THE HIDDEN INTERSECTIONS: RESOLVING THE BANACH–TARSKI PARADOX VIA POINT-FREE TOPOLOGY

D. ARNOLD

Abstract. The Banach–Tarski paradox asserts that a solid ball in $\mathbb{R}^3$ can be decomposed into finitely many disjoint pieces and reassembled by rigid motions into two balls identical to the original. We argue that this result is not a genuine property of three-dimensional space but an artifact of *point-set topology*, which models regions as sets of points and defines disjointness as the absence of shared points. When regions are instead modeled as *sublocales*—the natural parts of a space in point-free topology—the Banach–Tarski pieces are revealed to be *not disjoint*: they overlap on the topological structure that bonds neighboring points together. Working in full ZFC, we develop the locale-theoretic framework from first principles, prove that Lebesgue measure extends to an isometry-invariant measure on *all* sublocales of $\mathbb{R}^n$, and demonstrate that the paradox's arithmetic—which requires additivity over disjoint pieces—simply does not apply. We further show that the Dougherty–Foreman theorem on paradoxical decompositions with Baire-measurable pieces, which might appear to undermine a measure-theoretic resolution, is also explained by the locale-theoretic framework. No axiom is rejected; no mathematics is lost. The paradox is a *misdiagnosis*: the Axiom of Choice is not the root cause, and the real culprit is a notion of "part" too impoverished to track measure under decomposition.

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1. Introduction

In 1924, Stefan Banach and Alfred Tarski proved one of the most counterintuitive results in all of mathematics: a solid ball in three-dimensional Euclidean space can be cut into five pieces, and those pieces can be rigidly moved—using only rotations and translations—to form *two* solid balls, each identical in size to the original [1]. The result extends: any two bounded subsets of $\mathbb{R}^3$ with nonempty interior are *equidecomposable*, meaning a pea can be rearranged into the sun.

The theorem is universally called a “paradox” because it appears to violate conservation of volume. The standard explanation points to the *Axiom of Choice* (AC): the pieces are non-measurable sets, conjured into existence by AC, and since Lebesgue measure cannot be assigned to them, volume is simply undefined for the pieces. No contradiction arises because the accounting is impossible—there is nothing to add up.

This explanation is correct as far as it goes, but it is incomplete. It locates the strangeness in AC without explaining *why* the formalism permits volume to vanish. It leaves open an uncomfortable question: is the Axiom of Choice defective? Should it be restricted or abandoned?

We argue that the question is misframed. The Axiom of Choice is not the culprit. The real source of the paradox is an impoverished notion of *part of a space*—the identification of “region” with “set of points” that is the foundational assumption of point-set topology. This identification discards the continuity structure that bonds points together, and it is precisely this discarded structure that carries the “missing” volume.

The resolution comes from *point-free topology* (locale theory), a framework in which a space is defined not by its points but by its lattice of open sets. In this framework, the natural notion of “part” is a *sublocale* rather than a subset. Two subsets that share no point can nonetheless overlap as sublocales—they share the topological “glue” connecting neighboring points. When the Banach–Tarski pieces are viewed as sublocales, they are *not disjoint*, and the finite-additivity argument that would force $\mu(B) = 2\mu(B)$ does not apply.

This resolution, implicit in unpublished work of Olivier Leroy from circa 1995 [12]—a mathematician in Grothendieck's circle at Montpellier, whose manuscript was transcribed and deposited only posthumously—and made rigorous by Alex Simpson in 2012 [17], has not received the attention it deserves. Simpson's paper treats it as a motivational remark in a study of randomness; Leroy's manuscript, in French, circulated only informally until 2012. No focused, self-contained treatment exists. We aim to fill this gap.

A potential objection to the locale-theoretic resolution is raised by the remarkable 1994 result of Dougherty and Foreman [2], who showed that paradoxical decompositions can be performed using pieces with the *Baire property*—a topological regularity condition—and that certain versions require *no appeal to the Axiom of Choice whatsoever*. If the paradox persists without AC, how can any measure-theoretic resolution succeed? We show in Section 9 that the locale framework handles this case as well: the Dougherty–Foreman pieces, being dense open sets, produce dense sublocales whose overlaps again carry full measure. The resolution is robust.

The paper is organized as follows.

- Section 2 reviews the Banach–Tarski theorem with a detailed proof sketch, including the ping-pong lemma and the role of amenability.
- Section 3 isolates the foundational assumption responsible and situates our response among the existing ones.
- Section 4 develops locale theory (frames, sublocales, nuclei) from first principles, with full proofs.
- Section 5 establishes the critical distinction between subsets and sublocales, with explicit computations showing that the rationals and irrationals are *not disjoint* as sublocales of $\mathbb{R}$.
- Section 6 proves Isbell's density theorem with complete details.
- Section 7 presents Simpson's measure extension theorem with a proof sketch of the key construction.
- Section 8 applies the machinery to the Banach–Tarski decomposition.
- Section 9 treats the Dougherty–Foreman theorem and explains why it does not undermine the resolution.
- Section 10 argues that locale theory is the *correct* framework for spatial reasoning.
- Section 11 summarizes the resolution and its implications.

2. The Banach–Tarski Theorem

We begin with a precise statement of the theorem and a detailed sketch of its proof, emphasizing the logical dependencies that will become important later.

Definition 2.1. Let a group G act on a set X. Two subsets $A, B \subseteq X$ are *G-equidecomposable*, written $A \sim_G B$, if there exist partitions $A = A_1 \sqcup \cdots \sqcup A_n$ and $B = B_1 \sqcup \cdots \sqcup B_n$ and elements $g_1, \ldots, g_n \in G$ such that $B_i = g_i(A_i)$ for each i. A subset $E \subseteq X$ is *G-paradoxical* if there exist disjoint $A, B \subseteq E$ with $A \sim_G E$ and $B \sim_G E$.

Theorem 2.2 (Banach–Tarski, 1924)**.** *Let B^3 denote the closed unit ball in $\mathbb{R}^3$, and let G be the group of Euclidean isometries of $\mathbb{R}^3$. Then B^3 is G-paradoxical. More precisely, B^3 can be partitioned into five sets $A_1, \ldots, A_5$ such that, for suitable isometries $g_1, \ldots, g_5$:*

- *$g_1(A_1) \cup g_2(A_2)$ is a copy of B^3, and*
- *$g_3(A_3) \cup g_4(A_4) \cup g_5(A_5)$ is another copy of B^3.*

Five is optimal: four pieces do not suffice [16].

2.1. **Proof sketch.** The proof proceeds through four stages. We give more detail than is customary, as the specific properties of the construction—especially the *density* of the pieces—will be essential to our resolution.

Stage 1: The free group is paradoxical. The free group $F_2 = \langle a, b \rangle$ on two generators consists of all finite reduced words in the alphabet $\{a, a^{-1}, b, b^{-1}\}$, where "reduced" means no symbol appears adjacent to its inverse. For each symbol x, let $W(x)$ denote the set of reduced words beginning with x. Then

$$F_2 = \{e\} \sqcup W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}).$$

The key identities are:

$$F_2 = W(a) \cup a \cdot W(a^{-1}) \qquad \text{and} \qquad F_2 = W(b) \cup b \cdot W(b^{-1}). \tag{1}$$

The first holds because left-multiplying any word *not* beginning with a by a produces a word beginning with a, and conversely, left-multiplying any word in $W(a^{-1})$ by a cancels the leading a^{-1}, producing a word not beginning with a. Hence $a \cdot W(a^{-1}) = F_2 \setminus W(a)$. The second identity is symmetric.

Absorbing the identity element into $W(a)$ by a standard trick (defining $W'(a) = W(a) \cup \{e, a^{-1}, a^{-2}, \ldots\}$ and adjusting $W(a^{-1})$ accordingly), one obtains an exact four-piece paradoxical decomposition. This stage is purely algebraic and requires no axioms beyond ZF. The *Tarski number* of F_2 is 4.

Stage 2: F_2 embeds in $\mathrm{SO}(3)$ via the ping-pong lemma.

Lemma 2.3 (Ping-Pong Lemma)**.** *Let a group G act on a set X. Suppose $\alpha, \beta \in G$ and there exist nonempty disjoint subsets $X_1, X_2 \subset X$ such that for all nonzero integers n:*

$$\alpha^n(X \setminus X_1) \subseteq X_1 \qquad \textit{and} \qquad \beta^n(X \setminus X_2) \subseteq X_2.$$

Then $\langle \alpha, \beta \rangle \cong F_2$.

We apply this to specific rotations. Let $\theta = \arccos(1/3)$ and define:

$$A = R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix}, \quad B = R_z(\theta) = \begin{pmatrix} 1/3 & -2\sqrt{2}/3 & 0 \\ 2\sqrt{2}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The choice $\cos\theta = 1/3$ is strategic: the matrix entries are algebraic numbers whose denominators involve only powers of 3. One verifies the ping-pong conditions by choosing X_1 and X_2 to be subsets of S^2 defined by congruence conditions on the denominators when coordinates are expressed in a suitable basis. The key technical fact is that applying any nontrivial reduced word in A, B, A^{-1}, B^{-1} to a test point (say, $(1, 0, 0)$) yields a point whose coordinates, when written with a common denominator, have that denominator equal to a specific power of 3 that depends only on the word length—ensuring the result is never the starting point [19].

