Science - 1 Assignment - 5

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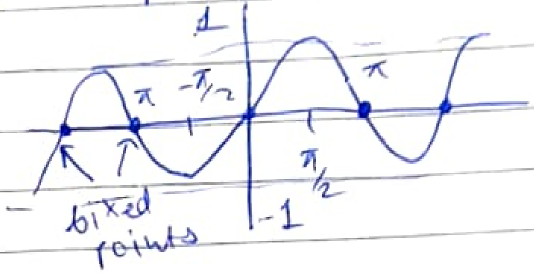
2.1) interpret $\dot{x} = \sin x$ as a flow on the line

→ 2.1.1
find all fixed points

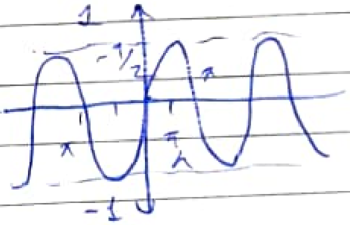
* for finding fixed points
$\dot{x} = 0$
$\sin x = 0$



∴ ∞ fixed pts.

→ 2.1.2
velocity $v = \dot{x}$
∴ hence greatest velocity to right would be when func. attains max positive value

for $x = (4n+1)\frac{\pi}{2}$
the velocity will be greatest to right.



→ 2.1.3
a) $\dot{x} = \sin x$

$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d^2x}{dt^2}$

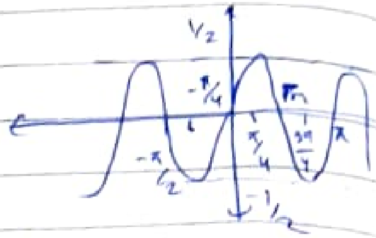
$= \frac{d \sin x}{dx} \times \frac{dx}{dt} = \sin x \cos x$

$= \frac{\sin 2x}{2}$

b) to maximise $\dot{x}$ we maximise $\frac{\sin 2x}{2}$

$2x = (4n+1)\frac{\pi}{2}$

$x = (4n+1)\frac{\pi}{4}, \; n \in \mathbb{Z}$

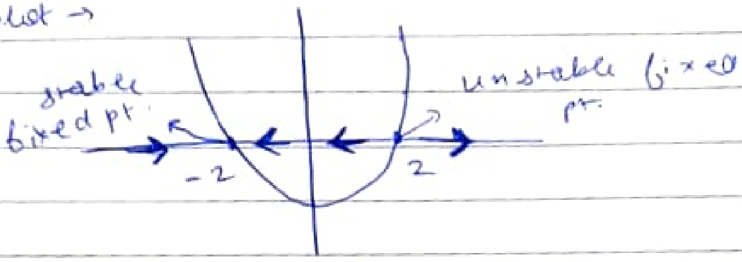


max value for $\dot{x}$ would be $\frac{1}{2}$

2.2) → 2.2.1

$\dot{x} = 4x^2 - 16$

plot →



vector field → vector field would go to right if graph lies above x-axis and left if it is under x-axis.

fixed pt. → points where $\dot{x} = 0$

$4x^2 - 16 = 0 \Rightarrow x = \pm 2$

stability → If arrow point towards fixed pt. it is stable fixed pt. otherwise it is unstable fixed pt.

graphically looking, $x = -2$ is fixed stable pt.
$x = 2$ is unstable pt.

Analytically

$\dot{x} = 4x^2 - 16$

$\frac{dx}{dt} = 4x^2 - 16$

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$$\int \frac{dx}{x^2-4} = \int 4\,dt$$

$$-\frac{1}{4}\int \frac{(x-2)-(x+2)}{(x-2)(x+2)} = \int 4\,dt$$

$$-\frac{1}{4}\left[\log|x+2| - \log|x-2|\right] = 4t + c$$

$$\left[\log\left(\left|\frac{x-2}{x+2}\right|\right)\right] = 16t + 4c$$

$$x(t) = \frac{2c\,x_0 e^{16t} + x_0 - 2e^{16t} + 2}{-x_0 e^{16t} + x_0 + 2e^{16t} + 2}$$

where $x(t) = x_0$ at $t = 0$

$$c = \frac{1}{4}\ln\left(\frac{x_0-2}{x_0+2}\right)$$

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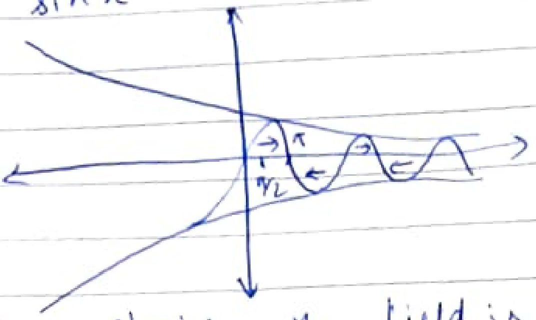
→ 2.2.4

$\dot{x} = e^{-x} \sin x$

plot →



For region above x, field is towards right and vice versa.

fixed points → $\dot{x} = 0$

$e^{-x} \sin x = 0$

$e^{-x} = 0$ or $\sin x = 0$

$\therefore \quad x = \pm n\pi \quad n \in Z^+$

$x = (2n+1)\pi \rightarrow$ stable

$x = 2n\pi \rightarrow$ unstable

$$\frac{d\dot{x}}{dx} = \frac{d}{dx}(e^{-x} \sin x)$$

$$= e^{-x}\cos x - e^{-x} \sin x$$

$\hookrightarrow x = 2n\pi, \quad \frac{dx}{dt} > 0$

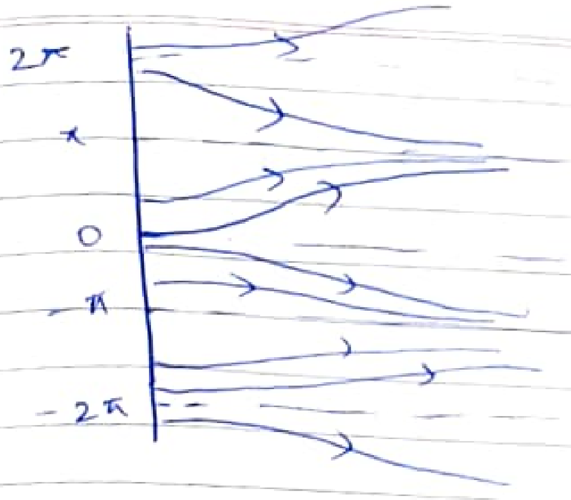
$x = 2n\pi + \pi \quad \frac{dx}{dt} < 0$

