\kappa-1)(a+b)+1)^2-4\kappa(\kappa-1)ab}}{2(\kappa-1)}.$$

For the DSBS, $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$, defined in (8.24) and (8.25), respectively can be written in closed form as

$$\underline{\Upsilon}_{\mathrm{LD}}(\alpha,\beta) = \mathsf{D}(h^{-1}(1-\alpha), h^{-1}(1-\beta)) \quad \text{and}$$
$$\overline{\Upsilon}_{\mathrm{LD}}(\alpha,\beta) = \mathsf{D}(h^{-1}(1-\alpha), 1-h^{-1}(1-\beta)),$$

where $h^{-1} : [0,1] \to [0,1/2]$ is the inverse of the binary entropy function h when its domain is restricted to $[0,1/2]$.

We plot the CL exponents achieved by Hamming balls, and the MD and LD exponents achieved by Hamming balls, spheres, or spherical

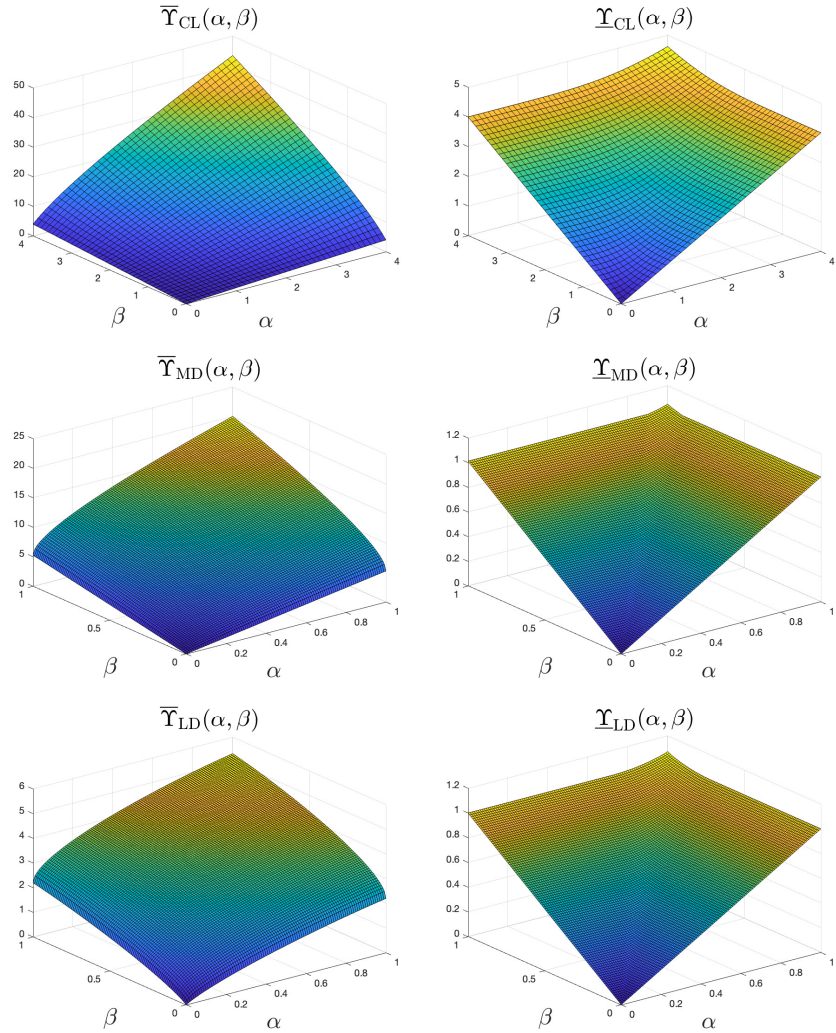


Figure 8.5: Forward and reverse CL, MD, and LD exponents induced by Hamming balls (or spheres) for $\rho = 0.9$. Observe that $\underline{\Upsilon}_{\mathrm{MD}}$ and $\underline{\Upsilon}_{\mathrm{LD}}$ appear to be convex while $\overline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ appear to be concave. The convexity and concavity of $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ respectively have implications for the OPS conjecture (Conjecture 8.1) whose resolution is provided in Section 8.5.

Table 8.1: Comparison of subcubes and Hamming balls or, equivalently, spheres

Regimes	Central limit		Moderate deviations	Large deviations
a, b	Fixed and large	Fixed and small	Subexp. vanishing	Exp. vanishing
Subcubes	Better	Worse	Worse	Worse
Balls/Spheres	Worse	Better	Better	Better

shells in Fig. 8.5. By the homogeneity property in (8.32) and (8.33), the surfaces corresponding to $\underline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{MD}}$ are formed by an infinite number of half-lines from the origin to infinity. Furthermore, by the relation between the MD, LD and CL exponents in (8.38)–(8.39) and (8.40)–(8.41), the surfaces of $\underline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{MD}}$ can be recovered from the surfaces of $\underline{\Upsilon}_{\mathrm{CL}}$ and $\overline{\Upsilon}_{\mathrm{CL}}$ by zooming them out, or recovered from $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ by zooming into a neighborhood of the origin. However, the surfaces of $\underline{\Upsilon}_{\mathrm{CL}}$ and $\overline{\Upsilon}_{\mathrm{CL}}$ as well as the surfaces of $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ *cannot* be recovered from those of $\underline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{MD}}$. In other words, $\underline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{MD}}$ contain much less information compared to $\underline{\Upsilon}_{\mathrm{CL}}$ and $\overline{\Upsilon}_{\mathrm{CL}}$ as well as $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$. This is not unexpected as the MD regime can be thought of a limiting case of the LD and CL regimes. Numerical results in Fig. 8.5 suggest that $\underline{\Upsilon}_{\mathrm{MD}}$ and $\underline{\Upsilon}_{\mathrm{LD}}$ are convex, and $\overline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ are concave, but $\underline{\Upsilon}_{\mathrm{CL}}$ and $\overline{\Upsilon}_{\mathrm{CL}}$ are neither convex nor concave. In Section 8.5, we discuss these issues rigorously in the context of the OPS conjecture (Conjecture 8.1).

We now compare the performances of subcubes, Hamming balls, and Hamming spheres (or spherical shells). We illustrate the forward joint probabilities achieved by subcubes and Hamming balls in Fig. 8.6. As the gaps between the probabilities are visually imperceptible, we also illustrate their differences on the right plot of Fig. 8.6. Based on the numerical comparisons, we observe that for large a and b, subcubes are better. However, for small a and b, Hamming balls are better. We summarize the performances of various geometric structures under different asymptotic regimes in Table 8.1. Based on these results, it is natural to ask whether subcubes are *optimal* for large a and b, and whether Hamming balls or spheres are *optimal* for small a and b. In the following sections, we provide answers to these questions.

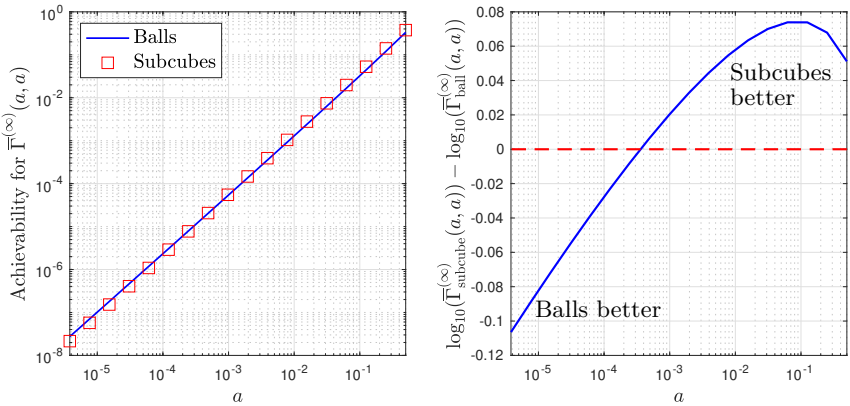


Figure 8.6: Left: The forward joint probabilities achieved by subcubes and Hamming balls with $a = b$ and $\rho = 0.5$; Right: The difference between the logarithms of the forward joint probabilities achieved by subcubes and Hamming balls which shows that subcubes outperform balls for large a and vice versa.

8.3 Converses in the Central Limit Regime

In this and the next two sections, we discuss the optimality of subcubes, Hamming balls, and spheres (or spherical shells) in the various asymptotic regimes for the forward and reverse joint probabilities. In this section, we consider the CL regime in which we are interested in determining whether subcubes are optimal in for the NICD problem for $a = b \in \{1/2, 1/4\}$. The case $a = b = 1/2$ is relatively well known and solved by Witsenhausen [178]. The case $a = b = 1/4$, however, is more challenging and, in fact, was posed as an open problem by E. Mossel in 2017 [119]; see also Mossel [120, Problem 2.6]. Here, we term the case $a = b = 1/4$, as the *mean*-1/4 *stability problem*. In the CL regime, it is also natural to ask whether Hamming balls are optimal for small but fixed a and b (i.e., $0 < a, b < 1/4$). Since this case behaves similarly to that in the MD regime, we will discuss it in the next section concerning the MD regime.

8.3.1 Case of $a = b = 1/2$: Maximal Correlation Method

We first consider the optimality of subcubes (or Boolean functions) for the case $a = b = 1/2$ in the NICD problem. By using the properties

of the *maximal correlation*, the non-asymptotic optimality of subcubes for this basic case was confirmed positively by Witsenhausen [178]. We recall from (1.2) in the introduction that the *Hirschfeld–Gebelein–Rényi* (or *HGR*) *maximal correlation* [63], [84], [144] between two random variables X and Y is defined as

$$\rho_{\mathrm{m}}(X;Y) := \sup_{f,g} \rho(f(X);g(Y)),$$

where $\rho(U;V)$ denotes the correlation coefficient between U and V (defined in (1.1)), and the supremum is taken over all real-valued functions f and g such that $0 < \mathrm{Var}(f(X)), \mathrm{Var}(g(Y)) < \infty$. It is well-known that the maximal correlation satisfies several desirable properties, including tensorization and the data processing inequality.

1. Tensorization: For a sequence of independent pairs of random variables $(X^n, Y^n) = \{(X_i, Y_i)\}_{i=1}^n$, we have
$$\rho_{\mathrm{m}}(X^n;Y^n) = \max_{i\in[n]} \rho_{\mathrm{m}}(X_i;Y_i). \tag{8.42}$$

2. Data processing inequality (DPI): For the Markov chain $U - X - Y - V$, we have
$$\rho_{\mathrm{m}}(U;V) \le \rho_{\mathrm{m}}(X;Y). \tag{8.43}$$

3. Binary random variables: For binary X and Y, we have
$$\rho_{\mathrm{m}}(X;Y) = |\rho(X;Y)|\,. \tag{8.44}$$

Using these properties, Witsenhausen [178] proved the following theorem.

Theorem 8.3. Let π_{XY} be the doubly symmetric binary distribution with correlation coefficient ρ as defined in (8.1). For any $\mathcal{A}$ and $\mathcal{B}$ with $\pi_X^n(\mathcal{A}) = a$ and $\pi_Y^n(\mathcal{B}) = b$,

$$ab - \rho\sqrt{a\bar{a}b\bar{b}} \le \pi_{XY}^n(\mathcal{A}\times\mathcal{B}) \le ab + \rho\sqrt{a\bar{a}b\bar{b}}. \tag{8.45}$$

Proof. Let $(X^n, Y^n) \sim \pi_{XY}^n$. Define $U := \mathbb{1}_{\mathcal{A}}(X^n)$ and $V := \mathbb{1}_{\mathcal{B}}(Y^n)$. Then we have the Markov chain $U - X^n - Y^n - V$. Consider,

$$\begin{aligned}\frac{|\pi_{XY}^n(\mathcal{A}\times\mathcal{B}) - ab|}{\sqrt{a\bar{a}}\sqrt{b\bar{b}}} &= |\rho(U;V)| \\ &= \rho_{\mathrm{m}}(U;V) && (8.46)\\ &\le \rho_{\mathrm{m}}(X^n;Y^n) && (8.47)\\ &= \rho_{\mathrm{m}}(X_1;Y_1) && (8.48)\\ &= \rho, && (8.49)\end{aligned}$$

where (8.46) and (8.49) follow from (8.44), (8.47) follows from the data processing inequality in (8.43), and (8.48) follows from the tensorization property in (8.42) (since all pairs of random variables are *identically distributed*, the max in (8.42) is simply $\rho_{\mathrm{m}}(X_1;Y_1)$). □

From Theorem 8.3, one deduces that for $a = b = 1/2$,

$$\frac{1-\rho}{4} \le \pi_{XY}^n(\mathcal{A}\times\mathcal{B}) \le \frac{1+\rho}{4}. \tag{8.50}$$

Based on the discussion around (8.13)–(8.14), the upper bound is achieved by a pair of identical dictator functions, i.e., $f(x^n) = g(x^n) = x_i$ (or $1 - x_i$) for all $i \in [n]$. Moreover, the lower bound is achieved by a pair anti-symmetric dictator functions, i.e., $f(x^n) = 1 - g(x^n) = x_i$ for all $i \in [n]$. Hence,

$$\overline{\Gamma}^{(n)}\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{1+\rho}{4} \quad \text{and} \quad \underline{\Gamma}^{(n)}\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{1-\rho}{4} \quad \text{for all } n \ge 1.$$

This result also can be proven by the hypercontractivity method and Fourier analysis; these are discussed in the next two subsections.

8.3.2 Case of $a = b = 1/2$: Hypercontractivity Method

The classic *hypercontractivity inequalities* form an important class of functional inequalities. These inequalities play a fundamental role in the NICD problem when the means of the Boolean functions are assumed to be either large or small. The forward and reverse parts of the hypercontractivity inequalities for the DSBS are stated in Theorem 8.4 which follow from Gross [69], Borell [27], and O'Donnell [131].

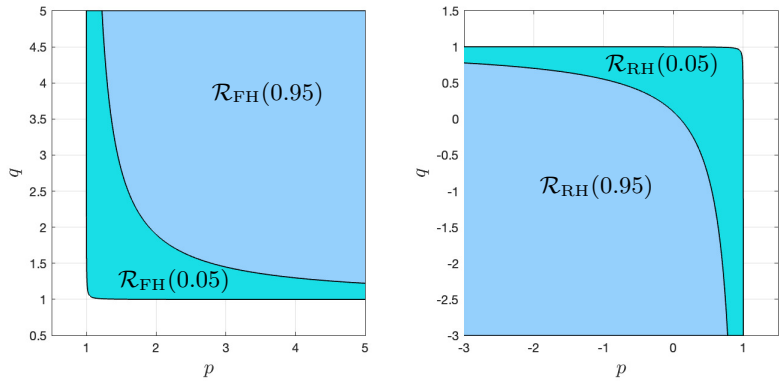


Figure 8.7: Plots of the forward (left) and reverse (right) hypercontractivity regions in (8.51) and (8.52) for $\rho = 0.05$ and 0.95

We commence with some definitions. For $f : \mathcal{X}^n \to [0, \infty)$ and $g : \mathcal{Y}^n \to [0, \infty)$, denote their *inner product*

$$\langle f, g \rangle := \mathbb{E}[f(X^n)g(Y^n)],$$

where the expectation is taken with respect to π_{XY}^n. Define the L^p*-norm* for $p \in [1, \infty)$ and the *pseudo* L^p*-norm* for $p \in (-\infty, 1)\backslash\{0\}$ as

$$\|f\|_p := (\mathbb{E}[f^p(X^n)])^{1/p}.$$

For $p \in \{0, \pm\infty\}$, $\|f\|_p$ is defined by its continuous extensions. Specifically,

$$\begin{aligned} \|f\|_0 &:= \mathrm{e}^{\mathbb{E}[\ln f(X^n)]}, \\ \|f\|_\infty &:= \max_{x^n \in \mathcal{X}^n} f(x^n), \quad \text{and} \\ \|f\|_{-\infty} &:= \min_{x^n \in \mathcal{X}^n} f(x^n), \end{aligned}$$

where $\|f\|_0$ is known as the *geometric mean* of f. Note that $\|f\|_p = 0$ for $p < 0$ if f is not positive π_X-almost everywhere.

For the DSBS $(X, Y) \sim \pi_{XY}$ with correlation coefficient ρ, define

$$\mathcal{R}_{\mathrm{FH}}(\rho) := \{(p, q) \in [1, \infty]^2 : (p-1)(q-1) \geq \rho^2\}, \quad \text{and} \tag{8.51}$$

$$\mathcal{R}_{\mathrm{RH}}(\rho) := \{(p, q) \in [-\infty, 1]^2 : (p-1)(q-1) \geq \rho^2\}. \tag{8.52}$$

These regions are respectively called the *forward* and *reverse hypercontractivity regions* for the DSBS and are illustrated in Fig. 8.7.

Theorem 8.4 (Hypercontractivity: DSBS and Two-Function Version)**.** Let $(X^n, Y^n) \sim \pi_{XY}^n$ be a source sequence generated by a DSBS with correlation coefficient ρ.

1. The inequality
$$\langle f, g \rangle \le \|f\|_p \|g\|_q \tag{8.53}$$
holds for all $f : \{0,1\}^n \to [0,\infty)$ and $g : \{0,1\}^n \to [0,\infty)$, if and only if $(p,q) \in \mathcal{R}_{\mathrm{FH}}(\rho)$.

2. The inequality
$$\langle f, g \rangle \ge \|f\|_p \|g\|_q \tag{8.54}$$
holds for all $f : \{0,1\}^n \to [0,\infty)$ and $g : \{0,1\}^n \to [0,\infty)$, if and only if $(p,q) \in \mathcal{R}_{\mathrm{RH}}(\rho)$.

These two inequalities (due to [27], [69], [131]) are known as the *two-function versions* of the hypercontractivity inequalities for the DSBS. These inequalities are equivalent to the following *single-function versions* of the hypercontractivity inequalities for the DSBS.

Before we describe these single-function versions, we introduce some additional notation. Denote $q' = \frac{q}{q-1}$ as the *Hölder conjugate* of q for $q \neq 1$; for $q = 1$, both $q = \pm\infty$ are Hölder conjugates of q. For a DSBS sequence $(X^n, Y^n) \sim \pi_{XY}^n = \pi_{X|Y}^n \times \pi_Y^n$ with correlation coefficient ρ, the *noise operator* or *conditional expectation operator* T_ρ (or $\pi_{X|Y}^n$) as
$$T_\rho f(y^n) := \mathbb{E}[f(X^n) \mid Y^n = y^n] = \sum_{x^n \in \mathcal{X}^n} f(x^n) \pi_{X|Y}^n(x^n|y^n). \tag{8.55}$$
One can easily check that $T_{\rho_1 \rho_2} = T_{\rho_1} T_{\rho_2}$ for all $\rho_1, \rho_2 \in [0,1]$.

Theorem 8.5 (Hypercontractivity: DSBS and Single-Function Version)**.** Let $(X^n, Y^n) \sim \pi_{XY}^n$ be a source sequence generated by a DSBS with correlation coefficient ρ.

1. The inequality
$$\|T_\rho f\|_q \le \|f\|_p \tag{8.56}$$
holds for all $f : \{0,1\}^n \to [0,\infty)$, if and only if $(p,q') \in \mathcal{R}_{\mathrm{FH}}(\rho)$ (with $1' := \infty$).

2. The inequality

$$\|T_\rho f\|_q \geq \|f\|_p \tag{8.57}$$

holds for all $f : \{0,1\}^n \to [0,\infty)$, if and only if $(p, q') \in \mathcal{R}_{\mathrm{RH}}(\rho)$ (with $1' := -\infty$).

Here we do not delve deeper into the equivalence between the single- and two-function versions of hypercontractivity inequalities, since we will discuss the equivalence in detail in Section 10.2.3.

By applying the hypercontractivity inequalities, Kamath and Anantharam [94, Eqns. (28) and (29)] provided the following bounds.

Theorem 8.6 (Hypercontractivity bound for the DSBS)**.** Define the function

$$\varphi_{a,b}(s,t,p) := \frac{(s^p a + \overline{a})^{\frac{1}{p}} (t^q b + \overline{b})^{\frac{1}{q}} - 1}{(s-1)(t-1)} - \frac{a}{t-1} - \frac{b}{s-1}$$

with $q := 1 + \rho^2/(p-1)$. Then, for any sets $\mathcal{A}$ and $\mathcal{B}$ with $\pi_X^n(\mathcal{A}) = a$ and $\pi_Y^n(\mathcal{B}) = b$,

$$\sup_{s,t>0, p:(s-1)(t-1)(p-1)<0} \varphi_{a,b}(s,t,p) \leq \pi_{XY}^n(\mathcal{A} \times \mathcal{B}) \tag{8.58}$$

$$\leq \inf_{s,t>0, p:(s-1)(t-1)(p-1)>0} \varphi_{a,b}(s,t,p). \tag{8.59}$$

Proof. This theorem follows by setting f and g in Theorem 8.4 to be $\{s,1\}$-valued and $\{t,1\}$-valued functions respectively. Note that changing the range of the functions f and g from $\{0,1\}$ to the sets $\{s,1\}$ and $\{t,1\}$ respectively does not affect the values of the probability masses of the joint distribution of $(f(X^n), g(Y^n))$. □

It can be shown analytically that the hypercontractivity bounds are no worse than the maximal correlation bounds in Theorem 8.3 for any $a, b \in [0,1]$; see Fig. 8.8 for a numerical comparison. Moreover, for $a = b = 1/2$, the hypercontractivity bounds in (8.58) and (8.59) reduce to the sharp bounds $\frac{1-\rho}{4} \leq \pi_{XY}^n(\mathcal{A} \times \mathcal{B}) \leq \frac{1+\rho}{4}$, which correspond to the bounds given by the maximal correlation technique in (8.50).

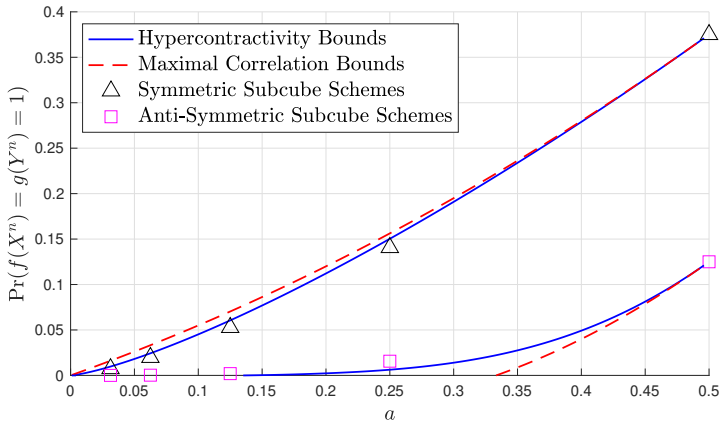


Figure 8.8: Illustration the maximal correlation bounds in (8.45), the hypercontractivity bounds in (8.58)–(8.59) as well as the performances of symmetric and anti-symmetric subcube schemes

8.3.3 Case of $a = b = 1/4$: Boolean Fourier Analysis

We now consider the case $a = b = 1/4$, and we answer the forward part of Mossel's mean-1/4 stability problem. Mossel's mean-1/4 stability problem [119], [120] consists in the determination of $\overline{\Gamma}^{(n)}(1/4, 1/4)$ (forward part) and $\underline{\Gamma}^{(n)}(1/4, 1/4)$ (reverse part) for $n \geq 2$, and also the optimal Boolean functions that attain the maximum and minimum that define these two quantities.

The forward part of this problem was resolved by the present authors in [198], [205] using elements of *Boolean Fourier analysis*. We recap some fundamentals of this study here. Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, its *Fourier coefficients* are defined as

$$\hat{f}_{\mathcal{S}} := \mathbb{E}[f(X^n)\chi_{\mathcal{S}}(X^n)] = \frac{1}{2^n} \sum_{x^n \in \{0,1\}^n} f(x^n)\, \chi_{\mathcal{S}}(x^n) \quad \text{for all } \mathcal{S} \subset [n],$$

where the *(Fourier) basis functions* are

$$\chi_{\mathcal{S}}(x^n) := (-1)^{\sum_{i \in \mathcal{S}} x_i} \quad \text{for all } x^n \in \{0,1\}^n,$$

and $X^n \sim \text{Unif}\{0,1\}^n$. The function f can be expressed in terms of the

Fourier coefficients as

$$f(x^n) = \sum_{\mathcal{S}\subset[n]} \hat{f}_{\mathcal{S}}\ \chi_{\mathcal{S}}(x^n) \quad \text{for all } x^n \in \{0,1\}^n,$$

which is known as the *Fourier expansion* of f. For $0 \le k \le n$, define the *degree-k Fourier weight* of f as

$$\mathbf{W}_k[f] := \sum_{\mathcal{S}\subset[n]:|\mathcal{S}|=k} \hat{f}_{\mathcal{S}}^2. \tag{8.60}$$

It is easy to check that if we define the *degree-k part of f* as $f_k(x^n) := \sum_{\mathcal{S}\subset[n]:|\mathcal{S}|=k} \hat{f}_{\mathcal{S}}\ \chi_{\mathcal{S}}(x^n)$, then $\mathbb{E}[f_k(X^n)^2] = \mathbf{W}_k[f]$. Hence, $\mathbf{W}_k[f]$ represents the "energy" of the degree-k part in f's Fourier expansion.

The Fourier weights satisfy the following properties. Proofs of these properties can be found in the delightful exposition of Boolean functions by O'Donnell [131].

Lemma 8.7. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$ with mean a,

$$\mathbf{W}_0[f] = a^2 \quad \text{and} \quad \sum_{k=0}^{n} \mathbf{W}_k[f] = a. \tag{8.61}$$

Furthermore, if $(X^n, Y^n) \sim \pi_{XY}^n$ is a source sequence of the DSBS with correlation coefficient ρ, then for any pair of Boolean functions $f, g : \{0,1\}^n \to \{0,1\}$,

$$\Pr(f(X^n) = g(Y^n) = 1) = \sum_{k=0}^{n} \rho^k \sum_{\mathcal{S}\subset[n]:|\mathcal{S}|=k} \hat{f}_{\mathcal{S}}\, \hat{g}_{\mathcal{S}} \quad \text{and}$$

$$\Pr(f(X^n) = f(Y^n) = 1) = \sum_{k=0}^{n} \mathbf{W}_k[f]\, \rho^k.$$

For $\rho \in (0,1)$, lower degree Fourier weights have a higher contribution to the joint probability $\Pr(f(X^n) = f(Y^n) = 1)$ than higher degree weights. Hence, to bound this joint probability, we can focus on bounding the lower degree Fourier weights of f. Observe from (8.61) that given the mean of f, the degree-0 Fourier weight is fully specified. Hence, it is instructive to estimate the *second most important* Fourier weight. In particular, we are interested in the *degree-1 Fourier weight* $\mathbf{W}_1[f]$ under the condition that the mean of f is specified. In the

literature, there exist several bounds on $\mathbf{W}_1[f]$. These include Chang's bound, which can be found in [131, Level-1 Inequality] and [35] and the linear programming (LP) bounds of Fu, Wei, and Yeung [59] and Yu and Tan [198]. In particular, the LP bounds state that

$$\mathbf{W}_1[f] \le \varphi(a) := \begin{cases} 2a(\sqrt{a} - a) & 0 \le a \le 1/4 \\ a/2 & 1/4 < a \le 1/2 \end{cases}. \tag{8.62}$$

By the Cauchy–Schwarz inequality, one easily observes that

$$\begin{aligned} &\Pr(f(X^n) = g(Y^n) = 1) \\ &\le \max\left\{\Pr(f(X^n) = f(Y^n) = 1), \Pr(g(X^n) = g(Y^n) = 1)\right\}. \end{aligned} \tag{8.63}$$

This inequality implies that in the determination of $\overline{\Gamma}^{(n)}(a, a)$ (the symmetric case in which $a = b$), it suffices to consider a pair of *identical* Boolean functions.

By combining the ideas in Lemma 8.7, the LP bound in (8.62) and (8.63), the present authors proved the following result [198], [205].

Theorem 8.8. For all $a \in [0, 1]$ and $n \ge 2$,

$$\overline{\Gamma}^{(n)}(a, a) \le a^2 + \rho\varphi(a) + \rho^2(a - a^2 - \varphi(a)).$$

Particularizing this upper bound to $a = b = 1/4$, we obtain $\overline{\Gamma}^{(n)}(1/4, 1/4) \le (\frac{1+\rho}{4})^2$. Per the discussion leading to (8.13), this upper bound is attained by a pair of identical $(n-2)$-subcubes. Hence,

$$\overline{\Gamma}^{(n)}\Big(\frac{1}{4}, \frac{1}{4}\Big) = \Big(\frac{1+\rho}{4}\Big)^2 \quad \text{for all } n \ge 2,$$

resolving the forward part of Mossel's mean-1/4 stability problem. However, the reverse part of the same problem (i.e., which Boolean functions attain $\underline{\Gamma}^{(n)}(1/4, 1/4)$) remains open.

8.4 Converse in the Moderate Deviations Regime

We now consider the optimality of Hamming balls and spheres in the MD regime and the CL regime with small a and b. To address this question, we resort to two key ideas, namely the hypercontractivity inequalities in Theorem 8.4 and the *small set expansion (SSE)* theorem.

A well-known result to address the optimality of Hamming balls and spheres in the MD regime and the CL regime with small a and b is the SSE theorem [124], [131], which is a consequence of the hypercontractivity inequalities in Theorem 8.4.

Theorem 8.9 (Small set expansion: DSBS version)**.** For any $n \geq 1$ and $\alpha, \beta > 0$,

$$\underline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta) \geq \underline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta) \quad \text{and}$$
$$\overline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta) \leq \overline{\Upsilon}_{\mathrm{MD}}(\alpha, \beta),$$

where $\underline{\Upsilon}_{\mathrm{MD}}$ and $\overline{\Upsilon}_{\mathrm{MD}}$ are expressed in closed form for the DSBS in (8.34) and (8.35) respectively.

The reader might wonder about the term "small set expansion" that is used to describe Theorem 8.9. This term refers to a curious phenomenon of the Hamming cube being a "small set expander" in the sense that any small subset $\mathcal{A} \subset \{0,1\}^n$ has an usually large (or expanded) boundary. Here, the Hamming cube is regarded as an edge-weighted complete graph, known as the *ρ-stable hypercube graph*, in which each edge (x^n, y^n) is assigned a weight equal to the probability $\pi^n_{XY}(x^n, y^n)$. The limiting case as $\rho \downarrow 0$ of this phenomenon is quantified by the edge-isoperimetric inequality which will be stated in Theorem 9.5. We refer readers to O'Donnell [131] for more intuition about the term "small set expansion".

Proof Sketch of Theorem 8.9. Substituting the indicator functions $f \leftarrow \mathbb{1}_{\mathcal{A}}$ and $g \leftarrow \mathbb{1}_{\mathcal{B}}$ into (8.53) and (8.54) respectively, and optimizing over (p, q), we obtain the inequalities as stated in the SSE theorem. □

Due to the equivalence among the CL, MD, and LD exponents for all $n \in \mathbb{N}$ (as discussed after Definition 8.2) and the homogeneity property in (8.32) and (8.33), $\underline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta)$ and $\overline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta)$ in Theorem 8.9 can be replaced by $\underline{\Upsilon}^{(n)}_{\mathrm{CL}}(\alpha, \beta)$ and $\overline{\Upsilon}^{(n)}_{\mathrm{CL}}(\alpha, \beta)$ respectively, or by $\underline{\Upsilon}^{(n)}_{\mathrm{LD}}(\alpha, \beta)$ and $\overline{\Upsilon}^{(n)}_{\mathrm{LD}}(\alpha, \beta)$ respectively.

The bounds in the SSE theorem are achieved by sequences of Hamming balls or spherical shells. Hence, these geometric objects are optimal in attaining the MD exponents.

8.5 Converse in the Large Deviations Regime

We now address the final asymptotic regime of interest, namely, the large deviations regime. First, we introduce some terminology. Let $\mathcal{I} \subset \mathbb{R}^d$ be a convex subset of d-dimensional Euclidean space. We recall that for a function $f : \mathcal{I} \to \mathbb{R}$, its *lower convex envelope* $\mathbb{L}[f]$ is the function defined at each point of $\mathcal{I}$ as the supremum of all convex functions that lie under f, i.e., for every $\mathbf{x} \in \mathbb{R}^d$, $\mathbb{L}[f](\mathbf{x}) := \sup\{g(\mathbf{x}) : g \text{ is convex}, g \le f \text{ on } \mathcal{I}\}$. By Carathéodory's theorem, equivalently,

$$\mathbb{L}[f](\mathbf{x}) = \inf_{\{\mathbf{x}_i\}_{i=1}^{d+1} \subset \mathcal{I},\ \{\lambda_i\}_{i=1}^{d+1}} \sum_{i=1}^{d+1} \lambda_i f(\mathbf{x}_i), \tag{8.64}$$

where $\{\lambda_i\}_{i=1}^{d+1}$ is a $(d+1)$-dimensional probability mass function with $\sum_{i=1}^{d+1} \lambda_i \mathbf{x}_i = \mathbf{x}$. The *upper concave envelope* $\mathbb{U}[f]$ is defined as $-\mathbb{L}[-f]$. The SSE theorem in Theorem 8.9 can be strengthened to the following result, known as the *strong SSE theorem*; see Yu, Anantharam, and Chen [195] and Yu [192].

