Def of Invariant

$$P: A \longrightarrow \text{Bool}$$

$$P: A \longrightarrow \mathbb{N}$$

If $w \longrightarrow v$

then $\boxed{P(w) = P(v)}$

Exle: 1) $\text{Odd}(w) : A \to \text{Bool}$

2) $\text{Even}(w) : A \to \text{Bool}$

3) $A \longrightarrow \mathbb{N}$

$w \mapsto \#a \bmod 2$

Assume: $aaa \longrightarrow$ only this rule

Then: $\#a \bmod 3$ is an invariant

For instance: $\underbrace{aaaaabb}$ $\longrightarrow$ $\underbrace{aabb}$

—

$\# a \bmod 3$ $\# a \bmod 3$

2 2

does not change

therefore $\# a \bmod 3$ is an invariant

Back to the exle from the lecture on invariants

aa →

b →

first delete b's then delete aa's

Q: In what sense are Even / Odd the "only" invariants?

Even is a $\boxed{\text{complete}}$ invariant

Invariants allow you to make <u>distinctions</u>

How do we know that

abaaababba $\cancel{\overset{*}{\longleftrightarrow}}$ baabaabba?

1) only works if normal forms are unique

1) reduce to different normal forms

2) use the invariant

ab → ba
ba → ab } # a is different (6a left, 5a right)

what if we add

aa →
b →

a mod 2 is 0 left

a mod 2 is 1 right

Intuitively: An invariant is complete if it allows you to make all distinctions.

Exle: In our example

* Even
* Odd
* # a mod 2

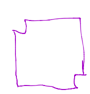
} each of them is a complete invariant

If I ask to find all invariants
I am looking for one
complete invariant

<u>Def</u> An invariant I is
<u>complete</u> if

$$w \overset{*}{\longleftrightarrow} v \iff I(w) = I(v)$$

<u>equivalence</u> | <u>equality</u>
<u>computation</u> $\longrightarrow$ | <u>no computation</u>
| only math

Recall the mutilated chess board  :

algorithms for tiling the plane vs # black tiled = # white tiled

One invariant

vs

A <u>set</u> of invariants ?

We can always combine invariants into one.

The generic way of doing this is :

Assume we have:

$$\frac{\begin{array}{l} I_1 : A \to \mathbb{N} \\ I_2 : A \to Bool \end{array}}{}$$

Combine:

$$\begin{array}{l} A \to \mathbb{N} \times Bool \\ a \mapsto (I_1(a), I_2(a)) \end{array}$$

How can we know that

Even

is complete

for $ab \to ba$

$ba \to ab$

$aa \to$

$b \to$

?

$w \overset{*}{\longleftrightarrow} v \iff Even(w) = Even(v)$

$\Rightarrow$ ✓ (Invariant)

the same

$\boxed{\text{If } Even(w) \text{ and not } Even(v) \text{ then not } w \overset{*}{\longleftrightarrow} v.}$

" $\Leftarrow$ "

$\boxed{\text{If } Even(w) = Even(v) \text{ then } w \overset{*}{\longleftrightarrow} v}$

Proof:

eliminate all b

eliminate all aa

$$
\begin{tikzcd}
aababa \arrow[d] & & aabbb \arrow[d] \\
aaaa \arrow[dr] & & aa \arrow[dl] \\
& {[\ ]} &
\end{tikzcd}
$$

[ ] empty string

This proves $w \overset{*}{\Leftrightarrow} v$ if w, v have even #a.

A similar argument works in case w, v have odd #a:

$$
\begin{tikzcd}
aaaaba \arrow[d] & abaa \arrow[d] \\
aaaaa & aaa
\end{tikzcd}
$$

$$\searrow a \swarrow$$

$$P : A \longrightarrow Bool$$

Def 1

I added a section on weak invariants to the lecture notes.

$$w \longrightarrow v \implies P(w) = true \implies P(v) = true$$

Def 2

We may call this strong invariants

$$w \longrightarrow v \implies P(w) = P(v)$$

The orange highlights how

s

$$P(w) = true \iff P(v) = true$$

The question of whether Even on its own is a complete invariant depends on whether we use

Def 1 or Def 2

$aaabb \to$

$aaab \to$

$bab \to$

$\to ab$

$\to a$

$aa \to b$

$bb \to b$

$ab \to a$

$ab \to ab$

$ba \to ba$

aa aaa bb $\xrightarrow{*}$ replace all aa by b

abbbb $\xrightarrow{*}$ the a eats all b

a

---

1) only even a's result b

2) only b's result b

3) odd a's + some b's result a

4) even a's + some b's