Termination

ARS $(A, \rightarrow)$ model of computation

$\longrightarrow$ a one-step computation

flexible: "one step" big or small

- one operation on the machine
- the body of a whole loop
- ...

$A \xrightarrow{\varphi} \mathbb{N}$ measure function

$a \longrightarrow b \implies \varphi(a) > \varphi(b)$

Exles of programs and used measure functions to prove termination.

Let us turn to a theoretical question.

Theory is practice on a higher level.

Practice is application of a method.
Theory is development of a method.

Practice: proving termination of your $\boxed{\text{program}}$
if $\exists$ measure function $\Rightarrow$ terminating singular

Theory: What is a method to prove termination for $\boxed{\text{programs}}$ in general.

plural

Question: $\boxed{\text{If program is terminating} \Rightarrow \exists \text{ measure function}}$?

theoretical question for a programmer

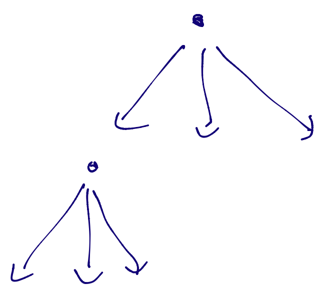
practical question for a developer of methods to verify termination

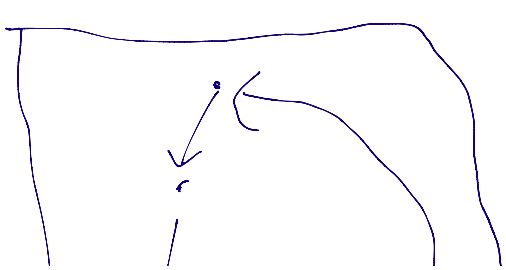
Thm: If terminating & finitely branching

then $\exists$ measure function

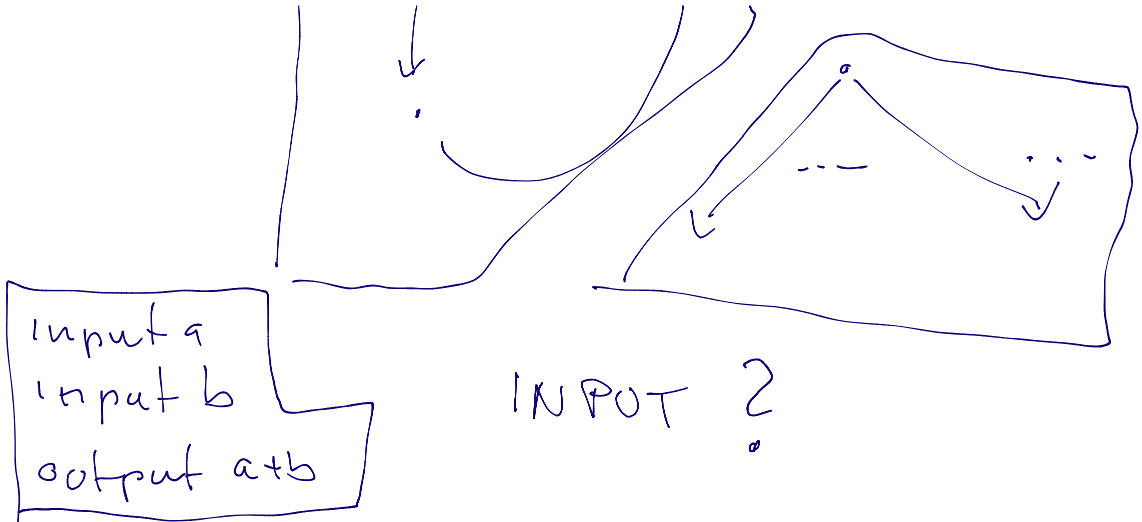
Q finitely branching

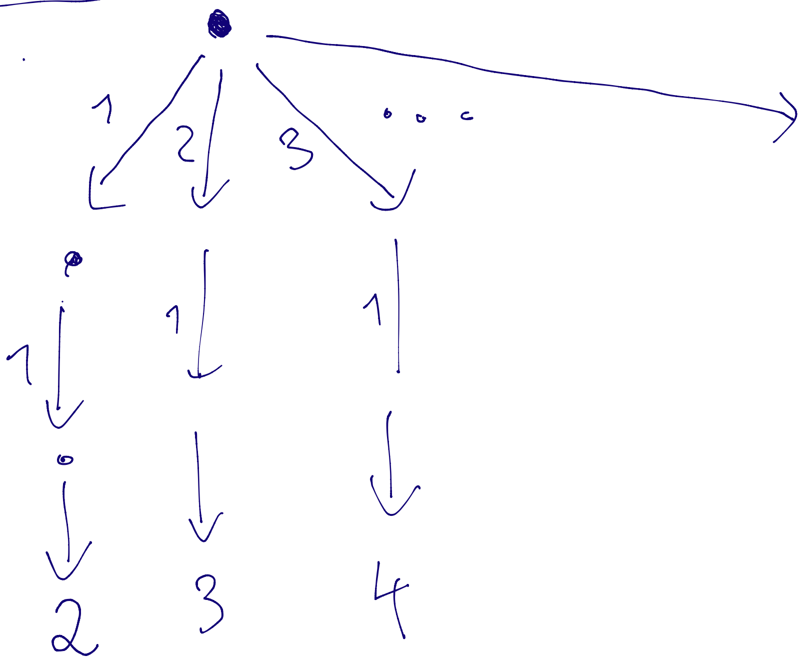
$\boxed{\text{all } a \in A \text{ there are only finitely } b \text{ such that } a \to b}$





loop





Claim: terminat. & fin. br. $\Rightarrow$ $\exists$ meas. fun.

