Differentiation.

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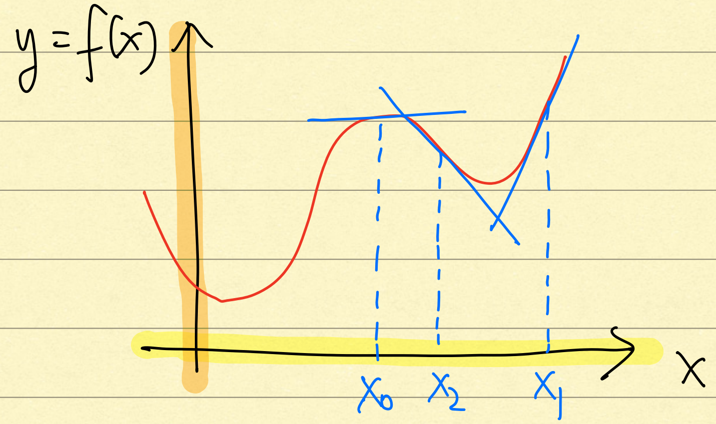


$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $\mathbb{R}$: set of all real numbers

real number

Def: The derivative of f at $a \in \mathbb{R}$ is defined as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Derivative may not exist. $\qquad$ $f'(a) = 1$

Eg 1: Find the derivative of $f(x) = x$ at $x = 1$

Consider, $\quad f'(a) = \lim_{x \to a} \frac{x-a}{x-a} = \lim_{x \to a} \frac{1}{1} = \lim_{x \to a} 1 = 1.$

$\forall a \in \mathbb{R}, \quad f'(a) = 1 \Rightarrow f'(1) = 1.$

$\{x : 0 \le x < \infty\}.$

Eg 2: $\quad f(x) = \sqrt{x} \qquad f : [0, \infty) \to \mathbb{R}$

Consider

$$f'(a) = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{\cancel{\sqrt{x}-\sqrt{a}}}{(\cancel{\sqrt{x}-\sqrt{a}})(\sqrt{x}+\sqrt{a})}$$

$$= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}} \quad ///.$$

Recall: $a^2 - c^2 = (a-c)(a+c)$

$$x-a = (\sqrt{x})^2 - (\sqrt{a})^2 = (\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})$$

Eg 3: $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=x^2 \quad f'(a) = 2a.$

$$f'(a) = \lim_{x\to a} \frac{\overset{f(x)}{x^2} - a^2 {\color{red}= f(a)}}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{x-a}}$$

$$= \lim_{x\to a} (x+a) = 2a$$

Differentiation Rules.

constant

i) Constant Multiple: $\frac{d}{dx}\left(kf(x)\right) = k\frac{df}{dx}(x)$

ii) Sum: $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{df}{dx}(x) + \frac{dg}{dx}(x).$

$\Rightarrow$ Differentiation is linear $\qquad a, b, k \in \mathbb{R}.$

scalar constants

$$\frac{d}{dx}\left(af(x) + bg(x)\right) = a\frac{df}{dx}(x) + b\frac{dg}{dx}(x).$$

iii) Difference $\quad \frac{d}{dx}\left(f(x) - g(x)\right) = \frac{df}{dx}(x) - \frac{dg(x)}{dx}$

Eg: Differentiation of a polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i = \cancel{a_0} + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$\frac{df}{dx}(x) = a_1 + 2a_2 x^1 + 3a_3 x^2 + \cdots + n a_n x^{n-1}$$

$$= \sum_{i=1}^{n} a_i \, i \, x^{i-1}$$

Question: $$g(x) = \sum_{i=0}^{\infty} a_i x^i$$

How to differentiate $g(x)$ wrt x?

Product Rule $\qquad f(x) = g(x) h(x)$

We know $g'(x)$ & $h'(x)$

$$f'(x) = g'(x) h(x) + g(x) h'(x).$$

diff fix + fix diff.