Theorem 8.10 (Strong small set expansion: DSBS version)**.** For any $n \ge 1$ and $\alpha, \beta \in (0, 1]$,

$$\underline{\Upsilon}_{\mathrm{LD}}^{(n)}(\alpha, \beta) \ge \mathbb{L}[\underline{\Upsilon}_{\mathrm{LD}}](\alpha, \beta) \quad \text{and} \tag{8.65}$$

$$\overline{\Upsilon}_{\mathrm{LD}}^{(n)}(\alpha, \beta) \le \mathbb{U}[\overline{\Upsilon}_{\mathrm{LD}}](\alpha, \beta). \tag{8.66}$$

The proof of this theorem (and also its generalization to the finite alphabet case in Theorem 8.11) will be provided in Section 10.3. The proof is based on the information-theoretic characterizations of hypercontractivity inequalities (also discussed in Section 10).

By Carathéodory's representation of the lower convex and upper concave envelopes in (8.64), the bounds in Theorem 8.10 can be asymptotically achieved by "time-sharing" at most three (since $d = 2$ in our case) concentric or anti-concentric Hamming spheres (or balls) for each length n. Specifically, let $(\lambda_1, \lambda_2, \lambda_3)$ be a PMF, i.e., $\lambda_i \ge 0$ for all $i \in [3]$ and $\sum_{i=1}^{3} \lambda_i = 1$. For each blocklength $n \in \mathbb{N}$, this strategy uses certain concentric or anti-concentric Hamming spheres $\mathbb{S}^{(i)}$ for a period of length $\lfloor n\lambda_i \rfloor, i \in [3]$. Since time-sharing of certain Hamming

spheres is optimal in the LD regime, this confirms a weaker version of the OPS conjecture (Conjecture 8.1) in which the convexification and concavification operations in (8.65) and (8.66) respectively are permitted.

Theorem 8.10 is known as the *strong* SSE theorem because the bounds given in Theorem 8.10 are asymptotically sharp in the LD regime. This is in contrast to the ones given in the vanilla SSE theorem (Theorem 8.9) which are not sharp in the LD regime. Furthermore, both these two theorems are asymptotically sharp in the MD regime, since the bounds in the strong SSE theorem reduce to the ones in the SSE theorem, as shown in (8.38) and (8.39). Hence, Theorem 8.10 is stronger than the SSE theorem (Theorem 8.9), in the sense that for all $\alpha, \beta \in [0,1]$ and $\gamma > 0$,

$$\begin{aligned}&\mathbb{L}[\underline{\Upsilon}_{\mathrm{LD}}](\alpha,\beta) \geq \underline{\Upsilon}_{\mathrm{MD}}(\gamma\alpha,\gamma\beta) \quad \text{and}\\ &\mathbb{U}[\overline{\Upsilon}_{\mathrm{LD}}](\alpha,\beta) \leq \overline{\Upsilon}_{\mathrm{MD}}(\gamma\alpha,\gamma\beta).\end{aligned}$$

To prove the OPS conjecture, we need to remove the operations of taking the lower convex and upper concave envelopes in the strong SSE theorem. This was done by the first author of this monograph [193]. In particular, he showed that $\underline{\Upsilon}_{\mathrm{LD}}$ is convex and $\overline{\Upsilon}_{\mathrm{LD}}$ is concave. Combining this result with the strong SSE theorem (Theorem 8.10) allows us to conclude that the OPS conjecture is unconditionally true and that Hamming balls or spheres (without time-sharing) are optimal in the LD regime [193]. That is, for the DSBS and $\alpha, \beta \in (0,1)$,

$$\underline{\Upsilon}^{(\infty)}_{\mathrm{LD}}(\alpha,\beta) = \underline{\Upsilon}_{\mathrm{LD}}(\alpha,\beta) \quad \text{and} \quad \overline{\Upsilon}^{(\infty)}_{\mathrm{LD}}(\alpha,\beta) = \overline{\Upsilon}_{\mathrm{LD}}(\alpha,\beta). \tag{8.67}$$

Several special cases of (8.67) were established in the literature prior to the most general result of Yu [193]. The limiting cases as $\rho \downarrow 0$ and $\rho \uparrow 1$ were shown by Ordentlich, Polyanskiy, and Shayevitz [133]. The "symmetric" special case with $\alpha = \beta$ was shown by Kirshner and Samorodnitsky [97]. We introduce these results in Section 10, since they are consequences of strengthened versions of the hypercontractivity inequalities.

We summarize all converse results discussed in Sections 8.3–8.5 and techniques used to prove them in Table 8.2.

Table 8.2: Converse (optimality) results and techniques for the 2-user NICD problem in the CL, MD, and LD regimes

Regimes	Central Limit		Moderate Deviations	Large Deviations
	Fixed and large a, b	Fixed but small a, b	Subexp. vanishing a, b	Exp. vanishing a, b
Maximal Correlation	Sharp for $a = b = 1/2$	Not sharp	Not sharp	Not sharp
Fourier Analysis	Sharp for $a = b = 1/2$ and $a = b = 1/4$	Not sharp	Not sharp	Not sharp
SSE	Not sharp	Essentially sharp	Sharp	Not sharp
Strong SSE	Not sharp			Sharp

8.6 Extensions to Sources Beyond the DSBS

Thus far, we have only considered the DSBS. Can the results in Sections 8.2-8.5 be extended to other bivariate memoryless sources? Indeed, the SSE and strong SSE theorems, can be extended to sources on Polish spaces (separable completely metrizable topological space). We refer the reader to [192] for details. Here for simplicity, we discuss analogues of the preceding results for the finite alphabet and bivariate Gaussian cases. The NICD problem for the latter case has been completely solved by Borell [28] and Mossel and Neeman [121].

8.6.1 Finite Alphabets

In this section, we generalize the NICD problem to the finite alphabet case in which $\mathcal{X}$ and $\mathcal{Y}$ are finite sets. Let $\pi_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$. For simplicity, we assume that the supports of π_X and π_Y are $\mathcal{X}$ and $\mathcal{Y}$ respectively. Given π_X and π_Y, define their maximum exponents of "atomic events" as

$$\begin{aligned} \alpha_{\max}(\pi_X) &:= \max_{x \in \mathcal{X}} \log \frac{1}{\pi_X(x)} \qquad \text{and} \\ \beta_{\max}(\pi_Y) &:= \max_{y \in \mathcal{Y}} \log \frac{1}{\pi_Y(y)}. \end{aligned} \tag{8.68}$$

For $n \geq 1$, $\alpha \in (0, \alpha_{\max}(\pi_X)]$ and $\beta \in (0, \beta_{\max}(\pi_Y)]$, re-define the *forward* and *reverse LD exponents* respectively as

$$\underline{\Upsilon}_{\mathrm{LD}}^{(n)}(\alpha, \beta) := -\frac{1}{n} \log \max_{\substack{\mathcal{A} \subset \mathcal{X}^n, \mathcal{B} \subset \mathcal{Y}^n: \\ \pi_X^n(\mathcal{A}) \leq 2^{-n\alpha}, \pi_Y^n(\mathcal{B}) \leq 2^{-n\beta}}} \pi_{XY}^n(\mathcal{A} \times \mathcal{B}) \quad \text{and} \tag{8.69}$$

$$\overline{\Upsilon}_{\mathrm{LD}}^{(n)}(\alpha, \beta) := -\frac{1}{n} \log \min_{\substack{\mathcal{A} \subset \mathcal{X}^n, \mathcal{B} \subset \mathcal{Y}^n: \\ \pi_X^n(\mathcal{A}) \geq 2^{-n\alpha}, \pi_Y^n(\mathcal{B}) \geq 2^{-n\beta}}} \pi_{XY}^n(\mathcal{A} \times \mathcal{B}). \tag{8.70}$$

Let $\underline{\Upsilon}_{\mathrm{LD}}^{(\infty)}$ and $\overline{\Upsilon}_{\mathrm{LD}}^{(\infty)}$ be their pointwise limits as $n \to \infty$. These are the same as the forward and reverse LD exponents in (8.9) and (8.10) but here, π_{XY} is no longer restricted to be a DSBS.

Theorem 8.11 (Strong small set expansion: General version)**.** For any joint distribution on a finite alphabet π_{XY}, any blocklength $n \geq 1$, $\alpha \in (0, \alpha_{\max}(\pi_X)]$, and $\beta \in (0, \beta_{\max}(\pi_Y)]$, (8.65) and (8.66) remain true, with $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ defined in (8.24) and (8.25) for π_{XY}, i.e.,

$$\underline{\Upsilon}_{\mathrm{LD}}(\alpha, \beta) = \min_{Q_X, Q_Y : D(Q_X \| \pi_X) \geq \alpha, D(Q_Y \| \pi_Y) \geq \beta} \mathsf{D}(Q_X, Q_Y \| \pi_{XY})$$

and analogously for $\overline{\Upsilon}_{\mathrm{LD}}$. Moreover, the inequalities in (8.65) and (8.66) remain asymptotically tight in the limit as $n \to \infty$.

However, in general, $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ are not necessarily convex and concave, respectively. Hence, unlike the case of the DSBS, for sources on finite alphabets, the operations of taking the lower convex and upper concave envelopes in (8.65) and (8.66) cannot be removed in general. Nevertheless, the bounds $\mathbb{L}[\underline{\Upsilon}_{\mathrm{LD}}](\alpha, \beta)$ and $\mathbb{U}[\overline{\Upsilon}_{\mathrm{LD}}](\alpha, \beta)$ can be asymptotically attained by *time-sharing* the use of at most three type classes (cf. the discussion after Theorem 8.10).

Theorem 8.11 was first proven by Yu, Anantharam, and Chen [195] by using information-theoretic and coupling techniques. In this monograph, we will provide a simple proof of Theorem 8.11, which is based on the information-theoretic characterizations of hypercontractivity inequalities as discussed in Section 10.3.

Similarly, one can generalize the DSBS-specific definitions in (8.11) and (8.12) to an arbitrary distribution π_{XY} on a finite alphabet. Then, the SSE theorem (Theorem 8.9) can be also generalized to the finite alphabet case.

Theorem 8.12 (Small set expansion: General version)**.** For any $n \geq 1$ and $\alpha, \beta > 0$,

$$\underline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta) \geq \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{L}[\underline{\Upsilon}_{\mathrm{LD}}](\epsilon\alpha, \epsilon\beta) \quad \text{and} \tag{8.71}$$

$$\overline{\Upsilon}^{(n)}_{\mathrm{MD}}(\alpha, \beta) \leq \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{U}[\overline{\Upsilon}_{\mathrm{LD}}](\epsilon\alpha, \epsilon\beta). \tag{8.72}$$

Moreover, the inequalities in (8.71) and (8.72) are asymptotically tight in the limit as $n \to \infty$.

Since, in general, $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ are not necessarily convex and concave, respectively, the operations of taking the lower convex and upper concave envelopes in (8.71) and (8.72) cannot be removed as well. As a consequence, for this case, (8.71) and (8.72) cannot be written as in the variational expressions that appear on right-hand sides of (8.29) and (8.30).

8.6.2 Gaussian Sources

We next consider memoryless bivariate Gaussian sources with correlation coefficient $\rho \in (-1, 1) \setminus \{0\}$. For such sources, the NICD problem was completely solved by Borell [28] (for the symmetric cases in which $a = b$) and Mossel and Neeman [121] (for the asymmetric cases) for all $(a, b) \in [0, 1]^2$ and *non-asymptotically*, i.e., for all n. Let π_{XY} be the bivariate Gaussian distribution with mean $(0, 0)$ and covariance matrix $\mathbf{K}$ given in (8.17), where the correlation coefficient $\rho \in (-1, 1) \setminus \{0\}$. As usual, let $(X^n, Y^n) \sim \pi^n_{XY}$.

Theorem 8.13 (Borell's isoperimetric theorem)**.** For any $n \geq 1$ and $a, b \in [0, 1]$,

$$\overline{\Gamma}^{(n)}(a, b) = \Lambda_\rho(a, b) \quad \text{and} \quad \underline{\Gamma}^{(n)}(a, b) = \Lambda_{-\rho}(a, b),$$

where the bivariate normal copula $\Lambda_\rho(\cdot, \cdot)$ is defined in (8.19).

Moreover, it has been shown by Mossel and Neeman [121] that the optimal subsets $(\mathcal{A}, \mathcal{B})$ attaining $\overline{\Gamma}^{(n)}$ or $\underline{\Gamma}^{(n)}$ must be equal to parallel halfspaces (almost everywhere).

Specialized to the case of $a = b = 1/2$, this theorem implies that

$$\overline{\Gamma}^{(n)}\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{1}{2} - \frac{\arccos\rho}{2\pi} \quad \text{and} \quad \underline{\Gamma}^{(n)}\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{\arccos\rho}{2\pi}. \tag{8.73}$$

The optimal $(\mathcal{A},\mathcal{B})$ attaining $\overline{\Gamma}^{(n)}(1/2,1/2)$ correspond to a pair of identical halfspaces through the origin. In contrast, the optimal $(\mathcal{A},\mathcal{B})$ attaining $\underline{\Gamma}^{(n)}(1/2,1/2)$ correspond to a pair of complementary halfspaces through the origin.

Next, we provide a proof sketch of Theorem 8.13 which is due to Mossel and Neeman [121]. In fact, they also proved the following equivalent form of Theorem 8.13.

Theorem 8.14. For any $n \geq 1$, any pair of measurable functions $f, g : \mathbb{R}^n \to [0,1]$, and any $0 < \rho < 1$,

$$\mathbb{E}[\Lambda_\rho(f(X^n), g(Y^n))] \leq \Lambda_\rho(\mathbb{E}[f(X^n)], \mathbb{E}[g(Y^n)]). \tag{8.74}$$

If $-1 < \rho < 0$, the inequality in (8.74) is reversed.

To see that Theorem 8.14 implies Theorem 8.13, set $f = \mathbb{1}_{\mathcal{A}}$ and $g = \mathbb{1}_{\mathcal{B}}$ for two sets $\mathcal{A},\mathcal{B} \subset \mathbb{R}^n$ such that $\mathbb{E}[f(X^n)] = a$ and $\mathbb{E}[g(Y^n)] = b$ in Theorem 8.14. Observe that $\Lambda_\rho(0,0) = \Lambda_\rho(1,0) = \Lambda_\rho(0,1) = 0$, and $\Lambda_\rho(1,1) = 1$. Therefore, $\Lambda_\rho(f(X^n), g(Y^n)) = \mathbb{1}_{\mathcal{A}\times\mathcal{B}}(X^n, Y^n)$, which implies that $\overline{\Gamma}^{(n)}(a,b) \leq \Lambda_\rho(a,b)$. Obviously, by definition, $\overline{\Gamma}^{(n)}(a,b) \geq \Lambda_\rho(a,b)$ follows by setting $\mathcal{A}$ and $\mathcal{B}$ to be two parallel halfspaces. Hence, $\overline{\Gamma}^{(n)}(a,b) = \Lambda_\rho(a,b)$.

We now argue that Theorem 8.13 implies Theorem 8.14. For this purpose, given $f, g : \mathbb{R}^n \to [0,1]$, define $\mathcal{A}$ and $\mathcal{B}$ (subsets of $\mathbb{R}^{n+1}$) to be the respective hypographs[1] of $\Phi^{-1} \circ f : \mathbb{R}^n \to \mathbb{R}$ and $\Phi^{-1} \circ g : \mathbb{R}^n \to \mathbb{R}$, where recall that $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian and $\Phi^{-1} : (0,1) \to \mathbb{R}$ is its inverse. It can be readily checked that

$$\begin{aligned}\mathbb{E}[\Lambda_\rho(f(X^n), g(Y^n))] &= \Pr\left(X_{n+1} \leq \Phi^{-1} \circ f(X^n), Y_{n+1} \leq \Phi^{-1} \circ g(Y^n)\right) \\ &= \pi_{XY}^{n+1}(\mathcal{A}\times\mathcal{B}),\end{aligned}$$

[1]The *hypograph* $\mathrm{hyp}(h)$ of a function $h : \mathcal{X} \to \mathbb{R}$ is the set of points of $\mathcal{X}\times\mathbb{R}$ lying on or below its graph, i.e., $\mathrm{hyp}(h) := \{(x,r) \in \mathcal{X}\times\mathbb{R} : r \leq h(x)\}$.

where $(X^{n+1}, Y^{n+1}) \sim \pi_{XY}^{n+1}$. On the other hand, $\mathbb{E}[f(X^n)] = \pi_X^{n+1}(\mathcal{A})$ and $\mathbb{E}[g(Y^n)] = \pi_Y^{n+1}(\mathcal{B})$, and hence, the right-hand side of (8.74) satisfies

$$\Lambda_\rho(\mathbb{E}[f(X^n)], \mathbb{E}[g(Y^n)]) = \Lambda_\rho(\pi_X^{n+1}(\mathcal{A}), \pi_Y^{n+1}(\mathcal{B})).$$

Thus, Theorem 8.13 in $n+1$ dimensions implies Theorem 8.14 in n dimensions.

Hence, to prove Theorem 8.13, it suffices to prove Theorem 8.14. In their proof of Theorem 8.14, Mossel and Neeman [121] first constructed an Ornstein–Uhlenbeck semigroup, then defined R_t, an auxiliary function for this semigroup that connects the two sides of (8.74) as limiting cases. Lastly, they showed that R_t is monotone. A similar idea was also used in Bakry and Ledoux [10].

Proof Sketch of Theorem 8.14. For every $t \geq 0$, define the operator P_t that acts on functions $f : \mathbb{R}^n \to [0,1]$ as

$$(P_t f)(x^n) := \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t}\, x^n + \sqrt{1 - \mathrm{e}^{-2t}}\, y^n\big)\, \mathrm{d}\pi_Y^n(y^n).$$

This operator is known as the *Ornstein–Uhlenbeck semigroup operator*. Note that $P_t f \to f$ pointwise as $t \to 0$ and $P_t f \to \mathbb{E}[f]$ pointwise as $t \to \infty$.

Let $f_t := P_t f$ and $g_t := P_t g$, and consider the quantity

$$R_t := \mathbb{E}\big[\Lambda_\rho(f_t(X^n), g_t(Y^n))\big]. \tag{8.75}$$

As $t \to 0$, R_t converges to the left-hand side of (8.74); as $t \to \infty$, R_t converges to the right-hand side of (8.74). Hence, to establish Theorem 8.14, it suffices to prove that $\mathrm{d}R_t/\mathrm{d}t \geq 0$ for all $t > 0$. This point can be checked by careful calculations, as shown in the following lemma due to Mossel and Neeman [121].

Lemma 8.15. The function $t \in [0, \infty) \mapsto R_t$, defined in (8.75), satisfies

$$\frac{\mathrm{d}R_t}{\mathrm{d}t} = \frac{\rho}{2\pi\sqrt{1-\rho^2}}\, \mathbb{E}\left[\exp\left(-\frac{v_t^2 + w_t^2 - 2\rho v_t w_t}{2(1-\rho^2)}\right)\right] \|\nabla v_t - \nabla w_t\|^2,$$

where $v_t := \Phi^{-1} \circ f_t : \mathbb{R}^n \to \mathbb{R}$, $w_t := \Phi^{-1} \circ g_t : \mathbb{R}^n \to \mathbb{R}$, and ∇ denotes the gradient operator. Hence, the derivative of R_t for $t \geq 0$ is nonnegative.

This completes the proof sketch of Theorem 8.14. □

9

q-Stability

In Section 8, we discussed the 2-user NICD problem. In this section, we extend the NICD problem to the multi-user case, and consider two versions of these extensions. In the *symmetric* version, we maximize the agreement probability of the random bits generated individually by the users. In the *asymmetric* version, we maximize the joint probability that all the random bits are equal to 1. These two maximization problems are equivalent in the 2-user setting (see discussion following (8.3)), but are *not* equivalent in the setting involving 3 or more users. This distinction results in the upcoming set of problems being significantly more challenging, but they provide more insight into the NICD and related problems.

Indeed, these extensions have inspired researchers to define a more general concept known as the *q-stability*. This is done by generalizing the number of users in the NICD problem from an integer k to an arbitrary real number $q \geq 1$. The *max q-stability* problem concerns the identification of Boolean functions that most "stable"—measured in terms of the q-stability—under the action of a noise operator. Such a problem not only significantly generalizes the 2-user NICD problem to a version parametrized by an arbitrary real number $q \geq 1$, but more

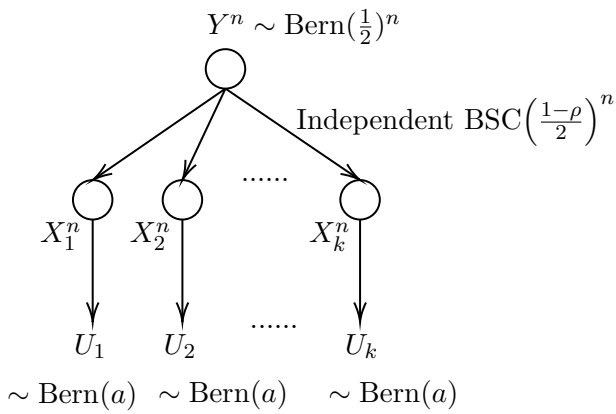
importantly, it seamlessly connects to several interesting contemporary conjectures in information theory and discrete probability, including the Mossel–O'Donnell conjecture [122], the Courtade–Kumar conjecture [40], and the Li–Médard conjecture [110]. Hence, the study of q-stability provides us a comprehensive and unified understanding of these conjectures.

Similar to Section 8, in this section, we focus mainly on the doubly symmetric binary source (DSBS) with correlation coefficient $\rho \in (-1, 1)$. In Section 9.1, we formulate the multi-user NICD problem for the DSBS. We define the asymmetric and symmetric forward joint probabilities, and also generalize them to various max q-stabilities by relaxing the number of users to an arbitrary real number $q \geq 1$. In Section 9.2, we introduce several important conjectures concerning the max q-stability problem for the case in that the Boolean functions in question are *balanced*. These include the Mossel–O'Donnell, Courtade–Kumar, and Li–Médard conjectures. In Section 9.3, we describe resolutions for the conjectures in the extreme cases in which the correlation coefficient $\rho \downarrow 0$ or $\rho \uparrow 1$. Interestingly, in these two extreme cases, the conjectures are characterized by the classic *edge-isoperimetric inequality* and the *maximal degree*-1 *Fourier weight*. Hence, related concepts in discrete geometry, e.g., influences and edge boundaries, will also be introduced. In Section 9.4, we describe recent progress on partial resolutions of these conjectures. In Section 9.5, we introduce the solutions to the max q-stability problem in the moderate and large deviations regimes. Finally, in Section 9.6, we discuss known results on the max q-stability problem for sources beyond the DSBS including bivariate Gaussian sources.

9.1 The Multi-User NICD Problem and q-Stability

9.1.1 Formulation

Before formally introducing the k-user NICD problem, we first introduce a class of Boolean functions, known as *majority functions*. For an odd number $m \in [n]$, let $\mathrm{Maj}_m : \{0,1\}^n \to \{0,1\}$ be the majority function on the first m bits which is given by $\mathrm{Maj}_m(x^n) := \mathbb{1}\{\sum_{i=1}^m x_i \geq m/2\}$ for each $x^n \in \{0,1\}^n$. Then, clearly, Maj_1 is a dictator function, and



$$\text{Asymmetric Version: } \max \Pr(U_1 = U_2 = \ldots = U_k = 1)$$

$$\text{Symmetric Version: } \max \Pr(U_1 = U_2 = \ldots = U_k)$$

Figure 9.1: The Non-Interactive Correlation Distillation problem with k users

Maj_n is the indicator of the Hamming ball $\mathbb{B}_{n/2}(1^n)$ (as introduced in Section 8.2.2). Hence, majority functions are generalizations of dictator functions and indicators of Hamming balls. Furthermore, we say that a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is *anti-symmetric* (or *odd*) if

$$f(x^n) + f(\bar{x}^n) = 1 \quad \text{for all } x^n \in \{0,1\}^n, \tag{9.1}$$

where $\bar{x}^n := 1^n - x^n$ is the bitwise negation of x^n. Equivalently, for an anti-symmetric Boolean function f, $\text{supp}(f)^c = 1^n - \text{supp}(f)$, where $1^n - \mathcal{A} := \{1^n - x^n : x^n \in \mathcal{A}\}$ for any set $\mathcal{A} \subset \{0,1\}^n$. By definition, for any odd $m \in [n]$, the majority function Maj_m is anti-symmetric.

The k-user NICD problem, which is illustrated in Fig. 9.1, was investigated by Mossel and O'Donnell [122] for the symmetric version, and by Li and Médard [110] for the asymmetric version. There are k correlated memoryless sources $X_1, X_2, \ldots, X_k$ generated from a common memoryless Bernoulli source $Y \sim \text{Bern}(\frac{1}{2})$ through k independent binary symmetric channels with crossover probability $p = (1-\rho)/2$; hence, $0 < \rho < 1$ is the correlation coefficient between $X_{j,i}$ and Y_i for all $j \in [k]$ and $i \in [n]$. A Boolean function $f_i : \{0,1\}^n \to \{0,1\}$ is applied to each source sequence[1] X_i^n to generate a random bit $U_i = f_i(X_i^n)$.

[1]Here, we use the notation X_i^n to denote the i^{th} (out of k) length-n correlated

Definition 9.1. For a dyadic rational $a = M/2^n \in [0,1]$ (in which $M \in \{0,1,\ldots,2^n\}$), define the *forward joint probability at mean* a as

$$\Gamma_\rho^{(k)}(a) := \max_{\substack{\text{Boolean } f_i, 1\le i\le k:\\ \Pr(f_i(X_i^n)=1)=a}} \Pr\left(f_1(X_1^n) = \ldots = f_k(X_k^n) = 1\right). \quad (9.2)$$

Since we do not consider the reverse counterpart of the forward joint probability in (9.2) throughout this section, we omit the overline on Γ (cf. the notation $\overline{\Gamma}^{(n)}$ used for the forward joint probability in (8.2)) but we make the number of users k and the correlation coefficient ρ explicit in the notation. To avoid notational overload, we also omit the superscript n that indexes the blocklength. Table 9.1 lists commonly encountered operational quantities in this section.

Table 9.1: Table of commonly used operational quantities in this section

Name	Symbol	Definition(s)
Forward joint probability at a	$\Gamma_\rho^{(k)}(a)$	(9.2), (9.7)
q-stability of f	$\mathbf{S}_\rho^{(q)}[f]$	(9.4)
Asymmetric max q-stability at a	$\Gamma_\rho^{(q)}(a)$	(9.8)
Symmetric max q-stability at a	$\breve{\Gamma}_\rho^{(q)}(a)$	(9.14)
Symmetric q-stability of f	$\breve{\mathbf{S}}_\rho^{(q)}[f]$	(9.15)
Symmetric forward joint probability at a	$\breve{\Gamma}_\rho^{(k)}(a)$	(9.19)
Φ-stability of f	$\mathbf{S}_\rho^{(\Phi)}[f]$	(9.22)
Φ-asymmetric max q-stability at a	$\Pi_\rho^{(q)}(a)$	(9.23)
Φ-symmetric max q-stability at a	$\breve{\Pi}_\rho^{(q)}(a)$	(9.24)
LD exponent	$\Upsilon_{q,\mathrm{LD}}^{(n)}(\alpha)$	(9.51)
MD exponent	$\Upsilon_{q,\mathrm{MD}}^{(n)}(\alpha)$	(9.52)

It clearly holds that every pair (X_j^n, X_ℓ^n) with $j \neq \ell$ is a source sequence generated by a DSBS with correlation coefficient ρ^2 (because $X_j - Y - X_\ell$). This implies that $\Gamma_\rho^{(2)}(a)$ corresponds to the forward joint probability defined in (8.2) for the DSBS with correlation coefficient ρ^2.

Due to the apparent symmetry of the problem, one may naturally wonder whether the k functions $f_1,\ldots,f_k$ that attain the forward joint

source sequences instead of the random vector $(X_i, X_{i+1},\ldots,X_n)$.

probability are necessarily identical. This is positively confirmed in the following proposition which can be proved using either the idea in [122, Proposition 3] or [110]. We provide a self-contained proof.

Proposition 9.1. Let $\mathcal{F}$ be any class of Boolean functions. Let $k, n \geq 1$ and $\rho \in (0,1)$. Every tuple of functions $(f_1, \ldots, f_k) \in \mathcal{F}^k$ that maximizes $\Pr\left(f_1(X_1^n) = \ldots = f_k(X_k^n) = 1\right)$ satisfies $f_1 = f_2 = \ldots = f_k$.

Proof. Since $\mathcal{F}$ is finite, we may enumerate its elements as $\mathcal{F} = \{g_j : j \in [M]\}$ where $M \geq 2$ to avoid the trivial case in which $M = 1$. Suppose that among the k users, g_j is used by kp_j of them. Then clearly, $\{p_j : j \in [M]\}$ forms a distribution on $\mathcal{F}$ or, isomorphically, on $[M]$. On the other hand, the joint probability induced by this scheme is

$$\Pr\left(f_1(X_1^n) = \ldots = f_k(X_k^n) = 1\right) = \mathbb{E}_{Y^n}\left[\prod_{j=1}^{M} (T_\rho g_j(Y^n))^{kp_j}\right], \quad (9.3)$$

where T_ρ is the noise operator defined in (8.55). On the other hand, given $a_1, \ldots, a_M > 0$, the map $(p_1, \ldots, p_M) \in \mathcal{P}([M]) \mapsto \prod_{j=1}^{M} a_j^{p_j}$ is convex. Hence, the expression in (9.3) is *convex* in $(p_1, \ldots, p_M)$. *Maximizing* (9.3) over $(p_1, \ldots, p_M)$ on the probability simplex $\mathcal{P}([M])$, we see that the maximum is attained at a vertex of $\mathcal{P}([M])$. This in turn implies that the maximum of $\Pr\left(f_1(X_1^n) = \ldots = f_k(X_k^n) = 1\right)$ over all $(f_1, \ldots, f_k) \in \mathcal{F}^k$ is attained by some $(f_1, \ldots, f_k)$ such that $f_1 = f_2 = \ldots = f_k$. The necessity of the identity of Boolean functions in attaining this maximum can also be verified; see Mossel and O'Donnell [122]. □

By particularizing $\mathcal{F}$ in Proposition 9.2 to be the set of Boolean functions with mean a, any tuple of k functions $(f_1, \ldots, f_k)$ that attains the forward joint probability necessarily satisfies $f_1 = f_2 = \ldots = f_k$. This observation draws our attention to the following related quantity known as the *q-stability* [52], [110].

Definition 9.2. For any $q \in [1, \infty)$ and a Boolean function $f : \{0,1\}^n \to \{0,1\}$, the *$q$-stability* of f is defined as

$$\mathbf{S}_\rho^{(q)}[f] := \mathbb{E}_{Y^n}\left[(T_\rho f(Y^n))^q\right]. \quad (9.4)$$

For $q = 2$, the q-stability reduces to the *correlation* $\mathbb{E}[f(X^n)f(\hat{X}^n)]$, or equivalently, the *joint probability* $\Pr(f(X^n) = f(\hat{X}^n) = 1)$, where $(X^n, \hat{X}^n)$ is a source sequence of the DSBS with correlation coefficient ρ^2. Hence, $\mathbb{E}[f(X^n)f(\hat{X}^n)]$ is termed the *noise stability* of the Boolean function f with parameter ρ^2, which is denoted as $\mathbf{S}_{\rho^2}[f]$. As mentioned in the discussion following (9.2),

$$\mathbf{S}_{\rho^2}[f] = \mathbf{S}_{\rho}^{(2)}[f]. \tag{9.5}$$

To better understand the concept of the q-stability, we now compute it for two functions.