Analytical solution

$\dot{x} = e^{-x} \sin x$

$$\frac{dx}{e^{-x} \sin x} = dt$$

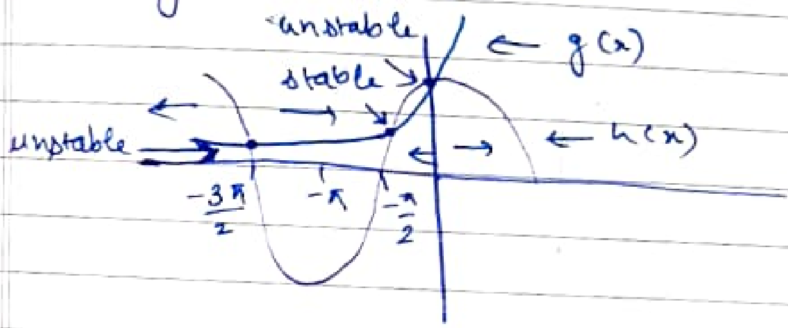
This cannot be solved analytically.



→ 2.2.7

we will plot each seperately an look at intersection.

$g(x) = e^x$ and $h(x) = \pm \cos x$



looking at equation → $\dot{x} = g(x) - h(x)$

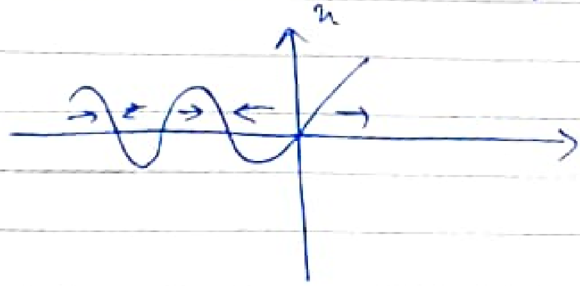
$\dot{x} > 0 \Rightarrow g(x) > h(x)$

$\dot{x} = 0 \Rightarrow g(x) = h(x)$

$\dot{x} < 0 \Rightarrow g(x) < h(x)$

pts of intersection ⇒ fixed pt.s as $\dot{x} = 0$

if on LHS of fixed pt. g(x) dominates and on RHS h(x) then that point and we have



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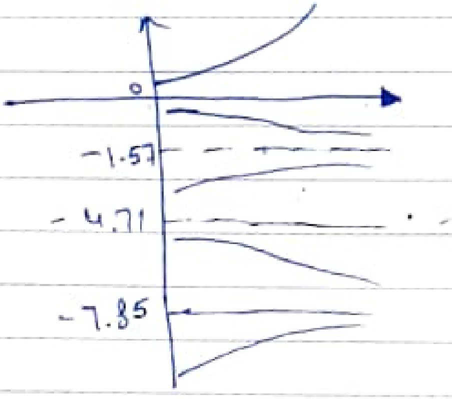
hence we have fixed pt. at $x \approx \pi\left[\frac{1}{2}-n\right], n \in \mathbb{N}$

Inf. number of points with alt. stability

$\frac{dx}{dt} = e^x - \cos x$, cannot be solved in closed form to guess trajectory, if particle start from point that lie b/w fixed pt. It moves to stable one. If it start from stable point it stays there.

If it start from unstable point, move to closest stable point.



2.3 → 2.3.1

$$\dot{N} = rN\left(1-\frac{N}{K}\right)$$

a) Separate variable and integrate, using partial function

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int r\,dt \rightarrow (1)$$

$$\frac{1}{N(1-\frac{N}{K})} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}}$$

$$\Rightarrow A\left(1-\frac{N}{K}\right) + BN = 1$$

$$A - \frac{AN}{K} + BN = 1$$

$$A = 1 \quad , \quad B = 1/K$$

$$\int \frac{dN}{N} + \frac{1}{K}\int \frac{dN}{1-\frac{N}{K}} = \int r\,dt$$

$$\ln N + \frac{1}{\cancel{K}} \times -\frac{1}{1/K} \ln(K-N) = \int r\,dt$$

$$\ln\left(\frac{N}{K-N}\right) = rt + c$$

$$c = \ln\left(\frac{N_0}{K-N_0}\right)$$

$$N(t) = \frac{N_0 K e^{rt}}{K - N_0 + N_0 e^{rt}}$$

b) change $x = 1/N$

$$\dot{N} = \frac{dN}{dt} = rN(1 - N/K)$$

$$N = 1/x$$

$$\frac{dN}{dt} = \frac{dN}{dx} \times \frac{dx}{dt} = -1/x^2 \cdot \frac{dx}{dt} = \frac{-\dot{x}}{x^2}$$

$$\dot{x} = r\left[1/K - x\right]$$

$$\int \frac{dx}{1/K - x} = \int r\,dt$$

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$$-\ln\left[\frac{1}{K} - x\right] = rt + c$$

$$x(t) = \frac{Kx_0 + e^{rt} - 1}{K \cdot e^{rt}}$$ where $x(t) = x_0$ at $t = 0$

verification $\to x = 1/N$

$$\frac{1}{N} = \frac{K/N_0 + e^{rt} - 1}{Ke^{rt}}, \quad N = \frac{N_0 e^{rt} K}{K + N_0 e^{rt} - N_0}$$

$\to$ 2.3.2

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$$

$$\dot{x} = k_1 a x - k_{-1} x^2$$

a) fixed pts can be found by equating flow eq^n to 0.

