IMPERIAL

Integrated Mathematical Modelling of Image Analysis, Deep Learning, and Data Analysis Using Category Theory Tools

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Introduction: Category Theory

Originally formulated in 1945 by Samuel Eilenberg and Saunders Mac Lane, category theory is a branch of mathematics focused on the study of structures and the relationships between them. Its foundation rests on two undefined elements—objects and morphisms—typically visualized as nodes and arrows. These elements are intentionally abstract, allowing their interpretation to shift with context: they may denote biological systems and their interactions, or purely theoretical constructs. This flexibility makes category theory a powerful tool for unifying disparate domains. In the present work, we draw on both physical and abstract interpretations to explore the underlying architecture of complex biomedical phenomena.

To build structure upon the foundational elements of category theory, two fundamental operations are introduced: the domain and the codomain. These assign to each morphism f12 an associated pair of objects, such that the domain of f12 is object ob1, and the codomain of f12 is object ob2. Conceptually, this can be visualized as a directed arrow originating from the domain object and terminating at the codomain object.

$$\begin{tikzcd} ob1 \arrow[d, "f12"] \\ ob2 \end{tikzcd}$$

Two additional operations are postulated: identity and composition. The identity operation assigns to each object an identity morphism, denoted as idob1 = 1ob: ob1 → ob1 representing a neutral transformation of the object onto itself. The composition operation combines two morphisms f12 and f23 to produce a new morphism f13, defined when the domain of f23 matches the codomain of f12, such that f23 ∘ f12 : dom(f12) → cod(f23). These operations obey two fundamental laws: Unity law: 1ob2 ∘ f12 = f12 and f23 ∘ 1ob2 = f23 and Associativity law: f12 ∘ (f23 ∘ f34) = (f12 ∘ f23) ∘ f34

$$\begin{tikzcd} ob1 \arrow[r, "f13"] \arrow[d, "f12"'] & ob3 \arrow[d, "f34"] \\ ob2 \arrow[loop left, "1ob2"] \arrow[ur, "f23"] \arrow[r] & ob4 \end{tikzcd}$$

We can introduce a functor F as a morphism between categories. For categories Cat1 and Cat2, a functor consists of two components: an object function and a morphism function. The object function assigns to each object in Cat1 an object in Cat2: F(ob1) in Cat2. The morphism function assigns to each morphism in Cat1 a morphism in Cat2: F(f12) : F(ob1) → F(ob2). A functor must satisfy two key conditions: Identity preservation: F(1ob1) = 1Fob1 and Composition preservation: F(f23 ∘ f12) = F(f23) ∘ F(f12). These conditions ensure that the functor respects the categorical structure, preserving the way objects and morphisms are related in the source category.

$$\begin{tikzcd} \text{Cat1} & \text{Cat2} \\ ob1 \arrow[loop left, "1ob1"] \arrow[r, "F"] \arrow[d, "f12"'] & Fob1 \arrow[loop right, "1Fob1"] \arrow[d, "Ff12"] \\ ob2 \arrow[r] & Fob2 \end{tikzcd}$$

We can also introduce a morphism between functor categories Cat2 and Cat3, called a natural transformation.

$$\begin{tikzcd} \text{Cat1} & \text{Cat2} & \text{Cat3} \\ ob1 \arrow[d, "f12"'] & Fob1 \arrow[r, "t", "tob1"'] \arrow[d, "Ff12"'] & Gob1 \arrow[d, "Gf12"] \\ ob2 & Fob2 \arrow[r, "tob2"'] & Gob2 \end{tikzcd}$$

Results: Imaging in Category Theory

We propose a categorical formulation of the imaging process, grounded in the foundational principles of category theory. In this framework, objects are treated as atomic and indivisible—they are not penetrable or decomposable. All structural properties and interactions are represented exclusively by morphisms. As such, the internal composition of an object is not accessible; all knowledge about an object arises from the morphisms associated with it. This perspective echoes the metaphysical concept of monads introduced by Leibniz in Monadology, wherein monads are entirely self-contained entities. Leibniz's view is even more restrictive: monads do not interact at all, yet their internal processes maintain perfect coherence in space and time across the system. In our model, a physical object is represented by an object in a category. In line with Leibnizian philosophy, we focus solely on endomorphisms—morphisms from an object to itself—denoted ef11. Given the potentially infinite range of properties and internal relations, we index these morphisms as ef11_1, ef11_2, and so on. To account for dynamic changes in these endomorphisms, we introduce endofunctors, labeled eF, structure-preserving mappings from a category to itself, that model the evolving internal states of the object. This approach offers a principled and abstract language for representing physical systems in imaging without invoking direct access to their internal structure. We define this structure as the Concrete Object1 Category, or ConObj1Cat.

$$\text{ConObj1Cat} \qquad \begin{tikzcd} ob1 \arrow[loop left, "ef11\_1"] \arrow[loop above, "ef11\_2"] \end{tikzcd} \qquad \begin{tikzcd} \text{ConObj1Cat} \arrow[loop left, "eF11\_1"] \arrow[loop above, "eF11\_2"] \end{tikzcd}$$