Example 9.1. For the dictator function Maj_1,

$$\begin{aligned}\mathbf{S}_{\rho}^{(q)}[\mathrm{Maj}_1] &= \mathbf{S}_{\rho}^{(q)}[X_1] \\ &= \mathbb{E}_{Y_1}\left[(\mathbb{E}[X_1|Y_1])^q\right] \\ &= \frac{1}{2}\Big(\frac{1+\rho}{2}\Big)^q + \frac{1}{2}\Big(\frac{1-\rho}{2}\Big)^q.\end{aligned}$$

Example 9.2. For the indicator of the Hamming ball Maj_n, it is not easy to derive the exact value of its q-stability for each dimension $n \in \mathbb{N}$. However, one can determine the limit of the q-stability of Maj_n as $n \to \infty$. By the (multivariate) central limit theorem,

$$\frac{2}{\sqrt{n}}\Big(\sum_{i=1}^{n}\begin{bmatrix}X_i\\Y_i\end{bmatrix} - \frac{n}{2}\begin{bmatrix}1\\1\end{bmatrix}\Big) \xrightarrow{\mathrm{d}} \mathcal{N}\Big(\begin{bmatrix}0\\0\end{bmatrix}, \mathbf{K}\Big),$$

where the covariance matrix $\mathbf{K}$ is defined in (8.17). Define the *Gaussian q-stability function* $\Lambda_{\rho}^{(q)} : [0,1] \to [0,1]$ as

$$\Lambda_{\rho}^{(q)}(a) := \mathbb{E}\big[\Pr(U \le \Phi^{-1}(a)|V)^q\big] = \mathbb{E}\Big[\Phi\Big(\frac{\Phi^{-1}(a) - \rho V}{\sqrt{1-\rho^2}}\Big)^q\Big], \tag{9.6}$$

where (U, V) is a pair of jointly Gaussian random variables with zero mean and covariance matrix $\mathbf{K}$. Therefore, for every $(\rho, q) \in (-1, 1) \times [0, 1]$, the limit of the q-stability of Maj_n is

$$\lim_{n\to\infty} \mathbf{S}_{\rho}^{(q)}[\mathrm{Maj}_n] = \Lambda_{\rho}^{(q)}(1/2).$$

We relate the q-stability to the NICD problem by observing that for an integer k, the forward joint probability in (9.2) can be rewritten as

$$\Gamma_{\rho}^{(k)}(a) = \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} \mathbf{S}_{\rho}^{(k)}[f]. \tag{9.7}$$

Hence, it is natural to term $\Gamma_\rho^{(k)}(a)$ as the *asymmetric max k-stability at mean a*. If we replace the integer k in (9.7) with an arbitrary real number $q \in [1,\infty)$, we can define the asymmetric max q-stability at mean a [52], [110] as follows.

Definition 9.3. For $q \in [1,\infty)$, the *asymmetric max q-stability at mean a* is defined as

$$\Gamma_\rho^{(q)}(a) := \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} \mathbf{S}_\rho^{(q)}[f] \tag{9.8}$$

$$= \max_{\mathcal{A}\subset\{0,1\}^n:\pi_X^n(\mathcal{A})=a} \mathbb{E}_{Y^n}\left[\pi_{X|Y}^n(\mathcal{A}|Y^n)^q\right]. \tag{9.9}$$

The equality in (9.9) follows because for a Boolean function f, $T_\rho f(y^n) = \pi_{X|Y}^n(\mathcal{A}|y^n)$ with $\mathcal{A}$ being the support of f; see (8.55).

A few remarks concerning this definition are in order. First, for fixed $a \in [0,1]$, the function $q \in [1,\infty) \mapsto \Gamma_\rho^{(q)}(a)$ is nonincreasing. Second, given $q \geq 1$ and two correlation coefficients $0 \leq \rho \leq \hat{\rho} \leq 1$, for any $y^n \in \{0,1\}^n$, we have

$$(T_\rho f)^q(y^n) = (T_{\rho/\hat{\rho}} T_{\hat{\rho}} f)^q(y^n) \leq T_{\rho/\hat{\rho}}(T_{\hat{\rho}} f)^q(y^n), \tag{9.10}$$

where the equality follows from the fact that $T_{\rho_1\rho_2} = T_{\rho_1} T_{\rho_2}$ for all $\rho_1, \rho_2 \in [0,1]$, and the inequality follows by Jensen's inequality ($x \mapsto x^q$ is convex for $q \geq 1$). From (9.10), we obtain

$$\mathbf{S}_\rho^{(q)}[f] \leq \mathbb{E}_{Y^n}\left[T_{\rho/\hat{\rho}}(T_{\hat{\rho}} f)^q(Y^n)\right] \tag{9.11}$$

$$= \mathbb{E}_{Z^n}\left[(T_{\hat{\rho}} f)^q(Z^n)\right] \qquad (Z^n \sim \text{Unif}\{0,1\}^n) \tag{9.12}$$

$$= \mathbf{S}_{\hat{\rho}}^{(q)}[f], \tag{9.13}$$

where (9.12) follows because if the input to a binary symmetric channel is uniform, so is its output.[2] Hence, given $q \geq 1$ and $a \in [0,1]$, the function $\rho \in [0,1] \mapsto \Gamma_\rho^{(q)}(a)$ is nondecreasing. Finally, if $\rho = 1$ (i.e., there is no noise), then $\Gamma_1^{(q)}(a) = a$. If instead $\rho = 0$, then $\Gamma_0^{(q)}(a) = a^q$.

[2] The block of inequalities in (9.11)–(9.13) can also be re-interpreted as follows. Given a DSBS (X,Y) with correlation coefficient $\rho \in [0,1]$, we can construct a Markov chain $X - Z - Y$ with correlation coefficient between X and Z being $\hat{\rho} \in [0,\rho]$ such that for any $q \geq 1$, we have $\mathbb{E}[\mathbb{E}[f(X)|Y]^q] \leq \mathbb{E}[\mathbb{E}[\mathbb{E}[f(X)|Z]^q]|Y]] = \mathbb{E}[\mathbb{E}[f(X)|Z]^q]$.

To find the solution to the asymmetric max q-stability problem (i.e., the equivalent optimization problems in Definition 9.3), we have to identify Boolean functions that are the "most stable" under the action of the noise operator T_ρ, with the stability being measured by the q-stability $\mathbf{S}_\rho^{(q)}[f]$.

Analogously to the asymmetric max q-stability at mean a, one can define a *symmetric* version of this stability notion by maximizing the sum of the q-stabilities of f and $1-f$.

Definition 9.4. For $q > 1$, the *symmetric max q-stability at mean a* is[3]

$$\breve{\Gamma}_\rho^{(q)}(a) := \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} \breve{\mathbf{S}}_\rho^{(q)}[f], \tag{9.14}$$

where

$$\breve{\mathbf{S}}_\rho^{(q)}[f] := \mathbf{S}_\rho^{(q)}[f] + \mathbf{S}_\rho^{(q)}[1-f] \tag{9.15}$$

is the *symmetric q-stability of* f.

Let f be an anti-symmetric Boolean function. Consider,

$$\begin{aligned}
\mathbf{S}_\rho^{(q)}[1-f] &= \mathbf{S}_\rho^{(q)}[f(1^n - \cdot)] && (9.16)\\
&= \frac{1}{2^n} \sum_{y^n \in \{0,1\}^n} \left(\mathbb{E}[f(\bar{X}^n)|Y^n = y^n]\right)^q \\
&= \frac{1}{2^n} \sum_{\bar{y}^n \in \{0,1\}^n} \left(\mathbb{E}[f(\bar{X}^n)|\bar{Y}^n = \bar{y}^n]\right)^q \\
&= \mathbf{S}_\rho^{(q)}[f], && (9.17)
\end{aligned}$$

where (9.16) follows from (9.1), and (9.17) follows because $(\bar{X}^n, \bar{Y}^n)$ has the same joint distribution as (X^n, Y^n). Hence, for an anti-symmetric Boolean function f,

$$\breve{\mathbf{S}}_\rho^{(q)}[f] = 2\,\mathbf{S}_\rho^{(q)}[f]. \tag{9.18}$$

Furthermore, similarly to the asymmetric case, the symmetric max q-stability also admits an important operational interpretation in the

[3] We use the breve accent on symbols that signify *symmetric* quantities (e.g., $\breve{\Gamma}_\rho^{(q)}$). The breve serves as a mnemonic as it is symmetric about a vertical axis.

k-user NICD problem; see (9.19). To describe this, we need to first introduce the following proposition, which is the symmetric counterpart of Proposition 9.1 and is due to Mossel and O'Donnell [122]. The proof is almost the same as that of Proposition 9.1 and hence, is omitted.

Proposition 9.2. Let $\mathcal{F}$ be any class of Boolean functions. Let $k, n \geq 1$ and $\rho \in (0,1)$. Every tuple of functions $(f_1, \ldots, f_k) \in \mathcal{F}^k$ that maximizes $\Pr\left(f_1(X_1^n) = \ldots = f_k(X_k^n)\right)$ satisfies $f_1 = f_2 = \ldots = f_k$.

By choosing $\mathcal{F}$ in Proposition 9.2 to be the set of Boolean functions with mean a, we deduce that the symmetric max q-stability with $q = k$ (an integer) satisfies

$$\breve{\Gamma}_{\rho}^{(k)}(a) = \max_{\substack{\text{Boolean } f_i, 1 \leq i \leq k: \\ \Pr(f_i(X_i^n)=1)=a}} \Pr\left(f_1(X_1^n) = \ldots = f_k(X_k^n)\right). \quad (9.19)$$

This is also called the *symmetric forward joint probability* in the k-user NICD problem. Thus, the symmetric max k-stability $\breve{\Gamma}_{\rho}^{(k)}$ quantifies the *maximum agreement probability* over all Boolean functions with a fixed mean in the k-user NICD problem (Fig. 9.1). In contrast, the forward joint probability or asymmetric max k-stability $\Gamma_{\rho}^{(k)}$ (in (9.2) and (9.7)) quantifies the maximum agreement probability *when the generated bits take on the value* 1.

9.1.2 Variants of q-Stabilities

The reader will notice that the definitions of the asymmetric and symmetric max q-stabilities in (9.9) and (9.14) are trivial for the case $q = 1$, since for this case, any Boolean f such that $\mathbb{E}[f(X^n)] = a$ satisfies

$$\mathbf{S}_{\rho}^{(1)}[f] = a \quad \text{and} \quad \breve{\mathbf{S}}_{\rho}^{(1)}[f] = 1. \quad (9.20)$$

Hence, the asymmetric and symmetric max 1-stabilities at mean a are attained by *any* Boolean functions with mean a. Are there any "more meaningful" notions of asymmetric and symmetric max q-stabilities for $q = 1$? We answer this question in the affirmative by defining variants of the max q-stabilities. These variants connect the q-stabilities to the *most informative Boolean functions* problem of Courtade and Kumar

[40], one of the most important open problems in information theory at the time of the writing of this monograph.

To introduce these variants, for $q \geq 1$, define[4] $\Phi_q, \breve{\Phi}_q : (0,1) \to \mathbb{R}$ as

$$\Phi_q(t) := t \cdot \frac{\ln_q(t)}{\ln 2} \quad \text{and} \quad \breve{\Phi}_q(t) := \Phi_q(t) + \Phi_q(1-t), \tag{9.21}$$

where $\ln_q : (0,\infty) \to \mathbb{R}$ is defined as

$$\ln_q(t) := \begin{cases} \ln(t) & q = 1 \\ \dfrac{t^{q-1}-1}{q-1} & q > 1 \end{cases}$$

and is known as the *q-logarithm* introduced by Tsallis [164], but with a slight reparameterization. Note that for Φ_q and $\breve{\Phi}_q$, the case of $q = 1$ is the continuous extension of the case $q > 1$.

Definition 9.5. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and another function $\Phi : (0,1) \to \mathbb{R}$, define the *$\Phi$-stability of f* with respect to a correlation parameter ρ as

$$\mathbf{S}_\rho^{(\Phi)}[f] = \mathbb{E}_{Y^n}\left[\Phi(T_\rho f(Y^n))\right]. \tag{9.22}$$

Thus, this definition is analogous to that of the q-stability (Definition 9.2) as we recover the latter when we instantiate $\Phi(t) = t^q$. We are, however, going to consider Φ to be the functions in (9.21).

Definition 9.6. Define the *Φ-asymmetric* and *Φ-symmetric max q-stabilities at mean a* as

$$\Pi_\rho^{(q)}(a) := \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} \mathbf{S}_\rho^{(\Phi_q)}[f] \quad \text{and} \tag{9.23}$$

$$\breve{\Pi}_\rho^{(q)}(a) := \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} \mathbf{S}_\rho^{(\breve{\Phi}_q)}[f]. \tag{9.24}$$

For $q > 1$, it is easy to verify that

$$\Pi_\rho^{(q)}(a) = \frac{\Gamma_\rho^{(q)}(a) - a}{(q-1)\ln 2} \quad \text{and} \quad \breve{\Pi}_\rho^{(q)}(a) = \frac{\breve{\Gamma}_\rho^{(q)}(a) - 1}{(q-1)\ln 2}, \tag{9.25}$$

[4] These functions are not to be confused with the Gaussian cumulative distribution function which is also denoted as $\Phi(\cdot)$.

where $\Gamma_\rho^{(q)}(a)$ and $\breve{\Gamma}_\rho^{(q)}$ are the asymmetric and symmetric max q-stabilities defined in (9.9) and (9.14) respectively. For $q = 1$, the Φ-asymmetric and Φ-symmetric max 1-stabilities at mean a can be expressed respectively as

$$\Pi_\rho^{(1)}(a) = \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} \mathbb{E}_{Y^n}\left[T_\rho f(Y^n) \log T_\rho f(Y^n)\right], \tag{9.26}$$

and

$$\breve{\Pi}_\rho^{(1)}(a) = \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} -H(f(X^n)|Y^n) \tag{9.27}$$

$$= \max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=a} I(f(X^n); Y^n) - h(a). \tag{9.28}$$

The objective function in (9.26) is known as the *entropy (functional)* of the noisy Boolean function $T_\rho f$, and the objective function in (9.27) is the negative *conditional Shannon entropy* of $f(X^n)$ given Y^n. For dictator functions f, the conditional Shannon entropy $H(f(X^n)|Y^n)$ is equal to $H(X_1|Y_1) = h((1-\rho)/2)$.

The maximization in (9.28) for $a = 1/2$ corresponds to the balanced version of the *most informative Boolean function* problem which was first studied in the papers of Courtade and Kumar [40], [102]. They conjectured that the maximum in (9.28) is attained by dictator functions (cf. Section 8.2.1). In the following section, we provide more details on this conjecture, and we also review several related conjectures concerning the q-stabilities. Observe from the one-to-one relationships in (9.25) that for $q > 1$, the original definitions of the asymmetric and symmetric max q-stabilities $\Gamma_\rho^{(q)}$ and $\breve{\Gamma}_\rho^{(q)}$ in (9.9) and (9.14) are "equivalent" to their Φ-versions $\Pi_\rho^{(q)}$ and $\breve{\Pi}_\rho^{(q)}$ defined respectively in (9.23) and (9.24), in the sense that once the former (resp. the latter) has been determined, the latter (resp. the former) will also be determined. Hence, throughout this section, for $q > 1$, we refer to $\Gamma_\rho^{(q)}$ and $\Pi_\rho^{(q)}$ interchangeably for the asymmetric case. We will also refer to $\breve{\Gamma}_\rho^{(q)}$ and $\breve{\Pi}_\rho^{(q)}$ interchangeably for the symmetric case. However, for $q = 1$, we only consider the quantities $\Pi_\rho^{(1)}$ and $\breve{\Pi}_\rho^{(1)}$ since the definitions of $\Gamma_\rho^{(1)}$ and $\breve{\Gamma}_\rho^{(1)}$ are trivial for this case; see (9.20).

9.2 Related Conjectures

In this section, we introduce several prominent conjectures on the max q-stabilities. We first consider the optimality of dictator functions in attaining the asymmetric and symmetric max q-stabilities at mean $a = 1/2$ (also called the *balanced* case). For ease of reference, we first state a corollary to Witsenhausen's classical result [178] in Theorem 8.3. This corollary implies that dictator functions are optimal in attaining asymmetric or symmetric max q-stabilities for $q = 2$ and $a = 1/2$ (the 2-user NICD problem in the CL regime with $a = b = 1/2$).

Corollary 9.1. For $q = 2$ and $\rho \in (0, 1)$, both $\Gamma_\rho^{(q)}(1/2)$ and $\breve{\Gamma}_\rho^{(q)}(1/2)$ are attained by dictator functions.

There was no further progress on the max q-stability problem for almost 30 years since Witsenhausen's seminal work [178] in 1975. In 2005, Mossel and O'Donnell [122] considered the symmetric max q-stability problem with $q \in \{3, 4, 5, \ldots\}$ and made progress on this problem. They resolved the case of $q = 3$ for the balanced case (i.e., $a = 1/2$) using a cute reduction argument.

Theorem 9.2. For $q = 3$ and $\rho \in (0, 1)$, $\breve{\Gamma}_\rho^{(q)}(1/2)$ is attained by dictator functions.

Proof. Theorem 9.2 can be proved by reducing the problem involving $q = 3$ to the (simpler) problem in which $q = 2$. By the equivalence between the NICD problem and max q-stability, we consider the 3-user NICD problem with 3 (possibly) distinct functions (f_1, f_2, f_3). For brevity, denote the values of the joint probability mass function of $(U_1, U_2, U_3) = (f_1(X_1^n), f_2(X_2^n), f_3(X_3^n))$ as $\{p_{000}, p_{001}, \ldots, p_{111}\}$. Then, we see that the following identity holds:

$$3 + \sum_{(i,j)\in[3]^2 : i\neq j} \Pr(U_i = U_j) = 5 + 4 \Pr(U_1 = U_2 = U_3). \quad (9.29)$$

This identity can be verified by bookkeeping the probability masses. For example, note that $\Pr(U_1 = U_2) = p_{000} + p_{001} + p_{110} + p_{111}$ and $\Pr(U_1 = U_2 = U_3) = p_{000} + p_{111}$. Having established this, leveraging the case for $q = 2$ (Corollary 9.1), we know that the left-hand side of (9.29)

is maximized by identical dictator functions over all balanced Boolean functions (i.e., $\mathbb{E}[f_i(X_i^n)] = 1/2$); hence, so is the right-hand side. □

Based on Corollary 9.1 and Theorem 9.2, one may naïvely conjecture that dictator functions are optimal in attaining the asymmetric or symmetric max q-stability at mean $1/2$ for any integer $q \geq 2$. However, this was disproved by Mossel and O'Donnell [122]. Specifically, using computer-assisted calculations, they constructed counterexamples, as shown in the following proposition, such that when $q = 10$, dictator functions are not optimal in attaining the asymmetric and symmetric max q-stabilities at mean $a = 1/2$.

Proposition 9.3. For $q = 10$ and $\rho = 0.48$, it holds that

$$\mathbf{S}_\rho^{(q)}[\mathrm{Maj}_3] > \max\{\mathbf{S}_\rho^{(q)}[\mathrm{Maj}_1], \mathbf{S}_\rho^{(q)}[\mathrm{Maj}_5]\} \quad \text{and} \tag{9.30}$$

$$\breve{\mathbf{S}}_\rho^{(q)}[\mathrm{Maj}_3] > \max\{\breve{\mathbf{S}}_\rho^{(q)}[\mathrm{Maj}_1], \breve{\mathbf{S}}_\rho^{(q)}[\mathrm{Maj}_5]\}. \tag{9.31}$$

Proof. By computer-assisted calculations, for $q = 10$ and $\rho = 0.48$, one finds that $\breve{\mathbf{S}}_\rho^{(q)}[\mathrm{Maj}_1] \leq 0.0493$, $\breve{\mathbf{S}}_\rho^{(q)}[\mathrm{Maj}_5] \leq 0.0488$, and $\breve{\mathbf{S}}_\rho^{(q)}[\mathrm{Maj}_3] \geq 0.0496$. Hence, the inequality in (9.31) holds. The inequality in (9.30) follows from (9.31) since $\breve{\mathbf{S}}_\rho^{(q)}[\mathrm{Maj}_m] = 2\,\mathbf{S}_\rho^{(q)}[\mathrm{Maj}_m]$ for any odd $m \leq n$; see (9.18). □

Since $\mathbf{S}_\rho^{(q)}[\mathrm{Maj}_3] > \mathbf{S}_\rho^{(q)}[\mathrm{Maj}_1]$ and Maj_1 is a dictator function, dictators are not optimal for $q = 10$ and $\rho = 0.48$. Furthermore, since the indicators of subcubes and the indicators of Hamming balls (or spheres) have been shown to be optimal or asymptotically optimal in several cases for the NICD problem (Sections 8.3–8.5), one may wonder whether the max q-stability is always exactly attained by these functions. The inequality $\mathbf{S}_\rho^{(q)}[\mathrm{Maj}_3] > \mathbf{S}_\rho^{(q)}[\mathrm{Maj}_5]$ implies a negative answer to this question. For $n = 5$, $q = 10$, $a = 1/2$, and $\rho = 0.48$, both the indicators of subcubes and Hamming balls in the 5-dimensional Hamming cube are not optimal. In fact, Maj_3 corresponds to the indicator of a set formed by multiplying Hamming balls in the 3-dimensional cube and the 2-dimensional cube.

Now things have become relatively clearer. For small q, e.g., $q = 2$ or $q = 3$, dictator functions are optimal in attaining the asymmetric

(for $q = 2$) or symmetric (for $q = 2, 3$) max q-stabilities at mean $a = 1/2$. On the other hand, for large q, e.g., $q = 10$, dictator functions are not optimal. Mossel and O'Donnell [122] conjectured that for all $q \in \{4, 5, \ldots, 9\}$, dictator functions maximize the symmetric q-stability $\breve{\Gamma}_{\rho}^{(q)}(1/2)$ over all balanced Boolean functions.

The symmetric max 1-stability problem at mean $a = 1/2$ (i.e., $\breve{\Pi}_{\rho}^{(1)}(1/2)$) was studied by Kumar and Courtade [102] and [40]. This question concerns the identification of the class of balanced Boolean functions that maximize the mutual information $I(f(Y^n); X^n)$; cf. (9.28). The authors conjectured that dictator functions maximize the symmetric 1-stability. We note that this is a weaker version of the original conjecture posed by Courtade and Kumar. In the original version of their conjecture, the Boolean functions are not restricted to be balanced. Along these lines, Li and Médard [110] conjectured that for $q \in (1, 2)$ (non-integer), the max asymmetric q-stability is still attained by dictator functions. Here we summarize and generalize this family of conjectures in the following two conjectures.

Conjecture 9.1 (Asymmetric max q-stability)**.** For $\rho \in [0, 1]$ and $q \in [1, 9]$, $\Pi_{\rho}^{(q)}(1/2)$ is attained by dictator functions.

Conjecture 9.2 (Symmetric max q-stability)**.** For $\rho \in [0, 1]$ and $q \in [1, 9]$, $\breve{\Pi}_{\rho}^{(q)}(1/2)$ is attained by dictator functions.

Observe that dictator functions are anti-symmetric. Hence, (9.18) holds for dictator functions, which implies that if Conjecture 9.1 is true, so is Conjecture 9.2. Conjectures 9.1 and 9.2 together consist of three (named) conjectures, as summarized in Table 9.2.

Barnes and Özgür [11] proved an interesting dichotomy concerning these conjectures.

Lemma 9.3. For $a = 1/2$, there are two thresholds $q_{\min}$ and $q_{\max}$ satisfying $1 \leq q_{\min} \leq 2 \leq q_{\max}$ such that dictator functions are optimal in attaining the asymmetric max q-stability with $q \geq 1$ if and only if $q \in [q_{\min}, q_{\max}]$. This statement also holds for the symmetric max q-stability but with possibly different thresholds $\breve{q}_{\min}$ and $\breve{q}_{\max}$ satisfying the same condition $1 \leq \breve{q}_{\min} \leq 2 \leq \breve{q}_{\max}$.

Table 9.2: Illustration of the various named conjectures on max q-stabilities; these constitute Conjectures 9.1 and 9.2

q	Are dictators optimal in attaining $\Pi_\rho^{(q)}(1/2)$ (or $\breve{\Pi}_\rho^{(q)}(1/2)$)?
$q = 1$	Courtade–Kumar conjecture (balanced version) [40]
$1 < q < 2$	Li–Médard conjecture [110]
$q = 2$ (2-User NICD)	True and shown by Witsenhausen [178] (cf. Section 8.3.1)
$2 < q \leq 9$	Mossel–O'Donnell conjecture [122]

Proof Sketch of Lemma 9.3. For any $q \in \mathbb{R}$ (not necessarily greater than or equal to 1), define

$$N_q(f) := 2^n \left\| T_\rho f \right\|_q^q = \sum_{y^n \in \{0,1\}^n} \left(T_\rho f(y^n) \right)^q.$$

Let f_0 be a dictator function, e.g., $f_0 = \mathrm{Maj}_1$. Define

$$g_f(q) := N_q(f) - N_q(f_0). \tag{9.32}$$

By using a result due to Laguerre [104], one can find that the sum of exponentials $g_f(q)$ has at most four roots. Observe that

$$g_f(0) = 0, \quad g_f(1) = 0, \quad g_f(2) \leq 0, \quad \text{and}$$
$$g_f(q),\ g_f(-q) > 0 \quad \text{for sufficiently large } q.$$

From these observations, we know that $g_f(q)$ has a root at $q_1 \geq 2$, another at $q_2 = 1$, and another at $q_3 = 0$. Moreover, the remaining root q_4 satisfies $q_4 \leq 2$. Hence, $g_f(q) \leq 0$ for all q in the interval $[\max\{q_4, 1\}, q_1]$; see Fig. 9.2. Taking the intersection of these intervals for all non-dictator functions f, we obtain the interval $[q_{\min}, q_{\max}]$, where $q_{\min}$ and $q_{\max}$ are the desired thresholds. The symmetric case follows similarly. □

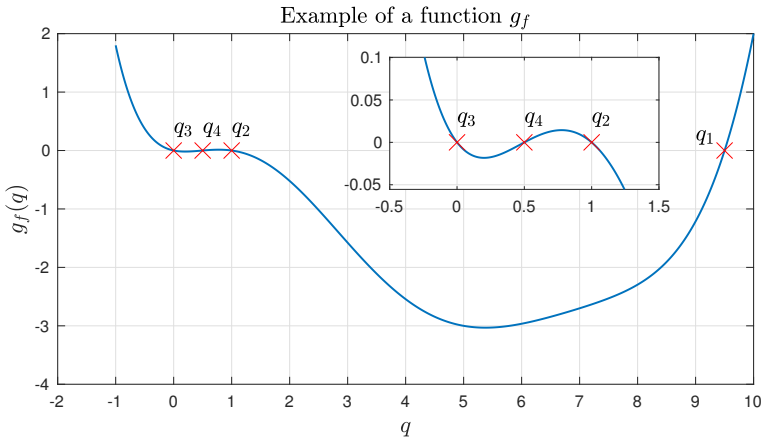


Figure 9.2: Example of a function g_f in (9.32).

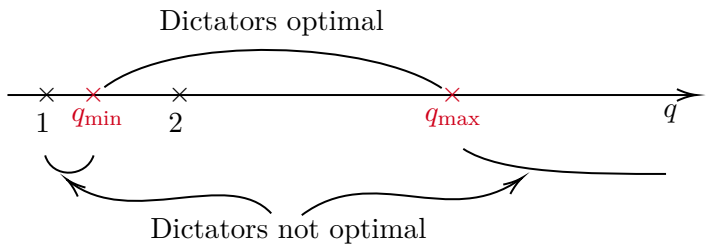


Figure 9.3: Illustration of Lemma 9.3.

This lemma is illustrated in Fig. 9.3.

Remark 9.1. This lemma has several important implications.

(a) Firstly, this lemma implies that both the Courtade–Kumar conjecture and the Li–Médard conjecture are equivalent to the statements that $q_{\min} = 1$ for the asymmetric version and $\breve{q}_{\min} = 1$ for the symmetric version. Hence, the Courtade–Kumar conjecture and the Li–Médard conjecture are also equivalent (to each other).

(b) Secondly, it also implies that the Mossel–O'Donnell conjecture is equivalent to the statements that $q_{\max} \geq 9$ for the asymmetric version and $\breve{q}_{\max} \geq 9$ for the symmetric version. On the other hand, from Proposition 9.3, we see that $\max\{q_{\max}, \breve{q}_{\max}\} < 10$.

(c) Lastly, by Lemma 9.3 and Theorem 9.2 (i.e., Conjecture 9.2 holds for $q = 3$), Conjecture 9.2 also holds for any $q \in [2, 3]$. In other words, Conjecture 9.2 is only open for $q \in [1, 2) \cup (3, 9]$.

Combining all points in Remark 9.1 yields that $2 \leq q_{\max} < 10$ and $3 \leq \breve{q}_{\max} < 10$. If Conjectures 9.1 and 9.2 hold, then the estimates of $q_{\max}$ and $\breve{q}_{\max}$ can be improved to $9 \leq q_{\max}, \breve{q}_{\max} < 10$, as shown in Fig. 9.4.

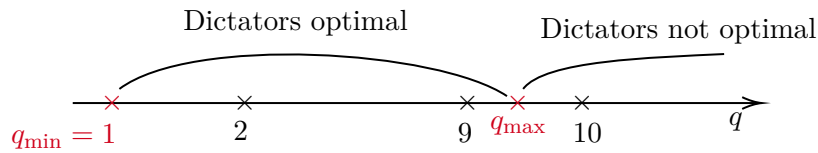


Figure 9.4: Illustration of the range of q for the optimality of dictator functions if Conjectures 9.1 and 9.2 are true.

9.3 Extreme Cases of the Correlation Coefficient

To better understand the max q-stabilities, and also to connect them to several well-known concepts in the analysis of Boolean functions, we first focus our attention on the extreme cases in which the correlation coefficient ρ tends to 0 or 1, but the dimension (or blocklength) n is kept fixed. To illustrate the intuition as to why some results hold, we introduce the concepts of *influences* and *edge-isoperimetric inequalities*.

9.3.1 Influences

For a vector $x^n \in \{0,1\}^n$, we denote the vector with the i^{th} bit flipped as $(x^n)^{\oplus i} := (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n)$. Denote the length-$(n-1)$ with the i^{th} component removed as $x^{\setminus i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$.

Definition 9.7. The *influence of coordinate* $i \in [n]$ on a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is defined as

$$\mathbf{I}_i[f] := \Pr\left(f(X^n) \neq f((X^n)^{\oplus i})\right),$$

where $X^n \sim \text{Unif}\{0,1\}^n$.

Let f be a Boolean function that depends only on $x^{\setminus i}$. This means that the value of f evaluated at every x^n is independent of the i^{th} component x_i. For such an f, clearly, $\mathbf{I}_i[f] = 0$. On the other hand, if f depends only on the i^{th} component, i.e., the dictator functions $f(x^n) = x_i$ or $1 - x_i$, then $\mathbf{I}_i[f] = 1$. Hence, the influence of coordinate i measures how much a function is influenced by the i^{th} coordinate of the input; this coincides with the literal meaning of "influence". Furthermore, the influence can be also expressed in terms of the discrete derivative operator as follows:

Definition 9.8. Let $x^{i\mapsto b} := (x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$. The i^{th} *discrete derivative operator* D_i maps a function $f : \{0,1\}^n \to \mathbb{R}$ to the function $D_i f : \{0,1\}^n \to \mathbb{R}$ defined as

$$D_i f(x^n) := f(x^{i\mapsto 1}) - f(x^{i\mapsto 0}).$$

One observes that

$$\mathbf{I}_i[f] = \mathbb{E}[D_i f(X^n)^2] = \|D_i f\|_2^2. \tag{9.33}$$

This formula enables us to generalize the definition of the influence from a Boolean function to an arbitrary real-valued function defined on $\{0,1\}^n$; see O'Donnell [131]. We do not discuss this generalization here, since we only mainly focus on Boolean functions.

Definition 9.9. The *total influence* (or *average sensitivity*) of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is defined as

$$\mathbf{I}[f] := \sum_{i=1}^{n} \mathbf{I}_i[f].$$

The quantities $D_i f$, $\mathbf{I}_i[f]$, and $\mathbf{I}[f]$ admit the following Fourier-analytic representations.