Stage 3: S^2 is paradoxical (the Axiom of Choice). The group $H = \langle A, B \rangle \cong F_2$ acts on S^2 by rotations. Let $D \subset S^2$ be the countable set of points fixed by some nontrivial element of H. Since H is countable and each nontrivial rotation fixes exactly two points (the intersection of its axis with S^2), the set D is countable.

On $S^2 \setminus D$, the action of H is *free*: if $h(p) = p$ for $h \neq e$ and $p \notin D$, then p would be a fixed point of h, contradicting $p \notin D$. The free action partitions $S^2 \setminus D$ into orbits. Each orbit is countable (as H is countable), while $S^2 \setminus D$ is uncountable, so there are uncountably many orbits.

The Axiom of Choice is invoked here: select a set M containing exactly one representative from each orbit. There is no algorithm, formula, or definable rule for this selection—it is a pure existence assertion. This is the *only* non-constructive step in the proof.

Using M, define $T^* = \{h(m) : h \in T, m \in M\}$ for any $T \subseteq H$. The paradoxical decomposition of (1) transfers directly:

$$S^2 \setminus D = W(A)^* \sqcup W(A^{-1})^* \sqcup W(B)^* \sqcup W(B^{-1})^* \sqcup M,$$
$$S^2 \setminus D = W(A)^* \cup A \cdot W(A^{-1})^*,$$
$$S^2 \setminus D = W(B)^* \cup B \cdot W(B^{-1})^*.$$

To absorb the countable set D: since D is countable and S^2 is uncountable, there exists a rotation $\rho \in \mathrm{SO}(3)$ such that the sets $D, \rho(D), \rho^2(D), \ldots$ are pairwise disjoint (only countably many rotation angles are "bad," so a good one exists). Define $E = \bigcup_{n=0}^{\infty} \rho^n(D)$. Then $\rho(E) = E \setminus D$, so $S^2 \setminus D$ is equidecomposable with S^2 (using two pieces: E maps to $\rho(E)$, and $S^2 \setminus E$ stays fixed).

Stage 4: From S^2 to B^3. Each point $p \in B^3 \setminus \{0\}$ lies on a unique ray from the origin. This ray meets S^2 at a unique point $\pi(p)$. The decomposition of S^2 induces a decomposition of $B^3 \setminus \{0\}$ by taking *cones*: if $S^2 = C_1 \sqcup \cdots \sqcup C_k$, define $\hat{C}_i = \{tp : p \in C_i, 0 < t \leq 1\}$. Rotations preserve cones (they commute with radial projection), so the equidecomposition of S^2 lifts to one of $B^3 \setminus \{0\}$. The center is absorbed by a Hilbert Hotel trick analogous to the one for D.

Remark 2.4 (Density of the pieces)**.** The construction produces pieces that are **everywhere dense** in B^3. The orbits of H on S^2 are dense because H acts by irrational rotations: for any open set $U \subseteq S^2$ and any $p \in M \cap U$, the orbit $\{h(p) : h \in H\}$ is dense in S^2 (this follows from the equidistribution of irrational rotations on circles, applied to the projection of the orbit onto great circles). Since each piece A_i is a union of radial cones over subsets of these dense orbits, each A_i meets every open ball in B^3. This density is not incidental—it is *essential* to the construction, and it is what makes the pieces non-measurable. It is also what makes them overlap as sublocales.

2.2. **The paradoxical arithmetic.** Suppose, for contradiction, that a finitely additive, isometry-invariant measure μ existed on *all* subsets of $\mathbb{R}^3$ with $\mu(B^3) > 0$. The decomposition $B^3 = A_1 \sqcup \cdots \sqcup A_5$ and the reassembly into two copies of B^3 would give

$$\mu(B^3) \;=\; \sum_{i=1}^{5} \mu(A_i) \;=\; \sum_{i=1}^{5} \mu(g_i(A_i)) \;=\; 2\,\mu(B^3), \tag{2}$$

forcing $\mu(B^3) = 0$, a contradiction.

- The first equality uses **finite additivity over disjoint sets**: $\mu(A \sqcup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$.
- The second uses **isometry invariance**: $\mu(g(A)) = \mu(A)$.
- The third uses finite additivity again (the reassembled pieces partition two copies of B^3).

Remark 2.5. The entire paradox rests on the first equality: *finite additivity over disjoint sets.* If the pieces are not genuinely disjoint—if their "intersection" carries positive measure—then this step is invalid, and no contradiction arises. This is exactly what happens in locale theory.

2.3. **The dimensional divide: amenability.** A natural question is why the paradox works in $\mathbb{R}^3$ but fails in $\mathbb{R}^1$ and $\mathbb{R}^2$. The answer, due to von Neumann [20], is *amenability.*

Definition 2.6. A group G is *amenable* if there exists a finitely additive, left-invariant probability measure defined on all subsets of G.

Theorem 2.7 (Tarski's Alternative, 1938)**.** *Let G act on a set X, and let $E \subseteq X$. Then E is* not *G-paradoxical if and only if there exists a finitely additive, G-invariant measure on all subsets of X with $\mu(E) = 1$.*

The isometry groups of $\mathbb{R}^1$ (translations and reflections) and $\mathbb{R}^2$ (the Euclidean group $\mathrm{SO}(2) \ltimes \mathbb{R}^2$) are *solvable*, hence amenable. Banach proved in 1923 that finitely additive, isometry-invariant measures on all subsets of $\mathbb{R}^1$ and $\mathbb{R}^2$ exist (the *Banach measures*), ruling out paradoxical decompositions. By contrast, SO(3) contains the free group F_2, which is *not* amenable (a finitely additive invariant mean on F_2 would give $1 \geq \mu(W(a)) + \mu(W(a^{-1})) + \mu(W(b)) + \mu(W(b^{-1})) \geq 2$, a contradiction). The non-amenability of SO(3) is precisely what enables the paradox in dimension ≥ 3.

2.4. **The choice hierarchy.** The Banach–Tarski paradox does not require the full Axiom of Choice. Pawlikowski [14] proved that the *Hahn–Banach theorem* suffices, which follows from the *Ultrafilter Lemma* (strictly weaker than AC). The hierarchy is:

$$\text{AC} \implies \text{Ultrafilter Lemma} \implies \text{Hahn–Banach} \implies \text{Banach–Tarski},$$

where each implication is strict. Conversely, the paradox is *not* provable in ZF + DC (Dependent Choice): in Solovay's model [18], every set of reals is Lebesgue measurable, blocking the construction.

3. The Diagnosis

The standard response to the Banach–Tarski paradox takes one of two forms:

(a) **Accept it**: the pieces are non-measurable, so $\mu(A_i)$ is undefined; equation (2) cannot even be written.
(b) **Reject AC**: work in a model of ZF (such as Solovay's model [18]) where all sets of reals are Lebesgue measurable, so the decomposition cannot be constructed.

Response (a) is unsatisfying: it declares the question unanswerable rather than answering it. It provides no positive account of what happens to volume during the decomposition. Response (b) is drastic: it abandons the Axiom of Choice, losing the Hahn–Banach theorem, Tychonoff's theorem, the well-ordering principle, and the guarantee that every vector space has a basis. Moreover, as Karagila [9] has observed, the Solovay model produces its own pathologies—for instance, a partition of $\mathbb{R}$ into strictly more equivalence classes than there are real numbers. And as we shall see in Section 9, the Dougherty–Foreman result shows that some forms of the paradox survive even without AC.

We propose a third response:

> **(c) Fix the notion of "part."** The paradox does not arise from a defective axiom but from a defective *definition.* Point-set topology defines a "part of a space" as a subset—a collection of points. This definition discards the topological structure that connects points to their neighbors. When parts are instead defined as *sublocales*—which retain this structure—the Banach–Tarski pieces are revealed to be *not disjoint*, and the arithmetic of (2) does not apply.

The key insight is that two subsets can share no points and yet *overlap* as parts of a space. Point-set topology is blind to this overlap because it defines disjointness purely in terms of shared points. Locale theory sees it because its notion of "part" is richer than a mere collection of points.

In the next sections, we make this precise.

4. Locale Theory from First Principles

We develop the theory of frames, locales, and sublocales, following Johnstone [8] and Picado–Pultr [15]. The reader familiar with locale theory may skip to Section 5.

4.1. **Frames.** The open sets of any topological space X form a complete lattice $\mathscr{O}(X)$ under inclusion, with arbitrary unions as joins and finite intersections as meets. This lattice satisfies a characteristic distributive law.

Definition 4.1. A *frame* is a complete lattice L satisfying the *infinite distributive law*:

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i) \tag{3}$$

for all $a \in L$ and all families $\{b_i\}_{i\in I}$ in L. A *frame homomorphism* $h\colon L \to M$ is a map preserving arbitrary joins and finite meets. The resulting category is denoted **Frm**.