Eg: $f(x) = \overset{g(x)}{x} \underbrace{\cos x}_{h(x)}$ $\qquad g'(x) = 1, \quad h'(x) = -\sin x.$

$$f'(x) = g'(x) h(x) + g(x) h'(x).$$

$$= 1 \cdot \cos x + x(-\sin x)$$

$$= \cos x - x \sin x.$$

///.

$$\begin{cases} f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \quad = \sum_{i=0}^{\infty} \frac{x^i}{i!} \end{cases}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{e^x - e^a}{x - a}$$

$$= \lim_{x \to a} \frac{e^a(e^{x-a}) - e^a}{x - a} = \lim_{x \to a} \frac{e^a(e^{x-a} - 1)}{x - a}$$

$$= e^a \lim_{x \to a} \frac{e^{x-a} - 1}{x - a} = e^a \lim_{x \to a} \frac{(\cancel{1} + (x-a) + \frac{(x-a)^2}{2!} + \cdots) - \cancel{1}}{x - a}$$

0, 0

$$= e^a \quad ////.$$

$$e^y = 1 + y + \frac{y^2}{2!} + \cdots$$

$$\frac{(x-a)^2}{2!(x-a)} = \frac{(x-a)^1}{2!} \to 0$$

<u>Quotient Rule</u> $\qquad f(x) = \frac{g(x)}{h(x)}$,

Need to make sure that $h(x) \neq 0$ $\forall x \in \mathbb{R}$. for all

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{h(x)^2}$$

This can be derived by using the product rule.

$$f(x) = (g(x)) \cdot \left(\frac{1}{h(x)}\right) \qquad \left(\frac{1}{h(x)}\right)' = \left((h(x))^{-1}\right)'$$

$$= -h(x)^{-2} h'(x)$$

fix · diff + fix · diff

$$f'(x) = g(x)\left(-\frac{h'(x)}{h(x)^2}\right) + \frac{1}{h(x)} \cdot g'(x)$$

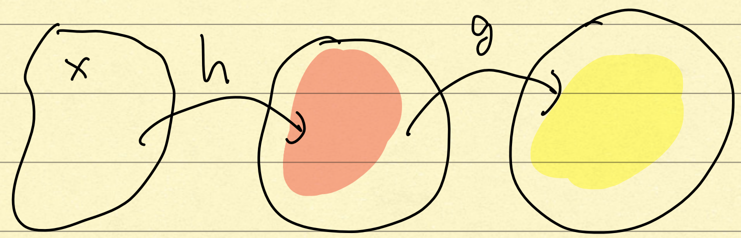
$$= \frac{h(x)\,g'(x) - g(x)\,h'(x)}{h(x)^2}$$

Product Rule $f(x) = g(x)\,h(x)$ (Product)

$$f'(x) = g'(x)\,h(x) + g(x)\,h'(x).$$

diff fix + fix diff.

Chain Rule $f(x) = g(h(x))$



Define $u(g(x), h(x)) = g(x)\,h(x) = f(x)$

$f(x) = g(h(x))$ Composition

$$f'(x) = g'(h(x)) \cdot h'(x)$$

Eg: $f(x) = \dfrac{1}{\sqrt{1+\cos x}}.$

Define $h(x) = 1+\cos x$, $g(y) = \frac{1}{\sqrt{y}} = y^{-1/2}$

$$g(h(x)) = \frac{1}{\sqrt{h(x)}} = \frac{1}{\sqrt{1+\cos x}}.$$

$$h'(x) = -\sin x, \quad g'(y) = -\frac{1}{2} y^{-3/2}$$

By the chain rule

$$\boxed{f'(x) = g'(h(x))\, h'(x)} = g'(1+\cos x)\ (-\sin x)$$

$$= -\frac{1}{2}(1+\cos x)^{-3/2}(-\sin x)$$

$$= \frac{\sin x}{2(1+\cos x)^{3/2}} \quad ////.$$

Eg: $f(x) = \tan x = \frac{\sin x}{\cos x}.$

$$f'(x) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Ex 1

1. $f(x) = \left(x^2 - \frac{2}{x^3}\right)^2$ $\qquad (-2x^{-3})' = 6x^{-4}$

$$f'(x) = 2\left(x^2 - \frac{2}{x^3}\right)^1 \left(2x + \frac{6}{x^4}\right)$$

2. $f(x) = e^{-x^2} = g(h(x)) = g(x^2)$

$$g(y) = e^{-y} \Rightarrow g'(y) = -e^{-y}$$

$$f'(x) = g'(h(x))\, h'(x)$$

$$= -e^{-x^2}(2x) = -2xe^{-x^2} \quad ////$$

3. $$f(x) = \frac{x^2+2x+2}{2x^3+x-1}$$

$$f'(x) = \frac{(2x^3+x-1)(2x+2) - (x^2+2x+2)(6x^2+1)}{(2x^3+x-1)^2}$$

4. $$f(x) = \cot x = \frac{\cos x}{\sin x}$$

$$f'(x) = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$= -\csc^2 x \qquad \csc x = \frac{1}{\sin x}$$