Theorem 9.4. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and $i \in [n]$,

$$D_i f(x^n) = -2 \sum_{\mathcal{S}\subset[n]:\mathcal{S}\ni i} \hat{f}_{\mathcal{S}} \cdot \chi_{\mathcal{S}\setminus\{i\}}(x^n), \tag{9.34}$$

$$\mathbf{I}_i[f] = 4 \sum_{\mathcal{S}\subset[n]:\mathcal{S}\ni i} \hat{f}_{\mathcal{S}}^2, \qquad \text{and} \tag{9.35}$$

$$\mathbf{I}[f] = 4 \sum_{\mathcal{S}\subset[n]} |\mathcal{S}| \hat{f}_{\mathcal{S}}^2 = 4 \sum_{k=0}^{n} k \cdot \mathbf{W}_k[f]. \tag{9.36}$$

where $\mathbf{W}_k[f]$ denotes the degree-k Fourier weight of f defined in (8.60).

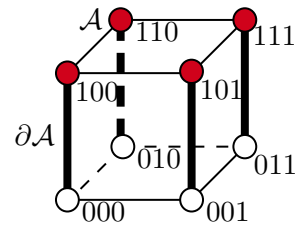


Figure 9.5: Hamming graph for $n = 3$. For the dictator function $\mathrm{Maj}_1(x^3) = x_1$, the set $\mathcal{A} = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$ (indicated in red balls). The edge boundary $\partial\mathcal{A}$ of $\mathcal{A}$ is indicated as the four thick edges. Each boundary edge is a dimension-1 edge.

Proof. Since D_i is a linear operator, (9.34) follows by expressing f in terms of its Fourier coefficients, and then applying the following identity

$$D_i \chi_{\mathcal{S}}(x^n) = \begin{cases} -2\,\chi_{\mathcal{S}\setminus\{i\}}(x^n) & i \in \mathcal{S} \\ 0 & i \notin \mathcal{S} \end{cases}.$$

The identity in (9.35) follows from (9.33) and (9.34), and the fact that for any sets $\mathcal{S}, \mathcal{T} \subset [n]$,

$$\mathbb{E}[\chi_{\mathcal{S}}(X^n)\chi_{\mathcal{T}}(X^n)] = \mathbb{1}\{\mathcal{S} = \mathcal{T}\}. \tag{9.37}$$

The identity in (9.36) follows from (9.35) and Definition 9.9. □

The quantities $\mathbf{I}_i[f]$ and $\mathbf{I}[f]$ also admit interesting graph-theoretic interpretations. Consider the undirected graph in which the vertices consist of all vectors in $\{0,1\}^n$, and two vertices $x^n, y^n \in \{0,1\}^n$ are joined by an edge if the Hamming distance between them is exactly 1, i.e., $d_{\mathrm{H}}(x^n, y^n) = 1$. This graph is known as the *Hamming graph*; see Fig. 9.5 for the Hamming graph when $n = 3$.

Definition 9.10. For a set $\mathcal{A} \subset \{0,1\}^n$, define its *edge boundary* $\partial\mathcal{A}$ as the set of edges in the Hamming graph such that one of its endpoints belongs to $\mathcal{A}$ while the other one belongs to $\mathcal{A}^{\mathrm{c}}$. Every edge that belongs to $\partial\mathcal{A}$ is called a *boundary edge*. An boundary edge $\{x^n, y^n\} \in \partial\mathcal{A}$ is known as a *dimension-i edge* if $y^n = (x^n)^{\oplus i}$, i.e., x^n and y^n are identical except in their i^{th} coordinates.

For a set $\mathcal{A} \subset \{0,1\}^n$, one observes the following facts.

1. The fraction of dimension-i edges that are boundary edges of $\mathcal{A}$ in the Hamming graph is equal to $\mathbf{I}_i[\mathbb{1}_\mathcal{A}]$.

2. The fraction of edges in the Hamming graph that are boundary edges of $\mathcal{A}$ is equal to $\frac{1}{n}\mathbf{I}[\mathbb{1}_\mathcal{A}]$. This implies that $|\partial\mathcal{A}| = 2^{n-1}\mathbf{I}[\mathbb{1}_\mathcal{A}]$, since the total number of edges in the Hamming graph is $n\,2^{n-1}$.

Example 9.3. Let $n = 3$. The Hamming graph is shown in Fig. 9.5. This graph has $3 \cdot 2^{3-1} = 12$ edges. Consider the dictator function $\mathrm{Maj}_1(x^3) = x_1$. The support of f is the set $\mathcal{A} = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$. Both $\mathcal{A}$ and $\partial\mathcal{A}$ are indicated in Fig. 9.5 and $|\partial\mathcal{A}| = 4$. For this dictator function, $\mathbf{I}_1[\mathbb{1}_\mathcal{A}] = 1$ and $\mathbf{I}_2[\mathbb{1}_\mathcal{A}] = \mathbf{I}_3[\mathbb{1}_\mathcal{A}] = 0$ (as discussed after Definition 9.7). Note from Fig. 9.5 that there are four dimension-1 edges and no dimension-2 and dimension-3 edges. Thus, the *fractions* of dimension-1, dimension-2 and dimension-3 edges that are boundary edges of $\mathcal{A}$ are 1, 0, and 0 respectively, corroborating Fact 1. Furthermore, $\mathbf{I}[\mathbb{1}_\mathcal{A}] = \sum_{i=1}^3 \mathbf{I}_i[\mathbb{1}_\mathcal{A}] = 1$ and the fraction of edges that belong to $\partial\mathcal{A}$ is $1/3 = 4/12$, corroborating Fact 2.

9.3.2 Edge-Isoperimetric Inequalities

From Fact 2, we see that the total influence of f is related to the cardinality of the edge boundary of its support set $\mathcal{A}$. A classical result due to Harper [74] quantifies this relation via the so-called *edge-isoperimetric inequality*.

Theorem 9.5 (Edge-isoperimetric inequality)**.** For $f : \{0,1\}^n \to \{0,1\}$ with $a = \min\{\mathbb{E}[f], 1 - \mathbb{E}[f]\}$,

$$\mathbf{I}[f] \geq 2a \log\left(\frac{1}{a}\right). \tag{9.38}$$

This inequality can be seen as a Boolean function version of the *log-Sobolev inequality*. The relationship between this edge-isoperimetric inequality and the real-valued function version of log-Sobolev inequalities will be discussed extensively in Section 10.4.

This inequality in (9.38) is sharp for $a = 2^{-k}$ with $1 \leq k \leq n$, since for this case, the indicator function of an $(n-k)$-subcube attains

the lower bound. If a, a dyadic rational, is the mean of f, $\mathbf{I}[f]$ is minimized when f is the indicator of a lexicographic set of size $2^n a$ (cf. Section 8.2.1). The edge-isoperimetric inequality will be used to resolve the extreme cases of the max q-stability problem via the following two theorems that establish a connection between the q-stability and the total influence.

Theorem 9.6. For $f : \{0,1\}^n \to \{0,1\}$,

$$\mathbf{S}_\rho^{(2)}[f] = \sum_{\mathcal{S}\subset[n]} \rho^{2|\mathcal{S}|}\ \hat{f}_{\mathcal{S}}^2 = \sum_{k=0}^{n} \rho^{2k}\ \mathbf{W}_k[f].$$

Proof. This theorem follows by (9.37) and the facts that

$$\mathbf{S}_\rho^{(2)}[f] = \langle T_\rho f, T_\rho f\rangle \qquad \text{and} \qquad (\widehat{T_\rho f})_{\mathcal{S}} = \sum_{\mathcal{S}\subset[n]} \rho^{|\mathcal{S}|}\ \hat{f}_{\mathcal{S}}, \tag{9.39}$$

where $\{(\widehat{T_\rho f})_{\mathcal{S}}\}_{\mathcal{S}\subset[n]}$ are the Fourier coefficients of $T_\rho f$. □

This theorem implies that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\mathbf{S}_\rho^{(2)}[f]\Big|_{\rho=1} &= \frac{\mathbf{I}[f]}{2},\\
\frac{\mathrm{d}}{\mathrm{d}\rho}\mathbf{S}_\rho^{(2)}[f]\Big|_{\rho=0} &= 0, \qquad \text{and}\\
\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\mathbf{S}_\rho^{(2)}[f]\Big|_{\rho=0} &= 2\,\mathbf{W}_1[f].
\end{aligned}$$

Theorem 9.6 pertains to $q = 2$. For general $q > 1$, the derivatives of $\mathbf{S}_\rho^{(q)}[f]$ at $\rho = 0$ and 1 are given in the following theorem which is due to Li and Médard [110].

Theorem 9.7. For $f : \{0,1\}^n \to \{0,1\}$ with mean $a \in (0,1]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\mathbf{S}_\rho^{(q)}[f]\Big|_{\rho=1} &= \frac{q}{4}\ \mathbf{I}[f], \qquad (9.40)\\
\frac{\mathrm{d}}{\mathrm{d}\rho}\mathbf{S}_\rho^{(q)}[f]\Big|_{\rho=0} &= 0, \qquad \text{and}\\
\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\mathbf{S}_\rho^{(q)}[f]\Big|_{\rho=0} &= q(q-1)\ a^{q-2}\ \mathbf{W}_1[f].
\end{aligned}$$

Proof. By using the Fourier-analytic relations in (9.39), we obtain

$$T_\rho f(y^n) = \sum_{\mathcal{S}\subset[n]} \rho^{|\mathcal{S}|}\, \hat{f}_{\mathcal{S}}\, \chi_{\mathcal{S}}(y^n).$$

By the definition of the q-stability in (9.4)

$$\mathbf{S}_\rho^{(q)}[f] = \mathbb{E}_{Y^n}\left[\left(\sum_{\mathcal{S}\subset[n]} \rho^{|\mathcal{S}|}\hat{f}_{\mathcal{S}}\chi_{\mathcal{S}}(Y^n)\right)^q\right].$$

Differentiating this with respect to ρ yields

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\mathbf{S}_\rho^{(q)}[f] = q\ \mathbb{E}_{Y^n}\left[(T_\rho f(Y^n))^{q-1} \sum_{\mathcal{S}\subset[n]:|\mathcal{S}|\geq 1} |\mathcal{S}|\rho^{|\mathcal{S}|-1}\hat{f}_{\mathcal{S}}\chi_{\mathcal{S}}(Y^n)\right].$$

Setting $\rho = 1$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}\mathbf{S}_\rho^{(q)}[f]\Big|_{\rho=1} &= q\ \mathbb{E}_{Y^n}\left[f(Y^n)^{q-1} \sum_{\mathcal{S}\subset[n]:|\mathcal{S}|\geq 1} |\mathcal{S}|\hat{f}_{\mathcal{S}}\chi_{\mathcal{S}}(Y^n)\right] \\
&= q\ \mathbb{E}_{Y^n}\left[f(Y^n) \sum_{\mathcal{S}\subset[n]:|\mathcal{S}|\geq 1} |\mathcal{S}|\hat{f}_{\mathcal{S}}\chi_{\mathcal{S}}(Y^n)\right] \qquad (9.41) \\
&= q \sum_{\mathcal{S}\subset[n]:|\mathcal{S}|\geq 1} |\mathcal{S}|\hat{f}_{\mathcal{S}}^2 \\
&= \frac{q}{4}\,\mathbf{I}[f], \qquad (9.42)
\end{aligned}$$

where (9.41) follows since f only takes values in $\{0,1\}$, and hence, $f^{q-1} = f$, and (9.42) follows from (9.36). This proves (9.40). The other equalities can be proved similarly. □

9.3.3 Max q-Stabilities in Extreme Cases of ρ

Based on the concept of the total influence and the results stated in Sections 9.3.1 and 9.3.2, we are now ready to analyze the extreme cases of the max q-stability as $\rho \downarrow 0$ and $\rho \uparrow 1$. We first state a lower bound on the derivative of the q-stability with respect to ρ evaluated at $\rho = 1$. This result is due to Mossel and O'Donnell [122] for integer q and Li and Médard [110] for real $q > 1$.

Theorem 9.8. Let $q > 1$. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$ with mean a,

$$\frac{\partial}{\partial \rho} \mathbf{S}_\rho^{(q)}[f]\Big|_{\rho=1} \geq \frac{q}{2}\, a \log\Big(\frac{1}{a}\Big).$$

This lower bound is attained if $a = 2^{-k}$ for any $1 \leq k \leq n$ and f is the indicator of an $(n-k)$-subcube.

This theorem follows by the edge-isoperimetric inequality in (9.38) and (9.40).

Note that if $\rho = 1$, then for any $f : \{0,1\}^n \to \{0,1\}$ with mean a, it holds that $\mathbf{S}_\rho^{(q)}[f] = a$ (cf. (9.20)). Hence, from Theorem 9.8, it is plausible, via "continuity arguments", that if $a = 2^{-k}$ for integer k and ρ is sufficiently close to 1, then $\mathbf{S}_\rho^{(q)}[f]$ is maximized by the indicator of an $(n-k)$-subcube. This can be proven rigorously using the fact that the number of Boolean functions for a given n is finite, some approximation arguments involving Taylor's theorem, and bounds on the derivative of $\mathbf{S}_\rho^{(q)}$ evaluated at $\rho = 1$ (Theorem 9.8). This is stated formally in the following theorem which is due to Mossel and O'Donnell [122] for integer q and Li and Médard [110] for real q.

Theorem 9.9. Fix $n \geq 1$, $q > 1$, and $a = 2^{-k}$ with $1 \leq k \leq n$. There exists an $\epsilon \in (0,1)$ such that for all $\rho \in [1-\epsilon, 1]$, $\Gamma_\rho^{(q)}(a)$ is attained by the indicator of an $(n-k)$-subcube.

Proof Sketch of Theorem 9.9. Fix a Boolean function f and $\rho \in (0,1)$. Using Taylor's theorem, we can write

$$\mathbf{S}_\rho^{(q)}[f] = \mathbf{S}_1^{(q)}[f] + (\rho - 1)\frac{\partial}{\partial \rho}\mathbf{S}_\rho^{(q)}[f]\Big|_{\rho=1} + \phi_f(\tilde{\rho})(\rho-1)^2,$$

where $\phi_f : [0,1] \to \mathbb{R}$ is a bounded function induced by f and $\tilde{\rho} \in (\rho, 1)$. Since n is fixed, the number of Boolean functions $f : \{0,1\}^n \to \{0,1\}$ is finite. From this fact, we deduce that $\phi(\rho) := \max_{f:\{0,1\}^n \to \{0,1\}} \phi_f(\rho)$, then ϕ is bounded, i.e., there is some constant c_2 such that $|\phi(\rho)| \leq c_2$ for all $\rho \in [0,1]$. Moreover, if f is not the indicator of an $(n-k)$-subcube, it holds that (cf. Theorem 9.8)

$$\frac{\partial}{\partial \rho} \mathbf{S}_\rho^{(q)}[f]\Big|_{\rho=1} > \frac{q}{2}\, a \log\Big(\frac{1}{a}\Big).$$

By again exploiting that fact that the number of Boolean functions is finite,

$$c_1 := \min_{f \in \mathcal{F}} \frac{\partial}{\partial \rho} \mathbf{S}_\rho^{(q)}[f] \Big|_{\rho=1} > \frac{q}{2}\, a\, \log\left(\frac{1}{a}\right),$$

where $\mathcal{F}$ denotes the set of Boolean functions $f : \{0,1\}^n \to \{0,1\}$ that *cannot* be written as the indicator of an $(n-k)$-subcube. Therefore, for any $f \in \mathcal{F}$,

$$\mathbf{S}_\rho^{(q)}[f] \leq a + c_1(\rho - 1) + c_2(\rho - 1)^2. \tag{9.43}$$

By Taylor's theorem, one can lower bound the q-stability for the indicator of an $(n-k)$-subcube $\mathbb{C}_{n-k}$ as

$$\mathbf{S}_\rho^{(q)}[\mathbb{1}_{\mathbb{C}_{n-k}}] \geq a + (\rho - 1)\frac{q}{2}\, a\, \log\left(\frac{1}{a}\right) + c_3(\rho - 1)^2, \tag{9.44}$$

where c_3 is an absolute constant independent of ρ. Comparing (9.43) and (9.44), we observe that there exists a constant $\epsilon > 0$ such that the right-hand side of (9.44) is larger than (9.43) for all $\rho \in [1-\epsilon, 1]$, concluding the proof sketch of Theorem 9.9. □

Concerning the other extreme case, i.e., the limiting case as $\rho \downarrow 0$, following the proof ideas used in Theorems 9.8 and 9.9, one can also show the following result, which is due to Mossel and O'Donnell [122] and Li and Médard [110].

Theorem 9.10. Fix $n \geq 1$, $q > 1$, and a dyadic rational $a \in (0,1)$. There exists an $\epsilon \in (0,1)$ such that for all $\rho \in [0, \epsilon]$, $\Gamma_\rho^{(q)}(a)$ is attained by some Boolean function that maximizes the degree-1 Fourier weight $\mathbf{W}_1$. In particular, for $a = 1/2$, there exists an $\epsilon > 0$ such that for all $\rho \in [0, \epsilon]$, $\Gamma_\rho^{(q)}(1/2)$ is attained by dictator functions.

Theorems 9.8, 9.9, and 9.10 can be extended to their symmetric counterparts of the max q-stability. For $q = 1$, they can also be extended to the Φ-versions of the max q-stabilities (cf. Definition 9.6); see Courtade and Kumar [40], Ordentlich, Shayevitz, and Weinstein [134], and Yang and Wesel [187].

9.4 The Balanced Case

In this section, we consider the balanced case, i.e., $a = 1/2$, and discuss recent progress on Conjectures 9.1 and 9.2. We first focus on the case $q = 1$, i.e., the balanced version of the Courtade–Kumar conjecture, which can be stated as follows.

Conjecture 9.3. For any $n \in \mathbb{N}$ and $\rho \in (0,1)$,

$$\max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=1/2} I(f(X^n);Y^n) \stackrel{?}{=} 1 - h\Big(\frac{1-\rho}{2}\Big). \tag{9.45}$$

In the original version of Courtade–Kumar conjecture, the Boolean function f is not required to satisfy $\mathbb{E}[f(X^n)] = 1/2$. It has been numerically verified to be true for all $n \le 7$ [40]. An old result by Witsenhausen and Wyner [176] (also see Erkip [53]) yields the following bound.

Proposition 9.4. It holds that

$$\max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=1/2} I(f(X^n);Y^n) \le \rho^2. \tag{9.46}$$

This proposition can be proved via the so-called Mrs. Gerber's lemma [184] or the hypercontractivity inequality in (8.56). Here, we provide a short justification based on the latter. By (8.56), we obtain that for $q > 1$ and any Boolean function f with mean a,

$$\mathbf{S}_\rho^{(q)}[f] \le a^{\frac{q}{1+(q-1)\rho^2}}.$$

In other words,

$$\Gamma_\rho^{(q)}(a) \le a^{\frac{q}{1+(q-1)\rho^2}} \quad \text{and} \quad \breve{\Gamma}_\rho^{(q)}(a) \le a^{\frac{q}{1+(q-1)\rho^2}} + \bar{a}^{\frac{q}{1+(q-1)\rho^2}}.$$

Substituting the latter into (9.25) and setting $a = 1/2$ yields

$$\breve{\Pi}_\rho^{(q)}(1/2) \le \frac{2^{\frac{(1-q)(1-\rho^2)}{1+(q-1)\rho^2}} - 1}{(q-1)\ln 2}.$$

Letting $q \downarrow 1$, we obtain $\breve{\Pi}_\rho^{(1)}(1/2) \le \rho^2 - 1$. Substituting this into (9.28) and noting that $h(1/2) = 1$ yields (9.46) as desired.

Considering small ρ, and using Fourier analysis and hypercontractivity, Ordentlich, Shayevitz, and Weinstein [134] improved the bound in (9.46) to the following.

Proposition 9.5. For $0 \le \rho \le 1/\sqrt{3}$,

$$\max_{\text{Boolean } f:\mathbb{E}[f(X^n)]=1/2} I(f(X^n);Y^n) \le \frac{\log e}{2}\rho^2 + 9\Big(1-\frac{\log e}{2}\Big)\rho^4. \quad (9.47)$$

The bounds in (9.46) and (9.47) are illustrated in Fig. 9.6. The bound in (9.47) is better than (9.46) in the range $0 < \rho < 1/3$. Moreover, the bound in (9.47) is asymptotically tight as $\rho \downarrow 0$, i.e., the ratio of the bound in (9.47) and the right-hand side of (9.45) converges to 1 as $\rho \downarrow 0$. This point can be seen from the fact that by Taylor's theorem, as $\rho \downarrow 0$,

$$1 - h\Big(\frac{1-\rho}{2}\Big) = \frac{\log e}{2}\rho^2 + \frac{\log e}{12}\rho^4 + O(\rho^6).$$

In 2016, Samorodnitsky [149] made a significant breakthrough on the Courtade–Kumar conjecture. Specifically, he proved the existence of a *dimension-independent* interval for which Conjecture 9.3 holds for all ρ in the interval.

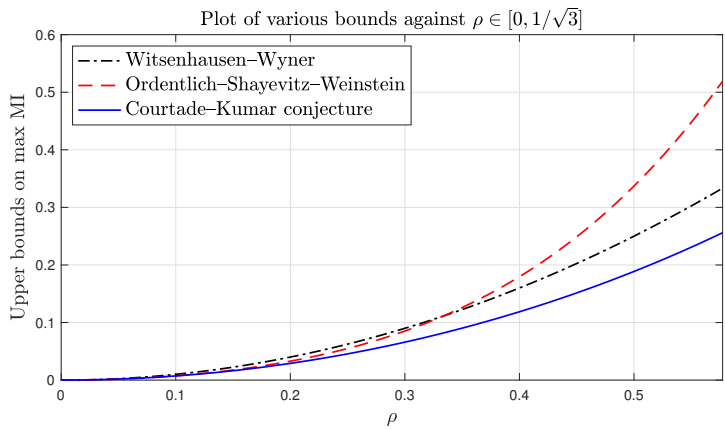


Figure 9.6: Illustration bounds on $\max I(f(X^n);Y^n)$ by Witsenhausen and Wyner [176] in (9.46), Ordentlich, Shayevitz, and Weinstein [134] in (9.47), and the Courtade–Kumar conjecture in (9.45)

Theorem 9.11. There exists a constant $0 < \rho_0 < 1$ (independent of n), such that (9.45) holds for any $n \in \mathbb{N}$ and any $\rho \in (0, \rho_0]$.

The proof by Samorodnitsky [149] is based on Fourier analysis, random restrictions, techniques in Ordentlich, Shayevitz, and Weinstein

[134], the Friedgut–Kalai–Naor (FKN) theorem [90], among others. Samorodnitsky's proof is highly technical, so we do not present it here. However, we should note that in the proof of Theorem 9.11, ρ_0, which is not explicitly provided, is assumed to be "sufficiently small". It is also worth noting that the conclusion that the value ρ_0 is independent of n (resulting in a *dimension-independent* interval $(0, \rho_0]$) is the crux of this theorem. Indeed, if we allow ρ_0 to vary with n, then the resulting theorem is merely an extension of Theorem 9.10 to the case $q = 1$, which can be proved by combining the bound in (9.47) by Ordentlich, Shayevitz, and Weinstein [134] and the discreteness of the space of Boolean functions; see [134, Corollary 1]. This fact can also be deduced using calculus [187].

Using Fourier analysis and optimization theory, the first author of this monograph [191] provided an explicit threshold for Theorem 9.11. Specifically, he showed that (9.45) holds for any n and any $\rho \in (0, \rho_1]$, where ρ_1 be the solution in $(0, 1)$ to the equation

$$(1+\rho^2)\log\Big(\frac{1+\rho}{2}\Big) - (1-\rho)^2\log\Big(\frac{1-\rho}{2}\Big) = 0.$$

The value of $\rho_1 \approx 0.461491$.

In the Courtade–Kumar conjecture, if the Boolean function is set to a dictator function $f(x^n) = x_1$ (say), then the objective function $I(f(X^n); Y^n) = I(X_1; Y_1)$. Motivated by this, in addition to the original Courtade–Kumar conjecture (in which f is an arbitrary Boolean function and not required to satisfy $\mathbb{E}[f(X^n)] = 1/2$), Courtade and Kumar also proposed a weaker version of this conjecture. They conjectured that for any $n \in \mathbb{N}$ and $\rho \in (0, 1)$,

$$\max_{\text{Boolean } f,g} I(f(X^n); g(Y^n)) = 1 - h\Big(\frac{1-\rho}{2}\Big). \tag{9.48}$$

This weaker version was proven by Pichler, Piantanida, and Matz [136] by using Fourier analysis and a novel partitioning technique.

Theorem 9.12. The equality in (9.48) holds for all $(n, \rho) \in \mathbb{N} \times (0, 1)$.

Since the Li–Médard conjecture was only recently posed (at the time of writing), there is less progress on it compared to the Courtade–Kumar

conjecture. Hence, we do not elaborate on it apart from mentioning some partial progress by Yu [191] for a certain set of (q, ρ).

Finally, we summarize some recent progress on the Mossel–O'Donnell conjecture, which states that dictator functions are optimal in attaining both the asymmetric and symmetric max q-stabilities for $2 < q \leq 9$ (and for any $n \in \mathbb{N}$ and any $\rho \in (0, 1)$). As discussed in Theorems 9.9 and 9.10, the limiting cases as $\rho \downarrow 0$ and $\rho \uparrow 1$ (with fixed n) were resolved in [122] for the symmetric case and in [110], [122] for the asymmetric case. However, for other intermediate values of ρ, there has been fairly limited progress. For the symmetric case, the best known result is Mossel and O'Donnell's result in Theorem 9.2; this result resolved the eponymous conjecture for $q = 3$ and for any $\rho \in (0, 1)$. Combining this with the result of Barnes and Özgür [11] (in Lemma 9.3) yields the conclusion the Mossel–O'Donnell conjecture holds for all $2 < q \leq 3$. There is even less progress for the asymmetric case in which the best known result remains that of Witsenhausen's result in Corollary 9.1 for the case $q = 2$. Recently, in [191], the first author of this monograph made some progress on the Mossel–O'Donnell conjecture. He showed that the symmetric version of the Mossel–O'Donnell conjecture holds for $2 < q \leq 5$, and the asymmetric version holds for $2 < q \leq 3$. These imply that $3 \leq q_{\max} < 10$ and $5 \leq \breve{q}_{\max} < 10$. The proofs are based on Fourier analysis and optimization theory.

9.5 Moderate and Large Deviations Regimes

In this section, we consider the max q-stabilities in the MD and LD regimes. Recall the definition of the asymmetric max q-stability with $q \in [1, \infty)$ in (9.9). It can be rewritten as

$$\Gamma_{\rho}^{(q)}(a) = \left(\max_{\mathcal{A} \subset \{0,1\}^n : \pi_X^n(\mathcal{A}) \leq a} \| \pi_{X|Y}^n(\mathcal{A}|Y^n) \|_q \right)^q, \tag{9.49}$$

where the maximization is over all subsets of $\{0, 1\}^n$. We now extend the asymmetric max q-stability to the case of $q \in (-\infty, 1) \backslash \{0\}$. For $q \in (-\infty, 1) \backslash \{0\}$, define

$$\Gamma_{\rho}^{(q)}(a) := \left(\min_{\mathcal{A} \subset \{0,1\}^n : \pi_X^n(\mathcal{A}) \geq a} \| \pi_{X|Y}^n(\mathcal{A}|Y^n) \|_q \right)^q. \tag{9.50}$$

We note that even though a min is present in (9.50), we still term this quantity as the asymmetric *max* q-stability.

We are now interested in the MD and LD asymptotics of (9.49) and (9.50). Similarly to the 2-user NICD problem, in the LD regime, the parameter a is assumed to vanish exponentially fast as $n \to \infty$, i.e., $a = 2^{-n\alpha}$ for some fixed constant $\alpha \in (0, 1)$. In the MD regime, a is assumed to vanish subexponentially fast, i.e., $a = 2^{-\theta_n \alpha}$ for an MD sequence $\{\theta_n\}_{n\in\mathbb{N}}$.

Definition 9.11. We define the LD and MD exponents corresponding to the quantities in (9.49) and (9.50) as follows.

1. For $n \geq 1$, $\alpha \in [0, 1]$, and $q \geq 1$, define the *LD exponent* as
$$\Upsilon^{(n)}_{q,\mathrm{LD}}(\alpha) := -\frac{1}{n} \log \max_{\mathcal{A}:\pi^n_X(\mathcal{A})\leq 2^{-n\alpha}} \|\pi^n_{X|Y}(\mathcal{A}|Y^n)\|_q. \tag{9.51}$$
For $q \in (-\infty, 1)\backslash\{0\}$, $\Upsilon^{(n)}_{q,\mathrm{LD}}(\alpha)$ is defined similarly but with the maximization in (9.51) replaced by a minimization, and the inequality reversed.

2. For $n \geq 1$, $\alpha \in [0, \infty)$, $q \geq 1$, and an MD sequence $\{\theta_n\}_{n\in\mathbb{N}}$, define the *MD exponent* as
$$\Upsilon^{(n)}_{q,\mathrm{MD}}(\alpha) := -\frac{1}{\theta_n} \log \max_{\mathcal{A}:\pi^n_X(\mathcal{A})\leq 2^{-\theta_n\alpha}} \|\pi^n_{X|Y}(\mathcal{A}|Y^n)\|_q. \tag{9.52}$$
For $q \in (-\infty, 1)\backslash\{0\}$, $\Upsilon^{(n)}_{q,\mathrm{MD}}(\alpha)$ is defined similarly but with the maximization in (9.52) replaced by a minimization, and the inequality reversed.

3. Define $\Upsilon^{(\infty)}_{q,\mathrm{MD}}$ and $\Upsilon^{(\infty)}_{q,\mathrm{LD}}$ as the pointwise limits of (9.51) and (9.52) as $n \to \infty$.

Note that in the definitions in (9.51)–(9.52), we remove the q^{th} power in (9.49)–(9.50). This slight modification will result in a multiplicative factor of q in the characterizations of these exponents. We deliberately choose such definitions since the bounds on the exponents in Definition 9.11 provided in the following two theorems will be consistent with the bounds for the 2-user NICD problem. We also remark that these

quantities depend on ρ but these dependencies are suppressed to avoid notational clutter in what follows.

For $q \in (1-\rho^{-2}, \infty)\backslash\{0\}$ and $\alpha > 0$, let

$$\Upsilon_{q,\mathrm{MD}}(\alpha) := \frac{\alpha}{1+(q-1)\rho^2}. \tag{9.53}$$

By using the single-function versions of hypercontractivity inequalities (Theorem 8.5), we can obtain the following result.

Theorem 9.13 (q-stability). Let $n \geq 1$ and $\alpha > 0$. For $q \geq 1$,

$$\Upsilon^{(n)}_{q,\mathrm{MD}}(\alpha) \geq \Upsilon_{q,\mathrm{MD}}(\alpha), \tag{9.54}$$

and for $q \in (1-\rho^{-2}, 1)\backslash\{0\}$,

$$\Upsilon^{(n)}_{q,\mathrm{MD}}(\alpha) \leq \Upsilon_{q,\mathrm{MD}}(\alpha). \tag{9.55}$$

Moreover, these two bounds are asymptotically tight, i.e., for $q \in (1-\rho^{-2}, \infty)\backslash\{0\}$,

$$\Upsilon^{(\infty)}_{q,\mathrm{MD}}(\alpha) = \Upsilon_{q,\mathrm{MD}}(\alpha). \tag{9.56}$$

Lastly, for $q \in (-\infty, 1-\rho^{-2}]$, $\Upsilon^{(\infty)}_{q,\mathrm{MD}}(\alpha) = \infty$. The equalities are achieved by sequences of Hamming balls or spherical shells.

The function $\Upsilon_{q,\mathrm{MD}}$, defined in (9.53), is plotted in Fig. 9.7.

Proof of Theorem 9.13. This theorem is a consequence of the classic hypercontractivity inequalities in (8.56) and (8.57). Substituting $f \leftarrow \mathbb{1}_{\mathcal{A}}$ into (8.56) and (8.57), we obtain for $q \geq 1$,

$$\|\pi^n_{X|Y}(\mathcal{A}|Y^n)\|_q \leq \pi^n_X(\mathcal{A})^{\frac{1}{1+(q-1)\rho^2}},$$

and for $q \in (1-\rho^{-2}, 1)\backslash\{0\}$,

$$\|\pi^n_{X|Y}(\mathcal{A}|Y^n)\|_q \geq \pi^n_X(\mathcal{A})^{\frac{1}{1+(q-1)\rho^2}}.$$

These inequalities immediate imply (9.54) and (9.55).