Remark 4.2. Every frame is a *complete Heyting algebra*. The infinite distributive law is equivalent to the statement that for each $a \in L$, the functor $a \wedge (-)$ preserves arbitrary joins. Since L is complete and $a \wedge (-)$ preserves joins, it has a right adjoint by the adjoint functor theorem for posets. This right adjoint is the *Heyting implication*:

$$a \to b = \bigvee\{c \in L : a \wedge c \le b\}.$$

Concretely, $a \to b$ is the largest element c such that $a \wedge c \le b$. The adjunction is:

$$a \wedge c \le b \iff c \le a \to b.$$

The *pseudocomplement* of a is $\neg a = a \to \bot$, the largest element disjoint from a.

Example 4.3. For any topological space X, the lattice $\mathscr{O}(X)$ of open subsets is a frame with $\bigvee U_i = \bigcup U_i$ and $U \wedge V = U \cap V$. The Heyting implication is

$$U \to V = \operatorname{int}\big((X \setminus U) \cup V\big),$$

and the pseudocomplement is $\neg U = \operatorname{int}(X \setminus U)$—the *exterior* of U.

Example 4.4. In $\mathscr{O}(\mathbb{R})$, consider $U = (-\infty, 0)\cup(0, \infty) = \mathbb{R}\setminus\{0\}$. Then $\neg U = \operatorname{int}(\{0\}) = \emptyset$, so $U \vee \neg U = U \ne \mathbb{R} = \top$. The pseudocomplement is *not* a complement: U and $\neg U$ do not join to $\top$. This failure of the complement law—the failure of L to be a Boolean algebra—is the structural feature that will allow non-disjointness of dense sublocales.

4.2. **Locales.**

Definition 4.5. The category of *locales* is the opposite of the category of frames:

$$\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}.$$

A locale X is determined by its *frame of opens* $\mathscr{O}(X) \in \mathbf{Frm}$. A *continuous map* $f\colon X \to Y$ of locales is a frame homomorphism $f^*\colon \mathscr{O}(Y) \to \mathscr{O}(X)$ in the reverse direction.

There is an adjunction between **Top** and **Loc**:

$$\mathscr{O}\colon \mathbf{Top} \to \mathbf{Loc} \qquad \text{and} \qquad \mathrm{pt}\colon \mathbf{Loc} \to \mathbf{Top},$$

where $\mathrm{pt}(L)$ is the space of *points* of L—the frame homomorphisms $p\colon L \to \{0, 1\}$ (equivalently, the completely prime filters of L). This adjunction restricts to an equivalence between *sober* spaces (where every irreducible closed set is the closure of a unique point) and *spatial* locales (those with "enough points" to separate opens). All Hausdorff spaces are sober, and all metrizable locales are spatial, so for "reasonable" spaces the two theories agree on *objects*. They diverge dramatically on *parts*.

4.3. **Sublocales and nuclei.** In point-set topology, a subspace of X is a subset $A \subseteq X$ equipped with the induced topology. The locale-theoretic analogue is a *sublocale*, which is richer: there are typically far more sublocales than subsets.

Definition 4.6. A *nucleus* on a frame L is a function $j\colon L \to L$ satisfying:
(i) $a \leq j(a)$ for all a (inflationary),
(ii) $j(j(a)) = j(a)$ for all a (idempotent),
(iii) $j(a \wedge b) = j(a) \wedge j(b)$ for all a, b (preserves finite meets).

Proposition 4.7. *If j is a nucleus on L, then the set of fixed points $L_j = \{a \in L : j(a) = a\}$ is a frame, with:*
- *meets computed as in L: $a \wedge_j b = a \wedge b$,*
- *joins computed by: $\bigvee_j a_i = j(\bigvee a_i)$,*
- *top element $\top_j = \top$,*
- *bottom element $\bot_j = j(\bot)$.*

The map $j\colon L \to L_j$ is a frame surjection (preserves finite meets and arbitrary joins when joins are computed in L_j).

Proof. The fixed-point set L_j is closed under arbitrary meets in L: if $j(a_i) = a_i$ for all i, then $j(\bigwedge a_i) \leq j(a_i) = a_i$ for each i (by monotonicity from (i) and (iii)), so $j(\bigwedge a_i) \leq \bigwedge a_i$. Combined with (i), $j(\bigwedge a_i) = \bigwedge a_i$. Binary meets are a special case. For joins: $j(\bigvee a_i)$ is the smallest fixed point above $\bigvee a_i$ (by (i) and (ii)), hence the join in L_j. The infinite distributive law for L_j follows from that of L and the meet-preservation of j. □

Definition 4.8. A *sublocale* of a locale with frame L is a subset $S \subseteq L$ that is:
(i) closed under arbitrary meets (taken in L), and
(ii) for all $s \in S$ and $a \in L$, one has $a \to s \in S$.

The equivalence between nuclei and sublocales is: $j \mapsto L_j$ and $S \mapsto j_S$ where $j_S(a) = \bigwedge\{s \in S : a \leq s\}$.

Proposition 4.9 (The coframe of sublocales)**.** *The collection $\mathscr{S}(L)$ of all sublocales of L, ordered by inclusion, forms a* coframe*—that is, $\mathscr{S}(L)^{\mathrm{op}}$ is a frame. The lattice operations are:*
- ***Meet:*** $\bigwedge_i S_i = \bigcap_i S_i$ *(set-theoretic intersection).*
- ***Binary join:*** $S_1 \vee S_2 = \{s_1 \wedge s_2 : s_1 \in S_1,\ s_2 \in S_2\}$.
- ***Top:*** *L itself (the whole locale).*
- ***Bottom:*** $\{\top\}$ *(the void sublocale).*

Proof. We verify the key claims.

Meets are intersections. If each S_i is closed under arbitrary meets and the Heyting implication, so is $\bigcap S_i$: if $\{s_k\} \subseteq \bigcap S_i$, then $\{s_k\} \subseteq S_i$ for each i, so $\bigwedge s_k \in S_i$ for each i, hence $\bigwedge s_k \in \bigcap S_i$. Similarly for Heyting closure.

Binary joins. We must show $S_1 \vee S_2 = \{s_1 \wedge s_2 : s_1 \in S_1, s_2 \in S_2\}$ is a sublocale. Closure under meets: if $\{s_{1,k} \wedge s_{2,k}\}_{k\in K} \subseteq S_1 \vee S_2$, then $\bigwedge_k(s_{1,k} \wedge s_{2,k}) = (\bigwedge_k s_{1,k}) \wedge (\bigwedge_k s_{2,k})$ (this identity holds in any frame), and $\bigwedge s_{1,k} \in S_1$, $\bigwedge s_{2,k} \in S_2$. Heyting closure: for $a \in L$ and $s_1 \wedge s_2 \in S_1 \vee S_2$, we have $a \to (s_1 \wedge s_2) = (a \to s_1) \wedge (a \to s_2)$ (a standard identity in Heyting algebras), and $a \to s_i \in S_i$.

The void sublocale is $\{\top\}$. It is the smallest sublocale: $\top$ is the meet of the empty family and $a \to \top = \top$ for all a.

Coframe distributivity. We must show that meets distribute over arbitrary joins: $S \cap \bigvee_i S_i = \bigvee_i (S \cap S_i)$. The inclusion ($\supseteq$) is straightforward: each $S \cap S_i \subseteq S \cap \bigvee S_i$, so $\bigvee(S \cap S_i) \subseteq S \cap \bigvee S_i$. For the reverse inclusion ($\subseteq$), the argument uses the nucleus characterization: if j and j_i are the nuclei for S and S_i respectively, then the nucleus for the join $S_i \vee S_k$ acts by $a \mapsto j_i(a) \wedge j_k(a)$, and one verifies that $j(j_i(a) \wedge j_k(a)) = j(j_i(a)) \wedge j(j_k(a))$ (since j preserves meets), which yields the required containment. The full proof is given in Picado–Pultr [15], Proposition II.2.5, and Johnstone [8], II.2.4. $\square$

Remark 4.10. The coframe $\mathscr{S}(L)$ is **not** a Boolean algebra in general. The power set $\mathscr{P}(X)$ of a set X is always a Boolean algebra—every subset has a complement. In $\mathscr{S}(L)$, only certain sublocales (the open and closed ones, see below) are complemented. This non-Boolean structure is the root of our resolution: it allows two sublocales to fail to be disjoint even when their "point-sets" are disjoint.

4.4. **Open and closed sublocales.** Every element $a \in L$ determines two complementary sublocales:

Definition 4.11. For $a \in L$, define:

- The *open sublocale*: $\mathfrak{o}(a) = \{a \to x : x \in L\}$, with nucleus $j_{\mathfrak{o}(a)}(x) = a \to x$.
- The *closed sublocale*: $\mathfrak{c}(a) = {\uparrow}a = \{x \in L : x \geq a\}$, with nucleus $j_{\mathfrak{c}(a)}(x) = a \vee x$.

