Solutions Practice Test Nov 2, 2020

Invariants for $ab \to ba$
$ba \to ab$

are $\#a$

$\#b$

ARS $(A, \to)$

Invariant $\#a : A \to \mathbb{N}$

$\#b : A \to \mathbb{N}$

- Guess Invariant $f : A \to B$
- Verify the Invariant

$a_1 \to a_2 \Rightarrow f(a_1) = f(a_2)$

$\to$ by going through all the rules

rules	# a before	# a after
$ab \to ba$	1	1
$ba \to ab$	1	1

$bbbb \to bab$ $\qquad\qquad$ $bb \to a$

$\quad\;\; \to b$ $\qquad\qquad\quad$ $ab \to$

the normal form of bbbb is b

the normal form of a is a

⋮

computations follow $\longrightarrow$

the normal form of aaa is [ ]

What is the equiv. class of [ ]?

Characterise those w such that $w \xrightarrow{*} [\,]$.

aaa → [ ]

ab → [ ]

aabbaabb $\xrightarrow[?]{*}$ [ ]

$\underline{\text{aaaabbbb}}$ → $\underline{\text{ab}}$bbb

→ bbb

→ ab

→ [ ]

mod2

aaa →

even

ab →

odd

bb → a

aa →

aa → b

mod3

$\boxed{\text{\# a mod 3 invariant for aaa →}}$

how to extend this to $ab \to$
$bb \to a$?
$aa \to b$

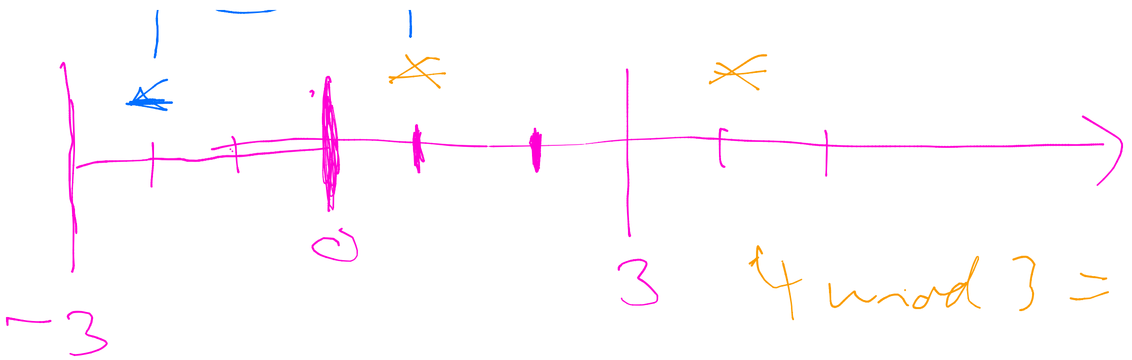
try: $(\#a + \#b) \bmod 3$ is not invariant wrt

	before	after
$ab \to [\,]$	$2 \neq$	0
	0	0

$$\boxed{(\#a - \#b) \bmod 3}$$

$bb \to a$	$-2 \bmod 3 = 1$	1
$aa \to b$	2	$-1 \bmod 3 = 2$

$-2 \bmod 3 = 1$



Same invariant without negative numbers

$$(\#a + 2\cdot\#b) \bmod 3$$

$\overset{1+1+1}{aaa} \rightarrow \quad = 0 \mod 3$

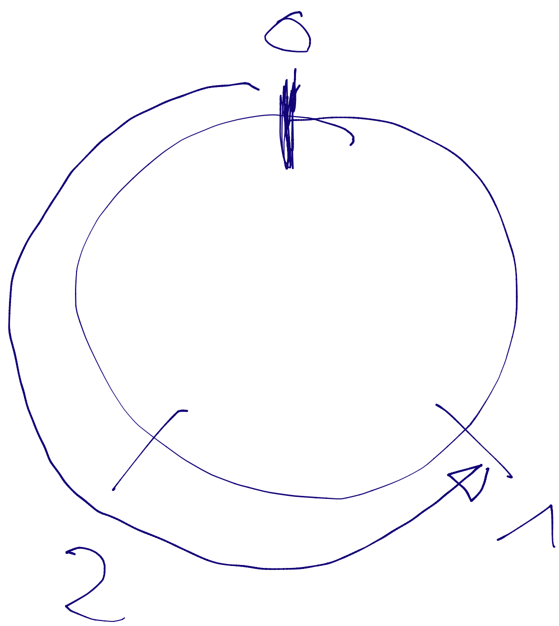
$\overset{1+2}{ab} \rightarrow \quad = 0 \mod 3$

$\overset{1+1}{aa} \rightarrow \overset{2}{b} \quad =$

$\overset{2+2}{bb} \rightarrow \overset{1}{a} \quad = \mod 3$

Trick: • Associate numbers to letters

• Concatenation as plus (or times?)