The asymptotic tightness of (9.54) and (9.55) can be verified by choosing the sets $\mathcal{A}$ in the definition of the MD exponent to be sequences of Hamming balls or spherical shells. The asymptotic tightness for $q \in (-\infty, 1-\rho^{-2}]$ follows by the monotonicity of the L^q-norm in q, and taking limits as $q \downarrow 1-\rho^{-2}$ in (9.56). We omit the details. □

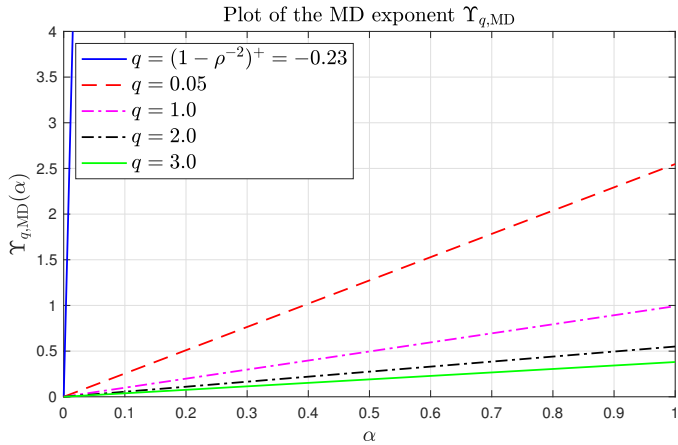


Figure 9.7: The MD exponent of the q-stability $\Upsilon_{q,\mathrm{MD}}$ for $\rho = 0.9$. Observe that $\Upsilon_{q,\mathrm{MD}}$ is linear given each $q \neq 0$ and diverges as $q \downarrow 1 - \rho^{-2} \approx -0.2346$.

We now turn our attention to the LD exponent. For $q \neq 0$, define

$$\theta_q(Q_X, Q_Y) := \mathsf{D}(Q_X, Q_Y \| \pi_{XY}) - \frac{D(Q_Y \| \pi_Y)}{q'}. \tag{9.57}$$

where q' is the Hölder conjugate of q Define

$$\Upsilon_{q,\mathrm{LD}}(\alpha) := \inf_{Q_X, Q_Y : D(Q_X \| \pi_X) \geq \alpha} \theta_q(Q_X, Q_Y) \tag{9.58}$$

for $q \geq 1$, and

$$\Upsilon_{q,\mathrm{LD}}(\alpha) := \begin{cases} \sup\limits_{Q_X : D(Q_X \| \pi_X) \leq \alpha} \inf\limits_{Q_Y} \theta_q(Q_X, Q_Y) & 0 < q < 1 \\ \sup\limits_{Q_X : D(Q_X \| \pi_X) \leq \alpha} \sup\limits_{Q_Y} \theta_q(Q_X, Q_Y) & q < 0 \end{cases} \tag{9.59}$$

for $q \in (-\infty, 1) \backslash \{0\}$. It can be verified that $\Upsilon_{q,\mathrm{LD}}(\alpha) \geq 0$ for all $q \neq 0$. Asymptotically tight bounds are provided in the following theorem, which is known as the *strong q-stability theorem* and was proved by the first author of this monograph [192].

Theorem 9.14 (Strong q-stability)**.** For any $n \geq 1$ and $\alpha \in (0, 1)$, it holds that for $q \geq 1$,

$$\Upsilon^{(n)}_{q,\mathrm{LD}}(\alpha) \geq \mathbb{L}[\Upsilon_{q,\mathrm{LD}}](\alpha), \tag{9.60}$$

and for $q \in (-\infty, 1)\backslash\{0\}$,

$$\Upsilon_{q,\mathrm{LD}}^{(n)}(\alpha) \le \mathbb{U}[\Upsilon_{q,\mathrm{LD}}](\alpha). \tag{9.61}$$

Moreover, these two bounds are asymptotically tight, i.e.,

$$\Upsilon_{q,\mathrm{LD}}^{(\infty)}(\alpha) = \mathbb{L}[\Upsilon_{q,\mathrm{LD}}](\alpha) \quad \text{and} \quad \Upsilon_{q,\mathrm{LD}}^{(\infty)}(\alpha) = \mathbb{U}[\Upsilon_{q,\mathrm{LD}}](\alpha), \tag{9.62}$$

and these equalities are achieved by sequences of Hamming balls or spheres.

It has been shown in [193] that for $q \ge 1$, $\Upsilon_{q,\mathrm{LD}}$ is convex, and for $q \in (-\infty, 1)\backslash\{0\}$, $\Upsilon_{q,\mathrm{LD}}$ is concave. Combining this result with the strong q-stability theorem (Theorem 9.14) tells us that the lower convex envelope and upper concave envelope operations in (9.62) can be removed and Hamming balls or spheres are optimal in the LD regime. That is, for the DSBS and $\alpha \in (0, 1)$,

$$\Upsilon_{q,\mathrm{LD}}^{(\infty)}(\alpha) = \Upsilon_{q,\mathrm{LD}}(\alpha) \quad \text{and} \quad \Upsilon_{q,\mathrm{LD}}^{(\infty)}(\alpha) = \Upsilon_{q,\mathrm{LD}}(\alpha).$$

This is parallel to the discussion of the resolution of the OPS conjecture in Section 8.5; also see (8.67). The asymptotically tight bound $\Upsilon_{q,\mathrm{LD}}$, defined in (9.59), is plotted in Fig. 9.8 for various q's.

9.6 Extensions to Sources Beyond the DSBS

Similarly to the discussion in Section 8.6, the q-stability and strong q-stability theorems can be extended to sources defined on arbitrary finite alphabets as well as jointly Gaussian sources. We discuss these extensions here.

9.6.1 Finite Alphabets

Let π_{XY} be a joint distribution defined on a finite alphabet. We now consider its q-stability. For $\alpha \in [0, \alpha_{\max}(\pi_X)]$ (defined in (8.68)), we reuse the definitions in (9.51)–(9.52) for $\Upsilon_{q,\mathrm{MD}}^{(n)}$ and $\Upsilon_{q,\mathrm{LD}}^{(n)}$, but with the underlying distribution set to be π_{XY}. The strong q-stability theorem (Theorem 9.13) can be extended to the following general version, which was first shown in [192], as a consequence of the strong version of hypercontractivity inequalities derived in [192]. We provide a simple proof of Theorem 9.15 in Section 10.3.

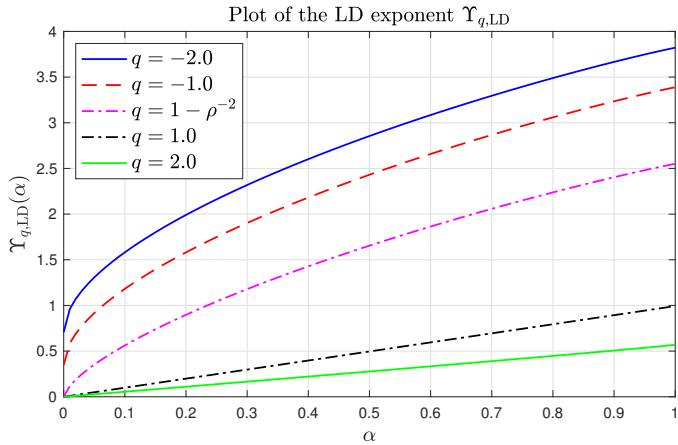


Figure 9.8: The LD exponent of the q-stability $\Upsilon_{q,\mathrm{LD}}$ for $\rho = 0.9$. Observe that $\Upsilon_{q,\mathrm{LD}}$ appears to be ("only slightly") convex for $q > 1$, concave for $q \in (-\infty, 1) \setminus \{0\}$, and linear for $q = 1$. Also $\Upsilon_{q,\mathrm{LD}}(0)$ vanishes when $q \leq 1 - \rho^{-2} \approx -0.2346$.

Theorem 9.15 (Strong q-stability: General version)**.** For any $n \geq 1$ and $\alpha \in (0, \alpha_{\max}(\pi_X)]$, (9.60) holds for $q \geq 1$, and (9.61) holds for $q \in (-\infty, 1) \setminus \{0\}$. Moreover, (9.60) and (9.61) are asymptotically tight, i.e., (9.62) holds.

The q-stability theorem (Theorem 9.14) can be also generalized to the finite alphabet case, but for general sources on finite alphabets, a limiting operation is needed.

Theorem 9.16 (q-Stability: General version)**.** For any $n \geq 1$, $\alpha > 0$, and $q \geq 1$,

$$\Upsilon^{(n)}_{q,\mathrm{MD}}(\alpha) \geq \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{L}[\Upsilon_{q,\mathrm{LD}}](\epsilon\alpha). \tag{9.63}$$

If instead $q \in (-\infty, 1) \setminus \{0\}$ and $\Upsilon_{q,\mathrm{LD}}(0) = 0$, then

$$\Upsilon^{(n)}_{q,\mathrm{MD}}(\alpha) \leq \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{U}[\Upsilon_{q,\mathrm{LD}}](\epsilon\alpha). \tag{9.64}$$

Moreover, the inequalities in (9.63)–(9.64) are asymptotically tight.

9.6.2 Gaussian Sources

Finally, we turn our attention to memoryless bivariate Gaussian sources with correlation coefficient $\rho \in (0, 1)$. For this class of sources, the

max q-stability problem was completely solved by Borell [28] for all $a \in [0,1]$. Let π_{XY} be a bivariate Gaussian distribution with zero mean and covariance matrix $\mathbf{K}$ given in (8.17). Let $(X^n, Y^n) \sim \pi_{XY}^n$. For this distribution, a real number $q > 1$, and $a \in [0,1]$, we define

$$\Gamma_\rho^{(q)}(a) := \sup \|\pi_{X|Y}^n(\mathcal{A}|Y^n)\|_q,$$

where the supremum runs over all measurable sets $\mathcal{A} \subset \mathbb{R}^n$ such that $\pi_X^n(\mathcal{A}) = a$. The following theorem is due to Borell [28].

Theorem 9.17 (Borell's q-stability theorem)**.** For any $n \geq 1$, $q > 1$, $0 \leq \rho < 1$, and $a \in [0,1]$, one has

$$\Gamma_\rho^{(q)}(a) = \Lambda_\rho^{(q)}(a), \tag{9.65}$$

where $\Lambda_\rho^{(q)}$, the Gaussian q-stability function, is defined in (9.6). Moreover, optimal subsets $\mathcal{A}$ (i.e., those attaining $\Gamma_\rho^{(q)}$) are parallel halfspaces.

The proof of this theorem can be found in Borell [28] and Eldan [52]. Moreover, the proof of this theorem with q being an integer can also be found in Isaksson and Mossel [87] and Neeman [129]. We remark that Neeman's proof in [129] is an extension of the one for Borell's isoperimetric theorem given in Section 8.6.2 to the multi-user case.

We now consider the Gaussian version of the Courtade–Kumar conjecture. Substituting (9.65) into the Φ-symmetric max q-stability in (9.24) and taking limits as $q \downarrow 1$, one can deduce that $\breve{\Pi}_\rho^{(1)}(a)$ is attained by halfspaces with π_X^n-probability a. This implies that

$$\max_{\substack{f:\mathbb{R}^n \to \{0,1\} \text{ measurable:} \\ \mathbb{E}[f(X^n)]=a}} -H\big(f(X^n)|Y^n\big)$$

$$= -H\big(\mathbb{1}\{X_1 \leq \Phi^{-1}(a)\}|Y_1\big) = -\mathbb{E}_{Y_1}\left[h\left(\Phi\Big(\frac{\Phi^{-1}(a)-\rho Y_1}{\sqrt{1-\rho^2}}\Big)\right)\right]. \tag{9.66}$$

That is, given $a \in [0,1]$, the indicator of any half-space with π_X^n-probability a (e.g., $(-\infty, \Phi^{-1}(a)] \times \mathbb{R}^{n-1}$) maximizes the mutual information between $f(X^n)$ and Y^n over all $\{0,1\}$-valued measurable functions f. This statement was also proved by Kindler, O'Donnell, and Witmer [96] using a different method. If we do not fix a, then

similarly to the original Courtade–Kumar conjecture for the DSBS, it is natural to conjecture that for this Gaussian version of Courtade–Kumar conjecture, the mutual information is also maximized at $a = 1/2$ for every $\rho \in (0, 1)$. This point can be confirmed numerically as shown in Fig. 9.9 in which we plot the right-hand side of (9.66) plus $h(a)$ as a function of $a \in [0, 1/2]$ for different ρ's. Note that we only focus on the case $a \in [0, 1/2]$ in Fig. 9.9, since the function considered is symmetric with respect to $a = 1/2$. It is easily seen that the maxima of these curves occur at $a = 1/2$.

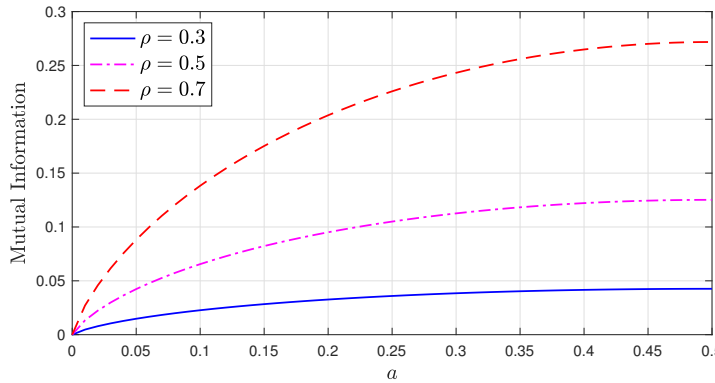


Figure 9.9: The mutual information, i.e., the right-hand side of (9.66) plus $h(a)$.

10

Functional Inequalities

In this section, we consider functional extensions of the NICD and the max q-stability problems as described in Sections 8 and 9 respectively. Recall that in the 2-user NICD problem, we optimize the probability of agreement between two random bits that are generated in a distributed manner via *Boolean functions* from a joint source (X^n, Y^n). In this section, we replace the Boolean functions $f, g \in \{0,1\}^n \to \{0,1\}$ with *arbitrary nonnegative functions*, and obtain corresponding functional inequalities. Specifically, we will introduce the Brascamp–Lieb inequalities, the hypercontractivity inequalities, and the log-Sobolev inequalities, as well as their strengthened counterparts. We provide information-theoretic characterizations of these inequalities, and also use them to prove the strong SSE theorem and the strong q-stability theorem stated respectively in Sections 8.6 and 9.6. Analogously to the forward and reverse joint probabilities in the NICD problem (Definition 8.1), the optimal constants or exponents in these inequalities can be also regarded as refinements of GKW's common information when the latter is equal to zero, but with the "information" measured by the *entropy* of a nonnegative function, rather than the Shannon entropy.

This section concerning *functional inequalities* (or inequalities involving functionals) starts by formally defining some convenient quantities, such as the minimum relative entropy region, in Section 10.1. Using these new definitions, we provide alternative representations of the forward and reverse large deviations exponents in the NICD and q-stability problems. These quantities are then used in Section 10.2 to express the hypercontractivity regions (which generalize and strengthen the classic Hölder inequalities) and Brascamp–Lieb exponents in terms of single-letter, information-theoretic quantities. We then connect these exponents to the NICD and q-stability problems in Section 10.3, leading to a short proof of the strong SSE theorem (Theorem 8.11). In Section 10.4, we discuss the log-Sobolev inequalities, provide single-letter expressions for their optimal constants, and use the results as a bridge to connect the hypercontractivity inequalities to their strengthened counterparts, which are presented in Section 10.5. In Section 10.5, our discussion culminates with expressions for the strong log-Sobolev constant and a strengthened hypercontractivity inequality for the DSBS. Throughout this section, we focus on *information-theoretic* characterizations of optimal constants and exponents in various functional inequalities.

As there are several interconnected results in this section and Sections 8 and 9, we illustrate their relationships by means of a graph in Fig. 10.1.

10.1 Preliminary Definitions

Throughout this section, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are finite sets and π_{XY} is a joint distribution on $\mathcal{X} \times \mathcal{Y}$.

Assumption 10.1 (Full support of marginals)**.** The supports of π_X and π_Y are $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Definition 10.1. Define the *minimum relative entropy region* with respect to a joint distribution $\pi_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ as

$$\mathcal{D}(\pi_{XY}) := \bigcup_{Q_X, Q_Y} \Big\{ \big(D(Q_X \| \pi_X), D(Q_Y \| \pi_Y), \mathsf{D}(Q_X, Q_Y \| \pi_{XY}) \big) \Big\},$$

where $\mathsf{D}(Q_X, Q_Y \| \pi_{XY})$ is the minimal relative entropy with respect to π_{XY} over all couplings of Q_X and Q_Y, defined in (8.23). Due to

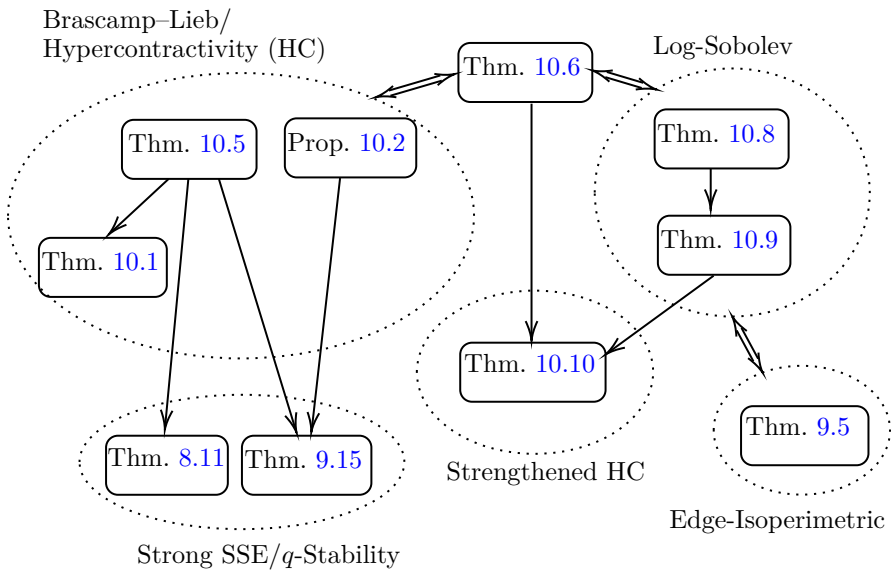


Figure 10.1: A graph of the main results in this and Sections 8 and 9, where $\longrightarrow$ denotes an implication and $\Longleftrightarrow$ denotes a close relationship.

Assumption 10.1, any Q_X and Q_Y defined on $\mathcal{X}$ and $\mathcal{Y}$ respectively are absolutely continuous with respect to π_X and π_Y respectively.

The minimum relative entropy region is the subset of $\mathbb{R}^3$ that is formed by the pair of relative entropies $(D(Q_X\|\pi_X), D(Q_Y\|\pi_Y))$ and the minimal relative entropy $\mathsf{D}(Q_X, Q_Y\|\pi_{XY})$ as Q_X and Q_Y run over all distributions that are absolutely continuous with respect to π_X and π_Y respectively.

Definition 10.2. For $(s,t) \in [0, \alpha_{\max}(\pi_X)] \times [0, \beta_{\max}(\pi_Y)]$ (refer to (8.68) for definitions), define the *upper* and *lower envelopes* of the minimal relative entropy region $\mathcal{D}(\pi_{XY})$ respectively as

$$\underline{\varphi}(s,t) := \min_{Q_X, Q_Y : D(Q_X\|\pi_X)=s, D(Q_Y\|\pi_Y)=t} \mathsf{D}(Q_X, Q_Y\|\pi_{XY}), \quad (10.1)$$

and

$$\overline{\varphi}(s,t) := \max_{Q_X, Q_Y : D(Q_X\|\pi_X)=s, D(Q_Y\|\pi_Y)=t} \mathsf{D}(Q_X, Q_Y\|\pi_{XY}). \quad (10.2)$$

Fix $(\alpha, \beta) \in [0, \alpha_{\max}(\pi_X)] \times [0, \beta_{\max}(\pi_Y)]$. Recall that the upper bound on the forward LD exponent, previously defined in (8.24), is

$$\underline{\Upsilon}_{\mathrm{LD}}(\alpha, \beta) = \min_{Q_X, Q_Y : D(Q_X \| \pi_X) \geq \alpha, D(Q_Y \| \pi_Y) \geq \beta} \mathsf{D}(Q_X, Q_Y \| \pi_{XY}), \quad (10.3)$$

and the lower bound on the reverse LD exponent, is

$$\overline{\Upsilon}_{\mathrm{LD}}(\alpha, \beta) = \max_{Q_X, Q_Y : D(Q_X \| \pi_X) \leq \alpha, D(Q_Y \| \pi_Y) \leq \beta} \mathsf{D}(Q_X, Q_Y \| \pi_{XY}). \quad (10.4)$$

Based on the functions presented in Definition 10.2, we may modify the definitions of $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ for the DSBS to a source $(X, Y) \sim \pi_{XY}$ defined on a finite alphabet as follows

$$\underline{\Upsilon}(\alpha, \beta) := \min_{s \geq \alpha, t \geq \beta} \mathbb{L}[\underline{\varphi}](s, t) \quad \text{and} \quad (10.5)$$

$$\overline{\Upsilon}(\alpha, \beta) := \max_{s \leq \alpha, t \leq \beta} \mathbb{U}[\overline{\varphi}](s, t). \quad (10.6)$$

Note that $\underline{\Upsilon}_{\mathrm{LD}}$ and $\overline{\Upsilon}_{\mathrm{LD}}$ in (10.3) and (10.4) may not be convex and concave respectively for arbitrary π_{XY} (see the discussion following Theorem 8.11). Hence, in the modified definitions in (10.5) and (10.6), we take the lower convex envelope for $\underline{\varphi}$ and the upper concave envelope for $\overline{\varphi}$. With these operations, $\underline{\Upsilon}(\alpha, \beta)$ is convex and nondecreasing in (α, β), and $\overline{\Upsilon}(\alpha, \beta)$ is concave and nondecreasing in (α, β).[1] Henceforth, we omit the subscript LD in (10.5) and (10.6).

Before presenting the next definition, we recall the definition of θ_q in (9.57) as

$$\theta_q(Q_X, Q_Y) := \mathsf{D}(Q_X, Q_Y \| \pi_{XY}) - \frac{D(Q_Y \| \pi_Y)}{q'},$$

but now, instead of being a DSBS, π_{XY} is an arbitrary distribution defined on the finite set $\mathcal{X} \times \mathcal{Y}$.

Definition 10.3. For $q \geq 1$ and for $s \in [0, \alpha_{\max}(\pi_X)]$, define

$$\varphi_q(s) := \min_{Q_X, Q_Y : D(Q_X \| \pi_X) = s} \theta_q(Q_X, Q_Y),$$

[1] We say a function of two variables is *nondecreasing* if it is nondecreasing in one argument when the other is fixed.

and for $q \in (-\infty, 1)\backslash\{0\}$, define

$$\varphi_q(s) := \begin{cases} \max\limits_{Q_X : D(Q_X\|\pi_X)=s} \min\limits_{Q_Y} \theta_q(Q_X, Q_Y) & 0 < q < 1 \\ \max\limits_{Q_X : D(Q_X\|\pi_X)=s} \max\limits_{Q_Y} \theta_q(Q_X, Q_Y) & q < 0 \end{cases}.$$

We denote Υ_q as the lower convex envelope of φ_q for $q \geq 1$ and the upper concave envelope of φ_q for $q \in (-\infty, 1)\backslash\{0\}$. Specifically, for $\alpha \in [0, \alpha_{\max}(\pi_X)]$,

$$\Upsilon_q(\alpha) := \begin{cases} \min\limits_{s \geq \alpha} \mathbb{L}[\varphi_q](s) & q \geq 1 \\ \max\limits_{s \leq \alpha} \mathbb{U}[\varphi_q](s) & q \in (-\infty, 1)\backslash\{0\} \end{cases}.$$

Observe that Υ_q is an alternative representation of $\Upsilon_{q,\mathrm{LD}}$ defined in (9.58) and (9.59). By definition, $\Upsilon_q(\alpha)$ is convex and nondecreasing in α for each $q \geq 1$, and concave and nondecreasing in α for each $q \in (-\infty, 1)\backslash\{0\}$.

To avoid having to deal with the undefined arithmetic operation $\infty - \infty$, we adopt the following convention.

Convention 10.1. When we write an optimization problem with distributions as decision variables, we implicitly require that the distributions satisfy the condition that all the integrals and relative entropies (appearing in the constraints and the objective function) to be *finite*. Otherwise, the value of the optimization problem is set to $+\infty$ if it is an infimization, and $-\infty$ if it is a supremization.

To keep notation uncluttered, we also adopt the following convention.

Convention 10.2. When we write an optimization over functions f and g, we implicitly require these functions to be *nonnegative*.

10.2 Classic Hypercontractivity and Brascamp–Lieb Inequalities

In this section, we introduce a class of functional inequalities, known as *Brascamp–Lieb (BL) inequalities*. We also review the well-known Hölder and hypercontractivity inequalities which are special cases of the BL inequalities. We introduced the hypercontractivity inequalities in the context of of the DSBS in Section 8.3.1. In contrast, here we study these inequalities for *arbitrary* sources defined on finite alphabets.

10.2.1 Hölder and Hypercontractivity Inequalities

We review the well-known *forward* and *reverse Hölder inequalities* here. Given a joint distribution π_{XY} and an extended real number $p \in \mathbb{R} \cup \{\pm\infty\}$, for any pair of nonnegative functions (f, g), the forward and reverse Hölder inequalities are respectively

$$\langle f, g \rangle \le \|f\|_p \|g\|_q \quad \text{if} \quad p \ge 1 \quad \text{and} \tag{10.7}$$

$$\langle f, g \rangle \ge \|f\|_p \|g\|_q \quad \text{if} \quad p \le 1, \tag{10.8}$$

where q is the Hölder conjugate of p. Since the (pseudo) L^q-norms $\|\cdot\|_q$ are nondecreasing in $q \in \mathbb{R} \cup \{\pm\infty\}$, the scalar q in (10.7) can be replaced by any $q \ge p'$. Similarly, q in (10.8) can be replaced by any $q \le p'$.

If $X = Y$ (i.e., $P_{Y|X}(\cdot|x)$ places all its mass at x for every $x \in \mathcal{X}$), then the forward and reverse Hölder inequalities are sharp in the following sense. If $p > 1$, then (10.7) becomes an equality if and only if $|f|^p$ and $g^{p'}$ are *linearly dependent*, i.e., there exist real numbers $a, b \ge 0$, not both zero, such that $a|f|^p = b|g|^{p'}$ holds (π_X-almost everywhere). If $p < 1$, $\langle f, g \rangle < \infty$ and $\|g\|_{p'} > 0$, then (10.8) is an equality if and only if the equality $|f|^p = a|g|^{p'}$ holds (π_X-almost everywhere) for some $a \ge 0$. Moreover, for the case $X = Y$, the parameters (p, q) in (10.7) and (10.8) cannot be improved in the sense that given $p \ge 1$, for any $q < p'$, there exists a pair of (f, g) that violates (10.7); similarly, given $p \le 1$, for any $q > p'$, there exists a pair of (f, g) that violates (10.8).

However, the Hölder inequalities are not sharp in general when $X \ne Y$ (which is the case of interest to us). If $X \ne Y$, then the parameters (p, q) in the Hölder inequalities can be "improved". Specifically, given a joint distribution π_{XY} and $p \ge 1$, we are interested in how *small* $q \in \mathbb{R} \cup \{\pm\infty\}$ can be such that for *any* nonnegative functions $f : \mathcal{X} \to [0, \infty)$ and $g : \mathcal{Y} \to [0, \infty)$, it holds that

$$\langle f, g \rangle \le \|f\|_p \|g\|_q. \tag{10.9}$$

By the forward Hölder inequality, the infimum of all such q's is at most p', the Hölder conjugate of p. Similarly, given $p \le 1$, we are interested in

how *large* $q \in \mathbb{R} \cup \{\pm\infty\}$ can be such that for any nonnegative functions f and g, it holds that

$$\langle f, g \rangle \geq \|f\|_p \|g\|_q. \tag{10.10}$$

For this case, the supremum of all such q's is at least p'. Inequalities (10.9) and (10.10) for the case $X \neq Y$ are respectively termed the *forward* and *reverse hypercontractivity inequalities*, since the forward and reverse Hölder inequalities in (10.7) and (10.8) respectively are regarded as the (usual) *contractivity* inequalities, and inequalities (10.9) and (10.10) with improved (p, q) are *strengthenings* of the forward and reverse Hölder inequalities.

Inequalities (10.9) and (10.10) motivate the following definitions.

Definition 10.4. The *forward* and *reverse hypercontractivity regions* [15], [112] are respectively defined as

$$\mathcal{R}_{\mathrm{FH}}(\pi_{XY}) := \{(p, q) \in [1, \infty)^2 : \langle f, g \rangle \leq \|f\|_p \|g\|_q, \ \forall\, f, g \geq 0\}$$

and

$$\mathcal{R}_{\mathrm{RH}}(\pi_{XY}) := \{(p, q) \in (-\infty, 1]^2 : \langle f, g \rangle \geq \|f\|_p \|g\|_q, \ \forall\, f, g \geq 0\}.$$

By definition, these two regions correspond to the sets of parameters (p, q) for which the forward or reverse hypercontractivity inequalities in (10.9) and (10.10) hold. We remark that the notion of *hypercontractivity ribbons* was introduced in Anantharam *et al.* [5, Eqn. (6.117)] and Kamath [92], prior to the hypercontractivity regions being introduced in Beigi and Gohari [15] and Liu [112]. The hypercontractivity ribbons correspond to the hypercontractivity regions apart from the exclusions of the Hölder regions $\{(p, q) \in [1, \infty)^2 : q \geq p'\}$ and $\{(p, q) \in (-\infty, 1]^2 : q \leq p'\}$, and that the Hölder conjugate of q is taken.

We can write $\mathcal{R}_{\mathrm{RH}}(\pi_{XY})$ as the disjoint union of four sets

$$\mathcal{R}^{++}_{\mathrm{RH}}(\pi_{XY}) := (0, 1]^2 \cap \mathcal{R}_{\mathrm{RH}}(\pi_{XY}), \tag{10.11}$$

$$\mathcal{R}^{+-}_{\mathrm{RH}}(\pi_{XY}) := ((0, 1] \times (-\infty, 0)) \cap \mathcal{R}_{\mathrm{RH}}(\pi_{XY}), \tag{10.12}$$

$$\mathcal{R}^{-+}_{\mathrm{RH}}(\pi_{XY}) := ((-\infty, 0) \times (0, 1]) \cap \mathcal{R}_{\mathrm{RH}}(\pi_{XY}), \quad \text{and} \tag{10.13}$$

$$\mathcal{R}^{--}_{\mathrm{RH}}(\pi_{XY}) := (-\infty, 0]^2.$$

The forward hypercontractivity region and the first three subregions of the reverse hypercontractivity region in (10.11), (10.12), and (10.13) admit the following information-theoretic characterizations; see [1], [16], [33], [92], [112], [192].

Theorem 10.1 (Information-theoretic characterizations of hypercontractivity regions)**.** The forward hypercontractivity region $\mathcal{R}_{\mathrm{FH}}(\pi_{XY})$ can be expressed in terms of the minimal relative entropy as the set of $(p, q) \in [1, \infty)^2$ such that

$$\mathsf{D}(Q_X, Q_Y \| \pi_{XY}) \geq \frac{1}{p} D(Q_X \| \pi_X) + \frac{1}{q} D(Q_Y \| \pi_Y).$$

In addition, $\mathcal{R}^{++}_{\mathrm{RH}}(\pi_{XY})$ is the set of all $(p, q) \in (0, 1]^2$ such that

$$\mathsf{D}(Q_X, Q_Y \| \pi_{XY}) \leq \frac{1}{p} D(Q_X \| \pi_X) + \frac{1}{q} D(Q_Y \| \pi_Y).$$

Finally, $\mathcal{R}^{+-}_{\mathrm{RH}}(\pi_{XY})$ is the set of all $(p, q) \in (0, 1] \times (-\infty, 0)$ such that

$$\min_{Q_Y} \left\{ \mathsf{D}(Q_X, Q_Y \| \pi_{XY}) - \frac{1}{q} D(Q_Y \| \pi_Y) \right\} \leq \frac{1}{p} (Q_X \| \pi_X). \tag{10.14}$$

By symmetry, $\mathcal{R}^{-+}_{\mathrm{RH}}(\pi_{XY})$ can be characterized in an analogous manner to $\mathcal{R}^{+-}_{\mathrm{RH}}(\pi_{XY})$ in (10.14). The proof of Theorem 10.1 is provided in Section 10.2.2, since it is a special case of the information-theoretic characterizations of the BL inequalities, which we present therein. Theorem 10.1 can be specialized to Theorem 8.4 (the two function version of the hypercontractivity inequalities for the DSBS); see [127], [128].