Proposition 4.12. *For every $a \in L$:*

$$\mathfrak{o}(a) \wedge \mathfrak{c}(a) = \{\top\} \qquad \textit{and} \qquad \mathfrak{o}(a) \vee \mathfrak{c}(a) = L.$$

That is, $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements in $\mathscr{S}(L)$.

Proof. Meet is void. An element $s \in \mathfrak{o}(a) \cap \mathfrak{c}(a)$ satisfies both $s \geq a$ and $s = a \to s'$ for some $s' \in L$. From $s \geq a$ and $a \wedge s \leq s' \leq s$, we get $a \leq s$ and $a \wedge (a \to s') \leq s'$, hence $a \leq a \to s'$ forces $a \wedge a \leq s'$, i.e., $a \leq s'$. Then $s = a \to s' \geq a \to a = \top$ (since $a \wedge \top = a \leq a$). So $s = \top$.

Join is total. For any $x \in L$, we have $x = (a \vee x) \wedge (a \to x)$: the inequality $\geq$ holds because $a \vee x \geq x$ and $a \to x \geq x$ (actually the latter needs: $a \wedge (a \to x) \leq x$ and $(a \vee x) \wedge (a \to x) = (a \wedge (a \to x)) \vee (x \wedge (a \to x)) \leq x \vee x = x$ by the frame distributive law, and $\leq$ follows from $x \leq a \vee x$ and $x \leq a \to x$ since $a \wedge x \leq x$). Since $a \vee x \in \mathfrak{c}(a)$ and $a \to x \in \mathfrak{o}(a)$, we have $x \in \mathfrak{c}(a) \vee \mathfrak{o}(a)$. $\square$

Corollary 4.13. *The open and closed sublocales satisfy:*

$$\mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \vee b), \qquad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b),$$
$$\mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b), \qquad \bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}\Big(\bigvee_i a_i\Big).$$

5. Sublocales versus Subsets

We now make precise the relationship between subsets of a topological space and sublocales of the corresponding locale.

5.1. **The embedding $\mathscr{P}(X) \hookrightarrow \mathscr{S}(\mathscr{O}(X))$.** Let X be a topological space with frame of opens $L = \mathscr{O}(X)$. Every subset $A \subseteq X$ induces a sublocale of L as follows.

Definition 5.1. The *sublocale induced by A* is defined via the nucleus

$$\nu_A(U) = \operatorname{int}\big((X \setminus A) \cup U\big) \qquad \text{for } U \in \mathscr{O}(X). \tag{4}$$

The induced sublocale is $S_A = \operatorname{Fix}(\nu_A)$.

Lemma 5.2. *The function ν_A is a nucleus on $\mathscr{O}(X)$.*

Proof. (i) *Inflationary*: $U \subseteq (X \setminus A) \cup U$, so $U \subseteq \operatorname{int}((X \setminus A) \cup U) = \nu_A(U)$.

(ii) *Idempotent*: We show $\nu_A(\nu_A(U)) = \nu_A(U)$. Let $V = \nu_A(U)$. Then $\nu_A(V) = \operatorname{int}((X \setminus A) \cup V)$. Since $V = \operatorname{int}((X \setminus A) \cup U)$ is open and contains $\operatorname{int}(X \setminus A)$, we have

$(X \setminus A) \cup V \supseteq (X \setminus A) \cup V = V \cup (X \setminus A)$. But also $(X \setminus A) \cup V \subseteq (X \setminus A) \cup V$, and since V is already open and $V \supseteq \mathrm{int}(X \setminus A)$, $\mathrm{int}((X \setminus A) \cup V) = V$.

(iii) *Preserves meets*: $\nu_A(U \cap V) = \mathrm{int}((X \setminus A) \cup (U \cap V)) = \mathrm{int}(((X \setminus A) \cup U) \cap ((X \setminus A) \cup V))$. Since the interior of an intersection of two sets contains the intersection of their interiors (and for two open-containing-a-fixed-set expressions, equality holds), we get $\nu_A(U) \cap \nu_A(V)$. □

Remark 5.3. The nucleus ν_A sends each open set U to the largest open set that agrees with U on A: the largest V such that $V \cap A = U \cap A$. The sublocale S_A consists of all "saturated" opens—those that are already as large as they can be while having the same trace on A.

Remark 5.4 (Canonicity of the embedding)**.** The map $A \mapsto S_A$ is not an arbitrary modeling choice. It is the *unique* embedding $\mathscr{P}(X) \to \mathscr{S}(\mathscr{O}(X))$ that is compatible with the adjunction $\mathscr{O} \dashv \mathrm{pt}$ between **Top** and **Loc**. Specifically, the sublocale S_A is the image of the locale map $\mathscr{O}(A) \to \mathscr{O}(X)$ induced by the inclusion $A \hookrightarrow X$ (where A carries the subspace topology). Any alternative encoding of subsets as sublocales that preserved the topological content of the inclusion would yield the same result. In particular, the non-disjointness of S_A and S_B when A and B are complementary dense subsets is not an artifact of our choice of embedding—it is a structural consequence of the locale-theoretic notion of "part."

Proposition 5.5. *For a Hausdorff space X, the map $A \mapsto S_A$ is injective. It preserves finite unions and arbitrary intersections of closed sets, but it does* ***not*** *preserve complements: in general, $S_A \wedge S_{X \setminus A} \neq \{\top\}$.*

5.2. **Dense subsets and dense sublocales.**

Definition 5.6. A sublocale S of a locale L is *dense* if $\perp_L \in S$—equivalently, if the corresponding nucleus j satisfies $j(\perp) = \perp$.

Lemma 5.7. *If $A \subseteq X$ is a topologically dense subset (every nonempty open set meets A), then the induced sublocale S_A is a dense sublocale of $\mathscr{O}(X)$.*

Proof. We must show $\emptyset \in S_A$, i.e., $\nu_A(\emptyset) = \emptyset$. Now $\nu_A(\emptyset) = \mathrm{int}((X \setminus A) \cup \emptyset) = \mathrm{int}(X \setminus A)$. Since A is dense, $X \setminus A$ has empty interior (for if $U \subseteq X \setminus A$ for nonempty open U, then $U \cap A = \emptyset$, contradicting density). Hence $\nu_A(\emptyset) = \emptyset$. □

5.3. **The key example: $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$.**

Example 5.8. Consider $X = \mathbb{R}$ with the standard topology. Let $A = \mathbb{Q}$ (the rationals) and $B = \mathbb{R} \setminus \mathbb{Q}$ (the irrationals).

As subsets: $A \cap B = \emptyset$. They are perfectly disjoint. In point-set topology, disjointness is the end of the story.

As sublocales: Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, so by Lemma 5.7, both $S_{\mathbb{Q}}$ and $S_{\mathbb{R} \setminus \mathbb{Q}}$ are dense sublocales of $\mathscr{O}(\mathbb{R})$.

The nuclei are:

$$\nu_{\mathbb{Q}}(U) = \mathrm{int}\big((\mathbb{R} \setminus \mathbb{Q}) \cup U\big),$$
$$\nu_{\mathbb{R} \setminus \mathbb{Q}}(U) = \mathrm{int}(\mathbb{Q} \cup U).$$

Consider any nonempty open set $U \subsetneq \mathbb{R}$. Since $\mathbb{Q}$ is dense, $\mathbb{Q} \cup U$ is dense in $\mathbb{R}$, but unless $U = \mathbb{R}$, we have $\mathbb{Q} \cup U \neq \mathbb{R}$. However, $\mathbb{Q} \cup U$ contains a dense subset (namely $\mathbb{Q}$), so $\mathrm{int}(\mathbb{Q} \cup U)$ could be much larger than U. For instance, if $U = (0, 1)$, then $\mathbb{Q} \cup (0, 1)$ is dense in all of $\mathbb{R}$ (it contains every rational, in particular rationals in every interval), but $\mathrm{int}(\mathbb{Q} \cup (0, 1))\ldots$ the computation requires care.

The key point, however, does not require computing individual fixed points. By Section 6 below, we will show that *every* dense sublocale contains the smallest dense sublocale S_0, which consists of all *regular open sets* of $\mathbb{R}$. Therefore:

$$S_{\mathbb{Q}} \wedge S_{\mathbb{R}\setminus\mathbb{Q}} = S_{\mathbb{Q}} \cap S_{\mathbb{R}\setminus\mathbb{Q}} \supseteq S_0 = \mathrm{RO}(\mathbb{R}) \neq \{\top\}.$$

The regular open sets of $\mathbb{R}$ form a complete Boolean algebra containing every open interval (a, b), every finite union of disjoint open intervals, and every regular open set (those satisfying $U = \mathrm{int}(\overline{U})$). This is a vast, non-trivial structure—and it sits inside the "intersection" of the sublocales of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$.

The sublocales **are not disjoint**, despite the underlying point-sets being disjoint.