5. $$f(x) = \underline{x^2}\,\underline{e^x}$$

$$f'(x) = \underbrace{2x}_{(x^2)'} e^x + x^2 \underbrace{e^x}_{(e^x)'} = (2x+x^2)\, e^x.$$

6. $$f(x) = \underline{e^x}\,\underline{\cos x}$$

$$f'(x) = e^x(-\sin x) + \cos x \, e^x$$
$$= e^x(\cos x - \sin x). \quad //.$$

7. $f(x) = \underline{\sin^2 x +} \cos^2 x. = 1 \qquad f'(x) = 0.$ //

$$f'(x) = 2\sin x(\cos x) + 2\cos x(-\sin x) = 0.$$

8. $f(x) = x^3 - 4x^2 + 1$
$$f'(x) = 3x^2 - 8x. \quad //.$$

Parametric Differentiation.

$$x = v(t), \qquad y = u(t).$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{dx}.}$$

parameter

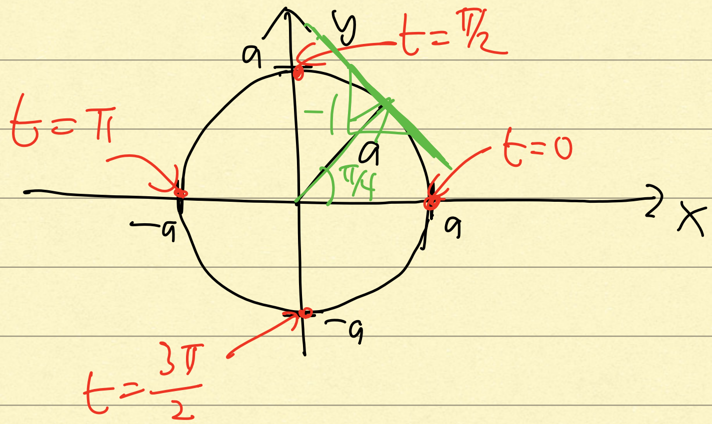
Example 8: $\underline{x = a\cos t, \ y = a\sin t.}$
Tangent to curve at $t = \pi/4$.

$t = 0, x = a, y = 0.$
$t = \frac{\pi}{2}, x = 0, y = a$
$t = \pi, x = -a, y = 0$

$$\cos t = \frac{x}{a}, \qquad \sin t = \frac{y}{a}.$$

$$\cos^2 t + \sin^2 t = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

$$\Rightarrow \quad x^2 + y^2 = a^2.$$



$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = a\cos t\left(\frac{1}{-a\sin t}\right) = -\frac{1}{\tan t}.$$

$$= -\sec t.$$

$x = a\cos t,\quad y = a\sin t.$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{dx}{dt} = -a\sin t \qquad \frac{dy}{dt} = a\cos t.$$

$$\Downarrow$$

$$\frac{dt}{dx} = \frac{1}{-a\sin t} \qquad \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\frac{1}{\tan\frac{\pi}{4}} = -1.$$

Implicit Differentiation $\quad y = f(x)$

$F(x,y) = 0.$

1. Treat y as a function of x in $F(x,y)=0$.
2. Differentiate wrt x.
   (Whenever you see y, make use of chain rule and retain dy/dx)

3. Solve for dy/dx.

Eg 7:

$$x^2+y^2-25=0 \quad = F(x,y)$$

$$\Downarrow$$

$$x^2+y(x)^2-25=0.$$

$$\Big\Downarrow \frac{d}{dx}$$

$$2x+2y\frac{dy}{dx}=0 \Longrightarrow \underline{\frac{dy}{dx}=-\frac{x}{y}}.$$

$$\frac{d}{dx}\underline{y(x)^2}=\frac{d}{dx}g(y(x))\overset{\text{Chain}}{=}g'(y(x))\,y'(x)=2y(x)y'(x)$$

$$g(y)=y^2,\quad g'(y)=2y$$

Example of implicit diff^{n}.

