IMPERIAL

Integrated Mathematical Modelling of Image Analysis, Deep Learning, and Data Analysis Using Category Theory Tools

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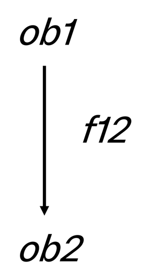
Introduction: Category Theory

Core Concepts

- Origin: Developed in 1945 by Eilenberg and Mac Lane to study structures and relationships.
- Basic Elements:
  - Objects (nodes)
  - Morphisms (arrows between objects)
  - Abstract and context-adaptable (e.g., biology, theory)
- Utility: Unifies concepts across domains; used here to model biomedical complexity.

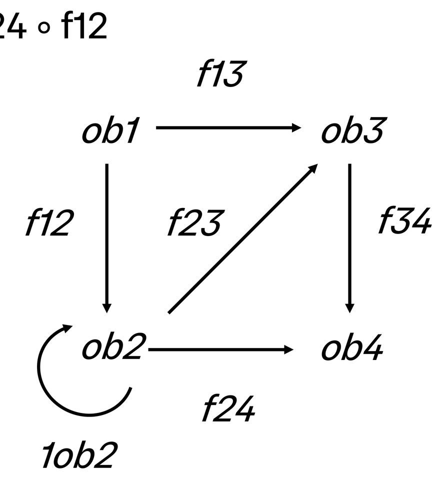


Foundational Operations

- Domain/Codomain: Each morphism f12 maps from:
  - Domain: object ob1
  - Codomain: object ob2
- Identity:
  - Each object has an identity morphism: 1ob : ob → ob
- Composition:
  - Combines two morphisms: f23 ∘ f12 : dom(f12) → cod(f23)
  - Defined when codomain of f12 = domain of f2

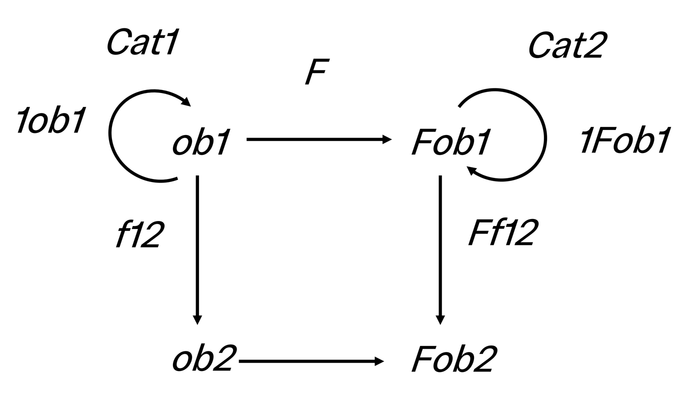
Laws and Diagrams

- Unity Law:
  1ob2 ∘ f12 = f12
  f23 ∘ 1ob2 = f23
- Associativity Law:
  f12 ∘ (f23 ∘ f34) = (f12 ∘ f23) ∘ f34
- Commutativity:
  f34 ∘ f13 = f24 ∘ f12



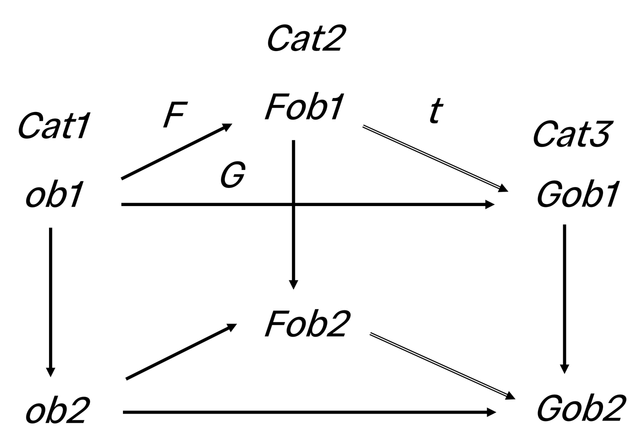
Functors

- Definition: A mapping between categories (e.g., Cat1 → Cat2)
- Components:
  - Object function: F(ob1) in Cat2
  - Morphism function: F(f12) : F(ob1) → F(ob2)
- Functor Laws:
  - Identity: F(1ob1) = 1Fob1
  - Composition: F(f23 ∘ f12) = F(f23) ∘ F(f12)



Natural Transformation

- A morphism between functors (e.g., between functor categories Cat2 and Cat3)