Assume: $(A, \to)$ is termination
$(A, \to)$ is fin. br.

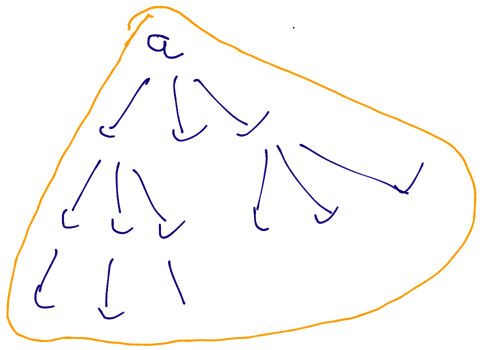
Show: $\exists$ measure function
$\varphi : A \to \mathbb{N}$

$|S|$

The number of elements of S

here an idea is needed

$$\varphi(a) \stackrel{def}{=} |\{b \in A \mid a \xrightarrow{*} b\}|$$



Show: φ is a measure function

Choose any a, b

$$a \to b \Rightarrow \varphi(a) > \varphi(b)$$

Assume $a \to b$

Show $\varphi(a) > \varphi(b)$

$$|\{c \mid a \xrightarrow{*} c\}| > |\{d \mid b \xrightarrow{*} d\}|$$

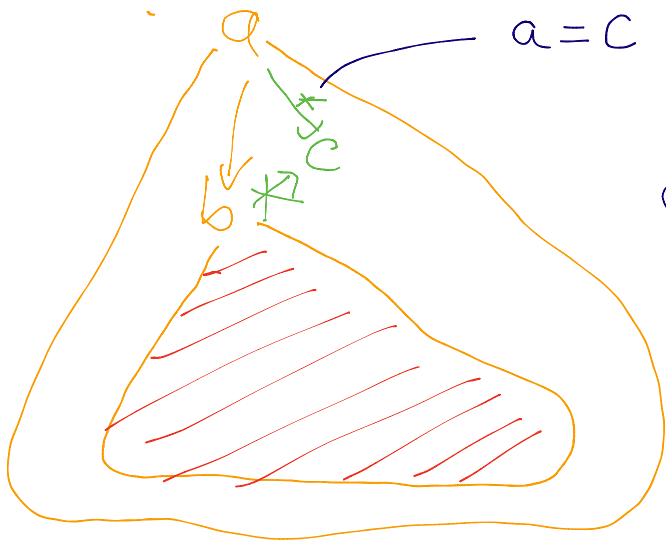
(1) $\{c \mid a \xrightarrow{*} c\} \supseteq \{d \mid b \xrightarrow{*} d\}$

(2) $\exists c$ st. $a \xrightarrow{*} c$, $b \not\xrightarrow{*} c$

Show(1) Use def $\xrightarrow{*}$

refl & $\boxed{\text{transitive closure}}$

of $\longrightarrow$



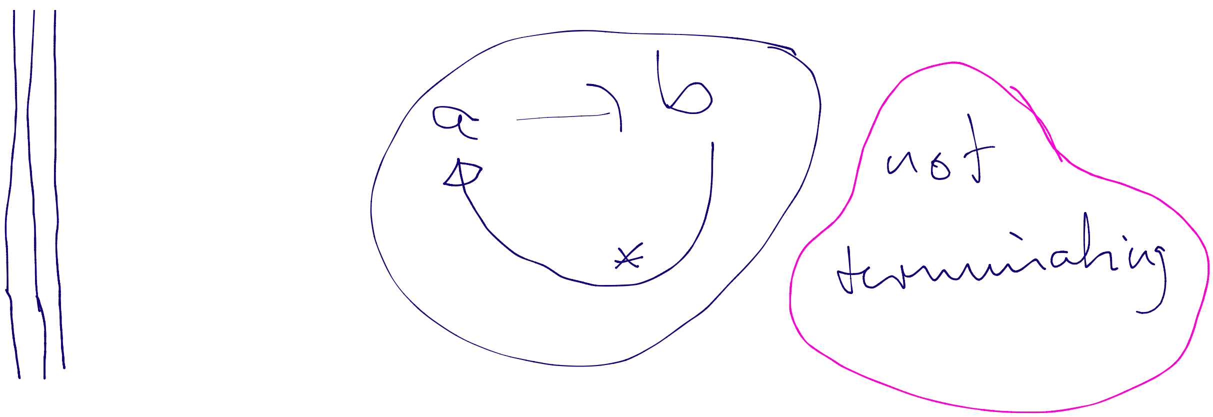
QED for (1)

Show(2) Idea needed

what could be c?

$c = a$

$$\boxed{\text{remains: } b \xrightarrow{*} a}$$



Useful observation

$$\longrightarrow \;\subseteq\; \xrightarrow{*}$$

General remark on proofs.
Any proof of

$$A \Longrightarrow B$$

has the structure

Assume A

⋮

B

Any proof of

not A

has the structure of

Assume A

⋮

contradiction