Find $\frac{dy}{dx}$, y' for $\underline{x^3y^5+3x=8y^3+1}$

$$\Big\Downarrow \frac{d}{dx}$$

$$x^3\left(5y^4\frac{dy}{dx}\right)+y^5(3x^2)+3=24y^2\frac{dy}{dx}$$

$$(5x^3y^4-24y^2)\frac{dy}{dx}=-3-3y^5x^2$$

$$\frac{dy}{dx}=\frac{-3-3y^5x^2}{5x^3y^4-24y^2}$$

Exercise 2

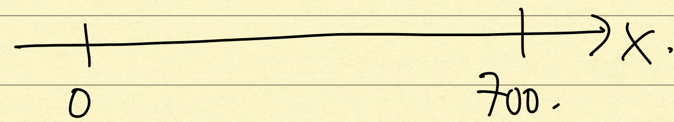
Sub $y=0$, $t=\sqrt{\frac{500}{16}}=5.59.$

1. $x=120t$

$y=-16t^2+500$

$x=120\times 5.59=670.8$

$<700.$



2 a) $x=t^2+1,\quad y=t^3+1$

$\frac{dx}{dt}=2t,\quad \frac{dy}{dt}=3t^2,\quad \frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=3t^2\cdot\frac{1}{2t}$

$=\frac{3}{2}t.$ //

b) $x=te^{2t},\quad y=2t^2+1$

$\frac{dx}{dt}=2te^{2t}+e^{2t}\cdot 1,\quad \frac{dy}{dt}=4t$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{4t}{2te^{2t}+e^{2t}}$$

3. $y^2=x^2+\sin(xy)$

$\Downarrow \frac{d}{dx}$

$$2y\frac{dy}{dx}=2x+\cos(xy)\frac{d}{dx}(xy)$$

$$=2x+\cos(xy)\left(x\frac{dy}{dx}+y\cdot 1\right).$$

$$\left(2y - x\cos(xy)\right)\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.$$

b) $y = x^x \Rightarrow \ln y = \ln(x^x) = x\ln x.$

Diff wrt x.

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + (\ln x)\cdot 1$$

$$\frac{dy}{dx} = y(1+\ln x) = \underline{\underline{x^x(1+\ln x)}}$$

$y = e^x \qquad \frac{dy}{dx} = e^x$

$y = x^{10} \qquad \frac{dy}{dx} = 10x^9$

If f has a derivative, then it is written as f'.
f' is itself a function
If f' has a derivative, then f is twice differentiable.
The derivative of f' is written as f''.

Ex: $s = f(t) = t^3 - 6t^2 + 9t.$ displacement

1. Velocity $f'(t) = 3t^2 - 12t + 9 = v(t)$.

Acceleration $f''(t) = a(t) \overset{= v'(t)}{=} 6t - 12$.

The acceleration at $t = 4$ is $6 \cdot 4 - 12 = 12\, m/s^2$.

2. Speeding up $\Rightarrow a(t) > 0$, $6t - 12 > 0 \Rightarrow \underline{t > 2}$.

Slowding down $\Rightarrow a(t) < 0$, $6t - 12 < 0$, $\underline{t < 2}$.

f'' : Second-order derivative $\quad f^{(2)}$

$f^{(n)}$: n^{th}-order derivative

Exercise 3:

1. $p(x) = x^3 + 37x^2 - 12x + 23$.

$p'(x) = 3x^2 + 74x - 12$

$\underline{p''(x) = 6x + 74}$.

2. $f(x) = e^x$

$f'(x) = e^x \qquad f''(x) = e^x$

3. $g(x) = \cos x$

$g'(x) = -\sin x.$ $\quad g''(x) = -\cos x$

4. $q(x) = (1+x)^3$

$q'(x) = 3(1+x)^2$ $\quad q''(x) = 6(1+x)$.