$$\dot{x} = k_1 a x - k_{-1} x^2 = 0$$

$$x(k_1 a - k_{-1} x) = 0$$

thus fixed pts. are $\to x_1^* = 0$, $x_2^* = \frac{a k_1}{k_{-1}}$

To solve analytically

$$\int \frac{dx}{x(k_1 a - k_{-1} x)} = \int dt$$

$$\frac{1}{x(k_1 a - k_{-1} x)} = \frac{B}{x} + \frac{C}{k_1 a - k_{-1} x}$$

$$1 = B(k_1 a - k_{-1} x) + Cx$$

$$B = \frac{1}{k_1 a} \qquad C = \frac{k_{-1}}{a k_1}$$

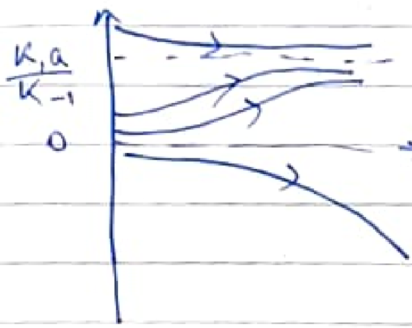
$\therefore$ sowing integral we get

$$\ln\left(\frac{x}{k_1 a - k_{-1} x}\right) = k_1 a t + c$$

$t=0,\ x = x_0$

$$c = \ln\left(\frac{x_0}{k_1 a - k_{-1} x_0}\right)$$

$$x(t) = \frac{x_0 k_1 a e^{k_1 a t}}{k_1 a - k_{-1} x_0 (1 - e^{k_1 a t})}$$



3.4

→ 3.4.1

Pitchfork bifurcation is kind of symmetrical bifurcation at which fixed points tend to appear and disappear in symmetrical pairs. If pitchfork stabilises fixed points then it is supercritical pitchfork bifurcation and if it destabilises, then it is subcritial pitchfork bifurcation.

$$\dot{x} = rx + 4x^3$$

$$\dot{x} = 0$$

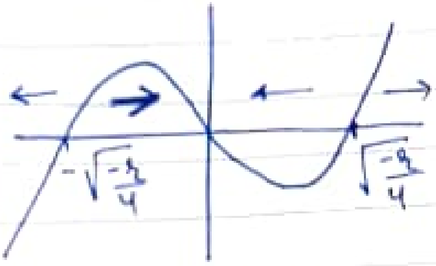
$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \pm\sqrt{\frac{-r}{4}}$$

if $r < 0$, 3 fixed pts exist

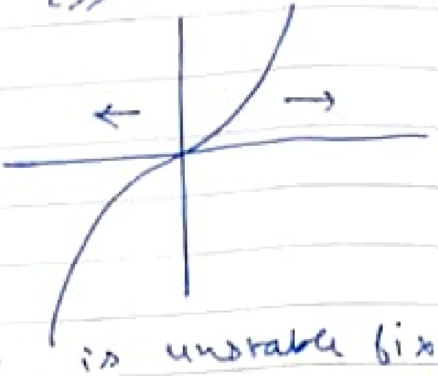
else only one fixed point $x=0$

∴ pitch fork bifurcation occurs at $\varepsilon = 0$.

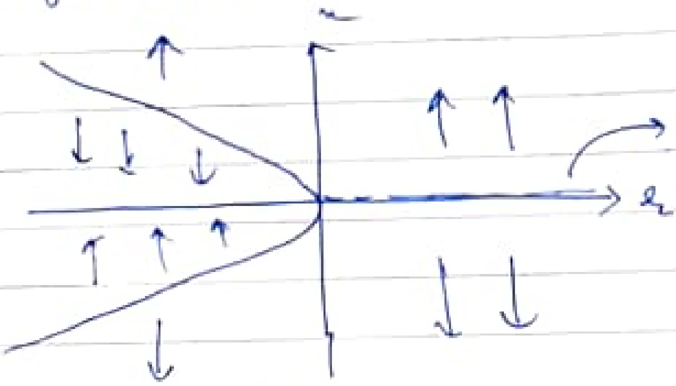
$\varepsilon < 0$



$\varepsilon \geq 0$



for $\varepsilon \geq 0$ we find $x=0$ is unstable fixed point and for $\varepsilon < 0$, $x=0$ is stable fixed pt. and 2 unstable fixed points are created at $x = \pm\sqrt{\frac{-\varepsilon}{4}}$. Hence this is subcritical bifurcation.



→ for $\varepsilon \geq 0$ $x=0$ is unstable

$x = -\sqrt{-\varepsilon/4}$

for $\varepsilon < 0$ $x = \pm\sqrt{-\varepsilon/4}$ is unstable.

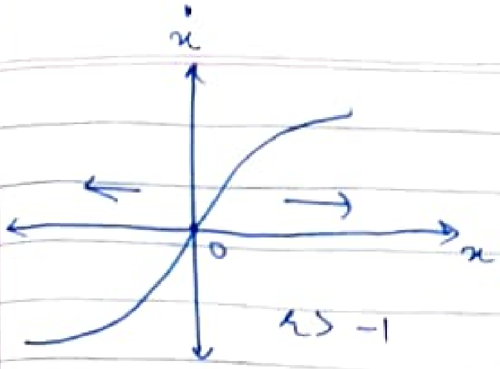
→ **3.4.4**

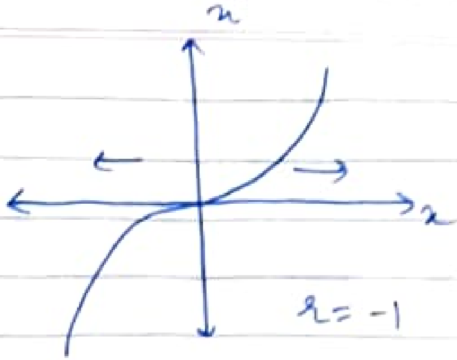
$$\dot{x} = x + \frac{\varepsilon x}{1+x^2}$$

fixed points, $\dot{x} = 0$

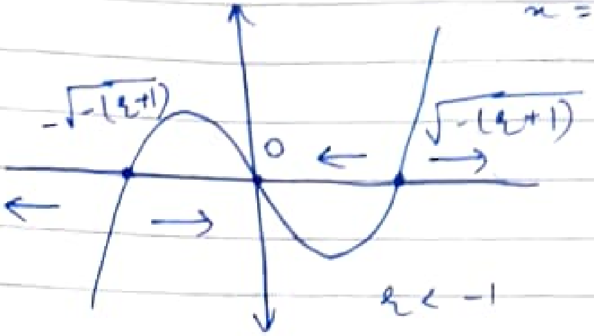
$$x\left(1 + \varepsilon/(1+x^2)\right) = 0$$

