Set-Theoretical Foundation of Imaging in Microscopy

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Abstract

In recent decades, we have observed the flourishing of different microscopic techniques, e.g. super-resolution microscopy and light-sheet microscopy which generate a substantial amount of data. An increasing number of sophisticated mathematical approaches are applied for image processing and data extraction. We propose to build a logically consistent link between imaging and mathematics formalising imaging in microscopy. To achieve this we use the Morse–Kelley (MK) set theory, which was first proposed by Kelley (Kelley, 1955), and construct different mathematical structures over sets, e.g. Von Neumann universes. At the end, we define an image as an indexed sets of physical and mathematical objects supplemented with mathematical structures. In addition, to demonstrate the power of the approach, we link biological objects and their hierarchy with Von Neumann universes.

Introduction: Mathematical Preliminaries

Morse–Kelley Set Theory

Any formal system starts from primitives – self-evident undefined terms (Tarski, 1946). In the Morse–Kelley set theory, it is class. Informally, we define class as any collection of objects which is free from logical paradoxes. Sets are defined as classes which are members of other classes. Equality is always used in a sense of logical identity (Kelley, 1955). Set, $\{s_1, s_2, \dots, s_n\}$, is depicted as s:

$$s = \{s_1, s_2, \dots, s_n\},$$

where s_i is a member of a set. We emphasize that a set is a collection of unique terms (Cantor, 1895). In the MK theory, it is possible to perform all the usual set-theoretical operations with sets, e.g. union, intersection. The symbol $\coloneqq$ is used to introduce definition. We define an ordered pair the same way as Kuratowski (Kuratowski, 1921):

$$(s_1, s_2) \coloneqq \{\{s_1\}, \{s_1, s_2\}\}.$$

The MK theory is a proper extension of Zermelo–Fraenkel with choice (ZFC) set theory and has nine axioms including the axiom of choice. The class which has all possible sets of discourse as members is called a universal class, which is von Neumann universe.

Von Neumann Universe

Let us define ranks over the finite set, s, as:

$$r_0 \coloneqq \{s_1, \dots, s_n\},$$

$$r_{\alpha+1} \coloneqq \wp(r_\alpha),$$

where α is an ordinal number and $\wp(\dots)$ is a powerset, which is a set of all subsets of the previous rank.

On top of ranks we define universes as the union of the indexed ranks:

$$U_\alpha(s) \coloneqq \bigcup_\alpha r_\alpha,$$

and the Von Neumann universe as:

$$U(s) \coloneqq \bigcup_{\alpha \to \infty} r_\alpha \,.$$

Von Neumann universe is usual model for set theories, e.g. ZFC and MK.

Image Formation

Inductive Example from Microscopy

In microscopy we usually have rational numbers after imaging; it is tempting to represent the result of imaging as a set. Suppose that we work with a one-pixel two-bit camera, so we have:

$$\{0\} \text{ or } \{1\},$$

which does not have any problems. If we have a two-pixel two-bit camera, we can obtain 4 different results:

$$\{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}.$$

For {0,0}, two members of the set are not unique, and the set in this case is equivalent to {0}:

$$\{0,0\} = \{0\}.$$

The same happens with {1,1}:

$$\{1,1\} = \{1\};$$

{0, 1} and {1, 0} are equivalent as well:

$$\{0,1\} = \{0,1\}.$$

As a result, we have only 3 sets:

$$\{0\}, \{0,1\}, \{1\},$$

and we lose information.

In order to improve the situation, we apply the axiom of choice and index each member of the set by members of the indexing set, e.g. natural numbers, as an ordered pair. For the two-pixel two-bit camera we define a pixel as:

$$p_1 \coloneqq (1, b) \text{ and } p_2 \coloneqq (2, b),$$

where *b* (result of imaging) can be 0 or 1. Now, we can keep information about the first pixel and the second pixel, and the imaging can be represented as set:

$$\{p_1, p_2\}.$$

The approach can be extended to the case of any finite camera with an arbitrary number of pixels and bits.