Hypercontractivity inequalities were investigated in [1], [25]–[27], [69], [95], [125], [152] among others. Information-theoretic characterizations of the hypercontractivity (and BL) inequalities can be traced back to the seminal work of Ahlswede and Gács [1] in which, instead of the hypercontractivity regions, the hypercontractivity *constants* (which are quantities induced by the hypercontractivity regions) were characterized in terms of relative entropies. The information-theoretic characterization of the forward hypercontractivity region is implied by the information-theoretic characterization of the forward BL inequalities on Euclidean spaces in Carlen and Cordero-Erausquin [33]; this was independently discovered later by Nair [126] in the case of finite alphabets.

An information-theoretic characterization of $\mathcal{R}_{\mathrm{RH}}^{++}(\pi_{XY})$ for finite alphabets was provided by Kamath [92]. Subsequently, an information-theoretic characterization of the entire reverse hypercontractivity region for finite alphabets was shown by Beigi and Nair [16]. Extensions of these characterizations to Polish spaces were studied by Liu [112] using a minimax theorem known as the *Fenchel–Rockafellar duality*.

As a consequence of Definitions 10.2, 10.3, and Theorem 10.1, the regions $\mathcal{R}_{\mathrm{FH}}(\pi_{XY})$, $\mathcal{R}_{\mathrm{RH}}^{++}(\pi_{XY})$, and $\mathcal{R}_{\mathrm{RH}}^{+-}(\pi_{XY})$ also admit the following equivalent characterizations:

$$\begin{aligned}
\mathcal{R}_{\mathrm{FH}}(\pi_{XY}) &= \Big\{(p,q) \in [1,\infty)^2 : \underline{\varphi}(\alpha,\beta) \geq \frac{\alpha}{p} + \frac{\beta}{q},\ \forall\, \alpha,\beta \geq 0\Big\} \\
&= \Big\{(p,q) \in [1,\infty)^2 : \underline{\Upsilon}(\alpha,\beta) \geq \frac{\alpha}{p} + \frac{\beta}{q},\ \forall\, \alpha,\beta \geq 0\Big\}, \\
\mathcal{R}_{\mathrm{RH}}^{++}(\pi_{XY}) &= \Big\{(p,q) \in (0,1]^2 : \overline{\varphi}(\alpha,\beta) \leq \frac{\alpha}{p} + \frac{\beta}{q},\ \forall\, \alpha,\beta \geq 0\Big\} \\
&= \Big\{(p,q) \in (0,1]^2 : \overline{\Upsilon}(\alpha,\beta) \leq \frac{\alpha}{p} + \frac{\beta}{q},\ \forall\, \alpha,\beta \geq 0\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_{\mathrm{RH}}^{+-}(\pi_{XY}) &= \Big\{(p,q) \in (0,1] \times (-\infty,0) : \varphi_{q'}(\alpha) \leq \frac{\alpha}{p},\ \forall\, \alpha \geq 0\Big\} \\
&= \Big\{(p,q) \in (0,1] \times (-\infty,0) : \Upsilon_{q'}(\alpha) \leq \frac{\alpha}{p},\ \forall\, \alpha \geq 0\Big\},
\end{aligned}$$

where q' is the Hölder conjugate of q.

10.2.2 Brascamp–Lieb Inequalities

The Brascamp–Lieb (BL) inequalities constitute a class of inequalities that generalizes the families of Hölder and hypercontractivity inequalities. The *forward* and *reverse BL inequalities* are defined as follows. Given a distribution π_{XY} and $p, q \in \mathbb{R}$, for any pair of nonnegative functions $f : \mathcal{X} \to [0,\infty)$ and $g : \mathcal{Y} \to [0,\infty)$,

$$\langle f, g\rangle \leq \overline{C}\, \|f\|_p \|g\|_q \quad \text{and} \tag{10.15}$$

$$\langle f, g\rangle \geq \underline{C}\, \|f\|_p \|g\|_q, \tag{10.16}$$

where $\overline{C} = \overline{C}_{p,q}$ and $\underline{C} = \underline{C}_{p,q}$ depend only on p and q given the distribution π_{XY}. The hypercontractivity inequalities in (10.9) and (10.10)

correspond to the BL inequalities with $\overline{C} = 1$ in (10.15) and $\underline{C} = 1$ in (10.16) respectively.

The forward version of the BL inequalities in (10.15) was originally studied in the 1970s by Brascamp and Lieb [29], who were motivated by problems in particle physics. The reverse version in (10.16) was initially studied by Barthe [12]. In fact, the inequalities in (10.15) and (10.16) are special cases of the original forward and reverse BL inequalities. We only discuss these special cases.

Definition 10.5. The (optimal) *forward* and *reverse BL constants* are respectively defined as

$$\overline{C}^*_{p,q}(X;Y) := \sup_{f,g:\|f\|_p\|g\|_q>0} \frac{\langle f,g\rangle}{\|f\|_p\|g\|_q} \quad \text{and}$$
$$\underline{C}^*_{p,q}(X;Y) := \inf_{f,g:\|f\|_p\|g\|_q>0} \frac{\langle f,g\rangle}{\|f\|_p\|g\|_q}.$$

Additionally, define the *forward* and *reverse BL exponents* respectively as

$$\underline{\Lambda}_{p,q}(X;Y) := -\log \overline{C}^*_{p,q}(X;Y) \quad \text{and}$$
$$\overline{\Lambda}_{p,q}(X;Y) := -\log \underline{C}^*_{p,q}(X;Y).$$

It is well-known that the forward and reverse BL exponents possess the important tensorization and the data processing properties.

Lemma 10.2 (Tensorization)**.** Let $(X^n, Y^n) = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ be a collection of pairs of random variables that are mutually independent. Then

$$\underline{\Lambda}_{p,q}(X^n;Y^n) = \sum_{i=1}^n \underline{\Lambda}_{p,q}(X_i;Y_i) \quad \text{and} \tag{10.17}$$
$$\overline{\Lambda}_{p,q}(X^n;Y^n) = \sum_{i=1}^n \overline{\Lambda}_{p,q}(X_i;Y_i). \tag{10.18}$$

Proof. The proof here is due to Beigi and Nair [16] and is based on applying the one-dimensional BL inequality in (10.15) to each pair of random variables iteratively. To prove (10.17), it suffices to show that if for each $i \in [n]$, there exist a constant $\overline{C}_i$ such that $\langle f_i, g_i\rangle \leq$

$\overline{C}_i\|f_i\|_p\|g_i\|_q$ holds for all nonnegative f_i and g_i defined on $\mathcal{X}$ and $\mathcal{Y}$, then $\langle f, g\rangle \le \overline{C}\|f\|_p\|g\|_q$ holds for all nonnegative f and g defined on $\mathcal{X}^n$ and $\mathcal{Y}^n$, where $\overline{C} = \prod_{i=1}^n \overline{C}_i$. This point can be shown as follows:

$$\begin{aligned}\langle f, g\rangle &= \mathbb{E}_{X^{n-1},Y^{n-1}}\left[\mathbb{E}_{X_n,Y_n}[f(X^n)g(Y^n) \mid X^{n-1}, Y^{n-1}]\right] \\ &\le \overline{C}_n\mathbb{E}_{X^{n-1},Y^{n-1}}\left[\|f(X^{n-1},\cdot)\|_p\|g(Y^{n-1},\cdot)\|_q\right] \\ &\le \overline{C}_n\overline{C}_{n-1}\mathbb{E}_{X^{n-2},Y^{n-2}}\left[\|f(X^{n-2},\cdot)\|_p\|g(Y^{n-2},\cdot)\|_q\right] \\ &\;\;\vdots \\ &\le \overline{C}\|f\|_p\|g\|_q.\end{aligned}$$

Hence, we have (10.17). The inequality in (10.18) follows similarly. □

Lemma 10.3 (Data processing inequalities)**.** Assume random variables U, X, Y, and V form a Markov chain $U - X - Y - V$ in this order. Then for $p, q \ge 1$,

$$\underline{\Lambda}_{p,q}(X;Y) \le \underline{\Lambda}_{p,q}(U;V), \tag{10.19}$$

and for $p, q \le 1$,

$$\overline{\Lambda}_{p,q}(X;Y) \ge \overline{\Lambda}_{p,q}(U;V). \tag{10.20}$$

Moreover, if U and V are deterministic functions of X and Y respectively, then the two inequalities hold for all $p, q \in \mathbb{R}$.

Proof. For any $f : \mathcal{U} \to [0,\infty)$ and $g : \mathcal{V} \to [0,\infty)$, let $\hat{f} : x \in \mathcal{X} \mapsto \mathbb{E}[f(U) \mid X = x]$ and $\hat{g} : y \in \mathcal{Y} \mapsto \mathbb{E}[g(V) \mid Y = y]$. Then we have $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, and by Jensen's inequality, $\|f\|_p \ge \|\hat{f}\|_p$ and $\|g\|_q \ge \|\hat{g}\|_q$ for $p, q \ge 1$, and the directions of these two inequalities are reversed for $p, q \le 1$. These facts establish (10.19) and (10.20). □

Similarly to the hypercontractivity regions (see Definition 10.4 and Lemma 10.1), the BL exponents also admit rather natural information-theoretic characterizations. Define the function

$$\begin{aligned}\phi(Q_X, Q_Y) := \inf_{R_X,R_Y}\Big\{&\mathsf{D}(R_X, R_Y\|\pi_{XY}) + \frac{1}{p}D(R_X\|Q_X) - \frac{1}{p}D(R_X\|\pi_X) \\ &+ \frac{1}{q}D(R_Y\|Q_Y) - \frac{1}{q}D(R_Y\|\pi_Y)\Big\},\end{aligned} \tag{10.21}$$

where according to Convention 10.1, the infimization is taken over all pairs of distributions $(R_X, R_Y) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$ such that all the relative entropies in the objective function are finite. Then we have the following information-theoretic characterizations of the forward and reverse BL exponents.

Proposition 10.1. For $p, q \in \mathbb{R}\backslash\{0\}$, if $(X, Y) \sim \pi_{XY}$, then

$$\underline{\Lambda}_{p,q}(X;Y) = \inf_{Q_X, Q_Y} \phi(Q_X, Q_Y) \quad \text{and} \tag{10.22}$$
$$\overline{\Lambda}_{p,q}(X;Y) = \sup_{Q_X, Q_Y} \phi(Q_X, Q_Y).$$

Proof. The proof leverages the following "duality" lemma.

Lemma 10.4 (Duality of Relative Entropy)**.** Let $\{P_i\}_{i=1}^n$ be n probability mass functions on a finite set $\mathcal{X}$. Let $\{s_i\}_{i=1}^n \subset \mathbb{R} \setminus \{0\}$ be nonzero real numbers such that $\sum_{i=1}^n s_i = 1$. Let $c : \mathcal{X} \to \mathbb{R}$ be a function. Define

$$\beta := \sum_{x \in \mathcal{X}} 2^{-c(x)} \Big(\prod_{i=1}^n P_i(x)^{s_i} \Big).$$

Then we have[2]

$$-\log \beta = \inf_{Q \ll P_i, \forall\, i \in [n]} \left\{ \sum_{i=1}^n s_i \, D(Q \| P_i) + \mathbb{E}_Q[c(X)] \right\}. \tag{10.23}$$

Moreover, if $0 < \beta < \infty$, the infimization in (10.23) is uniquely attained by the distribution

$$Q^*(x) = \frac{2^{-c(x)}}{\beta} \Big(\prod_{i=1}^n P_i(x)^{s_i} \Big) \quad \text{for all } x \in \mathcal{X}.$$

This lemma was stated by Shayevitz [154]. It can be proved by using Lagrange multipliers. The generalization of this lemma to arbitrary measurable spaces can be proven by using the nonnegativity of the relative entropy; see [112, Theorem 2.2.3] or [192].

We may assume, by homogeneity, that $\|f\|_p = \|g\|_q = 1$. Without loss of generality, we may also assume, due to Assumption 10.1, that $\operatorname{supp}(f) \subset \operatorname{supp}(\pi_X)$ and $\operatorname{supp}(g) \subset \operatorname{supp}(\pi_Y)$. Hence, we can write

$$f(x)^p = \frac{Q_X(x)}{\pi_X(x)} \quad \text{and} \quad g(y)^q = \frac{Q_Y(y)}{\pi_Y(y)}, \tag{10.24}$$

[2] We adopt the convention $\inf \emptyset = \infty$, $0 \cdot \infty = 0$, and $0^s = \infty$ for $s < 0$.

for some probability mass functions Q_X and Q_Y. Moreover, since f and g are finite on their supports, Q_X and π_X are mutually absolutely continuous if $p < 0$, and Q_Y and π_Y are mutually absolutely continuous if $q < 0$. From (10.24), we see that

$$\langle f, g\rangle = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \pi_{XY}(x,y)\Big(\frac{Q_X(x)}{\pi_X(x)}\Big)^{1/p}\Big(\frac{Q_Y(y)}{\pi_Y(y)}\Big)^{1/q}. \tag{10.25}$$

Now substituting (10.25) into the definitions of $\underline{\Lambda}_{p,q}$ and $\overline{\Lambda}_{p,q}$, and using Lemma 10.4 (with the identifications $c \leftarrow 0$, $s_1 \leftarrow 1$, $s_2 \leftarrow 1/p$, $s_3 \leftarrow -1/p$, $s_4 \leftarrow 1/q$, $s_5 \leftarrow -1/q$, and $P_1 \leftarrow \pi_{XY}$, $P_2 \leftarrow Q_X\pi_{Y|X}$, $P_3 \leftarrow \pi_{XY}$, $P_4 \leftarrow Q_Y\pi_{X|Y}$, $P_5 \leftarrow \pi_{XY}$), we obtain Proposition 10.1. □

Define the following linear combination of relative entropies

$$\theta(Q_X, Q_Y) := \mathsf{D}(Q_X, Q_Y\|\pi_{XY}) - \frac{1}{p}\,D(Q_X\|\pi_X) - \frac{1}{q}\,D(Q_Y\|\pi_Y).$$

By using the tensorization property, the BL exponents also can be written in the following alternative information-theoretic forms in terms of variational characterizations of $\theta(Q_X, Q_Y)$.

Theorem 10.5. For $p, q \in \mathbb{R}\backslash\{0\}$, if $(X, Y) \sim \pi_{XY}$, then

$$\underline{\Lambda}_{p,q}(X;Y) = \begin{cases} \inf\limits_{Q_X,Q_Y} \theta(Q_X, Q_Y) & p, q > 0 \\ -\infty & p < 0 \text{ or } q < 0 \end{cases} \tag{10.26}$$

and

$$\overline{\Lambda}_{p,q}(X;Y) = \begin{cases} \sup\limits_{Q_X,Q_Y} \theta(Q_X, Q_Y) & p, q > 0 \\ \sup\limits_{Q_X}\inf\limits_{Q_Y} \theta(Q_X, Q_Y) & q < 0 < p \\ \sup\limits_{Q_Y}\inf\limits_{Q_X} \theta(Q_X, Q_Y) & p < 0 < q \\ 0 & p, q < 0 \end{cases}. \tag{10.27}$$

For Euclidean spaces, the forward part of this theorem, i.e., (10.26), was derived in Carlen and Cordero-Erausquin [33]. The reverse part of this theorem, i.e., (10.27), for finite alphabets was derived in Beigi and Nair [16] for all $p, q \neq 0$, and also by Liu *et al.* [113] for $p, q > 0$.

The characterizations in (10.26) and (10.27) are consistent with the ones for the hypercontractivity regions given in Theorem 10.1. This can be seen observing that $\underline{\Lambda}_{p,q} \geq 1$ if and only if $(p,q) \in \mathcal{R}_{\mathrm{FH}}(\pi_{XY})$, and $\underline{\Lambda}_{p,q} \leq 1$ if and only if $(p,q) \in \mathcal{R}_{\mathrm{RH}}(\pi_{XY})$. Hence, Theorem 10.1 is indeed a consequence of Theorem 10.5.

Proof of Theorem 10.5. The characterization in (10.26) follows directly from (10.22) by swapping the two infima. We now prove the characterization in (10.27). We first consider the case of $p, q > 0$. On one hand, by setting (R_X, R_Y) in (10.21) to be (Q_X, Q_Y), we have that $\phi(Q_X, Q_Y) \leq \theta(Q_X, Q_Y)$. Hence,

$$\overline{\Lambda}_{p,q}(X;Y) \leq \sup_{Q_X, Q_Y} \theta(Q_X, Q_Y). \tag{10.28}$$

On the other hand, by the tensorization property stated in (10.18) in Lemma 10.2, for $(X^n, Y^n) \sim \pi_{XY}^n$,

$$\begin{aligned}
\overline{\Lambda}_{p,q}(X;Y) &= \frac{1}{n}\overline{\Lambda}_{p,q}(X^n, Y^n) \\
&= \sup_{f,g:\|f\|_p\|g\|_q>0} -\frac{1}{n}\log\frac{\langle f, g\rangle}{\|f\|_p\|g\|_q} \\
&\geq \max_{\mathcal{A}_n\subset\mathcal{X}^n, \mathcal{B}_n\subset\mathcal{Y}^n} -\frac{1}{n}\log\frac{\pi_{XY}^n(\mathcal{A}_n\times\mathcal{B}_n)}{\pi_X^n(\mathcal{A}_n)^{1/p}\pi_Y^n(\mathcal{B}_n)^{1/q}},
\end{aligned} \tag{10.29}$$

where in the last line, we restrict f and g to be the indicators of two non-empty sets $\mathcal{A}_n \subset \mathcal{X}^n$ and $\mathcal{B}_n \subset \mathcal{Y}^n$, respectively.

To further lower bound (10.29), we take $(\mathcal{A}_n, \mathcal{B}_n)$ therein to be a pair of type classes $(\mathcal{T}_{T_X^{(n)}}, \mathcal{T}_{T_Y^{(n)}})$ in which the sequence of pairs of types $\{(T_X^{(n)}, T_X^{(n)})\}_{n\in\mathbb{N}}$ converges to some pair of distributions (Q_X, Q_Y) as $n \to \infty$. Then, by Sanov's theorem (see Theorem 1.1),

$$\lim_{n\to\infty} -\frac{1}{n}\log\frac{\pi_{XY}^n\big(\mathcal{T}_{T_X^{(n)}}\times\mathcal{T}_{T_Y^{(n)}}\big)}{\pi_X^n\big(\mathcal{T}_{T_X^{(n)}}\big)^{1/p}\pi_Y^n\big(\mathcal{T}_{T_Y^{(n)}}\big)^{1/q}} = \theta(Q_X, Q_Y). \tag{10.30}$$

Hence, we obtain $\overline{\Lambda}_{p,q}(X;Y) \geq \theta(Q_X, Q_Y)$. Since (Q_X, Q_Y) is arbitrary, we have $\overline{\Lambda}_{p,q}(X;Y) \geq \sup_{Q_X,Q_Y}\theta(Q_X, Q_Y)$. Therefore, (10.27) holds.

We omit the proofs for other cases, since they are similar to the above argument. □

An interesting observation arising from this proof is the following. For $p, q > 0$, by combining (10.28) and (10.29), we obtain

$$\max_{\mathcal{A}_n \subset \mathcal{X}^n, \mathcal{B}_n \subset \mathcal{Y}^n} -\frac{1}{n} \log \pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n) + \frac{1}{np} \log \pi_X^n(\mathcal{A}_n) + \frac{1}{nq} \log \pi_Y^n(\mathcal{B}_n) \leq \sup_{Q_X, Q_Y} \theta(Q_X, Q_Y). \tag{10.31}$$

Furthermore, as shown in (10.30), by appealing to Sanov's theorem, this inequality is *asymptotically tight* (which means that as $n \to \infty$, the limits of the left- and right-hand sides are equal). Similarly, for $p, q > 0$, one can observe that

$$\min_{\mathcal{A}_n \subset \mathcal{X}^n, \mathcal{B}_n \subset \mathcal{Y}^n} -\frac{1}{n} \log \pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n) + \frac{1}{np} \log \pi_X^n(\mathcal{A}_n) + \frac{1}{nq} \log \pi_Y^n(\mathcal{B}_n) \geq \inf_{Q_X, Q_Y} \theta(Q_X, Q_Y), \tag{10.32}$$

and this inequality is also asymptotically tight by Sanov's theorem [49].

As a consequence of (10.31) and (10.32), we find that certain sequences of $\{0, 1\}$-valued functions attain the BL exponents. Hence, a BL inequality holds for all nonnegative functions if and only if any of its multi-dimensional extensions hold for any $\{0, 1\}$-valued functions. In addition to the set of $\{0, 1\}$-valued functions, one can also use the following construction of functions to assert the (asymptotic) optimality of a BL inequality. We can first identify a optimal pair (f^*, g^*) for the one-dimensional case. By the tensorization property of the BL exponents (Theorem 10.2), the n-fold product of (f^*, g^*) also constitutes an optimal pair that allows us to assert the optimality of a BL inequality. In contrast, the asymptotic optimality of $\{0, 1\}$-valued functions is advantageous in our quest to prove the strong SSE theorem (Theorem 8.11) as will be done in Section 10.3.

10.2.3 Single-Function Versions

The BL inequalities discussed in Section 10.2.2 involve *two* nonnegative functions. In the literature, there exist *single-function* versions of BL inequalities and they have been shown to be equivalent to their two-function counterparts (as was discussed in the context of the DSBS in Section 8.3.2). We now introduce the single-function versions of BL

inequalities. First recall from (8.55) that the *conditional expectation operator* induced by $\pi_{X|Y}$ is the operator that maps a function $f : \mathcal{X} \to \mathbb{R}$ to the function

$$y \in \mathcal{Y} \mapsto \pi_{X|Y=y}(f) := \mathbb{E}[f(X) \,|\, Y = y] = \sum_{x \in \mathcal{X}} \pi_{X|Y}(x|y) f(x).$$

Then, given a joint distribution π_{XY} and two real numbers p and q, for any nonnegative function $f : \mathcal{X} \to [0, \infty)$, the single-function versions of the BL inequalities read

$$\|\pi_{X|Y}(f)\|_q \le \overline{C}\, \|f\|_p \quad \text{and} \tag{10.33}$$

$$\|\pi_{X|Y}(f)\|_q \ge \underline{C}\, \|f\|_p \tag{10.34}$$

for some constants $\overline{C}$ and $\underline{C}$.

We remark that (10.33) and (10.34) are in fact equivalent to the *strong data processing inequalities* for the Rényi divergence [142]. The latter concerns the tradeoff between $D_p(Q_X \| \pi_X)$ and $D_q(Q_Y \| \pi_Y)$, where Q_Y represents the output distribution induced by the input distribution Q_X and the stochastic kernel $\pi_{Y|X}$, i.e., $Q_X \to \pi_{Y|X} \to Q_Y$. The equivalence follows since we can set $f = Q_X / \pi_X$ and observe that

$$\log \|f\|_p = \frac{1}{p'} D_p(Q_X \| \pi_X) \tag{10.35}$$

and

$$\begin{aligned} \log \|\pi_{X|Y}(f)\|_q &= \frac{1}{q} \log \sum_{y \in \mathcal{Y}} \Big(\sum_{x \in \mathcal{X}} \frac{Q_X(x)}{\pi_X(x)} \pi_{X|Y}(x|y) \Big)^q \pi_Y(y) \\ &= \frac{1}{q} \log \sum_{y \in \mathcal{Y}} \Big(\frac{Q_Y(y)}{\pi_Y(y)} \Big)^q \pi_Y(y) \\ &= \frac{1}{q'} D_q(Q_Y \| \pi_Y). \end{aligned}$$

For more details, see the papers by Raginsky [142] and Yu [192].

The promised equivalence between the single- and two-function versions of the BL inequalities is formalized in the following proposition.

Proposition 10.2. Inequality (10.33) for $q \ge 1$ holds if and only if (10.15) holds but with q in the latter replaced by its Hölder conjugate $q' = \frac{q}{q-1}$. Similarly, inequality (10.34) for $q \le 1$ holds if and only if (10.16) holds but with q in the latter replaced by its Hölder conjugate q'.

Proof. By Hölder's inequality, for any $\hat{g} : \mathcal{Y} \to [0, \infty)$, it holds that

$$\|\hat{g}\|_q = \begin{cases} \sup\limits_{g:\|g\|_{q'}>0} \dfrac{\langle \hat{g}, g\rangle}{\|g\|_{q'}} & q \geq 1 \\ \inf\limits_{g:\|g\|_{q'}>0} \dfrac{\langle \hat{g}, g\rangle}{\|g\|_{q'}} & q \leq 1 \end{cases},$$

where $1' = \infty$ and $1' = -\infty$ for the first and second clauses respectively. Setting $\hat{g}$ to be $\pi_{X|Y}(f)$, we obtain the following equivalences: For $q \geq 1$,

$$\sup_{f:\|f\|_p>0} \frac{\|\pi_{X|Y}(f)\|_q}{\|f\|_p} = \sup_{(f,g):\|f\|_p>0,\|g\|_{q'}>0} \frac{\langle f, g\rangle}{\|f\|_p\|g\|_{q'}}, \quad (10.36)$$

and for $q \leq 1$,

$$\inf_{f:\|f\|_p>0} \frac{\|\pi_{X|Y}(f)\|_q}{\|f\|_p} = \inf_{(f,g):\|f\|_p>0,\|g\|_{q'}>0} \frac{\langle f, g\rangle}{\|f\|_p\|g\|_{q'}}. \quad (10.37)$$

By the equivalence in (10.36), for $q \geq 1$, the single-function version of BL inequality in (10.33) is equivalent to the two-function version in (10.15) with $\overline{C}$ and p unchanged but with q replaced by its Hölder conjugate q'. Similarly, for $q \leq 1$, by the equivalence in (10.37), the single-function version of BL inequality in (10.34) is equivalent to the two-function version in (10.16) with $\underline{C}$ and p unchanged but with q replaced by q'. □

10.3 Connections to the NICD Problem and q-Stability

As observed in the proof of Theorem 10.5, certain sequences of $\{0,1\}$-valued functions attain the BL exponents. We now provide a detailed discussion on this observation. We also discuss the connections between the BL exponents and the NICD problem (Section 8), as well as the q-stability problem (Section 9).

Recall the general version of the strong SSE theorem (Theorem 8.11) and the general version of the strong q-stability theorem (Theorem 9.15). For the LD exponents $\underline{\Upsilon}^{(n)}_{\mathrm{LD}}$ and $\overline{\Upsilon}^{(n)}_{\mathrm{LD}}$ defined in (8.69) and (8.70), the

strong SSE theorem states that for π_{XY} defined on a finite alphabet, any $n \geq 1$, $\alpha \in (0, \alpha_{\max}(\pi_X)]$, and $\beta \in (0, \beta_{\max}(\pi_Y)]$, it holds that

$$\underline{\Upsilon}^{(n)}_{\mathrm{LD}}(\alpha, \beta) \geq \mathbb{L}[\underline{\Upsilon}_{\mathrm{LD}}](\alpha, \beta) \quad \text{and} \tag{10.38}$$

$$\overline{\Upsilon}^{(n)}_{\mathrm{LD}}(\alpha, \beta) \leq \mathbb{U}[\overline{\Upsilon}_{\mathrm{LD}}](\alpha, \beta). \tag{10.39}$$

Moreover, the inequalities in (10.38) and (10.39) are asymptotically tight in the limit as $n \to \infty$. We now provide a proof of the strong SSE theorem by leveraging its connections to the information-theoretic characterizations of BL exponents.

Proof of Theorem 8.11. Observe that (10.31) and (10.32) for $p, q > 0$ can be rewritten as follows. For all $\mathcal{A}_n \subset \mathcal{X}^n$ and $\mathcal{B}_n \subset \mathcal{Y}^n$,

$$\begin{aligned} &-\frac{1}{n}\log \pi^n_{XY}(\mathcal{A}_n \times \mathcal{B}_n) + \frac{1}{np}\log \pi^n_X(\mathcal{A}_n) + \frac{1}{nq}\log \pi^n_Y(\mathcal{B}_n) \\ &\quad \geq \inf_{s,t\geq 0} \underline{\varphi}(s,t) - \frac{s}{p} - \frac{t}{q} \geq \inf_{s,t\geq 0} \underline{\Upsilon}(s,t) - \frac{s}{p} - \frac{t}{q}, \end{aligned} \tag{10.40}$$

where $\underline{\varphi}$ and $\underline{\Upsilon}$ are defined in (10.1) and (10.5) respectively. Analogously, for all $\mathcal{A}_n \subset \mathcal{X}^n$ and $\mathcal{B}_n \subset \mathcal{Y}^n$,

$$\begin{aligned} &-\frac{1}{n}\log \pi^n_{XY}(\mathcal{A}_n \times \mathcal{B}_n) + \frac{1}{np}\log \pi^n_X(\mathcal{A}_n) + \frac{1}{nq}\log \pi^n_Y(\mathcal{B}_n) \\ &\quad \leq \sup_{s,t\geq 0} \overline{\varphi}(s,t) - \frac{s}{p} - \frac{t}{q} \leq \sup_{s,t\geq 0} \overline{\Upsilon}(s,t) - \frac{s}{p} - \frac{t}{q}, \end{aligned} \tag{10.41}$$

where $\overline{\varphi}$ and $\overline{\Upsilon}$ are defined in (10.2) and (10.6) respectively. For any $(\mathcal{A}_n, \mathcal{B}_n)$, set $a := -\frac{1}{n}\log \pi^n_X(\mathcal{A}_n)$ and $b := -\frac{1}{n}\log \pi^n_Y(\mathcal{B}_n)$. Let (u, v) be a subgradient[3] of $\underline{\Upsilon}$ at (a, b). Since $\underline{\Upsilon}$ is convex and *nondecreasing*, $u, v \geq 0$. Hence, by definition of the subgradient,

$$\inf_{s,t\geq 0} \underline{\Upsilon}(s,t) - us - vt = \underline{\Upsilon}(a,b) - ua - vb. \tag{10.42}$$

Substituting $p = 1/u$ and $q = 1/v$ into (10.40) and utilizing (10.42), we have

$$-\frac{1}{n}\log \pi^n_{XY}(\mathcal{A}_n \times \mathcal{B}_n) \geq \underline{\Upsilon}(a,b).$$

[3]Let $\mathcal{I} \subset \mathbb{R}^d$ be convex. A vector $\mathbf{g} \in \mathbb{R}^d$ is a *subgradient* of $f : \mathcal{I} \to \mathbb{R}$ at $\mathbf{x} \in \mathcal{I}$ if for all $\mathbf{z} \in \mathcal{I}$, $f(\mathbf{z}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{z} - \mathbf{x} \rangle$.

Similarly, by using (10.41), we have

$$-\frac{1}{n}\log \pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n) \le \overline{\Upsilon}(a,b).$$

Hence,

$$\underline{\Upsilon}^{(n)}(\alpha,\beta) \ge \min_{a\ge\alpha, b\ge\beta} \underline{\Upsilon}(a,b) = \underline{\Upsilon}(\alpha,\beta) \quad \text{and}$$
$$\overline{\Upsilon}^{(n)}(\alpha,\beta) \le \max_{a\le\alpha, b\le\beta} \overline{\Upsilon}(a,b) = \overline{\Upsilon}(\alpha,\beta).$$

Finally, the asymptotic tightness of (8.65) and (8.66) can be verified by appealing to Sanov's theorem (Theorem 1.1). □

The strong SSE theorem (Theorem 8.11) can be further strengthened if the *exact* values of the marginal probabilities are given, instead of only bounds as in the definitions of $\underline{\Upsilon}^{(n)}$ and $\overline{\Upsilon}^{(n)}$. It has been shown in [192] that for all $\mathcal{A}_n \subset \mathcal{X}^n$ and $\mathcal{B}_n \subset \mathcal{Y}^n$,

$$\underline{\Upsilon}(a,b) \le -\frac{1}{n}\log \pi_{XY}^n(\mathcal{A}_n \times \mathcal{B}_n) \le \mathbb{U}[\overline{\varphi}](a,b), \tag{10.43}$$

where $a = -\frac{1}{n}\log \pi_X^n(\mathcal{A}_n)$, $b = -\frac{1}{n}\log \pi_Y^n(\mathcal{B}_n)$, and $\overline{\varphi}$ was defined in (10.2). Moreover, the lower and upper bounds in (10.43) are asymptotically tight as $n \to \infty$.