5. $h(x) = e^{2x}$

$h'(x) = \boxed{2e^{2x}},$ $\quad h''(x) = 4e^{2x}.$

$h(x) = g(f(x)) = g(2x)$, $\quad g(y) = e^y \Rightarrow g'(y) = e^y$

$h'(x) = g'(f(x))f'(x) = e^{2x} \cdot 2 = \boxed{2e^{2x}}$

6. $h(x) = e^{2x+1}$

$= g(f(x)) = g(2x+1)$, $\quad g(y) = e^y$

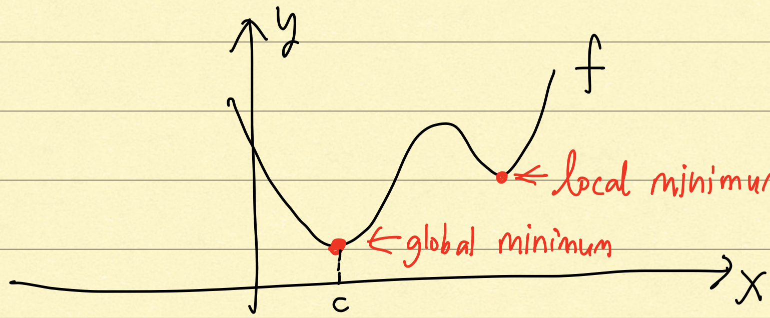
$g'(y) = e^y$

$h'(x) = g'(f(x))f'(x)$

$= e^{2x+1} \cdot 2 = 2e^{2x+1}$ //.

Finding maximum & minimum of a function.

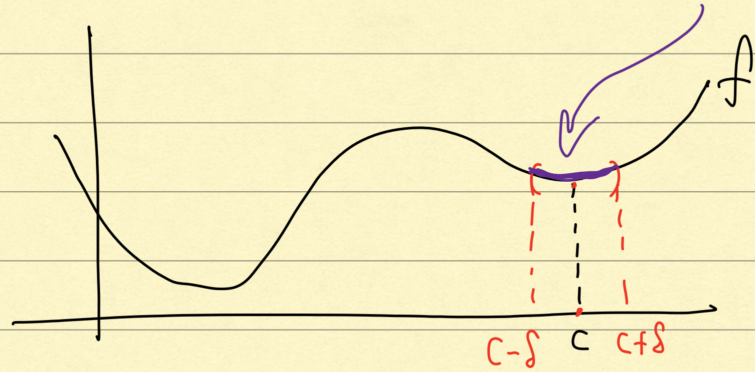
Def: A function $f: \mathbb{R} \to \mathbb{R}$ has a global minimum at $c \in \mathbb{R}$ if $f(c) \leq f(x)$ for all $x \in \mathbb{R}$.



Minimum: singular. Minima: plural.

Def: A function $f: \mathbb{R} \to \mathbb{R}$ has a local minimum at $c \in \mathbb{R}$ if there exist $\delta > 0$ such that

$$f(c) \leq f(x) \quad \text{for all } c-\delta \leq x \leq c+\delta .$$



Rmk: Every global minimum is a local minimum.

To find local minima/maxima, solve $f'(x) = 0$.
Each x s.t. $f'(x)=0$ is called a stationary point.

$f''(x) > 0$ $\Rightarrow$ x is a local <u>minimum</u>.

$f''(x) < 0$ $\Rightarrow$ x: local <u>maximum</u>.

$f''(x) = 0$ $\Rightarrow$ x: <u>pts of inflection</u>

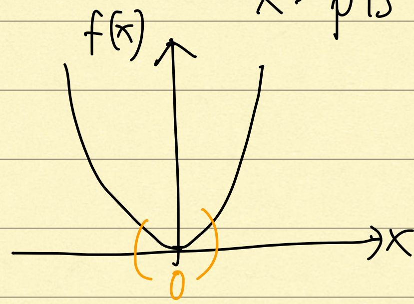
$\Downarrow$

local max/local min/neither.