In biology, we have a naturally developed hierarchy: population -> organism -> organ -> tissue -> cell -> organelle -> molecule. It can be represented as set of indexed members:

$$population_1 = \{organism_1, \dots, organism_k\}$$

…

$$cell_1 = \{orgnelle_1, \dots, organelle_l\}, and$$

$$organelle_1 = \{molecule_1, \dots, molecule_m\},$$

where *k, l, m* are natural numbers.

Segmenting our images, we associate certain pixels with biological objects that are members of the bellowed mentioned hierarchy. It will be an advantage to add to our images not only information about selection and identification, but also about the hierarchy. The Von Neumann universe is the right mathematical tool to develop this. From the set of pixels, we create ranks following our definition from the introduction:

$$r_0 \coloneqq \{p_1, p_2\},$$

$$r_1 \coloneqq \{\emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\},$$

$$r_2 \coloneqq \left\{\{\emptyset\}, \{\{p_1\}\}, \{\{p_2\}\}, \{\{p_1, p_2\}\}, \{\emptyset, \{p_1\}\}, \dots\right\},$$

where Ø is an empty set, {}. On top of the ranks, we construct universes:

$$U_0 \coloneqq r_0 = \{p_1, p_2\},$$

$$U_1 \coloneqq r_0 \bigcup r_1 = \{p_1, p_2, \emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\},$$

$$U_2 \coloneqq r_0 \bigcup r_1 \bigcup r_2 = \{p_1, p_2, \emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}, \{\emptyset\}, \{\{p_1\}\}, \{\{p_2\}\}, \{\emptyset, \{p_1\}\}, \dots\}.$$

The number of members of r_α is calculated:

$$\| r_{\alpha+1} \| = 2^{\|r_\alpha\|} \ or$$

$$\| r_{\alpha+1} \| = 2^{\cdot^{\cdot^{(2^{(2^{(2^n)})})}}} \ \alpha \text{ times},$$

where n is the number of members of the set. For the two-pixel camera we obtain: $\|r_0\|=2$, $\|r_1\|=2^2=4$, $\|r_1\|=2^4=16$; for 64-pixel camera - $\|r_0\|=64$, $\|r_1\|=2^{64}=1.8*10^{19}$; for a standard, 512x512-pixel camera - $\|r_0\|=262144$, $\|r_1\|=2^{262144}$. We can notice that the number of possible combinations of pixels in ranks increases very rapidly and the richness of universes is even larger.

Suppose we have the following cellular components (objects) and associated biological hierarchy:

$$organ_1 = \{tissue_1\},$$

$$tissue_1 = \{cell_1\},$$

$$cell_1 = \{cytoplasm_1, microtubule_1\}.$$

We can pick up members of the Von Neumann universe of pixels and associate them with individual biological objects forming ordered pairs:

$$(cytoplasm_1, \{p_1\}).$$

As for certain biological structures it is important to add directionality; we can add this to pixels by pairing:

$$(microtubule_1, (p_1, p_2)).$$

We can transfer biological hierarchy and organisation to pixels:

$$(cell_1, \{\{p_1, p_2\}\}) \text{ and } \left(organ_1, \left\{\left\{\{\{p_1, p_2\}\}\right\}\right\}\right).$$

To finish, it is possible to combine all formed structures together:

$$\left\{\left(organ_1, \left\{\left\{\{\{p_1, p_2\}\}\right\}\right\}\right), (cell_1, \{\{p_1, p_2\}\}), (cytoplasm_1, \{p_1\}), \left(microtubule_1, (p_1, p_2)\right)\right\}.$$

The richness of the Von Neumann universe allows us to create very complex descriptions of biological reality and relationships between components imaged by microscopes or any other imaging system. We should add that instead of pixels we can have any usual mathematical objects, such as vectors, tensors (tables), and geometric figures.