$x = 0$ or $x = \pm\sqrt{-(\varepsilon+1)}$





$x = 0$ is unstable



$x = 0$ is stable

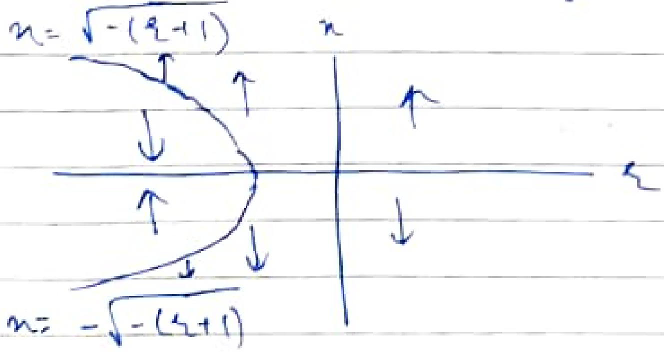
$x_0 = \pm\sqrt{-(r+1)}$ is unstable.

as we can see from $\dot{x}$ vs x plots, pitchfork bifurcation occurs when we go below $r < -1$, hence critical value for $r = -1$ a

We have a single fixed point for $r \geq -1$ & suddenly got 2 additional fixed points $r < -1$, hence displaying pitchfork bifurcation

Also, intially $x = 0$ was unstable when r goes below -1 $x = 0$ becomes stable and 2 new unstable fixed points arise at $x = \pm\sqrt{-(r+1)}$ hence this is

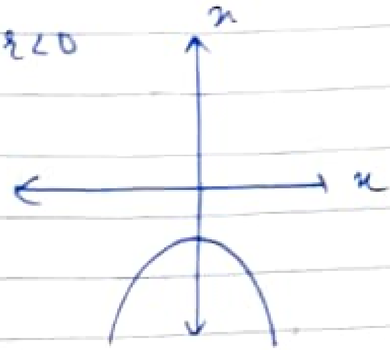
subcritical bifurcation.



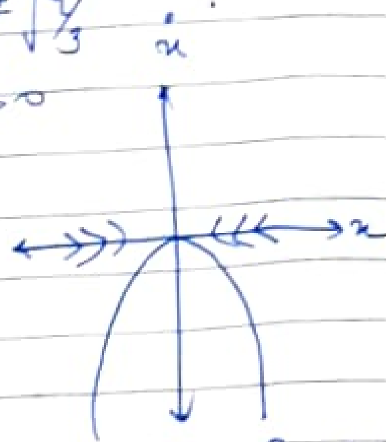
5) $\dot{x} = r - 3x^2$

fixed pts at $x = \pm\sqrt{r/3}$

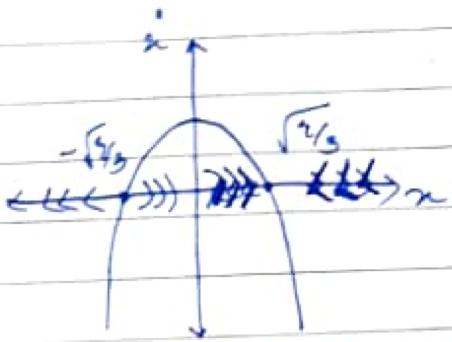
$r<0$



$r>0$



semi-stable at $x=0$

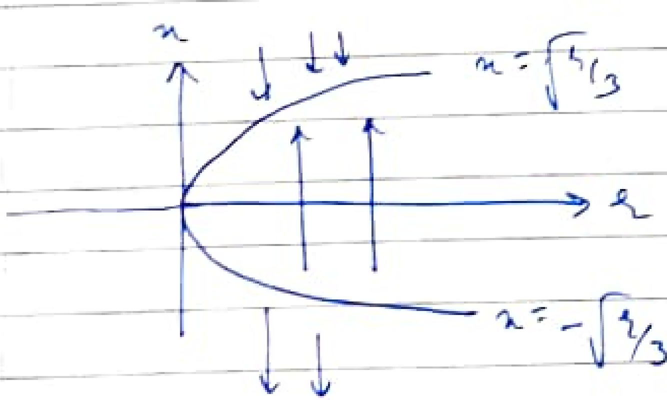


unstable at $x = -\sqrt{r/3}$

stable at $x = \sqrt{r/2}$

as we change r from -ve to +ve no. of fixed pts are $0\ (r<0)$, $1\ (r=0)$, $2\ (r>0)$

$\therefore$ This is saddle node bifurcation with critical value of $r=0$.

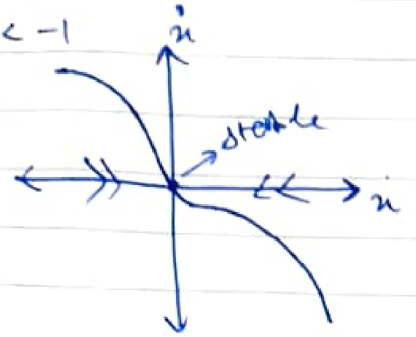


10) $\dot{x} = \epsilon x + \frac{x^3}{1+x^2}$

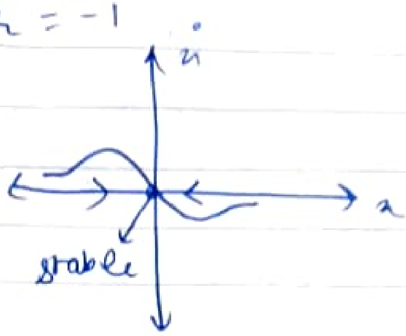
Fixed pts. $\Rightarrow$ $\epsilon x + \frac{x^3}{1+x^2} = 0$