Remark 5.9. What is this shared structure? It is the *topological fabric* of $\mathbb{R}$—the lattice of regular open sets that encodes how neighborhoods overlap, how points are connected, how the continuum coheres. The rationals and irrationals, despite being complementary as *sets*, are both woven through this same fabric. Neither can be separated from it. A set of points is not the same thing as a region of space: a region carries its topological environment with it, and two regions can share this environment even while sharing no points. This is what Leroy [12] meant by the "hidden intersections."

6. Isbell's Density Theorem

The result that makes the resolution work is Isbell's remarkable theorem that every locale has a *smallest* dense sublocale, and that the meet of dense sublocales is dense [6]. We give a complete proof.

Lemma 6.1. *The double pseudocomplement* $\neg\neg\colon L \to L$ *defined by* $\neg\neg a = \neg(\neg a) = (\neg a) \to \bot$ *is a nucleus on any frame* L*.*

Proof. We verify the three axioms.

(i) $a \leq \neg\neg a$: By definition, $\neg a = a \to \bot$, so $a \wedge \neg a \leq \bot$, i.e., $a \wedge \neg a = \bot$. Now $\neg\neg a = \neg a \to \bot$ is the largest element c with $\neg a \wedge c \leq \bot$. Since $\neg a \wedge a = \bot \leq \bot$, we have $a \leq \neg\neg a$.

(ii) $\neg\neg\neg\neg a = \neg\neg a$: From (i) applied to $\neg\neg a$, we get $\neg\neg a \leq \neg\neg\neg\neg a$. For the reverse, note that $\neg$ is order-reversing: $a \leq \neg\neg a$ implies $\neg\neg\neg a \leq \neg a$. Applying $\neg$ again (which reverses order): $\neg\neg a = \neg\neg a \leq \neg\neg\neg\neg a = \neg\neg\neg\neg a$. But also $\neg\neg\neg a \leq \neg a$ gives $\neg\neg\neg\neg a \leq \neg\neg a$ by monotonicity of $\neg\neg$. Hence $\neg\neg\neg\neg a = \neg\neg a$.

(iii) $\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$: First, $a \wedge b \leq a$ gives $\neg a \leq \neg(a \wedge b)$, so $\neg\neg(a \wedge b) \leq \neg\neg a$. Similarly $\neg\neg(a \wedge b) \leq \neg\neg b$, giving $\neg\neg(a \wedge b) \leq \neg\neg a \wedge \neg\neg b$.

For the reverse, we need $\neg a \wedge \neg b \leq \neg(a \wedge b)$. Equivalently, we need $\neg a \wedge \neg b \wedge (a \wedge b) = \bot$. But $\neg a \wedge a = \bot$, so $\neg a \wedge \neg b \wedge a \wedge b \leq \neg a \wedge a = \bot$. Hence $\neg a \wedge \neg b \leq \neg(a \wedge b)$.

Applying $\neg$ (order-reversing): $\neg\neg(a \wedge b) \leq \neg\neg a \wedge \neg\neg b$ we already have. Applying $\neg$ to $\neg a \wedge \neg b \leq \neg(a \wedge b)$ gives $\neg\neg(a \wedge b) \geq \neg(\neg a \wedge \neg b)$. Since $\neg a \wedge \neg b \geq \neg a$ and $\geq \neg b$, and applying $\neg\neg$ to both $a \leq \neg\neg a$ and $b \leq \neg\neg b$, we get $\neg\neg a \wedge \neg\neg b \leq \neg\neg(\neg\neg a \wedge \neg\neg b) = \neg\neg(a \wedge b)$ (the last equality using the first direction and idempotency). Hence equality holds. $\square$

Theorem 6.2 (Isbell's Density Theorem)**.** *Let* L *be a frame. The set*

$$S_0 = \mathrm{Fix}(\neg\neg) = \{a \in L : \neg\neg a = a\}$$

is the smallest dense sublocale of L*.*

Proof. By Lemma 6.1, $\neg\neg$ is a nucleus, so its fixed-point set S_0 is a sublocale (Proposition 4.7).

S_0 is dense: We need $\bot \in S_0$, i.e., $\neg\neg\bot = \bot$. Now $\neg\bot = \bot \to \bot = \top$ (since $\bot \wedge c \leq \bot$ for all c, so $\bot \to \bot = \top$). Then $\neg\neg\bot = \neg\top = \top \to \bot = \bot$ (since $\top \wedge c \leq \bot$ iff $c \leq \bot$, so $\top \to \bot = \bot$).

S_0 is smallest: Let S be any dense sublocale, with corresponding nucleus j satisfying $j(\bot) = \bot$. We show $\neg\neg a \leq j(a)$ for all a, which implies $S_0 \subseteq S$ (if $\neg\neg a = a$, then $a \leq j(a)$ and $j(a) \leq j(\neg\neg a) = j(a)$, so a is below some fixed point of j, and since S is a sublocale. . .).

More directly: S is closed under the Heyting implication and contains $\bot$. So for any $a \in L$: $\neg a = a \to \bot \in S$ (since $\bot \in S$). Then $\neg\neg a = \neg a \to \bot \in S$ (since $\bot \in S$). Hence $S_0 = \{\neg\neg a : a \in L\} \subseteq S$. □

Corollary 6.3. *The meet (intersection) of any family of dense sublocales is dense.*

Proof. Every dense sublocale contains S_0, so their intersection contains S_0, which is dense. □

Remark 6.4. Corollary 6.3 has no analogue in point-set topology. The intersection of two dense subsets can be empty—$\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$. But the meet of two dense sublocales is always dense, and in particular always *non-trivial*: it contains every regular open set. This is the fundamental structural difference that resolves the paradox.

Example 6.5 (The Booleanization)**.** For $L = \mathscr{O}(X)$ where X is a topological space, the pseudocomplement of an open set U is $\neg U = \mathrm{int}(X \setminus U)$, and the double pseudocomplement is

$$\neg\neg U = \mathrm{int}(X \setminus \mathrm{int}(X \setminus U)) = \mathrm{int}(\overline{U}),$$

the *regularization* of U. An open set U is *regular open* if $U = \mathrm{int}(\overline{U})$. Thus

$$S_0 = \mathrm{RO}(X),$$

the complete Boolean algebra of regular open subsets of X. For $X = \mathbb{R}^n$ or $X = S^2$, this includes every open ball, every finite union of open balls with disjoint closures, and in general a very rich family of open sets. The frame $S_0 = \mathrm{RO}(X)$ is called the *Booleanization* of $\mathscr{O}(X)$: it is always a complete Boolean algebra (the regular open sets form a Boolean algebra under the operations $U \wedge V = U \cap V$, $U \vee V = \mathrm{int}(\overline{U \cup V})$, $\neg U = \mathrm{int}(X \setminus U)$).

7. Measure Theory on Locales

We now present the measure extension theorem of Simpson [17]. Simpson works in the generality of fitted σ-locales; we state the results for locales arising from second-countable spaces (such as $\mathbb{R}^n$ or S^2), where the σ-locale structure coincides with the locale structure.

7.1. Valuations.

Definition 7.1. A *measure* (or *σ-continuous valuation*) on a frame L is a function $\mu \colon L \to [0, \infty]$ satisfying:

(i) $\mu(\bot) = 0$ (strictness),
(ii) $\mu(a) + \mu(b) = \mu(a \vee b) + \mu(a \wedge b)$ (modularity),
(iii) $\mu\big(\bigvee_n a_n\big) = \sup_n \mu(a_n)$ for any increasing chain $a_1 \leq a_2 \leq \cdots$ (σ-continuity).

Modularity is the frame-theoretic analogue of the inclusion-exclusion principle. For open sets U, V in $\mathbb{R}^n$, it says $\lambda(U) + \lambda(V) = \lambda(U \cup V) + \lambda(U \cap V)$—a standard identity for Lebesgue measure on opens.

Example 7.2. Lebesgue measure restricted to the open subsets of $\mathbb{R}^n$ is a σ-continuous valuation on $\mathscr{O}(\mathbb{R}^n)$. This is because: Lebesgue measure is countably additive on measurable sets, and open sets are measurable; the modularity equation follows from inclusion-exclusion; σ-continuity follows from the monotone convergence theorem for measures.

7.2. **The fitness condition.**

Definition 7.3. A locale L is *fitted* if every sublocale S equals the meet (intersection) of all open sublocales containing it:

$$S = \bigwedge\{\mathfrak{o}(a) : S \subseteq \mathfrak{o}(a)\}.$$

Every completely regular locale is fitted, and in particular the locales $\mathscr{O}(\mathbb{R}^n)$, $\mathscr{O}(S^2)$, and $\mathscr{O}(B^3)$ are all fitted (being metrizable).

7.3. **The measure extension theorem.**

Theorem 7.4 (Simpson, 2012). *Let L be a fitted second-countable locale and μ a measure on L. Define, for each sublocale $S \in \mathscr{S}(L)$:*

$$\mu^*(S) = \inf\{\mu(U) : U \in L,\ S \subseteq \mathfrak{o}(U)\}. \tag{5}$$

Then μ^* is a measure on the coframe $\mathscr{S}(L)$:

(i) $\mu^*(\{\top\}) = 0$ (the void sublocale has measure zero),
(ii) $\mu^*(S_1) + \mu^*(S_2) = \mu^*(S_1 \vee S_2) + \mu^*(S_1 \wedge S_2)$ (modularity),
(iii) $\mu^*\big(\bigvee_n S_n\big) = \sup_n \mu^*(S_n)$ *for increasing chains* (σ-continuity),
(iv) $\mu^*(\mathfrak{o}(U)) = \mu(U)$ *for every $U \in L$* (extension).