General Case of Imaging

Let us develop more a general formalism. In our discourse we assume that physical objects, mathematical objects, and imaging are our primitive terms. We postulate that the physical object of interest, which can be imaged, is represented as a set and all such objects belong to class, PO^{Class}. We also add indexing class, IND^{Class}, and a class of mathematical objects, MO^{Class}. Usually IND^{Class} includes natural numbers, N, and MO^{Class} - rational numbers, Q. It is possible to add a finite number of classes as our thinking progresses. It is possible that sets can belong simultaneously to different classes, e.g. N.

We can select members of class, MO^{Class}, form a set, *mo*, and index them pairing with members of indexing class and forming a new set:

$$mo^i.$$

We construct the Von Neumann universe from mo^i and form an ordered pair with the original indexed set:

$$(mo^i, U(mo^i)).$$

Based on the axiom of choice, we select structures from the Von Neumann universe which are useful for us and construct a new set, $U^*(mo^i)$, and add to the ordered pair to form a triple:

$$T(mo^i) := (mo^i, U(mo^i), U^*(mo^i)).$$

In a similar manner, we form a triple for the physical objects:

$$T(po^i) := (po^i, U(po^i), U^*(po^i)).$$

We can merge them together:

$$(T(po^i), T(mo^i)).$$

We unify two selected sets and define:

$$U^*(po^i, mo^i) := U^*(po^i) \bigcup U^*(mo^i).$$

We repeat the procedure of formation of the Von Neumann universe for $U^*(po^i, mo^i)$, and we form the triple:

$$T(U^*(po^i, mo^i)) := \left(U^*(po^i, mo^i), U\left(U^*(po^i, mo^i)\right), U^*\left(U^*(po^i, mo^i)\right)\right).$$

At the end, we form the following structure:

$$I(po^i, mo^i) := (T(po^i), T(mo^i), T\left(U^*(po^i, mo^i)\right))$$

The structure has the capability to describe associations of physical objects with all possible indexed well-founded mathematical objects. We propose to use it as a new definition of image. In case we want to describe quantitative or semi quantitative images, we have to add a structure, *T(co^i)*, of imaged calibration objects co^i from the class of calibrations CO^{Class} and define:

$$U^*(po^i, mo^i, co^i) := U^*(po^i) \bigcup U^*(mo^i) \bigcup U^*(co^i).$$

Finally, we have:

$$QI(po^i, mo^i, co^i) := \left(T(po^i), T(mo^i), T(co^i), T\left(U^*(po^i, mo^i, co^i)\right)\right).$$

It is possible also to add triple for probability and statistical modelling; we skip all steps in this case as it is built analogically.

We can substantially simplify our constructions and structures combining into one set po^i and mo^i:

$$com^i := po^i \bigcup mo^i.$$

After that we can form a triple:

$$T(com^i) \coloneqq (com^i, U(com^i), U^*(com^i)).$$

All information which we can store in $I(po^i, mo^i)$ can also be stored in $T(com^i)$. Nevertheless, from an epistemological point of view, we prefer $I(po^i, mo^i)$.

Conclusion and Future Direction

We proposed to define image as an indexed set of physical and mathematical objects supplemented with mathematical structures. Such definition can accommodate not only the usual epifluorescence images or confocal images, but also super-resolution data and any complex imaging data. We believe that set theory and other branches of foundation of mathematics, e.g. type theory and category theory can be fruitful for digital transformation of imaging and microscopy. It is reasonable to expand the domain of our formalisation to include non-well-formed or cyclic structures, where we can have set equations: s={s}, and try to imbed our structures into a more broad range of logical constructs e.g. as set-theoretic multiverses (Hamkins, 2012) or hyperverses (Arrigoni, 2013). In order to perform usual mathematical operations, we have to imbed our structures (or parts of them) into universes of real numbers. From a practical point of view, we should develop a new imaging format which should include hierarchy of biological objects and hierarchy of pixels.

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