In addition, we postulate that any concrete object under study possesses at least two distinct endomorphisms: a foreground process, representing the biological or molecular activity of primary interest for imaging and analysis, and a background process, which may contribute to signal or noise but lies outside the scope of direct investigation. Each of these processes is associated with a corresponding endofunctor acting on the relevant endomorphisms, preserving the internal structure of the system. We formalize our framework within category theory by embedding our categories of concrete objects into the category CAT—the category of all (small) categories—which is cartesian closed. This allows us to define not only endomorphisms and functors between categories but also categorical products, enabling formal representation of composite systems via the product of two categories.

$$\text{CAT}$$
$$\begin{tikzcd} \text{ConObj1Cat} \arrow[loop left, "ef11\_1"] \arrow[dr, "F"'] & & \text{ConObj2Cat} \arrow[loop left, "ef22\_1"] \arrow[dl, "F"] \\ & \text{ConObj1Cat X ConObj2Cat} \arrow[loop left, "ef1x2\_1"] & \end{tikzcd}$$

We postulate that imaging can be modeled as a functor from a category of concrete objects—or a product of such categories—to an indexed set category, which we denote as ImageCat. This functor maps each object to its corresponding image and translates endomorphisms—namely, the foreground and background processes—into the associated endomorphisms of the image.

$$\text{CAT}$$
$$\begin{tikzcd} \text{ConObj1Cat} \arrow[loop left, "fp11\_1"] \arrow[loop right, "bp11\_1"] \arrow[d, "F"'] \\ \text{ImageCat} \arrow[loop left, "fpImage\_1"] \arrow[loop right, "bpImage1\_1"] \end{tikzcd}$$

Image Analysis in Category Theory

Image analysis can be formally represented as a structured sequence of categorical transformations, modeled via a chain of categories and functors. The process begins with the category ConObj1Cat, which captures concrete objects under observation. A functor maps this category to ImageCat, describing indexed sets corresponding to acquired images. From ImageCat, another functor leads to GroupImageCat, encoding segmented representations of the image or its components. GroupImageCat is then functorially mapped to GeomModelCat, representing the geometric model of the image, which may be understood as the category of regions of interest or other mathematical abstractions of spatial features. GeomModelCat, together with ImageCat, is subsequently mapped to DataCat, which is further transformed into MarkovCat, a category responsible for statistical analysis and inference over the data.

$$\begin{tikzcd} \text{Con Obj1 Cat} \arrow[r] & \text{Image Cat} \arrow[r] & \text{Group Image Cat} \arrow[r] & \text{Geom Model Cat} \arrow[r] & \text{Data Cat} \arrow[r] & \text{Markov Cat} \end{tikzcd}$$

Additionally, DataCat may also receive information from ImageCat.

$$\begin{tikzcd} \text{Con Obj1 Cat} \arrow[r] & \text{Image Cat} \arrow[r] \arrow[rrr, bend right] & \text{Group Image Cat} \arrow[r] & \text{Geom Model Cat} \arrow[r] & \text{Data Cat} \arrow[r] & \text{Markov Cat} \end{tikzcd}$$

The categories introduced here are defined in a general sense, with their precise instantiation depending on the specific objectives of the image analysis. For instance, ImageCat may take the form of VectorCat, MatrixCat, or GraphCat; similarly, GroupImageCat could be realized as VectorCat, MatrixCat, GraphCat, or even GroupCat when algebraic symmetries are relevant. Meanwhile, GeomModelCat may be instantiated as a Topos category for sheaf-theoretic interpretations or as a Manifold category when smooth geometric structures are required. GroupImageCat may also be instantiated by a chain of AI-related categories—particularly neural network architectures—that perform segmentation and classification

While the proposed model may resemble a conventional image analysis pipeline in its sequential structure, category theory provides a deeper formalism for evaluating and comparing distinct analytical pathways. Crucially, it enables the identification of transformations between such pathways that preserve essential outcomes—allowing us to recognize when different computational strategies are equivalent up to isomorphism.

$$\begin{tikzcd} \text{Con Obj1 Cat} \arrow[r] & \text{Image Cat} \arrow[r, bend left, ""{name=A}] \arrow[r, ""'{name=B}] \arrow[from=B, to=A, Rightarrow, "t1"] & \text{Group Image Cat} \arrow[r, bend left, ""{name=C}] \arrow[r, ""'{name=D}] \arrow[from=D, to=C, Rightarrow, "t2"] & \text{Geom Model Cat} \arrow[r] & \text{Data Cat} \arrow[r] & \text{Markov Cat} \end{tikzcd}$$

Conclusion

We have presented a mathematically and methodologically consistent framework for modeling imaging processes across complex physical and biological systems, grounded in the formal apparatus of category theory. This approach enables the integration of advanced mathematical structures and physical models into imaging pipelines in a composable and principled way. Importantly, the proposed formalism supports practical implementation through modern functional programming paradigms, including Haskell, functional Python, and JAX—allowing direct application to AI-driven imaging workflows. The resulting model offers a coherent foundation for both algorithm cataloguing and educational use, promoting transparency, reproducibility, and theoretical depth in imaging science.