$$\boxed{f(x) = x^4}$$

$$f'(x) = 4x^3 = 0$$

$$x = 0$$



$$f''(x) = 12x^2\Big|_{x=0} = \underline{0} \quad \text{inconclusive}$$

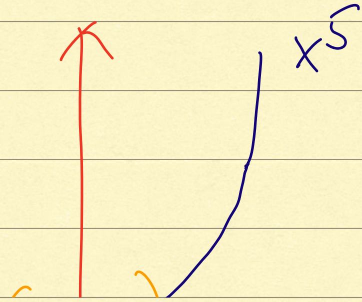
Ex:

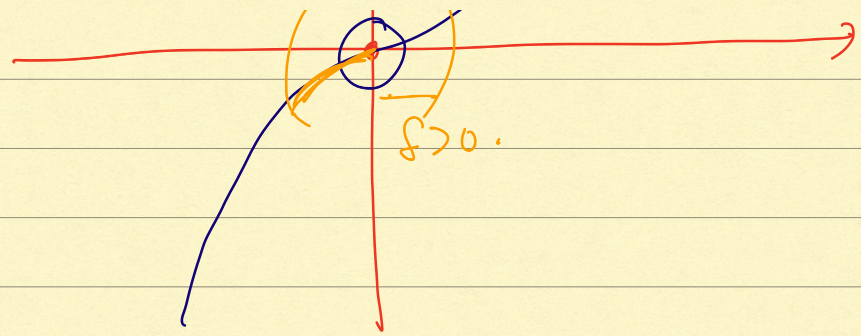
$$y = f(x) = x^5, \quad f'(x) = 0 \Rightarrow 5x^4 = 0 \Rightarrow \boxed{x = 0}$$

stationary pt.

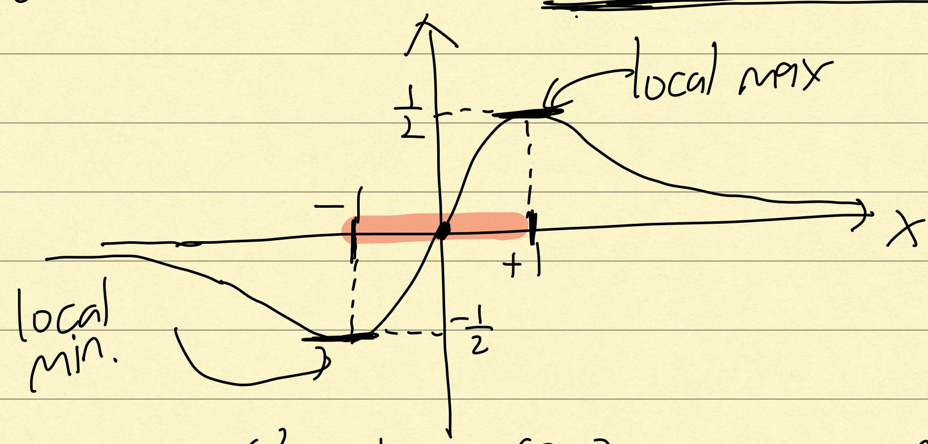
$$f''(x) = 20x^3\Big|_{x=0} = \underline{0}.$$

x is a pt of inflexion.





Eg 9: $y = f(x) = \dfrac{x}{x^2+1} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$



$$f'(x) = \frac{(x^2+1)\cdot 1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2} = 0$$

$x = \pm 1$ stationary pts.

$$f''(x) = \frac{(1+x^2)^2(-2x) - (1-x^2)2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{\cancel{(1+x^2)}\left[-2x(1+x^2) - 4x(1-x^2)\right]}{(1+x^2)^{\cancel{4}\cdot 3}}$$

$$= \frac{-6x+2x^3}{(1+x^2)^3}$$

At the stationary pt $x=1$, $f''(1)<0$ ✓
$\Rightarrow$ this is a local maximum!

At the stationary pt $x=-1$, $f''(-1)>0$ ✓
$\Rightarrow$ this is a local minimum.

Claim:
At $x=1$, f attains its global max.