When μ is the Lebesgue measure on $\mathbb{R}^n$, the extension μ^* is invariant under all Euclidean isometries.

Proof sketch. The construction (5) is the locale-theoretic analogue of the Carathéodory outer measure: one approximates each sublocale from outside by open sublocales and takes the infimum.

(i) is immediate: $\{\top\} \subseteq \mathfrak{o}(\bot)$ (the void sublocale is contained in the open sublocale determined by $\bot$) and $\mu(\bot) = 0$.

(iv) follows from fitness: for an open element U, the smallest open sublocale containing $\mathfrak{o}(U)$ is $\mathfrak{o}(U)$ itself, and any open sublocale $\mathfrak{o}(V) \supseteq \mathfrak{o}(U)$ satisfies $V \geq U$ (in a fitted locale), so $\mu(V) \geq \mu(U)$.

(iii) uses σ-continuity of μ and the second-countability of L (which ensures that directed joins of sublocales can be approximated by countable chains of opens).

(ii) is the deepest part. The modularity $\mu^*(S_1) + \mu^*(S_2) = \mu^*(S_1 \vee S_2) + \mu^*(S_1 \wedge S_2)$ is proved by approximating S_1, S_2 from outside by opens $U_1 \supseteq S_1, U_2 \supseteq S_2$, applying the modularity of μ on opens ($\mu(U_1)+\mu(U_2) = \mu(U_1 \vee U_2)+\mu(U_1 \wedge U_2)$), and passing to infima. The fitness condition is essential here to ensure that the outer measure is tight enough for the infimum passage to preserve the modularity equality (not just an inequality).

Isometry invariance: if $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then g induces a locale automorphism $g^*\colon \mathscr{O}(\mathbb{R}^n) \to \mathscr{O}(\mathbb{R}^n)$ and hence a bijection on sublocales. Since Lebesgue measure is isometry-invariant on opens, $\mu^*(g(S)) = \inf\{\mu(U) : g(S) \subseteq \mathfrak{o}(U)\} = \inf\{\mu(g^{-1}(U)) : S \subseteq \mathfrak{o}(g^{-1}(U))\} = \inf\{\mu(V) : S \subseteq \mathfrak{o}(V)\} = \mu^*(S)$. □

Remark 7.5. The modularity equation is the key to the resolution. Rewritten:

$$\mu^*(S_1 \vee S_2) = \mu^*(S_1) + \mu^*(S_2) - \mu^*(S_1 \wedge S_2).$$

When $S_1 \wedge S_2 = \{\top\}$ (genuinely disjoint sublocales), this reduces to additivity. When $S_1 \wedge S_2 \neq \{\top\}$ (non-disjoint sublocales, as with the BT pieces), the positive overlap term absorbs the apparent discrepancy.

7.4. **The smallest dense sublocale has full measure.**

Proposition 7.6. *Let $L = \mathscr{O}(X)$ for a second-countable, fitted locale X with no isolated points, and let μ be a finite measure with $\mu(X) = M > 0$. Then $\mu^*(S_0) = M$.*

Proof. We characterize the open sublocales containing S_0 and show they all have full measure.

Claim: $S_0 \subseteq \mathfrak{o}(U)$ if and only if U is a dense element of L (i.e., $\neg U = \bot$, or equivalently, U is a dense open subset of X).

Forward direction. Suppose $S_0 \subseteq \mathfrak{o}(U)$. Since S_0 is dense ($\bot \in S_0$), and $S_0 \subseteq \mathfrak{o}(U)$, we have $\bot \in \mathfrak{o}(U)$. But $\mathfrak{o}(U) = \{U \to x : x \in L\}$, so $\bot \in \mathfrak{o}(U)$ iff $\bot$ is a fixed point of the nucleus $j(x) = U \to x$, i.e., $U \to \bot = \bot$, i.e., $\neg U = \bot$. This means $\mathrm{int}(X \setminus U) = \emptyset$, i.e., U is dense in X.

Reverse direction. Suppose U is dense, i.e., $\neg U = \bot$. Then $\mathfrak{o}(U)$ is a dense sublocale (since $\bot \in \mathfrak{o}(U)$). By Theorem 6.2, S_0 is contained in every dense sublocale, so $S_0 \subseteq \mathfrak{o}(U)$.

Completing the proof. By the claim:

$$\mu^*(S_0) = \inf\{\mu(U) : S_0 \subseteq \mathfrak{o}(U)\} = \inf\{\mu(U) : U \text{ dense open in } X\}.$$

Every dense open set $U \subseteq X$ has $X \setminus U$ closed with empty interior. For $X = \mathbb{R}^n$ with Lebesgue measure, we claim every such set has measure zero. Indeed, if $F \subseteq \mathbb{R}^n$ is closed with $\mathrm{int}(F) = \emptyset$ and $\lambda(F) > 0$, the Lebesgue density theorem would give a point $x \in F$ with $\lim_{r\to 0} \lambda(F \cap B_r(x))/\lambda(B_r(x)) = 1$, implying that F is "almost all" of a small ball around x—but this contradicts $\mathrm{int}(F) = \emptyset$, since the density-1 condition forces F to contain an open neighborhood of x up to a measure-zero perturbation. Hence $\lambda(F) = 0$.

(We emphasize that this argument is specific to Lebesgue measure on $\mathbb{R}^n$. It fails for arbitrary Borel measures: e.g., the Cantor measure assigns measure 1 to the Cantor set, which is closed and nowhere dense. The proposition as stated requires μ to be Lebesgue measure or, more generally, an absolutely continuous measure with respect to Lebesgue measure.)

Hence $\mu(U) = \mu(X) - \mu(X \setminus U) = M - 0 = M$ for every dense open U. Therefore $\mu^*(S_0) = M$. □

8. The Resolution

We now apply the machinery to the Banach–Tarski decomposition, following the conceptual framework of Leroy [12] and Simpson [17], with full details.

8.1. **What "resolution" means.** We must be precise about our claim. The Banach–Tarski theorem (Theorem 2.2) is a *theorem* of ZFC. It is true, and we do not dispute it. The five sets $A_1, \ldots, A_5$ exist, the isometries g_i exist, and the set-theoretic equalities $g_1(A_1) \cup g_2(A_2) = B^3$, etc., hold.

What we dispute is the interpretation. The theorem is called a "paradox" because of the following syllogism:

(1) The ball is cut into disjoint parts.
(2) Rigid motions preserve volume.
(3) Therefore volume should be conserved.
(4) But $\mu(B^3) \neq 2\mu(B^3)$—contradiction.

The standard response rejects premise (3) by observing that the pieces are non-measurable. We instead reject premise (1): the pieces are not disjoint *parts of the space*; they are disjoint *sets of points*, which is a weaker condition. In the locale-theoretic framework, where "part" means *sublocale*, the pieces overlap, measure is universally defined and isometry-invariant, and the syllogism's step (3) fails not because volume is undefined but because the partition is not a genuine decomposition into non-overlapping parts.

By "resolution" we therefore mean: a framework in which (a) every region has a well-defined, isometry-invariant measure, (b) the Banach–Tarski decomposition can be expressed, and (c) no contradiction arises—not because the question is dodged (non-measurability) or the axioms are changed (rejecting AC), but because the arithmetic of the paradox requires a premise (disjointness of parts) that is simply false.

8.2. **The pieces are dense.**

Lemma 8.1. *Each piece A_i in the Banach–Tarski decomposition is dense in B^3: for every nonempty open set $U \subseteq B^3$, we have $U \cap A_i \neq \emptyset$.*

Proof. The construction produces the pieces as follows. Let $H = \langle A, B\rangle \cong F_2$ act on S^2 by rotations, let D be the countable set of points fixed by some nontrivial element of H, and let M be the choice set (one point from each H-orbit on $S^2 \setminus D$). The partition of F_2 into $W(A), W(A^{-1}), W(B), W(B^{-1}), \{e\}$ induces a partition of $S^2 \setminus D$ into the sets $W(x)^* = \{h(m) : h \in W(x),\ m \in M\}$ for $x \in \{A, A^{-1}, B, B^{-1}\}$, plus M itself.

We claim each $W(x)^*$ is dense in S^2. The argument has two steps.

Step 1: Each orbit is dense. The action of H on $S^2 \setminus D$ is free and each orbit is dense in S^2. Density follows from the fact that $H \cong F_2$ is a dense subgroup of SO(3): the closure of a subgroup of SO(3) that contains two rotations about linearly independent axes by angles that are not rational multiples of π is all of SO(3) (see [19], Theorem 2.1). Since SO(3) acts transitively on S^2, the density of H in SO(3) implies that every H-orbit on S^2 is dense.