$x = 0$ or $x = \pm\sqrt{-\epsilon/_{\epsilon+1}}$

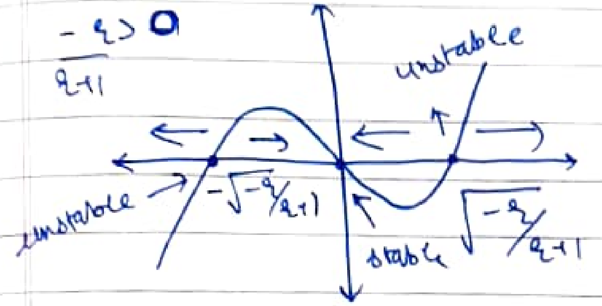
$\epsilon < -1$



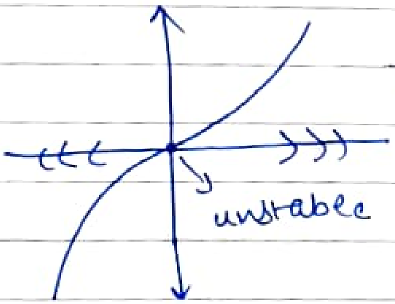
$\epsilon = -1$



$\frac{-\epsilon}{\epsilon+1} > 0$



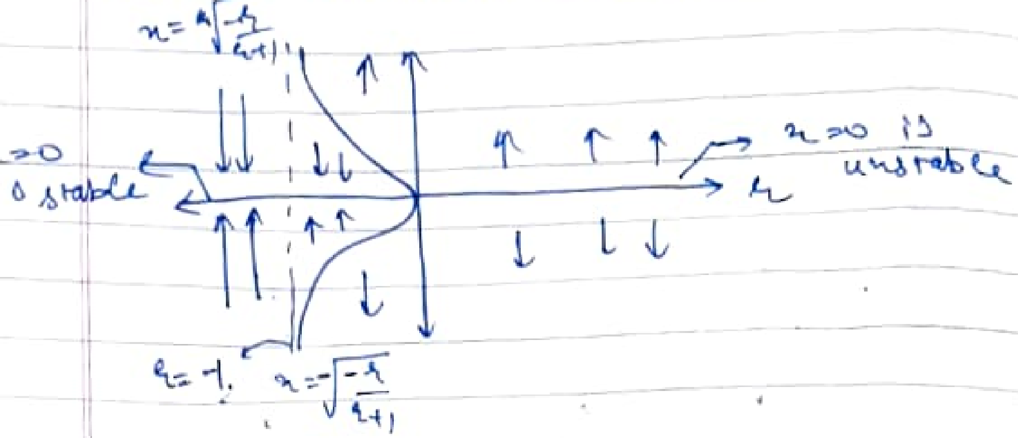
2more fixed points appear at $-\frac{\epsilon}{\epsilon+1} > 0$



$-\frac{\epsilon}{\epsilon+1} < 0$

critical values at $\epsilon = 0, -1$

as we see from $\dot{x}$ v/s x plots there was a single stable fixed pt at $x=0$, when value ϵ was in range $1 < \epsilon < 0$, 2 more fixed pts appear indicating pitchfork bifurcation. Both of these find points are unstable when value of ϵ goes above 0, we are again left with a single fixed point at $x=0$ but it is unstable. Hence this is subcritical bifurcation.



15) $\dot{x} = rx + x^3 - x^5$ in system with potential $V(x)$

$$\frac{dV}{dx} = -\dot{x}$$

To find min of potentials $\frac{dV}{dx} = 0$

$\Rightarrow \dot{x} = 0$

$$rx + x^3 - x^5 = 0 \Rightarrow x(r + x^2 - x^4) = 0$$

$$x = 0, \quad x^2 = \frac{1 \pm \sqrt{1+4r}}{2}$$

for ~~[illegible]~~ $V(x)$ to have some values at min, first compute $V(x)$

$$\frac{dV}{dx} = x^5 - x^3 - rx$$

$$\int dV = \int (x^5 - x^3 - rx)\, dx$$

$$V(x) = \frac{x^6}{6} - \frac{x^4}{4} - \frac{rx^2}{2} + C$$

at $x=0$, $V(x) = 0$

$$\therefore V(x) = \frac{x^6}{6} - \frac{x^4}{4} - \frac{rx^2}{2}$$

$$\frac{d^2V}{dx^2} = 5x^4 - 3x^2 - r$$

at $x = 0$, $\frac{d^2V}{dx^2} = -r$

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for $x=0$ to be min, $\frac{d^2V}{dx^2}>0 \Rightarrow q<0$

for value at minima to be same
$V(x)$ at $x^2 = \frac{1 \pm \sqrt{1+4q}}{2} = V(x)$ at $x=0$

$2x^6 - 3x^4 - 6qx^2 = 0$

$x^2 = 0$	$2x^4 - 3x^2 - 6q = 0$
$\frac{1 \pm \sqrt{1+4q}}{2} = 0$	at $x^2 = \frac{1 \pm \sqrt{1+4q}}{2}$
$\sqrt{1+4q} = \pm 1$	$4q - 1 + \sqrt{1+4q} = 12q$
$q = 0$	

$8q+1 = \pm\sqrt{1+4q}$
$64q^2 + 12q = 0$
$q = 0, -3/16$

we have to check their validity by ensuring condition that there are exactly 3 minimas

at $q=0$

$x^2 = \frac{1 \pm \sqrt{1+4q}}{2}$

$x = 0, 1, -1$

$\frac{d^2V}{dx^2} = 5x^4 - 3x^2$

at $x=0$
$\frac{d^2V}{dx^2} = 0$
inflection point

at $x=1$
$\frac{d^2V}{dx^2} = 2 > 0$
minima

at $x=-1$
$\frac{d^2V}{dx^2} = 2 > 0$
minima

$\therefore$ we get 2 minima only

at $q = -3/16$

$x^2 = \frac{1 \pm 1/2}{2}$, $x = \pm\sqrt{3/4}, \pm\sqrt{1/4}$

$\frac{d^2V}{dx^2} = 5x^4 - 3x^2 + 3/16$

at $x=0$, $\frac{d^2V}{dx^2} = 3/16 > 0$
$\hookrightarrow$ minima

at $x = -\frac{1}{2}$, $\frac{d^2V}{dx^2} = -1/4 < 0$
$\hookrightarrow$ maxima

at $x = 1/2$, $\frac{d^2V}{dx^2} = -1/4$, maxima

at $x = \sqrt{3}/2$
$\frac{d^2V}{dx^2} = 3/4 > 0 \hookrightarrow$ minima

at $x = -\sqrt{3}/2$
$\frac{d^2V}{dx^2} = 3/4 > 0 \hookrightarrow$ minima

∴ We get 3 minimas.