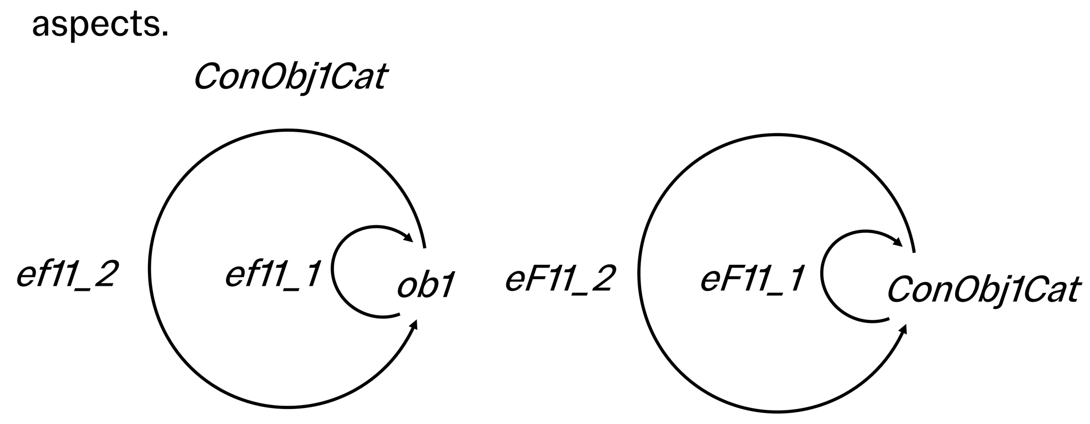
Results: Imaging in Category Theory

Categorical Imaging Framework

- Objects are atomic and indivisible; internal structure is not accessible.
- All information arises from morphisms between or within objects.
- Echoes Leibniz's monads: self-contained entities; internal processes remain coherent without direct interaction.
- A physical object is modeled as an object in a category, focusing on endomorphisms (e.g., ef11_1, ef11_2).

Dynamic Structure via Endofunctors

- Endofunctors (eF) represent internal evolution: mappings from a category to itself preserving structure.
- Endomorphisms are indexed to account for multiple internal aspects.

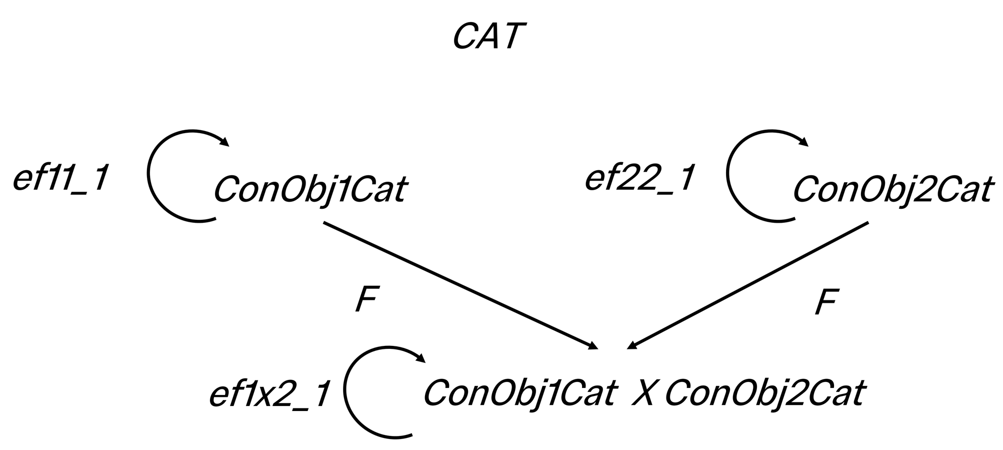


Foreground & Background Processes

- Every object has at least two key endomorphisms:
  - Foreground: represents the biological/molecular process of interest.
  - Background: represents confounding or peripheral processes.
- Each process is associated with an endofunctor preserving its structure.

Formal Structure & Embedding

- Define a category of concrete objects: ConObjCat.
- Embed this category into CAT (the category of all small categories), which is cartesian closed.
- Enables use of:
  - Endomorphisms
  - Functors
  - Categorical products (for representing composite systems)



Imaging as a Functor

- Imaging process modeled as a functor:
  - From: a category (or product of categories) of concrete objects
  - To: an indexed set category called ImageCat
- This functor:
  - Maps each object to its image
  - Translates endomorphisms (foreground/background) into corresponding image processes