A similar relation can be found between the single-function version of the BL exponents and the notion of q-stability discussed in Section 9.1.1. Following steps similar to the proof for the strong SSE theorem, one can prove the strong q-stability theorem (Theorem 9.15). We omit the details here. Furthermore, similarly to the strong SSE theorem, the strong q-stability theorem can be further strengthened if the marginal probabilities are specified. For details, see [192].

10.4 Logarithmic Sobolev Inequalities

We discuss the *logarithmic Sobolev* (or *log-Sobolev*) *inequalities* in this section. It will be seen (from Theorem 10.6) that such inequalities turn out to be *equivalent*, in sense to be made precise, to the hypercontractivity inequalities (cf. Section 10.2.1). We will also focus on

information-theoretic characterizations of certain log-Sobolev inequalities. For more details on the classical aspects of this rich topic, the reader is referred to Raginsky and Sason [143] and Ledoux [105]. The results in this section serve as important elements of the proofs of the main results in Section 10.5 in which the classic hypercontractivity inequalities are strengthened. This section thus forms a bridge between the classic hypercontractivity inequalities and their strengthened versions.

10.4.1 Preliminaries on Dirichlet forms and Entropies

Let $\mathcal{X} = \mathcal{Y}$. As assumed, $\mathcal{X}$ is a finite set. Let $\mathbf{L}$ be a $|\mathcal{X}| \times |\mathcal{X}|$ matrix (a linear operator acting on real-valued functions defined on $\mathcal{X}$) such that $L_{x,y} \geq 0$ for $x \neq y$ and $\sum_{y \in \mathcal{X}} L_{x,y} = 0$ for all x. Let $T_t := \mathrm{e}^{t\mathbf{L}}$ $(t \geq 0)$ be a matrix induced by $\mathbf{L}$, where $\mathrm{e}^{\mathbf{A}}$ denotes the *matrix exponential* of $\mathbf{A}$. The operator T_t is known as a *Markov operator*, which is one that sends a real-valued function on $\mathcal{X}$ to another real-valued function on $\mathcal{X}$. In addition, $\{T_t\}_{t \geq 0}$ forms a *Markov semigroup*, since it satisfies the semigroup property, namely that $T_{t+s} = T_t T_s = T_s T_t$ for all $s, t \geq 0$. For more details on Markov operators and Markov semigroups, the reader is referred to Bakry, Gentil, and Ledoux [9] and Rudnicki, Pichór, and Tyran-Kamińska [145].

Let π be a stationary distribution corresponding to $\{T_t\}_{t \geq 0}$, i.e., $\pi = \pi T_t$ for all $t \geq 0$ or, equivalently, $\pi \mathbf{L} = \mathbf{0}$. We can regard π and T_t (for a fixed $t \geq 0$) as corresponding to π_Y and $\pi_{X|Y}$ respectively. As such, the y^{th} row of the matrix T_t is $\pi_{X|Y}(\cdot|y)$. As usual, denote the inner product for two real-valued functions f and g defined on $\mathcal{X}$ as $\langle f, g \rangle_\pi := \mathbb{E}_\pi[fg] = \sum_{x \in \mathcal{X}} \pi(x) f(x) g(x)$.

Definition 10.6. The *Dirichlet form* of $\{T_t\}_{t \geq 0}$ is

$$\mathcal{E}(f, g) := -\sum_{(x,y) \in \mathcal{X}^2} L_{x,y} f(y) g(x) \pi(x) = -\langle \mathbf{L} f, g \rangle_\pi, \tag{10.44}$$

where $(\mathbf{L}f)(x) := \sum_{y \in \mathcal{X}} L_{x,y} f(y)$. The *normalized Dirichlet form* of $\{T_t\}_{t \geq 0}$ is

$$\overline{\mathcal{E}}(f, g) := \frac{\mathcal{E}(f, g)}{\langle f, g \rangle_\pi}.$$

We now extend the definitions of the Dirichlet form and its normalized version to the n-dimensional Cartesian product space $\mathcal{X}^n$. Let $T_t^{\otimes n}$ be the product semigroup on $\mathcal{X}^n$ induced by T_t. Recall from Section 9.3.1, that given a vector $x^n \in \mathcal{X}^n$, let $x^{\backslash k} := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathcal{X}^{n-1}$ be the subvector of x^n with the k^{th} coordinate removed. For two real-valued functions f and g defined on $\mathcal{X}^n$, let

$$\psi(x^{\backslash k}) := \mathcal{E}\big(f(x^{\backslash k}, \cdot), g(x^{\backslash k}, \cdot)\big)$$

be the action of the Dirichlet form $\mathcal{E}$ on the k^{th} coordinates of f and g with other coordinates held fixed. Then, the Dirichlet form of f and g and its normalized version are respectively given by

$$\mathcal{E}_n(f,g) := \sum_{k=1}^{n} \sum_{x^{\backslash k} \in \mathcal{X}^{n-1}} \psi(x^{\backslash k}) \prod_{j \in [n] \backslash \{k\}} \pi(x_j) \quad \text{and}$$

$$\overline{\mathcal{E}}_n(f,g) := \frac{\mathcal{E}_n(f,g)}{\langle f, g \rangle_{\pi^n}}.$$

In addition to the Dirichlet form, the other quantity involved in log-Sobolev inequalities is the *entropy* of a nonnegative function f.

Definition 10.7. For a nonnegative function f, the *entropy* and the *normalized entropy* of f are respectively defined as

$$\mathrm{Ent}(f) := \mathbb{E}_\pi[f \ln f] - \mathbb{E}_\pi[f] \ln \mathbb{E}_\pi[f] \quad \text{and} \quad \overline{\mathrm{Ent}}(f) := \frac{\mathrm{Ent}(f)}{\mathbb{E}_\pi[f]}.$$

Note that these notions of entropy and normalized entropy are commonly encountered in functional analysis; see, for example, Ledoux [105]. They are related to, but not the same as the Shannon entropy in classical information theory. Indeed, they bear more similarity to the relative entropy, in the sense that if f is the Radon–Nikodym derivative $\mathrm{d}Q/\mathrm{d}\pi$ of a distribution Q with respect to π (i.e., the function $x \in \mathcal{X} \mapsto Q(x)/\pi(x)$ for the finite alphabet case), then the entropy (and also the normalized entropy) of f is equal to the relative entropy of Q from π, i.e., $D(Q\|\pi)$. By Jensen's inequality, both the entropy and the normalized entropy are nonnegative.

10.4.2 Log-Sobolev Inequalities and Their Properties

The *log-Sobolev* inequalities quantify the relation between the Dirichlet form of a Markov semigroup for an arbitrary nonnegative function f and a composite function $g = \varphi \circ f$ for some given $\varphi : [0, \infty) \to [0, \infty)$, and the entropy of f. For $p \in \mathbb{R} \setminus \{0, 1\}$, let

$$c_p := \frac{p^2}{4(p-1)}.$$

Following the definitions in Mossel, Oleszkiewicz, and Sen [125], we define log-Sobolev inequalities as follows.

Definition 10.8. For $p \in \mathbb{R}\backslash\{0, 1\}$, the *$p$-log-Sobolev inequality with constant C* is

$$\mathrm{Ent}(f^p) \le C\, c_p\, \mathcal{E}(f, f^{p-1}) \tag{10.45}$$

for nonnegative f if $p > 1$ and for positive f if $p < 1$.For $p = 1$, the *1-log-Sobolev inequality with constant C* for positive f is

$$\mathrm{Ent}(f) \le \frac{C}{4}\, \mathcal{E}(f, \ln f).$$

For $p = 0$, the *0-log-Sobolev inequality with constant C* for positive f is

$$\mathrm{Var}(\ln f) \le -\frac{C}{2}\, \mathcal{E}\Big(f, \frac{1}{f}\Big).$$

The cases corresponding to $p = 0$ and $p = 1$ of the p-log-Sobolev inequality are the limiting cases of the p-log-Sobolev inequality for $p \in \mathbb{R}\backslash\{0, 1\}$ with the same constant C.

We now connect the log-Sobolev inequality and the classic hypercontractivity inequalities in (10.33) and (10.34) with $\overline{C}$ and $\underline{C}$ set to 1. Indeed, we will see from Theorem 10.6 that the log-Sobolev inequalities are *differential versions* of the hypercontractivity inequalities evaluated at $t = 0$. This theorem is a classical result due to Gross [69], and various proofs can be found in [6]–[8], [69], [125].

Here we provide a short self-contained proof.

Theorem 10.6 (Differential relationship between log-Sobolev and hypercontractivity inequalities)**.** Let C be a positive constant. Let $q : [0, \infty) \to \mathbb{R}$ be defined as

$$q(t) = 1 + (p-1)\, \mathrm{e}^{4t/C}. \tag{10.46}$$

(a) Fix $p > 1$. If for any $r \in [p, \infty)$, the r-log Sobolev inequality is satisfied with constant C, then for any $t > 0$,

$$\|T_t f\|_{q(t)} \le \|f\|_p \quad \text{for all } f \ge 0, \tag{10.47}$$

where $(T_t f)(x) = \sum_y T_t(x, y) f(y)$.

(b) Fix $p < 1$. If for any $r \in (-\infty, p]$, the r-log-Sobolev inequality is satisfied with constant C, then for any $t > 0$,

$$\|T_t f\|_{q(t)} \ge \|f\|_p \quad \text{for all } f \ge 0. \tag{10.48}$$

(c) Conversely, if (10.47) holds for $p > 1$ or (10.48) holds for $p < 1$, then the p-log-Sobolev inequality is satisfied with constant C.

The inequalities in (10.47) and (10.48) are respectively equivalent to the fact that[4] $(p, q(t)')$ belongs to the forward and reverse hypercontractivity regions of the joint distributions (Definition 10.4) induced by (T_t, π) for any $t > 0$. In fact, the relations between the BL inequalities and generalized p-log-Sobolev inequalities can also be established. For details, the reader is referred to [6]–[8], [69], [125].

Proof Sketch of Theorem 10.6. We first prove Statement (a) in which we assume that $p > 1$. Define the function $\zeta : [0, \infty)^2 \to \mathbb{R}$ as

$$\zeta(t, s) := \ln \|T_t f\|_{\frac{1}{s}}.$$

Then, one can check by direct differentiation that

$$\frac{\partial \zeta}{\partial s} = -\overline{\mathrm{Ent}}\big((T_t f)^{\frac{1}{s}}\big) \quad \text{and} \quad \frac{\partial \zeta}{\partial t} = -\overline{\mathcal{E}}_n\big(T_t f, (T_t f)^{\frac{1}{s}-1}\big). \tag{10.49}$$

We define $\xi(t) := 1/q(t)$, and hence, $\xi(0) = 1/p$ (refer to (10.46)). Therefore, by (10.49) and the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t, \xi(t)) = -\overline{\mathrm{Ent}}\Big((T_t f)^{\frac{1}{\xi(t)}}\Big)\xi'(t) - \overline{\mathcal{E}}_n\Big(T_t f, (T_t f)^{\frac{1}{\xi(t)}-1}\Big). \tag{10.50}$$

Observe that

$$\xi'(t) = \frac{-4(p-1)\mathrm{e}^{4t/C}}{C\, q(t)^2}.$$

[4] Here $q(t)'$ denotes the the Hölder conjugate of $q(t)$. In contrast, we use $q'(t)$ to denote the derivative of q evaluated at t.

It also holds that $\xi'(t) \geq -1/(C\, c_{q(t)})$ for all $t \geq 0$. Combining this with (10.50) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t,\xi(t)) \leq \overline{\mathrm{Ent}}\big((T_t f)^{\frac{1}{\xi(t)}}\big)\frac{1}{C\, c_{q(t)}} - \overline{\mathcal{E}}_n\big(T_t f, (T_t f)^{\frac{1}{\xi(t)}-1}\big). \quad (10.51)$$

On the other hand, by assumption, for any $r \in [p,\infty)$, the r-log-Sobolev inequality holds, i.e.,

$$\overline{\mathrm{Ent}}(g^r) \leq C\, c_r\, \overline{\mathcal{E}}_n(g, g^{r-1}) \quad \text{for all } g \geq 0. \quad (10.52)$$

Substituting g and r for $T_t f$ and $q(t) = 1/\xi(t)$ respectively into (10.52) and combining the resultant inequality with (10.51) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t,\xi(t)) \leq 0 \quad \text{for all } t \geq 0. \quad (10.53)$$

Finally, by integrating both sides of (10.53) from 0 to t, we have

$$\ln \|T_t f\|_{\frac{1}{\xi(t)}} - \ln \|f\|_{\frac{1}{\xi(0)}} \leq 0,$$

which is precisely the hypercontractivity inequality in (10.47).

Statement (b) follows analogously but the directions of the inequalities above are reversed. Statement (c) follows by first differentiating the hypercontractivity inequalities and evaluating them at $t = 0$. Then, we can recover the p-log-Sobolev inequalities from them. □

It is also well-known (see, for example, Ledoux [105]) that the p-log-Sobolev inequality satisfies the *tensorization property*.

Proposition 10.3. If a certain p-log-Sobolev inequality with constant C holds for (T_t, π), then the p-log-Sobolev inequality with the *same* constant holds for the product semigroup $(T_t^{\otimes n}, \pi^n)$.

Proof. We provide an information-theoretic proof for this proposition. We start by characterizing the optimal constant in the p-log-Sobolev inequality in terms of information-theoretic quantities. For the product semigroup $(T_t^{\otimes n}, \pi^n)$, the optimal constant is for $p \in \mathbb{R}\backslash\{0,1\}$

$$C^*_{p,n} := \sup_{f\,:\,c_p\,\mathcal{E}_n(f,f^{p-1})>0} \frac{\mathrm{Ent}(f^p)}{c_p\,\mathcal{E}_n(f, f^{p-1})}.$$

Since (T_t, π) is a special case of $(T_t^{\otimes n}, \pi^n)$ with n set to 1, $C^*_{p,1}$ is the optimal constant for the semigroup (T_t, π).

For a given Q_{X^n}, define the k^{th} "likelihood ratio"

$$\ell_k(y, x^{\setminus k}) := \frac{Q_{X_k|X^{\setminus k}}(y|x^{\setminus k})}{\pi(y)} \quad \text{for all } (y, x^{\setminus k}) \in \mathcal{X} \times \mathcal{X}^{n-1}.$$

If we write $f^p/\mathbb{E}[f^p] = Q_{X^n}/\pi^n$ for a distribution $Q_{X^n} \ll \pi^n$, then

$$\operatorname{Ent}(f^p) = D(Q_{X^n} \| \pi^n) \quad \text{and} \tag{10.54}$$

$$\mathcal{E}_n(f, f^{p-1}) = \sum_{k=1}^n \mathbb{E}_{\pi^{n-1}}[\eta(X^{\setminus k})], \tag{10.55}$$

where

$$\eta(x^{\setminus k}) := -\frac{Q_{X^{\setminus k}}(x^{\setminus k})}{\pi^{n-1}(x^{\setminus k})} \sum_{x,y} L_{x,y} \big(\ell_k(y, x^{\setminus k})\big)^{1/p} \, \big(\ell_k(x, x^{\setminus k})\big)^{1/p'} \, \pi(x).$$

Uniting (10.54) and (10.55), one can obtain the following information-theoretic characterization of $C^*_{p,n}$.

Lemma 10.7. For $n \in \mathbb{N}$ and $p \in \mathbb{R}\setminus\{0, 1\}$, it holds that

$$C^*_{p,n} = \sup_{Q_{X^n} \,:\, c_p \sum_{k=1}^n \mathbb{E}_{\pi^{n-1}}[\eta(X^{\setminus k})] > 0} \frac{D(Q_{X^n} \| \pi^n)}{c_p \sum_{k=1}^n \mathbb{E}_{\pi^{n-1}}[\eta(X^{\setminus k})]}. \tag{10.56}$$

Continuing the proof of Proposition 10.3, we notice that, on one hand, by the data processing inequality for the relative entropy, we have

$$\begin{aligned} D(Q_{X^n} \| \pi^n) &= \sum_{k=1}^n D(Q_{X_k|X^{k-1}} \| \pi | Q_{X^{k-1}}) \\ &\le \sum_{k=1}^n D(Q_{X_k|X^{\setminus k}} \| \pi | Q_{X^{\setminus k}}) \\ &= n D(Q_{X_K|X^{\setminus K}K} \| \pi | Q_{X^{\setminus K}|K} Q_K) \\ &= n D(Q_{X|U} \| \pi | Q_U), \end{aligned} \tag{10.57}$$

where $K \sim Q_K := \operatorname{Unif}[n]$ is independent of X^n and $U := (X^{\setminus K}, K)$.

On the other hand, using the definitions of η and ℓ_k, consider,

$$\begin{aligned}
&\sum_{k=1}^{n} \mathbb{E}_{\pi^{n-1}}[\eta(X^{\setminus k})] \\
&= -\sum_{k=1}^{n} \mathbb{E}_{Q_{X^{\setminus k}}}\Big[\sum_{x,y} L_{x,y}\big(\ell_k(y, X^{\setminus k})\big)^{1/p}\big(\ell_k(x, X^{\setminus k})\big)^{1/p'}\pi(x)\Big] \\
&= -n\mathbb{E}_{Q_K}\Big[\mathbb{E}_{Q_{X^{\setminus K}}}\Big[\sum_{x,y} L_{x,y}\big(\ell_K(y, X^{\setminus K})\big)^{1/p}\big(\ell_K(x, X^{\setminus K})\big)^{1/p'}\pi(x)\Big|K\Big]\Big] \\
&= n\sum_{u} Q_U(u)\mathcal{E}\left(\Big(\frac{Q_{X|U=u}}{\pi}\Big)^{1/p}, \Big(\frac{Q_{X|U=u}}{\pi}\Big)^{1/p'}\right), \qquad (10.58)
\end{aligned}$$

where the penultimate equality follows from the uniformity of K and the final equality follows from the definition of the Dirichlet form in (10.44) and that of $U = (X^{\setminus K}, K)$. From (10.57) and (10.58), we conclude that the objective function in (10.56) satisfies

$$\begin{aligned}
&\frac{D(Q_{X^n}\|\pi^n)}{c_p\sum_{k=1}^{n}\mathbb{E}_{\pi^{n-1}}\big[\eta(X^{\setminus k})\big]} \\
&\quad\le \frac{D(Q_{X|U}\|\pi|Q_U)}{c_p\sum_u Q_U(u)\mathcal{E}\Big(\big(\frac{Q_{X|U=u}}{\pi}\big)^{1/p}, \big(\frac{Q_{X|U=u}}{\pi}\big)^{1/p'}\Big)} \\
&\quad\le \max_u \frac{D(Q_{X|U=u}\|\pi)}{c_p\,\mathcal{E}\Big(\big(\frac{Q_{X|U=u}}{\pi}\big)^{1/p}, \big(\frac{Q_{X|U=u}}{\pi}\big)^{1/p'}\Big)} \qquad (10.59) \\
&\quad\le C^*_{p,1}, \qquad (10.60)
\end{aligned}$$

where in (10.59), the maximum is over all u in the alphabet of U (i.e., $\mathcal{X}^{n-1}\times[n]$) such that the denominator is positive, and (10.60) follows from Lemma 10.7 (with n set to 1). Therefore, $C^*_{p,n} \le C^*_{p,1}$ for any n and $p \in \mathbb{R}\setminus\{0,1\}$.

On the other hand, setting Q_{X^n} to be a product distribution Q_X^n in Lemma 10.7, we find that $C^*_{p,n} \ge C^*_{p,1}$ for all $n \in \mathbb{N}$. Combining these two bounds yields $C^*_{p,n} = C^*_{p,1}$ for $p \in \mathbb{R}\setminus\{0,1\}$. By taking limits in p (toward 0 and 1), one deduces that $C^*_{p,n} \ge C^*_{p,1}$ for $p \in \{0,1\}$. Hence, the tensorizaton property holds. □

The information-theoretic method employed in the proof of Proposition 10.3 can be also used to study certain *nonlinear* versions of

log-Sobolev inequalities. These inequalities were proposed as a topic for research by Kalai and Linial [91] in 1995. However, there was no progress for over twenty years since the initial proposal of these inequalities until recent works by Samorodnitsky [148], Samorodnitsky [149], and Polyanskiy and Samorodnitsky [139].

Note that (10.45) delineates a certain *linear* relationship between the entropy $\mathrm{Ent}(f^p)$ and the Dirichlet form $\mathcal{E}(f, f^{p-1})$. For $\alpha \geq 0$ and $n \in \mathbb{N}$, let

$$\mathcal{F}_\alpha^{(n)} := \{f : c_p \, \overline{\mathcal{E}}_n(f, f^{p-1}) = n\alpha\}.$$

To study the *nonlinear* tradeoff between the normalized Dirichlet form and the normalized entropy, we define the *log-Sobolev function* as

$$\Xi_p(\alpha) := \sup_{f \in \mathcal{F}_\alpha^{(1)}} \overline{\mathrm{Ent}}(f^p). \tag{10.61}$$

Extending the definition of $\Xi_p(\alpha)$ from T_t to $T_t^{\otimes n}$, we define

$$\Xi_p^{(n)}(\alpha) := \sup_{f \in \mathcal{F}_\alpha^{(n)}} \frac{1}{n} \overline{\mathrm{Ent}}(f^p). \tag{10.62}$$

It would be useful to provide a tight dimension-independent bound for $\Xi_p^{(n)}(\alpha)$. As mentioned in Section 9.4, by *dimension-independent*, we mean that the bound on $\Xi_p^{(n)}(\alpha)$ does not depend on n; in information theory parlance, this is known as a *single-letter* bound. A tight dimension-independent bound for $\Xi_p^{(n)}(\alpha)$ was shown by Polyanskiy and Samorodnitsky [139] in the following theorem.

Theorem 10.8. It holds that for $p \in \mathbb{R} \backslash \{0, 1\}$,

$$\Xi_p^{(n)}(\alpha) \leq \mathbb{U}[\Xi_p](\alpha). \tag{10.63}$$

Moreover, this upper bound is asymptotically tight as $n \to \infty$, which means that

$$\lim_{n \to \infty} \Xi_p^{(n)}(\alpha) = \mathbb{U}[\Xi_p](\alpha). \tag{10.64}$$

If additionally, Ξ_p in (10.61) is concave, then the upper bound in (10.63) is also tight for all finite $n \geq 1$.

This theorem is a strengthening of the (linear) p-log-Sobolev inequality in (10.45) with optimal constant $C_p^* := C_{p,1}^*$ given in (10.56). Here

we set n in (10.56) to 1 since the tensorization property holds. Since the function $\mathbb{U}[\Xi_p]$ is nonlinear in general, the inequality in (10.63) is known as the *nonlinear p-log-Sobolev inequality*.

To appreciate the relation between the linear and nonlinear p-log-Sobolev inequalities, one can demonstrate that the optimal constant C_p^* in the (linear) p-log-Sobolev inequality in (10.45) is the right-derivative of $\mathbb{U}[\Xi_p](\alpha)$ at $\alpha = 0$ if $\Xi_p(0) = 0$. If $\Xi_p(0) > 0$, then the linear p-log-Sobolev inequality in (10.45) does not hold for any finite C.

Proof of Theorem 10.8. We follow the same steps as in the proof of Lemma 10.3 up to (10.58). Then, combining these steps with the definition of $\Xi_p^{(n)}(\alpha)$ in (10.62), we find that

$$\begin{aligned}\Xi_p^{(n)}(\alpha) &\le \sup_{Q_{XU}\,:\,c_p\,\mathbb{E}_{Q_U}\left[\mathcal{E}\left(\left(\frac{Q_{X|U}}{\pi}\right)^{1/p},\left(\frac{Q_{X|U}}{\pi}\right)^{1/p'}\right)\right]=\alpha} D(Q_{X|U}\|\pi_X|Q_U)\\ &= \mathbb{U}[\Xi_p](\alpha).\end{aligned}$$

The asymptotic tightness of (10.64) can be verified by a time-sharing argument (cf. the discussion after Theorem 8.10). □

10.5 Strengthened Hypercontractivity Inequalities

The tools we reviewed in the preceding sections serve as ingredients for the culmination of this section—namely, a strengthened version of the hypercontractivity inequality. For the sake of clarity, we focus on the DSBS. Before doing so, we provide explicit expressions for the linear and nonlinear p-log-Sobolev inequalities particularized to the DSBS.

For the DSBS, $\mathcal{X} = \{0,1\}$, $\pi = \pi_Y = \mathrm{Bern}(1/2)$ and T_t is the Markov operator induced by $\pi_{X|Y}$ and is given by

$$T_t f(y) = f(y)\frac{1+\mathrm{e}^{-t}}{2} + f(1-y)\frac{1-\mathrm{e}^{-t}}{2} \qquad y \in \{0,1\}. \tag{10.65}$$

Note that the operator $T_t^{\otimes n}$ is the same as T_ρ in Section 8 with $\rho = \mathrm{e}^{-t}$. From (10.65), we know that $L_{x,y} = \mathbb{1}\{x \neq y\} - 1/2$ which is obtained

by differentiating T_t with respect to t and evaluating the derivative at $t = 0$. Moreover, the Dirichlet form for this case is

$$\mathcal{E}_n(f, g) = -\frac{1}{2}\langle \Delta f, g\rangle \quad \text{and} \tag{10.66}$$

$$\mathcal{E}_n(f, f) = \frac{2^{-n}}{4} \sum_{(x^n, y^n): x^n \sim y^n} \big(f(x^n) - f(y^n)\big)^2 = \frac{1}{4}\,\mathbf{I}[f]\,, \tag{10.67}$$

where $\Delta f(x^n) := \sum_{y^n : y^n \sim x^n} (f(y^n) - f(x^n))$, and $x^n \sim y^n$ means that $x^n, y^n \in \{0,1\}^n$ differ in exactly one coordinate. Recall that here $\mathbf{I}[f]$ denotes the total influence of f; see Definition 9.9.

For the DSBS, the optimal constant C in the (linear) p-log-Sobolev inequality is 2; see Gross [69]. The (linear) p-log-Sobolev inequality with optimal constant 2 can be derived from the single-function version of the hypercontractivity inequalities for the DSBS given in Theorem 8.5. This can be done by differentiating both sides of the hypercontractivity inequalities with respect to ρ and evaluating the derivative at $\rho = 1$.

10.5.1 Strong Log-Sobolev Inequalities

Polyanskiy and Samorodnitsky [139] proved the dimension-independent *nonlinear* p-log-Sobolev inequalities as stated in the next theorem. Before introducing these inequalities, we define $b_p : [0, \ln 2] \to [0, \infty)$ to be the convex increasing function given by

$$b_p(t) := \begin{cases} \dfrac{\operatorname{sign}(p-1)}{2}\left(1 - y^{\frac{1}{p}}(1-y)^{1-\frac{1}{p}} - y^{1-\frac{1}{p}}(1-y)^{\frac{1}{p}}\right) & p \neq 0, 1 \\ \left(\dfrac{1}{2} - y\right)\ln\dfrac{1-y}{y} & p = 1 \end{cases},$$

where $y(t) := h^{-1}(\ln 2 - t)$ and $h^{-1} : [0, \ln 2] \to [0, 1/2]$ is the inverse of the binary entropy function h *with base* e when its domain is restricted to $[0, 1/2]$.

Theorem 10.9 (p-log-Sobolev Inequality for the DSBS)**.** Let $p \in \mathbb{R} \setminus \{0, 1\}$. For all $f : \{0,1\}^n \to [0, \infty)$ (and $f > 0$ if $p < 1$), it holds that

$$\frac{1}{n}\operatorname{sign}(p-1)\;\overline{\mathcal{E}}_n(f, f^{p-1}) \geq b_p\Big(\frac{1}{n}\overline{\operatorname{Ent}}(f^p)\Big)\,, \tag{10.68}$$

where the normalized Dirichlet form is given by (10.66). Let $p = 1$. For all $f : \{0,1\}^n \to (0,\infty)$, it holds that

$$\frac{1}{n}\overline{\mathcal{E}}_n(f, \ln f) \geq b_1\Big(\frac{1}{n}\overline{\mathrm{Ent}}(f)\Big), \tag{10.69}$$

The inequality in (10.69) is the limiting case of (10.68) as $p \to 1$. Moreover, these inequalities are sharp in the sense that given p, there exists a nonnegative function f such that the equality holds.

Proof Sketch of Theorem 10.9. Theorem 10.9 follows directly from Theorem 10.8 by observing that b_p is the inverse of Ξ_p when π_{XY} is particularized to the DSBS. Moreover, Ξ_p is concave and increasing. The monotonicity and concavity of Ξ_p (or equivalently, the monotonicity and convexity of b_p) can be shown by calculating the first and second derivatives of Ξ_p (or b_p); see Polyanskiy and Samorodnitsky [139] for more details. Furthermore, the asymptotic sharpness of (10.68) follows directly from the asymptotic sharpness of (10.63). □

Based on Theorem 10.9, we are almost ready to introduce a strengthened version of the forward hypercontractivity inequality shown by Polyanskiy and Samorodnitsky [139]. Before doing so, we would like to discuss an intimate relationship between the linear and nonlinear log-Sobolev inequalities and the edge-isoperimetric inequality given in Theorem 9.5, as promised below (9.38). We first consider the linear log-Sobolev inequality. Consider the DSBS and the case $p = 2$. The linear 2-log-Sobolev inequality with optimal constant $C = 2$ reduces to

$$\mathrm{Ent}(f^2) \leq 2\,\mathcal{E}(f, f) \quad \text{for all } f \geq 0. \tag{10.70}$$

By the tensorization property and utilizing (10.67),

$$\mathrm{Ent}(f^2) \leq 2\,\mathcal{E}_n(f, f) = \frac{1}{2}\,\mathbf{I}[f].$$

Setting f to be a Boolean function with mean a, we obtain

$$\mathbf{I}[f] \geq 2a\,\ln\Big(\frac{1}{a}\Big). \tag{10.71}$$

Note that in the sharp edge-isoperimetric inequality in (9.38), the logarithm used is log (to the base 2), instead of ln. Hence, in (10.71), there is a multiplicative factor $\log \mathrm{e}$ off from the sharp inequality in (9.38).

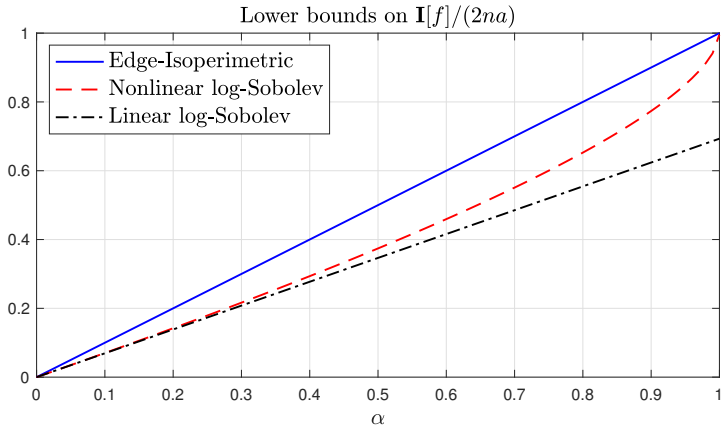


Figure 10.2: Comparison of the distinct parts α (edge-isoperimetric), $\alpha \ln 2$ (linear log-Sobolev) and $2b_2(\alpha)$ (nonlinear log-Sobolev) when a is set to $2^{-\alpha}$.

We next consider the nonlinear log-Sobolev inequality for the DSBS and $p = 2$. For this case, Theorem 10.9 reduces to the statement that for all $f \geq 0$, it holds that

$$\frac{1}{n}\overline{\mathcal{E}}_n(f, f) \geq b_2\Big(\frac{1}{n}\overline{\mathrm{Ent}}(f^2)\Big)\,, \tag{10.72}$$

where $b_2 : [0, \ln 2] \to [0, \infty)$ is the convex increasing function given by

$$b_2(t) = \frac{1 - 2\sqrt{y(1-y)}}{2}$$

with $y = h^{-1}(\ln 2 - t)$ (coinciding with the general definition of b_p). By (10.67) and setting f to be a Boolean function with mean a, we obtain

$$\mathbf{I}[f] \geq 4an\, b_2\Big(\frac{1}{n}\ln\Big(\frac{1}{a}\Big)\Big). \tag{10.73}$$

This inequality is tighter than (10.71), but looser than (9.38). This point can be observed from the facts that b_2 is convex and increasing, $b_2'(0) = 1/2$, $b_2(0) = 0$, $b_2(\ln 2) = 1/2$, and hence, $t/2 \leq b_2(t) \leq \frac{t}{2\ln 2}$ for all $t \in [0, \ln 2]$. If we consider the case $a = 2^{-\alpha}$, then the bounds in (9.38), (10.71), and (10.73) are $2na\,\alpha$, $4na\,b_2(\alpha)$, and $2na\,\alpha\ln 2$ respectively. Omitting the common factor $2na$, we plot α, $2b_2(\alpha)$, and $\alpha \ln 2$ in Fig. 10.2, which depicts the relations among these three inequalities.