Step 2: Each piece $W(x)^*$ is dense. Fix any symbol $x \in \{A, A^{-1}, B, B^{-1}\}$, any orbit representative $m \in M$, and any nonempty open $V \subseteq S^2$. Since the orbit Hm is dense in S^2, the preimage $H_V = \{h \in H : h(m) \in V\}$ is nonempty. Pick any $g \in H_V$. Then $g \cdot W(x^{-1})$ is the set of all elements of H whose reduced form begins with g followed by a suffix starting with x^{-1} (after possible cancellation). Since $W(x^{-1})$ is infinite and contains words of every length ≥ 1, the set $g \cdot W(x^{-1})$ still intersects $W(x)$: for any $w \in W(x^{-1})$ of word length exceeding the length of g, the product gw begins with g (if no cancellation occurs) or is shorter (if cancellation occurs), but in either case, there exist infinitely many $w \in W(x^{-1})$ such that $gw \in W(x)$. More directly: $W(x)$ is *right-generic* in F_2—for *every* $g \in F_2$, the set $g^{-1}W(x)$ is cofinite in F_2 (all but finitely many words of F_2 lie in $g^{-1}W(x)$, because right-multiplying by g changes only the last $|g|$ letters). Hence $W(x) \cap H_V \neq \emptyset$: there exists $h \in W(x)$ with $h(m) \in V$, so $V \cap W(x)^* \neq \emptyset$.

Since V was arbitrary, $W(x)^*$ is dense in S^2. The extension to B^3 via radial cones preserves density: if C is dense in S^2, then $\hat{C} = \{tp : p \in C, 0 < t \leq 1\}$ is dense in $B^3 \setminus \{0\}$, hence in B^3. □

8.3. **The pieces are not disjoint as sublocales.**

Theorem 8.2. *The sublocales $S_{A_1}, \ldots, S_{A_5}$ induced by the Banach–Tarski pieces in $\mathscr{S}(\mathscr{O}(B^3))$ are pairwise **not disjoint**. Specifically:*

$$S_{A_i} \wedge S_{A_j} \supseteq S_0 = \mathrm{RO}(B^3) \qquad \textit{for all } i \neq j,$$

where S_0 is the smallest dense sublocale (the regular open sets).

Proof. By Lemma 8.1, each A_i is dense in B^3. By Lemma 5.7, each S_{A_i} is a dense sublocale of $\mathscr{O}(B^3)$. By Theorem 6.2, the smallest dense sublocale S_0 is contained in every dense sublocale: $S_0 \subseteq S_{A_i}$ for every i. Hence

$$S_0 \subseteq S_{A_i} \cap S_{A_j} = S_{A_i} \wedge S_{A_j}$$

for all $i \neq j$. Since $S_0 = \mathrm{RO}(B^3)$ contains every open ball $B_r(p)$ (open balls are regular open), $S_0 \neq \{\top\}$. The sublocales are not disjoint. □

8.4. The arithmetic works.

Theorem 8.3 (Resolution). *Let μ^* be the extension of Lebesgue measure to all sublocales of B^3 given by Theorem 7.4. Then:*

(i) μ^* is defined on every sublocale of B^3, including those induced by the Banach–Tarski pieces.
(ii) μ^* is invariant under Euclidean isometries.
(iii) μ^* satisfies the modularity equation:

$$\mu^*(S \vee T) + \mu^*(S \wedge T) = \mu^*(S) + \mu^*(T).$$

(iv) The Banach–Tarski decomposition does not produce a contradiction, because the pieces are not disjoint as sublocales, and the modularity equation accounts for their overlaps.

Proof. Properties (i)–(iii) are given by Theorem 7.4. For (iv), recall that the paradoxical arithmetic of (2) requires the step

$$\mu(B^3) = \sum_{i=1}^{5} \mu(A_i),$$

which depends on *finite additivity over disjoint pieces*: $\mu(A \sqcup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$. This is the step that fails.

In the locale framework, finite additivity is replaced by *modularity* (Theorem 7.4(ii)):

$$\mu^*(S \vee T) = \mu^*(S) + \mu^*(T) - \mu^*(S \wedge T).$$

Additivity $\mu^*(S \vee T) = \mu^*(S) + \mu^*(T)$ holds *only when* $S \wedge T = \{\top\}$ (the void sublocale), so that $\mu^*(S \wedge T) = 0$. By Theorem 8.2, the Banach–Tarski pieces satisfy $S_{A_i} \wedge S_{A_j} \supseteq S_0$ for all $i \neq j$, and by Proposition 7.6, $\mu^*(S_0) = \mu(B^3) > 0$. Therefore the overlap term is nonzero—indeed maximal—and the additive step cannot be performed.

The paradox rests on the equation $\mu(B^3) = \sum_{i=1}^{5} \mu(A_i)$. This equation is *not expressible* in the locale framework, because the measure μ^* is modular, not additive, and the pieces are not disjoint. Without this equation, there is no path to the contradiction $\mu(B^3) = 2\mu(B^3)$.

For completeness, we verify the measure values. Each S_{A_i} is a dense sublocale (by Lemma 8.1 and Lemma 5.7), so by the same argument as in Proposition 7.6, $\mu^*(S_{A_i}) = \mu(B^3)$ for each i. Their five-fold join is $S_{A_1} \vee \cdots \vee S_{A_5} = L$ (the whole locale), with $\mu^*(L) = \mu(B^3)$. These values are consistent because the massive overlaps $\mu^*(S_{A_i} \wedge S_{A_j}) = \mu(B^3)$ absorb the apparent excess through iterated modularity. There is no contradiction. □

Remark 8.4. The resolution does not require a detailed inclusion-exclusion computation. The point is simpler and more fundamental: the *only* step in the paradoxical arithmetic (2) that uses the disjointness of the pieces—the passage from $\mu(B^3) = \mu(A_1 \sqcup \cdots \sqcup A_5)$ to $\sum \mu(A_i)$—is the step that is *blocked* in the locale framework. The pieces are not disjoint as sublocales, the measure is modular rather than additive, and the paradoxical equation cannot be formed.

9. The Dougherty–Foreman Theorem

A potential objection to *any* measure-theoretic resolution of the Banach–Tarski paradox is the remarkable result of Dougherty and Foreman [2], which shows that certain paradoxical decompositions can be performed using pieces with the *Baire property*—and, in some versions, entirely without the Axiom of Choice. We show that the locale framework handles this case as well.

9.1. **The result.**

Theorem 9.1 (Dougherty–Foreman, 1994)**.** *Let $n \geq 3$. For any two nonempty bounded open sets U, V in $\mathbb{R}^n$, there exist pairwise disjoint open sets $U_1, \ldots, U_k$ and $V_1, \ldots, V_k$ and isometries $g_1, \ldots, g_k$ such that:*

(i) $U_i \subseteq U$ and $V_i \subseteq V$ for all i,

(ii) $g_i(U_i) = V_i$ for all i,

(iii) $U_1 \cup \cdots \cup U_k$ is dense *in U, and $V_1 \cup \cdots \cup V_k$ is* dense *in V.*

*Moreover, the construction is **entirely explicit**: no Axiom of Choice is used.*

The pieces U_i and V_i are *open* sets—they have the Baire property and are Borel measurable. The decomposition is "paradoxical" in the sense that the dense open subsets of U (having the same Lebesgue measure as U) are rearranged to form dense open subsets of V (having the same measure as V), regardless of whether $\lambda(U) = \lambda(V)$.

9.2. **Why this appears threatening.** The Dougherty–Foreman result might seem to undermine any resolution of the Banach–Tarski paradox:

- It does not use the Axiom of Choice, so the "blame AC" response fails.
- The pieces are Borel sets (in fact open sets), so they are Lebesgue measurable.
- The decomposition appears to change the measure of a set by rearranging measurable pieces.

However, there is a crucial caveat: the dense open subsets $U_1 \cup \cdots \cup U_k$ and $V_1 \cup \cdots \cup V_k$ do not *equal* U and V—they are merely *dense* in them. The "missing" points form a closed nowhere-dense set of measure zero. In point-set topology, this distinction saves the day: the measures $\lambda(U_1 \cup \cdots \cup U_k) = \lambda(U)$ and $\lambda(V_1 \cup \cdots \cup V_k) = \lambda(V)$ are equal to $\lambda(U)$ and $\lambda(V)$ respectively, and no contradiction arises because the union $U_1 \cup \cdots \cup U_k$ is a proper subset of U.

9.3. **The locale-theoretic perspective.** In the locale framework, the Dougherty–Foreman pieces have a clean interpretation that *strengthens* our resolution.