$-2 \bmod 3$

$=$

1

"nice" answer:

w is equivalent to [ ]

if and only if

$$\boxed{(\#a + 2\cdot\#b) \bmod 3 = 0}$$

- for a long word we can predict whether it is equiv. to [ ] without doing the computation

does not mention →

w is in the equ. class of $\overset{b}{a}$ if and only if

$$(\#a + 2\cdot\#b) \bmod 3 = \overset{2}{1}$$

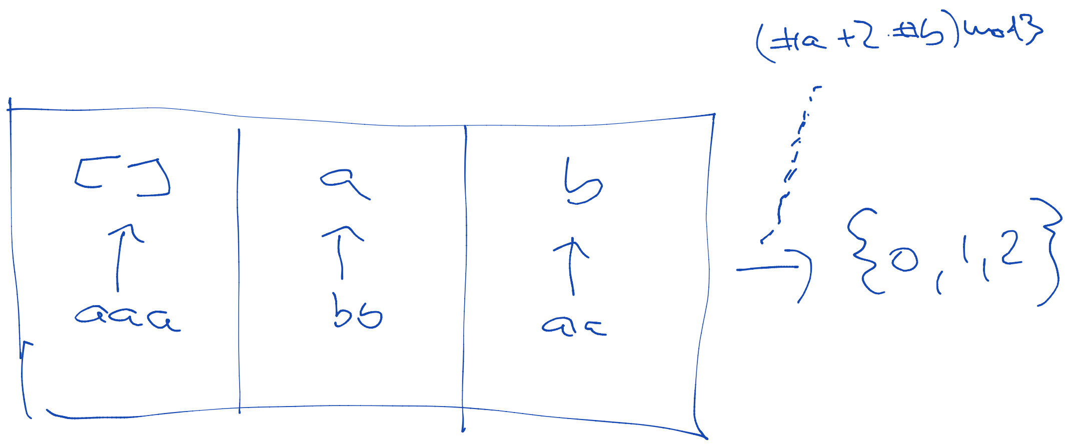
because $aa \to b$

$ab \to$

$bb \to a$

all words with two or more letters are equivalent to a word with 0 or 1 letter only.

Hence, there cannot be other equ.-classes than those containing [ ], a, b.



Does not terminate

because there is some

computation that does

not terminate, for example,

$ab \to ba \to ab \to ba \to \dots$

Because of $ab \to$

we should keep $\boxed{ba \to ab}$

but we do not need

$ab \to ba$.

We drop the rule $ab \to ba$.

What is the measure

function?

- Guess
- Verify

$\boxed{A \to \mathbb{N}}$

$\boxed{\text{base } 10}$

$$\overset{2}{b}\overset{1}{a} \xrightarrow{>} \overset{1}{a}\overset{2}{b}$$

$$\overset{1}{a}\overset{1}{a}\overset{1}{a} \xrightarrow{>} \varepsilon$$

$$\overset{1}{a}\overset{2}{b} \xrightarrow{>} \varepsilon$$

$$\overset{2}{b}\overset{2}{b} \xrightarrow{>} \overset{1}{a}$$

$$\overset{1}{a}\overset{1}{a} \xrightarrow{>} \overset{2}{b}$$

$$\mathrm{phi}([a, b, a, b, a, a]) =$$

$$[1, 2, 1, 2, 1, 1]$$

order this as base 10 numbers

Q3

$$((A\ B)\ C)\ D$$

$$\lambda x.(\lambda y.(\lambda z.(A\ B)\ C))$$

$$(\lambda f.(\lambda x.(f\ (f\ x))))\ (\lambda x.x)$$

$$(\lambda f.(\lambda x.((f\ f)\ x)))\ (\lambda x.(\lambda y.(x\ y)))$$

$$(1 + (2 * 3)) + (4 * 5)$$

$$(\lambda f . (\lambda x . (f\ f\ x)))$$

$$((A\ B)\ C)$$

$$\lambda f . (\quad\quad)$$

PARSING PART

as far to the right as possible

$$((A\ B)\ C)$$

left

$$\lambda x . (\lambda y . (f\ y))$$

right

Q4

$$(\lambda f . \lambda x . f (f x)) (\lambda x . x) \rightarrow_\beta$$

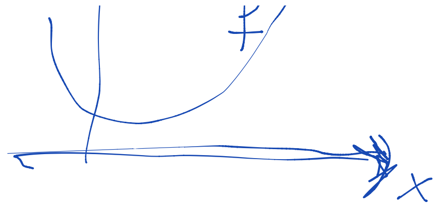
$$\lambda x . (\lambda x . x) ((\lambda x . x) x) \rightarrow_\beta$$

$$\lambda x . ((\lambda x . x) \; x) \rightarrow_\beta$$

$$\lambda x . x$$

$(\lambda x . x)$ is the identity

$$f(x) = a x^2 + b x + c$$



$$\lambda x.\ ax^2 + bx + c$$

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parabola (int x) {
    return a*x*x + b*x + c }
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$$(\lambda x.\, xx)(\lambda x.\, xx) \longrightarrow_{\beta}$$

$$(\lambda x.\, xx)(\lambda x.\, xx) \longrightarrow_{\beta}$$

$$\dots$$

λ-calculus
is not terminating