∴ $r = \frac{-3}{16}$ is the desired value

3.5

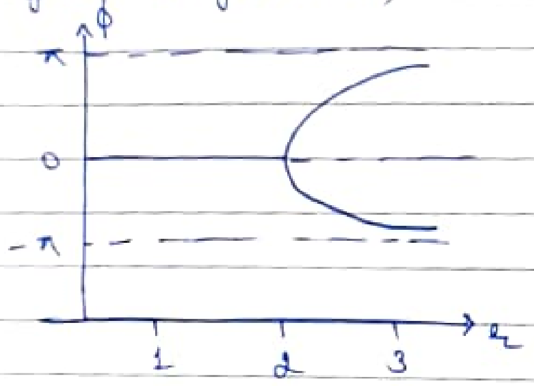
2) The eq^n is:

$$\dot{\phi} = \frac{d\phi}{dt} = f(\phi) = -\sin\phi + \gamma\sin\phi\cos\phi$$

$$= \sin\phi\,(\gamma\cos\phi - 1), \quad -\pi < \phi < \pi$$

$\gamma \geq 0$ ($\gamma = \frac{r\omega^2}{g}$ on pg. 63)

given diagram is,



Linear stability analysis is done to decide stability of fixed points by looking at sign of $\frac{d}{d\phi}(\dot{\phi}(\phi^*))$

where ϕ^* is a fixed pt.

$\frac{d}{d\phi}(\dot{\phi}(\phi^*)) < 0 \rightarrow$ stable

$\frac{d}{d\phi}(\dot{\phi}(\phi^*)) > 0 \rightarrow$ unstable

now fixed pts are obtained at $\dot{\phi} = 0$

$\Rightarrow \sin\phi\,(\gamma\cos\phi - 1) = 0$

$\sin\phi = 0$	$\cos\phi = 1/\gamma$
$\phi = 0, \pi$	$\phi = \pm\cos^{-1}(1/\gamma)$
[as $-\pi \leq \phi < \pi$]	[as $-\pi \leq \phi \leq \pi$]

hence fixed points are

$$0, \pi, \pm \cos^{-1}(1/\gamma)$$

Now consider

$$\frac{d}{d\phi}(\dot{\phi}) = \frac{d}{d\phi}\left[\sin\phi\,(\gamma\cos\phi - 1)\right]$$

$$= \frac{d}{d\phi}\left[\frac{\gamma}{2}\sin 2\phi - \sin\phi\right]$$

$$= \gamma\cos 2\phi - \cos\phi$$

at $\phi^* = 0$

$$\frac{d\dot{\phi}}{d\phi} = \gamma - 1$$

ϕ^* is unstable if $\gamma > 1$
ϕ^* is stable if $\gamma < 1$

at $\phi^* = \pi$

$$\frac{d\dot{\phi}}{d\phi} = \gamma + 1$$

$\phi^* = \pi$ is always unstable as $\gamma \geq 0$

at ϕ^* $\cos^{-1}(1/\gamma)$

$$\frac{d\dot{\phi}}{d\phi} = \frac{1-\gamma^2}{\gamma}$$

at $\phi^* = -\cos^{-1}(1/\gamma)$

$$\frac{d\dot{\phi}}{d\phi} = \frac{1-\gamma^2}{\gamma}$$

$$\frac{1-\gamma^2}{\gamma} = \frac{(1-\gamma)(1+\gamma)}{\gamma}$$

We need to consider only $\gamma > 1$ for sign change as $\gamma \geq 0$

$$\frac{(1-\gamma)(1+\gamma)}{\gamma} > 0 \qquad \gamma < 1$$

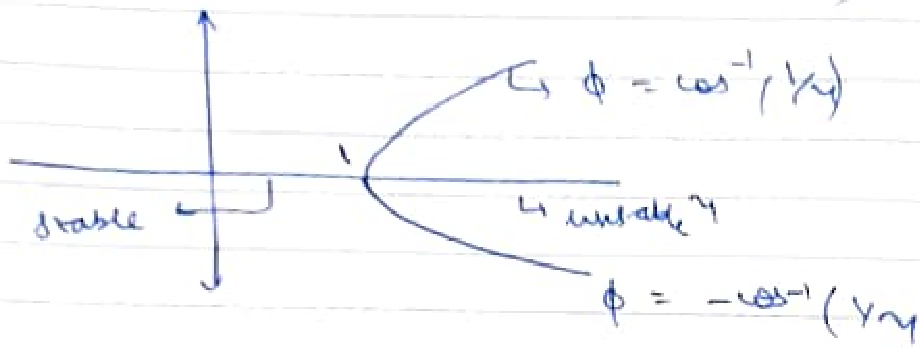
$$\frac{(1-\gamma)(1+\gamma)}{\gamma} < 0 \qquad \gamma > 1$$

hence $\phi^* = \pm\cos^{-1}\left(\frac{1}{\gamma}\right)$ are stable $\gamma>1$

unstable $\gamma<1$

but for $\gamma = 1$, ϕ^* is not defined,

$\therefore \gamma \in [1, \infty)$ for $\phi^* = \pm\cos^{-1}\left(\frac{1}{\gamma}\right)$



which is same as given graph.

$\therefore$ Given graph is correct.

5) $$\frac{d\phi}{dt} = \sin\phi\,(\gamma\cos\phi - 1)$$

$$= \left(\left[2\phi - \frac{(2\phi)^3}{3!} + \frac{(2\phi)^5}{5!} \cdots\cdots\right]\times \frac{\gamma}{2} - \left[\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \cdots\cdots\right]\right)$$

$$= (\gamma - 1)\phi - \frac{(4\gamma - 1)\phi^3}{3!} + O(\phi^5)$$

$$= A\phi - B\phi^3 + O(\phi^5)$$

comparing coefficient, $A = \gamma - 1$, $B = \frac{4\gamma - 1}{6}$

→ 3.7

3) Given system $\dot{N} = rN\left(1 - \frac{N}{K}\right) - H$, $H > 0$ is const.

