CS 40 Homework 5

Viraj Zaveri

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1 Cubes

Let $P(n)$ denote the proposition that the sum of the first n cubes of the natural numbers is $(\frac{n^2+n}{2})^2$. Our conjecture is that $P(n)$ is true for all natural numbers n.

Base case: $P(0)$ is true, $(0)^3 = \left(\frac{(0)^2+0}{2}\right)^2 = 0$. This is true because the sum of the zeroth cubed natural number is indeed 0.

Inductive case: To complete the inductive case, we must show that the proposition $P(i) \rightarrow P(i+1)$ is true for all natural numbers i. We first assume the inductive hypothesis $P(i)$. Therefore,

$$0 + 1 + 8 + 27 + 64 + ... + i^3 = \left(\frac{i^2+i}{2}\right)^2$$

$P(i+1)$, based on the given formula, should hypothetically be represented

$$\left(\frac{(i+1)^2+(i+1)}{2}\right)^2$$

We know that the sum of the first $i+1$ cubes of natural numbers is as follows

$$0 + 1 + 8 + 27 + 64 + ... + i^3 + (i+1)^3$$

From the inductive hypothesis, we know that i^3 is $\left(\frac{i^2+i}{2}\right)^2$. Therefore, the above summation is equal to $\left(\frac{i^2+i}{2}\right)^2 + (i+1)^3$. Adding the two terms and simplifying, we obtain

$$\frac{i^4 + 6i^3 + 13i^2 + 12i + 4}{4}$$

This is equal to the assumption we made about $P(i+1)$ in the beginning. Thus, $P(i+1)$ is true. Because we've completed the base case and inductive case, we can conclude that $P(n)$ is true for all natural numbers n through mathematical induction.

2 Collatz Lite

Goal: Prove that $P(n)$ will converge to zero for all positive numbers defined by the procedure below.

$$f(x) = \begin{cases} 2x - 2 & \text{if } x \text{ is odd} \\ \frac{x}{2} & \text{if } x \text{ is even} \end{cases} \tag{1}$$

Base case: $P(1)$ is true; 2(1) - 2 = 0

Inductive case: To complete the inductive case, we must show that the proposition $P(k) \rightarrow P(k+1)$ is true for all positive integers n. We first assume the inductive hypothesis $P(n)$ is true for any n from 1 to k.

Case (i): $k + 1$ is even. If $k + 1$ is even, then following the procedure, we know that we must divide by 2 and obtain $\frac{k+1}{2}$. This result is less than k. From our inductive hypothesis, we know that $P(n)$ is true for any k from 1 to k. Therefore, $P(k + 1)$ is true when it's even.

Case (ii): $k + 1$ is odd. In this case, we must complete 2 steps. Following the procedure, $k+1$ becomes $2(k+1)-2 = 2k$. $2k$ is even since it's a multiple of 2, so we apply the procedure again with the first case of $f(x)$. We then obtain k. From the inductive hypothesis, we know that $P(k)$ is true. Therefore, $P(k + 1)$ is true when it's odd.

Since $P(k + 1)$ is true in the case that it's even and odd, and the base case is true, we can conclude that $P(n)$ is true for all positive integers through strong induction.

3 Casting

We are not using partial functions since there is a subset of A that will not contain an image in R. Therefore, we will instead map R to A.

a. F = $\{x|x : R \rightarrow A$ and x is injective$\}$

b. G = $\{x|x : R \rightarrow A$ and x need not be injective$\}$

4 Permutations

Let $P(n)$ denote the proposition that a set with cardinality n has $n!$ distinct permutations. Our conjecture is that $P(n)$ is true for all n such that $n > 0$.

Base case: $P(1)$ is true; $1! = 1$, there is only one way to map a single element to itself.

Inductive case: To complete the inductive case, we must show that the proposition $P(k) \to P(k+1)$ is true for all possible cardinalities of S. We first assume the inductive hypothesis $P(k)$ is true for all non-negative integers k.

We can denote $k = |S|$. Suppose we add an arbitrary element x to the set S. Therefore, we take $S \cup \{x\}$. The cardinality of S now becomes $k+1$. There are $k+1$ ways to add this element to each existing permutation of the elements of S. Therefore, there are $k+1(k!)$ total ways to arrange the elements of $S \cup \{x\}$.

By the definition of a factorial, this expression becomes $(k+1)!$. Because we've proved the base and inductive cases are true, we can conclude that $P(n)$ is true for all possible cardinalities n of S.

5 Bracelets

8575

There are 7^5 permutations of choosing 5 beads from 7 colors. We must account for the fact that there exist symmetric permutations. To be precise, there are 7^3 of these symmetric permutations. This is because the first through third elements are free, but the last two depend on the first two. We can then subtract the two quantities to find the non-symmetric permutations, but we must divide by 2 to get the unique permutations. We then obtain 8232. We can then add back the unique symmetric permutations to get 8575.

6 Teamwork

Method: Direct proof

(1) The total number of possible 3-person groups from a pool of 6 people is $\binom{6}{3} = 20$.

(2) However, since each student switches groups after each project, we must find the number of distinct groups that this student is present in.

(3) This would be $\binom{5}{2} = 10$ since there are 5 remaining students from which we need to choose 2 to form a group of 3.

(4) We are given that there are a total of 11 projects. From (3), however, there are only 10 distinct groups per student.

(5) Through the Pigeonhole Principle, it is therefore impossible for a student to avoid being in the same group twice.

Thus, since the number of projects is greater than the number of distinct groups per student, students must be in the same group at least twice.