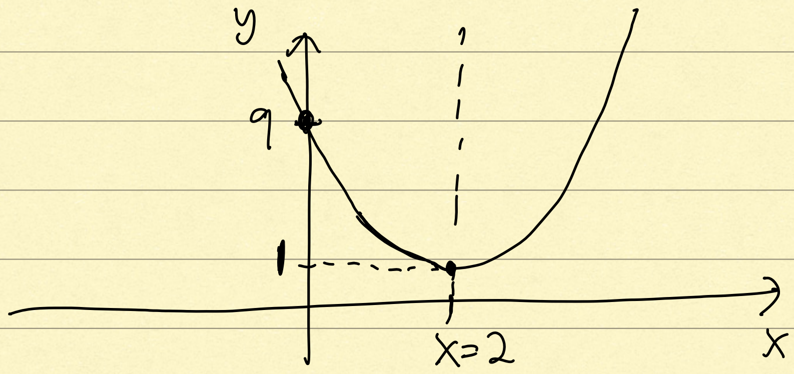
$$f(1) = \left.\frac{x}{x^2+1}\right|_{x=1} = \frac{1}{2}.$$

Ex 4

1a) $y = 2x^2 - 8x + 9$

$$\frac{dy}{dx} = 4x - 8 = 0, \quad x = 2$$

$$\frac{d^2y}{dx^2} = 2 > 0 \Rightarrow \text{local min}$$



b) $y = x^3 + x^2 - 8x + 5$

$$\frac{dy}{dx} = 3x^2 + 2x - 8 = 0.$$

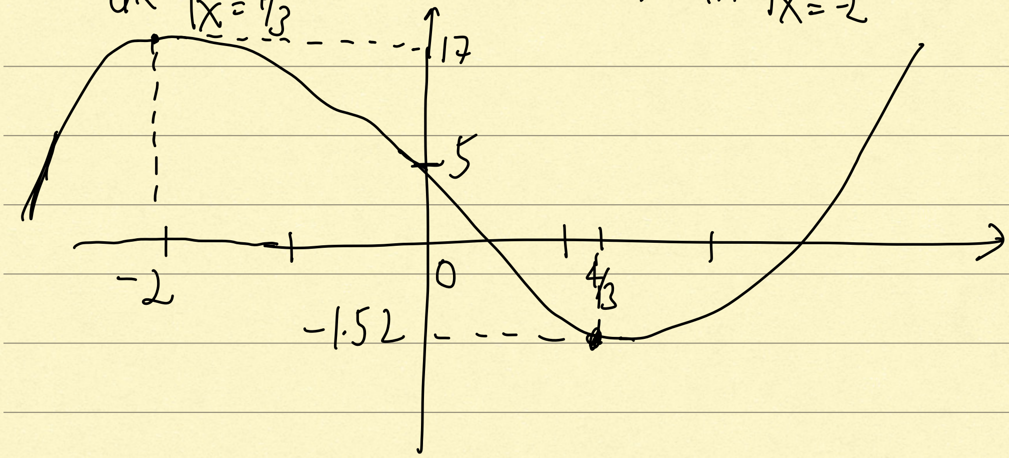
$$(3x-4)(x+2)=0$$

$f(x) = -1.52 \Leftarrow \underline{x = 4/3}$, $\underline{x = -2} \Rightarrow f(x) = -8+4+16+5 = 17$

local min   local max.

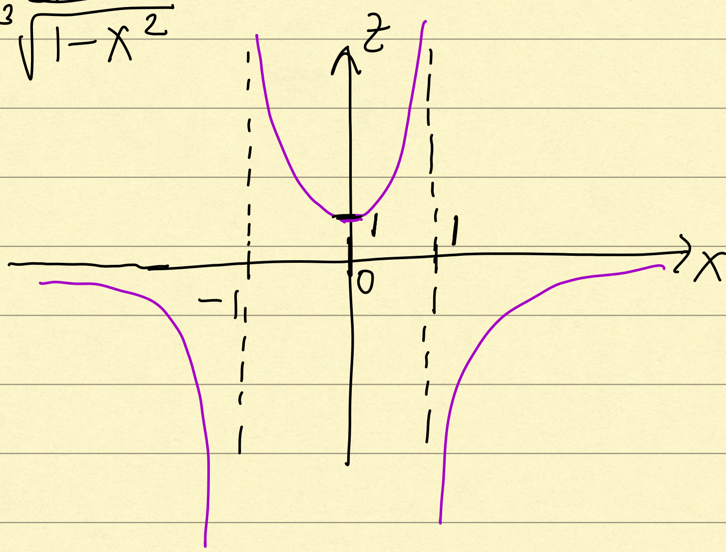
$$\frac{d^2y}{dx^2} = 6x+2$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=4/3} = 6\left(\frac{4}{3}\right)+2 > 0, \quad \left.\frac{d^2y}{dx^2}\right|_{x=-2} = 6(-2)+2 < 0.$$