to get dimensionless form, put $x = N/K$

$$\dot{x} = \dot{N}/K \quad \text{or} \quad \dot{N} = \dot{x}K$$

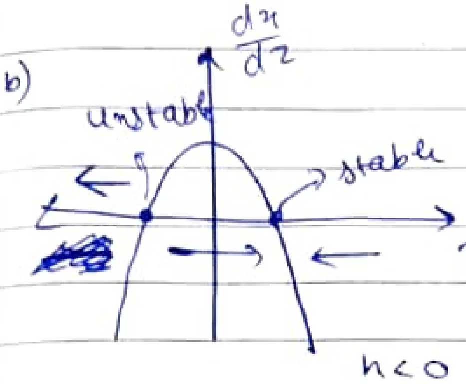
$rKx(1-x) - H = K\dot{x}$

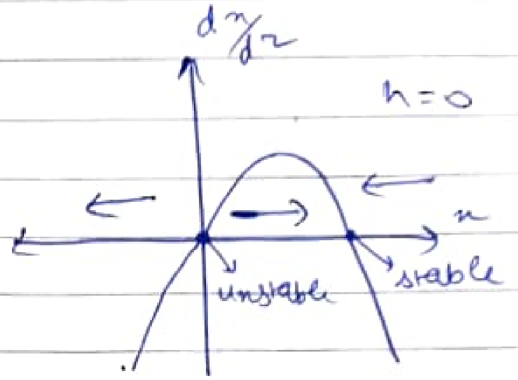
$\dot{x} = rx(1-x) - H/K$

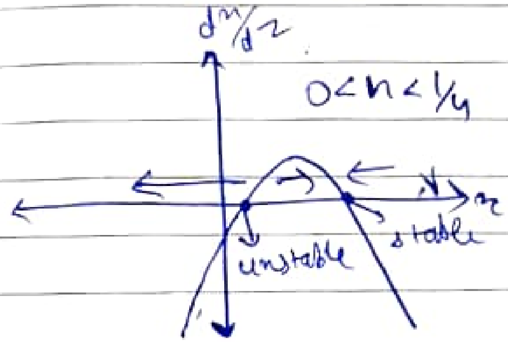
$\frac{dx}{r\,dt} = x(1-x) - H/rK$, $\tau = rt$, $h = \frac{H}{rK}$

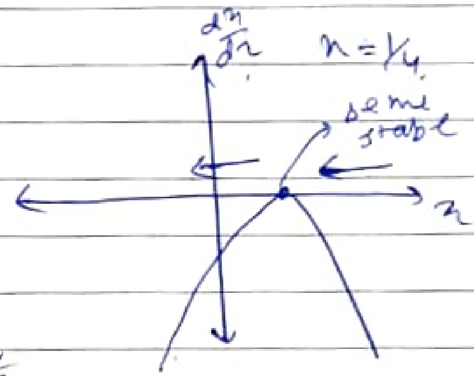
$\frac{dx}{d\tau} = x(1-x) - h$

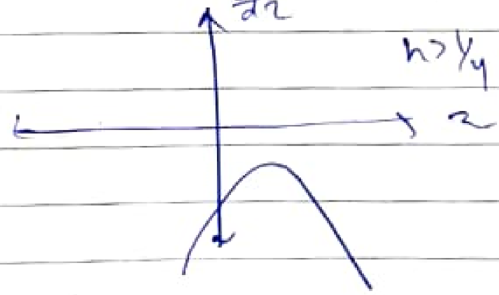
b)











c) This is saddle bifurcation because after a point (critical value of h is h_c) we go from no fixed pts. to 2, 1 unstable and 1 stable.

critical value $h_c = 1/4$

This can be obtained by checking for condition of obtaining a single fixed pt. $\frac{dx}{d\tau} = 0$ has equal roots.

$x(1-x) - h = 0$ has equal roots

$\therefore h_c = 1/4$

d) When $h < h_c$, there is a possibility of maintaining fish population in check in a way that effects of population growth and fishing are partially counteracted as rate of pop. growth is +ve, when population is relatively lower, ~~and~~ → rate is -ve, when there is too much population. It is stable when $h < h_c$.

When $h > h_c$, rate is always -ve.
∴ Population will become as in case there is too much fishing. So system will collapse.

5) $$\dot{g} = k_1 S_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

a) consider $x = g/k_4 \Rightarrow \frac{dx}{dt} = \frac{\dot{g}}{k_4}$

$$\dot{g} = k_4 \dot{x} \Rightarrow k_1 S_0 - k_2 g + \frac{k_3 g^2/k_4^2}{1 + g^2/k_4^2}$$

$$= k_4 \dot{x}$$

$$k_4 \dot{x} = k_1 S_0 - k_2 k_4 x + \frac{k_3 x^2}{1+x^2}$$

$$\frac{k_4}{k_3} \dot{x} = \frac{k_1 S_0}{k_3} - \frac{k_2 k_4 x}{k_3} + \frac{x^2}{1+x^2}$$

put $S = \frac{k_1 S_0}{k_3}$, $r = \frac{k_2 k_4}{k_3}$

$$\frac{k_4}{k_3} \dot{x} = S - rx + \frac{x^2}{1+x^2}$$

$$\frac{dx}{d\tau} = S - rx + \frac{x^2}{1+x^2} \quad \text{where } \tau = \frac{k_3}{k_4} t$$

b) $S=0 \Rightarrow \frac{dx}{d\tau} = -rx + \frac{x^2}{1+x^2} \Rightarrow$ fixed pts at $\frac{dx}{d\tau} = 0$

$-rx + \frac{x^2}{1+x^2} = 0 \quad , \rightarrow x = 0 \, , \frac{1 \pm \sqrt{1-4r^2}}{2r}$

$\hookrightarrow$ fixed pts:

fixed points $\frac{1 \pm \sqrt{1-4r^2}}{2r}$ are +ve because $r>0$

and they exist only when $1-4r^2 \geq 0$

i.e. $0 < r \leq 1/2$ as $r>0$ from initial conditions.