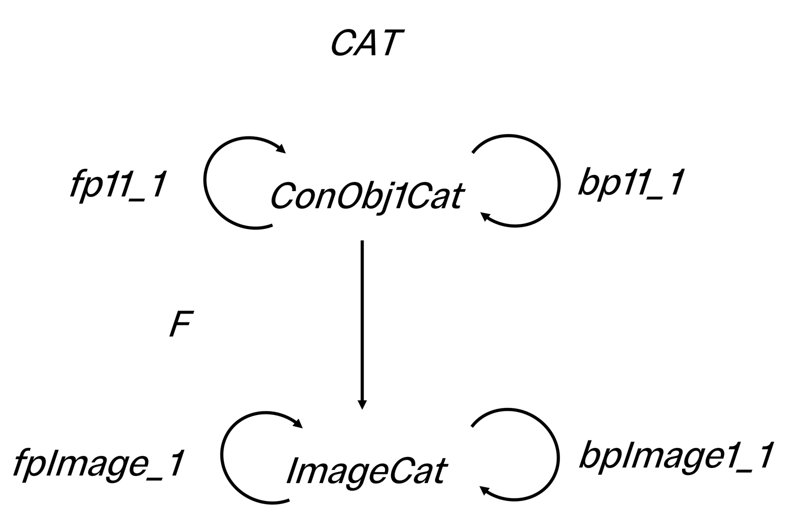
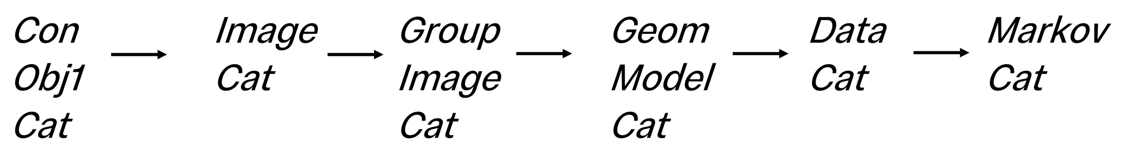


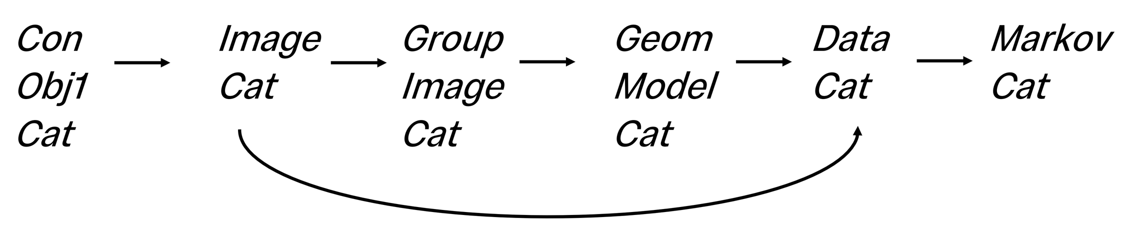
Image Analysis in Category Theory

Categorical Image Analysis Pipeline

- ConObj1Cat: Category of concrete objects under observation.
- ImageCat: Functor maps ConObj1Cat to ImageCat (indexed sets of images).
- GroupImageCat: Functor maps ImageCat to segmented representations.
- GeomModelCat: Functor maps GroupImageCat to geometric/spatial models.
- DataCat: Receives functorial input from both GeomModelCat
- MarkovCat: Statistical inference is modeled by a functor from DataCat.



- DataCat may also receive input directly from ImageCat.

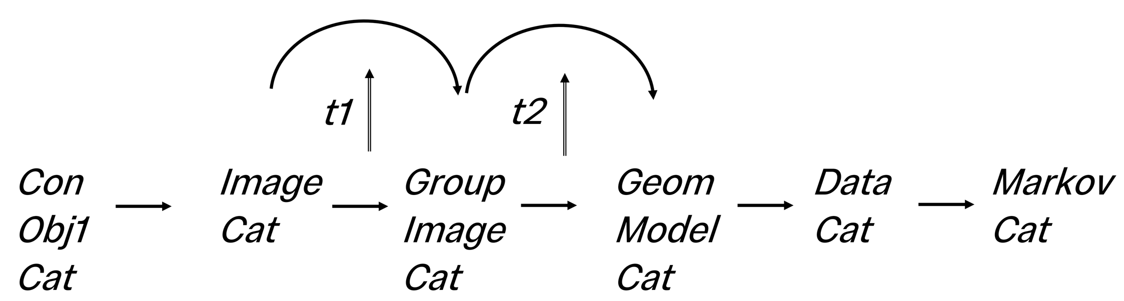


Flexible Category Instantiations

- ImageCat: May instantiate as VectorCat, MatrixCat, or GraphCat.
- GroupImageCat: Can be VectorCat, MatrixCat, GraphCat, GroupCat, or AI-related categories (e.g., neural networks).
- GeomModelCat: Can be ToposCat (sheaf theory) or ManifoldCat (geometry).

Theoretical Advantages

- Category theory provides a rigorous structure to:
  - Represent diverse analysis steps formally.
  - Enable comparison of distinct pipelines.
  - Identify isomorphic transformations between strategies—revealing functional equivalence despite differing implementations.



Conclusion

- Introduced a consistent, category-theory-based framework for modeling imaging in complex physical and biological systems.
- Enables structured integration of mathematical models into imaging pipelines.
- Supports implementation via functional programming (e.g., Haskell, functional Python, JAX).
- Suitable for AI-driven workflows in imaging.
- Provides a solid foundation for:
  - Algorithm cataloguing
  - Educational applications
- Promotes transparency, reproducibility, and theoretical clarity in imaging science. Represent diverse analysis steps formally.