We note that it makes eminent sense that (10.71) and (10.73) are looser than (9.38). This is because that the former two inequalities are derived from the linear and nonlinear log-Sobolev inequalities in (10.70) and (10.72) which are valid not only for Boolean functions, but for *any nonnegative functions*. In contrast, the edge-isoperimetric inequality in (9.38), which is derived by a combinatorial method, is specific to and sharp for Boolean functions.

10.5.2 Strengthened Version of Hypercontractivity Inequalities

We now introduce a strengthened version of forward hypercontractivity inequality due to Polyanskiy and Samorodnitsky [139]. We first introduce an additional definition. For a nonnegative function $f : \mathcal{X}^n \to [0, \infty)$, the *p-entropy* of f [192] is defined as

$$\overline{\mathrm{Ent}}_p(f) := \frac{p}{p-1} \log \frac{\|f\|_p}{\|f\|_1}.$$

In fact, if f is the Radon–Nikodym derivative of Q with respect to π, i.e., $f = \mathrm{d}Q/\mathrm{d}\pi$, then $\overline{\mathrm{Ent}}_p(f) = D_p(Q\|\pi)$; also see (10.35). Basic properties of the p-entropy, such as its continuity and monotonicity, can be found in Yu [192]. Let $g : [0, \ln 2] \to [2, 2/\ln 2]$ be defined as

$$g(t) := \frac{2 - 4\sqrt{y(1-y)}}{\ln 2 - h(y)},$$

where $y = h^{-1}(\ln 2 - t)$.

Theorem 10.10. Fix two numbers $1 < p < \infty$ and $0 \le \alpha \le \ln 2$. Then the differential equation in $u : [0, \infty) \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = g\Big(\frac{\alpha}{p'}(1 + \mathrm{e}^{-u(t)})\Big)$$

with initial solution $u(0) = \ln(p-1)$ has a unique solution on $[0, \infty)$. Furthermore, for any $f : \{0,1\}^n \to [0, \infty)$ with $\frac{1}{n}\overline{\mathrm{Ent}}_p(f) \ge \alpha$, we have

$$\|T_t^{\otimes n} f\|_{q(t)} \le \|f\|_p \quad \text{where} \quad q(t) = 1 + \mathrm{e}^{u(t)}. \tag{10.74}$$

The core idea of the proof of Theorem 10.10 is to integrate both sides of the nonlinear p-log-Sobolev inequality in Theorem 10.9. It is

similar to the proof of Theorem 10.6, and hence, omitted. Theorem 10.10 was used by Ordentlich, Polyanskiy, and Shayevitz [133] to prove the limiting cases as $\rho \downarrow 0$ and $\rho \uparrow 1$ of the NICD problem in the LD regime.

As remarked by Polyanskiy and Samorodnitsky [139], the function g is a smooth, convex, and strictly increasing bijection. Consequently, the function $q(t)$ in (10.74) is smooth and satisfies $q(t) > 1 + (p-1)\mathrm{e}^{2t}$ for all $t > 0$. Note that the maximum (and hence best possible) parameter $q(t)$ for the classic forward hypercontractivity is equal to $1 + (p-1)\mathrm{e}^{2t}$; see (8.56). Hence, the inequality in (10.74) strictly improves the classic forward hypercontractivity inequality in (8.56). Furthermore, $q(t)$ also satisfies

$$q(t) = p + q'(0)t + \frac{1}{2}q''(0)t^2 + o(t^2), \qquad \text{as } t \to 0\,,$$

where

$$\begin{aligned} q'(0) &= (p-1)g(\alpha), \quad \text{and} \\ q''(0) &= (p-1)\Big(g(\alpha)^2 - g'(\alpha)g(\alpha)\frac{\alpha}{p}\Big). \end{aligned}$$

Since the strengthened version of the hypercontractivity inequality in (10.74) is obtained by integrating both sides of the *sharp* nonlinear p-log-Sobolev inequality in Theorem 10.9, one can observe that (10.74) is locally sharp at $t = 0$ in the following sense. For every $\hat{q}(t)$ such that $\hat{q}(0) = p$ and $\hat{q}'(0) > q'(0)$ there exists a function f with $\frac{1}{n}\overline{\mathrm{Ent}}_p(f) \geq \alpha$ such that $\|T_t^{\otimes n} f\|_{\hat{q}(t)} > \|f\|_p$ holds for any sufficiently small t. However, the inequality in (10.74) does not appear to be *globally* asymptotically sharp in the sense that there exists a function $\hat{q} : [0,\infty) \to \mathbb{R}$ such that $\hat{q}(t) > q(t)$ for all $t > 0$, and $\|T_t^{\otimes n} f\|_{\hat{q}(t)} \leq \|f\|_p$ holds for all $f : \{0,1\}^n \to [0,\infty)$ with $\frac{1}{n}\overline{\mathrm{Ent}}_p(f) \geq \alpha$. Recently, a globally sharp inequality was derived by the first author of this monograph [192], who showed that given q, the minimum p such that the inequality $\|T_t^{\otimes n} f\|_q \leq \|f\|_p$ holds for any $f : \{0,1\}^n \to [0,\infty)$ with $\frac{1}{n}\overline{\mathrm{Ent}}_p(f) \geq \alpha$ is $\alpha/\varphi_q(\alpha)$ (where φ_q is defined in Definition 10.3), in which $\rho = \mathrm{e}^{-t}$. Moreover, this sharp bound is asymptotically attained by indicators of Hamming spheres. We compare the sharp bound given by Yu [192] and the bound in Theorem 10.10 in Fig. 10.3. This figure indicates that (10.74) is close to optimal as $p \downarrow 1$.

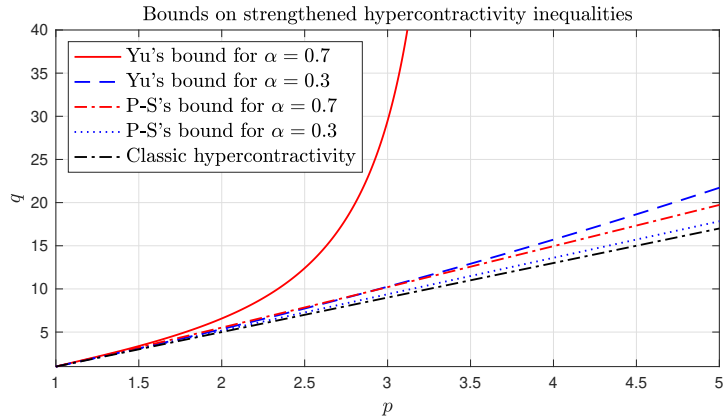


Figure 10.3: Comparison of the sharp bound derived by Yu [192] and the bound in Theorem 10.10 derived by Polyanskiy and Samorodnitsky [139] (P-S), where $\rho = 0.5$. Note that we use q and p to denote $q(t)$ and p respectively in Theorem 10.9.

Along the same lines, it is natural to investigate sharper versions of BL inequalities. Indeed, Polyanskiy posed a conjecture concerning the asymptotically sharp BL inequalities in 2016. This conjecture is stated in Kirshner and Samorodnitsky [97] and reproduced here.

Conjecture 10.1. Fix $\rho \in (0,1)$, $q > 1$, and a scalar $\alpha \in (0, \ln 2)$. Then, there exists

$$p_0 < 1 + \rho^2(q-1)$$

such that for any $p \geq p_0$ the maximum of $\frac{1}{n} \ln \frac{\|T_\rho f\|_q}{\|f\|_p}$ over all nonnegative functions f with $\frac{1}{n}\overline{\mathrm{Ent}}_p(f) \geq \alpha$ is asymptotically attained by a sequence of functions that are indicators of Hamming spheres with radii converging to some constant as $n \to \infty$.

This conjecture was confirmed in the affirmative by Kirshner and Samorodnitsky [97] for the case $q = 2$. As stated in [97], Conjecture 10.1 for all $q > 1$ was proved by Polyanskiy in an unpublished work [137]. It was also proven independently by the first author of this monograph in [192], [193]. In particular, it was shown that Conjecture 10.1 holds even for $p = 1$. Indeed, the asymptotically sharp bound on $q(t)$ such that $\|T_t^{\otimes n} f\|_{q(t)} \leq \|f\|_p$ holds for any $f : \{0,1\}^n \to [0,\infty)$ with $\frac{1}{n}\overline{\mathrm{Ent}}_p(f) \geq \alpha$ can be characterized by the strong BL inequality derived in the same references. Readers may refer to [192], [193] for the details.

11

Open Problems

We have taken a whirlwind tour of classic and contemporary notions related to the common information between two random variables. In this final section, we list some open problems that represent fertile grounds for future research.

11.1 Open Problems Related to Wyner's Common Information

We now introduce two open problems related to extensions of Wyner's common information.

11.1.1 Rényi Common Information for all Orders

As shown in Part II, the (unnormalized and normalized) Rényi common information forms a bridge between Wyner's common information and the exact common information (see Fig. 5.1). The latter two quantities correspond to the Rényi common information of order 1 (normalized) and order ∞ (unnormalized) respectively. Hence, the Rényi common information of order $\alpha \in [0, \infty]$ is a natural generalization of these quantities. However, the complete characterization of Rényi common information of order α remains open for a large range of α and sources.

Here by "complete characterization", we refer to providing "single-letter expressions". In Theorem 4.3, we present upper and lower bounds on the Rényi common information for $\alpha \in [0,2] \cup \{\infty\}$. For $\alpha \in (0,1]$, the unnormalized and normalized Rényi common information of order α are both shown to be equal to Wyner's common information, and hence, it has been completely characterized. However, for $\alpha \in (1,\infty]$, the upper and lower bounds given in Theorem 4.3 only coincide for some special cases, e.g., the case for the DSBS and $\alpha = \infty$, and the case of sources with Wyner-product distributions (cf. Definition 4.3). The complete characterization of Rényi common information for all discrete and continuous sources and for all orders $\alpha \in (1,\infty]$ is a major open problem on this topic. An interesting special case of this open problem is Conjecture 5.1, which concerns the determination of the exact common information (or the unnormalized Rényi common information of order ∞) for jointly Gaussian sources.

11.1.2 Exact Rényi Common Information for all Orders

Another interesting observation from Part II is that the *exact* Rényi common information of order α (originally defined in (7.15))

$$T_{\mathrm{Ex}}^{(\alpha)}(\pi_{XY}) := \lim_{n\to\infty} \frac{G_\alpha(\pi_{XY}^n)}{n}. \tag{11.1}$$

connects the exact common information and the nonnegative rank of a matrix; see Corollary 7.2 and Fig. 7.1. Specifically, the exact common information corresponds to the exact Rényi common information of order 1. Given a bivariate source π_{XY}, if we write its distribution as a matrix $\mathbf{M}$, then the asymptotic exponent of the nonnegative rank of $\mathbf{M}^{\otimes n}$ is the exact Rényi common information of order 0. Hence, the exact Rényi common information of order α in (11.1) simultaneously generalizes both the concepts of the exact common information and the nonnegative rank. This inspires us to define the *nonnegative α-rank*

$$\mathrm{rank}_+^{(\alpha)}(\mathbf{M}) := 2^{G_\alpha(\pi_{XY})}$$

in (7.16) in Section 7.4. This notion extends the concept of common information beyond the realm of information theory. The complete characterization of the exact Rényi common information of order α

remains open. In Corollary 7.2, we provide a single-letter expression for the exact Rényi common information only for the order ∞. Since the exact common information for the DSBS has been completely characterized, the exact Rényi common information of order 1 for the DSBS is completely characterized as well. The complete characterization of the exact Rényi common information of orders $\alpha \in [0, \infty) \setminus \{1\}$ for the DSBS and for $\alpha \in [0, \infty)$ for other sources remains open.

11.2 Open Problems Related to Gács–Körner–Witsenhausen's Common Information

In this section, we introduce several interesting open problems on the extensions of GKW's common information. These extensions mainly concern the q-stability as discussed in Section 9. Recall from Section 9.1.1 that the *noise stability* for a Boolean function $f : \mathcal{X}^n \to \{0, 1\}$ with respect to ρ is

$$\mathbf{S}_\rho[f] = \mathbb{E}[f(X^n)f(Y^n)],$$

where (X^n, Y^n) is a source sequence generated by a DSBS with correlation coefficient $\rho \in [0, 1]$. This concept can be extended to real-valued functions $f : \mathcal{X}^n \to \mathbb{R}$. For the Gaussian source with correlation coefficient $\rho \in [0, 1]$, the noise stability of f can be defined similarly. When there is no ambiguity, for Gaussian sources, we also denote the noise stability of a real-valued function f as $\mathbf{S}_\rho[f]$. For both the DSBS and the Gaussian source, the noise stability and the q-stability satisfy the relation

$$\mathbf{S}_{\rho^2}[f] = \mathbf{S}_\rho^{(2)}[f],$$

for any f and $\rho \in (0, 1)$. The same equation for the DSBS and Boolean functions is given in (9.5).

We classify open problems related to GKW's common information into three sets according to the underlying sources, namely, the DSBS, the Gaussian source, and the so-called ball- and sphere-noise source.

11.2.1 The Doubly Symmetric Binary Source

We introduce four open problems concerning the DSBS.

Determination of $q_{\min}, q_{\max}, \breve{q}_{\min}$ and $\breve{q}_{\max}$

One of main open problems on the q-stability is the determination of the thresholds $q_{\min}$ and $q_{\max}$ for the asymmetric max q-stability and $\breve{q}_{\min}$ and $\breve{q}_{\max}$ for the symmetric max q-stability given in Lemma 9.3 due to Barnes and Özgür [11]. Weaker versions of this open problem are stated in Conjectures 9.1 and 9.2, namely, the symmetric and asymmetric versions of the Mossel–O'Donnell, the Courtade–Kumar, and the Li–Médard conjectures. Although these conjectures for certain ranges of (q, ρ) have been resolved as discussed in Section 9.4, other cases remain wide open. These conjectures are significant since they connect several different fields including discrete Fourier analysis, information theory, discrete probability, etc. Among these conjectures, the Courtade–Kumar conjecture is regarded as one of the most fundamental conjectures at the interface of information theory and the analysis of Boolean functions.

Optimality of Majorities

The general open problem as discussed above on the q-stability appears to be intractable with the current set of analytical tools. A possible strategy to make some progress is to first find the structure of the optimal solutions attaining the max q-stability, and according to this observation, to prove that the optimal solutions belong to a small class of functions. If functions in this class are well-behaved, it is then relatively easy to deduce which Boolean functions *in this class* maximize the q-stability. It has been observed that for odd dimensions n and mean $a = 1/2$, the family of majority functions (defined in Section 9.1.1), which is well-behaved, may be a plausible candidate, since both dictator functions and indicators of Hamming balls are majority functions. Hence, it was conjectured by Mossel and O'Donnell [122] that Maj_n minimizes the symmetric q-stability over all *anti-symmetric* Boolean functions. We state this formally as follows.

Conjecture 11.1 (Optimality of majorities)**.** Consider the DSBS with correlation coefficient $\rho \in (0, 1)$. Fix $q > 1$ and n odd. Then, $\breve{\mathbf{S}}_{\rho}^{(q)}[f]$ is maximized among anti-symmetric Boolean functions f by a majority function Maj_m for some odd number $m \in [n]$.

In the original conjecture [122], q was restricted to be a positive integer. Conjecture 11.1 is weaker than what we hope to resolve the max q-stability problem, since only *anti-symmetric* Boolean functions are considered in this conjecture, instead of all balanced Boolean functions. Indeed, one can consider a more general question whether $\breve{\mathbf{S}}_\rho^{(q)}[f]$ is maximized by a majority function Maj_m among all *balanced* Boolean functions. If the answer is affirmative, it would have significant implications in addressing the max q-stability problem in the sense that it allow us to focus our attention *only on majority functions*.

Stability of Majorities under Bounds on Coefficients

It is also interesting to investigate the noise stability for a specific class of Boolean functions, e.g., the class of functions whose influences or Fourier coefficients are constrained. It is well known that for the majority function Maj_n, all of its Fourier coefficients vanish as $n \to \infty$; see, e.g., [131]. On the other hand, the noise stability $\mathbf{S}_\rho[\mathrm{Maj}_n]$ of Maj_n for the DSBS with correlation coefficient $\rho \in (0,1)$ satisfies

$$\lim_{n\to\infty} \mathbf{S}_\rho[\mathrm{Maj}_n] = \frac{1}{4} + \frac{\arcsin \rho}{2\pi}. \tag{11.2}$$

This can be shown similarly to (8.18) and (8.73) in which a and b are set to $1/2$ and one uses the central limit theorem to approximate the DSBS by a jointly Gaussian source. It has been conjectured by Mossel, O'Donnell, and Oleszkiewicz [123] that for all balanced Boolean functions f with small Fourier coefficients, the noise stability $\mathbf{S}_\rho[f]$ cannot exceed the right-hand side of (11.2) "by too much". This is quantified in the following conjecture.

Conjecture 11.2 (Majority is most stable under bounds on the Fourier coefficients)**.** Consider the DSBS π_{XY} with correlation coefficient $\rho \in (0,1)$ and let $(X^n, Y^n) \sim \pi_{XY}^n$. Let $f : \{0,1\}^n \to \{0,1\}$ be an arbitrary balanced function, i.e., $\mathbb{E}[f] = 2^{-n} \sum_{x^n \in \{0,1\}^n} f(x^n) = 1/2$. Then,

$$\mathbf{S}_\rho[f] \le \frac{1}{4} + \frac{\arcsin \rho}{2\pi} + \varepsilon_\rho\Big(\max_{S \subset [n]} |\hat{f}_S| \Big),$$

where $\varepsilon_\rho(\delta) \downarrow 0$ as $\delta \downarrow 0$ for each fixed $\rho \in (0,1)$.

A weaker version of this conjecture in which $\max_{S\subset[n]} |\hat{f}_S|$ is replaced by the maximal influence $\max_{i\in[n]} \mathbf{I}_i[f]$ was resolved in [123].

Extracting a Constant or Sublinear Number of Bits

In GKW's common information, the number of bits that is required to be extracted from a source $(X, Y) \sim \pi_{XY}$ is *linear* in the dimension or blocklength n. In contrast, in the Non-Interactive Correlation Distillation (NICD) or the max q-stability problem, only a *single* or a *pair* of random bits is to be extracted. A natural generalization of these two problems is to consider the "intermediate regime" in which the number of bits that one hopes to extract is more than one but sublinear in n. For example, one may wish to extract (a constant) $\ell \geq 2$ bits by using a function $f : \{0,1\}^n \to \{0,1\}^\ell$ applied to X^n and Y^n where (X^n, Y^n) is a source sequence generated by a DSBS. We require f to be balanced (i.e., $f(X^n) \sim \mathrm{Unif}\{0,1\}^\ell$), and at the same time, we aim to maximize the *agreement probability*

$$\Pr\left(f(X^n) = f(Y^n)\right) = \sum_{u^\ell\in\{0,1\}^\ell} \pi_{XY}^n(\mathcal{A}_{u^\ell} \times \mathcal{A}_{u^\ell}), \tag{11.3}$$

where $\mathcal{A}_{u^\ell} = f^{-1}(u^\ell) = \{x^n \in \{0,1\}^n : f(x^n) = u^\ell\}$ for $u^\ell \in \{0,1\}^\ell$ and π_{XY} is the distribution of the DSBS with correlation coefficient $\rho \in (0,1)$. In other words, we wish to find a partition $\{\mathcal{A}_{u^\ell}\}_{u^\ell\in\{0,1\}^\ell}$ of $\{0,1\}^n$ such that each subset $\mathcal{A}_{u^\ell}$ has the same cardinality and (11.3) is maximized. If we naïvely output the first ℓ bits x^ℓ by using the function $f(x^n) = x^\ell$—an indicator of an $(n-\ell)$-subcube (cf. Section 8.2.1)—then the induced agreement probability is exactly $\left(\frac{1+\rho}{2}\right)^\ell$. Indeed, as a consequence of our solution to (the forward part of) Mossel's mean-1/4 stability problem given in Section 8.3.3, for $\ell = 2$, the function f that outputs the first two bits attains the maximum of the agreement probability for this problem. In addition, by using the hypercontractivity inequalities in Theorem 8.5, Bogdanov and Mossel [24] showed that the maximal agreement probability

$$\max_{f:\{0,1\}^n\to\{0,1\}^\ell} \Pr\left(f(X^n) = f(Y^n)\right) \leq 2^{-\left(\frac{1-\rho}{1+\rho}\right)\ell}.$$

This upper bound is asymptotically tight as $\ell \to \infty$ in the sense that there exists $f : \{0,1\}^n \to \{0,1\}^\ell$ such that

$$\Pr\left(f(X^n) = f(Y^n)\right) \geq 0.003 \sqrt{\frac{2}{(1-\rho)\,\ell}}\, 2^{-\left(\frac{1-\rho}{1+\rho}\right)\ell}.$$

Thus, the exponents of the lower and upper bounds coincide and are equal to $\frac{1-\rho}{1+\rho}$. Determining the *exact* value of the maximum agreement probability over all balanced $\{0,1\}^\ell$-valued functions with fixed $\ell \geq 3$ remains open.

11.2.2 Gaussian Sources

We next introduce two open problems for bivariate Gaussian sources.

Standard Simplex Conjecture

We now consider a Gaussian version of the noise stability problem for balanced $[m]$-valued functions. This problem is analogous to its DSBS counterpart for $\{0,1\}^\ell$-valued functions. In the Gaussian version, we extract a pair of random variables $U = f(X^n)$ and $V = f(Y^n)$ by using a deterministic (measurable) map $f : \mathbb{R}^n \to [m]$ from a pair of length-n vectors (X^n, Y^n) drawn from a bivariate Gaussian source with correlation coefficient $\rho \in (0,1)$. We require f to be balanced in the sense that U, or equivalently V, is uniformly distributed on $[m]$. We aim to maximize the *agreement probability*

$$\Pr(U = V) = \sum_{i=1}^{m} \pi_{XY}^n(\mathcal{A}_i \times \mathcal{A}_i) \tag{11.4}$$

with $\mathcal{A}_i = f^{-1}(i)$ for $i \in [m]$. For this $[m]$-valued function version of Gaussian NICD problem, Isaksson and Mossel [87] posed the *standard simplex conjecture*. Before stating it, we have to introduce some terminology.

A *flat* or *simplex partition* $\{\mathcal{A}_i\}_{i=1}^m$ of $\mathbb{R}^n$ is one in which there exists vectors $\mathbf{a}_0 \in \mathbb{R}^n$ and $\{\mathbf{a}_i\}_{i=1}^m \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

- for all $i, j \in [m]$ such that $i \neq j$, $\mathbf{a}_i$ is not a positive multiple of $\mathbf{a}_j$;

- for all $i \in [m]$,

$$\mathcal{A}_i = \mathbf{a}_0 + \Big\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}_i, \mathbf{x}\rangle = \max_{j\in[m]} \langle \mathbf{a}_j, \mathbf{x}\rangle\Big\}.$$

A *standard simplex partition* is a flat partition $\{\mathcal{A}_i\}_{i=1}^m$ where $\|\mathbf{a}_i\|_2 = 1$ for all i and $\langle \mathbf{a}_i, \mathbf{a}_j\rangle = -\frac{1}{m-1}$ for all $i \neq j$.

Conjecture 11.3 (Standard simplex conjecture)**.** Consider the bivariate Gaussian source $(X, Y) \sim \pi_{XY}$ with correlation coefficient $0 < \rho < 1$. Then, among all partitions $\{\mathcal{A}_i\}_{i=1}^m$ of $\mathbb{R}^n$ into $3 \leq m \leq n+1$ measurable parts of equal π_{XY}^n-probability (i.e., $\pi_{XY}^n(\mathcal{A}_i) = 1/m$ for all $i \in [m]$), the agreement probability in (11.4) is *maximized* by standard simplex partitions. Furthermore, for $-1 < \rho < 0$, standard simplex partitions *minimize* the agreement probability in (11.4).

This conjecture was confirmed positively by Heilman [82] for the case $m = 3$ and in the low correlation (i.e., small ρ) regime. Specifically, for $m = 3$ and $n \geq 2$, there exists a function $\rho_0(n) > 0$ such that the conjecture holds for $0 < \rho < \rho_0(n)$. However, for other cases, Conjecture 11.3 remains open.

Symmetric Gaussian Problem

Recall that in the NICD and the max q-stability problem for the Gaussian case with mean $a = 1/2$ (cf. Sections 8.6.2 and 9.6.2), indicators of parallel halfspaces attain the forward joint probability and the max q-stability. Indicators of halfspaces are *anti-symmetric* (or *odd*) in the sense that $f(x^n) = 1 - f(-x^n)$ for almost all $x^n \in \mathbb{R}^n$; see the analogous definition for functions defined on $\{0,1\}^n$ in Section 9.1.1. It is interesting to ask which *symmetric* (or *even*) functions i.e., those that satisfy $f(x^n) = f(-x^n)$ for almost all $x^n \in \mathbb{R}^n$, maximize the joint probability in the NICD problem or the q-stability in the max q-stability problem. For Gaussian sources, Chakrabarti and Regev [34] posed the following problem.

Problem 11.1 (Symmetric Gaussian problem)**.** Fix $0 < \rho, a, b < 1$. Let $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$ have Gaussian measures a and b, respectively. Furthermore, suppose $\mathcal{A}$ is *centrally symmetric*, i.e., $\mathcal{A} = -\mathcal{A}$. What is

the maximum possible value of $\Pr(X^n \in \mathcal{A}, Y^n \in \mathcal{B})$, where X^n and Y^n are ρ-correlated n-dimensional standard Gaussian vectors?

Even though the problem statement requires that $\mathcal{A}$ is centrally symmetric, this problem is equivalent to requiring that *both* $\mathcal{A}$ and $\mathcal{B}$ are centrally symmetric [58]. Indeed, given a set $\mathcal{A}$, the optimal $\mathcal{B}$ that maximizes $\pi_{XY}(\mathcal{A} \times \mathcal{B})$ under the constraint $\pi_Y(\mathcal{B}) = b$ is the set of y such that $\frac{\mathrm{d}\pi_{Y|X}(y|\mathcal{A})}{\mathrm{d}\pi_Y(y)} \geq \lambda$ for some $\lambda > 0$. Hence, if $\mathcal{A}$ is centrally symmetric, so is the optimal $\mathcal{B}$ since for this case,

$$\frac{\mathrm{d}\pi_{Y|X}(y|\mathcal{A})}{\mathrm{d}\pi_Y(y)} = \frac{\mathrm{d}\pi_{Y|X}(-y|\mathcal{A})}{\mathrm{d}\pi_Y(-y)}.$$

It was conjectured in Chakrabarti and Regev [34] and O'Donnell [130] that $\Pr(X^n \in \mathcal{A}, Y^n \in \mathcal{B})$ is maximized by $(\mathbb{B}_r, \mathbb{B}_s)$ or by $(\mathbb{B}_r^{\mathrm{c}}, \mathbb{B}_s^{\mathrm{c}})$ for some appropriate $r, s > 0$, where $\mathbb{B}_r = \{x^n \in \mathbb{R}^n : \|x^n\|_2 \leq r\}$ denotes the ball centered at the origin with radius r. This conjecture was, however, disproved by Heilman [83] in dimensions two and higher.

11.2.3 Ball- and Sphere-Noise Sources

Up to this point, only *memoryless* sources or, equivalently, *product* distributions have been discussed. Extending the NICD and q-stability problems to sources with memory is a more challenging but fruitful endeavor, which may provide unique insights. For simplicity, here we consider *ball-* and *sphere-noise sources* since they behave similarly to memoryless sources in some sense. Hence, results on memoryless sources can be applied to ball- and sphere-noise sources.

NICD for Ball- and Sphere-Noise Sources

We first consider the ball-noise stability problem. Let $\pi_{X^nY^n}$ be a joint distribution on $\{0,1\}^n \times \{0,1\}^n$ such that $\pi_{X^n} = \mathrm{Unif}\{0,1\}^n$ and

$$Y^n = X^n \oplus Z^n = (X_i \oplus Z_i)_{i \in [n]},$$

where $Z^n \sim \mathrm{Unif}(\mathbb{B}_r)$ is independent of X^n and $\oplus$ denotes the modulo-2 sum. Here, $\mathbb{B}_r$ is the Hamming ball centered at 0^n with radius r (cf. Section 8.2.2). The distribution $\pi_{X^nY^n}$ is no longer of a product form

since the coordinates of (X^n, Y^n) are correlated through the entries of Z^n. For (n, r, M) such that $1 \le r \le n$ and $1 \le M \le 2^n$, define the *forward joint probability for* $\pi_{X^nY^n}$ (or *maximal ball-noise stability*) as

$$\Gamma^{(n)}_{\mathrm{Ball}}(M, r) := \max_{\substack{f:\{0,1\}^n \to \{0,1\} \\ \Pr(f(X^n)=1)=a}} \Pr(f(X^n) = f(Y^n) = 1),$$

where $(X^n, Y^n) \sim \pi_{X^nY^n}$ and $a = M/2^n$. For fixed $a, \beta \in (0, 1)$, define their upper and lower limits for *even* radii as $n \to \infty$ as

$$\overline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,even}}(a, \beta) := \limsup_{n\to\infty} \Gamma^{(n)}_{\mathrm{Ball}}\Big(\lfloor a2^n \rfloor, 2\Big\lfloor \frac{\beta n}{2} \Big\rfloor\Big) \quad \text{and} \tag{11.5}$$

$$\underline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,even}}(a, \beta) := \liminf_{n\to\infty} \Gamma^{(n)}_{\mathrm{Ball}}\Big(\lfloor a2^n \rfloor, 2\Big\lfloor \frac{\beta n}{2} \Big\rfloor\Big). \tag{11.6}$$

The limits for *odd* radii, denoted by $\overline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,odd}}$ and $\underline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,odd}}$, can be defined analogously but with $2\lfloor \frac{\beta n}{2} \rfloor$ in (11.5) and (11.6) replaced by $2\lfloor \frac{\beta n}{2} \rfloor + 1$. In many information-theoretic problems (e.g., error exponents for channel coding), when the dimension n is sufficiently large, the uniform distribution on the Hamming ball $\mathbb{B}_r$ can be thought of as the n-fold product of the Bernoulli distribution $\mathrm{Bern}(r/n)$. This inspires the first author of this monograph to pose the following conjecture in [194].

Conjecture 11.4 (NICD for ball-noise sources)**.** For $a, \beta \in (0, 1/2)$,

$$\underline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,even}}(a, \beta) = \overline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,even}}(a, \beta) = \overline{\Gamma}^{(\infty)}(a, a), \tag{11.7}$$

where $\overline{\Gamma}^{(\infty)}$ is the asymptotic forward joint probability for the DSBS with correlation coefficient $\rho = 1 - 2\beta$; see its definition in (8.6).

Conjecture 11.4 pertains only to even radii. For odd radii, Yu [194] showed that for $a, \beta \in (0, 1/2)$,

$$\underline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,odd}}(a, \beta) = \overline{\Gamma}^{(\infty)}_{\mathrm{Ball,\,odd}}(a, \beta) = \overline{\Gamma}^{(\infty)}(a, a). \tag{11.8}$$

The ball-noise stability problem can be interpreted as an isoperimetric problem in the r^{th} power of the Hamming graph [194]. The edge-isoperimetric inequality in Theorem 9.5 is a special case of this isoperimetric problem with $r = 1$.

In addition, similar questions can be posed when we replace the ball-noise with the sphere-noise. That is, we keep all things unchanged apart from the fact that $Z^n \sim \mathrm{Unif}(\mathbb{B}_r)$ is replaced by $Z^n \sim \mathrm{Unif}(\mathbb{S}_r)$, where $\mathbb{S}_r$ is the Hamming sphere centered at 0^n (cf. Section 8.2.3). For this case, the equalities for the odd case in (11.8) still holds. However, Yu [194] conjectured that the term $\overline{\Gamma}^{(\infty)}(a,a)$ in (11.7) should be replaced by $\frac{1}{2}\overline{\Gamma}^{(\infty)}(2a,2a)$. For more details, please refer to [194].

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