Also

$$0 \leq 1 \pm \sqrt{1-4r^2} \leq 2$$

$\therefore$

$$0 \leq \frac{1 \pm \sqrt{1-4r^2}}{2r} \leq 1/r$$

$\therefore$ both fixed points are +ve

critical pt is when there is a single fixed pt is obtained. i.e. $D=0 \Rightarrow 1-4r^2=0$

$r = 1/2$ (as $r>0$)

for $r > 1/2 \rightarrow$ no roots, $\therefore$ no fixed pts. except $x=0$

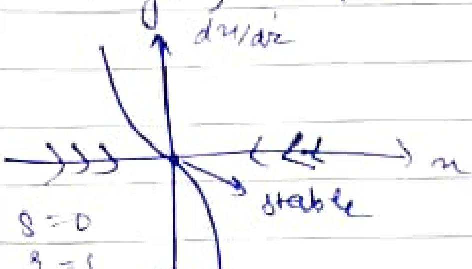
$\therefore$ 2 +ve fixed points exist if $r < r_c$, $r_c = 1/2$

$\therefore$ Proved.

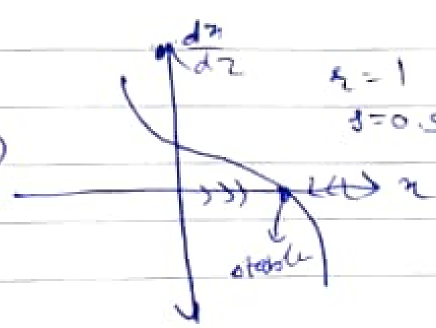
c) Initially, $g(0) = 0$

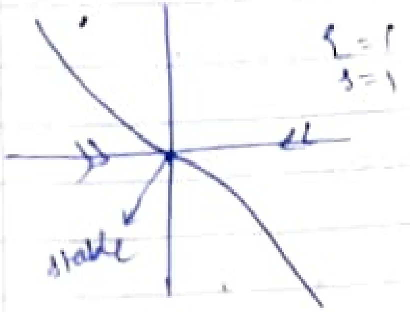
$x(0) = 0$

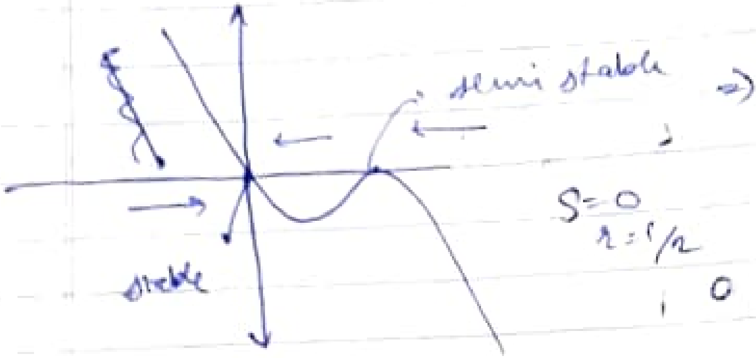
$g(t) = k_4 x(t)$

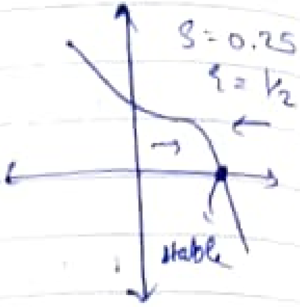


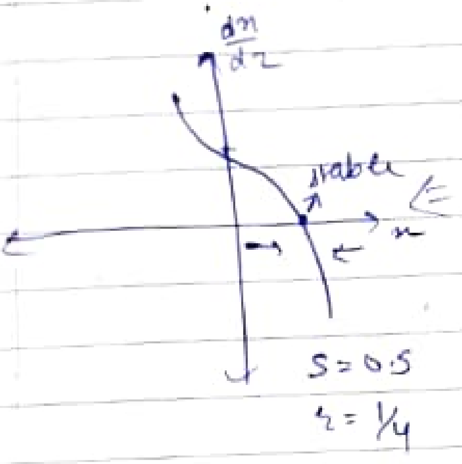
$\Rightarrow$

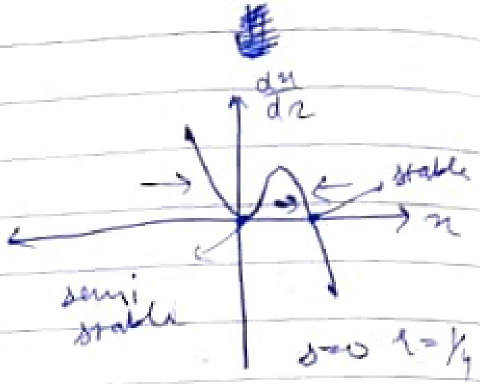












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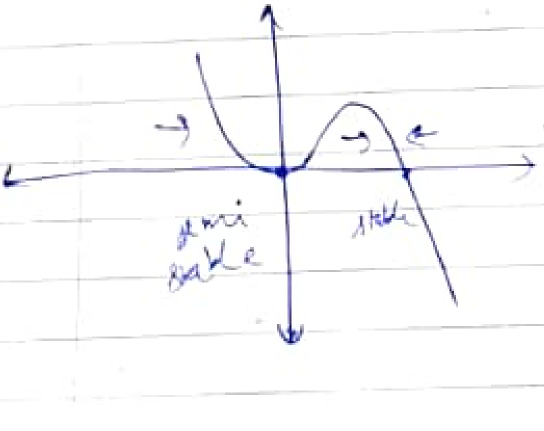

since present condition in which sys is is semi stable point, It can either stay there itself or return to original position.

Three cases regarding stability:

$r > r_c = 1/2$ system, returns to initial state

$r = r_c = 1/2$, system may or may not return to initial state.

$r < r_c = 1/2$, system will mostly be in new state.

d) We can observe that we get two more fixed points from a single one when value ~~from a~~ of r goes below $r_c = 1/2$. No matter value of r as long as $r < r_c$, Hence it is saddle node bifurcation.

In saddle node bifurcation at critical point we have single extra fixed point that is semi stable. This condition is given by $\frac{d}{dx}\left(\frac{dx}{dt}\right) = 0$

$$\frac{d}{dx}\left(s - rx + \frac{x^2}{1+x^2}\right) = 0 \Rightarrow r = \frac{2x}{(1+x^2)^2}$$

We also have $\frac{dx}{dt} = 0$ at fixed pt.

$$s - rx + \frac{x^2}{(1+x^2)} = 0$$

$$\Rightarrow s = \frac{(1-x^2)x^2}{(1+x^2)^2}$$

e)