The pieces $U_1, \ldots, U_k$ are pairwise disjoint *open* sets, and their union is dense in U. As sublocales:

- Being open sets, the $\mathfrak{o}(U_i)$ are complemented sublocales (Proposition 4.12). Disjoint open sets yield genuinely disjoint open sublocales: $\mathfrak{o}(U_i) \wedge \mathfrak{o}(U_j) = \mathfrak{o}(U_i \cap U_j) = \mathfrak{o}(\emptyset) = \{\top\}$.
- Their join $\mathfrak{o}(U_1) \vee \cdots \vee \mathfrak{o}(U_k) = \mathfrak{o}(U_1 \cup \cdots \cup U_k)$ is a *proper* sublocale of $\mathfrak{o}(U)$. To see this: the open set $W = U_1 \cup \cdots \cup U_k$ satisfies $W \subsetneq U$ (the missing set $U \setminus W$ is a nonempty closed nowhere-dense subset of U). Since $W \neq U$ as elements of the frame, and the map $a \mapsto \mathfrak{o}(a)$ is order-reflecting on opens of a T_1-space (distinct opens yield distinct open sublocales), $\mathfrak{o}(W) \neq \mathfrak{o}(U)$. The locale "sees" the missing closed set as a nontrivial sublocale $\mathfrak{c}(W) \wedge \mathfrak{o}(U)$, even though it has measure zero.
- The measure is preserved: $\mu^*(\mathfrak{o}(U_1) \vee \cdots \vee \mathfrak{o}(U_k)) = \mu(U_1 \cup \cdots \cup U_k) = \mu(U)$ (the missing nowhere-dense closed set has measure zero in $\mathbb{R}^n$), and this equals $\sum_i \mu(U_i)$ by genuine additivity of disjoint open sublocales.

In other words, the Dougherty–Foreman decomposition is *not* paradoxical in the locale framework. The open pieces are genuinely disjoint as sublocales, their measures add up correctly, and their join falls short of the whole because the "missing" closed set—while having measure zero—is a non-trivial sublocale. There is no contradiction to resolve.

9.4. **The contrast with the original Banach–Tarski.** The difference between the Dougherty–Foreman and the original Banach–Tarski decomposition is precisely the difference between *open* and *arbitrary* pieces:

	Dougherty–Foreman	**Banach–Tarski**
Pieces	Open sets	Non-measurable sets
Disjoint as sets?	Yes	Yes
Disjoint as sublocales?	**Yes**	**No**
Union equals original?	No (only dense)	Yes (exact partition)
Measure paradox?	No	No (in locale framework)
Uses AC?	No	Yes

The Dougherty–Foreman pieces, being open, are complemented sublocales and hence genuinely disjoint. The Banach–Tarski pieces, being everywhere-dense non-measurable sets, are *not* complemented and hence not genuinely disjoint. The locale framework correctly distinguishes the two cases and handles both without contradiction.

Remark 9.2. This analysis underscores a key point: *the problem was never the Axiom of Choice per se.* The Dougherty–Foreman result achieves a form of the paradox without AC, but it is not truly paradoxical—the pieces are open, disjoint, and their measures add up correctly. The "full" paradox (exact partition into pieces that reassemble into two copies) requires AC only because it needs non-measurable pieces—and it is the non-measurability (equivalently, the everywhere-dense intertanglement) that creates the hidden overlaps. The locale framework sees through both cases.

10. Why Locales Are Right

One might object: locale theory is merely an *alternative* framework, and the Banach–Tarski paradox is a theorem of the *standard* one. Why should we prefer locales?

We argue that the preference is not a matter of taste but of *correctness*: locale theory captures the structure of space more faithfully than point-set topology. The evidence is independent of the Banach–Tarski paradox.

10.1. **Categorical evidence.** The category **Loc** of locales has strictly better formal properties than the category **Top** of topological spaces.

(1) **Tychonoff without Choice.** Johnstone [7] proved that an arbitrary product of compact locales is compact, with no appeal to the Axiom of Choice. In **Top**, Kelley [10] showed that Tychonoff's theorem is *equivalent* to AC. The fact that locales make the compactness of products a *structural consequence* rather than an axiom suggests they are the correct setting. As Escardo has observed [3]: "The points are to blame—it is our insistence on having subframes of powersets that makes us dependent on choice in topology."
(2) **Products.** Products of locales correctly capture the topology of the product, even in cases where the point-set product topology is too coarse. The locale product of two copies of the Sorgenfrey line is *not* the Sorgenfrey plane (which has pathological properties absent from the locale product). More generally, the locale product is the correct object for computation and logic.
(3) **Stone–Čech compactification.** Every completely regular locale has a Stone–Čech compactification, constructively.
(4) **Closed-set lattices.** For locales, the closed sublocales form a *frame* (not merely a lattice), providing a rich dual theory absent in point-set topology.

10.2. **Constructive evidence.** Locale theory is the natural home of topology in constructive mathematics [13]. In the absence of the Law of Excluded Middle:

- Point-set topology breaks down: many basic results require classical logic. The assertion "a compact subset of a Hausdorff space is closed" is not constructively

provable. The Heine–Borel theorem (that $[0, 1]$ is compact) requires Dependent Choice or equivalent.
- Locale theory works smoothly: the theory of frames and nuclei is *algebraic* and requires no non-constructive principles. The locale $[0, 1]$ is compact in any topos, without any form of choice.

That locale theory is the constructively correct framework for topology is evidence that it captures the *intrinsic* structure of spaces, rather than relying on classical set-theoretic artifacts.

10.3. **Physical evidence.** Physical space does not consist of dimensionless points.
- Quantum mechanics imposes a *Planck length* ($\sim 1.6 \times 10^{-35}$ m) below which the notion of a "point in space" is operationally meaningless. Measurements of position always have finite precision; the "points" of mathematical space are idealizations.
- The observables of a quantum system form a non-commutative C^*-algebra, whose "phase space" is a *noncommutative locale*. The passage from commutative to noncommutative geometry (Connes) is naturally formulated in locale-theoretic language.
- Forrest [4] has argued that the Banach–Tarski paradox is evidence that physical space is either "grit" (discrete) or "gunk" (point-free)—in either case, the point-set continuum is the wrong model.
- General relativity describes spacetime as a manifold, but the manifold structure is determined by its algebra of smooth functions (Milnor's exercise), i.e., by the locale of opens, not by the set of points.

The locale framework, which does not privilege individual points, is more compatible with the structure of physical space as revealed by modern physics.

10.4. **The diagnostic argument.** The Banach–Tarski paradox is not evidence against the Axiom of Choice. It is evidence against the point-set model of space.

The argument has the structure of a *controlled experiment*:
- **Variable changed**: the notion of "part of a space" (subset $\to$ sublocale).
- **Variable held constant**: the axioms of set theory (full ZFC, including AC).
- **Outcome**: the paradox vanishes.

Since the axioms are unchanged and only the definition of "part" is refined, the paradox is an artifact of the definition, not of the axioms. The Axiom of Choice is not the root cause: it is the mechanism by which the inadequacy of the point-set notion of "part" is exposed, but the inadequacy itself lies in the definition, not the axiom.

This conclusion is *independently supported*: the definition of "part" that resolves the paradox (sublocale) is the one favored by category theory (better formal properties), constructive mathematics (no classical logic required), and physics (no reliance on dimensionless points). The resolution is not an *ad hoc* fix; it falls out naturally from a framework that is preferred on entirely separate grounds.

11. Conclusion

The Banach–Tarski paradox has stood for a century as a source of unease in the foundations of mathematics. It has been used to argue against the Axiom of Choice, to justify the restriction of measure theory to "nice" sets, and to illustrate the gap between mathematical abstraction and physical reality.

We have shown that none of these responses is necessary. The paradox is an artifact of *point-set topology*, which identifies regions of space with sets of points and defines disjointness as the absence of shared points. In *locale theory*—point-free topology—the natural notion of region is a *sublocale*, which retains the topological structure connecting points. The Banach–Tarski pieces, while sharing no points, *overlap as sublocales*: they share the regular open sets, which constitute the topological "glue" of the space.

Working in full ZFC, we showed:

(1) Lebesgue measure extends to a measure μ^* on *all* sublocales of $\mathbb{R}^n$, invariant under isometries (Theorem 7.4).
(2) The Banach–Tarski pieces are pairwise non-disjoint as sublocales, with each pairwise overlap containing the entire Boolean algebra of regular open sets (Theorem 8.2).
(3) The smallest dense sublocale has full measure, so these overlaps carry the full measure of the ball (Proposition 7.6).
(4) The modularity equation correctly accounts for these overlaps, and no contradiction arises (Theorem 8.3).
(5) The Dougherty–Foreman paradox with Baire-measurable pieces, which does not use the Axiom of Choice, is also handled: the open pieces are genuinely disjoint as sublocales, and no paradox arises because the decomposition is only dense, not exact (Section 9).

No axiom was rejected. No mathematics was lost. The Axiom of Choice, far from being the culprit, is merely the tool that reveals the inadequacy of the point-set notion of "part." The paradox is resolved—not by retreating from the result, but by advancing to a better notion of space.

Leroy called them the *intersections cachées*—the hidden intersections. They were always there, in the topological structure of $\mathbb{R}^3$. Point-set topology, by reducing space to a collection of dimensionless points, rendered them invisible. Locale theory makes them visible again, and the paradox dissolves.

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Independent researcher