CS 40 Homework 4

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1 Types

a. $f : \mathbb{R}^+ \to \mathbb{R}$

b. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$

c. $f : \mathbb{R} \to [-1, 1]$

d. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{B}$

e. $f : \mathbb{Z} \to \wp(\mathbb{Z}^+)$

f. $f : \mathbb{N} \to \mathbb{Z}^+$

2 Relations

Goal: Prove $\sim$ is an equivalence relation.
Method: Direct proof

(1) By definition, if $\sim$ is an equivalence relation, it satisfies the properties of being reflexive, symmetric, and transitive.

(2) Given: a $\sim$ b $\leftrightarrow$ $f(a) = f(b)$

(3) Let $\exists$(a, b, c) $\in$ A(a $\neq$ b $\wedge$ b $\neq$ c $\wedge$ a $\neq$ c)

(4) Separating (2) into two implication statements, we know the following: if a $\sim$ b, then $f(a) = f(b)$. Thus, if a $\sim$ a, then $f(a) = f(a)$. This implies that $\sim$ is reflexive.

(5) We also know that if b $\sim$ a, then $f(b) = f(a)$. Since the = operator is symmetric, this means that $f(a) = f(b)$. Hence, $\sim$ is symmetric.

(6) Finally, if a $\sim$ b and b $\sim$ c, then $f(a) = f(b)$ and $f(b) = f(c)$. The = operator is also transitive; therefore, we can deduce that $f(a) = f(c)$. Thus, $\sim$ is transitive.

By the definition of an equivalence relation, since $\sim$ is reflexive, symmetric, and transitive, it is therefore an equivalence relation.

3 Cancellation

Goal: Prove the below inference rules are valid iff f is a bijection.

$$f \circ g = f \circ h \quad \vdash \quad g = h$$
$$g \circ f = h \circ f \quad \vdash \quad g = h$$

Method: Proof by cases
We can conduct a proof by cases with 4 cases.

Case (i): f is both injective and surjective. If f is both injective and surjective, then f^{-1} must exist. If we apply f^{-1} to both sides of the first inference rule, we obtain $f^{-1}(f(g(x)) = f^{-1}(f(h(x))$. This provides the identity ι. Therefore, f^{-1} and f cancel and we are left with $g(x) = h(x)$. The same case applies to the second inference rule; however, the inverse is still applied to f and we obtain ι again.

Case (ii): f is surjective but not injective. We can define $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\} = x^2$. f is therefore surjective since every positive real number and zero can be mapped to by a real number. However, the function isn't injective since not every element in the domain has a distinct element in the codomain. Furthermore, we define $g(x) = x$ and $h(x) = |x|$. $f(g(x))$ and $f(h(x))$ will always produce the same output; however, $g(x) \neq h(x)$.

Case (iii): f is injective but not surjective. Since f isn't surjective, there exists at least one element in the codomain of f that isn't mapped to by an element in its domain. Thus, g and h aren't bounded by that element, so g and h can diverge from there.

Case (iv): f is neither injective nor surjective. Since cases (ii) and (iii) don't hold, the combination of them will not hold either. Therefore, in this case, the two inference rules are invalid.

4 Counting

(1) Firstly, we want to prove that $|\mathbb{Z}| = |\mathbb{N}|$.

(2) We can prove this by defining a function $f : \mathbb{Z} \to \mathbb{N}$:

$$f(x) = \left\{ \begin{array}{ll} 2x - 1, & \text{x } > 0 \\ -2x, & \text{x} \leq 0 \end{array} \right\} \tag{1}$$

Since $f(x)$ is both injective and surjective, it is bijective. Through the property of bijectivity, $|\mathbb{Z}| = |\mathbb{N}|$. Therefore, the set of all integers is countably infinite.

(3) Next, we can define a set S as the set of all square numbers. We can also define a function $g : \mathbb{N} \to S$ where $g(x) = x^2$

(4) $g(x)$ is a bijective function since there is a unique element in the codomain for each element in the domain and every element in the codomain can be mapped to by some element in the domain.

(5) Therefore, $|S| = |\mathbb{N}|$.

(6) By transitivity from steps (2) and (5), we can therefore state that $|\mathbb{Z}| = |S|$.

Thus, we can conclude that the cardinality of the set of integers is equal to the cardinality of the set of square numbers.

5 Cantor's Theorem

(1) Assume for the sake of contradiction that $f : S \to \wp(S)$ is a surjective function.

(2) Since f is surjective, $\exists x \in S(f(x) = D)$.

(3) We also know from the property of surjection that $|S| >= |\wp(S)|$.

(4) Assume that $x \in D$. This means that $x \in f(x)$ which means that $x \notin D$ from the definition of the construction of D through its predicate. We've reached a contradiction that x is an element of D yet not an element too.

(5) Let's assume the opposite then that $x \notin D$. This means that $x \notin f(x)$ which means that $x \in D$. We've reached the same contradiction where x is simultaneously in D and not in D.

(6) Therefore, we've shown that there is no $x \in S$ such that $f(x) = D \subseteq S$. f cannot be surjective then.

(7) Our initial assumption of f being surjective was therefore invalid.

Thus, by proof by contradiction, $|S| < |\wp(S)|$.