Termination

ARS $(A, \rightarrow)$ model of computation

$\rightarrow$ a one-step computation

flexible: "one step" big or small

- one operation on the machine
- the body of a whole loop
- ...

$A \xrightarrow{\varphi} \mathbb{N}$ measure function

$a \rightarrow b \Rightarrow \varphi(a) > \varphi(b)$

Exles of programs and used measure functions to prove termination.

Let us turn to a theoretical question.

Theory is practice on a higher level.

Practice is application of a method.

Theory is development of a method.

Practice: proving termination of your $\boxed{\text{program}}$

if $\exists$ measure function $\Rightarrow$ terminating singular

Theory: What is a method to prove termination for $\boxed{\text{programs}}$ in general.

plural

Question: $\boxed{\text{If program is terminating} \Rightarrow \exists \text{ measure function}}$?

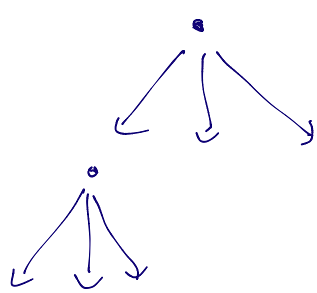
- theoretical question for a programmer
- practical question for a developer of methods to verify termination

Thm: If terminating & finitely branching

then $\exists$ measure function

Q finitely branching

$\boxed{\text{all } a \in A \text{ there are only finitely } b \text{ such that } a \to b}$

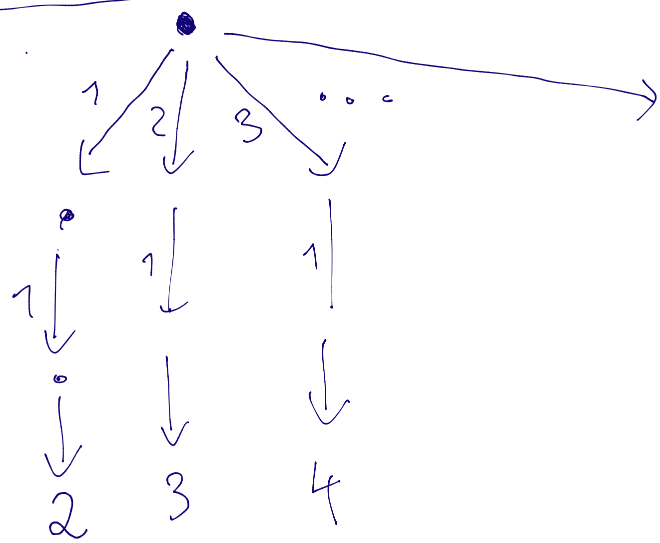


What would be an example of

a non finitely branching ARS ?

input a
input b
output a+b

INPUT ?



Coming back to the proof:

Claim: $\boxed{\text{terminat. \& fin. br.} \Rightarrow \exists \text{ meas. fun.}}$

Assume: $(A, \to)$ is termination
$(A, \to)$ is fin. br.

Show: $\exists$ measure function

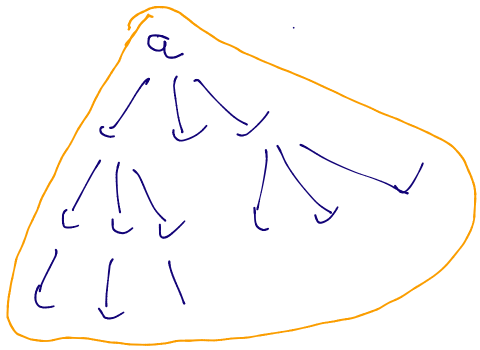
$\varphi : A \to \mathbb{N}$

$|S|$

The number of elements of S

here an idea is needed

$$\varphi(a) \stackrel{\text{def}}{=} |\{b \in A \mid a \xrightarrow{*} b\}|$$



Show: φ is a measure function

Choose any a, b

$$a \to b \Rightarrow \varphi(a) > \varphi(b)$$

Assume $a \to b$

Show $\varphi(a) > \varphi(b)$

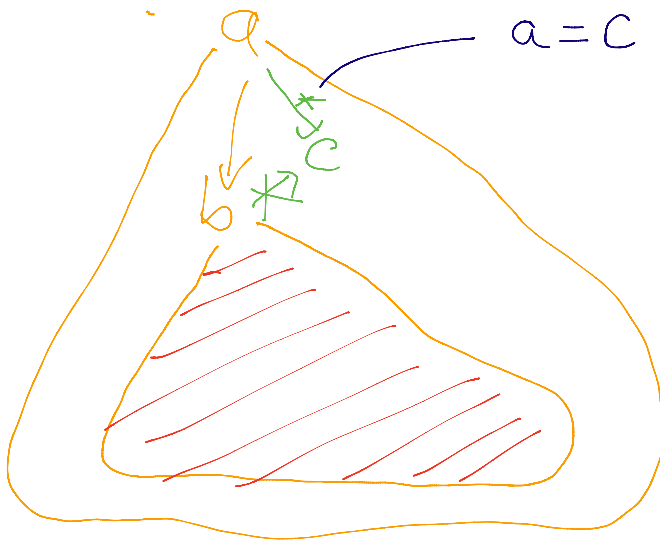
$$|\{c \mid a \xrightarrow{*} c\}| > |\{d \mid b \xrightarrow{*} d\}|$$

(1) $\{c \mid a \xrightarrow{*} c\} \supseteq \{d \mid b \xrightarrow{*} d\}$

(2) $\exists c$ s.t. $a \xrightarrow{*} c$, $b \not\xrightarrow{*} c$

Show(1) Use def $\xrightarrow{*}$

refl & $\boxed{\text{transitive closure}}$ of $\longrightarrow$



$a = c$

QED for (1)

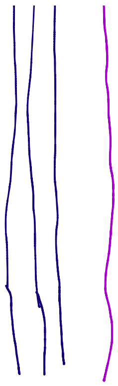
Show(2) Idea needed

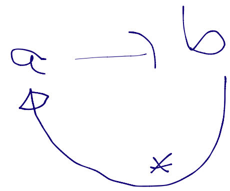
what could be c?

Define: $c = a$

Show: not $b \xrightarrow{*} a$

Assume: $b \xrightarrow{*} a$





$a \to b$ and $b \xrightarrow{*} a$
are assumptions

it follows $a \xrightarrow{*} a$

contradiction to terminating

Further Remarks :

1) Useful observation

$$\longrightarrow \;\subseteq\; \xrightarrow{*}$$

2) General remark on proofs.

2a) Any proof of

$$A \Longrightarrow B$$

has the structure

Assume A

⋮

B

2b) Any proof of

not A

has the structure of

Assume A

⋮

contradiction