c) $y = \dfrac{1}{\sqrt[3]{1-x^2}}$

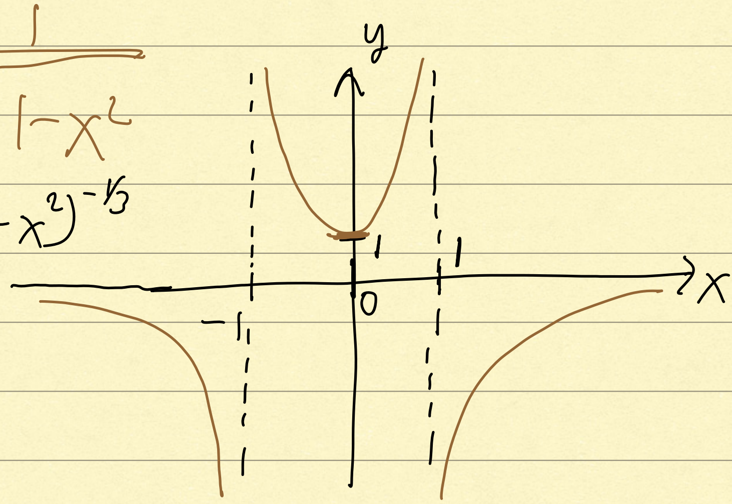
$z = \dfrac{1}{1-x^2}$



$y = z^{1/3}$

$$y = \frac{1}{\sqrt[3]{1-x^2}}$$

$$= (1-x^2)^{-1/3}$$



$$\frac{dy}{dx} = -\frac{1}{3}(1-x^2)^{-4/3}(-2x) = 0 \Rightarrow x = 0$$

stationary pt.

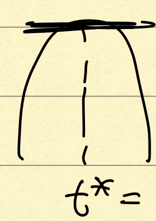
$$\parallel$$

$$\frac{2}{3}x(1-x^2)^{-4/3}$$

$$\frac{d^2y}{dx^2} = \cancelto{0}{\frac{2}{3}x\left[-\frac{4}{3}(1-x^2)^{-7/3}(-2x)\right]} + \underline{(1-x^2)^{-4/3}\,\frac{2}{3}}.$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{2}{3} > 0 \Rightarrow 0 \text{ is a local min.}$$

2. $s = -\frac{1}{2}gt^2 + v_0 t + s_0$



$$s'(t) = -gt + v_0 = 0$$

$$t^* = \frac{v_0}{g}$$

$$s''(t) = -g < 0 \Rightarrow t^* = \frac{v_0}{g} \text{ is local max}$$

$$s(t^*) = -\frac{1}{2} g \left(\frac{v_0}{g}\right)^2 + v_0 \left(\frac{v_0}{g}\right) + S_0$$

$$= \frac{v_0^2}{2g} + S_0 . \quad //$$

3. $$n = \frac{a}{x-c} + b(100-x)$$

Profit $p(x) =$ Money obtained − Manufacturing cost

= # sold × Selling price.

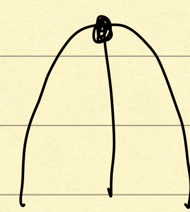
− # sold × manufacturing cost per bag.

$$= \left(\frac{a}{x-c} + b(100-x)\right) \cdot x$$

$$- \left(\frac{a}{x-c} + b(100-x)\right) \cdot c.$$

$$= \left(\frac{a}{x-c} + b(100-x)\right)(x-c)$$

$$p(x) = a + b(100-x)(x-c)$$



$$\frac{dp(x)}{dx} = b\left[(100-x)(1) + (x-c)(-1)\right] = 0$$

$$100 - x - x + c = 0.$$

$$x = \frac{100+c}{2} \quad ///$$

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https://vyftan.github.io/math_refresher.pdf.